4) A continuous random variable X has the pdf given by $f(x) = \begin{cases} k(1+x)^2, & -2 \le x \le 0 \\ 4k & 0 < x \le \frac{4}{3} \\ 0, & elsewhere \end{cases}$ Find the value of the constant k hence compute $P(X > 1)$ and $P(-1 < X < 1)$

5) A continuous random variable X has the pdf given by $f(x) = \begin{cases} kx, & 0 \le x \le 2 \\ k(4-x) & 2 < x \le 4 \\ 0, & elsewhere \end{cases}$ Find the value of the constant k hence compute $P(X > 3)$ and $P(1 < X < 3)$

1.4 Distribution Function of a Random Variables

Definition: For any random variable X, we define the **cumulative distribution function (CDF)**, *F(x)* as $F(x) = P(X \le x) = \begin{cases} \sum_{t=-\infty}^{x} f(t) & \text{If X is discrete} \\ \int_{-\infty}^{x} f(t)dt & \text{If X is continuous} \end{cases}$ for every x.

\\ Again here t is introduced to facilitate summation /integration//

Properties of any cumulative distribution function

- $\lim_{x \to \infty} F(x) = 1$ and $\lim_{x \to -\infty} F(x) = 0$
- F(x) is a non-decreasing function.
- F (x) is a right continuous function of *x*. In other words $\lim_{t \to x} F(t) = F(x)$

Reminder If the c.d.f. of X is F(x) and the p.d.f. is f(x), then differentiate F(x) to get f(x), and integrate f(x) to get F(x);

Theorem: For any random variable X and real values a < b, $P(a \le X \le b) = F(b) - F(a)$

Example 1

Let X be a discrete random variable with pmf given by $f(x) = \begin{cases} \frac{1}{20}(1+x) & \text{for } x = 1,2,3,4,5 \\ 0, & elsewhere \end{cases}$.

Determine the cdf of X hence compute $P(X > 3)$

Solution

$$F(x) = \sum_{t=-\infty}^{x} f(t) = \frac{1}{20}\sum_{t=1}^{x}(x+1) = \frac{1}{20}(2+3+....+x) = \frac{1}{20}\left\{\frac{x}{2}\left[4+(x-1)\right]\right\} = \frac{x(x+3)}{40}$$

$$F(x) = \begin{cases} 0 & \text{for } x < 1 \\ \frac{x(x+3)}{40} & \text{for } x = 1,2,3,4,5 \\ 1 & \text{for } x > 5 \end{cases} \quad \text{Recall for an AP } S_n = \frac{n}{2}\left[2a+(n-1)d\right]$$

$$P(X > 3) = 1 - P(X \le 3) = 1 - \frac{3 \times 6}{40} = \frac{11}{20}$$

Example 2

Suppose X is a continuous random variable whose pdf f(x) is given by

$f(x) = \begin{cases} \frac{1}{2}x, & 0 \le x \le 2 \\ 0, & elsewhere \end{cases}$. Obtain the cdf of X hence compute $P(X > \frac{2}{3})$

Solution

$$\mathrm{F(x)} = \int_{-\infty}^{x} \mathrm{f(t)dt} = \int_{0}^{x} \tfrac{1}{2}\,\mathrm{t dt} = \tfrac{1}{4}\left[t^2\right]_0^x = \tfrac{1}{4}x^2 \quad \text{thus } \mathrm{F(x)} = \begin{cases} 0, & x<0 \\ \frac{x^2}{4}, & 0 \le x \le 2 \\ 1, & x>2 \end{cases}$$

$$P\left(X > \tfrac{2}{3}\right) = 1 - P\left(X \le \tfrac{2}{3}\right) = 1 - \tfrac{1}{4}\left(\tfrac{2}{3}\right)^2 = \tfrac{8}{9}$$

Example 3

A continuous random variable X has a probability density function given by

$$\mathrm{f(x)} = \begin{cases} 0.25, & 0 < x < 2 \\ 0.5x + c, & 2 < x < 3 \\ 0, & elsewhere \end{cases} \text{ Find the cdf of X hence compute } P(1.5 \le \mathrm{X} > 2.5).$$

Solution

$$\mathrm{F(x)} = \int_{-\infty}^{x} \mathrm{f(t)dt} = \begin{cases} \int_0^x \frac{1}{4}\mathrm{dt} = \frac{\mathrm{x}}{4} \\ \int_2^x \left(\frac{1}{2}\mathrm{t} - \frac{3}{4}\right)\mathrm{dt} + k = \frac{x^2 - 3\mathrm{x}}{4} + \frac{1}{2} + k \end{cases}$$

Under the two levels, F(2) must be the same. (reason for introducing k)

$$\Rightarrow \mathrm{F(2)} = \tfrac{1}{2} = k \quad \therefore \ \mathrm{F(x)} = \begin{cases} 0 \text{ for } x<0 \\ \frac{x}{4} \text{ for } 0 \le x \le 2 \\ \frac{x^2 - 3\mathrm{x}}{4} + 1 \text{ for } 2 \le x \le 3 \\ 1, \text{ for } x > 2 \end{cases}$$

Exercise

1. 1. The pdf of a continuous random variable X is given by $\mathrm{f(x)} = \begin{cases} C/\sqrt{x}, & 0 \le x \le 4 \\ 0, & elsewhere \end{cases}$ Find the value of the constant C, the cdf of X and $P(X \ge 1)$
2. 2. The pdf of a random variable X is given by $\mathrm{g(x)} = \begin{cases} kx(1-x), & 0 \le x \le 1 \\ 0, & elsewhere \end{cases}$ Find the value of the constant k, the cdf of X and the value of m such that $\mathrm{G(x)} = \frac{1}{2}$
3. 3. Find the cdf of a random variable Y whose pdf is given by;

   a) $\mathrm{f(x)} = \begin{cases} 1/3, & 0 \le x \le 1 \\ 1/3, & 2 \le x \le 4 \\ 0, & elsewhere \end{cases}$ b) $\mathrm{f(x)} = \begin{cases} x/2, & 0 \le x \le 1 \\ 1/2, & 1 \le x \le 2 \\ (3-x)/2, & 2 \le x \le 3 \\ 0, & elsewhere \end{cases}$
4. 4. If the cdf of a random variable Y is given by $\mathrm{F(x)} = 1 - \dfrac{9}{y^2}$ for $Y \ge 3$ and $\mathrm{F(x)} = 0$ for $Y < 3$, find $P(X \le 5)$, $P(X > 8)$ and the pdf of X

1.5 Derived Random Variables

Give the pdf of a random variable say X, we can obtain the distribution of a second random variable say Y provided that we know some functional relationship between X and Y say $\mathrm{Y} = u(X)$. Eg $\mathrm{Y} = 2X + 3$, $\mathrm{Y} = X^3$, $\mathrm{Y} = \sqrt{X} + c$ etc

For a 1-1 relationship between X and Y eg $Y = 2X + 3$, f(x) and g(y) yields exactltly the same probabilities only the random variable and the set of values it can assume changes.

Example 1

Give the pmf of a random variable X as $f(x) = \begin{cases} x/6 & \text{for } x = 1,2,3 \\ 0, & \textit{elsewhere} \end{cases}$ find the pmf of $Y = X^2$

Solution

The only values of Y with non zero probabilities are $Y = 1$, $Y = 4$ and $Y = 9$. Now

$P(Y=1) = P(X^2 = 1) = P(X = 1) = \frac{1}{6}$ $\qquad$ $P(Y=4) = P(X^2 = 4) = P(X = 2) = \frac{1}{3}$ and

$P(Y=9) = P(X^2 = 9) = P(X = 3) = \frac{1}{2}$

In some cases several values of X will give rise to the same value of Y. The procedure is just the same as above but it is necessary to add the several probabilities that are associated with each value x that provides a unique value y.

Example 2

Give the pmf of a r.v X as $f(x) = \begin{cases} \dfrac{x+1}{15} & \text{for } x = 0,1,2,3,4 \\ 0, & \textit{elsewhere} \end{cases}$ find the pmf of $Y = (X-2)^2$

Solution

x	0	1	2	3	4
y	4	1	0	1	4

$P(Y=0) = P(X=2) = \frac{1}{5}$ $\qquad$ $P(Y=1) = P(X=1) + P(X=3) = \frac{2}{15} + \frac{4}{15} = \frac{2}{5}$

$P(Y=1) = P(X=0) = P(X=4) = \frac{1}{15} + \frac{1}{3} = \frac{2}{5}$ Therefore the pmf of Y can be written as

y	0	1	4
$P(Y = y)$	$\frac{2}{5}$	$\frac{2}{5}$	$\frac{2}{5}$

\

Exercise

1. Suppose the pmf of a r.v X is given by $f(x) = \begin{cases} 1/6 & \text{for } x = 1,2,3,4,5,6 \\ 0, & \textit{elsewhere} \end{cases}$, Obtain the pmf of $Y = 2X^2$ and $Z = X - 3$
2. Let the pmf of a r.v X be given by $f(x) = \begin{cases} \dfrac{x^2+1}{18} & \text{for } x = 0,1,2,3 \\ 0, & \textit{elsewhere} \end{cases}$, determine the pmf of $Y = X^2 + 1$
3. Suppose the pmf of a r.v X is given by $f(x) = \begin{cases} x/10 & \text{for } x = 0,1,2,3,4 \\ 0, & \textit{elsewhere} \end{cases}$, Obtain the pmf of $Y = |X - 2|$