Topics in Matrix Analysis

Roger A. Horn, Charles R. Johnson

Book DOI: http://dx.doi.org/10.1017/CBO9780511840371

Online ISBN: 9780511840371

Hardback ISBN: 9780521305877

Paperback ISBN: 9780521467131

Chapter

Hints for problems pp. 561-583

Chapter DOI: http://dx.doi.org/10.1017/CBO9780511840371.008

Cambridge University Press

Hints for problems

Section (1.0)

2. $\rho(A+B) \leq |||A+B|||_2$, where the matrix norm $|||\cdot|||_2$ is the spectral norm (see Section (5.6) of [HJ]). How are $\rho(A)$ and $|||A|||_2$ related when A is normal?

Section (1.2)

4. Consider $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

15. Show by example that they are not the same, but by (1.2.12) they both must be "angularly" the same. Thus, the only difference can be stretching or contracting along rays from the origin.

17. The orthogonal complement of the intersection of the nullspaces of A and A^* is the span of the union of the column spaces of A and A^*. Let $U \equiv [P \;\; P'] \in M_n$ be unitary, let $U^*x \equiv \begin{bmatrix} \xi \\ \eta \end{bmatrix}$ with $\xi \in \mathbb{C}^k$, and show that $x^*Ax = \xi^*(P^*AP)\xi$.

20. Permute the basis to $\{e_1, e_4, e_3, e_2\}$ and use (1.2.10).

22. $x^*A^{-1}x = z^*A^*z$ for $z \equiv A^{-1}x$.

23. (a) If $Ax = \lambda x$, then $x^*Ax = \lambda x^*x$ and $x^*A^*x = \bar{\lambda} x^*x$; add.

25. (a) Let $B = V\Sigma W^*$ be a singular value decomposition of B and show that $|x^*By| \leq \sigma_1(B)\|x\|_2\,\|y\|_2$, with equality when x and y are the first columns of V and W, respectively.

Section (1.3)

1. This may be done by applying the Gram-Schmidt procedure to a basis of $\mathbb{C}^n$ whose first two vectors are x and y if x and y are independent, or whose first vector is just one of them if they are dependent and one is nonzero. There is nothing to do if both are zero. Householder transformations can also be used. Try both approaches.

4. Work back from the special form, using the transformations that produced it.

13. Reduce A to upper triangular form by a unitary similarity and use (1.3.6).

14. Use (1.3.6).

17. (c) Direct computation or recall the Newton identities in Section (1.2) of [HJ].

Section (1.4)

1. (a) For the first four, let $F(A) \equiv \mathrm{Co}(\sigma(A))$.
(b) For all but (1.2.4), let $F(A) \equiv \{z: z \in F(A)$ and Re z is a minimum$\}$
(c) For all but (1.2.3), let $F(A) \equiv \mathrm{Co}(F(A) \cup \{\mathrm{tr}\, A\})$ or, even simpler, $F(A) = \phi$
(d) For all but (1.2.2), let $F(A)$ be the boundary of $F(A)$
(e) For the last four, let $F(A)$ be the interior of $F(A)$, when $F(A)$ has an interior, and let $F(A) = F(A)$ otherwise.

Section (1.5)

5. Consider 2-by-2 examples of the form $\begin{bmatrix} 0 & a \\ b & 0 \end{bmatrix}$.

15. For each $x = [x_1, \ldots, x_n]^T \in \mathbb{C}^n$, define $x^{(i)} \equiv [x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n]^T \in \mathbb{C}^{n-1}$ and consider $(x^{(1)*}A_1x^{(1)} + \cdots + x^{(n)*}A_nx^{(n)})/n$.

16. Let a denote the smallest, and b the largest, eigenvalue of A, and show that the area ratio in (1.5.17) is exactly $(\beta - \alpha)/(a - b)$; we have $\beta \geq \alpha > 0$ and we want $a > 0$.

23. (l) $|x^*Ax| \leq |x|^T|A|\,|x|$
(m) $|x^*Ax| \leq |x|^TA|x| = |x|^T\mathrm{H}(A)|x|$

(o) Use (g) and (d).

24. If $x \in \mathbb{C}^n$ is a unit vector such that $|||A|||_2 = |x^*Ax|$, use the Cauchy-Schwarz inequality and look carefully at the case of equality.

29. Notice that $J_{2k}(0)$ is a principal submatrix of $J_{2k+1}(0)$, and that $J_{2k}(0)^k = \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix}$ with $I \in M_k$. Use Problem 25 in Section (1.2) and the power inequality in Problem 23(d).

Section (1.6)

1. Use the unitary similarity invariance property to reduce to the case of an upper triangular matrix and take the two eigenvectors into account.

2. Consider the eigenvalues of H(A) and S(A). Write the eigenvalues λ_1, λ_2 in terms of the entries a_{ij} using the quadratic formula and observe that, upon performing a unitary similarity to triangular form, the absolute value of the off-diagonal entry is $(\Sigma_{i,j} |a_{ij}|^2 - |\lambda_1|^2 - |\lambda_2|^2)^{\frac{1}{2}}$.

6. Consider (1.6.8) and think geometrically.

8. Consider the proof of (1.6.3).

12. Eigenvectors of A^* can be thought of as left eigenvectors of A, while those of A are right eigenvectors.

21. (b) $(I - P^2)^{\frac{1}{2}}$ and P are simultaneously (unitarily) diagonalizable and hence they commute. Alternatively, use the singular value decomposition for A.
(c) Use the singular value decomposition of A.
(f) See the 1982 paper by Thompson and Kuo cited at the end of the section for an outline of a calculation to verify the general case.
(g) If $U = \begin{bmatrix} A & B \\ * & * \end{bmatrix}$ is unitary, then $B \in M_{k,n-k}$ and $B^*B = I - A^*A$, so rank B = rank$(I - A^*A)$.

23. See Problem 11 of Section (1.2) and explicitly compute both sides of the asserted identity.

26. Suppose $p(\cdot)$ has degree k and let $B \in M_{(k+1)n}$ be unitary and such that $B^m = \begin{bmatrix} A^m & * \\ * & * \end{bmatrix}$ for $m = 1, 2, \ldots, k$. Then we have $p(B) = \begin{bmatrix} p(A) & * \\ * & * \end{bmatrix}$. But $p(B)$ is a contraction by the spectral theorem, so $|||p(A)|||_2 \le |||p(B)|||_2 \le 1$.

27. If A is row stochastic, show that every power A^m is row stochastic and hence $\{A^m: m = 1, 2, \ldots\}$ is uniformly bounded, which is not possible if the Jordan canonical form of A contains any Jordan blocks $J_k(1)$ with $k > 1$.

28. If all the points in $\sigma(A)$ are collinear, then either all of the line segment $L \equiv \mathrm{Co}(\sigma(A))$ lies in the boundary of $F(A)$ or the relative interior of L lies in the interior of $F(A)$.

30. Use (3.1.13) in [HJ], the direct sum property (1.2.10), and (1.6.6).

31. Use the preceding problem and the characterization of inner products given in Problem 14 of Section (7.2) of [HJ].

32. Consider $A = I_{n-2} \oplus \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$. Show that a matrix in M_2 is spectral if and only if it is radial if and only if it is normal. Show that for every $n \geq 3$ there are spectral matrices in M_n that are not radial.

35. Use the fact that $\rho(A) = \lim \| A^k \|^{1/k}$ as $k \rightarrow \infty$ for any norm $\|\cdot\|$ on M_n and the power inequality $r(A^k) \leq r(A)^k$; see Problem 23 in Section 1.5.

40. (c) If $Q = \begin{bmatrix} A & B \\ * & * \end{bmatrix}$ is complex orthogonal, then $B \in M_{k,n-k}$ and $B^TB = I - A^TA$, so rank $B \geq \mathrm{rank}(I - A^TA)$.

Section (1.7)

2. Show that $[\mathrm{H}(A)]^2 - \mathrm{H}(A^2)$ is positive semidefinite and hence $\lambda_{max}[\mathrm{H}(A^2)] \leq \lambda_{max}[\mathrm{H}(A)]^2$.

11. Use (1.5.2).

14. Use Schur's unitary triangularization theorem (2.3.1) in [HJ] to show that one may assume without loss of generality that A is upper triangular. Use the results of the exercises after (2.4.1) in [HJ] to show that it is then sufficient to consider the 2-by-2 case.

17. Use (1.7.11) and pick t_0 so that $F(A + t_0I)$ lies to the right of the imaginary axis.

18. Recall from Theorem (2.5.5) in [HJ] that a commuting family of normal matrices is simultaneously unitarily diagonalizable.

20. Use (1.7.9).

24. (a) implies that $(e^{i\theta}S)A^* = A(e^{i\theta}S)$ and $A(e^{i\theta}S)^* = (e^{i\theta}S)^*A^*$ so

$AH(e^{i\theta}S) = H(e^{i\theta}S)A^*$. Now use (1.3.5) to choose $\theta \in \mathbb{R}$ so that $P \equiv H(e^{i\theta}S)$ is positive definite, show that $K = P^{-\frac{1}{2}}AP^{\frac{1}{2}} = P^{\frac{1}{2}}A^*P^{-\frac{1}{2}}$ is Hermitian, and note that $A = P(P^{-\frac{1}{2}}KP^{-\frac{1}{2}})$.

25. See Problem 24. When is a normal matrix similar to a Hermitian matrix?

26. Use $A = U^*A^*U$ to show that A commutes with U^2. Let $\{\lambda_1,\ldots,\lambda_n\} = \sigma(U)$. Since $\sigma(e^{i\theta}U) \subset RHP$ for some $\theta \in \mathbb{R}$, the interpolation problem $p(\lambda_i^2) = \lambda_i$, $i = 1,\ldots, n$, has a polynomial solution $p(t)$ (of degree at most $n-1$), so $U = p(U^2)$ commutes with A.

27. (a) If $\zeta \in F(B)$ then $|\zeta| \le r(B) \le |||B|||_2$.

28. Perform a simultaneous unitary similarity of A, B, and C by U that diagonalizes A and C; the diagonal entries of UBU^* are in $F(B)$.

Section (2.1)

3. Use induction. If $p(t) = t^k + \gamma_1 t^{k-1} + \gamma_2 t^{k-2} + \cdots + \gamma_{k-1}t + \gamma_k$ is a real polynomial with $(-1)^m\gamma_m > 0$, $m = 1,\ldots, k$, show that the coefficients of the products $p(t)(t-\lambda)$ and $p(t)(t^2 - at + b)$ both have the same strict sign-alternation pattern if λ, a, $b > 0$.

5. Use the result of Problem 2 to construct examples in $M_2(\mathbb{R})$.

7. Let $z_0 = re^{i\theta}$ be in the complement of W_n and consider the value of the characteristic polynomial $p_A(z_0) = z_0^n - E_1(A)z_0^{n-1} + E_2(A)z_0^{n-2} - E_3(A)z_0^{n-3} + \cdots$. What is $p_A(0)$? If $r > 0$, what is the sign of $\mathrm{Im}[(-1)^k E_k(A)z_0^{n-k}]$?

8. Perform a congruence on A via $\begin{bmatrix} I & -B^{-1}C \\ 0 & I \end{bmatrix}$.

Section (2.2)

2. (b) Consider (a) with $x_i(t) = \exp(\lambda_i t)$, $a = -\infty$, $b = 0$.

3. Use Problem 2 and Theorem (5.2.1).

5. Consider a small perturbation A_ϵ of A that is positive stable and diagonalizable.

9. Consider $\int_0^\infty \frac{d}{dt}P(t)\, dt$

Section (2.3)

2. See Problem 2 of Section (2.1).

3. If A satisfies (a) or (b), then so does DA; explain why it suffices to show that (a) and (b) imply that A is positive stable. Use (2.3.2); check that $E_1(A)E_2(A) - E_3(A) > 0$ if and only if $(2xyz - ac\beta - \alpha\gamma b) + (x+y)(xy - a\alpha) + (x+z)(xz - b\beta) + (y+z)(yz - c\gamma) > 0$.

Section (2.4)

2. The equation $GA + A^*G = I$ has a positive definite solution G, so $(GH)K + K(GH)^* = I$.

7. If Hermitian G, H, with H positive definite, satisfy (2.4.1), show that G^2 is positive definite and that G is a Lyapunov solution for each $\alpha G + (1\text{-}\alpha)A$, $0 \leq \alpha \leq 1$. Conclude that $i(\alpha G + (1\text{-}\alpha)A)$ is constant, $0 \leq \alpha \leq 1$, since no $\alpha G + (1\text{-}\alpha)A$ may have an eigenvalue with zero real part, and thus that $i(A)$, $\alpha = 0$, is the same as $i(G)$, $\alpha = 1$.

8. Carefully follow the development (2.4.1) through (2.4.6).

9. Give a diagonal Lyapunov solution with positive semidefinite right-hand side and use (2.4.7) or Problem 8.

10. Show that there is a positive diagonal Lyapunov solution with positive definite right-hand side.

11. If B is positive or negative definite, the assertion follows immediately from (1.7.8). If B is nonsingular and indefinite, let α be a given point in $F(A)$, let $A(t) \equiv (1-t)\alpha I + tA$, and observe that $F(A(t)) \subset F(A) \subset RHP$ for $0 \leq t \leq 1$. Use (1.7.8) to argue that $\sigma[A(t)B]$ varies continuously with t and lies in the union of two fixed disjoint angular cones, so the number of eigenvalues of $A(t)B$ in each cone is constant as t varies from 0 to 1. Reduce the singular case to the nonsingular case as in the proof of (2.4.15).

12. Set $G \to H^{-1}$ and $A \to HA$.

Section (2.5)

3. Use (2.5.3.17).

5. Write $\alpha A + (1-\alpha)B = \alpha B[B^{-1}A + (\alpha^{-1} - 1)I]$.

6. (a) Use Theorem 2.5.4(a) and (2.5.3.12); $A = B - R$ with $R \geq 0$, so $B^{-1}A = I - B^{-1}R \in Z_n$. If $x > 0$ and $Ax > 0$, then $B^{-1}Ax > 0$. Use Problem 5.

9. Use the fact that, for an index set $\alpha \subseteq \{1, 2, \ldots, n\}$ and a nonsingular matrix $A \in M_n$, $\det A^{-1}(\alpha) = \det A(\alpha')/\det A$ (see (0.8.4) in [HJ]).

10. Use (2.5.3.13) and (2.5.12). Prove the same implication if A is the inverse of an H-matrix.

11. Use the result of Problem 10.

12. (c) Use (2.5.3.13) and (2.5.12) to find a positive diagonal $D = \mathrm{diag}(d_1, \ldots, d_n)$ such that BD is strictly row diagonally dominant and $(BD)^{-1}$ is strictly diagonally dominant of its column entries. Now take $E \equiv \mathrm{diag}(\beta_{11}/d_1, \ldots, \beta_{nn}/d_n)$ and check that $(EBD)^{-1}$ satisfies the conditions in (a). Then $\det[A \circ (EBD)^{-1}] \geq \det A$. Use Hadamard's inequality for an inverse M-matrix in Problem 9.

15. Partition A^{-1} conformally with A and use the presentation of the partitioned inverse given in (0.7.3) in [HJ]; the Schur complement of A_{11} is the inverse of the block $(A^{-1})_{22}$. If A is an M-matrix, then $A^{-1} \geq 0$. Conversely, use the alternate presentation for the block

$$(A^{-1})_{11} = (A_{11})^{-1} + (A_{11})^{-1}A_{12}[A_{22} - A_{21}(A_{11})^{-1}A_{12}]^{-1}A_{21}(A_{11})^{-1}$$

to show that if A_{11} and its Schur complement are M-matrices then $A^{-1} \geq 0$.

17. (a) Using the notation of the proof of Theorem (2.5.12), first suppose A is strictly row diagonally dominant and all $a_{ii} > 0$, $\det A_{ii} > 0$. Show that $r_2(A)\det A_{11} + \epsilon \det A_{12} \geq 0$ by incorporating the factor $r_2(A)$ into the first column of A_{11} in the same way ϵ was incorporated into the first column of A_{12}.

18. Use the Laplace expansion for $\det A$ (see (0.3.1) in [HJ]) and (2.5.6.1).

20. (b) Show that there is some $\epsilon > 0$ such that $(1-\epsilon)I - \epsilon A \geq 0$ and, for such an ϵ, define $B_\epsilon \equiv [(1-\epsilon)I - \epsilon A]/[(1-\epsilon) - \epsilon\tau(A)]$. Show that $B_\epsilon \geq 0$, $\rho(B_\epsilon) = 1$, and $\lambda_\epsilon \equiv [(1-\epsilon) - \epsilon\lambda]/[(1-\epsilon) - \epsilon\tau(A)] \in \sigma(B_\epsilon)$ whenever $\lambda \in \sigma(A)$. Use the result of Problem 22c of Section (1.5) (a consequence of the Kellogg-Stephens theorem) to show that λ_ϵ lies in the closed wedge L_n defined by (1.5.19) and deduce that $\lambda - \tau(A) = re^{i\theta}$ with $r \geq 0$ and $\theta \leq \pi/2 - \pi/n$.

22. Let $\lambda(t) \equiv 1 + t - t\lambda$, and show that one may choose $t > 0$ small enough

so that $\lambda(t) - 1 = t(1 - \lambda)$ lies in the closed polygon whose n vertices are the nth roots of unity $\{e^{2\pi ik/n}: k = 1,\ldots, n\}$. Then write $\lambda(t) = \Sigma_k \alpha_k e^{2\pi ik/n}$ with all $\alpha_k \geq 0$ and $\alpha_1 + \cdots + \alpha_n = 1$. Use Problem 21 to conclude that $\lambda(t)$ is an eigenvalue of a doubly stochastic (circulant) matrix $B \in M_n$ whose eigenvalues lie in the wedge L_n defined by (1.5.19). Set $A \equiv [(1 + t)I - B]/t$ and show that A is an M-matrix with $\tau(A) = 1$ and $\lambda \in \sigma(A)$.

23. If $\lambda > 0$, consider λI. Suppose λ is nonreal and define $A(\gamma) \equiv [a_{ij}(\gamma)] \in M_n$ by $a_{i,i+1} = 1$ for $i = 1,\ldots, n-1$, $a_{n,1} = \gamma$, and all other entries are zero. If n is *odd*, let $\gamma = +1$ and show that the characteristic polynomial of $\beta[A(1) + \alpha I]$ is $\beta^n[(t-\alpha)^n - 1]$. Consider the coefficients and show that if $\alpha, \beta > 0$, $\beta[A(1) + \alpha I]$ is a P-matrix with $\beta[e^{i(n\pm1)\pi/n} + \alpha]$ in its spectrum. If $\pm\mathrm{Im}\,\lambda > 0$, show that one can choose $\alpha, \beta > 0$ so that $\lambda = \beta[e^{i(n\pm1)\pi/n} + \alpha]$. If n is *even*, let $\gamma = -1$, show that the characteristic polynomial of $\beta[A(-1) + \alpha I]$ is $\beta^n[(t-\alpha)^n + 1]$, show that $\beta[A(-1) + \alpha I]$ is a P-matrix with the points $\beta[e^{i(n\pm1)\pi/n} + \alpha]$ in its spectrum, and conclude that λ can be represented as such a point for some $\alpha, \beta > 0$.

25. See Problem 3 of Section (2.1) for the necessity of this coefficient condition. For the sufficiency, consider the polynomial $p_\epsilon(t) \equiv p(t + \epsilon) = t^n + b_1(\epsilon)t^{n-1} + \cdots + b_{n-1}(\epsilon)t + b_n(\epsilon)$ for $\epsilon > 0$ small enough that $(-1)^k b_k(\epsilon) > 0$ for all $k = 1,\ldots, n$. Consider the matrix $A_\epsilon \equiv [a_{ij}] \in M_n(\mathbb{R})$ with $a_{i,i+1} = 1$ for $i = 1,\ldots, n-1$, $a_{ii} = \epsilon$ for $i = 1,\ldots, n-1$, $a_{n,n-i+1} = -b_i(\epsilon)$ for $i = 1,\ldots, n-1$, $a_{nn} = -b_1(\epsilon) + \epsilon$, and all other entries zero. Show that A_ϵ is a P-matrix whose characteristic polynomial is $p(t)$; see Problem 11 of Section (3.3) of [HJ] regarding the *companion matrix*.

28. Use Theorem (2.5.4(b)) and Problem 19.

29. Provide justification for the following identities and estimate:

$$\tau(DA) = \rho(A^{-1}D^{-1})^{-1} \geq \left[\rho\left[A^{-1}(\max_{1\leq i\leq n} d_i^{-1})I\right]\right]^{-1}$$

$$= \left[\rho(A^{-1}) \max_{1\leq i\leq n} d_i^{-1}\right]^{-1} = \tau(A) \min_{1\leq i\leq n} d_i$$

31. Construct a diagonal unitary matrix D such that the main diagonal entries of DB are positive, let $A = \alpha I - P$ with $\alpha > \rho(P)$, and set $R \equiv \alpha I - DB$. Check that $|R| \leq P$, so $\rho(R) \leq \rho(|R|) \leq \rho(P) < \alpha$. Then $|(DB)^{-1}| = |\alpha^{-1}(I - \alpha^{-1}R)^{-1}| = |\Sigma_k \alpha^{-k-1} R^k| \leq \Sigma_k \alpha^{-k-1} P^k = A^{-1}$. The induction argument for Theorem (2.5.4(c)) works here, too; the crucial inequality is $(\det A_{11})/\det A = (A^{-1})_{nn} \geq |(B^{-1})_{nn}| = |\det B_{11}/\det B|$.

33. If $\text{Re}\,\lambda < 0$, then $|\lambda + t|$ is a strictly decreasing function of t for small $t > 0$, but $\det(A + tI) = |\lambda_1 + t| \cdots |\lambda_n + t|$ is a strictly increasing function of t for all $t > 0$.

34. (a) Write $B = \alpha I - P$ as in Problem 19, and use Fan's theorem to show that the spectrum of A is contained in the union of discs of the form $\{|z - a_{ii}|: \rho(P) - p_{ii}\}$. Then show that $\rho(P) - p_{ii} = b_{ii} - \tau(B)$.
(b) Since $\tau(B) > 0$, the hypotheses ensure that the discs in (a) with $\text{Re}\, a_{ii} < 0$ lie to the left of the line $\text{Re}\, z = -\tau(B)$, and those with $\text{Re}\, a_{ii} > 0$ lie to the right of the line $\text{Re}\, z = \tau(B)$; use the argument in Theorem (6.1.1) in [HJ].

35. Use induction on n.

36. Use Sylvester's determinant identity (0.8.6) in [HJ].

Section (3.0)

4. (b) Using the singular value decomposition, it suffices to consider the case in which $A = \begin{bmatrix} S & 0 \\ 0 & 0 \end{bmatrix}$ with a positive diagonal $S \in M_r$, with $r = \text{rank}\, A$. Write $B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$ with blocks conformal to those of A, and use the criterion to show that $B_{12} = 0$, $B_{21} = 0$, $SB_{11}^* = B_{11}S$, and $B_{11}^*S = SB_{11}$. Conclude that B_{11} is Hermitian and commutes with S.

Section (3.1)

5. (f) Using the notation of (3.1.8(b)(3)), consider E_r and $\frac{1}{2}[(I_r \oplus I_{n-r}) + (I_r \oplus (-I_{n-r}))]$, where $I_k \in M_k$ is an identity matrix.

6. $|x^*Ay| \leq \|x\|_2 \|Ay\|_2 \leq \|x\|_2 \|y\|_2 \sigma_1(A)$. Use the singular value decomposition $A = V\Sigma W^*$ to find vectors x, y for which this inequality is an equality.

7. Let $A = V\Sigma W^*$ be a singular value decomposition and consider $C_r = WI_rV^*$, where $I_r = \text{diag}\,(1,\ldots, 1, 0,\ldots, 0) \in M_n$ has r ones.

8. (a) If $Ax = x$, then $\|x - A^*x\|_2^2 = \|x - A^*Ax\|_2^2 = \|x\|_2^2 - 2\|Ax\|_2^2 + \|A^*Ax\|_2^2 \leq 0$.

19. Use the reduction algorithm in the proof of Theorem (2.3.1) in [HJ] to construct a unitary triangularization of A; use the argument in the proof of Theorem (3.1.1) to show that the resulting upper triangular matrix is actually diagonal.

25. Write $A = I + iB$ for some Hermitian $B \in M_n$ and compute A^*A.

26. For any $x \in \mathbb{C}^n$ with $\|x\|_2 = 1$, show that $4 = 4\|Ux\|_2^2 = \|Ax\|_2^2 + 2x^*\mathrm{H}(A^*B)x + \|Bx\|_2^2 \leq 4$. Conclude that A and B are unitary and $\mathrm{H}(A^*B) = I$. Use Problem 25.

27. (a) If $A = V\Sigma W^*$ is a singular value decomposition of $A \in \mathcal{B}_n$ and if $\sigma_k(A) < 1$, choose $\epsilon > 0$ so $B = V(\Sigma + \epsilon E_{kk})W^*$ and $C = V(\Sigma - \epsilon E_{kk})W^*$ are both in $\mathcal{B}_n$, where $E_{kk} \in M_n$ has entry 1 in position k,k and all other entries are zero. Then $A = \frac{1}{2}(B + C)$, so A is not an extreme point of $\mathcal{B}_n$.

28. Use the discussion preceding Theorem (3.1.2) and the Cauchy-Schwarz inequality to show that $\sigma_i(A) = |x_i^*Ax_i| \leq \|Ax_i\|_2 \leq \sigma_i(A)$ and hence $x_i = c_i(Ax_i)$ with $|c_i| = 1$.

31. Compute $(A + \alpha I)^*(A + \alpha I)$ and use Corollary (4.3.3) in [HJ].

32. See Problem 26 in Section (4.4) of [HJ].

33. (a) Write $U = [U_{ij}]$ as a block matrix conformal with D. Notice that

$$\begin{aligned}\|U_{11}\|_2^2 + \|U_{21}\|_2^2 + \cdots + \|U_{r1}\|_2^2 &= n \\ &= \|U_{11}\|_2^2 + \|U_{12}\|_2^2 + \cdots + \|U_{1r}\|_2^2 \\ &= \|U_{11}\|_2^2 + |d_2/d_1|^2\,\|U_{21}\|_2^2 + \cdots + |d_r/d_1|^2\,\|U_{r1}\|_2^2\end{aligned}$$

(b) If $B = S\Lambda S^{-1}$, $\Lambda = \mathrm{diag}(\lambda_1,\ldots,\lambda_n)$, require $p(\lambda_i)^2 = \lambda_i$, $i = 1,\ldots,n$.

37. (e) Let $U \in M_n$ be a unitary matrix such that $U(A + B) \succeq 0$ and compute $|A + B| = \mathrm{H}(U(A + B))$.

39. Compute $\mathrm{tr}\, A^*A$ with $A = \mathrm{H}(A) + \mathrm{S}(A)$.

45. Using the notation of Theorem (4.5.9) in [HJ], $\lambda_k(AA^*) = \lambda_k(S[BS^{-1}S^{-*}B^*]S^*) = \theta_k\lambda_k(BS^{-1}S^{-*}B^*) = \theta_k\lambda_k(S^{-*}[B^*B]S^{-1})$.

Section (3.2)

4. (c) Write $[s_1 \; \ldots \; s_n]^T = Q[\sigma_1 \; \ldots \; \sigma_n]^T$, where Q is doubly substochastic and hence is a finite convex combination of generalized permutation matrices.

Section (3.3)

4. First generalize Theorem (3.3.4).

8. The largest singular values of a matrix are unaffected if zero rows are deleted.

9. Using the notation of the proof of (3.3.2), apply (3.3.4) to the product $U_k^* A U_k = \Delta_k$.

10. (a) $|\operatorname{tr} X| = |\Sigma_i \lambda_i(X)| \le \Sigma_i |\lambda_i(X)|$.

12. The lower bounds are trivial if B is singular; if not, $A = (AB)B^{-1}$.

14. See Problem 19 in Section (3.1).

18. (c) Expand $[A - B, A - B]_F$ and use (b).

19. (c) Use the argument given to prove (3.3.17).

20. (a) Let $A = U\Delta U^*$ be a unitary upper triangularization of A, so $\mathrm{H}(A) = U\mathrm{H}(\Delta)U^*$. What are the main diagonal entries of $\mathrm{H}(\Delta)$? Apply the Schur majorization theorem (4.3.26) in [HJ] to $\mathrm{H}(\Delta)$, whose eigenvalues are the same as those of $\mathrm{H}(A)$.
(b) Use Theorem (4.3.32) in [HJ] to construct $B = B^T = [b_{ij}] \in M_n(\mathbb{R})$ with $b_{ii} = \operatorname{Re} \lambda_i$, $i = 1,\ldots, n$, and eigenvalues $\{\eta_i\}$. Let $Q^TBQ = \operatorname{diag}(\operatorname{Re} \lambda_1,\ldots, \operatorname{Re} \lambda_n)$ for $Q \in M_n(\mathbb{R})$, $Q^T = Q^{-1}$. Let $C = [c_{ij}] \in M_n$ have $c_{ii} = \lambda_i$, $c_{ij} = 2b_{ij}$ if $j > i$, and $c_{ij} = 0$ if $j < i$, and take $A \equiv Q^TCQ$.

21. (a) Let D be a diagonal unitary matrix such that DA has nonnegative main diagonal entries, and let $B_k \in M_k$ be the principal submatrix of DA corresponding to the main diagonal entries $|a|_{[1]},\ldots, |a|_{[k]}$. Argue that $|\operatorname{tr} B_k| \le \sigma_1(B_k) + \cdots + \sigma_k(B_k) \le \sigma_1(A) + \cdots + \sigma_k(A)$.

26. (c) Use induction and Theorem (3.3.14); Lemma (3.3.8); $p = 2$.

30. Use Theorem (3.3.21) and (3.3.18).

Section (3.4)

7. (a) Simultaneously unitarily diagonalize the family $\{P_i\}$ with $P_i = U\Lambda_i U^*$, Λ_i a direct sum of zero blocks and an identity matrix. Do the same with the family $\{Q_i\}$, $Q_i = VD_iV^*$, D_i a direct sum of zero blocks and an identity matrix. Then $U^*\hat{A}V = \Lambda_1(U^*AV)D_1 + \cdots + \Lambda_m(U^*AV)D_m$.

Use Problem 6.

8. $A + \alpha I$, $B + \beta I$, and $A + B + (\alpha + \beta)I$ are all positive definite for some $\alpha, \beta \geq 0$.

9. (b) Use successive interchanges of pairs of entries to transform Py to y, and invoke (a) at each step.

Section (3.5)

3. Note that $E_{11}^2 = E_{11}$ and use Corollary (3.5.10).

7. Write $A = B^*B$ and partition $B = [B_1 \;\; B_2]$, $B_1 \in M_{n,k}$, $B_2 \in M_{n,n-k}$. Now write A and $\det A_{\|\cdot\|}$ in terms of B_1 and B_2 and apply (3.5.22) to show that $\det A_{\|\cdot\|} \geq 0$. Problem 6 in Section (7.5) of [HJ] gives a counterexample in the block 4-by-4 case (the blocks are 1-by-1 matrices). For a 3-by-3 block counterexample, partition this same matrix so that $A_{11} = A_{33} = [10]$ and $A_{22} = \mathrm{diag}(10, 10)$. Compute the 3-by-3 norm compression obtained with the spectral norm and show that it has a negative eigenvalue.

8. Use the majorization relation between the main diagonal entries and the eigenvalues of AA^*.

13. Use Corollaries (3.1.3) and (3.5.9).

14. In this case, $\sigma(A) \geq \sigma(B)$ entrywise.

16. To prove the triangle inequality for $\nu(\cdot)$, use Problem 14 and the matrix-valued triangle inequality (3.1.15).

20. (c) Use the fact (Section (5.1) of [HJ]) that a norm $\nu(\cdot)$ on a real or complex vector space V is derived from an inner product if and only if it satisfies the parallelogram identity $\nu(x)^2 + \nu(y)^2 = \frac{1}{2}[\nu(x+y) + \nu(x-y)]$ for all $x, y \in V$. Define $g(x) = \|\mathrm{diag}(x)\|$ for all $x \in \mathbb{R}^n$ and use (b).

22. (a) Let $f(t_1, t_2) = t_1 + t_2$ in Corollary (3.5.11) and use (3.5.8).
(b) Write $A = U\Lambda U^*$ with $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$, $\lambda_1 \geq \cdots \geq \lambda_n \geq 0$, and a unitary $U \in M_n$. For a given $k \in \{1, \ldots, n\}$, let $\Lambda_1 \equiv \mathrm{diag}(\lambda_1, \ldots, \lambda_k, 0, \ldots, 0)$, $\Lambda_2 \equiv \mathrm{diag}(0, \ldots, 0, \lambda_{k+1}, \ldots, \lambda_n)$, $B \equiv U\Lambda_1 U^*$, $C \equiv U\Lambda_2 U^*$, and partition $B = [B_{ij}]_{i,j=1}^2$ and $C = [C_{ij}]_{i,j=1}^2$ conformally with A. Then $A = B + C$ and

$$N_k(A) = N_k(B) = \mathrm{tr}\, B = \mathrm{tr}\, B_{11} + \mathrm{tr}\, B_{22} = N_k(B_{11}) + N_k(B_{22})$$

$$\leq N_k(B_{11}+C_{11})+N_k(B_{22}+C_{22})=N_k(A_{11})+N_k(A_{22})$$

Note that $\operatorname{tr} B_{22}=\lambda_1(B_{22})+\cdots+\lambda_r(B_{22})=N_k(B_{22})$ for $r\equiv \operatorname{rank} B_{22}\leq k$ $=\operatorname{rank} B$, and recall the monotonicity theorem, Corollary (4.3.3) in [HJ].
(c) See Problem 7 in Section (3.4). In fact, the same argument shows that $\|\hat{X}\|\leq\|X\|$ for any unitarily invariant norm $\|\cdot\|$ on M_n and any pinching $\hat{X}$ of $X\in M_n$, even if X is not positive semidefinite.

Section (3.6)

1. If (b) holds, let Δ be a real upper triangular matrix with eigenvalues $\sqrt{\lambda_i}$ and singular values $\sqrt{\sigma_i}$; take $A=\Delta^T\Delta$. If (c) holds, let $A=\Delta^*\Delta$ and show that $\operatorname{diag}\Delta=(\sqrt{\lambda_1},\ldots,\sqrt{\lambda_n})$ and $\sigma_i(\Delta)=\sigma_i(A)^{\frac{1}{2}}$.

Section (3.7)

3. Apply Theorem (6.4.1) in [HJ] to the block matrix in Problem 46 of Section (3.1).

Section (3.8)

2. For each choice of indices j, l, m, p one has $E_{jl}\bullet E_{mp}=\Sigma_{i,q}\,\gamma_{jlmpiq}\,E_{iq}$. Show that $E_{ij}\,A\,E_{lm}\,B\,E_{pq}=a_{jl}b_{mp}E_{iq}$ if $A=\Sigma_{j,l}\,a_{jl}E_{jl}$, $B=\Sigma_{m,p}b_{mp}E_{mp}$.

Section (4.2)

10. If the ith row of A is nonzero, consider the ith diagonal block $(AA^*)_{ii}BB^*=(A^*A)_{ii}B^*B$ from the identity $AA^*\otimes BB^*=A^*A\otimes B^*B$.

13. Use the formula for $\operatorname{rank}(A\otimes B)$ given in Theorem (4.2.15).

15. Apply (4.2.13) repeatedly.

16. Realize that $\Pi(A)$ is a principal submatrix of $A^{\otimes n}$.

19. Use the same strategy as in the proof of Theorem (4.2.12).

24. Consider $(A-B)\otimes C+B\otimes(C-D)$.

29. Consider the Ky Fan 2-norm $N_2(X)\equiv\sigma_1(X)+\sigma_2(X)$ and $\frac{1}{2}N_2(X)$, $A=\operatorname{diag}(2,1)$, $B=\operatorname{diag}(3,2)$.

Section (4.3)

5. Express the coefficient matrix $K(T)$ of this linear transformation, as in Lemma (4.3.2), in Kronecker form (4.3.1b).

14. To describe the Jordan structure of $A \circ B$ associated with a zero eigenvalue, it suffices to determine rank $(A \circ B)^k = (\text{rank } A^k)(\text{rank } B^k)$, and these values are easily determined from the Jordan structures of A and B.

16. Use Lemma (4.3.1).

22. Use (4.3.9b).

Section (4.4)

8. Convert this equation to one involving the appropriate Kronecker sum and examine its eigenvalues.

9. Use the formula in (4.4.14).

13. First use the Schur triangularization theorem to reduce A to upper triangular form with the respective eigenvalues λ_i grouped together.

18. (d) $A \circ I_n$ and $I_n \circ A$ are normal and commute; Theorem (4.4.5).

20. (a) dim $C(B) = \dim C(B^k)$, $k = 1, 2,\ldots$ when all the eigenvalues of B are positive.
(b) If $B^2 = A = C^2$ and all the eigenvalues of B and C are positive, notice that the sizes of the Jordan blocks of A, B, and C are the same; in particular, B and C are similar. If $C = SBS^{-1}$ and $B^2 = C^2$, then S commutes with B^2 and hence with B.

Section (4.5)

2. See Problem 12 of Section (4.2).

6. If $B \in M_n$ and tr $B = n$, then $|||B||| \geq \rho(B) \geq 1$.

10. Use Theorem (4.5.4) to write $A = BC$ in which B and C have distinct positive spectra; B and C are products of two positive definites (Problem 9 in Section (7.6) of [HJ]). If $P_1P_2P_3P_4 = \alpha I$ and all P_i are positive definite, then $P_1P_2 = \alpha P_3^{-1}P_4^{-1}$ and $\alpha > 0$ (each product has positive spectrum). If $A = \alpha I$ and α is not positive, $A = (\alpha P^{-1})P$ for a non-scalar positive definite P.

11. If det $A = 1$, use (4.5.4) to write $A = BC$ with $\sigma(B) = \{1, b_1, b_1^{-1},\ldots, b_k, b_k^{-1}\}$ when $n = 2k+1$ and analogously for C.

12. Use (4.5.4) again.

15. (b) (γ) If (β), let $X_1 = XY$, $X_2 = X^{-1}$, $X_3 = Y^{-1}B$.
(c) $B^{-1}A = XYX^{-1}Y^{-1}$.
(d) (γ) If (β), then $A = (Y^{-1}B)(B^{-1}YX)B(B^{-1}Y)(X^{-1}Y^{-1}B)$.
(d) (ϵ) If (δ), let $X_2 = C^{-1}AB = DBC^{-1}$.

Section (5.1)

5. $|||C|||_2 = \max\{|x^*Cy|: x \text{ and } y \text{ are unit vectors}\}$. Recall that $|||C|||_2$ is an upper bound for any Euclidean row or column length of C.

Section (5.2)

6. $x^*(A \circ B)x$ is the sum of the entries of $(D_x^*AD_x) \circ B$.

10. $(a_{11} \cdots a_{nn})\det B \geq \det AB = \det(A \circ B) \geq (a_{11} \cdots a_{nn})\det B$.

13. Let $B = VMV^*$ with $M = \text{diag}(\mu_1,\ldots,\mu_n)$ and $V = [v_{ij}]$ unitary, and let $x = [x_i] \in \mathbb{C}^n$. Show that $x^*(A \circ B)x = \Sigma_{r,s}\lambda_r\mu_s|\Sigma_i x_iu_{ir}v_{is}|^2 \leq x^*(|A| \circ |B|)x$ and use the monotonicity theorem (4.3.3) in [HJ].

14. $\max\{\lambda_1(A \circ B), -\lambda_n(A \circ B)\} = \max\{|x^*(A \circ B)x|: x \in \mathbb{C}^n, \|x\|_2 = 1\}$ $\leq \max\{x^*(\text{abs}(A) \circ \text{abs}(B))x: \|x\|_2 = 1\}$.

15. (a) If tr $AB = 0$, then $\text{tr}[(UAU^*)(UBU^*)] = 0$ for every unitary U. Choose U so that $UAU^* = \Lambda_1 \oplus 0$ for a positive diagonal $\Lambda_1 \in M_r$. If $\text{tr}[(\Lambda_1 \oplus 0)(UBU^*)] = 0$, the first r main diagonal entries of the positive semidefinite matrix UBU^* are zero, so $UBU^* = 0 \oplus B_2$ with $B_2 \in M_{n-r}$.
(b) To show that (2) $\Rightarrow$ (1), apply (a) to $(D_x^*AD_x)$ and B^T.
(c) $(A \circ B)x = 0 \Rightarrow (D_x^*AD_x)B^T = 0 \Rightarrow (D_x^*AD_x) = 0 \Rightarrow x = 0$.

Section (5.3)

2. Partition S by columns and use the fact that the spectral norm $|||\cdot|||_2$ on M_n is compatible with the Euclidean norm $\|\cdot\|_2$ on $\mathbb{C}^n$. You may wish to refer to facts in Chapter 5 of [HJ].

3. Let $R = (A^{\frac{1}{2}}B^{\frac{1}{2}})^{-1}$ and $S = A^{\frac{1}{2}}B^{\frac{1}{2}}B^{-\frac{1}{2}}D_xB^{\frac{1}{2}T}$.

Section (5.4)

4. Use (5.4.2b(i)) and (5.5.1).

Section (5.5)

8. (b) If $A = U \circ \overline{U}$, write $U = [u_{ij}] = [u_1 \ u_2 \ u_3]$ with $u_i \in \mathbb{C}^3$ and show that $u_1 \perp u_2$ implies $u_{21}u_{22} = 0$, which contradicts $a_{21}a_{22} \neq 0$.
(d) A convex combination of unitary matrices is a contraction.

Section (5.7)

2. Consider the role of the irreducible components, determine the case of equality in Lemma (5.7.8), and use (5.7.10).

5. Either (1) mimic the proof for positive definite matrices in [HJ], noting the differences in this case, or (2) apply the monotonicity of det for M-matrices [Theorem (2.5.4(c))] and the fact that an M-matrix may be scaled so that its inverse is diagonally dominant of its column entries, [(2.5.3.13) and (2.5.12)].

7. Apply the weighted arithmetic-geometric mean inequality (Appendix B of [HJ]) to the product

$$\prod_{\substack{i,\,j=1\\ p_{ij}>0}}^{n} \left[\frac{q_{ij}v_iu_j/v^TQu}{p_{ij}y_ix_j/y^TPx}\right]^{\frac{p_{ij}y_ix_j}{y^TPx}}$$

and simplify, as in the proof of Lemma (5.7.28).

10. Use the result of Problem 7.

15. (b) $A \circ B \geq A \star B$. Use Problems 31 and 19 in Section (2.5) and Corollary (5.7.4.1).

Section (6.1)

2. $P(0) = 0$ implies $A_0 = 0$, $P'(0) = 0$ implies $A_1 = 0$, and so forth.

9. If $P(\lambda)x = 0$ and $x \neq 0$, consider $x^*P(\lambda)x$. What can you say about the roots of $at^2 + bt + c = 0$ if a, b, and c are all positive?

15. Use (6.1.41) to evaluate $p(s,B)$. Use it again with each of the polynomials $p_{0,l}(s,\eta_j)$, $l = 0, 1,\dots, \beta_j$, $j = 1,\dots, \nu$, to evaluate $p_{0,l}(A,\eta_j)$. Then use (6.1.40.4) and the commutativity hypothesis.

22. Calculate $\Delta^2 f(a,b,(1-t)a + tb)$ and use (6.1.44). For the converse, use (6.1.46) with $a = \xi - h$, $b = \xi + h$; divide by h^2 and let $h \to 0$.

Section (6.2)

11. Consider the scalar-valued stem function $f(t) = (1-t)/(1+t)$. What is $|f(it)|$ if $t \in \mathbb{R}$? What is Re $f(t)$ if $|t| = 1$?

18. If $b \neq a$, explicitly diagonalize the matrix and use (6.2.1). To handle the case in which $b = a$, either pass to the limit, use (6.2.7), or use the power series for e^t to compute

$$e^{\begin{bmatrix} a & c \\ 0 & a \end{bmatrix}} = e^{aI + \begin{bmatrix} 0 & c \\ 0 & 0 \end{bmatrix}} = e^a e^{\begin{bmatrix} 0 & c \\ 0 & 0 \end{bmatrix}} = e^a\left[I + \begin{bmatrix} 0 & c \\ 0 & 0 \end{bmatrix}\right]$$

22. $\det e^X = e^{\operatorname{tr} X}$ for any $X \in M_n$; see Problem 4.

37. Define $f(J_2(0))$ in the obvious way by continuity and show that if $f(J_2(\lambda))$ were a Lipschitz continuous function of $\lambda \in (0,1)$ with respect to a given norm $\|\cdot\|$, then $\|f(J_2(\lambda)) - f(J_2(0))\| \le L\lambda$ for some $L > 0$.

40. Let $A = SJS^{-1}$ be a Jordan canonical form for A. Show that B is similar to $C = [f_{ij}(J)]$ via the similarity $S \oplus \cdots \oplus S$. Construct a permutation matrix P such that the diagonal blocks of P^TCP are $[f_{ij}(\lambda_1)],\dots, [f_{ij}(\lambda_n)]$ and observe that P^TCP is block upper triangular.

45. Use (6.5.7).

Section (6.3)

1. See the problems at the end of Section (7.5) of [HJ].

2. Consider $\lim_{\alpha \to 0} A^{(\alpha)}$.

10. Let $a \equiv \inf\{t > 0:\ \phi(t) = 0\}$ and consider $A \equiv [\phi(t_i - t_j)] \in M_3$ with $t_1 = a$, $t_2 = a/2$, and $t_3 = 0$.

11. $1/t^\alpha = \int_0^\infty e^{-ts} s^{\alpha-1}\, ds/\Gamma(\alpha)$ for any α, $t > 0$.

12. Show that $\Delta B = -I$ and that $A = CC^T$, where $C = [c_{ij}]$ is lower triangular and $c_{ij} = 1/i$ for $j = 1,\ldots, i$. Show that det $A = 1/(n!)^2$.

13. Show that $a_{ij} = i^{-1}j^{-1}/(1-\alpha_{ij})^{-1}$ with $\alpha_{ij} = (i-1)(j-1)/ij$. Then $\log \alpha_{ij} = -\log i - \log j + \Sigma_{k=1}^\infty (\alpha_{ij})^k/k$.

14. The doubly stochastic matrix B has exactly one positive eigenvalue, which is its Perron root, and $Be = \rho(B)e$. If $x \in \mathbb{R}^n$ and $x \perp e$, then x is a linear combination of eigenvectors of B corresponding to nonpositive eigenvalues, so $x^TBx \leq 0$.

Section (6.4)

7. Compute A^2 and try to solve the four equations. It simplifies the calculation to notice that all the eigenvalues of A are zero (since all the eigenvalues of A^2 are zero), so tr $A = a + d = 0$. What is the 1,2 entry of A^2?

13. What are the allowed forms for the $(m-1)$-tuples $\Delta_1 \equiv (k_1 - k_2, k_2 - k_3, \ldots, k_{m-1} - k_m), \ldots$?

23. B is similar to a symmetric matrix C (Theorem (4.4.9) in [HJ]), so the symmetric matrices A and C^2 are similar. Use Corollary (6.4.19).

26. Use the argument in the proof of Theorem (1.3.20) in [HJ] to show that rank $\begin{bmatrix} AA^T & 0 \\ A^T & 0 \end{bmatrix}^k = \text{rank} \begin{bmatrix} 0 & 0 \\ A^T & A^TA \end{bmatrix}^k$, $k = 1,\ldots, n$. Conclude that the nonsingular Jordan blocks of AA^T and A^TA are identical. What does condition (a) say about the singular Jordan blocks in AA^T and A^TA? Use Corollary (6.4.19).

27a. Calculate $C \equiv \text{Log}\, U$ by (6.4.21) and use Theorem (6.4.20) to show that $C^* = -C$; take $B = C/i$. One can also use the spectral theorem.

27b. Use the classical polar decomposition (see (7.3.3) in [HJ]) to write $A = PU$ and use Problem 27a.

27c. Calculate $C \equiv \text{Log}\, U$ by (6.4.21) and use Theorem (6.4.20) to show that $C = C^T$ and C is purely imaginary.

27d. Use Problem 27c to determine a real symmetric matrix B such that $U^TU = e^{2iB}$, and set $Q = Ue^{-iB}$. Show that Q is unitary as well as orthogonal, and conclude that it is real.

27e. Calculate $C \equiv \text{Log } P$ by (6.4.21) with $\theta = \pi$ and use Theorem (6.4.20) to show that $C = -C^T$ and C is purely imaginary.

27f. Use Problem 27e to determine a real skew-symmetric B such that $P^*P = e^{2iB}$, and set $Q = Pe^{-iB}$. Show that Q is both unitary and orthogonal, and conclude that it is real.

32. Use Problem 27f to write $P = Qe^{iB}$, where $Q, B \in M_n(\mathbb{R})$, $QQ^T = I$, $B^T = -B$, and B is a polynomial in P^*P. Since P is normal, use Theorem (7.3.4) in [HJ] to show that Q commutes with e^{iB} and compute $P = P^*$ to show that $Q^T = Q$. Then B is a polynomial in $P^2 = (e^{iB})^2$, so Q commutes with B. Let $C \equiv e^{iB}$, which is positive definite, and write $P = C^{\frac{1}{2}}[C^{-\frac{1}{2}}QC^{\frac{1}{2}}]C^{\frac{1}{2}}$, so P is congruent to a matrix that is similar to Q. Since P and Q have the same signature, $Q = I$ if P is positive definite.

33. Set $B = A^TX$, so rank B = rank A, $A = XB$, $A^T = BX^{-1}$, and $A^TA = B^2$.

34. (a) If $[A \;\; 0] = QS$, write $Q = [Q_1 \;\; Q_2]$ with $Q_1 \in M_{m,n}$ and $S = \begin{bmatrix} S_{11} & S_{12} \\ S_{12}^T & S_{22} \end{bmatrix}$ with $S_{11} \in M_m$. Then $A = Q_1S_{11} + Q_2S_{12}^T$, $0 = Q_1S_{12} + Q_2S_{22}$, $Q_1^TQ_1 = I$, and $Q_1^TQ_2 = 0$. Now show that $S_{12} = 0$, so $A = Q_1S_{11}$.

35. If range A = range B (that is, the column spaces are the same), then rank A = rank $B = r$ and there are permutation matrices $P, Q \in M_n$, matrices $C_1, C_2 \in M_{r,n-r}$, and a nonsingular $S_1 \in M_r$ such that $AP = [A_1 \;\; A_2]$, $BQ = [B_1 \;\; B_2]$ with $A_1, B_1 \in M_{n,r}$ having full rank r, $A_2 = A_1C_1$, $B_2 = B_1C_2$, $B_1 = A_1S_1$. Notice that $[A_1 \;\; A_1C_1] = [A_1 \;\; 0]\begin{bmatrix} I & C_1 \\ 0 & I \end{bmatrix}$.

37. (a) Denote the entries in the first row of K by $\alpha_0, \alpha_1, \ldots, \alpha_{m-1}$. Explicitly compute the entries in the first rows of $K^2 = J_m(\lambda)^2$, and solve sequentially for $\alpha_0, \alpha_1, \ldots, \alpha_{m-1}$.
(c) See Example (6.2.14). If $C = SJS^{-1}$, then $C^2 = SJ^2S^{-1} = A$, so $B = p(A) = Sp(J^2)S^{-1}$. Note that $K \equiv p(J^2)$ satisfies the hypotheses in (b). But $B^2 = SK^2S^{-1} = SJ^2S^{-1}$, so $K = J$ and $B = C$.

Section (6.5)

4a. Let $X^T(t) = [x_1(t) \; x_2(t) \; \dots \; x_n(t)]$, where each $x_i(t) \in \mathbb{C}^n$. Note that $\det X(t) = \det X^T(t)$ and show that

$$\begin{aligned}\tfrac{d}{dt}\det X(t) = \tfrac{d}{dt}\det X^T(t) &= \det\left[\frac{dx_1}{dt}\; x_2 \;\dots\; x_n\right] + \cdots\\ &= \det\Big[[a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n] \; x_2 \;\dots\; x_n\Big] + \cdots\\ &= \det[a_{11}x_1 \; x_2 \;\dots\; x_n] + \cdots = a_{11}(t)\det X^T(t) + \cdots\\ &= \operatorname{tr} A(t) \det X^T(t) = \operatorname{tr} A(t) \det X(t)\end{aligned}$$

Show that this first-order ordinary differential equation has the solution (6.5.40).

4e. Use the power series for the matrix exponential to show necessity. For sufficiency, show that $aI + A \geq 0$ for some $a > 0$, so that $e^A = e^{-aI+aI+A} = (e^{-aI})e^{aI+A} = e^{-a}\,e^{aI+A} \geq 0$.

5b. Integrate (6.5.42) to show that

$$\int_0^t \tfrac{d}{ds}X(s)\, ds = X(t) - C = -A\left[\int_0^t X(s)\, ds\right] - \left[\int_0^t X(s)\, ds\right]B$$

Now let $t \to \infty$.

5c. Use (6.2.6) to express e^{-tA} and show that all of the Jordan blocks of $e^{\tau(A)t}\,e^{-tA}$ remain bounded as $t \to \infty$.

8. As in Theorem (6.5.35), show that $X(0) = I$ and $\frac{d}{dt}X(t) = AX(t)$ if $X(t) = q_0(t)I + q_1(t)A + \cdots + q_{k-1}(t)A^{k-1}$.

15. See Problem 18 in Section (7.3) of [HJ].

16. Use (2.5.3.2) to write $A = \alpha I - P$ with $P \geq 0$ and $\alpha > \rho(P)$. Then

$$\left[\tfrac{1}{\alpha}A\right]^{1/m} = \left[I - \tfrac{1}{\alpha}P\right]^{1/m} = \sum_{k=0}^{\infty}(-1)^k\begin{bmatrix}\frac{1}{m}\\ k\end{bmatrix}\left[-\tfrac{1}{\alpha}P\right]^k = I - Q$$

where $Q \geq 0$ and $\rho(Q) < 1$, so $A^{1/m} = \alpha^{1/m}\left[\tfrac{1}{\alpha}A\right]^{1/m}$ is an M-matrix. It may be helpful to use (5.6.10) in [HJ] to guarantee that there is a matrix norm $|||\cdot|||$ for which $|||\frac{1}{\alpha}P||| < 1$; recall that $\binom{t}{k} = t(t-1)(t-2)\cdots(t-k+1)/k!$, even when t is not a positive integer.

19. $e^{tA}e^{tB}e^{-tA}e^{-tB} = e^{\log[I-(I-e^{tA}e^{tB}e^{-tA}e^{-tB})]} = \cdots$ as in (6.5.16-17).

21. It suffices to consider only diagonal $A = \Lambda = \mathrm{diag}(\lambda_1,\ldots,\lambda_n)$. Why? Compute $\mathrm{tr}(\Lambda^\alpha B^{1-\alpha})$ and use Hölder's inequality.

26. Use (6.5.54a) and see the proof of Corollary (6.5.22).

28. Replace X by $e^{A/2m}e^{B/2m}$ in (6.5.56) and use (6.5.19) and Problem 27.

29. Write $\mathrm{tr}\, e^{A+A^*} = e^{\lambda_1} + \cdots + e^{\lambda_n}$.

30. Show that (6.5.23) becomes $e^B - e^A = (B-A)\,e^A\int_0^1 e^{t(B-A)}\,dt$ when A and B commute.

33. Apply (6.5.25) to (6.5.24) and integrate.

34. Differentiate $A(t) \equiv e^{-tmA}(e^{tA})^m$.

35. (f) See Problem 12 in Section (2.4) of [HJ].

36. Differentiate $P(t)^m = P(t)$ and left-multiply by $P(t)$.

42. See the preceding problem for necessity. For sufficiency, use Theorem (4.3.32) in [HJ] to construct a real symmetric $B = [b_{ij}] \in M_n(\mathbb{R})$ with main diagonal entries $b_{ii} = \mathrm{Re}\,\beta_i$ and $\sigma(B) = \{\eta_i\}$. Let $B = PHP^T$ with a real orthogonal P and a diagonal $H = [\eta_{ij}]$ with $\eta_{ii} = \eta_i$. Form an upper triangular $\Delta = [d_{ij}]$ with $d_{ii} = \beta_i$ and $d_{ij} = 2b_{ij}$ for all $j > i$. Check that $A \equiv P\Delta P^T$ has the asserted properties.

43. Use (6.5.20(2)) with the absolute eigenvalue product functions $\varphi_m(\cdot)$ defined in Corollary (6.5.22). Notice that

$$\prod_{i=1}^m \lambda_i(e^{\frac{1}{2}(A+A^*)}\,e^{\frac{1}{2}(B+B^*)}) \le \prod_{i=1}^m \sigma_i(e^{\frac{1}{2}(A+A^*)}\,e^{\frac{1}{2}(B+B^*)})$$

$$\le \prod_{i=1}^m \sigma_i(e^{\frac{1}{2}(A+A^*)})\sigma_i(e^{\frac{1}{2}(B+B^*)}) = \prod_{i=1}^m \lambda_i(e^{\frac{1}{2}(A+A^*)})\lambda_i(e^{\frac{1}{2}(B+B^*)})$$

for $m = 1,\ldots, n$.

44. See Corollaries (6.5.22) and (3.5.10).

Section (6.6)

6. Use Corollary (6.2.11)

9. Consider det $[\Delta f(t_i, t_j)]^2_{i,j=1}$.

10. Use the definition (6.6.33).

17. Using the result in Problem 12, it suffices to consider only the functions $g_u(t) \equiv t/(t+u)$ for $u \geq 0$.

22. Use a Taylor expansion to show that $h^2 e^h \geq (e^h - 1)^2$ is false for all sufficiently small $h > 0$.

28. Show that $\Delta^2 f(s,t,u) = (stu)^{-\frac{1}{2}}(\sqrt{s}+\sqrt{u})^{-1}(\sqrt{t}+\sqrt{u})^{-1} + (st)^{-\frac{1}{2}}(\sqrt{s}+\sqrt{u})^{-1}(\sqrt{t}+\sqrt{u})^{-1}(\sqrt{s}+\sqrt{t})^{-1}$, then show that the sum in (6.6.54) reduces to $K\circ[(Z\circ X)D(Z\circ X)] + K\circ Z\circ(Z\circ X)^2$, where $K \equiv [(t_it_j)^{-\frac{1}{2}}]^n_{i,j=1}$, $Z \equiv [(\sqrt{t_i}+\sqrt{t_j})^{-1}]^n_{i,j=1}$, $D = \mathrm{diag}(t_1^{-1},\ldots, t_n^{-1})$, $t_1 \geq \cdots \geq t_n > 0$, and X is nonsingular and Hermitian. Although both terms are positive semidefinite, the first can be singular; see Problem 40. Use Corollary (5.3.7) [as in the third exercise following Theorem (6.6.52)] and Theorem (5.2.1) to show that the term $K\circ[Z\circ(Z\circ X)^2]$ is always positive definite.

29. Consider det $[\Delta^2 f(t_i, t_j, t_1)]^2_{i,j=1}$. Note (6.1.44).

31. Use the criterion (6.6.61), then use a Taylor series expansion to show that $e^h(e^h - 1 - h)h^2 \geq 2(e^h - 1 - he^h)^2$ is false for all sufficiently small $h > 0$.

37. Consider

$$A = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda - \epsilon \end{bmatrix} \text{ and } B = \begin{bmatrix} \lambda + \epsilon & \sqrt{2}\epsilon \\ \sqrt{2}\epsilon & \lambda + \epsilon \end{bmatrix}$$

for $\lambda \in (a,b)$ and small $\epsilon > 0$.

39. Use an upper bound from Theorem (5.5.18) and a lower bound from Theorem (5.3.4). Recall that $\xi^* X \xi \geq \lambda_{min}(X)$ for any Hermitian matrix X and unit vector ξ.

40. Use Corollary (5.3.7) and Theorem (5.2.1).

43. Use the identity (6.6.64) and Corollary (6.6.19).