Calculus II

COOPERATIVE UNIVERSITY OF KENYA
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Purpose of the course

To introduce the student to basic integration methods and their applications

Expected Learning Outcomes

By the end of the course the learner should be able to:

i) State the fundamental theorem of integral calculus

ii) Use various integration methods to integrate functions

iii) Apply integration to find the length of an arc, area under a curve and the volume of solid of revolution

iv) Evaluate definite integrals by trapezoidal and Simpson's rules

Course Content

Anti-derivative, Fundamental theorem of integration. Techniques of integration: Rules of integration, Methods of integration: Algebraic substitution, powers of trigonometric functions, Use of factor and t-formulae, use of inverse trigonometric and hyperbolic functions, Integration by partial fractions. Integration by parts, Reduction formulae.

Applications of Integration; arc length, area and volume of solids of revolution, mean value and root-mean square value.

Multiple Integrals: Double integrals, change of the order of integration and the Fubini's theorem. Polar and curvilinear coordinates. Moment of inertia. Change of the coordinate system. Triple integrals, space coordinate systems.
Power series: Maclaurin's and Taylor's theorems and their applications to singular integrals.

Teaching / Learning Methodologies: Lectures; Tutorials; Class discussion

Instructional Materials and Equipment: Handouts; White board

Course Assessment: Examination - 70%; Continuous Assessments (Exercises and Tests) - 30%;

Total - 100%

Recommended Text Books

1. Tom M. Apostol (2007); Calculus, Volume I, 2nd Ed; Wiley
2. Alex (2009); Calculus Ideas and Applications; John Wiley and Sons
3. Thomas G.B., and Finney R.L (2009) 11th Edition; *Calculus and Analytical geometry*, Wesley
4. Hunt Richard A (2009); Calculus; (2nd Edition); Harper Collins College Publishers

Text Books for further Reading

1. Bradley Smith (2009); Calculus; Prentice Hall- Gale.
2. Stein Sheran K. (2008); Calculus with analytic Geometry; McGraw Hill.
3. Boyce William E and DiPrima, Richard C.(2008); Calculus; John Wiley.
4. Hirst K. E and Hirst K. E (2005); *Calculus of One Variable*; Springer

Integral Calculus

Integration is the process by which we determine functions from their derivatives. Integration and differentiation are intimately connected. They are reverse processes of each other.

The process of determining a function from its derivative $f(x)$ and one of its known values has two steps. The first is to find a formula that gives us all the functions that could possibly have $f(x)$ as a derivative. These functions are the so – called antiderivatives of f, and the formula that gives them all is called the indefinite integral of f. The second step is to use the known function value to select the particular antiderivative needed from the indefinite integral.

Finding Antiderivaties – Indefinite Integrals

Definitions

A function $F(x)$ is an **antiderivative** of a function $f(x)$ if

$$F'(x) = f(x)$$

for all x in the domain of f. The set of all antiderivatives of f is the **indefinite integral** of f with respect to x denoted by

$$\int f(x)dx.$$

The symbol $\int$ is **an integral sign**. The function f is the **integrand** of the integral and x is the **variable of integration**. In integral notation we write

$$\int f(x)\,dx = F(x) + C. \qquad (1)$$

The constant C is the **constant of integration** or the **arbitrary constant**. Equation (1) is read, “The indefinite integral of f with respect to x is $F(x) + C$". When we find $F(x) + C,$ we say that we have **integrated** f and **evaluated** the integral.

Example

Evaluate $\int 2x\,dx$.

Solution

an antiderivative of 2x

$$\int 2x\,dx = x^2 + C$$

The formula $x^2 + C$ generates all the antiderivatives of the function 2x. The functions $x^2 + 1,\ x^2 - \pi,$ and $x^2 + \sqrt{2}$ are all antiderivatives of the function 2x as can be verified by differentiation.

The process of integration reverses the process of differentiation. In differentiation, if $f(x) = 2x^2$ then $f'(x) = 4x$. Thus the integral of 4x is $2x^2$, i.e. integration is the process of moving from $f'(x)$ to $f(x)$. By similar reasoning, the integral of 2t is t^2.

In differentiation, the differential coefficient $\frac{dy}{dx}$ indicates that a function of x is being differentiated with respect to x, the ‘dx’ indicating that it is ‘with respect to x’. In integration, the variable of integration is shown by adding d (the variable) after the function to be integrated.

Thus $\int 4x\,dx$ means ‘the integral of 4x with respect to x’, and $\int 2t\,dt$ means ‘the integral of 2t with respect to t’.

As stated above, the differential coefficient of $2x^2$ is 4x, hence $\int 4xdx = 2x^2$. However, the differential coefficient of $2x^2 + 7$ is also 4x. Hence $\int 4xdx$ is also equal to $2x^2 + 7$. To allow for the possible presence of a constant, whenever the process of integration is performed, a constant ‘c’ is added to the result. Thus $\int 4x\,dx = 2x^2 + c$ and $\int 2t\,dt = t^2 + c$

‘c’ is called the **arbitrary constant of integration**.

The general solution of integrals of the form $\boldsymbol{ax^n}$

The general solution of integrals of the form $\int ax^n\,dx$ where a and n are constants is given by:

$$\int \boldsymbol{ax^n\,dx = a\frac{x^{n+1}}{n+1} + c}$$

This rule is true when n is fractional, zero, or a positive or negative integer, with the exception of $n = -1$.

Using this rule gives:

(i) $\int 3x^4\,dx = \frac{3x^{4+1}}{4+1} + c = \boldsymbol{\frac{3}{5}x^5 + c}$

(ii) $\int \frac{2}{x^2}\,dx = \int 2x^{-2}\,dx = \frac{2x^{-2+1}}{-2+1} + c = \frac{2x^{-1}}{-1} + c = \boldsymbol{\frac{-2}{x} + c}$, and

(iii) $\int \sqrt{x}\,dx = \int x^{\frac{1}{2}}\,dx = \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + c = \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + c = \boldsymbol{\frac{2}{3}\sqrt{x^3} + c}$

Each of these three results may be checked by differentiation.

(a) The integral of a constant k is $kx + c$.

For example,

$$\int 8\,dx = \boldsymbol{8x + c}$$

(b) When a sum of several terms is integrated the result is the sum of the integrals of the separate terms.

For example,

$$\int (3x + 2x^2 - 5)\,dx = \int 3x\,dx + \int 2x^2\,dx - \int 5\,dx$$

$$= \boldsymbol{\frac{3x^2}{2} + \frac{2x^3}{3} - 5x + c}$$

Standard integrals

Since integration is the reverse process of differentiation the standard integrals listed in the table below may be deduced and readily checked by differentiation.

Table 1: Standard integrals

(i)	$\int ax^n\,dx = \frac{ax^{n+1}}{n+1} + c$ $(except\ when\ n = -1)$
(ii)	$\int \cos ax\,dx = \frac{1}{a}\sin ax + c$
(iii)	$\int \sin ax\,dx = -\frac{1}{a}\cos ax + c$
(iv)	$\int sec^2 ax\,dx = \frac{1}{a}\tan ax + c$
(v)	$\int cosec^2 ax\,dx = -\frac{1}{a}\cot ax + c$
(vi)	$\int \sec ax tan\,ax\,dx = \frac{1}{a}\sec ax + c$
(vii)	$\int cosec\ ax cot\ ax\,dx = -\frac{1}{a}cosec\ ax + c$
(viii)	$\int e^{ax}\,dx = \frac{1}{a}e^{ax} + c$
(ix)	$\int \frac{1}{x}dx = \ln x + c$
(x)	$\int \tan ax\,dx = \frac{1}{a}\ln\lvert\sec ax\rvert + c$
(xi)	$\int \cot ax\,dx = \frac{1}{a}\ln\lvert\sin ax\rvert + c$
(xii)	$\int \sec ax\,dx = \frac{1}{a}\ln\lvert\sec ax + \tan ax\rvert + c$
(xiii)	$\int \mathrm{cosec}\,ax\,dx = -\frac{1}{a}\ln\lvert\mathrm{cosec}\,ax + \cot ax\rvert + c$
(xiv)	$\int \frac{1}{\sqrt{(a^2-x^2)}}dx = sin^{-1}\frac{x}{a} + c$
(xv)	$\int \frac{\mathbf{1}}{(\boldsymbol{a^2+x^2})}dx = \frac{1}{a}tan^{-1}\frac{x}{a} + c$

Example 1. Determine (a) $\int 5x^2\,dx\ (b)\ \int 2t^3\,dt.$

The standard integral, $\int ax^n\,dx = \frac{ax^{n+1}}{n+1} + c$

(a) When a = 5 and n = 2 then

$$\int 5x^2\,dx = \frac{5x^{2+1}}{2+1} + c = \boldsymbol{\frac{5x^3}{3} + c}$$

(b) When a = 2 and n = 3 then

$$\int 2t^3\,dt = \frac{2t^{3+1}}{3+1} + c = \frac{2t^4}{4} + c = \boldsymbol{\frac{1}{2}t^4 + c}$$

Each of these results may be checked by differentiating them.

Example 2. Determine $\int\left(4 + \frac{3}{7}x - 6x^2\right)dx.$

$\int \left(4 + \frac{3}{7}x - 6x^2\right) dx$ may be written as $\int 4\, dx + \int \frac{3}{7}x\, dx - \int 6x^2\, dx$, i.e. each term is

integrated separately. (This splitting up of terms only applies, however, for addition and subtraction.)

Hence $\int \left(4 + \frac{3}{7}x - 6x^2\right) dx = 4x + \left(\frac{3}{7}\right)\frac{x^{1+1}}{1+1} - (6)\frac{x^{2+1}}{2+1} + c = 4x + \left(\frac{3}{7}\right)\frac{x^2}{3} - (6)\frac{x^3}{3} + c$

$$= \mathbf{4x + \frac{3}{14}x^2 - 2x^3 + c}$$

Note that when an integral contains more than one term there is no need to have an arbitrary constant for each; just a single constant at the end is sufficient.

Example3. Determine $(a) \int \frac{2x^3 - 3x}{4x} dx$ $(b) \int (1-t)^2\, dt$.

(a) Rearranging into standard integral form gives:

$$\int \frac{2x^3 - 3x}{4x} dx = \int \left(\frac{2x^3}{4x} - \frac{3x}{4x}\right) dx = \int \left(\frac{x^2}{2} - \frac{3}{4}\right) dx = \left(\frac{1}{2}\right)\frac{x^{2+1}}{2+1} - \frac{3}{4}x + c$$

$$= \left(\frac{1}{2}\right)\frac{x^3}{3} - \frac{3}{4}x + c = \mathbf{\frac{1}{6}x^3 - \frac{3}{4}x + c}$$

(b) Rearranging $\int (1-t)^2\, dt$ gives:

$$\int (1 - 2t + t^2)\, dt = t - \frac{2t^{1+1}}{1+1} + \frac{t^{2+1}}{2+1} + c = t - \frac{2t^2}{2} + \frac{t^3}{3} + c$$

$$= \mathbf{t - t^2 + \frac{1}{3}t^3 + c}$$

This problem shows that functions often have to be rearranged into the standard form of

$\int ax^n\, dx$ before it is possible to integrate them.

Example 4. Determine$\int \frac{3}{x^2} dx$.

$$\int \frac{3}{x^2} dx = \int 3x^{-2}\, dx.$$

Using the standard integral, $\int ax^n\, dx$ when a = 3 and n = – 2 gives:

$$\int 3x^{-2}\, dx = \frac{3x^{-2+1}}{-2+1} + c = -3x^{-1} + c = \mathbf{\frac{-3}{x} + c}$$

Example 5. Determine $\int 3\sqrt{x}\,dx$.

For fractional powers it is necessary to appreciate $\sqrt[n]{a^m} = a^{\frac{m}{n}}$

$$\int 3\sqrt{x}\,dx = \int 3x^{\frac{1}{2}}\,dx = \frac{3x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + c = \frac{3x^{\frac{3}{2}}}{\frac{3}{2}} + c = 2x^{\frac{3}{2}} + c$$

$$= \mathbf{2\sqrt{x^3} + c}$$

Example 6. Determine $\int \frac{-5}{9\sqrt[4]{t^3}}\,dt$.

$$\int \frac{-5}{9\sqrt[4]{t^3}}\,dt = \int \frac{-5}{9t^{\frac{3}{4}}}\,dt = \int \frac{-5}{9}t^{-\frac{3}{4}}\,dt = \left(-\frac{5}{9}\right)\frac{t^{-\frac{3}{4}+1}}{-\frac{3}{4}+1} + c = \left(-\frac{5}{9}\right)\frac{t^{\frac{1}{4}}}{\frac{1}{4}} + c$$
$$= \left(-\frac{5}{9}\right)\left(\frac{4}{1}\right)t^{\frac{1}{4}} + c = -\mathbf{\frac{20}{9}\sqrt[4]{t} + c}$$

Example 7. Determine $\int \frac{(1+\theta)^2}{\sqrt{\theta}}\,d\theta$.

$$\int \frac{(1+\theta)^2}{\sqrt{\theta}}\,d\theta = \int \frac{(1+2\theta+\theta^2)}{\sqrt{\theta}}\,d\theta = \int \left(\frac{1}{\theta^{\frac{1}{2}}} + \frac{2\theta}{\theta^{\frac{1}{2}}} + \frac{\theta^2}{\theta^{\frac{1}{2}}}\right)d\theta = \int (\theta^{-\frac{1}{2}} + 2\theta^{1-\frac{1}{2}} + \theta^{2-\frac{1}{2}})d\theta$$
$$= \int (\theta^{-\frac{1}{2}} + 2\theta^{\frac{1}{2}} + \theta^{\frac{3}{2}})d\theta = \frac{\theta^{\left(-\frac{1}{2}\right)+1}}{-\frac{1}{2}+1} + \frac{2\theta^{\left(\frac{1}{2}\right)+1}}{\frac{1}{2}+1} + \frac{\theta^{\left(\frac{3}{2}\right)+1}}{\frac{3}{2}+1} + c$$
$$= \frac{\theta^{\frac{1}{2}}}{\frac{1}{2}} + \frac{2\theta^{\frac{3}{2}}}{\frac{3}{2}} + \frac{\theta^{\frac{5}{2}}}{\frac{5}{2}} + c = 2\theta^{\frac{1}{2}} + \frac{4}{3}\theta^{\frac{3}{2}} + \frac{2}{5}\theta^{\frac{5}{2}} + c = \mathbf{2\sqrt{\theta} + \frac{4}{3}\sqrt{\theta^3} + \frac{2}{5}\sqrt{\theta^5} + c}$$

Example 8. Determine $(a)\ \int 4\cos 3x\,dx\ \ (b)\ \int 5\sin 2\theta\,d\theta$.

(a) From the table of standard integrals,

$$\int 4\cos 4x\,dx = (4)\left(\frac{1}{3}\right)\sin 3x + c = \mathbf{\frac{4}{3}\sin 3x + c}$$

(b) From the table of standard integrals,

$$\int 5\sin 2\theta\,d\theta = (5)\left(-\frac{1}{2}\right)\cos 2\theta + c = -\mathbf{\frac{5}{2}\cos 2\theta + c}$$

Example 9. Determine $(a) \int 7\,sec^2 4t\,dt\ \ (b)\ 3\int cosec^2 2\theta\,d\theta$.

(a) From the table of standard integrals,

$$\int 7\, sec^2 4t\, dt \ = (7)\left(\frac{1}{4}\right)\tan 4t + c = \frac{\mathbf{7}}{\mathbf{4}}\mathbf{tan\, 4}\boldsymbol{t} + \boldsymbol{c}$$

(b) From the table of standard integrals,

$$3\int cosec^2 2\theta\, d\theta = (3)\left(-\frac{1}{2}\right)\cot 2\theta + c = -\frac{\mathbf{3}}{\mathbf{2}}\mathbf{cot\, 2}\boldsymbol{\theta} + \boldsymbol{c}$$

Example 10. Determine $(a) \int 5e^{3x}\, dx \ \ (b) \int \frac{2}{3e^{4t}} dt$.

(a) From the table of standard integrals,

$$\int 5e^{3x}\, dx \ = (5)\left(\frac{1}{3}\right)e^{3x} + c = \frac{\mathbf{5}}{\mathbf{3}}\boldsymbol{e^{3x}} + \boldsymbol{c}$$

(b) $\int \frac{2}{3e^{4t}} dt = \int \frac{2}{3} e^{-4t}\, dt = \left(\frac{2}{3}\right)\left(-\frac{1}{4}\right)e^{-4t} + c = -\frac{1}{6}e^{-4t} + c = -\frac{\mathbf{1}}{\mathbf{6}\boldsymbol{e^{4t}}} + \boldsymbol{c}$

Example 11. Determine $(a) \int \frac{3}{5x} dx \ \ (b) \int (\frac{2m^2+1}{m})\, dm$.

(a) $\int \frac{3}{5x} dx \ = \int \left(\frac{3}{5}\right)(\frac{1}{x})\, dx = \frac{\mathbf{3}}{\mathbf{5}}\mathbf{ln}\, \boldsymbol{x} + \boldsymbol{c}$ (From the table of standard integrals)

(b) $\int (\frac{2m^2+1}{m})\, dm = \int (\frac{2m^2}{m} + \frac{1}{m})\, dm = \int (2m + \frac{1}{m})\, dm = \frac{2m^2}{2} + \ln m + c = \boldsymbol{m^2} + \mathbf{ln}\, \boldsymbol{m} + \boldsymbol{c}$

The Fundamental Theorem of Calculus

The fundamental theorem of calculus reduces the problem of integration to anti-differentiation, i.e., finding a function F such that$F' = f$.

Statement of the Fundamental Theorem

Theorem - Fundamental Theorem of Calculus: Suppose that the function F is differentiable everywhere on [a, b] and that F' is integrable on [a, b]. Then $\int_a^b F'(x)\, dx = F(b) - F(a)$.

In other words, if f is integrable on [a, bJ and F is an antiderivative for f, i.e., if $F' = f$, then $\int_a^b f(x)\, dx = F(b) - F(a)$.

This theorem makes it easy to evaluate integrals.

Examples

1**.** Using the fundamental theorem of calculus, compute$\int_a^b x^2\, dx$.

Solution We begin by finding an antiderivative F(x) for $f(x) = x^2$; from the power rule, we may take $F(x) = \frac{1}{3}x^3$. Now, by the fundamental theorem, we have

$\int_a^b x^2\,dx = \int_a^b f(x)\,dx = F(b) - F(a) = \frac{1}{3}b^3 - \frac{1}{3}a^3$

We conclude that $\int_a^b x^2\,dx = \frac{1}{3}b^3 - \frac{1}{3}a^3$.

Expressions of the form F(b) - F(a) occur so often that it is useful to have a special notation for them: $F(x)|_a^b$ means F(b) - F(a). One also writes $F(x) = \int f(x)\,dx$ for the antiderivative (also called an indefinite integral). In terms of this notation, we can write the formula of the fundamental theorem of calculus in the form:

$$\int_a^b f(x)\,dx = F(x)|_a^b or \int_a^b f(x)\,dx = \left(\int f(x)\,dx\right)\Big|_a^b$$

where F is an antiderivative of f on [a. b] .

2. Find $\int_2^6 (x^2+1)dx$.

Solution By the sum and power rules for anti-derivatives, an anti-derivative for x^2+1 is $\frac{1}{3}x^3 + x$. By the fundamental theorem $\int_2^6 (x^2+1)dx = \left(\frac{1}{3}x^3 + x\right)\Big|_2^6 = \frac{6^3}{3} + 6 - \left(\frac{2^3}{3} + 2\right)$

$= 78 - 4\frac{2}{3} = 73\frac{1}{3}$

Exercises

1. Evaluate $\int_0^1 x^4\,dx$.

2. Find $\int_0^3 (x^2+3x)\,dx$.

3. Find $\int_{-1}^1 (t^4 + t^9)\,dt$.

4. Find $\int_0^{10} \frac{t^4}{100} - t^2)\,dt$.

Definite integrals

Integrals containing an arbitrary constant c in their results are called **indefinite integrals** since their precise value cannot be determined without further information. **Definite integrals** are those in which limits are applied. If an expression is written as $[X]_a^b$, ‘b’ is called the upper limit and ‘a’ the lower limit.

The operation of applying the limits is defined as $[X]_a^b$ = (b) − (a).

The increase in the value of the integral x^2 as x increases from 1 to 3 is written as $\int_1^3 x^2\,dx$.

Applying the limits gives:

$$\int_1^3 x^2\,dx = \left[\frac{x^3}{3}+c\right]_1^3 = \left(\frac{3^3}{3}+c\right)-\left(\frac{1^3}{3}+c\right) = (9+c)-\left(\frac{1}{3}+c\right) = \mathbf{8\frac{2}{3}}$$

Note that the ‘c’ term always cancels out when limits are applied and it need not be shown with definite integrals.

Examples

1. Evaluate

(a) $\int_1^2 3x\,dx \quad (b) \int_{-2}^3 (4-x^2)dx$.

(a) $\int_1^2 3x\,dx = \left[\frac{3x^2}{2}\right]_1^2 = \left\{\frac{3}{2}(2)^2\right\}-\left\{\frac{3}{2}(1)^2\right\} = 6-1\frac{1}{2} = \mathbf{4\frac{1}{2}}$

(b) $\int_{-2}^3 (4-x^2)dx = \left[4x-\frac{x^3}{3}\right]_{-2}^3 = \left\{4(3)-\frac{3^3}{3}\right\}-\left\{4(-2)-\frac{(-2)^3}{3}\right\}$

$$= \{12-9\} - \left\{-8-\frac{-8}{3}\right\} = \{3\}-\left\{-5\frac{1}{3}\right\} = \mathbf{8\frac{1}{3}}$$

2. Evaluate $\int_1^4 \frac{\theta+2}{\sqrt{\theta}}\,d\theta = \int_1^4 (\frac{\theta}{\sqrt{\theta}}+\frac{2}{\sqrt{\theta}})$, taking positive square roots only.

$\int_1^4 \frac{\theta+2}{\sqrt{\theta}}\,d\theta = \int_1^4 \left(\frac{\theta}{\theta^{\frac{1}{2}}}+\frac{2}{\theta^{\frac{1}{2}}}\right)d\theta = \int_1^4 \left(\theta^{\frac{1}{2}}+2\theta^{-\frac{1}{2}}\right)d\theta = \left[\frac{\theta^{\left(\frac{1}{2}+1\right)}}{\frac{1}{2}+1}+\frac{2\theta^{\left(\frac{-1}{2}\right)+1}}{\frac{-1}{2}+1}\right]_1^4 = \left[\frac{\theta^{\frac{3}{2}}}{\frac{3}{2}}+\frac{2\theta^{\frac{1}{2}}}{\frac{1}{2}}\right]_1^4 = \left[\frac{2}{3}\sqrt{\theta^3}+4\sqrt{\theta}\right]_1^4$

$$= \left[\frac{2}{3}\sqrt{4^3}+4\sqrt{4}\right]-\left[\frac{2}{3}\sqrt{(1)^3}+4\sqrt{1}\right]$$

$$= \left\{\frac{16}{3}+8\right\}-\left\{\frac{2}{3}+4\right\} = 5\frac{1}{3}+8-\frac{2}{3}-4 = \mathbf{8\frac{2}{3}}$$

3. Evaluate $\int_0^{\frac{\pi}{2}} 3\sin 2x\,dx$.

$\int_0^{\frac{\pi}{2}} 3\sin 2x\,dx = \left[(3)\left(-\frac{1}{2}\right)\cos 2x\right]_0^{\frac{\pi}{2}} = \left[-\frac{3}{2}\cos 2x\right]_0^{\frac{\pi}{2}} = \left\{-\frac{3}{2}\cos 2(\frac{\pi}{2})\right\} - \left\{-\frac{3}{2}\cos 0\right\} =$
$\left\{-\frac{3}{2}\cos\pi\right\} - \left\{-\frac{3}{2}\cos 0\right\} = \left\{-\frac{3}{2}(-1)\right\} - \left\{-\frac{3}{2}(1)\right\} = \frac{3}{2} + \frac{3}{2} = \mathbf{3}$

4. Evaluate $\int_1^2 4\cos 3t\,dt$.

$$\int_1^2 4\cos 3t\,dt = \left[4(\frac{1}{3})\sin 3t\right]_1^2 = \left[\frac{4}{3}\sin 3t\right]_1^2 = \left\{\frac{4}{3}\sin 6\right\} - \left\{\frac{4}{3}\sin 3\right\}$$

Note that limits of trigonometric functions are always expressed in radians—thus, for example, sin 6 means the sine of 6 radians = −0.279415...

Hence $\int_1^2 4\cos 3t\,dt = \left\{\frac{4}{3}(-0.279415\right\} - \left\{\frac{4}{3}(0.141120)\right\} = (-0.37255) - (0.18816)$

$$= \mathbf{-0.5607}$$

5. Evaluate

(a) $\int_1^2 4e^{2x}\,dx$ $(b)\int_1^4 \frac{3}{4u}du$, each correct to 4 significant figures.

(a) $\int_1^2 4e^{2x}\,dx = \left[\frac{4}{2}e^{2x}\right]_1^2 = [2e^{2x}]_1^2 = 2(e^4 - e^2) = 2(54.5982 - 7.3891) = \mathbf{94.42}$

(b) $\int_1^4 \frac{3}{4u}du = \left[\frac{3}{4}\ln u\right]_1^4 = \frac{3}{4}(\ln 4 - \ln 1) = \frac{3}{4}(1.3863 - 0) = \mathbf{1.040}$

Techniques of integration

Integration is a reverse of the process of differentiation. Given a derived function

$$\frac{dy}{dx} = f(x)$$

$$dy = f(x)dx$$

Integrating both sides, $\int dy = \int f(x)\,dx$

$y(x) = \int f(x)\,dx + c$

Functions which require integrating are not always in the ‘standard form’ shown in Table 1. However, it is often possible to change a function into a form which can be integrated by using either:

(i) an algebraic substitution,

(ii) a trigonometric or hyperbolic substitution,

(iii) partial fractions,

(iv) the t = tanθ/2 substitution,

(v) integration by parts, or

(vi) reduction formulae.

Power rule

$\int ax^n\,dx = \frac{ax^{n+1}}{n+1} + c, a$ being a constant.

Examples

1. $\int \frac{x^6-2x+4}{x^4}\,dx = \int(x^2 - 2x^{-3} + 4x^{-4})\,dx = \frac{x^3}{3} - \frac{2x^{-2}}{-2} + \frac{4x^{-5}}{-5} + c = \boldsymbol{\frac{x^3}{3} + x^{-2} - \frac{4}{3}x^{-3} + c}$

2. $\int(\sqrt{x} - 3\sqrt[4]{x^5})\,dx = \int(x^{\frac{1}{2}} - 3x^{\frac{5}{4}})\,dx = \frac{x^{1\frac{1}{2}}}{1\frac{1}{2}} - \frac{3x^{\frac{9}{4}}}{\frac{9}{4}} + c = \boldsymbol{\frac{2}{3}x^{\frac{3}{2}} - \frac{4}{9}x^{\frac{9}{4}} + c}$

Exercise

1. $\int(x^3+3)^2\,dx$

2. $(\frac{\sqrt[3]{x}-6\sqrt{x^3}+1}{x^3})dx$

Substitution Technique

In this method, the substitution usually made is to let u be equal to some function f(x) such that the integrand is reduced to a standard integral.

It is found that integrals of the forms,

$$k\int[f(x)]^n f'(x)\,dx \text{ and } k\int\frac{f'(x)}{[f(x)]^n}\,dx$$

(where k and n are constants) can both be integrated by substituting u for f(x).

Examples on integration using substitution

1. Determine $\int \cos(3x+7)\ dx$.

$\int \cos(3x+7)\ dx$ is not a standard integral of the form shown in Table 1, thus an algebraic substitution is made.

Let u=3x+7 then $\frac{du}{dx} = 3$ and rearranging gives $dx = \frac{du}{3}$. Hence,

$\int \cos(3x+7)\ dx = \int (\cos u)\frac{du}{3} = \int \frac{1}{3}\cos u\, du$, which is a standard integral

$= \frac{1}{3}\sin u + c$

Rewriting u as (3x+7) gives:

$$\int \cos(3x+7)\ dx = \mathbf{\frac{1}{3}\sin(3x+7)} + \boldsymbol{c},$$

which may be checked by differentiating it.

2. Determine $\int x\cos(3x^2-4)\ dx$.

Let $u = 3x^2 - 4$ then $\frac{du}{dx} = 6x$ and $dx = \frac{du}{6x}$

Hence $\int x\cos(3x^2-4)\ dx = \int x\cos u \frac{du}{6x} = \frac{1}{6}\int \cos u\, du = \frac{1}{6}\sin u + c = \mathbf{\frac{1}{6}\sin(3x^2-4) + c}$

Practice

Determine $(a) \int \cos(4x+5)\, dx$

$$(b) \int (x^2+1)\cos(3x^3+9x-5)\, dx\ (Hint\colon let\ u = 3x^3+9x-5)$$

(c) $\int \frac{\cos\sqrt{x}}{\sqrt{x}} dx\ \left(Let\ u = \sqrt{x}\right)$ $(d) \int e^{-3x}\cos(4+e^{-3x})\, dx\ (Let\ u = 4+e^{-3x})$

3.Determine $\int \sin 6x\, dx$.

Let u = 6x, then $\frac{du}{dx} = 6\ and\ dx = \frac{du}{6}$

Thus $\int \sin 6x\, dx = \int \sin u \frac{du}{6} = \frac{1}{6}\int \sin u\, du = -\frac{1}{6}\cos u + c = \mathbf{-\frac{1}{6}\cos 6x + c}$

4. Determine $\int x\sin(6-3x^2)\, dx$.

Let u = $6-3x^2$, then $\frac{du}{dx} = -6x\ and\ dx = \frac{du}{-6x}$

Hence $\int xsin\,(6-3x^2)\,dx = \int xsin\,u\frac{du}{-6x} = \frac{1}{6}\int -\sin u\,du$

$$= \frac{1}{6}\cos u + c = \mathbf{\frac{1}{6}\cos(6-3x^2) + c}$$

Practice

Determine $(a)\int(\pi x+1)\sin(\pi x^2+2x-5)\,dx\ (Let\ u = \pi x^2+2x-5)$

(b) $\int(x^2+1)\sin(x^3+3x-5)\,dx\ (Let\ u = x^3+3x-5)$

(c) $\int e^{2x}\sin(5-e^{2x})\,dx\ (Let\ u = 5-e^{2x})$

5. Find $\int x(1+3x)^5\,dx$.

Let $u = 1+3x \ldots (a)$

$$\frac{du}{dx} = 3, hence\ \frac{du}{3} = dx$$

Substituting into the question, $\int x(1+3x)^5\,dx = \int x(u)^5\frac{du}{3}du = \int\frac{1}{3}xu^5\,du$

From equation (a), $u = 1+3x$;

Thus $x = \frac{u-1}{3}\ and\ hence\ \int\frac{1}{3}xu^5\,du = \int\frac{1}{3}\left(\frac{u-1}{3}\right)u^5\,du = \int\frac{1}{3}\left(\frac{u^6}{3}-\frac{u^5}{3}\right)du$

$$= \frac{1}{3}\left\{\frac{1}{3}(\frac{u^7}{7}) - \frac{1}{3}(\frac{u^6}{6})\right\} + c = \frac{1}{3}\left\{\frac{1}{21}(1+3x)^7 - \frac{1}{18}(1+3x)^6\right\} + c$$

$$= \mathbf{\frac{1}{63}(1+3x)^7 - \frac{1}{54}(1+3x)^6 + c}$$

Practice

Find $\int x\sqrt{2x+1}\,dx$

6.Find $\int(2x-5)^7\,dx$.

(2x−5) may be multiplied by itself 7 times and then each term of the result integrated. However, this would be a lengthy process, and thus an algebraic substitution is made.

Let u = (2x−5) then $\frac{du}{dx} = 2\ and\ dx = \frac{du}{2}$

Hence$\int (2x-5)^7\,dx = \int u^7 \frac{du}{2} = \frac{1}{2}\int u^7\,du = \frac{1}{2}\left(\frac{u^8}{8}\right) + c = \frac{1}{16}u^8 + c$

Rewriting u as (2x−5) gives:

$$\int (2x-5)^7\,dx = \frac{\mathbf{1}}{\mathbf{16}}(\mathbf{2x-5})^{\mathbf{8}} + \boldsymbol{c}$$

7.Find $\int \frac{4}{(5x-3)}\,dx$.

Let u = 5x – 3, then $\frac{du}{dx} = 5\ and\ dx = \frac{du}{5}$

Hence $\int \frac{4}{(5x-3)}\,dx = \int \frac{4}{u}\frac{du}{5} = \frac{4}{5}\int \frac{1}{u}du = \frac{4}{5}\ln u + c = \frac{\mathbf{4}}{\mathbf{5}}\mathbf{ln}(\mathbf{5x-3}) + \boldsymbol{c}$

Practice

(i) Determine $(a)\int \frac{x}{2+3x^2}\,dx\ (Let\ u = 2+3x^2)\ (b)\int \frac{2x}{\sqrt{(4x^2-1)}}\,dx\ (Let\ u = 4x^2-1)$

$$Answer: \frac{\mathbf{1}}{\mathbf{6}}\mathbf{ln}(\mathbf{2+3x^2}) + \boldsymbol{c}\,;\ \frac{\mathbf{1}}{\mathbf{2}}\sqrt{(\mathbf{4x^2-1})} + \boldsymbol{c}$$

(ii) Determine $(a)\int \frac{x^3}{5+x^4}\,dx\ (Let\ u = 5+x^4)\ \ (b)\int \frac{x+1}{x^2+2x-5}\,dx(Let\ u = x^2+2x-5)$

(c) $\int \frac{x}{1-3x^2}\,dx\ (let\ u = 1-3x^2)$ (d) $\int \frac{e^{3x}}{2+e^{3x}}\,dx(Hint\colon let\ u = 2+e^{3x})$

(e) $\int \frac{\cos 3x}{1+\sin 3x}\,dx\ (Hint\colon let\ u = 1+\sin 3x)\ \ (f)\int \frac{sec^2 2x}{1+\tan 2x}\,dx\ (Hint\colon let\ u = 1+\tan 2x)$

8.Evaluate $\int_0^1 2e^{6x-1}\,dx$, correct to 4 significant figures.

Let u = 6x−1 then $\frac{du}{dx} = 6\ and\ dx = \frac{du}{6}$

Hence $\int 2e^{6x-1}\,dx = \int 2e^{u}\frac{du}{6} = \frac{1}{3}\int e^u\,du = \frac{1}{3}e^u + c = \frac{1}{3}e^{6x-1} + c$

Thus

$\int_0^1 2e^{6x-1}\,dx = \frac{1}{3}\left[e^{6x-1}\right]_0^1 = \frac{1}{3}\left[e^5 - e^{-1}\right] = \mathbf{49.35}$,

correct to 4 significant figures.

Practice

Determine $(a) \int e^{3x}\, dx$ (b) $\int x^2 e^{2x^3-4}\, dx$ $(Hint: Let\ u = 2x^3 - 4)$

$$(c) \int \cos 4x e^{\sin 4x}\, dx \ (Hint: Let\ u = \sin 4x)$$

9. Determine $\int 3x(4x^2 + 3)^5\, dx.$

Let u = $(4x^2$ + 3) then $\frac{du}{dx} = 8x\ and\ dx = \frac{du}{8x}$

Hence

$\int 3x(4x^2 + 3)^5\, dx = \int 3x(u)^5 \frac{du}{8x} = \frac{3}{8} \int u^5\, du$, by cancelling x.

The original variable 'x' has been completely removed and the integral is now only in terms of u and is a standard integral.

Hence $\frac{3}{8} \int u^5\, du = \frac{3}{8}\left(\frac{u^6}{6}\right) + c = \frac{1}{16} u^6 + c = \boldsymbol{\frac{1}{16}(4x^2 + 3)^6 + c}$

Practice

Determine $(a) \int x^2(1 + 4x^3)^3\, dx$ $(b) \int \frac{x+1}{\sqrt{2x^2+4x-1}} dx$ $(Hint: Let\ u = 2x^2 + 4x - 1)$

10. Evaluate $\int_0^{\frac{\pi}{6}} 24\, sin^5\theta \cos\theta\, d\theta.$

Let u = sinθ then $\frac{du}{d\theta} = \cos\theta\ and\ d\theta = \frac{du}{\cos\theta}$

Hence $\int 24\, sin^5\theta \cos\theta\, d\theta = \int 24u^5 \cos\theta \frac{du}{\cos\theta} = \int 24u^5\, du = 24\frac{u^6}{6} + c = 4u^6 + c$

$$= 4sin^6\theta + c$$

Thus $\int_0^{\frac{\pi}{6}} 24\, sin^5\theta \cos\theta\, d\theta = [4sin^6\theta]_0^{\frac{\pi}{6}} = 4\left\{\left(\sin\frac{\pi}{6}\right)^6 - (\sin 0)^6\right\} = 4\left[\left(\frac{1}{2}\right)^6 - 0\right]$

$$= \boldsymbol{\frac{1}{16}\ or\ 0.0625}$$

11. Evaluate $\int sec^2(7x - 1)\, dx$

Let u = 7x – 1, then $\frac{du}{dx} = 7\ and\ dx = \frac{du}{7}$

$$\int sec^2(7x-1)\,dx = \int sec^2u\frac{du}{7} = \frac{1}{7}\int sec^2udu = \frac{1}{7}\tan u + c = \mathbf{\frac{1}{7}\tan(7x-1)+c}$$

Practice

Determine $\int \frac{sec^2(\sqrt{x})}{\sqrt{x}}dx\ (Let\ u=\sqrt{x})$ $\quad Answer: \mathbf{2\tan\sqrt{x}+c}$

12. Evaluate $\int x^2cosec^2(1-3x^3)\,dx$

$$Let\ u = 1-3x^3, then\ \frac{du}{dx} = -9x^2\ and\ dx = \frac{du}{-9x^2}$$

Hence $\int x^2cosec^2(1-3x^3)\,dx = \int x^2cosec^2u(\frac{du}{-9x^2}) = -\frac{1}{9}\int cosec^2u\,du = \frac{1}{9}\cot u + c$

$= \mathbf{\frac{1}{9}\cot(1-3x^3)+c}$

Practice

Evaluate $(a)\int cosec^2(4x)\,dx\ (b)\int \frac{cosec^2(\sqrt{x})}{\sqrt{x}}dx\ (Let\ u=\sqrt{x})$

$$(c)\int (x+1)cosec^2(2x^2+4x-5)\,dx(Let\ u = 2x^2+4x-5)$$

13(a).Show that $\int \tan\theta\,d\theta = \ln(\sec\theta)+c$

$$\int \tan\theta\,d\theta = \int\frac{\sin\theta}{\cos\theta}d\theta$$

$$Let\ u = \cos\theta, then\ \frac{du}{d\theta} = -\sin\theta\ \ and\ d\theta = \frac{du}{-\sin\theta}$$

Hence $\int\frac{\sin\theta}{\cos\theta}d\theta = \int\frac{\sin\theta}{u}(\frac{du}{-\sin\theta}) = -\int\frac{du}{u} = -\ln u + c = -\ln(\cos\theta)+c = 0-\ln(\cos\theta)+$ $c = \ln 1 - \ln(\cos\theta)+c = \ln(\frac{1}{\cos\theta})+c$

Hence $\mathbf{\int \tan\theta\,d\theta = \ln|sec\,\theta| + c}$

Practice

Evaluate the indefinite integrals:

i.$\int\tan(5x-2)\,dx$,

ii.$\int x^2 \tan(5-x^3)\, dx$,

iii,$\int (x^2-1)\tan(3x^3-9x-7)\, dx$,

iv.$\int \frac{\tan\sqrt{x}}{\sqrt{x}} dx$.

13(b) Show that $\int \cot\theta\, d\theta = \ln|\sin\theta| + c$

Practice

Evaluate the indefinite integrals:

i.$\int \cot(5x)\, dx$,

ii.$\int x^2 \tan(2-x)\, dx$,

iii,$\int (x-1)\tan(x^2-2x-1)\, dx$,

iv.$\int \frac{\cot\sqrt{x}}{\sqrt{x}} dx$.

14(a) Show that $\int \sec x\, dx = \ln|\sec x + \tan x| + k$.

Multiply through the numerator and denominator by $(\sec x + \tan x)$. Thus

$\int \sec x\, dx = \int \frac{\sec x.(\sec x+\tan x)}{\sec x+\tan x} dx$….(i)

Let u = $\sec x + \tan x$ ....(ii)

$$\frac{du}{dx} = \sec x \tan x + sec^2 x$$

$\frac{du}{\sec x \tan x + sec^2 x} = dx$ .....(iii)

Substituting (ii) and (iii) in (i),

$$\int \sec x\, dx = \int \frac{\sec x.(\sec x + \tan x)}{u} \frac{du}{\sec x \tan x + sec^2 x} = \int \frac{1}{u} du = \ln|u| + k$$

Therefore, $\int \sec x\, dx = \ln|\sec x + \tan x| + k$.

Practice

Evaluate the indefinite integrals:

i.$\int 2x\sec(4x^2+5)\,dx,$ $\quad Ans\ \frac{1}{4}\ln|\sec(4x^2+5)+\tan(4x^2+5)|+k$

ii.$\int 2x^2\sec(4\pi x^3+6)\,dx.$ $\quad Ans\ \frac{1}{6\pi}\ln|\sec(4\pi x^3+6)+\tan(4\pi x^3+6)|+k$

14(b) Show that $\int \operatorname{cosec} x\,dx = -\ln|\operatorname{cosec} x+\cot x|+k.$

Multiply through the numerator and denominator by $(\operatorname{cosec} x+\cot x)$. Thus

$\int \operatorname{cosec} x\,dx = \int \frac{\operatorname{cosec} x\,(\operatorname{cosec} x+\cot x.)}{\operatorname{cosec} x+\cot x}dx$....(i)

Let u = $(\operatorname{cosec} x+\cot x)$ ....(ii)

$$\frac{du}{dx} = -\operatorname{cosec} x\cot x - cosec^2x$$

$\frac{du}{-\operatorname{cosec} x\cot x - cosec^2x} = dx$ .....(iii)

Substituting (ii) and (iii) in (i),

$$\int \operatorname{cosec} x\,dx = \int \frac{\operatorname{cosec} x.(\operatorname{cosec} x+\cot x)}{u}\frac{du}{-\operatorname{cosec} x\cot x - cosec^2x} = \int -\frac{1}{u}du$$

$$= -\int \frac{1}{u}du = -\ln|u|+k$$

Therefore, $\int \operatorname{cosec} x\,dx = -\ln|\operatorname{cosec} x+\cot x|+k$

Practice

Evaluate the indefinite integrals:

i.$\int \operatorname{cosec}(2x+3)\,dx,$

ii.$\int (x+2)\operatorname{cosec}(x^2+4x-3)\,dx.$

Change of limits

When evaluating definite integrals involving substitutions, it is sometimes more convenient to **change the limits** of the integral as shown in Problems 14 and 15 below.

15. Evaluate $\int_1^3 5x\sqrt{(2x^2+7)}dx$, taking positive values of square roots only.

$$Let\ u = 2x^2 + 7, then\ \frac{du}{dx} = 4x\ and\ dx = \frac{du}{4x}$$

It is possible in this case to change the limits of integration. Thus when x = 3, u = $2(3)^2 + 7 = 25$ and when x = 1,u = $2(1)^2 + 7 = 9$.

Hence $\int_1^3 5x\sqrt{(2x^2+7)}dx = \int_9^{25} 5x\sqrt{u}\frac{du}{4x} = \frac{5}{4}\int_9^{25}\sqrt{u}\,du = \frac{5}{4}\int_9^{25} u^{\frac{1}{2}}\,du$

Thus the limits have been changed, and it is unnecessary to change the integral back in terms of x.

Thus $\int_1^3 5x\sqrt{(2x^2+7)}dx = \frac{5}{4}\left[\frac{u^{\frac{3}{2}}}{\frac{3}{2}}\right]_9^{25} = \frac{5}{6}\left[\sqrt{u^3}\right]_9^{25} = \frac{5}{6}\left[\sqrt{25^3} - \sqrt{9^3}\right] = \frac{5}{6}(125-27) = \mathbf{81\frac{2}{3}}$

16. Evaluate $\int_0^2 \frac{3x}{\sqrt{(2x^2+1)}}dx$, taking positive values of square roots only.

Let $u = 2x^2 + 1, then\ \frac{du}{dx} = 4x\ and\ dx = \frac{du}{4x}$

When x = 2, u = $2(2)^2 + 1 = 9$ and when x = 0, u = 1.

Thus $\int_0^2 \frac{3x}{\sqrt{(2x^2+1)}}dx = \int_1^9 \frac{3x}{\sqrt{u}}\frac{du}{4x} = \frac{3}{4}\int_1^9 \frac{1}{\sqrt{u}}du = \frac{3}{4}\int_1^9 u^{-\frac{1}{2}}\,du = \frac{3}{4}\left[\frac{u^{\frac{1}{2}}}{\frac{1}{2}}\right]_1^9 = \frac{3}{2}\left[\sqrt{9} - \sqrt{1}\right] = \mathbf{3},$

taking positive values of square roots only.

Powers of trigonometric functions

Integrals Involving $\boldsymbol{sin^2\theta, cos^2\theta. tan^2\theta\ andcot^2\theta}$

In such cases we use the identities:

$$(a) cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$$

$$(b) sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)$$

Notice that both are obtained by elimination from the trigonometric identities:

$$(i) cos^2\theta + sin^2\theta = 1$$

$$(ii) cos^2\theta - sin^2\theta = \cos 2\theta$$

Since $1 + tan^2\theta = sec^2\theta\ and\ cot^2\theta + 1 = cosec^2\theta$, then:

$$(c) tan^2\theta = sec^2\theta - 1$$

$$(d) cot^2\theta = cosec^2\theta - 1$$

Examples

1. Evaluate $\int_0^{\frac{\pi}{4}} 2cos^2 4t\ dt$.

$$cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)\ and\ cos^2 4t = \frac{1}{2}(1 + \cos 8t)$$

Hence, $\int_0^{\frac{\pi}{4}} 2cos^2 4t\ dt = 2\int_0^{\frac{\pi}{4}} \frac{1}{2}(1 + \cos 8t)dt = \left[t + \frac{\sin 8t}{8}\right]_{0}^{\frac{\pi}{4}} dt = \left[\frac{\pi}{4} + \frac{\sin 8(\frac{\mu}{4})}{8}\right] - \left[0 + \frac{\sin 0}{8}\right] = \boldsymbol{\frac{\pi}{4}}$

Or **0.7854**

2.Determine $\int cos^4 2x\ dx$.

$$m^4 = (m^2)^2$$

Therefore $cos^4 2x = (cos^2 2x)^2 = \left[\frac{1}{2}(1 + cos4x)\right]^2$

Now, $(a + b)^2 = a^2 + 2ab + b^2$.

Thus, $\left[\frac{1}{2}(1 + cos4x)\right]^2 = \frac{1}{4}(1 + 2cos4x + cos^2 4x) = \frac{1}{4}\left\{1 + 2cos4x + \frac{1}{2}(1 + \cos 8x)\right\}$

Therefore, $\int cos^4 2x\ dx = \frac{1}{4}\int dx + \frac{2}{4}\int \cos 4x\ dx + \frac{1}{8}\int dx + \frac{1}{8}\int \cos 8x dx$

$$= \frac{x}{4} + \frac{1}{2}(\frac{1}{4}\sin 4x) + \frac{1}{8}x + \frac{1}{8}(\frac{1}{8}\sin 8x) + c = \boldsymbol{\frac{3}{8}x + \frac{1}{8}\sin 8x + \frac{1}{64}\sin 8x + c}$$

Practice

Evaluate $\int_0^{\frac{\pi}{4}} 4cos^4\theta d\theta$, correct to 4 significant figures. Answer **2.178**

3. Determine $\int sin^2 3x\ dx$.

$$sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)\ and\ sin^2 3x = \frac{1}{2}(1 - \cos 6x)$$

Hence, $\int sin^2 3x\, dx = \int \frac{1}{2}(1-\cos 6x)\, dx = \boldsymbol{\frac{1}{2}\left(x - \frac{\sin 6x}{6}\right) + c}$

4. Find $\int 3tan^2 4x\, dx$.

$$tan^2\theta = sec^2\theta - 1$$

Thus, $\int 3tan^2 4x\, dx = 3\int (sec^2 4x - 1)\, dx = 3\left(\frac{\tan 4x}{4} - x\right) + c = \boldsymbol{3\frac{\tan 4x}{4} - 3x + c}$

5.Evaluate $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{2} cot^2 2\theta\, d\theta$.

$$cot^2\theta = cosec^2\theta - 1$$

Hence, $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{2} cot^2 2\theta\, d\theta = \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} (cosec^2 2\theta - 1)\, d\theta = \frac{1}{2}\left[\frac{-\cot 2\theta}{2} - \theta\right]_{\frac{\pi}{6}}^{\frac{\pi}{3}}$

$= \frac{1}{2}\left\{\left(\frac{-\cot 2(\frac{\pi}{3})}{2} - \frac{\pi}{3}\right) - \left(\frac{-\cot 2(\frac{\pi}{6})}{2} - \frac{\pi}{6}\right)\right\}$ =[(0.2887−1.0472) − (−0.2887−0.5236)]

= **0.0269**

Exercise

Evaluate the following integrals:

1.$\int sin^2 2x\, dx,$

2.$\int 3cos^2 3x\, d,$

3.$\int 5tan^2 2t\, dt,$

4.$\int_0^{\frac{\pi}{4}} cos^2 4x\, dx,$

5.$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} cot^2\theta\, d\theta.$

More examples on powers of sines and cosines

6. Determine $\int sin^5\theta\, d\theta$.

Since $cos^2\theta + sin^2\theta = 1$ then $sin^2\theta = 1 - cos^2\theta$.

Hence$\int sin^5\theta\, d\theta = \int \sin\theta (sin^2\theta)^2\, d\theta = \int \sin\theta (1 - cos^2\theta)^2\, d\theta$

$$= \int \sin\theta(1 - 2cos^2\theta + cos^4\theta)d\theta = \int (\sin\theta - 2\sin\theta cos^2\theta + \sin\theta cos^4\theta)d\theta$$

$$= -\boldsymbol{\cos\theta} + \frac{\mathbf{2}\boldsymbol{cos^3\theta}}{\mathbf{3}} - \frac{\boldsymbol{cos^5\theta}}{\mathbf{5}} + \boldsymbol{c}$$

Whenever a power of a cosine is multiplied by a sine of power 1, or vice-versa, the integral may be determined by inspection as shown above.

In general, $\int cos^n\theta \sin\theta\, d\theta = -\frac{\boldsymbol{cos^{n+1}\theta}}{\boldsymbol{n+1}} + \boldsymbol{c}$ and $\int sin^n\theta \cos\theta\, d\theta = \frac{\boldsymbol{sin^{n+1}\theta}}{\boldsymbol{n+1}} + \boldsymbol{c.}$

7. Evaluate $\int_0^{\frac{\pi}{2}} sin^2x cos^3x\, dx$.

$$\int_0^{\frac{\pi}{2}} sin^2x cos^3x\, dx = \int_0^{\frac{\pi}{2}} sin^2x cos^2x cos\, x\, dx \int_0^{\frac{\pi}{2}} sin^2x(1 - sin^2x) \cos x\, dx$$

$$= \int_0^{\frac{\pi}{2}} (sin^2x cosx - sin^4x cos\, x)\, dx = \left[\frac{sin^3x}{3} - \frac{sin^5x}{5}\right]_0^{\frac{\pi}{2}}$$

$$= \left[\frac{(\sin\frac{\pi}{2})^3}{3} - \frac{(\sin\frac{\pi}{2})^5}{5}\right] - [0 - 0] = \frac{1}{3} - \frac{1}{5} = \frac{\mathbf{2}}{\mathbf{15}}$$

Or **0.1333**

8. Find $\int sin^2t cos^4t\, dt$.

$$\int sin^2t cos^4t\, dt = \int sin^2t(cos^2t)^2\, dt = \int (\frac{1 - \cos 2t}{2})(\frac{1 + \cos 2t}{2})^2\, dt$$

$$= \frac{1}{8}\int (1 - \cos 2t)(1 + 2\cos 2t + cos^2 2t)\, dt$$

$$= \frac{1}{8}\int (1 + 2\cos 2t + cos^2 2t - \cos 2t - 2cos^2 2t - cos^3 2t)\, dt$$

$$= \frac{1}{8}\int (1 + \cos 2t - cos^2 2t - cos^3 2t)\, dt$$

$$= \frac{1}{8}\int (1 + \cos 2t - \left(\frac{1 + \cos 4t}{2}\right) - \cos 2t(1 - sin^2 2t)\, dt$$

$$= \frac{1}{8}\int (\frac{1}{2} - \frac{\cos 4t}{2} + \cos 2t sin^2 2t)\, dt = \boldsymbol{\frac{1}{8}\left(\frac{t}{2} - \frac{\sin 4t}{8} + \frac{sin^3 2t}{6}\right) + c}$$

Exercise

Integrate each of the following functions with respect to the variable

1.$sin^3\theta$

2.$2cos^3 2x$

3.$2sin^3 t cos^3 t$

4. $sin^3 x cos^2 x$

Products of sines and cosines

$$\sin(A + B) = \sin A \cos B + \cos A \sin B \ldots.(1)$$

$$\sin(A - B) = \sin A \cos B - \cos A \sin B \ldots.(2)$$

Adding (1) and (2) gives $\boldsymbol{\sin A \cos B = \frac{1}{2}[\sin(A + B) + \sin(A - B)]}$.

Subtracting (2) from (1) gives$\boldsymbol{\cos A \sin B = \frac{1}{2}[\sin(A + B) - \sin(A - B)]}$.

Thus a product of sine and cosine can be replaced by the respective right hand side.

Similarly,

$$\cos(A + B) = \cos A \cos B - \sin A \sin B \ldots.(3)$$

$$\cos(A - B) = \cos A \cos B + \sin A \sin B \ldots.(4)$$

Adding (3) and (4) gives $\boldsymbol{\cos A \cos B = \frac{1}{2}[\cos(A + B) + \cos(A - B)]}$, thus a product of cosines can be replaced by the right hand side.

Subtracting (4) from (3) gives $\boldsymbol{\sin A \sin B = -\frac{1}{2}[\cos(A + B) - \cos(A - B)]}$, thus a product of sines can be replaced by the right hand side.

Examples

1.Determine $\int \sin 3t \cos 2t \, dt$.

$$\int \sin 3t \cos 2t \, dt = \int \frac{1}{2}[\sin(3t+2t)+\sin(3t-2t)]\, dt = \frac{1}{2}\int (\sin 5t + \sin t)\, dt$$

$$= \mathbf{\frac{1}{2}(\frac{-\cos 5t}{5} - \cos t) + c}$$

2.Find$\int \frac{1}{3}(\cos 5x \sin 2x)\, dx$.

$$\int \frac{1}{3}(\cos 5x \sin 2x)\, dx = \frac{1}{3}\int \frac{1}{2}[\sin(5x+2x)-\sin(5x-2x)]\, dx = \frac{1}{6}\int (\sin 7x - \sin 3x)\, dx$$

$$= \mathbf{\frac{1}{6}\left(\frac{-\cos 7x}{7} + \frac{\cos 3x}{3}\right) + c}$$

3.Evaluate$\int_0^1 2\cos 6\theta \cos \theta \, d\theta$, correct to 4 decimal places.

$$\int_0^1 2\cos 6\theta \cos\theta \, d\theta = 2\int_0^1 \frac{1}{2}[\cos(6\theta+\theta)+\cos(6\theta-\theta)]\, d\theta = \int_0^1 (\cos 7\theta + \cos 5\theta)\, d\theta$$

$$= \left[\left(\frac{\sin 7\theta}{7} + \frac{\sin 5\theta}{5}\right)\right]_0^1 = \left[\frac{\sin 7}{7} + \frac{\sin 5}{5}\right] - \left[\frac{\sin 0}{7} + \frac{\sin 0}{5}\right] = \left[\frac{\sin 7}{7} + \frac{\sin 5}{5}\right].$$

‘sin 7’ means ‘the sine of 7 radians’ and sin 5 ‘the sine of 5 radians’.

Hence $\int_0^1 2\cos 6\theta \cos\theta \, d\theta$ = (0.09386 + (−0.19178)) − (0)

= **−0.0979**, correct to 4 decimal places.

4.Find $3\int \sin 5x \sin 3x \, dx$.

$$3\int \sin 5x \sin 3x \, dx = 3\int -\frac{1}{2}[\cos(5x+3x) - \cos(5x-3x)]\, dx$$
$$= -\frac{3}{2}\int (\cos 8x - \cos 2x)\, dx = -\frac{3}{2}\left(\frac{\sin 8x}{8} - \frac{\sin 2x}{2}\right) + c$$
$$= \mathbf{\frac{3}{16}(4\sin 2x - \sin 8x) + c}$$

Exercise

Integrate each of the following functions with respect to the variable

1.sin 5t cos 2t.

2.2 sin 3x sin x.

3.3 cos 6x cos x.

4. $\frac{1}{2}\cos 4\theta \sin 2\theta$.

Integration using $\sin\theta$ substitution

Examples

1.Determine $\int \frac{1}{\sqrt{(a^2-x^2)}}dx$

Let $x = a\sin\theta, then\ \frac{dx}{d\theta} = a\cos\theta\ and\ dx = a\cos\theta d\theta$.

Hence $\int \frac{1}{\sqrt{(a^2-x^2)}}dx = \int \frac{1}{\sqrt{(a^2-a^2sin^2\theta)}} a\cos\theta d\theta = \int \frac{a\cos\theta d\theta}{\left[\sqrt{a^2(1-sin^2\theta)}\right]} = \int \frac{a\cos\theta d\theta}{\sqrt{(a^2cos^2\theta)}}$,

$$since\ cos^2\theta + sin^2\theta = 1.$$

$$= \int \frac{a\cos\theta d\theta}{acos\theta} = \int d\theta = \theta + c$$

Since $x = a\sin\theta, then \sin\theta = \frac{x}{a}\ and\ \theta = sin^{-1}\frac{x}{a}$.

Hence $\int \frac{1}{\sqrt{(a^2-x^2)}}dx = \boldsymbol{sin^{-1}\frac{x}{a} + c}$.

2. Evaluate $\int_0^3 \frac{1}{\sqrt{(9-x^2)}}dx$.

From the above example, $\int_0^3 \frac{1}{\sqrt{(9-x^2)}}dx = \left[\boldsymbol{sin^{-1}\frac{x}{3}}\right]_0^3, since\ a = 3$.

$$= (sin^{-1}1 - sin^{-1}0) = \frac{\pi}{2} - 0 = \boldsymbol{\frac{\pi}{2}\ or\ 1.5708}.$$

3.Find $\int \sqrt{(a^2-x^2)}\,dx$.

Let $x = a\sin\theta, then\ \frac{dx}{d\theta} = a\cos\theta\ and\ dx = a\cos\theta d\theta$.

Hence $\int \sqrt{(a^2-x^2)}\,dx = \int \sqrt{(a^2-a^2sin^2\theta)}\,a\cos\theta d\theta = \int \sqrt{(a^2(1-sin^2\theta)}\,a\cos\theta d\theta =$

$$= \int \sqrt{(a^2(cos^2\theta)} \, a\cos\theta d\theta$$

$$= \int (a\cos\theta)\,(a\cos\theta d\theta) = a^2 \int cos^2\theta \, d\theta$$

$$= a^2 \int (\frac{1+\cos 2\theta}{2}) d\theta \,, since\ (\cos 2\theta = 2cos^2\theta - 1)$$

$$= \frac{a^2}{2}\left(\theta + \frac{\sin 2\theta}{2}\right) + c = \frac{a^2}{2}\left(\theta + \frac{2\sin\theta\cos\theta}{2}\right) + c = \frac{a^2}{2}(\theta + \sin\theta\cos\theta) + c$$

Since $x = a\sin\theta, then\ \sin\theta = \frac{x}{a}\ and\ \theta = sin^{-1}\frac{x}{a}$.

Also, $cos^2\theta + sin^2\theta = 1, from\ which\ \cos\theta = \sqrt{1 - sin^2\theta} = \sqrt{\left[1 - (\frac{x}{a})^2\right]} = \sqrt{(\frac{a^2-x^2}{a^2})}$

$$= \frac{\sqrt{(a^2 - x^2)}}{a}$$

Thus $\int \sqrt{(a^2 - x^2)}\, dx = \frac{a^2}{2}(\theta + \sin\theta\cos\theta) + c = \frac{a^2}{2}\left[sin^{-1}\frac{x}{a} + \left(\frac{x}{a}\right)\frac{\sqrt{(a^2-x^2)}}{a}\right] + c$

$$= \boldsymbol{\frac{a^2}{2} sin^{-1}\frac{x}{a} + \frac{x}{2}\sqrt{(a^2 - x^2)} + c}$$

4.Evaluate $\int_0^4 \sqrt{(16 - x^2)}\, dx$

From above example, $\int_0^4 \sqrt{(16 - x^2)}\, dx = \left[\boldsymbol{\frac{16}{2} sin^{-1}\frac{x}{4} + \frac{x}{2}\sqrt{(16 - x^2)}}\right]_0^4$

$= \left[8sin^{-1}1 + 2\sqrt{(0)}\right] - [8sin^{-1}0 + 0] = 8sin^{-1}1 = 8\left(\frac{\pi}{2}\right) =$ **4π or 12.57**

Exercise

1.Determine $\int \frac{5}{\sqrt{4-t^2}} dt$.

2.Determine $\int \frac{3}{\sqrt{9-x^2}} dx$.

3.Determine $\int \sqrt{4 - x^2}\, dx$.

4. Determine $\int \sqrt{16 - 9x^2}\, dx$.

5.Evaluate $\int_0^4 \sqrt{16 - x^2}\, dx$.

Integration using $tan\theta$ substitution

Examples

1.Determine$\int \frac{1}{(a^2+x^2)} dx$.

Let $x = \text{atan}\,\theta$ $then$ $\frac{dx}{d\theta} = asec^2\theta$ and $dx = asec^2\theta d\theta$.

Hence $\int \frac{1}{(a^2+x^2)} dx = \int \frac{1}{(a^2+a^2tan^2\theta)}(asec^2\theta d\theta) = \int \frac{asec^2\theta d\theta}{a^2(1+tan^2\theta)} = \int \frac{asec^2\theta d\theta}{a^2sec^2\theta}$,

$$since\ 1 + tan^2\theta = sec^2\theta.$$

$$= \int \frac{1}{a} d\theta = \frac{1}{a}\theta + c.$$

Since $x = \text{atan}\,\theta$, $\theta = tan^{-1}\frac{x}{a}$.

Hence $\int \frac{\mathbf{1}}{(\boldsymbol{a^2+x^2})} \boldsymbol{dx} = \frac{\mathbf{1}}{\boldsymbol{a}} \boldsymbol{tan^{-1}} \frac{\boldsymbol{x}}{\boldsymbol{a}} + \boldsymbol{c}$.

2. Evaluate $\int_0^2 \frac{1}{(4+x^2)} dx$

From the above example, $\int_0^2 \frac{1}{(4+x^2)} dx = \frac{1}{2}\left[tan^{-1}\frac{x}{2}\right]_0^2$ since a = 2.

$= \frac{1}{2}(tan^{-1}1 - tan^{-1}0) = \frac{1}{2}\left(\frac{\pi}{4} - 0\right) = \boldsymbol{\frac{\pi}{8}}$ $\boldsymbol{or\ 0.3927}$

3.Evaluate $\int_0^1 \frac{5}{(3+2x^2)} dx$, correct to 4 decimal places.

$\int_0^1 \frac{5}{(3+2x^2)} dx = \int_0^1 \frac{5}{2\left[\frac{3}{2}+x^2\right]} dx = \frac{5}{2}\int_0^1 \frac{1}{\left[(\sqrt{\frac{3}{2}})^2+x^2\right]} dx = \frac{5}{2}\left[\frac{1}{\sqrt{\frac{3}{2}}} tan^{-1}\frac{x}{\sqrt{\frac{3}{2}}}\right]_0^1$, since $a = \sqrt{\frac{3}{2}}$

$$= \frac{5}{2}\left(\frac{1}{\sqrt{\frac{3}{2}}}\right)\left[tan^{-1}\frac{x}{\sqrt{\frac{3}{2}}}\right]_0^1 = \frac{5}{2}\sqrt{\frac{2}{3}}\left[tan^{-1}\sqrt{\frac{2}{3}}x\right]_0^1$$

$= \frac{5}{2}\sqrt{(\frac{2}{3})}\left[tan^{-1}\sqrt{(\frac{2}{3})} - tan^{-1}0\right] = (2.0412)[0.6847 - 0] = \mathbf{1.3976}$, correct to 4 decimal places.

Exercise

1.Determine $\int \frac{3}{4+t^2} dt$.

2.Determine $\int \frac{5}{16+9\theta^2} d\theta$.

3. Evaluate $\int_0^1 \frac{3}{1+t^2} dt$.

4.Evaluate $\int_0^3 \frac{5}{4+x^2} dx$.

Integration using $\sinh\theta$ substitution

Examples

1.Determine $\int \frac{1}{\sqrt{(x^2+a^2)}} dx$

Let $x = a\sinh\theta, then\ \frac{dx}{d\theta} = a\cosh\theta\ and\ dx = a\cosh\theta d\theta$

Hence $\int \frac{1}{\sqrt{(x^2+a^2)}} dx = \int \frac{1}{\sqrt{(a^2 sinh^2\theta+a^2)}} (a\cosh\theta d\theta) = \int \frac{a\cosh\theta}{\sqrt{a^2 cosh^2\theta}} d\theta,$

$$since\ cosh^2\theta - sinh^2\theta = 1.$$

$$= \int d\theta = \theta + c = \boldsymbol{sinh^{-1}\frac{x}{a} + c}\ , since\ x = a\sinh\theta.$$

Inverse hyperbolic functions may be evaluated most conveniently when expressed in a logarithmic form.

Let $y = sinh^{-1}\frac{x}{a}, then\ \frac{x}{a} = \sinh y$.

From definition of hyperbolic functions, $e^y = \cosh y + \sinh y$

But $\cosh y = \sqrt{sinh^2 y + 1}$.

Therefore $e^y = \sqrt{(\frac{x}{a})^2 + 1} + \frac{x}{a} = \frac{\sqrt{x^2+a^2}}{a} + \frac{x}{a} = \frac{x+\sqrt{x^2+a^2}}{a}$

Thus taking natural logarithms, $y = \ln\frac{x+\sqrt{x^2+a^2}}{a}$ or $sinh^{-1}\frac{x}{a} = \ln\frac{x+\sqrt{x^2+a^2}}{a}$.

Hence $\int \frac{1}{\sqrt{(x^2+a^2)}} dx = \ln\left\{\frac{x+\sqrt{x^2+a^2}}{a}\right\} + c$

2.Evaluate $\int_0^2 \frac{1}{\sqrt{x^2+4}} dx$, correct to 4 decimal places.

$\int_0^2 \frac{1}{\sqrt{x^2+4}} dx = \left[sinh^{-1}\frac{x}{2}\right]_0^2\ or\ \left[\ln\left\{\frac{x+\sqrt{x^2+4}}{2}\right\}\right]_0^2$, since a = 2.

Using the logarithmic form, $\int_0^2 \frac{1}{\sqrt{x^2+4}}dx = \left[\ln(\frac{2+\sqrt{8}}{2}) - \ln(\frac{0+\sqrt{4}}{2})\right] = \ln 2.4142 - \ln 1 = \mathbf{0.8814}$

3.Evaluate $\int_1^2 \frac{2}{x^2\sqrt{1+x^2}}dx$, correct to 3 significant figures

Since the integral contains a term of the form $\sqrt{a^2+x^2}$, then let $x = a\sinh\theta, from\ which\ \frac{dx}{d\theta} = a\cosh\theta\ and\ dx = a\cosh\theta d\theta$

Hence $\int \frac{2}{x^2\sqrt{1+x^2}}dx = \int \frac{2(\cosh\theta d\theta)}{sinh^2\theta\sqrt{(1+sinh^2\theta)}} = 2\int \frac{\cosh\theta d\theta}{sinh^2\theta\cosh\theta}, since\ cosh^2\theta - sinh^2\theta = 1$

$$= 2\int \frac{d\theta}{sinh^2\theta} = 2\int cosech^2\theta\, d\theta = -2\coth\theta + c$$

$$\coth\theta = \frac{\cosh\theta}{\sinh\theta} = \frac{\sqrt{(1+sinh^2\theta)}}{\sinh\theta} = \frac{\sqrt{1+x^2}}{x}$$

Hence $\int_1^2 \frac{2}{x^2\sqrt{1+x^2}}dx = -[2\coth\theta]_1^2 = -2\left[\frac{\sqrt{1+x^2}}{x}\right]_1^2 = -2\left[\frac{\sqrt{5}}{2} - \frac{\sqrt{2}}{1}\right] = 0.592,$

correct to 3 significant figures.

4.Find $\int \sqrt{(x^2+a^2)}\,dx$

Let $x = a\sinh\theta, then\ \frac{dx}{d\theta} = a\cosh\theta\ and\ dx = a\cosh\theta d\theta$

Hence $\int \sqrt{(x^2+a^2)}\,dx = \int \sqrt{(a^2sinh^2\theta + a^2)}(a\cosh\theta d\theta) = \int \sqrt{a^2(sinh^2\theta + 1)}\,(a\cosh\theta d\theta)$

$$= \int \sqrt{a^2cosh^2\theta}\,(a\cosh\theta d\theta) = \int (a\cosh\theta)\,(a\cosh\theta)d\theta = a^2\int cosh^2\theta\, d\,\theta$$

$$= a^2\int \left(\frac{1+\cosh 2\theta}{2}\right)d\theta = \frac{a^2}{2}\left(\theta + \frac{\sinh 2\theta}{2}\right) + c = \frac{a^2}{2}(\theta + \sinh\theta\cosh\theta) + c,$$

Since $\sinh 2\theta = 2\sinh\theta\cosh\theta$.

Since $x = \mathrm{a}\sinh\theta, then \sinh\theta = \frac{x}{a}\ and\ \theta = sinh^{-1}\frac{x}{a}$

Also since $cosh^2\theta - sinh^2\theta = 1$, then $\cosh\theta = \sqrt{(1+sinh^2\theta)} = \sqrt{\left[1+(\frac{x}{a})^2\right]} = \sqrt{(\frac{a^2+x^2}{a^2})}$

$$= \frac{\sqrt{a^2+x^2}}{a}$$

Hence $\int \sqrt{(x^2+a^2)}\,dx = \frac{a^2}{2}\left[sinh^{-1}\frac{x}{a} + (\frac{x}{a})\frac{\sqrt{a^2+x^2}}{a}\right] + c = \boldsymbol{\frac{a^2}{2} sinh^{-1}\frac{x}{a} + \frac{x}{2}\sqrt{a^2+x^2} + c}$

Exercise

1.Find $\int \frac{2}{\sqrt{(x^2+16)}}dx,$

2.Find$\int \sqrt{(x^2+9)}\,dx,$

3.Evaluate $\int_0^3 \frac{4}{\sqrt{(t^2+9)}}dt,$

4.Evaluate $\int_0^1 \sqrt{(16+9\theta^2)}\,d\theta.$

Integration using cosh $\boldsymbol{\theta}$ substitution

1.Determine $\int \frac{1}{\sqrt{x^2-a^2}}dx.$

Let $x = a\cosh\theta \; then \; \frac{dx}{d\theta} = \text{a}\sinh\theta \; and \; dx = \text{a}\sinh\theta d\theta$

Hence $\int \frac{1}{\sqrt{x^2-a^2}}dx = \int \frac{1}{\sqrt{(a^2cosh^2\theta - a^2)}}(\text{a}\sinh\theta d\theta) = \int \frac{\text{a}\sinh\theta d\theta}{\sqrt{a^2(cosh^2\theta-1)}} = \int \frac{\text{a}\sinh\theta d\theta}{\sqrt{a^2sinh^2\theta}},$

$$since\ cosh^2\theta - sinh^2\theta = 1.$$

$$= \int \frac{\text{a}\sinh\theta d\theta}{\text{a}\sinh\theta} = \int d\theta = \theta + c = \boldsymbol{cosh^{-1}\frac{x}{a} + c}, since\ x = \text{a}\cosh\theta.$$

It can be shown that $cosh^{-1}\frac{x}{a} = \ln\left\{\frac{x+\sqrt{(x^2-a^2)}}{a}\right\}$

Hence $\int \frac{1}{\sqrt{x^2-a^2}}dx = \boldsymbol{\ln\left\{\frac{x+\sqrt{(x^2-a^2)}}{a}\right\} + c}$ provides an alternative solution.

2.Determine $\int \frac{2x-3}{\sqrt{x^2-9}}dx$

$$\int \frac{2x-3}{\sqrt{x^2-9}}dx = \int \frac{2x}{\sqrt{x^2-9}}dx - \int \frac{3}{\sqrt{x^2-9}}dx$$

The first integral is determined using the algebraic substitution $u = x^2 - 9$, and the second integral is of the form$\int \frac{1}{\sqrt{x^2-a^2}}dx.$

Hence $\int \frac{2x}{\sqrt{x^2-9}}dx - \int \frac{3}{\sqrt{x^2-9}}dx = \boldsymbol{2\sqrt{(x^2-9} - 3cosh^{-1}\frac{x}{3} + c}$

3.Determine $\int \sqrt{x^2-a^2}\,dx$

Let $x = a\cosh\theta$ $then$ $\frac{dx}{d\theta} = \mathrm{asinh}\,\theta$ and $dx = \mathrm{asinh}\,\theta d\theta$

Hence $\int \sqrt{x^2 - a^2}\,dx = \int \sqrt{a^2cosh^2\theta - a^2}\,(\mathrm{asinh}\,\theta d\theta) = \int \sqrt{a^2(cosh^2\theta - 1)}\,(\mathrm{asinh}\,\theta d\theta)$

$$= \int \sqrt{a^2sinh^2\theta}\,(\mathrm{asinh}\,\theta d\theta) = a^2 \int sinh^2\theta d\theta = a^2 \int (\frac{\cosh 2\theta - 1}{2})\,d\theta,$$

since $\cosh 2\theta = 1 + 2sinh^2\theta$.

$$= \frac{a^2}{2}\left[\frac{\sinh 2\theta}{2} - \theta\right] + c = \frac{a^2}{2}[\sinh\theta\cosh\theta - \theta] + c, since\ \sinh 2\theta = 2\sinh\theta\cosh\theta.$$

Since $x = \mathrm{acosh}\,\theta$ $then$ $\cosh\theta = \frac{x}{a}$ and $\theta = cosh^{-1}\frac{x}{a}$.

Also, since $cosh^2\theta - sinh^2\theta = 1$, $then$ $\sinh\theta = \sqrt{(cosh^2\theta - 1)} = \sqrt{(\frac{x}{a})^2 - 1} = \frac{\sqrt{(x^2-a^2)}}{a}$

Hence $\int \sqrt{x^2 - a^2}\,dx = \frac{a^2}{2}\left[\frac{\sqrt{(x^2-a^2)}}{a}\left(\frac{x}{a}\right) - cosh^{-1}\frac{x}{a}\right]$+c $= \boldsymbol{\frac{x}{2}\sqrt{(x^2 - a^2)} - \frac{a^2}{2}cosh^{-1}\frac{x}{a} + c}$

4.Evaluate $\int_2^3 \sqrt{x^2 - 4}\,dx$.

From above example, $\int_2^3 \sqrt{x^2 - 4}\,dx = \left[\frac{x}{2}\sqrt{(x^2 - 4)} - \frac{4}{2}cosh^{-1}\frac{x}{2}\right]_2^3$, $when\ a = 2$.

$$= \left(\frac{3}{5}\sqrt{5} - 2cosh^{-1}\frac{3}{2}\right) - (0 - 2cosh^{-1}1)$$

Since $cosh^{-1}\frac{x}{a} = \ln\left\{\frac{x+\sqrt{(x^2-a^2)}}{a}\right\}$, $then\ cosh^{-1}\frac{3}{2} = \ln\left\{\frac{3+\sqrt{(3^2-2^2)}}{2}\right\} = \ln 2.6180 = 0.9624$

Similarly, $cosh^{-1}1 = 0$.

Hence $\int \sqrt{x^2 - a^2}\,dx = \left[\frac{3}{2}\sqrt{5} - 2(0.9624)\right] - (0) = \mathbf{1.429}$, correct to 4 significant figures

Exercise

1.Find $\int \frac{1}{\sqrt{t^2-16}}dt$,

2.Find $\int \frac{3}{\sqrt{(4x^2-9)}}dx$,

3.Evaluate $\int_1^2 \frac{2}{\sqrt{(x^2-1)}}dx$,

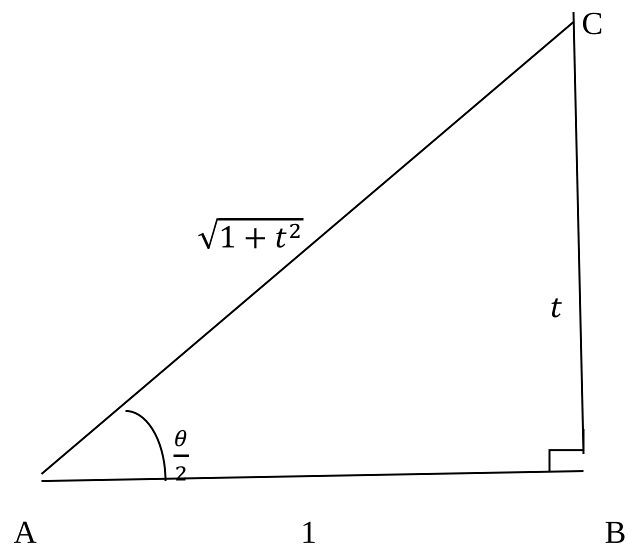
4.Evaluate $\int_2^3 \sqrt{(t^2 - 4)}\,dt$.

The $t = \tan\frac{\theta}{2}$ substitution

Integrals of the form $\int \frac{1}{a\cos\theta + b\sin\theta + c}\,d\theta$, where a, b and c are constants, may be determined by using the substitution $t = \tan\frac{\theta}{2}$.

. The reasoning is as explained below.

If angle A in the right-angled triangle ABC shown in the figure below is made equal to $\frac{\theta}{2}$ then, since tangent = $\frac{opposite}{adjacent}$, if BC = t and AB = 1, then $\tan\frac{\theta}{2}$ = t.



By Pythagoras' theorem, AC = $\sqrt{1+t^2}$

Therefore,

$$\sin\frac{\theta}{2} = \frac{t}{\sqrt{1+t^2}} \; and \cos\frac{\theta}{2} = \frac{1}{\sqrt{1+t^2}}$$

Since sin 2x = 2 sin x cos x(from double angle formulae), then $\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$

$$= 2\left(\frac{t}{\sqrt{1+t^2}}\right)\left(\frac{1}{\sqrt{1+t^2}}\right) = \frac{2t}{1+t^2}$$

i.e. $\boldsymbol{\sin\theta = \frac{2t}{1+t^2}}$ **…**(1)

Since cos 2x = $cos^2x - sin^2x$, then cos $\theta = cos^2\frac{\theta}{2} - sin^2\frac{\theta}{2} = (\frac{1}{\sqrt{1+t^2}})^2 - (\frac{t}{\sqrt{1+t^2}})^2$

i.e. $\boldsymbol{\cos\theta = \frac{1-t^2}{1+t^2}}$ **…**(2)

Also, since t = tan$\frac{\theta}{2}$, $\frac{dt}{d\theta} = \frac{1}{2}sec^2\frac{\theta}{2} = \frac{1}{2}(1 + tan^2\frac{\theta}{2})$ from trigonometric identities,

i.e. $\frac{dt}{d\theta} = \frac{1}{2}(1 + t^2)$ from which $\boldsymbol{d\theta = \frac{2dt}{1+t^2}}$ …(3)

Equations (1), (2) and (3) are used to determine integrals of the form $\int \frac{1}{a\cos\theta + b\sin\theta + c}d\theta$

where a, b or c may be zero.

Examples

1.Determine $\int \frac{1}{\sin\theta}d\theta$.

If $t = \tan\frac{\theta}{2}$, then $\sin\theta = \frac{2t}{1+t^2}$ and $d\theta = \frac{2dt}{1+t^2}$ from equations (1) and (3).

Thus $\int \frac{1}{\sin\theta}d\theta = \int \frac{1}{\frac{2t}{1+t^2}}\left(\frac{2dt}{1+t^2}\right) = \int \frac{1}{t}dt = \ln t + c$

Hence $\int \frac{1}{\sin\theta}d\theta = \mathbf{ln(tan}\boldsymbol{\frac{\theta}{2}}) + \boldsymbol{c}$

2.Determine $\int \frac{dx}{\cos x}$.

If $t = \tan\frac{x}{2}$ then $\cos x = \frac{1-t^2}{1+t^2}$ and $dx = \frac{2dt}{1+t^2}$ from equations (2) and (3).

Thus $\int \frac{dx}{\cos x} = \int \frac{1}{\frac{1-t^2}{1+t^2}}\left(\frac{2dt}{1+t^2}\right) = \int \frac{2}{1-t^2}dt.$

$\frac{2}{1-t^2}$ may be resolved into partial fractions.

Let $\frac{2}{1-t^2} \equiv \frac{2}{(1-t)(1+t)} \equiv \frac{A}{1-t} + \frac{B}{1+t} \equiv \frac{A(1+t)+B(1-t)}{(1-t)(1+t)}$

Hence 2 ≡ $A(1 + t) + B(1 - t)$.

When t = 1, 2 = 2A, from which,A = 1

When t = −1, 2 = 2B, from which,B = 1

Hence $\int \frac{2}{1-t^2}dt = \int \left\{\frac{1}{1-t} + \frac{1}{1+t}\right\}dt = -\ln(1-t) + \ln(1+t) + c = \ln\left\{\frac{(1+t)}{(1-t)}\right\} + c$

Thus $\int \frac{dx}{\cos x} = \mathbf{ln}\left\{\frac{1+\tan\frac{x}{2}}{1-\tan\frac{x}{2}}\right\} + \boldsymbol{c}$.

Note that since $\tan\frac{\pi}{4} = 1$, the above result may be written as:

$$\int \frac{dx}{\cos x} = \ln\left\{\frac{\tan\frac{\pi}{4} + \tan\frac{x}{2}}{1 - \tan\frac{\pi}{4}\tan\frac{x}{2}}\right\} + c = \mathbf{ln}\left\{\mathbf{tan}\left(\frac{\boldsymbol{\pi}}{\mathbf{4}} + \frac{\boldsymbol{x}}{\mathbf{2}}\right)\right\} + \boldsymbol{c}$$

3.Determine $\int \frac{dx}{1+\cos x}$.

If $t = \tan\frac{x}{2}$ then $\cos x = \frac{1-t^2}{1+t^2}$ and $dx = \frac{2dt}{1+t^2}$ from equations (2) and (3).

Thus$\int \frac{dx}{1+\cos x} = \int \frac{1}{1+\frac{1-t^2}{1+t^2}}\left(\frac{2dt}{1+t^2}\right) = \int \frac{1}{\frac{(1+t^2)+(1-t^2)}{1+t^2}}\left(\frac{2dt}{1+t^2}\right) = \int dt$

Hence $\int \frac{dx}{1+\cos x} = t + c = \mathbf{tan}\frac{\boldsymbol{x}}{\mathbf{2}} + \boldsymbol{c}$

4.Determine$\int \frac{d\theta}{5+4\cos\theta}$.

If $t = \tan\frac{\theta}{2}$, then $\cos\theta = \frac{1-t^2}{1+t^2}$ and $d\theta = \frac{2dt}{1+t^2}$ from equations (1) and (3).

Thus $\int \frac{d\theta}{5+4\cos\theta} = \int \frac{\frac{2dt}{1+t^2}}{5+4\left(\frac{1-t^2}{1+t^2}\right)} = \int \frac{\frac{2dt}{1+t^2}}{\frac{5(1+t^2)+4(1-t^2)}{(1+t^2)}} = 2\int \frac{dt}{t^2+9} = 2\int \frac{dt}{t^2+3^2} = 2\left(\frac{1}{3}tan^{-1}\frac{t}{3}\right) + c$

Hence $\int \frac{\boldsymbol{d\theta}}{\mathbf{5+4\cos\boldsymbol{\theta}}} = \frac{\mathbf{2}}{\mathbf{3}}\boldsymbol{tan^{-1}}\left(\frac{\mathbf{1}}{\mathbf{3}}\mathbf{tan}\frac{\boldsymbol{\theta}}{\mathbf{2}}\right) + \boldsymbol{c}$

5.Determine $\int \frac{dx}{\sin x+\cos x}$.

If $t = \tan\frac{x}{2}$ then $\sin x = \frac{2t}{1+t^2}$, $\cos x = \frac{1-t^2}{1+t^2}$ and $dx = \frac{2dt}{1+t^2}$ from equations (1), (2) and (3).

Thus $\int \frac{dx}{\sin x+\cos x} = \int \frac{\frac{2dt}{1+t^2}}{\frac{2t}{1+t^2}+\frac{1-t^2}{1+t^2}} = \int \frac{\frac{2dt}{1+t^2}}{\frac{2t+1-t^2}{1+t^2}} = \int \frac{2dt}{1+2t-t^2} = \int \frac{-2dt}{t^2-2t-1} = \int \frac{-2dt}{(t-1)^2-2}$

$$= \int \frac{2dt}{(\sqrt{2})^2 - (t-1)^2} = 2\left[\frac{1}{2\sqrt{2}}\ln\left\{\frac{\sqrt{2}+(t-1)}{\sqrt{2}-(t-1)}\right\}\right] + c$$

i.e. $\int \frac{\boldsymbol{dx}}{\mathbf{\sin\boldsymbol{x}+\cos\boldsymbol{x}}} = \frac{\mathbf{1}}{\sqrt{\mathbf{2}}}\mathbf{ln}\left\{\frac{\sqrt{\mathbf{2}}-\mathbf{1}+\mathbf{tan}\frac{\boldsymbol{x}}{\mathbf{2}}}{\sqrt{\mathbf{2}}+\mathbf{1}-\mathbf{tan}\frac{\boldsymbol{x}}{\mathbf{2}}}\right\} + \boldsymbol{c}$ (See problem 9 under Integration using partial fractions)

6.Determine $\int \frac{dx}{7-3\sin x+6\cos x}$.

From equations (1), (2) and (3),

$$\int \frac{dx}{7-3\sin x+6\cos x}=\int \frac{\frac{2dt}{1+t^2}}{7-3\left(\frac{2t}{1+t^2}\right)+6(\frac{1-t^2}{1+t^2})}=\int \frac{\frac{2dt}{1+t^2}}{\frac{7(1+t^2)-3(2t)+6(1-t^2)}{1+t^2}}$$

$$=\int \frac{2dt}{7+7t^2-6t+6-6t^2}=\int \frac{2dt}{t^2-6t+13}=\int \frac{2dt}{(t-3)^2+2^2}=2\left[\frac{1}{2}tan^{-1}(\frac{t-3}{2})\right]+c$$

Hence $\int \frac{dx}{7-3\sin x+6\cos x}=\boldsymbol{tan^{-1}\left(\frac{\tan\frac{x}{2}-3}{2}\right)+c}$

7.Determine $\int \frac{d\theta}{4\cos\theta+3\sin\theta}$.

From equations (1) to (3),

$$\int \frac{d\theta}{4\cos\theta+3\sin\theta}=\int \frac{\frac{2dt}{1+t^2}}{4\left(\frac{1-t^2}{1+t^2}\right)+3(\frac{2t}{1+t^2})}=\int \frac{2dt}{4-4t^2+6t}=\int \frac{dt}{2+3t-2t^2}$$

$$=-\frac{1}{2}\int \frac{dt}{t^2-\frac{3}{2}t-1}=-\frac{1}{2}\int \frac{dt}{(t-\frac{3}{4})^2-\frac{25}{16}}=\frac{1}{2}\int \frac{dt}{(\frac{5}{4})^2-(t-\frac{3}{4})^2}$$

$$=\frac{1}{2}\left[\frac{1}{2(\frac{5}{4})}\ln\left\{\frac{\frac{5}{4}+(t-\frac{3}{4})}{\frac{5}{4}-(t-\frac{3}{4})}\right\}+c\right]=\frac{1}{5}\ln\left\{\frac{\frac{1}{2}+t}{2-t}\right\}+c$$

(See problem 9 under Integration using partial fractions)

Hence $\int \frac{d\theta}{4\cos\theta+3\sin\theta}=\boldsymbol{\frac{1}{5}\ln\left\{\frac{\frac{1}{2}+\tan\frac{\theta}{2}}{2-\tan\frac{\theta}{2}}\right\}+c}$ **or** $\boldsymbol{\frac{1}{5}\ln\left\{\frac{1+2\tan\frac{\theta}{2}}{4-2\tan\frac{\theta}{2}}\right\}+c}$

Partial Fractions

By algebraic addition,

$$\frac{1}{x-2}+\frac{3}{x+1}=\frac{(x+1)+3(x-2)}{(x-2)(x+1)}=\frac{4x-5}{x^2-x-2}$$

The reverse process of moving from $\frac{4x-5}{x^2-x-2}$ to $\frac{1}{x-2}+\frac{3}{x+1}$ is called resolving into **partial fractions**.

In order to resolve an algebraic expression into partial fractions:

(i) the denominator must factorize (in the above example, $x^2 - x - 2$ factorizes as (x−2)(x+1)), and

(ii) the numerator must be at least one degree less than the denominator (in the above example

(4x−5) is of degree 1 since the highest powered x term is x^1 and ($x^2 - x - 2$) is of degree 2).

When the degree of the numerator is equal to or higher than the degree of the denominator, the

numerator must be divided by the denominator until the remainder is of less degree than the denominator(see Problems (c) and (d)).

There are basically three types of partial fraction and the form of partial fraction used is summarized in the following table , where f(x) is assumed to be of less degree than the relevant denominator and A, B and Care constants to be determined.

(In the latter type in the table , $ax^2 + bx + c$ is a quadratic expression which does not factorize

without containing surds or imaginary terms.)

Resolving an algebraic expression into partial fractions is used as a preliminary to integrating certain functions.

Type	Denominator containing	Expression	Form of Partial Fraction
1	Linear factors	$\frac{f(x)}{(x+a)(x-b)(x+c)}$	$\frac{A}{(x+a)} + \frac{B}{(x-b)} + \frac{C}{(x+c)}$
2	Repeated linear factors	$\frac{f(x)}{(x+a)^3}$	$\frac{A}{(x+a)} + \frac{B}{(x+a)^2} + \frac{C}{(x+a)^3}$
3	Quadratic factors	$\frac{f(x)}{(ax^2+bx+c)(x+d)}$	$\frac{Ax+B}{(ax^2+bx+c)} + \frac{C}{(x+d)}$

Examples

(a)Resolve$\frac{11-3x}{x^2+2x-3}$ into partial fractions.

The denominator factorizes as (x−1) (x+3) and the numerator is of less degree than the denominator. Thus $\frac{11-3x}{x^2+2x-3}$ may be resolved into partial fractions.

Let $\frac{11-3x}{x^2+2x-3} \equiv \frac{11-3x}{(x-1)(x+3)} \equiv \frac{A}{(x-1)} + \frac{B}{(x+3)}$

Where A and Bare constants to be determined, i.e. $\frac{11-3x}{x^2+2x-3} \equiv \frac{A(x+3)+B(x-1)}{(x-1)(x+3)}$ by algebraic addition.

Since the denominators are the same on each side of the identity then the numerators are equal to each other.

Thus, $11-3x \equiv A(x+3)+B(x-1)$

To determine constants A and B, values of x are chosen to make the term in A or B equal to zero.

When x=1, then $11-3(1) \equiv A(1+3)+B(0)$

i.e. $8 = 4A$

i.e. $A = 2$

When $x = -3$, then $11-3(-3) \equiv A(0)+B(-3-1)$

i.e. $20 = -4B$

i.e. $B = -5$

Thus, $\frac{11-3x}{x^2+2x-3} = \frac{2}{(x-1)} + \frac{-5}{(x+3)} = \frac{2}{(x-1)} - \frac{5}{(x+3)}$

Check: The right hand side simplifies to the result on the left by simple subtraction of fractions.

(b) Convert $\frac{2x^2-9x-35}{(x+1)(x-2)(x+3)}$ into the sum of three partial fractions.

Let $\frac{2x^2-9x-35}{(x+1)(x-2)(x+3)} \equiv \frac{A}{(x+1)} + \frac{B}{(x-2)} + \frac{C}{(x+3)} \equiv \frac{A(x-2)(x+3)+B(x+1)(x+3)+C(x+1)(x-2)}{(x+1)(x-2)(x+3)}$

by algebraic addition.

Equating the numerators gives:

$2x^2 - 9x - 35 \equiv A(x-2)(x+3) + B(x+1)(x+3) + C(x+1)(x-2)$

Let $x = -1$. Then

$2(-1)^2 - 9(-1) - 35 \equiv A(-3)(2) + B(0)(2) + C(0)(-3)$

i.e. $-24 = -6A$

i.e. $A = 4$

Let $x = 2$. Then

$2(2)^2 - 9(2) - 35 \equiv A(0)(5) + B(3)(5) + C(3)(0)$

i.e. −45 = 15B

i.e. B = −3

Let x = −3. Then

$2(-3)^2 - 9(-3) - 35 \equiv A(-5)(0) + B(-2)(0) + C(-2)(-5)$

i.e. 10 = 10 C

i.e. C = 1

Thus $\frac{2x^2-9x-35}{(x+1)(x-2)(x+3)} \equiv \frac{4}{(x+1)} - \frac{3}{(x-2)} + \frac{1}{(x+3)}$

(c)Resolve $\frac{x^2+1}{x^2-3x+2}$ into partial fractions.

The denominator is of the same degree as the numerator. Thus dividing out gives:

$$\begin{array}{r} 1 \\ x^2 - 3x + 2 \overline{) x^2 + \quad\quad 1} \\ \underline{x^2 - 3x + 2} \\ 3x - 1 \end{array}$$

Hence $\frac{x^2+1}{x^2-3x+2} \equiv 1 + \frac{3x-1}{x^2-3x+2} \equiv 1 + \frac{3x-1}{(x-1)(x-2)}$.

Let $\frac{3x-1}{(x-1)(x-2)} \equiv \frac{A}{(x-1)} + \frac{B}{(x-2)} \equiv \frac{A(x-2)+B(x-1)}{(x-1)(x-2)}$.

Equating numerators gives:

$3x-1 \equiv A(x-2) + B(x-1)$

Let x=1.Then 2 = −A

i.e. A = −2

Let x = 2.Then 5 = B

Hence $\frac{3x-1}{(x-1)(x-2)} \equiv \frac{-2}{(x-1)} + \frac{5}{(x-2)}$.

Thus $\frac{x^2+1}{x^2-3x+2} \equiv 1 - \frac{2}{(x-1)} + \frac{5}{(x-2)}$.

(d)Express $\frac{x^3-2x^2-4x-4}{x^2+x-2}$ in partial fractions.

The numerator is of higher degree than the denominator. Thus dividing out gives:

$$
\begin{array}{r}
x-3 \\
x^2+x-2\overline{)x^3-2x^2-4x-4} \\
\underline{x^3+x^2-2x} \\
-3x^2-2x-4 \\
\underline{-3x^2-3x+6} \\
x-10
\end{array}
$$

Thus$\frac{x^3-2x^2-4x-4}{x^2+x-2} \equiv x-3+\frac{x-10}{x^2+x-2} \equiv x-3+\frac{x-10}{(x+2)(x-1)}$

Let $\frac{x-10}{(x+2)(x-1)} \equiv \frac{A}{(x+2)} + \frac{B}{(x-1)} \equiv \frac{A(x-1)+B(x+2)}{(x+2)(x-1)}$

Equating the numerators gives:

x − 10 ≡ A(x−1) + B(x+2)

Let x = −2.Then −12 = −3A

i.e. A = 4

Let x = 1.Then −9 = 3B

i.e. B = −3

Hence $\frac{x-10}{(x+2)(x-1)} \equiv \frac{4}{(x+2)} - \frac{3}{(x-1)}$

Thus $\frac{x^3-2x^2-4x-4}{x^2+x-2} \equiv x-3+\frac{4}{(x+2)} - \frac{3}{(x-1)}$

(e)Resolve $\frac{2x+3}{(x-2)^2}$ into partial fractions.

The denominator contains a repeated linear factor, $(x-2)^2$.

Let $\frac{2x+3}{(x-2)^2} \equiv \frac{A}{(x-2)} + \frac{B}{(x-2)^2} \equiv \frac{A(x-2)+B}{(x-2)^2}$

Equating the numerators gives:

$2x + 3 \equiv A(x-2) + B$

Let x = 2. Then 7 = A(0) + B

i.e. B = 7

$2x + 3 \equiv A(x-2) + B \equiv Ax - 2A + B$

Since an identity is true for all values of the unknown, the coefficients of similar terms may be equated. Hence, equating the coefficients of x gives:2 = A.

[Also, as a check, equating the constant terms gives: 3 = −2A + B

When A= 2 and B = 7,

R.H.S. = −2(2) + 7 = 3 = L.H.S.]

Hence $\frac{2x+3}{(x-2)^2} \equiv \frac{2}{(x-2)} + \frac{7}{(x-2)^2}$

(f)Express $\frac{5x^2-2x-19}{(x+3)(x-1)^2}$ as the sum of three partial fractions.

The denominator is a combination of a linear factor and a repeated linear factor.

Let $\frac{5x^2-2x-19}{(x+3)(x-1)^2} \equiv \frac{A}{(x+3)} + \frac{B}{(x-1)} + \frac{C}{(x-1)^2} \equiv \frac{A(x-1)^2+B(x+3)(x-1)+C(x+3)}{(x+3)(x-1)^2}$ by algebraic addition.

Equating the numerators gives:

$5x^2 - 2x - 19 \equiv A(x-1)^2 + B(x+3)(x-1) + C(x+3)$ (1)

Let x = −3. Then

$5(-3)^2 - 2(-3) - 19 \equiv A(-4)^2 + B(0)(-4) + C(0)$

i.e. 32 = 16A, or A=2

Let x = 1. Then

$5(1)^2 - 2(1) - 19 \equiv A(0)^2 + B(4)(0) + C(4)$

i.e. −16 = 4C or C = −4

Without expanding the RHS of equation (1) it can be seen that equating the coefficients of x^2 gives:5 = A + B, and since A = 2, B = 3.

Hence $\frac{5x^2-2x-19}{(x+3)(x-1)^2} \equiv \frac{2}{(x+3)} + \frac{3}{(x-1)} - \frac{4}{(x-1)^2}$.

Practice

Resolve $\frac{3x^2+16x+15}{(x+3)^3}$ into partial fractions.

Ans $\frac{3}{(x+3)} - \frac{2}{(x+3)^2} - \frac{6}{(x+3)^3}$

(g) Express $\frac{7x^2+5x+13}{(x^2+2)(x+1)}$ in partial fractions.

The denominator is a combination of a quadratic factor, $(x^2 + 2)$, which does not factorize without introducing imaginary surd terms, and a linear factor, (x+1). Let,

$$\frac{7x^2 + 5x + 13}{(x^2 + 2)(x + 1)} \equiv \frac{Ax + B}{(x^2 + 2)} + \frac{C}{(x + 1)} \equiv \frac{(Ax + B)(x + 1) + C(x^2 + 2)}{(x^2 + 2)(x + 1)}$$

Equating numerators gives:

$7x^2 + 5x + 13 \equiv (Ax+B)(x+1) + C(x^2 + 2)$ (1)

Let x = −1.Then

$7(-1)^2 + 5(-1) + 13 \equiv (Ax+B)(0) + C(1+2)$

i.e. 15 = 3C or C = 5.

Identity (1) may be expanded as:

$7x^2 + 5x + 13 \equiv Ax^2 + Ax + Bx + B + Cx^2 + 2C$

Equating the coefficients of x^2 terms gives:

7 = A + C, and since C = 5, A = 2.

Equating the coefficients of x terms gives:

5 = A + B, and since A = 2, B = 3.

Hence $\frac{7x^2+5x+13}{(x^2+2)(x+1)} \equiv \frac{2x+3}{(x^2+2)} + \frac{5}{(x+1)}$.

Practice

Resolve $\frac{3+6x+4x^2-2x^3}{x^2(x^2+3)}$ into partial fractions.

Ans $\frac{2}{x}+\frac{1}{x^2}+\frac{3-4x}{x^2+3}$

Integration using partial fractions

The process of expressing a fraction in terms of simpler fractions—called **partial fractions**—is referred to as resolving the given fraction into partial fractions.

Certain functions have to be resolved into partial fractions before they can be integrated as demonstrated in the following worked problems.

1.Determine $\int \frac{11-3x}{x^2+2x-3}dx$.

Now $\frac{11-3x}{x^2+2x-3}=\frac{11-3x}{(x-1)(x+3)}\equiv\frac{2}{(x-1)}-\frac{5}{(x+3)}$.

Hence$\int \frac{11-3x}{x^2+2x-3}dx=\int\left\{\frac{2}{(x-1)}-\frac{5}{(x+3)}\right\}dx=\int\frac{2dx}{(x-1)}-\int\frac{5dx}{(x+3)}=\mathbf{2\ln(x-1)-5\ln(x+3)+c}$

Or $\mathbf{\ln\left\{\frac{(x-1)^2}{(x+3)^5}\right\}+c}$ by use of the laws of logarithms.

2.Find $\int\frac{2x^2-9x-35}{(x+1)(x-2)(x+3)}dx$.

$$\frac{2x^2-9x-35}{(x+1)(x-2)(x+3)}\equiv\frac{4}{(x+1)}-\frac{3}{(x-2)}+\frac{1}{(x+3)}$$

Hence $\int\frac{2x^2-9x-35}{(x+1)(x-2)(x+3)}dx=\int\left\{\frac{4}{(x+1)}-\frac{3}{(x-2)}+\frac{1}{(x+3)}\right\}dx=\int\frac{4dx}{(x+1)}-\int\frac{3dx}{(x-2)}+\int\frac{1dx}{(x+3)}$

$$=\mathbf{4\ln(x+1)-3\ln(x-2)+\ln(x+3)+c\ \textit{or}\ \ln\left\{\frac{(x+1)^4(x+3)}{(x-2)^3}\right\}+c}$$

3.Determine $\int\frac{x^2+1}{x^2-3x+2}dx$.

By dividing out (since the numerator and denominator are of the same degree) and resolving into partial fractions it was shown in (c) that $\frac{x^2+1}{x^2-3x+2}\equiv 1-\frac{2}{(x-1)}+\frac{5}{(x-2)}$.

Hence $\int\frac{x^2+1}{x^2-3x+2}dx=\int\left[1-\frac{2}{(x-1)}+\frac{5}{(x-2)}\right]dx=\mathbf{x-2\ln(x-1)+5\ln(x-2)+c}$ or

$$=\mathbf{x+\ln\left\{\frac{(x-2)^5}{(x-1)^2}\right\}+c}$$

4.Evaluate $\int_2^3\frac{x^3-2x^2-4x-4}{x^2+x-2}dx$, correct to 4 significant figures.

By dividing out and resolving into partial fractions it was shown in (d) that

$$\frac{x^3-2x^2-4x-4}{x^2+x-2} \equiv x-3+\frac{4}{(x+2)}-\frac{3}{(x-1)}$$

Hence $\int_2^3 \frac{x^3-2x^2-4x-4}{x^2+x-2}\,dx = \int_2^3 \left\{x-3+\frac{4}{(x+2)}-\frac{3}{(x-1)}\right\} dx$

$$= \left[\frac{x^2}{2}-3x+4\ln(x+2)-3\ln(x-1)\right]_2^3$$

$$= (\frac{9}{2}-9+4\ln 5-3\ln 2)-(2-6+4\ln 4-3\ln 1) = \mathbf{-1.687},$$

correct to 4 significant figures.

5.Determine $\int \frac{2x+3}{(x-2)^2}\,dx$.

It was shown in (e) that $\frac{2x+3}{(x-2)^2} \equiv \frac{2}{(x-2)}+\frac{7}{(x-2)^2}$.

Thus $\int \frac{2x+3}{(x-2)^2}\,dx = \int \left\{\frac{2}{(x-2)}+\frac{7}{(x-2)^2}\right\} dx = \boldsymbol{2\ln(x-2)-\frac{7}{(x-2)}+c}$

6.Find $\int \frac{5x^2-2x-19}{(x+3)(x-1)^2}\,dx$.

It was shown in (f) that $\frac{5x^2-2x-19}{(x+3)(x-1)^2} \equiv \frac{2}{(x+3)}+\frac{3}{(x-1)}-\frac{4}{(x-1)^2}$.

Hence $\int \frac{5x^2-2x-19}{(x+3)(x-1)^2}\,dx = \int \left\{\frac{2}{(x+3)}+\frac{3}{(x-1)}-\frac{4}{(x-1)^2}\right\} dx = 2\ln(x+3)+3\ln(x-1)+\frac{4}{(x-1)}+c$
or $\boldsymbol{\ln\{(x+3)^2(x-1)^3\}+\frac{4}{(x-1)}+c}$.

Practice

Find $\int \frac{x^2+1}{x^3+2x^2+x}\,dx$.

7.Evaluate $\int_{-2}^{1} \frac{3x^2+16x+15}{(x+3)^3}\,dx$, correct to 4 significant figures.

From the practice question above, $\frac{3x^2+16x+15}{(x+3)^3} \equiv \frac{3}{(x+3)}-\frac{2}{(x+3)^2}-\frac{6}{(x+3)^3}$.

Hence $\int_{-2}^{1} \frac{3x^2+16x+15}{(x+3)^3}\,dx = \int_{-2}^{1} \left\{\frac{3}{(x+3)}-\frac{2}{(x+3)^2}-\frac{6}{(x+3)^3}\right\} dx = \left[3\ln(x+3)+\frac{2}{(x+3)}+\frac{3}{(x+3)^2}\right]_{-2}^{1}$

$= \left(3\ln 4 + \frac{2}{4} + \frac{3}{16}\right) - \left(3\ln 1 + \frac{2}{1} + \frac{3}{1}\right) = \mathbf{-0.1536}$, correct to 4 significant figures.

8.Find $\int \frac{3+6x+4x^2-2x^3}{x^2(x^2+3)} dx$.

From the practice question above, $\frac{3+6x+4x^2-2x^3}{x^2(x^2+3)} \equiv \frac{2}{x} + \frac{1}{x^2} + \frac{3-4x}{x^2+3}$.

Thus$\int \frac{3+6x+4x^2-2x^3}{x^2(x^2+3)} dx = \int \left(\frac{2}{x} + \frac{1}{x^2} + \frac{3-4x}{x^2+3}\right) dx = \int \left\{\frac{2}{x} + \frac{1}{x^2} + \frac{3}{x^2+3} - \frac{4x}{x^2+3}\right\} dx$

$$\int \frac{3}{x^2+3} dx = 3\int \frac{1}{x^2+(\sqrt{3})^2} dx = \frac{3}{\sqrt{3}} tan^{-1} \frac{x}{\sqrt{3}}$$

$\int \frac{4x}{x^2+3} dx$ is determined using the algebraic substitution $u = x^2 + 3$.

Hence $\int \left\{\frac{2}{x} + \frac{1}{x^2} + \frac{3}{x^2+3} - \frac{4x}{x^2+3}\right\} dx = 2\ln x - \frac{1}{x} + \frac{3}{\sqrt{3}} tan^{-1} \frac{x}{\sqrt{3}} - 2\ln(x^2+3) + c$

$$= \mathbf{\ln\left(\frac{x}{x^2+3}\right)^2 - \frac{1}{x} + \sqrt{3}\,tan^{-1}\frac{x}{\sqrt{3}} + c}$$

9.Determine $\int \frac{1}{a^2-x^2} dx$.

Using partial fractions, let

$$\frac{1}{a^2-x^2} \equiv \frac{1}{(a-x)(a+x)} \equiv \frac{A}{(a-x)} + \frac{B}{(a+x)} \equiv \frac{A(a+x)+B(a-x)}{(a-x)(a+x)}$$

Then 1 ≡ A(a + x) + B(a − x)

Let x = a then A = $\frac{1}{2a}$. Let x = −a then B = $\frac{1}{2a}$

Hence $\int \frac{1}{a^2-x^2} dx = \int \frac{1}{2a}\left[\frac{1}{(a-x)} + \frac{1}{(a+x)}\right] dx = \frac{1}{2a}[-\ln(a-x) + \ln(a+x)] + c$

$$= \mathbf{\frac{1}{2a}\ln\left(\frac{a+x}{a-x}\right) + c}$$

Exercise

Determine $\int \frac{1}{x^2-a^2} dx$.

Integration by parts

From the product rule of differentiation:

$$\frac{d}{dx}(uv) = v\frac{du}{dx} + u\frac{dv}{dx},$$

where u and v are both functions of x.

Rearranging gives: $u\frac{dv}{dx} = \frac{d}{dx}(uv) - v\frac{du}{dx}$.

Integrating both sides with respect to x gives:

$$\int u\frac{dv}{dx}dx = \int \frac{d}{dx}(uv)\,dx - \int v\frac{du}{dx}dx$$

i.e. $\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$ or $\int \boldsymbol{u\,dv} = \boldsymbol{uv} - \int \boldsymbol{v\,du}$

This is known as the integration by parts formula and provides a method of integrating such

products of simple functions as $\int xe^x\,dx, \int t\sin t\,dt, \int e^\theta \cos\theta\,d\theta$ and $\int x\ln x\,dx$.

Given a product of two terms to integrate the initial choice is: 'which part to make equal to u' and 'which part to make equal to v'. The choice must be such that the 'u part' becomes a constant after successive differentiation and the 'dv part' can be integrated from standard integrals. Invariable, the following rule holds: If a product to be integrated contains an algebraic term (such as x, t^2or 3θ) then this term is chosen as the u part. An exception

to this rule is when a 'lnx' term is involved; in this case lnx is chosen as the 'upart'.

As a general guide, the acronym LIATE is used being abbreviation for the following functions:

L – Logarithmic function

I – Inverse functions

A – Algebraic functions

T – Trigonometric functions

E – Exponential Functions

The preceding function is taken as u while the remaining terms and dx form dv.

Examples

1. Determine $\int x\cos x\,dx$.

From the integration by parts formula,

$$\int u\,dv = uv - \int v\,du$$

Let u = x, from which $\frac{du}{dx} = 1\ i.e. du = dx. Let\ dv = \cos x dx, from\ which\ v = \int \cos x\,dx$

$$= \sin x.$$

Expressions for u, du and v are now substituted into the 'by parts' formula as shown below.

$$\int u\,dv \quad = \quad uv \quad - \quad \int v\,du$$

$$\int x\cos x\,dx = (x)\,(\sin x) - \int (\sin x)(\,dx)$$

i.e.

$$\int x\cos x\,dx = x\sin x - (-\cos x) + c$$

$$= \boldsymbol{x\sin x + \cos x + c}$$

[This result may be checked by differentiating the right hand side]

2. Find$\int 3te^{2t}\,dt$.

Let u = 3t, from which $\frac{du}{dt} = 3\ i.e. du = 3dt. Let\ dv = e^{2t}\,dt, from\ which\ v = \int e^{2t}\,dt = \frac{1}{2}e^{2t}$.

Substituting into

$$\int u\,dv = uv - \int v\,du$$

gives:

$$\int 3te^{2t}\,dt = (3t)\left(\frac{1}{2}e^{2t}\right) - \int \left(\frac{1}{2}e^{2t}\right)(3dt) = \frac{3}{2}te^{2t} - \frac{3}{2}\int e^{2t}\,dt = \frac{3}{2}te^{2t} - \frac{3}{2}\left(\frac{1}{2}e^{2t}\right) + c$$

$$= \frac{3}{2}te^{2t} - \frac{3}{2}\left(\frac{e^{2t}}{2}\right) + c$$

Hence$\int \boldsymbol{3te^{2t}\,dt = \frac{3}{2}e^{2t}\left(t - \frac{1}{2}\right) + c}$, which may be checked by differentiating.

3.Evaluate $\int_0^{\frac{\pi}{2}} 2\theta \sin\theta \, d\theta$.

$$Let\ u = 2\theta, from\ which, \frac{du}{d\theta} = 2, i.e.\ du = 2d\theta\ and\ dv = \sin\theta d\theta, from\ which$$

$$v = \int \sin\theta \, d\theta = -\cos\theta$$

Substituting into $\int u\, dv = uv - \int v\, du$ gives:

$$\int 2\theta \sin\theta \, d\theta = (2\theta)(-\cos\theta) - \int (-\cos\theta)(2d\theta) = -2\theta\cos\theta + 2\int \cos\theta \, d\theta$$

$$= -2\theta\cos\theta + 2\sin\theta + c$$

Hence $\int_0^{\frac{\pi}{2}} 2\theta \sin\theta \, d\theta = [-2\theta\cos\theta + 2\sin\theta]_0^{\frac{\pi}{2}} = \left[-2\left(\frac{\pi}{2}\right)\cos\frac{\pi}{2} + 2\sin\frac{\pi}{2}\right] - [0 + 2\sin 0]$

$$= (-0 + 2) - (0 + 0) = 2\ since \cos\frac{\pi}{2} = 0\ and\ \sin\frac{\pi}{2} = 1.$$

4.Evaluate $\int_0^1 5xe^{4x}\, dx$, correct to 3 significant figures.

Let u = 5x, from which $\frac{du}{dx} = 5$, i.e. du = 5dx and let $dv = e^{4x}dx$, from which, $v = \int e^{4x}\, dx$

$$= \frac{1}{4}e^{4x}$$

Substituting into $\int u\, dv = uv - \int v\, du$ gives:

$$\int 5xe^{4x}dx = (5x)\left(\frac{1}{4}e^{4x}\right) - \int \left(\frac{e^{4x}}{4}\right)(5dx) = \frac{5}{4}xe^{4x} - \frac{5}{4}\int e^{4x}dx = \frac{5}{4}xe^{4x} - \frac{5}{4}\left(\frac{e^{4x}}{4}\right) + c$$

$$= \frac{5}{4}e^{4x}\left(x - \frac{1}{4}\right) + c$$

Hence$\int_0^1 5xe^{4x}\, dx = \left[\frac{5}{4}e^{4x}\left(x - \frac{1}{4}\right)\right]_0^1 = \left[\frac{5}{4}e^4\left(1 - \frac{1}{4}\right)\right] - \left[\frac{5}{4}e^0\left(0 - \frac{1}{4}\right)\right] = \left(\frac{15}{16}e^4\right) - \frac{15}{16}$

$= 51.186 - 0.313 = 51.499 = \mathbf{51.5}$, correct to 3 significant figures

5.Determine $\int x^2 \sin x\, dx$.

Let u = x^2, from which, $\frac{du}{dx} = 2x$, i.e. du = 2xdx, and let dv = sinx dx, from which,

$$v = \int \sin x\, dx = -\cos x$$

Substituting into $\int u\,dv = uv - \int v\,du$ gives:

$$\int x^2 \sin x\,dx = (x^2)(-\cos x) - \int (-\cos x)(2x\,dx) = -x^2 \cos x + 2\int \mathrm{x}\cos x\,dx$$

The integral, $\int \mathrm{x}\cos x\,dx$, is not a 'standard integral' and it can only be determined by using the integration by parts formula again.

From example 1, $\int \mathrm{x}\cos x\,dx = x\sin x + \cos x$

Hence $\int x^2 \sin x\,dx = -x^2 \cos x + 2(x \sin x + \cos x) + c$

$$= -x^2 \cos x + 2x \sin x + 2\cos x + c = \boldsymbol{(2 - x^2)\cos x + 2x\sin x + c}$$

In general, if the algebraic term of a product is of power n, then the integration by parts formula is applied n times.

6.Find $\int x \ln x\,dx$.

The logarithmic function is chosen as the 'u part'. Thus when u = lnx, then $\frac{du}{dx} = \frac{1}{x}$, i.e. $du = \frac{dx}{x}$

Letting $dv = xdx$ gives $v = \int x\,dx = \frac{x^2}{2}$

Substituting into $\int u\,dv = uv - \int v\,du$ gives:

$$\int x \ln x\,dx = (\ln x)\left(\frac{x^2}{2}\right) - \int \left(\frac{x^2}{2}\right)\frac{dx}{x} = \frac{x^2}{2}\ln x - \frac{1}{2}\int x\,dx = \frac{x^2}{2}\ln x - \frac{1}{2}\left(\frac{x^2}{2}\right) + c$$

Hence $\int x \ln x\,dx = \boldsymbol{\frac{x^2}{2}(\ln x - \frac{1}{2}) + c}$

Practice

1.Determine $\int \ln x\,dx$

2.Find $\int x^3 \ln x\,dx$

7. Determine $\int x^2 e^x\,dx$.

Let $u = x^2, \frac{du}{dx} = 2x,\; du = 2xdx \; and \; dv = e^x dx \; or \; v = \int e^x\,dx = e^x$.

Substituting into $\int u\,dv = uv - \int v\,du$ gives:

$$\int x^2 e^x\,dx = x^2 e^x - \int e^x\,(2xdx) = x^2 e^x - 2\int x e^x\,dx$$

The integral, $\int xe^x\,dx$, is not a 'standard integral' and it can only be determined by using the integration by parts formula again.

Let $u = x, \frac{du}{dx} = 1,\ du = dx\ and\ dv = e^x dx\ or\ v = \int e^x\,dx = e^x$.

$$\int xe^x\,dx = xe^x - \int e^x\,dx = xe^x - e^x$$

Therefore $\int x^2e^x\,dx = x^2e^x - 2(xe^x - e^x) + c = \boldsymbol{(x^2 - 2x - 1)e^x + c}$

Note that the integration by parts formula is applied twice

Reduction Formulae

When using integration by parts, an

integral such as $\int x^2e^x\,dx$ requires integration by parts twice. Similarly, $\int x^3e^x\,dx$ requires integration by parts three times. Thus, integrals like $\int x^5e^x\,dx, \int x^6\cos x\,dx\ and\ \int x^8\sin x\,dx$

for example, would take a long time to determine using integration by parts. Reduction formulae provide a quicker method for determining such integrals and the method is demonstrated below.

Using reduction formulae for integrals of the form $\boldsymbol{\int x^ne^x\,dx}$

To determine $\int x^ne^x\,dx$ using integration by parts, let $u = x^n$ from which,

$$\frac{du}{dx} = nx^{n-1}\ and\ du = nx^{n-1}dx; dv = e^x dx\ from\ which\ v = \int e^x\,dx = e^x$$

Thus, $\int x^ne^x\,dx = x^ne^x - \int e^x nx^{n-1}\,dx$, using the integration by parts formula, i.e.

$$\int x^ne^x\,dx = x^ne^x - n\int x^{n-1}e^x\,dx.$$

The integral on the far right is seen to be of the same form as the integral on the left-hand side, except that n has been replaced by n−1.

Thus, if we let, $\int x^ne^x\,dx = I_n, then\ \int x^{n-1}e^x\,dx = I_{n-1}$.

Hence $\int x^ne^x\,dx = x^ne^x - n\int x^{n-1}e^x\,dx$ can be written as :

$$I_n = x^ne^x - nI_{n-1} \dots (1)$$

Equation (1) is an example of a reduction formula since it expresses an integral in n in terms of the same integral in n−1.

Examples

1. Determine$\int x^2e^x\,dx$ using a reduction formula.

Using equation (1) with n = 2 gives:

$$\int x^2e^x\,dx = I_2 = x^2e^x - 2I_1;$$

and $I_1 = x^1e^x - 1I_0$

$I_0 = \int x^0e^x\,dx = \int e^x\,dx = e^x + c'$

Hence $I_2 = x^2e^x - 2[x^1e^x - 1I_0]$

$$= x^2e^x - 2[x^1e^x - 1(e^x + c')] = x^2e^x - 2xe^x + 2e^x + c$$

Thus$\int x^2e^x\,dx = (\boldsymbol{x^2 - 2x - 1})\boldsymbol{e^x} + \boldsymbol{c}$

Notice that the constant of integration is added at the last step with indefinite integrals.

2. Use a reduction formula to determine $\int x^3e^x\,dx$.

From equation (1), $I_n = x^ne^x - nI_{n-1}$.

Hence $\int x^3e^x\,dx = I_3 = x^3e^x - 3I_2$

$$I_2 = x^2e^x - 3I_1$$

$$I_1 = x^1e^x - 1I_0$$

and $I_0 = \int x^0e^x\,dx = \int e^x\,dx = e^x$

Thus $\int x^3e^x\,dx = x^3e^x - 3[x^2e^x - 2I_1] = x^3e^x - 3[x^2e^x - 2(x^1e^x - 1I_0)]$

$$= x^3e^x - 3[x^2e^x - 2(x^1e^x - 1e^x)] = x^3e^x - 3x^2e^x + 6(x^1e^x - 1e^x)$$

$$= x^3e^x - 3x^2e^x + 6(x^1e^x - 1e^x) = x^3e^x - 3x^2e^x + 6xe^x - 6e^x$$

i.e. $\int x^3e^x\,dx = x^3e^x - 3x^2e^x + 6xe^x - 6e^x + c = \boldsymbol{e^x}(\boldsymbol{x^3 - 3x^2 + 6x - 6}) + \boldsymbol{c}$

Using reduction formulae for integrals of the form$\int \boldsymbol{x^n\cos x\,dx}$ and $\int \boldsymbol{x^n\sin x\,dx}$

(a) $\int \boldsymbol{x^n\cos x\,dx}$

Let $I_n = \int x^n\cos x\,dx$ then, using integration by parts:

if $u = x^n$ $then$ $\frac{du}{dx} = nx^{n-1} and\ du = nx^{n-1}dx\ and\ if\ dv = \cos x dx\ then\ v = \int \cos x\,dx$

$$= \sin x$$

Using the integration by parts formula $\int u\,dv = uv - \int v\,du$,

$$I_n = x^n \sin x - \int (\sin x) n x^{n-1} dx = x^n \sin x - n \int x^{n-1} \sin x\,dx$$

Using integration by parts again, this time with $u = x^{n-1}$

$$\frac{du}{dx} = (n-1)x^{n-2}, and\ dv = \sin x dx\ from\ which\ v = \int \sin x\,dx = -\cos x$$

Hence $I_n = x^n \sin x - n[x^{n-1}(-\cos x) - \int (-\cos x)(n-1)x^{n-2} dx]$

$$= x^n \sin x + n x^{n-1} \cos x - n(n-1) \int x^{n-2} \cos x\,dx$$

i.e. $\boldsymbol{I_n = x^n \sin x + n x^{n-1} \cos x - n(n-1) I_{n-2}}$ ... (2)

Example

Use a reduction formula to determine $\int x^2 \cos x\,dx$.

Using the reduction formula of equation (2):

$$\int x^2 \cos x\,dx = I_2 = x^2 \sin x + 2x^1 \cos x - 2(1) I_0\ \ and\ I_0 = \int x^0 \cos x\,dx = \int \cos x\,dx$$
$$= sinx$$

Hence $\int x^2 \cos x\,dx = \boldsymbol{x^2 \sin x + 2x \cos x - 2 \sin x + c}$

(b) $\boldsymbol{\int x^n sin\ x\,dx}$

Let $I_n = \int x^n \sin x\,dx$ then, using integration by parts:

if $u = x^n\ then\ \frac{du}{dx} = n x^{n-1} and\ du = n x^{n-1} dx\ and\ if\ dv = \sin x dx\ then\ v = \int \sin x\,dx$

$$= -\cos x$$

Using the integration by parts formula $\int u\,dv = uv - \int v\,du$,

$$\int x^n \sin x\,dx = I_n = x^n\,(-\cos x) - \int (-\cos x) n x^{n-1} dx = -x^n \cos x + n \int x^{n-1} \cos x\,dx$$

Using integration by parts again, this time with $u = x^{n-1}$

$$\frac{du}{dx} = (n-1)x^{n-2}, and\ dv = \cos x dx\ from\ which\ v = \int \cos x\,dx = \sin x$$

Hence $I_n = -x^n \cos x + n[x^{n-1}(\sin x) - \int(\sin x)(n-1)x^{n-2}dx]$

$$= -x^n \cos x + nx^{n-1}\sin x - n(n-1)\int x^{n-2}\sin x\,dx$$

i.e. $\boldsymbol{I_n = -x^n \cos x + nx^{n-1}\sin x - n(n-1)I_{n-2}}$ ... (3)

Example

Use a reduction formula to determine $\int x^3 \sin x\, dx$.

Using equation (3),

$$\int x^3 \sin x\,dx = I_3 = -x^3\cos x + 3x^2 \sin x - (3)(2)I_1 \;\; and\; I_1$$
$$= -x^1 \cos x + 1x^0 \sin x = -xcos\,x + \sin x$$

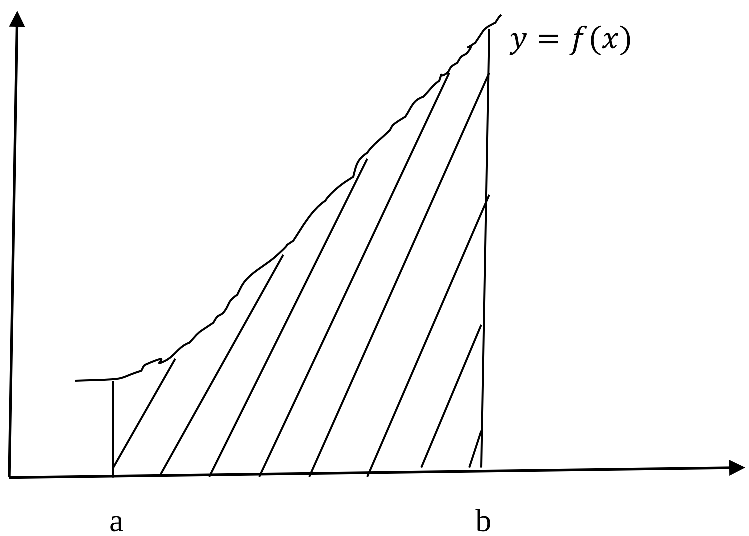
Hence $\int x^3 \sin x\,dx = -x^3 \cos x + 3x^2 \sin x - 6[-xcos\,x + \sin x]$

$$\boldsymbol{= -x^3\cos x + 3x^2 \sin x + 6x\cos x - 6\sin x + c}$$

Applications of Integration

There are a number of applications of integral calculus.

Areas under and between curves



The shaded area under the curve $y = f(x)$ between the ordinates $x = a\; and\; x = b$ and the x – axis is given by the integral $\int_a^b f(x)\,dx$.

Examples

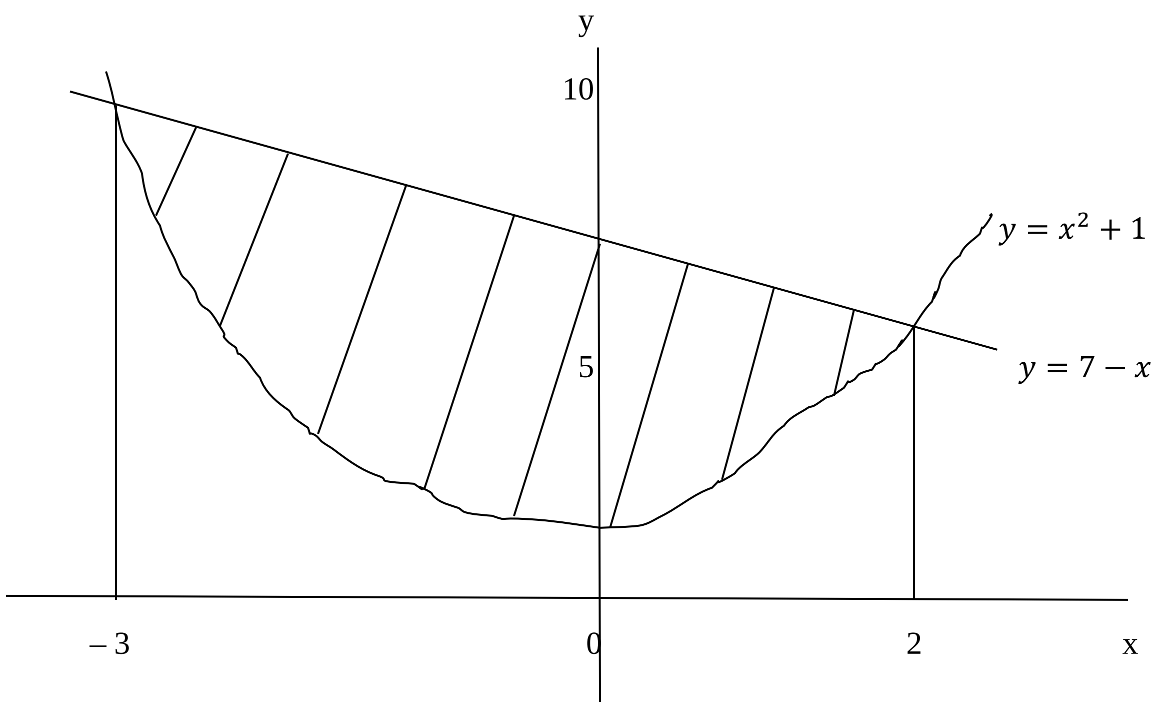
1.Determine the area enclosed by the curve $y = x^3 + 5$ and the x – axis between the ordinates $x = 1 \; and \; x = 3$.

Area $= \int_1^3 (x^3 + 5)\,dx = \left[\frac{x^4}{4} + 5x\right]_1^3 = \left[\frac{3^4}{4} + 5(3)\right] - \left[\frac{1}{4} + 5\right] = \frac{81}{4} + 15 - \frac{1}{4} - 5 = \frac{80}{4} - 10$

Thus, area $= 10$ square units.

2. Determine the area enclosed between the curves $y = x^2 + 1 \; and \; y = 7 - x$.

A sketch diagram of the area enclosed is as shown below



At the points of intersection the curves are equal. Thus, equating the y values of each curve gives:

$$x^2 + 1 = 7 - x$$

from which, $x^2 + x - 6 = 0$.

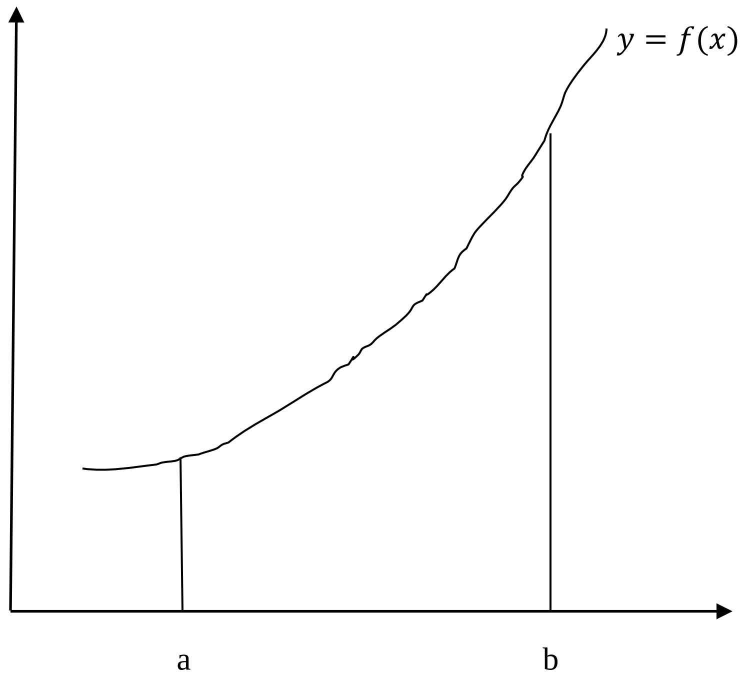
Factorising gives (x − 2)(x + 3) = 0 from which x = 2 and x = −3

Shaded area $= \int_{-3}^{2} (7 - x)\,dx - \int_{-3}^{2} (x^2 + 1)\,dx = \int_{-3}^{2} [(7 - x) - (x^2 + 1)]\,dx$

$$= \int_{-3}^{2} (6 - x - x^2)\, dx = \left[6x - \frac{x^2}{2} - \frac{x^3}{3}\right]_{-3}^{2} = \left(12 - 2 - \frac{8}{3}\right) - \left(-18 - \frac{9}{2} + 9\right)$$

$$= \left(7\frac{1}{3}\right) - \left(-13\frac{1}{2}\right) = \mathbf{20\frac{5}{6}}\ \boldsymbol{square\ units}$$

The Length of a curve



The length of a curve $y = f(x)$ between the ordinates $x = a\ and\ x = b$ is given by

$$L = \int_a^b \sqrt{1 + (\frac{dy}{dx})^2}\, dx = \int_a^b \sqrt{1 + (f'(x))^2} dx$$

Example

1.Find the length of the curve $y = \frac{4\sqrt{2}}{3} x^{\frac{3}{2}} - 1, 0 \le x \le 1.$

We use the above equation for L with a = 0, b = 1 and $y = \frac{4\sqrt{2}}{3} x^{\frac{3}{2}} - 1$

$$\frac{dy}{dx} = \frac{4\sqrt{2}}{3} . \frac{3}{2} x^{\frac{1}{2}} = 2\sqrt{2} x^{\frac{1}{2}}$$

$$(\frac{dy}{dx})^2 = (2\sqrt{2} x^{\frac{1}{2}})^2 = 8x.$$

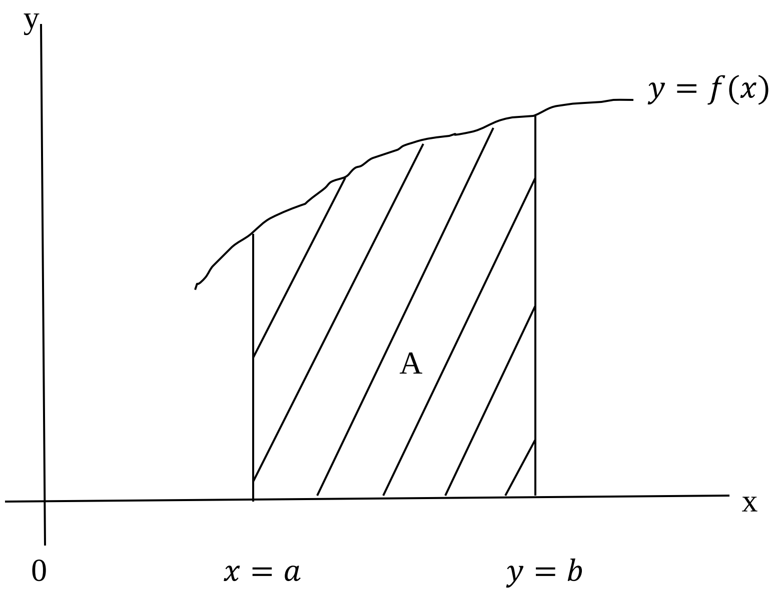
The length of the curve from $x = 0\ to\ x = 1$ is

$$L = \int_0^1 \sqrt{1 + (\frac{dy}{dx})^2}\, dx = \int_0^1 \sqrt{1+8x}\, dx = \frac{2}{3} . \left[\frac{1}{8}(1+8x)^2\right]_0^1 = \frac{13}{6}\ units.$$

Exercise

Find the length of the curve $y = \left(\frac{1}{3}\right)(x^2+2)^{\frac{3}{2}}$ from $x = 0\ to\ x = 3$.

Volumes of solids of revolution



With reference to the above diagram, the volume of revolution,V, obtained by rotating area A through one revolution about the x-axis is given by:

$$\boldsymbol{V = \int_a^b \pi y^2\, dx}$$

Example

The curve $y = x^2 + 4$ is rotated one revolution about the x-axis between the limits x = 1 and

x = 4. Determine the volume of solid of revolution produced.

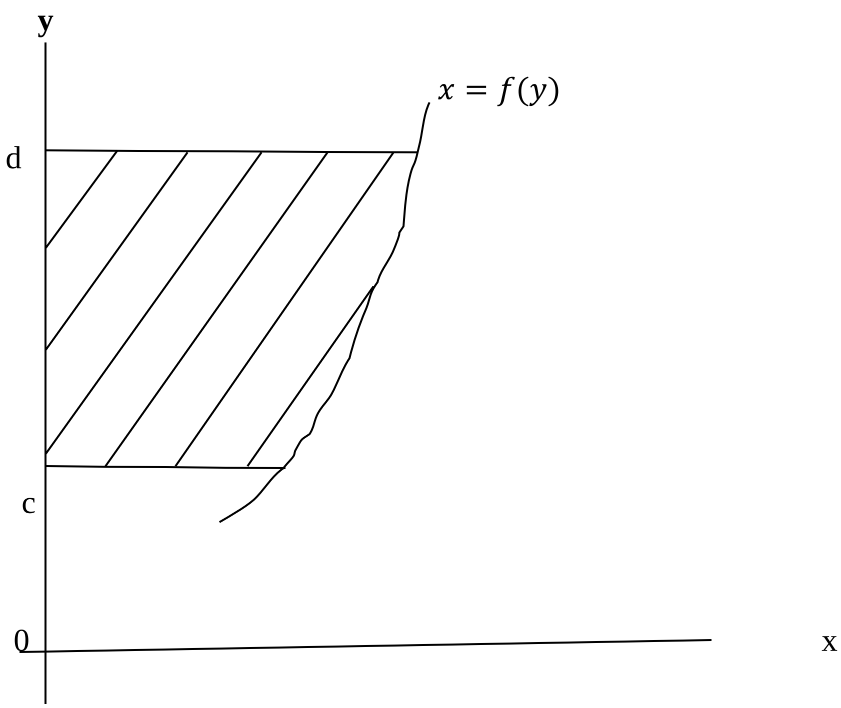
Revolving the area generated between the given limits, 360° about the x-axis produces a solid of revolution with volume given by:

Volume $= \int_1^4 \pi y^2\, dx = \int_1^4 \pi(x^2+4)^2\, dx = \int_1^4 \pi(x^4+8x^2+16)\, dx = \pi\left[\frac{x^5}{5}+\frac{8x^3}{3}+16x\right]_1^4$

$= \pi[(204.8+170.67+64)-(0.2+2.67+16)] = \mathbf{420.6\pi}$ **cubic units**

If a curve $x = f(y)$ is rotated 360° about the y-axis between the limits y = c and y = d then the volume generated,V, is given by:

$$\boldsymbol{V = \int_c^d \pi x^2\, dy}$$



Example

The curve $y = 2x^2$ is rotated 360° about the y – axis. Calculate the volume of the solid generated between $y = 0\ and\ y = 18$.

$$\boldsymbol{V = \int_c^d \pi x^2\, dy} = \int_0^{18} \pi\left(\frac{y}{2}\right) dy = \pi\left[\frac{y^2}{4}\right]_0^{18} = \pi\left[\frac{18^2}{4} - 0\right] = \boldsymbol{81\pi\ cubic\ units}$$

Multiple Integrals

The problems we can solve by integrating functions of two and three variables are similar to the problems solved by single – variable integration, but more general. We can perform the necessary calculations by drawing on our experience with functions of a single variable.

Double Integrals

We now show how to integrate a continuous function $f(x, y)$ over a bounded region in the xy – plane. There are many similarities between "double" integrals and "single" integrals for functions of a single variable. Every double integral can be evaluated in stages, using the single – integration methods already covered.

Fubini's Theorem

If $f(x, y)$ is continuous on the region $R: a \leq x \leq b, c \leq y \leq d$, then

$$\iint_R f(x,y)\, dA = \iint\limits_{ca}^{db} f(x,y)\, dxdy = \iint\limits_{ac}^{bd} f(x,y) dydx.$$

Fubini's theorem says that double integrals over rectangles can be calculated in turns. This means we can evaluate a double integral by integrating with respect to one variable at a time. Fubini's theorem also says that we may calculate the double integral by integrating in either order, a genuine convenience as seen in the second example.

Examples

1.Calculate $\iint_R f(x,y)\, dA$ for $f(x,y) = 1 - 6x^2y \; and \; R: 0 \leq x \leq 2, -1 \leq y \leq 1.$

By Fubini's theorem,

$$\iint_R f(x,y)\, dA = \int_{-1}^{1}\int_0^2 (1 - 6x^2y) dxdy = \int_{-1}^{1} [x - 2x^3y]_0^2 dy$$
$$= \int_{-1}^{1} (2 - 16y) dy = [2y - 8y^2]_{-1}^{1} = 4$$

Reversing the order of integration gives the same answer.

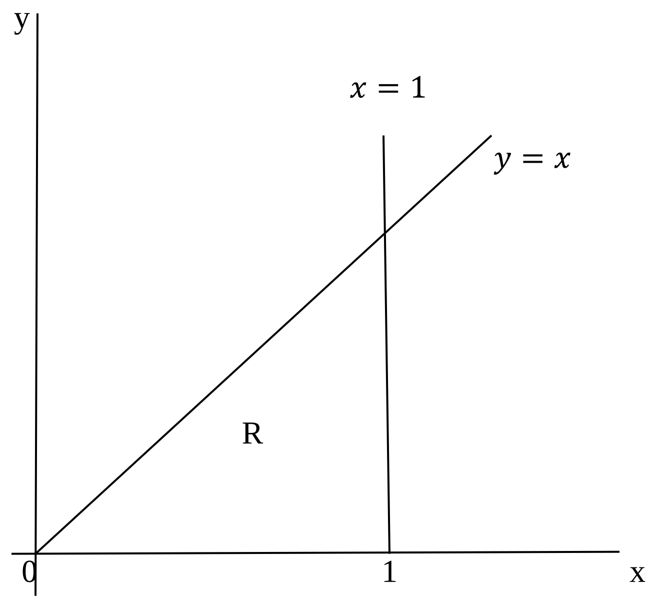
$$\int_0^2 \int_{-1}^{1} (1 - 6x^2y) dydx = \int_0^2 [y - 3x^2y^2]_{y=-1}^{y=1} dx = \int_0^2 [(1 - 3x^2) - (-1 - 3x^2)]\, dx$$
$$= \int_0^2 2\, dx = 4.$$

2.Calculate $\iint_R \frac{\sin x}{x} dA$, where R is the triangle in the xy – plane bounded by the x – axis, the line y = x, and the line x = 1.

If we integrate first with respect to y and then with respect to x, we find

$$\int_0^1 \left(\int_0^x \frac{sin\ x}{x} dy \right) dx = \int_0^1 (\left[y \frac{sin\ x}{x} \right] \begin{matrix} y = x \\ y = 0 \end{matrix}) \, dx = \int_0^1 sin\ x\, dx = -cos(1) + 1 \approx 0.46$$

If we reverse the order of integration and attempt to calculate $\int_0^1 \int_y^1 \frac{\sin x}{x} dxdy$, we are stopped by the fact that $\int (\frac{\sin x}{x})\, dx$ cannot be expressed in terms of elementary functions.



The region of integration in the above example

In examples like this, there is no general rule for predicting which order of integration will be right. If the one chosen does not work, turn to the other.