TOPIC 2 : TECHNIQUES OF INTEGRATION:

1 SUBSTITUTION (u-substitution)

Simplify i) $\int 4x^4 e^{x^5} dx$

ii) $\int x^2 e^{x^3} dx$

iii) $\int e^{\cos\theta} \sin\theta \, d\theta$

soution:

Let u be x^5

$\frac{du}{dx} = 5x^4$

$\therefore \frac{du}{5} = \int \frac{4}{5} e^u du = \frac{4}{5} \int e^u du$

$\int \frac{1}{4} \sin u du$

examples

very simple

$\int \sin \underset{u}{4t} \, dt$

let $u = 4t$

$\frac{du}{dt} = 4$

$du = 4dt$

$\frac{du}{4} = dt$

$\int \sin 4t \, dt = \int \sin u \frac{du}{4} = \int \frac{1}{4} \sin u$

$= \frac{1}{4} \int \sin u \, du$

$= -\frac{1}{4} \cos u + C.$

$= -\frac{1}{4} \cos 4t + C.$

29th September, 2025:

ii) $\int \sec 2x \tan 2x \, dx$??

iii) $\int \cot 5x \csc 5x \, dx$

$-\csc u = \int \csc u \cot u \, du$

Solution:

Let $u = 5x$, $du = 5\,dx$, $\frac{du}{5} = dx$

$= \int \cot u \csc u \frac{du}{5} = \frac{1}{5} \int \cot u \csc u \, du$

$= -\frac{1}{5} \csc u + C = -\frac{1}{5} \csc(5x) + C$

$= -\frac{1}{5} \csc(5x) + C$

iv) $\int 3x^2 \cos 4x^3 \int dx$??

$\int dx$

(u = substitution)

v) $\int \sqrt{x} \cos(3\sqrt[3]{x})\,dx$

let $u = 3\sqrt[3]{x} = 3x^{\frac{1}{3}}$ ....(i) ✓

$\frac{du}{dx} = 3 \cdot \frac{1}{3} x^{\frac{1}{3}-1} = x^{-\frac{2}{3}} = \left(x^{-\frac{1}{3}}\right)^2$

$= \left(\frac{1}{x^{\frac{1}{3}}}\right)^2$

$= \left(\frac{1}{\sqrt[3]{x}}\right)^2$

$\frac{du}{dx} = \frac{1}{(\sqrt[3]{x})^2}$: $du = \frac{1}{(\sqrt[3]{x})^2} dx$ ...

$\int 3\sqrt{x} \cos(3\sqrt[3]{x})\,dx = \int 3\sqrt{x} \cos u \,(\sqrt[3]{x})^2\,du$

$a^2 \cdot a^1 = a^{1+2} = a^3$ $= \int (3\sqrt{x})^3 \cos u\,du$

$u = 3\sqrt[3]{x}$ $\frac{u}{3} = 3\sqrt{x}$

Requires integration by parts $=$ $\left(\frac{u}{3}\right)^3 = (3\sqrt{x})^3$

$= \int \frac{u^3}{27} \cos u\,du$

$= \frac{1}{27} \int u^3 \cos u\,du$

N/B: $u \longrightarrow$ should be under the power | highest power

vi) $\int \frac{\cos 2t}{\sin^3 2t}\, dt = \int \frac{\cos 2t}{[\sin 2t]^3}\, dt$

let $u = \sin 2t \quad du = 2\cos 2t\, dt$

$\frac{du}{2} = \cos 2t\, dt$

comes out since it is a constant

$= \int \frac{1}{u^3} \cdot \frac{du}{2} = \frac{1}{2} \int \frac{1}{u^3}\, du$

??? $= \frac{1}{2} \int u^{-3}\, du = \frac{1}{2} \frac{u^{-3+1}}{-3+1} + C$

$= \frac{1}{2} \cdot \frac{u^{-2}}{-2} + C = \frac{-1}{4} \frac{1}{u^2} + C$

$= \frac{1}{4} \cdot \frac{1}{(\sin 2t)^2} + C$

$= -\frac{1}{4 \sin^2 (2t)} + C$

$= \frac{1}{4\sin^2 (2t)} + C$

19x

-3

$\int \sec^2 x \tan x \, dx$

$du = \sec x \tan x \, dx$

$du = \sec^2 x \, dx$

$u = \tan x \qquad \frac{du}{dx} = \sec^2 x$

$du = \underbrace{\sec^2 x \, dx}$

$$\int \underbrace{\sec^2 x}_{du} \overbrace{\tan x}^{u} \, dx = \int u \, du = \frac{u^2}{2} + C$$

$$= \frac{\tan^2 x}{2} + C.$$

$$\int \sec^2 x \tan x \, dx = \int \sec x \sec x \tan x \, dx$$

let $u = \sec x$

$du = \sec x \tan x \, dx$

$\frac{1}{2}\sec \quad = \sec$

xviii) $\int \tan^3 x \sec^2 x \, dx$

xiv) $\int \sin^3 x \cos x \, dx$

$\int \underbrace{\tan^3 x}_{u} \underbrace{\sec^2 x \, dx}_{du}$

$du = \sec^2 x \, dx$

$du = \sec x \tan x \, dx$

Let $u = \tan x$

$\therefore du = \sec^2 x \, dx$

$= \int u^3 du = \frac{u^4}{4} + C = \frac{\tan^4 x}{4} + C$

$\int \underbrace{\sin^3 x}_{u} \underbrace{\cos x \, dx}_{du}$

$du = \cos x \, dx$ $\quad u = \sin x$

$= \sin x \, dx$ $\quad u = \cos x$

Let $u = \sin x$

$du = \cos x \, dx$

$= \int u^3 du = \frac{u^4}{4} + C = \frac{\sin^4 x}{4} + C$

xv) $\displaystyle\int \overbrace{\frac{3t}{(t^2+1)^2}\, dt}^{du/2}$

Let $u = t^2 + 1$

$du = 2t\, dt$

$\dfrac{du}{2} = t\,dt$

$$\int \frac{3 \cdot \frac{du}{2}}{u^2} = \int \frac{3}{2}\frac{du}{u^2} = \frac{3}{2}\int u^{-2}\,du$$

$$= \frac{3}{2}\,\frac{u^{-2+1}}{-2+1} + c$$

$$= \frac{3}{2}\,\frac{u^{-1}}{-1} + C$$

$$= \frac{3}{2}\,\frac{1}{-u^1} + c$$

$$= \frac{-3}{2} \cdot \frac{1}{t^2+1} + C$$

$$= \frac{-3}{2(t^2+1)} + c$$

Nov 10, 2025:

The problem:

$$\int (ax^2+bx+c)\,dx$$

or

$$\int \frac{1}{ax^2+bx+c}\,dx \longrightarrow \text{Special technique}$$

Basic Integration:
(Quadratic in the Numerator) (Power Rule)

1. $$\int (ax^2+bx+c)\,dx = a\int x^2\,dx + b\int x\,dx + c$$

* (Integrating using power rule)

You get:

$$= a\frac{x^3}{3} + b\frac{x^2}{2} + cx + C$$

* Where C is the Constant of Integration.

2. Completing the Square method:

$1/(ax^2+bx+c)$

Trick: complete the square

ax^2+bx+c

After completing the square, you can always write it as $\boxed{a(x+d)^2+k}$

$a(x+d)^2 + k$

where $d = \frac{b}{2a}$ and $k = c - \frac{b^2}{4a}$.

matches one of these: * } solve with relevant formula: arctangent, logarithmic or hyperbolic.

$$\int \frac{1}{u^2 + A^2}\, du$$

$$\int \frac{1}{u^2 - A^2}\, du$$

$\frac{b}{2a} = \frac{4}{2}$

Example:

$$\int \frac{1}{x^2 + 4x + 8}\, dx$$

$8 - \left(\frac{4^2}{4}\right)$ $\frac{4}{4}$ $\frac{\cancel{16}\,4}{4}$

1 Complete the square

$$x^2 + 4x + 8 = \underline{(x+2)^2 + 4}$$

u-substitution

$u = x + 2 \rightarrow du \rightarrow dx$

Now integral is $\int \frac{1}{u^2 + 4}\, du$

use the arctangent integral formula.

$$\int \frac{1}{u^2 + a^2}\, du = \frac{1}{a} \arctan \frac{u}{a} + C.$$

$$a = 2$$

substitute back

$$\int \frac{1}{x^2 + 4x + 8}\, dx = \frac{1}{2} \arctan\left(\frac{x+2}{2}\right) + C$$

Example Questions:

i) $\displaystyle\int \frac{dx}{x^2 + x - 2}$

ii) $\displaystyle\int \frac{x^3}{x - 3x + 2}\, dx$

iii) $\displaystyle\int \frac{x^2 - 3x^2 - 2x}{x^2 + 1}\, dx$

iv) $\displaystyle\int \frac{x^2}{(x+2)^3}\, dx$

v) $\displaystyle\int \frac{x^2 - x - 2}{3x^3 - x^2 + 3x - 1}\, dx$

vi) $\displaystyle\int \frac{2x + 4}{x^3 - 2x^2}\, dx$

vii) $\displaystyle\int \frac{dx}{x^2 + x - 2}\, dx$

→ FRACTIONAL DECOMPOSITION

Partia fraction deconposition:

Rewuniting a rational ~~fraction~~ function (a fraction where both numerator and denominator are polynomeals) as a sum of simplar fractions with easie denominators

Steps:

1. Factor denominator into linear and irreducible quadratic pieces.

2. Decomposition form.

$$\frac{P(x)}{(x-a)(x^2+bx+c)} = \frac{A}{x-a} + \frac{Bx+C}{x^2+bx+C}$$

multiply thro denominator

Expand and match coeffient

Example:

$$\frac{5x+7}{(x-1)(x^2+4)}$$

Decompose as:

$$\frac{5x+7}{(x-1)(x^2+4)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+4}$$

Decompose:

$$\frac{x^2+2x+3}{(x^2+1)^2}$$

It cannot be factored any further.

$$\frac{x^2+2x+3}{(x^2+1)^2} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{(x^2+1)^2}$$

Clear the denominator:

$$x^2+2x+3 = (Ax+B)(x^2+1) + (Cx+D)$$

Expand
RHS

$$(Ax+B)(x^2+1)+(Cx+D) = Ax^3 + Ax + Bx^2 + B + Cx + D.$$

Group powers:

$$= \cancel{A}x^3 + Bx^2 + (A+C)x + (B+D)$$

Match Coefficients:

x^3 : No x^3 on the left, so $A = 0$.
x^2 : $B = 1$
x : $A + C = 2$, but $A = 0$ so $C = 2$.

Constants : $B + D = 3$ $B = 1$, so $D = 2$

Step 5: write the decomposition.

Plug your solved constants in:

$$\frac{x^2 + 2x + 3}{(x^2+1)} = \frac{1}{x^2+1} + \frac{2x+2}{(x^2+1)^2}$$

* FUNDAMENTAL THEORY OF CALCULUS.

$$\int_a^b f(x)\,dx = f(b) - f(a)$$

i) $\int_2^3 \frac{1}{x}\,dx =$

ii) $\int_0^{\pi/4} \cos x\,dx$

iii) $\int_0^{\pi} x\cos(2x - \frac{\pi}{2})\,dx$

iv) $\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \cos^2\theta \sin\theta\,dx$

v) $\int_9^4 \frac{1-\sqrt{x}}{\sqrt{x}}\,dx$

vi) $\int_2^3 \frac{1}{3x-1}\,dx$

vii) $\left(\int_2^3 \frac{1}{x}\,dx = \ln|x| + c\right)$

IMPROPER INTEGRAL:

<u>Type 1</u> (First Kind)

$\tan^{-1} \frac{1}{\tan}$

$\tan 40 = y$

$\frac{1}{y} = \tan^{-1}$

$y \neq 0$

$\tan^{-1} = 0$

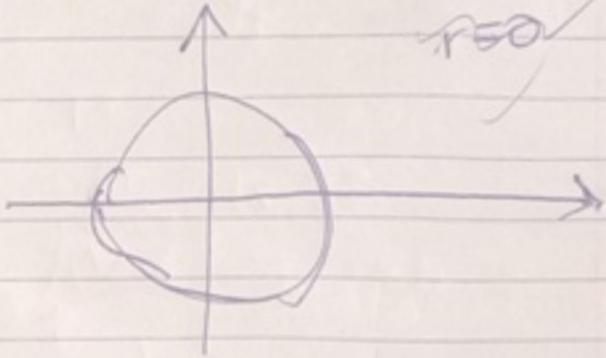
Infinite Integral

Determine wheather the following integeral converges.

$\frac{1}{r} = 0$

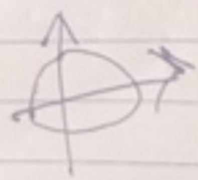
~~$r=0$~~

i) $\int_1^{\infty} \frac{1}{x^2}\,dx$ ii) $\int_1^{\infty} \frac{1}{x}\,dx$



iii) $\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx$

$$\int_1^{\infty} \frac{1}{x^2}\,dx = \lim_{t\to\infty} \int_1^{t} \frac{1}{x^2}\,dx =$$



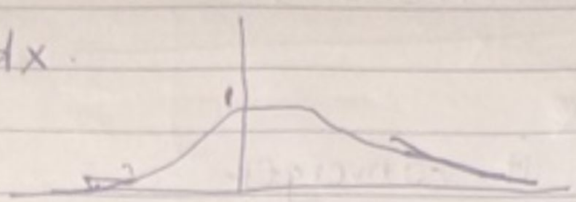
$$\lim_{t\to\infty} \int_1^{t} x^{-2}\,dx = \left[\frac{x^{-2+1}}{-2+1}\right]_1^t =$$

(Converges)

$$\lim_{t\to\infty} \left.\frac{-1}{x}\right]_1^t = \lim_{t\to\infty} \left[\frac{1}{t} - \frac{1}{1}\right] =$$

$$\lim_{t\to\infty} \left(-\frac{1}{t} + 1\right) = 0 + 1 = 0$$

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx$$



$$\int_{-\infty}^{0} \frac{1}{1+x^2}\,dx + \int_{0}^{\infty} \frac{1}{1+x^2}\,dx$$

$$\lim_{t \to -\infty} \int_{t}^{0} \frac{1}{1+x^2}\,dx + \lim_{k \to \infty} \int \frac{1}{1+x^2}\,dx$$

C.ST

A

$$\lim_{t \to -\infty} \left[\tan^{-1} x\right]_{t}^{0} + \lim_{k \to \infty} \left[\tan^{-1} x\right]_{0}^{k}$$

$$\lim_{t \to -\infty} \left[\tan^{-1} 0 - \tan^{-1} t\right] + \lim_{k \to \infty} \left[\tan^{-1} k - \tan^{-1} 0\right]$$

$(\therefore \tan^{-1} 0 = 0)$ $r = 0$

$$= \lim_{t \to -\infty} (0 \ [\tan^{-1} t] + \lim_{k \to \infty} [\tan^{-1} k]$$

let $\tan^{-1} k$ be y

$$\left(-\frac{\pi}{2}\right) + \frac{\pi}{2} = \pi$$

* It converges

DISCONTINUOUS INTEGERALS

Finite limits.

Integerate if it converges.

i) $\displaystyle\int_{2}^{13} \frac{1}{x-2}\,dx$ ii) $\displaystyle\int_{-\pi/3}^{\pi/2} \frac{1}{\cos x}\,dx$

$$\int_{2}^{13} \frac{1}{x-2}\,dx = \lim_{t \to 2^+} \int_{t}^{13} \frac{1}{x-2}\,dx =$$

$\ln 0 =$ undefined

$$\lim_{t \to 2^+} \Big[\ln x-2 \Big]_{t}^{13} = \lim_{t \to 2^+} \Big[(\ln|13-2|) - \ln(t-2) \Big]$$

$$= \lim_{t \to 2^+} \Big[\ln 11 - \ln(t-2) \Big] =$$

$$= \ln 11 - \lim_{t \to 2^+} \ln|t-2| \quad \to 0$$

= undefined / diverges

$t \to 0$

ii) $\int \frac{1}{\cos x}\, dx = \int_{\pi/3}^{\pi/2} \sec x\, dx = \lim_{t \to \frac{\pi}{2}^-} \int_{\pi/3}^{t} \frac{1}{\cos x}\, dx$

$$\lim_{t \to \frac{\pi}{2}^-} \int_{\pi/3}^{t} \sec x\, dx = \lim_{t \to \frac{\pi}{2}} \ln|\sec x \tan x\Big]_{\pi/3}^{t}$$

$$= \lim_{t \to \frac{\pi}{2}^-} \left[\ln|\sec t + \tan t| - \ln|\sec\left(\frac{\pi}{3} + \tan\frac{\pi}{3}\right)\right]$$

$$= \lim_{t \to \frac{\pi}{2}} \left[|\ln \overset{+\infty}{\sec t} + \overset{\infty}{\tan t}\right]^{+\infty} - \ln(2 + \sqrt{3})$$

$$= \lim_{t \to \frac{\pi}{2}} \left[\infty - \ln(2 + \sqrt{3})\right)$$

* undefined

$$\int_0^3 \frac{1}{\sqrt{3-x}}\,dx \qquad \int_0^3 \frac{1}{x^2}\,dx.$$

Approximation of INTEGERALS.

i) Simpson's Rule

$\frac{1}{3}$ - Simpson's Rule

Area under $f(x)$ between $x=a$ and $x=b$ is given by.

$$I = \int_a^b f(x)\,dx = \frac{1}{3}h\left\{f_0 + f_a + 4\underbrace{(f_1 + f_3 + f_5 + \cdots)}_{\text{odds}} + 2\underbrace{(f_2 + f_4 + \cdots)}_{\text{evens}}\right.$$

(Research)

* $\frac{3}{8}$ - simpson rule

Trapezium Rule

$I = \int [illegible] dx = \frac{h}{2}\left[[illegible] \right]$

Examples

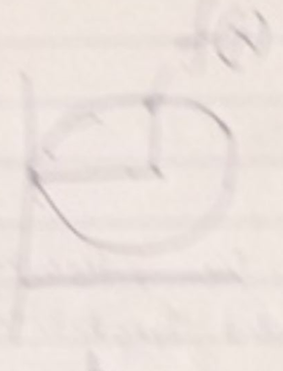
Use [illegible] strips [illegible] the following integrals using Composite Rule and Trapezium rule.

(i) $\int [illegible] \, dx$

(ii) $\int \sqrt{[illegible]} \, dx$

(iii) $\int_0^{\pi/4} \sqrt{1 + [illegible]} \, dx$

Volume of Revolution



$$A = \int_a^b \pi [f(x)]^2 \, dx$$

$$A = \int_c^d \pi (f(y))^2 \, dy$$

Example:

Find the volume of the solid obtained when the region under $\frac{1}{\sqrt{x}}$ over the interval $[1, 4]$ is revolved about x axis

Find the volume of the solid obtained when region between [illegible] and [illegible] is revolved about

1) $x =$ [illegible]
2) $y =$ [illegible]
3) $y =$ [illegible]

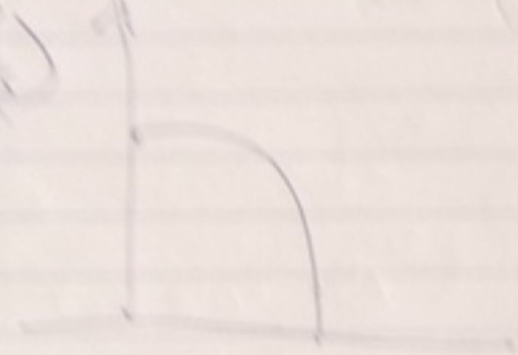
Volume of cylindrical shells

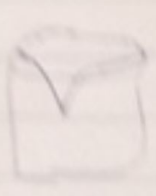
y-axis

(By washers)

(y-axis)

(cylindrical shells)





* CAT

$V = 2\pi \cdot r$

$V = 2\pi \int x f(x)\,dx = \int_a^b 2\pi x f(x)\,dx$

Use cylindrical solids to find the volume of the solid generated when the region enclosed between $y = \sqrt{x}$, $x = 1$ and $x = 4$ and x-axis is revolved about y-axis.

Arc length.

The length of the arc $f(x)$ from $x = a$ to $x = b$ is given by

$$\int_a^b 1 + \left(\frac{df}{dx}\right)^2 dx$$

CAT 15th DEC
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