Integration by Substitution

1. Evaluate.

i) $\int \sin 4t \, dt$

ii) $\int \frac{\cos 2t}{\sin^3 2t} \, dt$

iii) $\int 4x^3(\underbrace{x^4-1}_{u}) \, dx$

iv) $\int x^2\sqrt{\underbrace{4+x^3}_{u}} \, dx$

v) $\int \frac{1}{\sqrt{x}\,(\underline{1+\sqrt{x}})^2} \, dx$

vi) $\int \frac{x}{\sqrt{4-x^2}} \, dx$

vii) $\int \frac{3t}{(t^2+1)^2} \, dt$

viii) $\int (x^2-3)(x^3+2) \, dx$

open bracket apply substitution

ix) $\int \frac{x^3+5}{x^5} \, dx$

$\int \left(\frac{1}{x^2} + \frac{5}{x^5}\right) dx$
$= \int [x^{-2} + 5x^{-5}] \, dx$

x) $\int 3x^2 \cos(x^3) \, dx$

xi) $\int x \cos x^2 \, dx$

xii) $\int \sqrt[3]{x} \cos(3\sqrt[3]{x}) \, dx$

like us $\int \sin(7x+5) \, dx$

xiii) $\int x(x-3)^5 \, dx$

xiv) $\int \sin^3 x \cos x \, dx$

xv) $\int x \sec^2 x^2 \, dx$

xvi) $\int \sec 2x \tan 2x \, dx$

like us

xvii) $\int \sec^2 x \tan x \, dx$

xviii) $\int \tan^3 x \sec^2 x \, dx$

(xix) $\int \sqrt{1+\sqrt{x}} \, dx$

i) $\int \sin \underbrace{4t}_{u} \, dt$ let $u = 4t$

$\sin u$

$\frac{du}{dt} = 4$, $du = 4\,dt$

$\frac{du}{4} = dt$

$$\int \sin 4t \, dt = \int \sin u \frac{du}{4} = \int \frac{1}{4} \sin u \, du$$

$$= \frac{1}{4} \int \sin u \, du$$

$$= -\frac{1}{4} \cos u + c$$

$$= -\frac{1}{4} \cos 4t + c$$

xvi) $\int \cot 5x \csc 5x \, dx$ $-\csc u = \int \csc u \cot u \, du$

let $u = 5x$, $du = 5\,dx$, $\frac{du}{5} = dx$

$$= \int \cot u \csc u \frac{du}{5} = \frac{1}{5} \int \cot u \csc u \, du$$

$$= -\frac{1}{5} \csc u + c = -\frac{1}{5} \csc(5x) + c$$

xi) $\int x \cos x^2 \, dx$

Let $u = x^2$ $\frac{du}{dx} = 2x$, $\frac{du}{2x} = dx$

$du = 2x\,dx$

$$\int x \cos x^2 \, dx = \int \cancel{x} \cos u \, \frac{du}{2\cancel{x}} = \int \tfrac{1}{2} \cos u \, du$$

$$= \tfrac{1}{2} \int \cos u \, du = \tfrac{1}{2} \sin u + c$$

$$= \tfrac{1}{2} \sin x^2 + c$$

$\rightarrow$ $u = x^2$, $du = 2\underline{\underline{x\,dx}}$; $\frac{du}{2} = x\,dx$

$$\int x \cos x^2 \, dx = \int \cos u \, \frac{du}{2} = \tfrac{1}{2} \int \cos u \, du$$

$$= \frac{1}{2} \sin u + c = \tfrac{1}{2} \sin x^2 + c$$

$$\int \sqrt[3]{x} \cdot \cos\left(3\sqrt[3]{x}\right) dx$$

let $u = 3\sqrt[3]{x} = 3\left(x^{1/3}\right)$

$$\frac{du}{dx} = 3 \cdot \tfrac{1}{3} x^{\frac{1}{3}-1} = x^{-2/3} = \left(x^{-1/3}\right)^2$$

$$= \left(\frac{1}{x^{1/3}}\right)^2$$

$$= \left(\frac{1}{\sqrt[3]{x}}\right)^2$$

$$\frac{du}{dx} = \frac{1}{\left(\sqrt[3]{x}\right)^2} \quad . \quad du = \frac{1}{\left[\sqrt[3]{x}\right]^2} dx$$

$$\left(\sqrt[3]{x}\right)^2 du = dx$$

Power 1

$$\int \sqrt[3]{x}\cos\left(3\sqrt[3]{x}\right)dx = \int \sqrt[3]{x}\cos u\left(\sqrt[3]{x}\right)^2 du$$

$a^2 \cdot a^1 = a^{1+2} = a^3$

$$= \int \left(\sqrt[3]{x}\right)^3 \cos u\, du$$

$u = 3\sqrt[3]{x}$ $\qquad \frac{u}{3} = \sqrt[3]{x}$

$$\left(\frac{u}{3}\right)^3 = \left(\sqrt[3]{x}\right)^3$$

$=$

$$\int \frac{u^3}{27}\cos u\, du$$

$$= \frac{1}{27}\int u^3 \cos u\, du$$

Requires integration by parts.

1) $$\int \frac{\cos 2t}{\sin^3 2t}\,dt = \int \frac{\cos 2t}{[\sin 2t]^3}\,dt$$

let $u = \sin 2t$ $\qquad du = 2\cos 2t\, dt$

Trig u $\Rightarrow$ u' Trig' u $\qquad \frac{du}{2} = \cos 2t\, dt$

$$\int \frac{du}{2\cancel{\cos 2t}} \cdot \frac{\cancel{\cos 2t}}{(\underbrace{\sin 2t}_{u})^3} = \int \frac{1}{u^3}\cdot\frac{du}{2} = \frac{1}{2}\int \frac{1}{u^3}\,du$$

$$= \frac{1}{2}\int u^{-3}\,du = \frac{1}{2}\,\frac{u^{-3+1}}{-3+1} + c$$

$$= \frac{1}{2} \cdot \frac{u^{-2}}{-2} + c = -\frac{1}{4} \frac{1}{u^2} + c$$

$$= -\frac{1}{4} \cdot \frac{1}{(\sin 2t)^2} + c$$

$$= -\frac{1}{4\sin^2(2t)} + c$$

$$\int \sec^2 x \tan x \, dx$$

$$du = \sec^2 x \, dx$$

$u = \tan x \qquad \frac{du}{dx} = \sec^2 x$

$$du = \underbrace{\sec^2 x \, dx}$$

$$\int \underbrace{\sec^2 x}_{du} \overbrace{\tan x}^{u} \underbrace{dx} = \int u \, du = \frac{u^2}{2} + c$$

$$= \frac{\tan^2 x}{2} + c$$

$$du = \sec x \tan x \, dx$$

$$\int \sec^2 x \tan x \, dx = \int \sec x \; \sec x \tan x \, dx$$

let $u = \sec x$

$$du = \sec x \tan x \, dx$$

$$\int u\,du = \frac{u^2}{2} + c$$

$$= \frac{\sec^2 x}{2} + c$$

xviii) $\int \tan^3 x \sec^2 x \, dx$ xix) $\int \sin^3 x \cos x \, dx$

$$\int \tan^3 x \underbrace{\sec^2 x \, dx}_{du}$$

$u = \tan x$
$du = \sec^2 x \, dx$

$$\boxed{\begin{array}{l} du = \sec x \tan x \, dx \\ u = \sec x \end{array}}$$

Let $u = \tan x$

$du = \sec^2 x \, dx$

$$= \int u^3 du = \frac{u^4}{4} + c = \frac{\tan^4 x}{4} + c$$

$$\int \sin^3 x \underbrace{\cos x \, dx}_{du}$$

$u = \sin x$
$du = \cos x \, dx$

$u = \cos x$
$du = -\sin x \, dx$

Let $u = \sin x$ $du = \cos x \, dx$

$$\int u^3\,du = \frac{u^4}{4} + c = \frac{\sin^4 x}{4} + c$$

$du/2$

$$\int \frac{3t}{(t^2+1)^2}\,dt$$

let $u = t^2+1$

$du = 2t\,dt$

$\frac{du}{2} = t\,dt$

$$\int \frac{3\cdot\frac{du}{2}}{u^2} = \int \frac{3}{2}\frac{du}{u^2} = \frac{3}{2}\int u^{-2}\,du$$

$$= \frac{3}{2}\frac{u^{-2+1}}{-2+1} + c$$

$$= \frac{3}{2}\frac{u^{-1}}{-1} + c$$

$$= \frac{3}{2}\frac{1}{-1u} + c$$

$$= -\frac{3}{2(t^2+1)} + c$$

$$\int \frac{x}{\sqrt{4-x^2}}\,dx$$

let $u = 4-x^2$

$du = -2x\,dx$

$\frac{du}{-2} = x\,dx$

$\frac{du}{-2x} = dx$

$$\int \frac{\cancel{x}}{\sqrt{4-x^2}}\,\frac{du}{-2\cancel{x}}$$

$$= \int \frac{\frac{du}{-2}}{\sqrt{u}} = \int -\frac{1}{2}\,\frac{du}{u^{1/2}} = -\frac{1}{2}\int u^{-1/2}\,du$$

$$= -\frac{1}{2}\,\frac{u^{-1/2+1}}{-1/2+1} + c$$

$$= -\frac{1}{2}\,\frac{u^{1/2}}{1/2} + c = -\frac{1}{2} \times \frac{2}{1}\sqrt{u} + c$$

$$= -\sqrt{4-x^2} + c$$

$$\int \frac{1}{u}\,du = \ln|u| + c \qquad = \int \frac{\cos x}{\sin x}\,dx$$

$$\int e^u\,du = e^u + c \qquad \int x e^{x^2}\,dx$$