XYmodel

Consider the following Hamiltonian describing a chain of N spin-1/2

$$H = \sum_i J_{xy}(S_i^x S_{i+1}^x + S_i^y S_{i+1}^y) + \sum_i h S_i^z$$

where $S_i^x = \frac{1}{2} X_i, S_i^y = \frac{1}{2} Y_i, S_i^z = \frac{1}{2} Z_i$.
Denote $a_i^\dagger = S_i^x + i S_i^y$ and $a_i = S_i^x - i S_i^y$, then $S_i^x = \frac{1}{2}(a_i^\dagger + a_i)$ and $S_i^y = \frac{1}{2i}(a_i^\dagger - a_i)$.

$$a_i^\dagger a_i = (S_i^x + i S_i^y)(S_i^x - i S_i^y) = \frac{1}{2} + S_i^z$$

$$a_i a_i^\dagger = (S_i^x - i S_i^y)(S_i^x + i S_i^y) = \frac{1}{2} - S_i^z$$

$$\begin{aligned} H &= \sum_i \frac{1}{4} J_{xy}(a_i^\dagger + a_i)(a_{i+1}^\dagger + a_{i+1}) - \frac{1}{4} J_{xy}(a_i^\dagger - a_i)(a_{i+1}^\dagger - a_{i+1}) + \sum_i h a_i^\dagger a_i + \frac{Nh}{2} \\ &= \frac{1}{2} \sum_i J_{xy}(a_i^\dagger a_{i+1} + a_i a_{i+1}^\dagger) + \sum_i h a_i^\dagger a_i + \frac{Nh}{2} \end{aligned}$$

Jordan-Wigner transformation:

$$c_i = (\prod_{j=1}^{i-1} Z_j) a_i, c_i^\dagger = (\prod_{j=1}^{i-1} Z_j) a_i^\dagger$$

We have $a_i^\dagger a_i = c_i^\dagger c_i$ and

$$c_i^\dagger c_{i+1} = a_i^\dagger Z_i a_{i+1} = (S_i^x + i S_i^y) Z_i a_{i+1} = (-i S_i^y - S_i^x) a_{i+1} = -a_i^\dagger a_{i+1}$$

$$c_{i+1}^\dagger c_i = a_{i+1}^\dagger Z_i a_i = a_{i+1}^\dagger Z_i (S_i^x - i S_i^y) = a_{i+1}^\dagger (i S_i^y - S_i^x) = -a_{i+1}^\dagger a_i$$

Therefore $a_i^\dagger a_{i+1} = -c_i^\dagger c_{i+1}$ and $a_{i+1}^\dagger a_i = -c_{i+1}^\dagger c_i$.

$$H = -\frac{1}{2} \sum_i J_{xy}(c_i^\dagger c_{i+1} + c_{i+1}^\dagger c_i) + \sum_i h c_i^\dagger c_i + \frac{Nh}{2}$$

We ignore the constant term $\frac{Nh}{2}$ and denote H in matrix form:

$$H = c^\dagger A c$$

where

$$c = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_N \end{pmatrix}, A = \begin{pmatrix} h & -\frac{1}{2}J_x & 0 & \dots & 0 \\ -\frac{1}{2}J_x & h & -\frac{1}{2}J_x & \dots & 0 \\ 0 & -\frac{1}{2}J_x & h & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & -\frac{1}{2}J_x & h \end{pmatrix}.$$

For periodic boundary condition, we have

$$A = \begin{pmatrix} h & -\frac{1}{2}J_x & 0 & \dots & -\frac{1}{2}J_x \\ -\frac{1}{2}J_x & h & -\frac{1}{2}J_x & \dots & 0 \\ 0 & -\frac{1}{2}J_x & h & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\frac{1}{2}J_x & 0 & 0 & -\frac{1}{2}J_x & h \end{pmatrix}$$

Suppose the eigendecomposition of A is $A = U\Lambda U^\dagger$, then we can transform c into the momentum space $c = U\eta$ and the Hamiltonian becomes

$$H = c^\dagger Ac = \eta^\dagger U^\dagger U\Lambda U^\dagger U\eta = \eta^\dagger\Lambda\eta = \sum_k \Lambda_k \eta_k^\dagger \eta_k$$

Also, we have $\eta = U^\dagger c$ and

$$\eta_k = \sum_j U_{jk}^* c_j, \eta_k^\dagger = \sum_j U_{jk} c_j^\dagger$$

Floquet Hamiltonian

We apply a periodic magnetic field in the x direction on the first spin, the Hamiltonian of this magnetic field is

$$H_1 = \Omega\cos(\omega t)S_1^x = \frac{1}{2}\Omega(\cos(\omega t)S_1^x + \sin(\omega t)S_1^y) + \frac{1}{2}\Omega(\cos(\omega t)S_1^x - \sin(\omega t)S_1^y)$$

According to the rotating-wave approximation, we can omit the second term. The Hamiltonian becomes

$$\begin{aligned} H_1 &= \frac{1}{2}\Omega(\cos(\omega t)S_1^x + \sin(\omega t)S_1^y) \\ &= \frac{1}{4}\Omega(\cos(\omega t)(c_1^\dagger + c_1) + \frac{1}{i}\sin(\omega t)(c_1^\dagger - c_1)) \\ &= \frac{1}{4}\Omega\cos(\omega t)\sum_k (U_{1k}^*\eta_k^\dagger + U_{1k}\eta_k) + \frac{1}{4i}\Omega\sin(\omega t)\sum_k (U_{1k}^*\eta_k^\dagger - U_{1k}\eta_k) \\ &= \frac{1}{4}\Omega\cos(\omega t)\sum_k ((r_k - is_k)\eta_k^\dagger + (r_k + is_k)\eta_k) + \frac{1}{4i}\Omega\sin(\omega t)\sum_k ((r_k - is_k)\eta_k^\dagger - (r_k + is_k)\eta_k) \\ &= \frac{1}{4}\Omega\sum_k (r_k\cos(\omega t) - s_k\sin(\omega t))(\eta_k^\dagger + \eta_k) + \frac{1}{4i}\Omega\sum_k (r_k\sin(\omega t) + s_k\cos(\omega t))(\eta_k^\dagger - \eta_k) \end{aligned}$$

where r_k and s_k are the real and imaginary parts of U_{1k}.

We focus on a single mode k, regard the subsystem as a two-level system. Then $(\eta_k^\dagger + \eta_k)$ is the X operator and $i(\eta_k^\dagger - \eta_k)$ is the Y operator. The Hamiltonian becomes

$$\begin{aligned} H_{k1} &= \frac{1}{4}\Omega(r_k \cos(\omega t) - s_k \sin(\omega t))X - \frac{1}{4}\Omega(r_k \sin(\omega t) + s_k \cos(\omega t))Y \\ &= \frac{1}{4}\Omega\sqrt{r_k^2 + s_k^2}\cos(\omega t + \phi_k)X - \frac{1}{4}\Omega\sqrt{r_k^2 + s_k^2}\sin(\omega t + \phi_k)Y \end{aligned}$$

where $\cos(\phi_k) = r_k/\sqrt{r_k^2 + s_k^2}$ and $\sin(\phi_k) = s_k/\sqrt{r_k^2 + s_k^2}$.
And $H_k = \Lambda_k \eta_k^\dagger \eta_k = \frac{1}{2}\Lambda_k Z$ (ingore the constant term $\frac{1}{2}\Lambda_k$).
Toy model of two subsystems with an anti-commute term in the Hamiltonian:

$$H = \Delta_1 Z_1 + \Omega_1 X_1 + \Delta_2 Z_2 + \Omega_2 Z_1 X_2$$

Rotation frame(Nilsen and Chuang, 7.7.2)

We can denote the two level system hamiltonian $H_k + H_{k1}$ by

$$H_{\text{two level}} = \frac{\omega_0}{2} Z + g(X\cos(\omega t) - Y\sin(\omega t))$$

where

$$\omega_0 = \Lambda_k, g = \frac{1}{4}\Omega\sqrt{r_k^2 + s_k^2}.$$

The state of the system is $|\chi(t)\rangle$ and define the state in the rotation frame as $|\varphi(t)\rangle = e^{-i\omega t Z/2}|\chi(t)\rangle$, such that the Schrödinger equation

$$i\partial_t|\chi(t)\rangle = H_{\text{two level}}|\chi(t)\rangle$$

can be re-expressed as

$$i\partial_t|\varphi(t)\rangle = \left[e^{-i\omega Zt/2} H_{\text{two level}} e^{i\omega Zt/2} + \frac{\omega}{2} Z\right]|\varphi(t)\rangle.$$

Since

$$e^{-i\omega Zt/2} X e^{i\omega Zt/2} = (X\cos\omega t + Y\sin\omega t)$$

and

$$e^{-i\omega Zt/2} Y e^{i\omega Zt/2} = (-X\sin\omega t + Y\cos\omega t)$$

we have

$$i\partial_t|\varphi(t)\rangle = \left[\frac{\omega_0 + \omega}{2} Z + gX\right]|\varphi(t)\rangle,$$

where the terms on the right multiplying the state can be identified as the effective rotation frame Hamiltonian. The solution to this equation is

$$|\varphi(t)\rangle = e^{-i\left[\frac{\omega_0+\omega}{2}Z+gX\right]t}|\varphi(0)\rangle.$$

The concept of resonance arises from the behavior of this solution, which can be understood to be a single qubit rotation about the axis

$$\hat{n} = \frac{\hat{z} + \frac{2g}{\omega_0+\omega}\hat{x}}{\sqrt{1+\left(\frac{2g}{\omega_0+\omega}\right)^2}}$$

by an angle

$$|\vec{n}| = t\sqrt{\left(\frac{\omega_0+\omega}{2}\right)^2 + g^2}.$$

Therefore,

$$|\chi(t)\rangle = e^{i\omega tZ/2}|\varphi(t)\rangle = e^{i\omega tZ/2}e^{-i\left[\frac{\omega_0+\omega}{2}Z+gX\right]t}|\varphi(0)\rangle = e^{i\omega tZ/2}e^{-i\left[\frac{\omega_0+\omega}{2}Z+gX\right]t}|\chi(0)\rangle$$

Energy level with periodic boundary condition

For the periodic boundary condition, the transformation matrix is the quantum fourier transformation:

$$\eta_k = \frac{1}{\sqrt{N}}\sum_j e^{2\pi ikj/N}c_j, \eta_k^\dagger = \frac{1}{\sqrt{N}}\sum_j e^{-2\pi ikj/N}c_j^\dagger.$$

$$c_j = \frac{1}{\sqrt{N}}\sum_k e^{-2\pi ikj/N}\eta_k, c_j^\dagger = \frac{1}{\sqrt{N}}\sum_k e^{2\pi ikj/N}\eta_k^\dagger.$$

$$c_j^\dagger c_{j+1} = \frac{1}{N}\sum_{k_1,k_2} e^{2\pi i(k_1j-k_2j-k_2)/N}\eta_{k_1}^\dagger\eta_{k_2},$$

$$c_{j+1}^\dagger c_j = \frac{1}{N}\sum_{k_1,k_2} e^{2\pi i(k_1j-k_2j+k_1)/N}\eta_{k_1}^\dagger\eta_{k_2},$$

$$c_j^\dagger c_j = \frac{1}{N}\sum_{k_1,k_2} e^{2\pi i(k_1-k_2)j/N}\eta_{k_1}^\dagger\eta_{k_2},$$

$$
\begin{aligned}
H &= \frac{1}{2}\sum_j (c_j^\dagger c_{j+1} + c_{j+1}^\dagger c_j) + \sum_j c_j^\dagger c_j + \frac{N}{2} \\
&= \frac{1}{2}\sum_j (\frac{1}{N}\sum_{k_1,k_2} e^{2\pi i(k_1 j - k_2 j - k_2)/N}\eta_{k_1}^\dagger \eta_{k_2} + \frac{1}{N}\sum_{k_1,k_2} e^{2\pi i(k_1 j - k_2 j + k_1)/N}\eta_{k_1}^\dagger \eta_{k_2}) + \sum_j \frac{1}{N}\sum_{k_1,k_2} e^{2\pi i(k_1-k_2)j/N}\eta_{k_1}^\dagger \eta_{k_2} + \frac{N}{2} \\
&= \frac{1}{2N}\sum_{j,k_1,k_2} e^{2\pi i(k_1-k_2)j/N}(\eta_{k_1}^\dagger \eta_{k_2} e^{-2\pi i k_2/N} + \eta_{k_1}^\dagger \eta_{k_2} e^{2\pi i k_1/N}) + \frac{1}{N}\sum_{j,k_1,k_2} e^{2\pi i(k_1-k_2)j/N}\eta_{k_1}^\dagger \eta_{k_2} + \frac{N}{2} \\
&= \frac{1}{2N}\sum_{k_1,k_2}(\eta_{k_1}^\dagger \eta_{k_2} e^{-2\pi i k_2/N} + \eta_{k_1}^\dagger \eta_{k_2} e^{2\pi i k_1/N})\sum_j e^{2\pi i(k_1-k_2)j/N} + \frac{1}{N}\sum_{k_1,k_2}\eta_{k_1}^\dagger \eta_{k_2}\sum_j e^{2\pi i(k_1-k_2)j/N} + \frac{N}{2} \\
&= \frac{1}{2N}\sum_{k_1,k_2}(\eta_{k_1}^\dagger \eta_{k_2} e^{-2\pi i k_2/N} + \eta_{k_1}^\dagger \eta_{k_2} e^{2\pi i k_1/N}) N\delta_{k_1,k_2} + \frac{1}{N}\sum_{k_1,k_2}\eta_{k_1}^\dagger \eta_{k_2} N\delta_{k_1,k_2} + \frac{N}{2} \\
&= \frac{1}{2}\sum_k (\eta_k^\dagger \eta_k e^{-2\pi i k/N} + \eta_k^\dagger \eta_k e^{2\pi i k/N}) + \sum_k \eta_k^\dagger \eta_k + \frac{N}{2} \\
&= \sum_k (\cos(2\pi k/N) + 1)\eta_k^\dagger \eta_k + \frac{N}{2}
\end{aligned}
$$

The ground state of H is the vacuum state of η_k and $|11...1\rangle$ in the original spin language. The ground state energy is $E_0 = N/2$.