Goals of this recitation

- Using Tables to perform Laplace Transform.
- Applying the Laplace Transform to differential equations.

Contents

1. Laplace Transform

1.1. **Motivation.**

1.2. **Using the transforms.** Test.

$f(t)$	$F(s) = \mathcal{L}[f(t)]$		Formula
$f(t) = 1$	$F(s) = \frac{1}{s}$	$s > 0$	A
$f(t) = e^{at}$	$F(s) = \frac{1}{(s-a)}$	$s > a$	B
$f(t) = t^n$	$F(s) = \frac{n!}{s^{(n+1)}}$	$s > 0$	C
$f(t) = \sin(at)$	$F(s) = \frac{a}{s^2+a^2}$	$s > 0$	D
$f(t) = \cos(at)$	$F(s) = \frac{s}{s^2+a^2}$	$s > 0$	E
$f(t) = \sinh(at)$	$F(s) = \frac{a}{s^2-a^2}$	$s > \lvert a\rvert$	F
$f(t) = \cosh(at)$	$F(s) = \frac{s}{s^2-a^2}$	$s > \lvert a\rvert$	G
$f(t) = t^n e^{at}$	$F(s) = \frac{n!}{(s-a)^{(n+1)}}$	$s > a$	H
$f(t) = e^{at}\sin(bt)$	$F(s) = \frac{b}{(s-a)^2+b^2}$	$s > a$	I
$f(t) = e^{at}\cos(bt)$	$F(s) = \frac{(s-a)}{(s-a)^2+b^2}$	$s > a$	J
$f(t) = e^{at}\sinh(bt)$	$F(s) = \frac{b}{(s-a)^2-b^2}$	$s - a > \lvert b\rvert$	K
$f(t) = e^{at}\cosh(bt)$	$F(s) = \frac{(s-a)}{(s-a)^2-b^2}$	$s - a > \lvert b\rvert$	L

That people sometimes use to solve problems. One first question is how?

They try to recognise where on the table their input is by seeing which operation more closely align to the operations they have. For instance, if I gave you the following equation for $f(t)$ which formula letter do you think it would apply to and why?

$$f(t) = t^5 e^{3t}$$

If we were to follow that row, what would $L(f(t))$ be?

$$L(f(t))$$

Try solve these problems in the same fashion.

(1) Let $f(t) = e^{7t}$. Then

$$L(f(t)) =$$

(2) Let $f(t) = \sin(8t)$. Then

$$L(f(t)) =$$

(3) Let $f(t) = e^{5t}\cos(6t)$. Then

$$L(f(t)) =$$

(4) Let $f(t) = t^2 e^{5t}$. Then

$$L(f(t)) =$$

(5) Suppose $L(f(t)) = \frac{2}{s^2+2^2}$

$$f(t) =$$

(6) Suppose $L(f(t)) = \frac{s}{s^2-6^2}$

$$f(t) =$$

1.3. **Applying Laplace Transform to differential equations with initial conditions.** There are three properties of the Laplace Transform that we use to convert our differential equations into algebra. For the following rules let $f(t), g(t)$ be functions, c stand for a constant and $F(s)$ be the Laplace transform of $f(t)$. Then the following rules are ones you should mememorise.

Laplace distributes with addition

$$\mathscr{L}(f(t)+g(t)) = \mathscr{L}(f(t)) + \mathscr{L}(g(t))$$

You can factor out constants

$$\mathscr{L}(cf(t)) = c\mathscr{L}(f(t))$$

The number of derivatives becomes the power on s

$$\mathscr{L}(f^{(n)}(t)) = s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - s^1f^{(n-2)}(0) - f^{(n-1)}(0)$$

Note that the inverse Laplace Transform follows the first two rules aswell. Note that the Laplace Transform **does not distribute with multiplication.** That is a common mistake with the Laplace transform **and you will receive zero marks if you do it**. Lets first talk about the last rule as that is a new rule we have not seen with any other operation.

Suppose we were given $f(0) = 2$ and $f'(0) = 1$. Lets try to figure out what $\mathscr{L}(f''(t))$.

First, how many times is $f''(t)$ being differentiated?

So what is our n in this case?

Write down $\mathscr{L}(f''(t))$ in terms of how derivatives become powers. Note how the powers decrease in terms of s while the derivatives increase.

Using the information at the start of this fill into that equation.

Figure out the Laplace Transform of the following equations.

(1) Let $f(0) = 2$ and $f'(0) = 3$. Find $\mathscr{L}(f''(t))$

(2) Let $f(0) = 1$, $f'(0) = 2$ and $f''(0) = -1$. Find $\mathscr{L}(f'''(t))$

(3) Let $f(0) = 0$, $f'(0) = 1$ and $f''(0) = -2$. Find $\mathscr{L}(3f'''(t))$

(4) Let $f(0) = 3$ and$f'(0) = 2$. Find $\mathscr{L}(f''(t) + 2f'(t))$

1.4. **Adjusting the formula for the inverse Laplace transform.**

Useful fact about fractions: We can "swap" nominators

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd} = \frac{c}{b} \cdot \frac{d}{b}$$

Lets try take the inverse Laplace transform of the following function.

$$\frac{5}{s^2 + 3^2}$$

So to do this we are going to take the approach we have taken before.

- What operations are there in this expression?
- How are they applied?

So lets ask ourselves, what operations do we see in this expression?

How are each of these applied?

If we are to look at our table of Laplace Transforms, which is the closest in terms of operations to this?

Is there an issue?

What should our problem solving strategy do:

Solve the following inverse Laplace Transforms.

(1)

$$F(s) = \frac{2}{s^2 + 4^2}$$

(2)

$$F(s) = \frac{2s}{s^2 + 3^2}$$

(3)

$$F(s) = \frac{3}{(s-2)^2 - 7^2}$$

1.5. **Solving differential equations using Laplace Transform.** We have all the pieces, lets actually use them. To do this I am going to have us work through this together in steps.

The process will go as follows.

1. Apply Laplace transform.

2. Apply algebra to get it into pieces that look like terms we can apply Laplace Transform to.

3. Reformat like in the last section to get the Laplace Transform correct.

4. Apply the Laplace Transform.

5. Simplify.

Let's try solve the following differential equation going through these steps.

$$y'' - y' - 6y = 0 \qquad y(0) = 2, \quad y'(0) = -1$$

Now you try solving this question

$$y'' + 3y' + 2y = 0$$

And this too.

$$y'' + 5^2 y = \cos(3t)$$

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