2.2 Continuous Distribution

2.2.1 Uniform (Rectangular) Distribution

A **uniform density function** is a density function that is constant, (*Ie all the values are* equally likely outcomes over the domain). Often referred as the *Rectangular distribution* because the graph of the pdf has the form of a rectangle, making it the simplest kind of density function. The uniform distribution lies between two values on the x-axis. The total area is equal to 1.0 or 100% within the rectangle

Definition: A random variable X has a uniform distribution over the range $[a, b]$ If

$$\mathrm{f(x)} = \begin{cases} \frac{1}{b-a}, & a \le x \le b \\ 0, & elsewhere \end{cases}$$ We denote this distribution by $X \sim U(a, b)$

where: a = smallest value the variable can assume and b = largest value.

The expected Value and the Variance of *X* are given by $\mu = \dfrac{a+b}{2}$ and $\sigma^2 = \dfrac{(b-a)^2}{12}$

respectively. The cdf F(x) is given by

$$\mathrm{F(x)} = \frac{1}{b-a}\int_a^x dt = \frac{x-a}{b-a} \qquad \Rightarrow \quad \mathrm{F(x)} = \begin{cases} 0 & x < a \\ \frac{x-a}{b-a}, & a \le x \le b \\ 1 & x > b \end{cases}$$

Example Prof Hinga travels always by plane. From past experience he feels that take off time is uniformly distributed between 80 and 120 minutes after check in. determine the probability that: a) he waits for more than 15 minutes for take off after check in. b) the waiting time will be between 1.5 standard deviation from the mean,

Solution

$$X \sim U(80,120) \quad \Rightarrow \quad \mathrm{f(x)} = \begin{cases} \frac{1}{40}, & 80 \le x \le 120 \\ 0, & elsewhere \end{cases}$$

$$\mathrm{P(X > 105) = 1 - P(X \le 105) = 1 -} \tfrac{105-80}{40} = \tfrac{3}{8}$$

$$\mathrm{P}(\mu - 1.5\sigma \le x \le \mu + 1.5\sigma) = \int_{\mu-1.5\sigma}^{\mu+1.5\sigma} \tfrac{1}{40}\, dx = \tfrac{1}{40}\left[x\right]_{\frac{1}{4\mu-1.5\sigma 0}}^{\mu-1.5\sigma} = \frac{3\sigma}{40} \qquad \text{But } \sigma = \frac{b-a}{\sqrt{12}} = \frac{40}{\sqrt{12}}$$

$$\mathrm{P}(\mu - 1.5\sigma \le x \le \mu + 1.5\sigma) = \frac{3\sigma}{40} = \frac{3}{\sqrt{12}}$$

Exercise

1. Uniform: The amount of time, in minutes, that a person must wait for a bus is uniformly distributed between 0 and 15 minutes, inclusive. What is the probability that a person waits fewer than 12.5 minutes? What is the probability that will be between 0.5 standard deviation from the mean,
2. Slater customers are charged for the amount of salad they take. Sampling suggests that the amount of salad taken is uniformly distributed between 5 ounces and 15 ounces. Let *x* = salad plate filling weight, find the expected Value and the Variance of *x*. What is the probability hat a customer will take between 12 and 15 ounces of salad?
3. The average number of donuts a nine-year old child eats per month is uniformly distributed from 0.5 to 4 donuts, inclusive. Determine the probability that a randomly selected nine-year old child eats an average of;
    a) more than two donuts
    b) more than two donuts given that his or her amount is more than 1.5 donuts.
4. Starting at 5 pm every half hour there is a flight from Nairobi to Mombasa. Suppose that none of these plane tickets are completely sold out and they always have room for

passagers. A person who wants to fly to Mombasa arrives at the airport at a random time between 8.45 AM and (.45 AM. Determine the probability that he waits for

a) At most 10 minutes   b) At least 15 minutes

2.2.2 Exponential Distribution

The exponential distribution is often concerned with the amount of time until some specific event occurs. For example, the amount of time (beginning now) until an earthquake occurs has an exponential distribution. Other examples include the length, in minutes, of long distance business telephone calls, and the amount of time, in months, a car battery lasts. It can be shown, too, that the amount of change that you have in your pocket or purse follows an exponential distribution. Values for an exponential random variable occur in the following way. There are fewer large values and more small values. For example, the amount of money customers spend in one trip to the supermarket follows an exponential distribution. There are more people that spend less money and fewer people that spend large amounts of money. The exponential distribution is widely used in the field of reliability. Reliability deals with the amount of time a product lasts

In brief this distribution is commonly used to model waiting times betweenoccurrences of rare events, lifetimes of electrical or mechanical devices

Definition: A RV X is said to have an exponential distribution with parameter $\lambda > 0$ if the pdf of X is: $\text{f(x)} = \begin{cases} \lambda e^{-\lambda x} \text{for x} \geq 0 \text{ and } \lambda > 0 \\ 0 \quad otherwise \end{cases}$ we abbreviate this as $\text{X} \sim \exp(\lambda)$

λ is called the *rate parameter*

The mean and variance of this distribution are $\mu = \dfrac{1}{\lambda}$ and $\sigma^2 = \dfrac{1}{\lambda^2}$ respectively

The cumulative distribution function is F(x) is given by $\text{F(x)} = \begin{cases} 1 - e^{-\lambda x} \text{for x} \geq 0 \\ 0 \quad otherwise \end{cases}$

Example Torch batteries have a lifespan T years with pdf $\text{f(t)} = \begin{cases} 0.01 e^{-0.01t}, \text{T} \geq 0 \\ 0 \quad otherwise \end{cases}$. Determine the probability that the battery; a) Falls before 25 hours. b) life is between 35 and 50 hours. *c)* life exceeds 120 hours. d) life exceeds the mean lifespan.

S*olution*

a) $\text{P(T} < 25) = \text{F(25)} = \int_0^{25} e^{-0.01t} dt = 1 - e^{-0.01(25)} \approx 0.2212$

b) $\text{P(35} \leq \text{T} \leq 50) = \int_{35}^{50} e^{-0.01t} dt = e^{-0.35} - e^{-0.50} \approx 0.0982$

c) $\text{P(T} > 120) = \int_{120}^{\infty} e^{-0.01t} dt = e^{-1.2} - 0 \approx 0.3012$

d) $\mu = \dfrac{1}{0.01} = 100 \Rightarrow \text{P(T} > 100) = \int_{100}^{\infty} e^{-0.01t} dt = e^{-1} \approx 0.3679$

Exercise:

1. Jobs are sent to a printer at an average of 3 jobs per hour.
   a) What is the expected time between jobs?
   b) What is the probability thatthe next job is sent within 5 minutes?
2. The time required to repair a machine is an exponential random variable with rate λ= 0.5 downs/hour
   a) what is the probability that a repair time exceeds 2 hours?

b) what is the probability that the repair time will take at least 4 hours given that the repair man has been working on the machine for 3 hours?

3. Buses arrive to a bus stop according to an exponential distribution with rate λ= 4 busses/hour. If you arrived at 8:00 am to the bus stop,
   a) what is the expected time of the next bus?
   b) Assume you asked one of the people waiting for the bus about the arrival time of the last bus and he told you that the last bus left at 7:40 am. What is the expected time of the next bus?
4. Break downs occur on an old car with rate λ= 5 break-downs/month. The owner of the car is planning to have a trip on his car for 4 days.
   a) What is the probability that he will return home safely on his car.
   b) If the car broke down the second day of the trip and the car was fixed, what is the probability that he doesn't return home safely on his car.
5. Suppose that the amount of time one spends in a bank is exponentially distributed with mean 10 minutes. What is the probability that a customer will spend more than 15 minutes in the bank? What is the probability that a customer will spend more than 15 minutes in the bank given that he is still in the bank after 10 minutes?
6. Suppose the lifespan in hundreds of hours, T, of a light bulb of a home lamp is exponentially distributed with lambda = 0.2. compute the probability that the light bulb will last more than 700 hours Also, the probability that the light bulb will last more than 900 hours
7. Let X = amount of time (in minutes) a postal clerk spends with his/her customer. The time is known to have an exponential distribution with the average amount of time equal to 4 minutes.
   a) Find the probability that a clerk spends four to five minutes with a randomly selected customer.
   b) Half of all customers are finished within how long? (Find median)
   c) Which is larger, the mean or the median?
8. On the average, a certain computer part lasts 10 years. The length of time the computer part lasts is exponentially distributed.
   a) What is the probability that a computer part lasts more than 7 years?
   b) On the average, how long would 5 computer parts last if they are used one after another?
   c) Eighty percent of computer parts last at most how long?
   d) What is the probability that a computer part lasts between 9 and 11 years?
9. Suppose that the length of a phone call, in minutes, is an exponential random variable with decay parameter = 1/12 . If another person arrives at a public telephone just before you, find the probability that you will have to wait more than 5 minutes. Let X = the length of a phone call, in minutes. What is median mean and standard deviation of X?

2.2.3 Gamma Distribution

Gamma Function

Let $\alpha > 0$ we define $\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}e^{-x}dx$ called the gamma function with parameter

Theorem

i) $\Gamma(\alpha) = (\alpha-1)\Gamma(\alpha-1)$ if $\alpha > 1$ ii) $\Gamma(\alpha) = (\alpha-1)!$ for $\alpha \in Z^+$

iii) $\Gamma(\alpha) = 2\int_0^\infty u^{2\alpha-1}e^{-u^2}du$ and iv) $\Gamma(\frac{1}{2}) = \sqrt{\pi}$

Proof

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}e^{-x}dx = -\left[x^{\alpha-1}e^{-x}\right]_0^\infty + (\alpha-1)\int_0^\infty x^{\alpha-2}e^{-x}dx = (\alpha-1)\Gamma(\alpha-1)$$

For $\alpha=1$, $\Gamma(1)=\int_0^\infty e^{-x}dx=-\left[e^{-x}\right]_0^\infty=1$ Now suppose it holds for $\alpha=k$ ie

$\Gamma(k)=(k-1)!$ for $k\in Z^+$ then $\Gamma(k+1)=k(k-1)!=k!$ for $k\in Z^+$. Thus if the results holds for $\alpha=k$ then they must also hold for $\alpha=k+1$. But the results are true for $\alpha=1$

Therefore the results are true for any $\alpha\in Z^+$

from $\Gamma(\alpha)=\int_0^\infty x^{\alpha-1}e^{-x}dx$ put $x=\mathrm{u}^2$ $\Rightarrow dx=2udu \Rightarrow \Gamma(\alpha)=\int_0^\infty u^{2(\alpha-1)}e^{-u^2}2udu=2\int_0^\infty u^{2\alpha-1}e^{-u^2}du$

Gamma Distribution

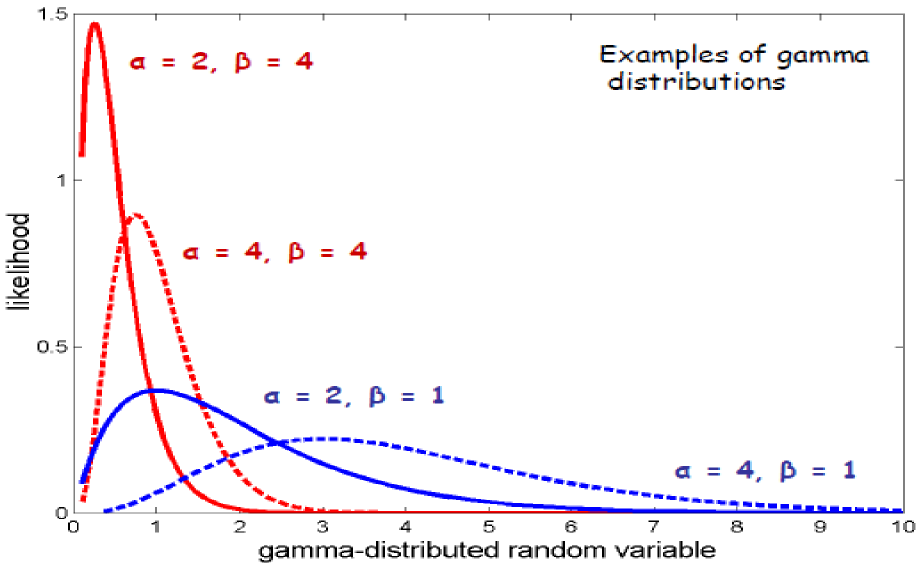
The $\mathrm{Gamma}(\alpha,\beta)$ distribution models the time required for *a* events to occur, given that the events occur randomly in a Poisson process with a mean time between events of β. For example, an insurance company observes that large commercial fire claims occur randomly in time with a mean of 0.7 years between claims. Not only in real life, the Gamma distribution is also wildly used in many scientific areas, like Reliability Assessment, Queuing Theory, Computer Evaluations, or biological studies. In a nut shell, this distribution is used to model total waiting time of a procedure that consists of α independent stages, each stage with a waiting time having a distribution $\mathrm{Exp}(\beta)$.Then the total time has a Gamma distribution with parameters α and β.

Definition A random variable X has *gamma density* if it pdf is given by

$$\mathrm{f(x)}=\begin{cases}\dfrac{\beta^\alpha}{\Gamma(\alpha)}x^{\alpha-1}e^{-\beta x}, \mathrm{x}\geq 0\\ 0 \quad otherwise\end{cases}$$ We say that the random variable $\mathrm{X}\sim\mathrm{Gamma}(\alpha,\beta)$

The parameter α is called the shape parameter while β is called the rate or scale parameter $\Gamma(\alpha)$ is the Gamma function.

Some examples of gamma distributions are plotted below. Notice that the modes shift to the right as the ratio of $\frac{\alpha}{\beta}$ increases.



Remarks

a) If $\beta=1$ then we have the standard gamma distribution.

b) If $\alpha=1$ then we have exponential density function.

The cdf, F(x) is of the form $F(x) = \frac{\beta^{\alpha}}{\Gamma(\alpha)}\int_0^x t^{\alpha-1}e^{-\beta t}dt$ and it's computation is not trivial.

Theorem: If X has a gamma distribution with parameters α and β, then

$$E(X) = \mu = \tfrac{\alpha}{\beta} \quad \text{and.} \quad Var\ (X) = \sigma^2 = \tfrac{\alpha}{\beta^2}$$

Definition: Let ν be a positive integer. A random variable X is said to have a **chi-square** distribution with ν degree of freedom if and only if is a **gamma-**distributed random variable with parameters $\alpha = \tfrac{\nu}{2}$ and $\beta = 2$

Theorem: If X is a chi-square random variable with degrees of freedom ν, then
$E(X) = \mu = \nu$ and. $Var\ (X) = \sigma^2 = 2\nu$

Example 1 Suppose the reaction time of a randomly selected individual to a certain stimulus has a standard gamma distribution with $\alpha = 2$ sec. Find the probability that reaction time will be (a) between 3 and 5 seconds (b) greater than 4 seconds

Solution

$\Gamma(2) = 1!$. Therefore $f(x) = xe^{-x}, x > 0$ and $f(x) = 0$ *elsewhere*

$$P(3 \le X \le 5) = \int_3^5 xe^{-x}dx = -(x+1)e^{-x}\Big|_3^5 = 4e^{-3} - 6e^{-5} \approx 0.1587$$

$$P(X > 4) = 1 - P(X \le 4) = 1 - \int_0^4 xe^{-x}dx = 1 - \left[-(x+1)e^{-x}\Big|_0^4\right] = 1 - \left[1 - 5e^{-4}\right] \approx 0.09158$$

Example 2 Suppose the survival time X in weeks of a randomly selected male mouse exposed to 240 rads of gamma radiation has a gamma distribution with $\alpha = 8$ and $\beta = 15$.

a) Find the expected value and the standard deviation of the survival time.
b) What is the probability that a mouse survives (i) between 60 and 120 weeks.(ii) at least 30 weeks

Solution

$$X \sim \Gamma(8,15) \quad \Rightarrow \quad f(x) = \begin{cases} \frac{1}{15^8\Gamma(8)} x^7 e^{-\frac{x}{15}}, \ x \ge 0 \\ 0 \quad otherwise \end{cases}$$

a) $E(X) = \mu) = (8)(15) = 120$ weeks $\qquad \sigma = \sqrt{8(15)^2} \approx 42.4264$ weeks

b) $$P(60 \le X \le 120) = \int_{60}^{120} \frac{1}{15^8\Gamma(8)} x^7 e^{-\frac{x}{15}}dx = \int_4^8 \frac{1}{\Gamma(8)} y^7 e^{-y}dy$$

$$= -\left(y^7 + 7y^6 + 42y^5 + 210y^4 + 840y^3 + 2520y^2 + 5040y + 5040\right)\frac{e^{-y}}{7!}\Bigg|_4^8$$

$$= \frac{261104e^{-4} - 6805296e^{-8}}{7!} \approx 0.4959$$

c) $$PX \ge 30) = \int_{30}^{\infty} \frac{1}{15^8\Gamma(8)} x^7 e^{-\frac{x}{15}}dx = \int_2^{\infty} \frac{1}{7!} y^7 e^{-y}dy$$

$$= -\left(y^7 + 7y^6 + 42y^5 + 210y^4 + 840y^3 + 2520y^2 + 5040y + 5040\right)\frac{e^{-y}}{7!}\Bigg|_2^{\infty}$$

$$= \frac{37200e^{-2} - 0}{7!} \approx 0.9989$$

Questionn The time between failures of a laser machine is exponentially distributed with a mean of 25,000 hours. What is the expected time until the second failure? What is the probability that the time until the third failure exceeds 50,000 hours?

2.2.4 Beta Distribution

Beta Function

We define a beat function as $\mathrm{B}(\alpha,\beta)$ as $\mathrm{B}(\alpha,\beta)=\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}dx$ with parameters $\alpha>0$ and $\beta>0$

Theorem:

i) $\mathrm{B}(\alpha,\beta)=\mathrm{B}(\beta,\alpha)$ ie by putting $y=1-x$ in the function

ii) $\mathrm{B}(\alpha,\beta)=\int_0^\infty \frac{u^{\alpha-1}}{(1+u)^{\beta-1}}du$ ie by putting $u=\frac{x}{1-x}$ or $x=\frac{u}{1+u}$ in the function

iii) $\mathrm{B}(\alpha,\beta)=2\int_0^{\pi/2}\cos^{2\alpha-1}t\sin^{2\beta-1}tdt$ ie by putting $x=\cos^2 t$ in the function

iv) $\mathrm{B}(\alpha,\beta)=\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$

v) $\Gamma(\tfrac{1}{2})=\sqrt{\pi}$ use $\mathrm{B}(\tfrac{1}{2},\tfrac{1}{2})=\frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2})}{\Gamma(1)}=[\Gamma(\tfrac{1}{2})]^2$ but

$\mathrm{B}(\tfrac{1}{2},\tfrac{1}{2})=2\int_0^{\pi/2}\cos^{2\times\frac{1}{2}-1}t\sin^{2\times\frac{1}{2}-1}tdt=2\int_0^{\pi/2}dt=\pi \quad\Rightarrow\quad \Gamma(\tfrac{1}{2})=\sqrt{\pi}$

Note $\Gamma(\tfrac{1}{2})=\int_0^\infty x^{-1/2}e^{-x}dx=2\int_0^\infty e^{-u^2}du=\sqrt{\pi}$. Simply $\int_0^\infty e^{-x^2}dx=\tfrac{1}{2}\sqrt{\pi}$

Beta Distribution

Definition: A random variable X is said to have a standard beta distribution with parameters α and β if it's probability density function is given by

$$\text{f(x)}=\frac{x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha,\beta)},0\le x\le 1 \text{ and f(x)} = 0 \textit{ elsewhere} \text{ we denote this as } X\sim Beta(\alpha,\beta)$$

Theorem: If X has a standard beta distribution with parameters α and β, then

$$\mathrm{E(X)}=\mu=\frac{\alpha}{\alpha+\beta} \quad \text{and.} \quad \mathrm{Var\ (X)}=\sigma^2=\frac{\alpha\beta}{(\alpha+\beta+1)(\alpha+\beta)^2}$$

3. MOMENTS AND MOMENT-GENERATING FUNCTIONS

Definition: The k^th^ **moment** of a r.v. X taken about zero, or about origin is defined to be $E[X^k]$ and denoted by μ_k' .

Definition: The k^th^ **moment** of a r.v. X taken about its mean, or the k^th^ **central moment** of X, is defined to be $E(X-\mu)^k$ and denoted by μ_k .

Definition: The **moment-generating function** (mgf), *m*(*t*), for a r.v. *X* is defined to be $M_x(t)$ or simply $M(t)=E[e^{tx}]$

We say that an mgf for *X* exists if there is *b* > 0 such that $M(t)<\infty$ for $|t|\le b$.