1. RANDOM VARIABLES

1.1 Introduction

In application of probability, we are often interested in a number associated with the outcome of a random experiment. Such a quantity whose value is determined by the outcome of a random experiment is called a **random variable**. It can also be defined as any quantity or attribute whose value varies from one unit of the population to another.

A **discrete** random variable is function whose range is finite and/or countable, Ie it can only assume values in a finite or countably infinite set of values. A **continuous** random variable is one that can take any value in an interval of real numbers. (There are *uncountably* many real numbers in an interval of positive length.)

1.2 Discrete Random Variables and Probability Mass Function

Consider the experiment of flipping a fair coin three times. The number of tails that appear is noted as a discrete random variable. *X*= number of tails that appear in 3 flips of a fair coin. There are 8 possible outcomes of the experiment: namely the sample space consists of

$$S = \{\text{HHH}, \text{HHT}, \text{HTH}, \text{HTT}, \text{THH}, \text{THT}, \text{TTH}, \text{TTT}\}$$

$$\text{X} = \{\ 0 \quad 1, \quad 1, \quad 2, \quad 1, \quad 2, \quad 2, \quad 3\}$$

are the corresponding values taken by the random variable *X*.

Now, what are the possible values that *X* takes on and what are the probabilities of *X* taking a particular value?

From the above we see that the possible values of *X* are the 4 values

$$\text{X} = \{0, 1, 2, 3\}$$

Ie the sample space is a disjoint union of the 4 events $\{X = j\}$ for j=0,1,2,3

Specifically in our example:

$\{\text{X} = 0\} = \{\text{HHH}\}$ $\quad\quad$ $\{\text{X} = 1\} = \{\text{HHT, HTH, THH}\}$

$\{\text{X} = 2\} = \{\text{TTH, HTT, THT}\}$ $\quad\quad$ $\{\text{X} = 3\} = \{\text{TTT}\}$

Since for a fair coin we assume that each element of the sample space is equally likely (with probability $\frac{1}{8}$, we find that the probabilities for the various values of *X,* called the *probability distribution* of *X* or the *probability mass function (pmf).* can be summarized in the following table listing the possible values beside the probability of that value

x	0	1	2	3
P(X=x)	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

Note: The probability that *X* takes on the value *x,* ie $\text{p}(\text{X} = \text{x})$, is defined as the sum of the probabilities of all points in *S* that are assigned the value *x*.

We can say that this pmf places mass $\frac{3}{8}$ on the value $\text{X} = 2$.

The "masses" (or probabilities) for a pmf should be between 0 and 1.

The total mass (i.e. total probability) must add up to 1.

Definition: The **probability mass function** of a discrete variable is a graph, table, or formula that specifies the proportion (or probabilities) associated with each possible value the random variable can take. The mass function $\text{P}(\text{X} = \text{x})$ (or just p(x) has the following properties:

$$0 \le \text{p}(\text{x}) \le 1 \text{ and } \sum_{\text{all x}} \text{p}(\text{x}) = 1$$

More generally, let X have the following properties

i) It is a discrete variable that can only assume values $x_1, x_2, \ldots. x_n$

ii) The probabilities associated with these values are
$P(X = x_1) = p_1$, $P(X = x_2) = p_2 \ldots\ldots P(X = x_n) = p_n$

Then X is a discrete random variable if $0 \le p_i \le 1$ and $\sum_{i=1}^{n} p_i = 1$

Remark: We denote random variables with capital letters while realized or particular values are denoted by lower case letters.

Example 1
Two tetrahedral dice are rolled together once and the sum of the scores facing down was noted. Find the pmf of the random variable 'the sum of the scores facing down.'
Solution

+	1	2	3	4
1	2	3	4	5
2	3	4	5	6
3	4	5	6	7
4	5	6	7	8

$X = \{1, 2, 3, 4, 5, 6, 7, 8\}$

Therefore t is given the pmf by the table below

x	2	3	4	5	6	7	8
P(X=x)	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{3}{16}$	$\frac{1}{4}$	$\frac{3}{16}$	$\frac{1}{8}$	$\frac{1}{16}$

This can also be written as a function

$$P(X = x) = \begin{cases} \frac{x-1}{16} \text{ for } x = 2, 3, 4, 5 \\ \frac{9-x}{16} \text{ for } x = 6, 7, 8 \end{cases}$$

Example 2
The pmf of a discrete random variable W is given by the table below

w	-3	-2	-1	0	1
P(W=w)	0.1	0.25	0.3	0.15	d

Find the value of the constant d, $P(-3 \le \text{w} < 0)$, $P(\text{w} > \text{-}1)$ and $P(-1 < \text{w} < 1)$
Solution

$$\sum_{\text{all w}} \text{p(W = w)} = 1 \quad \Rightarrow \quad 0.1 + 0.25 + 0.3 + 0.15 + d = 1 \quad \Rightarrow \quad d = 0.2$$

$$P(-3 \le \text{w} < 0) = P(W = -3) + P(W = -2) + P(W = -1) = 0.65$$

$$P(\text{w} > \text{-}1) = P(\text{w} = 0) + P(\text{w} = 1) = 0.15 + 0.2 = 0.35$$

$$P(-1 < \text{w} < 1) = P(W = 0) = 0.15$$

Example 3
A discrete random variable Y has a pmf given by the table below

y	0	1	2	3	4
P(Y=y)	c	2c	5c	10c	17c

Find the value of the constant c hence computes $P(1 \le \text{Y} < 3)$
Solution

$$\sum_{\text{ally}} \text{p(Y = y)} = 1 \quad \Rightarrow \quad c(1 + 2 + 5 + 10 + 17) = 1 \quad \Rightarrow \quad c = \tfrac{1}{35}$$

$$P(1 \le \text{Y} < 3) = P(Y = 1) + P(Y = 2) = \tfrac{2}{35} + \tfrac{5}{35} = \tfrac{1}{5}$$

Exercise
1. A die is loaded such that the probability of a face showing up is proportional to the face number. Determine the probability of each sample point.
2. Roll a fair die and let X be the square of the score that show up. Wtrie down the probability distribution of X hence compute $P(X < 15)$ and $P(3 \le \text{X} < 30)$

3. Let X be the random variable the number of fours observed when two dice are rolled together once. Show that X is a discrete random variable.
4. The pmf of a discrete random variable X is given by $P(X = x) = kx$ for $x = 1, 2, 3, 4, 5, 6$ Find the value of the constant k, $P(X < 4)$ and $P(3 \le X < 6)$
5. A fair coin is flip until a head appears. Let N represent the number of tosses required to realize a head. Find the pmf of N
6. A discrete random variable Y has a pmf given by $P(Y = y) = c\left(\frac{3}{4}\right)^x$ for $y = 0, 1, 2, ...$ Find the value of the constant c and $P(X < 3)$
7. Verify that $f(x) = \dfrac{2x}{k(k+1)}$ for $x = 0, 1, 2, ...k$ can serve as a pmf of a random variable X.
8. For each of the following determine c so that the function can serve as a pmf of a random variable X.
   a) $f(x) = cx$ for $x = 1, 2, 3, 4, 5$
   b) $f(x) = cx^2$ for $x = 0, 1, 2, ...k$
   c) $f(x) = c\left(\frac{1}{6}\right)^x$ for $x = 0, 1, 2, 3.....$
   d) $f(x) = c2^{-x}$ for $x =$ for $x = 0, 1, 2, ...$
9. A coin is loaded so that heads is three times as likely as the tails. For 3 independent tosses of the coin find the pmf of the total number of heads realized and the probability of realizing at most 2 heads.

1.3 Continuous Random Variables and Probability Density Function

A **continuous** random variable can assume any value in an interval on the real line or in a collection of intervals. The sample space is uncountable. For instance, suppose an experiment involves observing the arrival of cars at a certain period of time along a highway on a particular day. Let T denote the time that lapses before the 1st arrival, the T is a continuous random variable that assumes values in the interval $[0, \infty)$

Definition: A random variable *X* is *continuous* if there exists a nonnegative function f so that, for every interval *B*, $P(X \in B) = \int_B f(x)\, dx$. The function $f = f(x)$ is called the *probability density function* of *X*.

Definition: Let X be a continuous random variable that assumes values in the interval $(-\infty, \infty)$, The f(x) is said to be a probability density function (pdf) of X if it satisfies the following conditions

$$f(x) \ge 0 \text{ for all x }, \; p(a \le x \le b) = \int_a^b f(x)\, dx = 1 \text{ and } \int_{-\infty}^{\infty} f(x)\, dx = 1$$

The support of a continuous random variable is the smallest interval containing all values of x where f(x) >= 0.

Remark A crucial property is that, for any real number x, we have $P(X = x) = 0$ (implying there is no difference between $P(X \le x)$ and $P(X < x)$); that is it is not possible to talk about the probability of the random variable assuming a particular value. Instead, we talk about the probability of the random variable assuming a value within a given interval. The probability of the random variable assuming a value within some given interval from $x = a$ to $x = b$ is defined to be the area under the graph of the probability density function between $x = a$ and $x = b$.

Example 1

Let X be a continuous random variable. Show that the function