6 TECHNIQUES OF INTEGRATION

6.1 Powers of Trigonometric functions

Here we study the various methods of integrating some classes of trigonometric functions.

Case 1:

$\int sin^m x cos^n x dx$where either m or n is an odd integer. If m is odd, we substitute$sin^m x$with $sin^{m-1}x$, If n is odd, we substitute $cos^n x$with $cos^{n-1}x$. m and n can be positive or zero integers.

Example

1. $\int sin^6 x cos^3 x dx$

Solution

m=6 and n=3

Substituting $cos^3 x = cos^2 x cos x dx$

$$\int sin^6 x cos^3 x dx = \int sin^6 x \left(cos^2 x cos x\right) dx \tag{227}$$

$$= \int sin^6 x \left(1 - sin^2 x\right) cos x dx \tag{228}$$

$$= \int sin^6 x cos x dx - \int sin^8 x cos x dx \tag{229}$$

Let $v = sin x \Rightarrow dv = cos x dx$

$$= \int v^6 dv - \int v^8 dv \tag{230}$$

$$= \frac{v^7}{7} - \frac{v^9}{9} + c \tag{231}$$

$$= \frac{sin^7 x}{7} - \frac{sin^9 x}{9} + c \tag{232}$$

2. $\int cos^5 xdx$

m=0 and n=5

Substituting $cos^5 x = cos^4 xcosxdx$

$$\int cos^5 xdx = \int cos^4 xcosxdx \tag{233}$$

$$= \int \left(1 - sin^2 x\right)^2 cosxdx \tag{234}$$

$$= \int \left(1 - 2sin^2 x + sin^4 x\right) cosxdx \tag{235}$$

$$= \int cosxdx - 2\int sin^2 xcosxdx + \int sin^4 xcosxdx \tag{236}$$

Let $v = sinx \Rightarrow dv = cosxdx$

$$= \int vdv - 2\int v^2 dv + \int v^4 dv \tag{237}$$

$$= \frac{\cancel{v^2}}{\cancel{2}} - \frac{2v^3}{3} + \frac{v^5}{5} + c \tag{238}$$

$$= \frac{\cancel{sin^2 x}}{\cancel{2}} - \frac{2sin^3 x}{3} + \frac{sin^5 x}{5} + c \tag{239}$$

Case 2:

$\int sin^m xcos^n xdx$where both m and n are even integers. Here we use the identities:

a) $sin2x = 2sinxcosx$

b) $sin^2 x = \frac{1}{2}\left(1 - cos2x\right)$

c) $cos^2 x = \frac{1}{2}\left(1 + cos2x\right)$

b and c above are called the double angle formulas.

Example

$\int sin^6 xdx$

m=6 and n=0

$$\int sin^6 x dx = \int \left(sin^2 x\right)^3 dx \tag{240}$$

$$= \int \left(\frac{1-cos2x}{2}\right)^3 dx \tag{241}$$

$$= \frac{1}{8}\int \left(1 - 3cos(2x) + 3cos^2(2x) - cos^3(2x)\right) dx \tag{242}$$

$$= \frac{1}{8}\left[\int dx - 3\int cos2xdx + 3\int cos^2 2xdx - \int cos^3 2xdx\right] \tag{243}$$

$$= \frac{1}{8}\left[x - \frac{3}{2}sin2x + \frac{3}{2}\int (1+cos4x)\, dx - \int cos2x\left(1 - sin^2 2x\right) dx\right] \tag{244}$$

$$= \frac{1}{8}\left[x - \frac{3}{2}sin2x + \frac{3}{2}\left(x + \frac{1}{4}sin(4x)\right) dx - \frac{sin(2x)}{2} + \frac{sin^3(2x)}{6}\right] \tag{245}$$

$$= \frac{x}{8} - \frac{3}{16}sin2x + \frac{3}{16}x + \frac{3}{64}sin(4x) - \frac{1}{16}sin(2x) + \frac{1}{48}sin^3(2x) + c \tag{246}$$

Case 3:
$\int tan^n xdx$or $\int ctn^n xdx$or$\int sec^n xdx$
where n is an integer.
Here we use the following identities
$1 + tan^2 x = sec^2 x$
$ctn^2 x + 1 = csc^2 x$
Example

a) $\int tan^5xdx$
m=5

$$\int tan^5xdx = \int tan^3xtan^2xdx \tag{247}$$

$$= \int tan^3x\left(sec^2x - 1\right)dx \tag{248}$$

$$= \int tan^3xsec^2xdx - \int tan^3xdx \tag{249}$$

$$= \int tan^3xsec^2xdx - \int tanx\left(sec^2x - 1\right)dx \tag{250}$$

$$= \int tan^3xsec^2xdx - \int tanxsec^2xdx - \int tanxdx \tag{251}$$

Let $v = tanx \Rightarrow dv = sec^2xdx$

$$= \int v^3dv - \int vdv + \int tanxdx \tag{252}$$

$$= \frac{v^4}{4} - \frac{v^2}{2} + \int tanxdx \tag{253}$$

$$= \frac{tan^4x}{4} - \frac{tan^2x}{2} + \int \frac{sinx}{cosx}dx \tag{254}$$

$$= \frac{tan^4x}{4} - \frac{tan^2x}{2} + ln\left|cosx\right| + c \tag{255}$$

b) $\int cot^3 3xdx$
Solution

$$\int cot^3 3xdx = \int cot^2 3xcot(3x)dx \tag{256}$$

$$= \int \left(cosec^2 3x - 1\right) cot(3x)dx \tag{257}$$

$$= \int cosec^2 3x cot(3x)dx - \int cot(3x)dx \tag{258}$$

Let $u = cot(3x) \Longrightarrow \frac{du}{3} = cosec^2 3x$

$$= \frac{1}{3}\left[\int udu - \int cot(3x)dx\right] \tag{259}$$

$$= \frac{1}{3}\left[\frac{u^2}{2} - \int cot(3x)dx\right] \tag{260}$$

$$\frac{1}{3}\left[-\frac{cot^2(3x)}{2} - ln\left|sin(3x)\right|\right] + c \tag{261}$$

Case 4:

$\int tan^m x sec^n x dx$, where n is even. We use $1 + tan^2 x = sec^2 x$and break off a term of the form $sec^2 x$.

For $\int tan^m x sec^n x dx$,where m and n is odd, break off a term of the form $tanxsecx$.

For$\int tan^m x sec^n x dx$,where m is even and n is odd. Rewrite everything in terms of $secx$.

Example

a) $\int tan^3 x sec^5 x dx$

Solution

$$\int tan^3 x sec^5 x dx = \int tan^2 x sec^4 x \cdot secxtanxdx \tag{262}$$

$$= \int \left(sec^2 x - 1\right) sec^4 x secxtanxdx \tag{263}$$

$$= \int \left(sec^6 x - sec^4 x\right) secxtanxdx \tag{264}$$

Let $u = secx \Rightarrow du = secxtanxdx$

$$= \int \left(u^6 - u^4\right) du \tag{265}$$

$$= \frac{u^7}{7} - \frac{u^5}{5} + c \tag{266}$$

$$= \frac{sec^7 x}{7} - \frac{sec^5 x}{5} + c \tag{267}$$

b) $\int tan^3 xsec^4 xdx$

Solution

$$\int tan^3 xsec^4 xdx = \int tan^3 x \left(1 + tan^2 x\right) sec^2 xdx \tag{268}$$

$$= \int tan^3 xsec^2 xdx + \int tan^5 xsec^2 xdx \tag{269}$$

Let $u = tanx \Longrightarrow du = sec^2 xdx$

$$= \int u^3 du + \int u^5 du \tag{270}$$

$$= \frac{u^4}{4} + \frac{u^6}{6} + c \tag{271}$$

$$= \frac{tan^4 x}{4} + \frac{tan^6 x}{6} + c \tag{272}$$

Case 5:

To evaluate $\int cos(mx)cos(nx)dx$, $\int sin(mx)sin(nx)dx$or $\int sin(mx)cos(nx)dx$for $m \neq n$, we use the following identities:

$sin(mx)cos(nx) = \frac{1}{2}\left[sin(m+n)x + sin(m-n)x\right]$

$sin(mx)sin(nx) = \frac{1}{2}\left[cos(m-n)x - cos(m+n)x\right]$

$cos(mx)cos(nx) = \frac{1}{2}\left[cos(m+n)x + cos(m-n)x\right]$

Example:

a) $\int cos(4x)cos(3x)dx$
Solution

$$\int cos(4x)cos(3x)dx = \frac{1}{2}\int \left(cos(7x) + cos(x)\right)dx \tag{273}$$

$$= \frac{1}{2}\left[\int cos(7x)dx + \int cosxdx\right] \tag{274}$$

$$= \frac{1}{2}\left[\frac{sin7x}{7} + sinx\right] + c \tag{275}$$

$$= \frac{1}{14}sin(7x) + \frac{1}{2}sinx + c \tag{276}$$

b) $\int sin(5x)sinxdx$
Solution

$$\int sin(5x)sinxdx = \frac{1}{2}\int \left(cos(4x) - cos(6x)\right)dx \tag{277}$$

$$= \frac{1}{2}\left[\int cos(4x)dx - \int cos(6x)dx\right] \tag{278}$$

$$= \frac{1}{2}\left[\frac{sin(4x)}{4} - \frac{sin(6x)}{6}\right] + c \tag{279}$$

$$= \frac{1}{8}sin(4x) - \frac{1}{12}sin(6x) + c \tag{280}$$

6.2 Integration by substitution

In substitution we choose a parameter u, calculate du and substitute to change the initial form to an integration formula. It is uses to solve integrals of the following form:

(i) The numerator is a derivative of the denominator i.e. $\int \frac{f'(x)}{f(x)} = ln\left|f(x)\right|$

(ii) The numerator is a constant multiple of the derivative of the denominator e.g. $\int \frac{4x}{x^2+1}$

(iii) The numerator is a function of the derivative of the denominator.

Example

a) $\int \frac{x}{\sqrt{x^2+1}} dx$

Solution

Let $u = x^2 + 1 \Rightarrow du = 2x \Rightarrow \frac{du}{2} = xdx$

$$\int \frac{x}{\sqrt{x^2+1}} dx = \frac{1}{2} \int \frac{1}{u^{\frac{1}{2}}} du = \frac{1}{2} \int u^{-\frac{1}{2}} du = \frac{1}{2} \left[\frac{u^{\frac{1}{2}}}{\frac{1}{2}} \right] = u^{\frac{1}{2}} + c = \sqrt{x^2+1} + c \tag{281}$$

b) $\int \frac{1+x^2}{\sqrt[4]{3x+x^3}} dx$

Solution

Let $u = 3x + x^3 \Rightarrow du = (3 + 3x^2)\, dx \Rightarrow \frac{du}{3} = 1 + x^2$

$$\int \frac{1+x^2}{\sqrt[4]{3x+x^3}} dx = \int \frac{1+x^2}{u^{\frac{1}{4}}} \cdot \frac{du}{3(1+x^2)} = \frac{1}{3} \int u^{-\frac{1}{4}} du = \frac{4}{9} u^{\frac{3}{4}} + c = \frac{4}{9} \left(3x + x^3\right)^{\frac{3}{4}} + c \tag{282}$$

c) $\int tanx dx$

Solution

$$\int tanx dx = \int \frac{sinx}{cosx} dx \tag{283}$$

Let $u = cosx \Rightarrow du = -sinxdx$

$$= \int (-1) \cdot \frac{-1}{u} du = -ln\,|u| + c = -ln\,|cosx| + c \tag{284}$$

d) $\int \frac{sec^2 x}{1 - tanx} dx$

Solution

$$\int \frac{sec^2x}{1-tanx}dx = -\int \frac{-sec^2x}{1-tanx}dx = -ln\left|1 = tanx\right| + c \tag{285}$$

e) $\int x\sqrt{x+1}dx$

Solution

Let $u = x+1 \Rightarrow du = dx, x = u-1$

$$\int x\sqrt{x+1}dx = \int (u-1)\sqrt{u}du = \int \left(u^{\frac{3}{2}} - u^{\frac{1}{2}}\right)du = \frac{2}{5}u^{\frac{5}{2}} - \frac{2}{3}u^{\frac{3}{2}} + c = \frac{2}{5}(x+1)^{\frac{5}{2}} - \frac{2}{3}(x+1)^{\frac{3}{2}} + c \tag{286}$$

f) $\int \frac{dt}{1+e^t}$

Solution

$$\int \frac{dt}{1+e^t} = \int \frac{e^{-t}}{e^{-t}}\left(\frac{dt}{1+e^t}\right) = \int \frac{e^{-t}}{e^{-t}+1}dt \tag{287}$$

Let $u = e^{-t}+1 \Rightarrow du = -e^{-t}dt$

$$\int \frac{e^{-t}}{e^{-t}+1}dt = -\int \frac{e^{-t}}{u}\cdot\frac{1}{e^{-t}}du = -\int \frac{du}{u} = -ln\left|u\right| + c = -ln\left|e^{-t}+1\right| + c \tag{288}$$

g) $\int \frac{dx}{x^{\frac{1}{2}}+x^{\frac{1}{4}}}$

Choose the smallest integer divisible by 2 and 4.

Solution

Let $u = x^{\frac{1}{4}}, u^4 = x \Rightarrow 4u^3du = dx$

$$\int \frac{dx}{x^{\frac{1}{2}}+x^{\frac{1}{4}}} = \int \frac{4u^3du}{(u^4)^{\frac{1}{2}}+(u^4)^{\frac{1}{4}}} = \int \frac{4u^3}{u^2+u}du = \int \frac{4u^2}{u+1}du \tag{289}$$

By long division

$$\frac{4u^2}{u+1} = 4u - 4 + \frac{4}{u+1}$$

$$\int \frac{4u^2}{u+1}du = \int 4udu - \int 4du + \int \frac{4}{u+1}du \tag{290}$$

$$= 4\left[\frac{u^2}{2} - u + ln\left|u+1\right|\right] + c \tag{291}$$

$$= 4\left[\frac{x^{\frac{1}{2}}}{2} - x^{\frac{1}{4}} + ln\left|x^{\frac{1}{4}} + 1\right|\right] + c \tag{292}$$

Exercise

$$\int \frac{dx}{x^{\frac{1}{3}} + x^{\frac{1}{2}}} = 2x^{\frac{1}{2}} - 3x^{\frac{1}{3}} + 6x^{\frac{1}{6}} - 6ln\left|1 + x^{\frac{1}{6}}\right| + c$$

$$\int \frac{4dx}{x^{\frac{1}{3}} + 2x^{\frac{1}{2}}} = 4x^{\frac{1}{2}} - 3x^{\frac{1}{3}} + 3x^{\frac{1}{6}} - \frac{3}{2}ln\left|2x^{\frac{1}{6}} + 1\right| + c$$

Trigonometric substitution

Integrals containing terms of the form $\sqrt{a^2+x^2}$, $\sqrt{a^2-x^2}$and $\sqrt{x^2-a^2}$an be evaluated by use of trigonometric substitution. If the term is of the form:

(i) $\sqrt{a^2-x^2}$, then set $x = asin\theta$

(ii) $\sqrt{a^2+x^2}$,then set $x = atan\theta$

(iii)$\sqrt{x^2-a^2}$, then set $x = asec\theta$

Example

a) $\int \frac{dx}{\sqrt{a^2-x^2}}$

Let $x = asin\theta \Rightarrow x^2 = a^2sin^2\theta, dx = acos\theta d\theta$

$$\int \frac{dx}{\sqrt{a^2-x^2}} = \int \frac{acos\theta d\theta}{\sqrt{a^2-a^2sin^2\theta}} = \int \frac{cos\theta d\theta}{\sqrt{1-sin^2\theta}} = \int \frac{cos\theta}{cos\theta}d\theta = \int d\theta = \theta + c \tag{293}$$

But $\frac{x}{a} = sin\theta \Rightarrow \theta = sin^{-1}\left(\frac{x}{a}\right)$

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \theta + c = sin^{-1}\left(\frac{x}{a}\right) + c \tag{294}$$

b) $\int \sqrt{4 - t^2}dt$

Solution

Let $t = 2sin\theta \Rightarrow t^2 = 2^2 sin^2\theta, dt = 2cos\theta d\theta$

$$\int \sqrt{4 - t^2}dt = \int \sqrt{4 - 4sin^2\theta} \cdot 2cos\theta d\theta = \int 2cos\theta \cdot 2cos\theta d\theta = 4\int cos^2\theta d\theta \tag{295}$$

$$= 4\int \frac{1 + cos2\theta}{2} d\theta = 2\int (1 + cos2\theta)\, d\theta = 2\theta + sin2\theta + c \tag{296}$$

but $\frac{t}{2} = sin\theta \Rightarrow \theta = sin^{-1}\left(\frac{t}{2}\right)$

$sin2\theta = 2sin\theta cos\theta = tcos\theta = t\sqrt{1 - sin^2\theta} = t\sqrt{1 - \frac{t^2}{4}} = \frac{t}{2}\sqrt{4 - t^2}$

$$\int \sqrt{4 - t^2}dt = 2sin^{-1}\left(\frac{t}{2}\right) + \frac{t}{2}\sqrt{4 - t^2} + c \tag{297}$$

c) $\int \frac{dx}{\sqrt{9 - x^2}}$

Let $x = 3sin\theta \Rightarrow x^2 = 9sin^2\theta, dx = a3cos\theta d\theta$

$$\int \frac{dx}{\sqrt{9 - x^2}} = \int \frac{3cos\theta d\theta}{\sqrt{3^2 - 3^2 sin^2\theta}} = \int \frac{3cos\theta d\theta}{3\sqrt{1 - sin^2\theta}} = \int \frac{cos\theta}{cos\theta} d\theta = \int d\theta = \theta + c \tag{298}$$

But $\frac{x}{3} = sin\theta \Rightarrow \theta = sin^{-1}\left(\frac{x}{3}\right)$

$$\int \frac{dx}{\sqrt{9 - x^2}} = \theta + c = sin^{-1}\left(\frac{x}{3}\right) + c \tag{299}$$

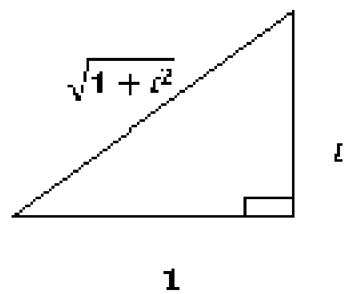
d) $\int \frac{dt}{\sqrt{1 + t^2}}$

Solution

Let $t = tan\theta \Rightarrow t^2 = tan^2\theta, dt = sec^2\theta d\theta$

$$\int \frac{dt}{\sqrt{1+t^2}} = \int \frac{sec^2\theta d\theta}{\sqrt{1+tan^2\theta}} \tag{300}$$

$$= \int \frac{sec^2\theta d\theta}{sec\theta} = \int sec\theta d\theta = \int \frac{sec\theta\left(sec\theta + tan\theta\right)d\theta}{sec\theta + tan\theta} = \int \frac{sec^2\theta + sec\theta tan\theta d\theta}{sec\theta + tan\theta} = ln\left|sec\theta + tan\theta\right| + \tag{301}$$



but $tan\theta = t \Rightarrow cos\theta = \frac{1}{\sqrt{1+t^2}} \Rightarrow sec\theta = \sqrt{1+t^2}$

$$ln\left|sec\theta + tan\theta\right| + c = ln\left|\sqrt{1+t^2} + t\right| + c \tag{302}$$

e) $\int \frac{dx}{a^2+b^2x^2}$

Solution

Let $bx = atan\theta \Rightarrow b^2x^2 = a^2tan^2\theta \Rightarrow dx = \frac{a}{b}sec^2\theta d\theta$

$$\frac{a}{b}\int \frac{sec^2\theta}{a^2\left(1+tan^2\theta\right)} = \frac{1}{ba}\int d\theta = \frac{\theta}{ba} + c \tag{303}$$

but $bx = atan\theta \Rightarrow \frac{bx}{a} = tan\theta \Rightarrow \theta = tan^{-1}\left(\frac{bx}{a}\right)$

$$\frac{\theta}{ba} + c = \frac{1}{ba} tan^{-1}\left(\frac{bx}{a}\right) \tag{304}$$

f) $\int \frac{dx}{\sqrt{x^2-4}}$

Solution

Let $x = 2sec\theta \Rightarrow x^2 = 4sec^2\theta, dx = 2sec\theta tan\theta$

$$\int \frac{dx}{\sqrt{x^2-4}} = \int \frac{2sec\theta tan\theta d\theta}{\sqrt{4sec^2\theta - 4}} = \int \frac{sec\theta tan\theta}{tan\theta} d\theta = \int sec\theta d\theta = ln\left|sec\theta + tan\theta\right| + c \tag{305}$$

But $sec\theta = \frac{x}{2}$

From the trigonometric triangle, $tan\theta = \frac{\sqrt{x^2-4}}{2}$

$$ln\left|sec\theta + tan\theta\right| + c = ln\left|\frac{x}{2} + \frac{\sqrt{x^2-4}}{2}\right| + c \tag{306}$$

g) $\int \frac{dx}{\sqrt{4x^2-1}}$

Solution

Let $2x = sec\theta \Rightarrow 4x^2 = sec^2\theta, 2dx = sec\theta tan\theta d\theta$

$$\int \frac{dx}{\sqrt{4x^2-1}} = \int \frac{\frac{1}{2}sec\theta tan\theta}{\sqrt{sec^2\theta - 1}} d\theta \tag{307}$$

$$= \frac{1}{2}\int sec\theta d\theta = \frac{1}{2} ln\left|sec\theta + tan\theta\right| + c \tag{308}$$

But $sec\theta = 2x$

From the trigonometric triangle,

$$\frac{1}{2} ln\left|sec\theta + tan\theta\right| + c = \frac{1}{2} ln\left|2x + \sqrt{4x^2-1}\right| + c \tag{309}$$

Exercise

Evaluate the following integrals

a) $\int \frac{2}{4+x^2} dx$

b) $\int \frac{3}{1+4x^2}dx$
c) $\int \frac{4}{\sqrt{9-x^2}}dx$
d)$\int \frac{1}{\sqrt{1-9x^2}}dx$
e) $\int \frac{1}{2+25x^2}dx$
f) $\int \frac{2}{\sqrt{3-4x^2}}dx$
g) $\int \frac{1}{3-2x+x^2}dx$
i) $\int \frac{5}{\sqrt{9-(x+2)^2}}dx$

6.3 Integration by Parts

If u and v are two functions of x, then

$$\frac{d}{dx}(uv) = v\frac{du}{dx} + u\frac{dv}{dx}$$

Integrating each side with respect to x

$$uv = \int v\frac{du}{dx}dx + \int u\frac{dv}{dx}dx$$

$$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$$

Example
a) Find$\int xcosxdx$
Solution

$$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$$

Let $u = x \Rightarrow du = dx$
Let $dv = cosx \Rightarrow v = \int cosx = sinx$

$$\int xcosxdx = xsinx - \int sinxdx \tag{310}$$

$$= xsinx + cosx + c \tag{311}$$

b) Determine $\int xsinxdx$

Solution

$$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$$

Let $u = x \Rightarrow du = dx$

Let $dv = sinx \Rightarrow v = \int sinx = -cosx$

$$\int xsinxdx = -xcosx + \int cosxdx \tag{312}$$

$$= -xcosx + sinx + c \tag{313}$$

c) $\int tan^{-1}xdx$

Solution

$$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$$

Let $u = tan^{-1}(x) \Rightarrow tan(u) = x, sec^2udu = dx \Rightarrow \frac{du}{dx} = \frac{1}{sec^2u} = \frac{1}{1+tan^2u} = \frac{1}{1+x^2}$

Let $dv = 1 \Rightarrow v = \int dv = x$

$$\int tan^{-1}xdx = xtan^{-1}x - \int x \cdot \frac{1}{1+x^2}dx \tag{314}$$

$$= xtan^{-1}x - \frac{1}{2}ln\left|1+x^2\right| + c \tag{315}$$

d) $\int x^2sinxdx$

Solution

$$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$$

Let $u = x^2 \Rightarrow du = 2xdx$and $dv = sinx \Rightarrow v = \int sinxdx = -cosx$

$$\int x^2 sinxdx = -x^2 cosx + 2\int xcosxdx \tag{316}$$

Let $u = x \Rightarrow du = 2dx$and $dv = cosx \Rightarrow v = \int cosxdx = sinx$

$$\int xcosxdx = xsinx - \int sinx = xsinx + cosx + c \tag{317}$$

$$\int x^2 sinxdx = -x^2 cosx + 2\left[xsinx + cosx\right] + c = -x^2 cosx + 2xsinx + 2cosx + c \tag{318}$$

Exercise

Evaluate the following integrals

a) $\int x^3 cos(2x)dx$

b) $\int xsinxcosxdx$

c) $\int xsin^{-1}xdx$

d) $\int utan^{-1}udu$

e) $\int tsin^2 tdt$

6.4 The Change of variable $t = tan\left(\frac{x}{2}\right)$

The change in variable $t = tan\left(\frac{x}{2}\right)$is applied to the trigonometric identity $tan2A = \frac{2tanA}{1-tan^2 A}$.

Let $A = \frac{x}{2} \Rightarrow 2A = x$

It is also possible to express sinx and cos x using the trigonometric triangle in terms of t to obtain

$sinx = \frac{2t}{1+t^2}$and $cosx = \frac{1-t^2}{1+t^2}$

These identities are also found in page two of the SMP tables.

When we make the change of variable $t = tan\left(\frac{x}{2}\right)$

$$\frac{dt}{dx}=\frac{1}{2}sec^2\frac{x}{2}\Rightarrow\frac{dx}{dt}=\frac{2}{sec^2\frac{x}{2}}=\frac{2}{1+tan^2\frac{x}{2}}=\frac{2}{1+t^2}$$

Example

a) $\int\frac{1}{1+cosx}dx$

Solution

$cosx=\frac{1-t^2}{1+t^2}$

$$\frac{dt}{dx}=\frac{1}{2}sec^2\frac{x}{2}\Rightarrow\frac{dx}{dt}=\frac{2}{sec^2\frac{x}{2}}=\frac{2}{1+tan^2\frac{x}{2}}=\frac{2}{1+t^2}$$

$$\int\frac{1}{1+cosx}dx=\int\frac{1}{1+\frac{1-t^2}{1+t^2}}\cdot\frac{2}{1+t^2}dt=\int\frac{1}{\frac{2}{1+t^2}}\cdot\frac{2}{1+t^2}dt=\int dt=t+c \tag{319}$$

But $t=tan\left(\frac{x}{2}\right)$

$$=tan\left(\frac{x}{2}\right)+c \tag{320}$$

Exercise

a)$\int\frac{1}{1+sinx}dx$

b) $\int\frac{1}{5+3cosx}dx$

c) $\int\frac{1}{3sinx+4cosx}dx$

d)$\int\frac{1}{1-sinx}dx$

6.5 The change of variable $t=tanx$

An integrand of the form sinx and cosx, particularly even powers of these my often be expressed as function of tanx and secx . In such a case the change of variable $t=tanx$is used.

Example

a) $\int\frac{1}{1+sin^2x}dx$

Solution

Dividing the numerator and denominator by cos^2x, we have

$$\int \frac{1}{1+sin^2x}dx = \int \frac{sec^2x}{sec^2x+tan^2x}dx \tag{321}$$

$$= \int \frac{sec^2x}{1+tan^2x+tan^2x}dx = \int \frac{sec^2x}{1+2tan^2x}dx = \int \frac{1+t^2}{1+2t^2}\cdot\frac{1}{1+t^2}dt \tag{322}$$

$$= \int \frac{1}{1+2t^2}dt \tag{323}$$

Let $\sqrt{2}t = tan\theta \Rightarrow \sqrt{2}dt = sec^2\theta d\theta, 2t^2 = tan^2\theta$

$$\int \frac{1}{1+2t^2}dt = \int \frac{sec^2\theta d\theta}{1+tan^2\theta} = \int \frac{sec^2\theta d\theta}{sec^2\theta} = \int d\theta = \theta + c \tag{324}$$

But $\sqrt{2}t = tan\theta \Rightarrow \theta = \frac{1}{\sqrt{2}}tan^{-1}t$

$$= \frac{1}{\sqrt{2}}tan^{-1}\sqrt{2}t + c \tag{325}$$

But $t = tanx$

$$= \frac{1}{\sqrt{2}}tan^{-1}\left(\sqrt{2}tanx\right) + c \tag{326}$$

Exercise
a) $\int \frac{1}{1+cos^2x}dx$
b) $\int \frac{2tanx}{cos2x}dx$
c) $\int \frac{1}{1+2sin^2x}dx$

6.6 Splitting the numerator

When a fractional integrand with a quadratic denominator cannot be written in simple partial fractions, it may be expresses as two fractions by splitting the numerator.

$$\int \frac{1+x}{1+x^2}dx = \int \left(\frac{1}{1+x^2} + \frac{x}{1+x^2}\right) dx$$

$$= tan^{-1}x + ln\left|\sqrt{1+x^2}\right| + c$$

The key to a more general application of this method is to express the numerator in two parts, one which is a multiple of the derivative of the denominator.

Example

a) Find $\int \frac{5x+7}{x^2+4x+8}dx$

Solution

Since$\frac{d}{dx}(x^2+4x+8) = 2x+4$

$$5x+7 = A(2x+4) + B$$

Therefore$A = \frac{5}{2}$and $B = -3$

$$\int \frac{5x+7}{x^2+4x+8}dx = \int \left[\frac{\frac{5}{2}(2x+4)}{x^2+4x+8} - \frac{3}{x^2+4x+8}\right] dx \tag{327}$$

$$= \frac{5}{2}ln\left|x^2+4x+8\right| - 3\int \frac{1}{(x+2)^2+4}dx \tag{328}$$

Let $x+2 = 2tan\theta \Rightarrow dx = 2sec^2\theta d\theta, (x+2)^2 = 4tan^2\theta$

$$\int \frac{1}{(x+2)^2+4}dx = \int \frac{2sec^2\theta d\theta}{4tan^2\theta+4} = \int \frac{2sec^2\theta d\theta}{4sec^2\theta d} = \frac{1}{2}\int d\theta = \frac{1}{2}\theta + c$$

But $x+2 = 2tan\theta \Rightarrow \theta = tan^{-1}\left(\frac{x+2}{2}\right)$

$$= \frac{1}{2}tan^{-1}\left(\frac{x+2}{2}\right) + c$$

$$= \frac{5}{2} ln \left| x^2 + 4x + 8 \right| - \frac{3}{2} tan^{-1} \left(\frac{x+2}{2} \right) + c \tag{329}$$

b) $\int \frac{2cosx+3sinx}{cosx+sinx} dx$

Solution

Let $2cosx + 3sinx = A(-sinx + cosx) + B(cosx + sinx)$

Therefore $A = -\frac{1}{2}$and $B = \frac{5}{2}$

$$\int \frac{2cosx + 3sinx}{cosx + sinx} dx = \int \frac{-\frac{1}{2}(-sinx + cosx)}{cosx + sinx} dx + \int \frac{\frac{5}{2}(cosx + sinx)}{cosx + sinx} dx \tag{330}$$

$$= -\frac{1}{2} ln \left| cosx + sinx \right| + \frac{5}{2} x + c \tag{331}$$

Exercise

a) $\int \frac{2x+3}{x^2+2x+10}$

b) $\int \frac{1-2x}{\sqrt{9-(x+2)^2}}$

c) $\int \frac{sinx}{cosx+sinx} dx$

d) $\int \frac{2cosx+9cosx}{3cosx+sinx} dx$

6.7 Further Integration by parts

Example

a) $\int e^{ax} cosbxdx$

Let $I = \int e^{ax} cosbxdx$

$$\frac{d}{dx}(uv) = v\frac{du}{dx} + u\frac{dv}{dx}$$

Let $u = cosbx \Rightarrow du = -bsinbx$and $dv = e^{ax} \Rightarrow v = \int dv = \frac{1}{a} e^{ax}$

$$I = \frac{1}{a} e^{ax} cosbx + \frac{b}{a} \int e^{ax} sinbxdx \tag{332}$$

Let $u = sinbx \Rightarrow du = bcosbx$and $dv = e^{ax} \Rightarrow v = \int dv = \frac{1}{a}e^{ax}$

$$\int e^{ax} sinbxdx = \frac{1}{a}e^{ax}sinbx - \int \frac{1}{a}e^{ax}bcosbx \tag{333}$$

$$\int e^{ax} sinbxdx = \frac{1}{a}e^{ax}sinbx - \frac{b}{a}\int e^{ax}cosbx \tag{334}$$

$$\int e^{ax} sinbxdx = \frac{1}{a}e^{ax}sinbx - \frac{b}{a}I \tag{335}$$

Replacing back,

$$I = \frac{1}{a}e^{ax}cosbx + \frac{b}{a}\left[\frac{1}{a}e^{ax}sinbx - \frac{b}{a}I\right] \tag{336}$$

$$I = \frac{1}{a}e^{ax}cosbx + \frac{b}{a^2}e^{ax}sinbx - \frac{b^2}{a^2}I \tag{337}$$

Eliminating the denominator by multiplying both sides by a^2,

$$a^2 I = ae^{ax}cosbx + be^{ax}sinbx - b^2 I \tag{338}$$

$$I\left(a^2 + b^2\right) = e^{ax}\left(acosbx + bsinbx\right) + c \tag{339}$$

$$I = \int e^{ax}cosbxdx = \frac{e^{ax}}{a^2+b^2}\left(acosbx + bsinbx\right) + c \tag{340}$$

Exercise
a) $\int e^{3x}cos(2x)dx$
b)$\int e^{4x}sin(3x)dx$
c)$\int e^{-t}cos\left(\frac{t}{2}\right)dx$
d) $\int e^{x}sin(2x+1)dx$
e) $\int e^{2\theta}cos^2\theta d\theta$

6.8 Integration by Partial fractions

Case 1: The denominator contains linear non-repeated factors only.

a) $\int \frac{3x}{(x-1)(x-2)(x-3)}dx$

Solution

$$\frac{3x}{(x-1)(x-2)(x-3)} = \frac{A}{(x-1)} + \frac{B}{(x-2)} + \frac{C}{(x-3)}$$

$$3x = A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2)$$

Solving for A, B and C simultaneously

$A = \frac{3}{2}, B = -6, C = \frac{9}{2}$

$$\int \frac{3x}{(x-1)(x-2)(x-3)}dx = \int \frac{\frac{3}{2}}{(x-1)} + \frac{-6}{(x-2)} + \frac{\frac{9}{2}}{(x-3)}dx \qquad (341)$$

$$= \frac{3}{2}\int \frac{dx}{(x-1)} - 6\int \frac{dx}{(x-2)} + \frac{9}{2}\int \frac{dx}{(x-3)} \qquad (342)$$

$$= \frac{3}{2}ln\,|x-1| - 6ln\,|x-2| + \frac{9}{2}ln\,|x-3| + c \qquad (343)$$

b) $\int \frac{x+4}{x(x-2)(x+5)}dx$

Solution

$$\frac{x+4}{x(x-2)(x+5)} = \frac{A}{x} + \frac{B}{(x-2)} + \frac{C}{(x+5)}$$

$$3x = A(x-2)(x+5) + B(x)(x+5) + C(x)(x-2)$$

Solving for A, B and C simultaneously

$A = -\frac{2}{5}, B = \frac{3}{7}, C = \frac{-1}{35}$

$$\int \frac{x+4}{x(x-2)(x+5)}dx = \int \frac{-\frac{2}{5}}{x} + \frac{\frac{3}{7}}{(x-2)} + \frac{-\frac{1}{35}}{(x+5)}dx \tag{344}$$

$$= \int \frac{-\frac{2}{5}}{x}dx + \int \frac{\frac{3}{7}}{(x-2)}dx + \int \frac{-\frac{1}{35}}{(x+5)}dx \tag{345}$$

$$= -\frac{2}{5}ln\left|x\right| + \frac{3}{7}ln\left|x-2\right| - \frac{1}{35}ln\left|x+5\right| + c \tag{346}$$

Case 2: The degree of the numerator is equal or higher to that of the denominator.

a) $\int \frac{x^3}{(x-1)(x-2)(x-3)}dx$

Solution

By long division

$$\int \frac{x^3}{(x-1)(x-2)(x-3)}dx = \int 1 + \frac{6x^2-11x+6}{(x-1)(x-2)(x-3)}dx$$

$$\frac{6x^2-11x+6}{(x-1)(x-2)(x-3)} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3}$$

$$6x^2-11x+6 = A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2)$$

Solving for A, B and C simultaneously

$A = \frac{1}{2}, B = -8, C = \frac{27}{2}$

$$\int \frac{x^3}{(x-1)(x-2)(x-3)}dx = \int \left\{1 + \frac{\frac{1}{2}}{x-1} - \frac{8}{x-2} + \frac{\frac{27}{3}}{x-3}\right\} dx \tag{347}$$

$$= x + \frac{1}{2}ln\left|x-1\right| - 8ln\left|x-2\right| + \frac{27}{2}ln\left|x-3\right| + c \tag{348}$$

Case 3: The denominator contains repeated roots

a) $\int \frac{3x-5}{(x+1)^2(3x-2)}dx$

Solution

$$\frac{3x-5}{(x+1)^2(3x-2)} = \frac{A}{(x+1)} + \frac{B}{(x+1)^2} + \frac{C}{(3x-2)}$$

$$3x-5 = A(x+1)(3x-2) + B(3x-2) + C(x+1)^2$$

Solving for A, B and C simultaneously

$A = \frac{9}{25}, B = \frac{8}{5}, C = \frac{-27}{25}$

$$\int \frac{3x-5}{(x+1)^2(3x-2)} = \int \frac{\frac{9}{25}}{(x+1)} + \frac{\frac{8}{5}}{(x+1)^2} + \frac{\frac{-27}{25}}{(3x-2)} \tag{349}$$

$$= \frac{9}{25}\int \frac{1}{(x+1)}dx + \frac{8}{5}\int \frac{1}{(x+1)^2}dx - \frac{27}{25}\int \frac{1}{(3x-2)}dx \tag{350}$$

$$= \frac{9}{25}ln\,|x+1| + \frac{8}{5}\cdot\frac{1}{x+1} - \frac{27}{25}\cdot\frac{1}{3}ln\,|3x-2| + c \tag{351}$$

$$= \frac{9}{25}ln\,|x+1| + \frac{8}{5}\cdot\frac{1}{x+1} - \frac{9}{25}ln\,|3x-2| + c \tag{352}$$

Case 4: The denominator containing quadratic functions

a) $\int \frac{x}{(x-1)(x^2+4)}dx$

Solution

$$\frac{x}{(x-1)(x^2+4)} = \frac{A}{(x-1)} + \frac{Bx+C}{(x^2+4)}$$

$$x = A(x^2+4) + (Bx+C)(x-1)$$

Solving for A and B simultaneously
$A = \frac{1}{5}, B = -\frac{1}{5}, C = \frac{4}{5}$

$$\int \frac{x}{(x-1)(x^2+4)} = \int \frac{\frac{1}{5}}{(x-1)} + \frac{-\frac{1}{5}x + \frac{4}{5}}{(x^2+4)} dx \tag{353}$$

$$\int \frac{x}{(x-1)(x^2+4)} = \frac{1}{5} \int \frac{1}{(x-1)} dx - \frac{1}{5} \int \frac{x-4}{(x^2+4)} dx \tag{354}$$

$$= \frac{1}{5} \int \frac{1}{(x-1)} dx - \frac{1}{10} \int \frac{2x}{(x^2+4)} dx + \frac{4}{5} \int \frac{1}{(x^2+4)} dx \tag{355}$$

$$= \frac{1}{5} ln\left|x-1\right| - \frac{1}{10} ln\left|x^2+4\right| + \frac{4}{5} \cdot \frac{1}{2} tan^{-1}\left(\frac{x}{2}\right) + c \tag{356}$$

$$= \frac{1}{5} ln\left|x-1\right| - \frac{1}{10} ln\left|x^2+4\right| + \frac{2}{5} tan^{-1}\left(\frac{x}{2}\right) + c \tag{357}$$

b) $\int \frac{x^3+1}{x^3-1} dx$
Solution

$$\int \frac{x^3+1}{x^3-1} dx = \int 1 + \frac{2}{(x-1)(x^2+x+1)} dx$$

$$\frac{2}{(x-1)(x^2+x+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+x+1}$$

$$2 = A(x^2+x+1) + (Bx+C)(x-1)$$

Solving for A, B and C simultaneously
$A = \frac{2}{3}, B = -\frac{2}{3}, C = -\frac{4}{3}$

$$\int \frac{x^3+1}{x^3-1} dx = \int \left\{ 1 + \frac{\frac{2}{3}}{x-1} + \frac{-\frac{2}{3}x - \frac{4}{3}}{x^2+x+1} \right\} dx \tag{358}$$

$$= x + \frac{2}{3}ln\,|x-1| - \frac{1}{3}\int \frac{2x+4}{x^2+x+1}dx \qquad (359)$$

$$= x + \frac{2}{3}ln\,|x-1| - \frac{1}{3}\int \frac{2x+1}{x^2+x+1}dx - \frac{3}{3}\int \frac{1}{x^2+x+1}dx \qquad (360)$$

$$= x + \frac{2}{3}ln\,|x-1| - \frac{1}{3}ln\left|x^2+x+1\right| - \int \frac{dx}{\left(x+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} \qquad (361)$$

$$= x + \frac{2}{3}ln\,|x-1| - \frac{1}{3}ln\left|x^2+x+1\right| - \frac{2}{\sqrt{3}}tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + c \qquad (362)$$

Exercise

Show that

a) $\int \frac{2x-3}{(x+6)(x-5)}dx = \frac{9}{11}ln(x+6) + \frac{13}{11}ln(x-5) + c$

b) $\int \frac{x^2}{x^2+7x+10}dx = \frac{4}{3}ln(x+2) - \frac{25}{3}ln(x+5) + c$

c) $\int \frac{x^2-3x-1}{x^3+x^2-2x}dx = ln\left|\frac{x^{\frac{1}{2}}(x+2)^{\frac{3}{2}}}{(x-1)}\right| + c$

d) $\int \frac{x^4}{(1-x)^3}dx = -\frac{1}{2}x^2 - 3x - ln(1-x)^6 - \frac{4}{1-x} + \frac{1}{2(1-x^2)} + c$

e) $\int \frac{2x^3}{(x^2+1)^2} = ln(x^2+1) + \frac{1}{x^2+1} + c$

f) $\int \frac{x^2+3x-4}{x^2-2x-8}dx = ln\,|x^2+4| - \frac{1}{2}arctan\left(\frac{1}{2}x\right) + \frac{4}{x^2+4} + c$

7 Solutions of First order Ordinary Differential equations by separation of variables

A differential equation is said to be separable if the variables can be separated. That is, a separable equation is one that can be written in the form

$$F(y)dy = G(x)dx$$

Once this is done, all that is needed is to solve the equation through integration of both sides. The method of solving separable equations can therefore be summarized as follows:

Step 1: separate the variables

Step 2: Integrate

Example

a) Solve the equation $2ydy = (x^2 + 1)\,dx$

Solution

$$\int 2ydy = \int \left(x^2 + 1\right) dx \tag{363}$$

$$\frac{2y^2}{2} = \frac{x^3}{3} + x + x \tag{364}$$

$$y^2 = \frac{x^3}{3} + x + x \tag{365}$$

b) Solve the equation $\frac{dy}{dx} = (1 + e^{-x})\,(y^2 - 1)$

Solution

$$\frac{dy}{y^2 - 1} = \left(1 + e^{-x}\right) dx \tag{366}$$

$$\int \frac{dy}{y^2 - 1} = \int \left(1 + e^{-x}\right) dx \tag{367}$$

$$\int \frac{dy}{y^2-1} = \int \frac{-\frac{1}{2}}{y+1} + \frac{\frac{1}{2}}{y-1}dy = -\frac{1}{2}ln\left|y+1\right| + \frac{1}{2}ln\left|y-1\right| + c$$

$$\int \left(1+e^{-x}\right)dx = x - e^{-x} + c$$

$$-\frac{1}{2}ln\left|y+1\right| + \frac{1}{2}ln\left|y-1\right| = x - e^{-x} + c \qquad (368)$$

c) $xydx - (x^2+1)\,dy = 0$
Solution

$$\frac{x}{x^2+1}dx = \frac{dy}{y} \qquad (369)$$

$$\int \frac{x}{x^2+1}dx = \int \frac{dy}{y} \qquad (370)$$

$$\frac{1}{2}ln\left|x^2+1\right| = ln\left|y\right| + c \qquad (371)$$

d) Solve $\frac{dy}{dx} = e^{3x-2y} + x^2e^{-2y}$
Solution

$$e^{2y}\frac{dy}{dx} = e^{2y}\left(e^{3x-2y} + x^2e^{-2y}\right) \qquad (372)$$

$$e^{2y}\frac{dy}{dx} = e^{3x} + x^2 \qquad (373)$$

$$\int e^{2y}dy = \int e^{3x} + x^2dx \qquad (374)$$

$$\frac{e^{2y}}{2} = \frac{1}{3}e^{3x} + \frac{x^3}{3} + c \qquad (375)$$

e) Solve $(1-x^2)(1-y)dx = xy(1+y)dy$
Solution

$$\frac{1-x^2}{x}dx = \frac{y(1+y)}{1-y}dy \tag{376}$$

$$\int \left(\frac{1}{x} - x\right) dx = \int \left(\frac{y}{1-y} + \frac{y^2}{1-y}\right) dy \tag{377}$$

$$ln\,|x| - \frac{x^2}{2} = \int \left[\left(-1 + \frac{1}{1-y}\right) + \left(-y - 1 + \frac{1}{1-y}\right)\right] dy \tag{378}$$

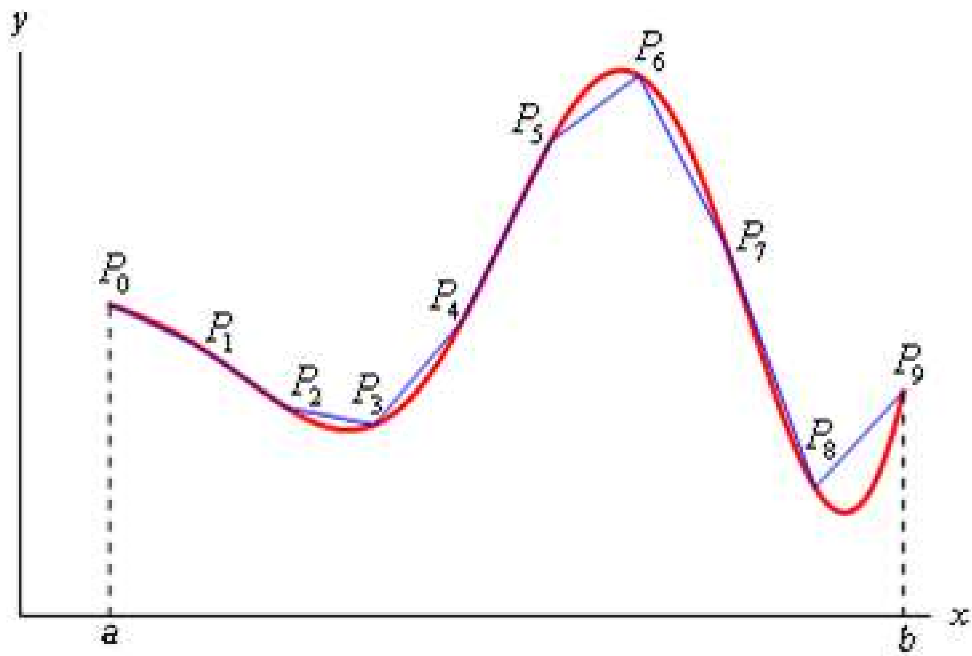
$$= -y - ln\,|1-y| - \frac{y^2}{2} - y - ln\,|1-y| + c \tag{379}$$

$$ln\,|x| - \frac{x^2}{2} = -2y - 2ln\,|1-y| - \frac{y^2}{2} + c \tag{380}$$

8 APPLICATION OF INTEGRATION

8.1 Arc Length

To determine the length of a continuous section$y = f(x)$on the interval $[a, b]$we'll need to estimate the length of the curve. We'll do this by dividing the interval up into n equal sub intervals each of width $\triangle x$and we'll denote the point on the curve at each point by Pi.We can then approximate the curve by a series of straight lines connecting the points. Here is a sketch of this situation for $n = 9$.



Now denote the length of each of these line segments by $|P_{i-1}P_i|$and the length of the curve will then be approximately,$L = \sum |P_{i-1}P_i|$.and we can get the exact length by taking n larger and larger. In other words, the exact length will be$\lim_{x\to\infty} \sum |P_{i-1}P_i|$

Now, let's get a better grasp on the length of each of these line segments. First, on each segment let's define

$$\triangle y_i = y_i - y_{i-1}$$

We can then compute directly the length of the line segments as follows:

$$|P_{i-1}P_i| = \sqrt{(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2} = \sqrt{(\triangle x)^2 + (\triangle y_i)^2}$$

By the mean value theorem, we know that on the interval $[x_{i-1}, x_i]$there is a point $x_i^\star$, so that

$$f(x_i) - f(x_{i-1}) = f'(x_i^\star)(x_i - x_{i-1})$$

$$\triangle y_i = f'\left(x_i^{\star}\right)\left(x_i - x_{i-1}\right)$$

$$\triangle y_i = f'\left(x_i^{\star}\right)\left(\triangle x\right)$$

Therefore the length of the curve may now be written as:

$$|P_{i-1}P_i| = \sqrt{\left(x_i - x_{i-1}\right)^2 + \left(y_i - y_{i-1}\right)^2} = \sqrt{\left(\triangle x\right)^2 + \left(\triangle y_i\right)^2} = \sqrt{\left(\triangle x\right)^2 + \left(f'\left(x_i^{\star}\right)\right)^2}\triangle x$$

The exact length of the curve is

$$\lim_{x\to\infty} \sum |P_{i-1}P_i| = \lim_{x\to\infty} \sqrt{\left(\triangle x\right)^2 + \left(f'\left(x_i^{\star}\right)\right)^2}\triangle x$$

However, using the definition of the definite integral, this is nothing more than

$$L = \int \sqrt{1 + \left(f'(x)\right)^2} dx \tag{381}$$

In a similar fashion we can also derive a formula for $x = h(y)$to obtain

$$L = \int \sqrt{1 + \left(f'(y)\right)^2} dy \tag{382}$$

Before we work any examples we need to make a small change in notation. Instead of having two formulas for the arc length of a function we are going to reduce it, in part, to a single formula. From this point on we are going to use the following formula for the length of the curve.

$$L = \int ds$$

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx,$$

if $y = f(x), a \leq x \leq b$

$$ds = \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy,$$

if $x = h(y), c \leq y \leq d$

Examples

a) Determine the length of the arc $y = ln\,(secx)$between $0 \leq x \leq \frac{\pi}{4}$

Solution

$$L = \int \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

$$\frac{dy}{dx} = \frac{tanxsecx}{secx} = tanx$$

$$L = \int \sqrt{1 + (tanx)^2} dx = \int secxdx = ln\,|secx + tanx| = ln\left|\sqrt{2} + 1\right| \tag{383}$$

b) Determine the length of $x = \frac{2}{3}\,(y-1)^{\frac{3}{2}}$between $1 \leq y \leq 4$

Solution

$$L = \int \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy$$

$$\frac{dx}{dy} = (y-1)^{\frac{1}{2}}$$

$$L = \int \sqrt{1 + \left((y-1)^{\frac{1}{2}}\right)^2} dy = \int \sqrt{y} dy = \frac{2}{3} y^{-\frac{3}{2}} = \frac{14}{3}$$

c) Find the perimeter of the astroid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$

Solution

$$\frac{2}{3}x^{-\frac{1}{3}} + \frac{2}{3}y^{-\frac{1}{3}}\frac{dy}{dx} = 0$$

or

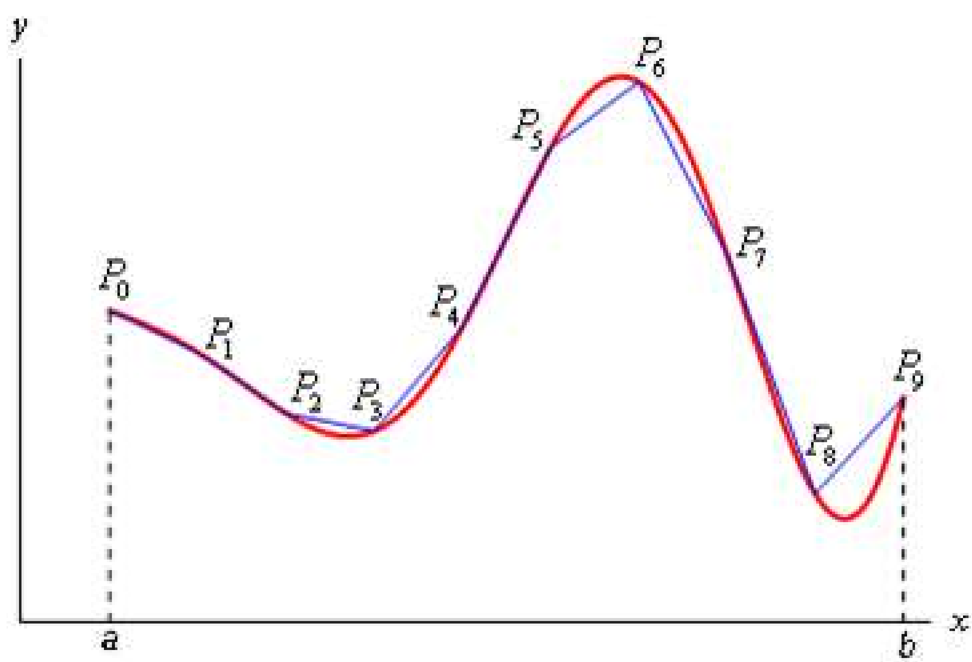
$$\frac{dy}{dx} = -\left(\frac{y}{x}\right)^{\frac{1}{3}}$$

$$L = 4\int \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

$$L = \int \sqrt{1 + \left(-\left(\frac{y}{x}\right)^{\frac{1}{3}}\right)^2} dx = 4\int \sqrt{\frac{x^{\frac{2}{3}} + y^{\frac{2}{3}}}{x^{\frac{2}{3}}}} dx = 4\int \frac{\sqrt{a^{\frac{2}{3}}}}{x^{\frac{2}{3}}} dx = 4a^{\frac{1}{3}} \cdot \frac{3}{2}\left[x^{\frac{2}{3}}\right] = 6a \tag{384}$$

8.2 Area

Consider the diagram below;



The area of the solid generated by revolving the area bounded by the curve $y = f(x)$ about the x-axis is given by:

$$s = 2\pi \int_{x=a}^{x=b} y \frac{ds}{dx} dx$$

If the revolution is about the y-axis, then;

$$s = 2\pi \int_{y=a}^{y=b} y \frac{ds}{dy} dy$$

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx,$$

if $y = f(x), a \leq x \leq b$

$$ds = \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy,$$

if $x = h(y), c \leq y \leq d$

EXAMPLE:

a) Determine the area obtained by revolving $y = \sqrt{9 - x^2}$bounded by y=-2 to y=2about the x axis,

$$s = 2\pi \int_{x=a}^{x=b} y \frac{ds}{dx} dx$$

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx,$$

$$\frac{dy}{dx} = \frac{1}{2}\left(9 - x^2\right)^{-\frac{1}{2}} \cdot -2x = \frac{x}{\sqrt{9 - x^2}}$$

$$ds = \sqrt{1 + \left(\frac{x}{\sqrt{9 - x^2}}\right)^2} = \sqrt{1 + \frac{x^2}{9 - x^2}} = \sqrt{\frac{9}{9 - x^2}} = \frac{3}{\sqrt{9 - x^2}}$$

$$s = 2\pi \int_{-2}^{2} \sqrt{9 - x^2} \cdot \frac{3}{\sqrt{9 - x^2}} dx$$

$$= 2\pi \int_{-2}^{2} 3dx$$

$$= 24\pi$$

(b) Determine the surface area generated by revolving about the y-axis, the curve $x = y^3$from y=0 to y=2.

$$s = 2\pi \int_{y=a}^{y=b} y \frac{ds}{dy} dy$$

$$ds = \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy,$$

$$\frac{dx}{dy} = 3y^2$$

$$ds = \sqrt{1 + 9y^4} dy$$

$$s = 2\pi \int_{y=0}^{y=2} y^3 \cdot \sqrt{1 + 9y^4} dy$$

$$= 2\pi \cdot \frac{1}{36} \int_{y=0}^{y=2} \left(1 + 9y^4\right)^{\frac{1}{2}} 36y^3 dy$$

Let $z = 1 + 9y^4 \implies dz = 36y^3 dy = dz$

$$= \frac{\pi}{18} \int z^{\frac{1}{2}} dz$$

$$= \frac{\pi}{18}\left(\frac{z^{\frac{3}{2}}}{\frac{3}{2}}\right) = \frac{\pi}{27}\left[145\sqrt{145}-1\right]$$

(c)Find the surface area of a solid generated when one arch of the cycloid $x = a\left(\theta + sin\theta\right)$, $y = a\left(1 - cos\theta\right)$revolves about the x-axis. The limits of half arch are $\theta = 0$to $\theta = \pi$

$$x = a\left(\theta + sin\theta\right) \Longrightarrow \frac{dx}{d\theta} = a\left(1 + cos\theta\right)$$

$$y = a\left(1 - cos\theta\right) \Longrightarrow \frac{dy}{d\theta} = asin\theta$$

$$\left(\frac{ds}{d\theta}\right)^2 = \left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2$$

$$\frac{ds}{d\theta} = \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}$$

$$= \sqrt{\left(a\left(1 + cos\theta\right)\right)^2 + \left(asin\theta\right)^2}$$

$$= \sqrt{a^2\left(2 + 2cos\theta\right)} = \sqrt{4a^2cos^2\frac{\theta}{2}} = 2acos\frac{\theta}{2}$$

$$s = 2\int_0^{\pi} 2\pi y ds$$

$$= 4\pi\int_0^{\pi} a\left(1 - cos\theta\right)\cdot 2acos\frac{\theta}{2}d\theta$$

$$= 8\pi a^2\int_0^{\pi} 2sin^2\frac{\theta}{2}cos\frac{\theta}{2}d\theta$$

$$= 32\pi a^2 \int_0^{\pi} sin^2\frac{\theta}{2}\left[\frac{1}{2}cos\frac{\theta}{2}\right] d\theta$$

$$= 32\pi a^2 \left[\frac{sin^3\frac{\theta}{2}}{3}\right]_0^{\pi} = \frac{32\pi a^2}{3} d\theta$$

(d) Find the surface of the solid generated by revolution of the ellipse $x^2 + 4y^2 = 16$, about the x axis. The limits are for x=-4 to x=4.

(e) Find the surface area of the solid generated by revolution of the curves $x = acos^3t$, $y = asin^3t$. The limits are from 0 to $\frac{\pi}{2}$.

8.3 Volume:

The area of the solid generated by revolving the area bounded by the curve $y = f(x)$ about the x-axis is given by:

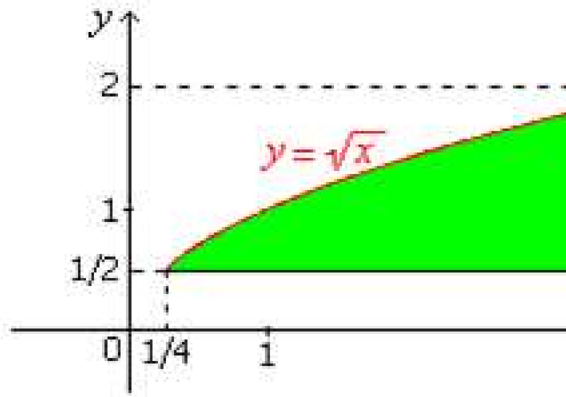
$$v = \pi \int_{x=a}^{x=b} y^2 dx$$

The area of the solid generated by revolving the area bounded by the curve $y = f(x)$ about the y-axis is given by:

$$v = \pi \int_{y=a}^{y=b} x^2 dy$$

Example:

a) Calculate the volume of the solid generated by revolving the plane re-



gion bounded by $y = \sqrt{x}$, $x = 4$and $y = \frac{1}{2}$about the x-axis

Solution

When $y = \frac{1}{2}, x = \left(\frac{1}{2}\right)^2 = \frac{1}{4}$

The volume obtained above has a cylindrical hole of radius $y = \frac{1}{2}$.

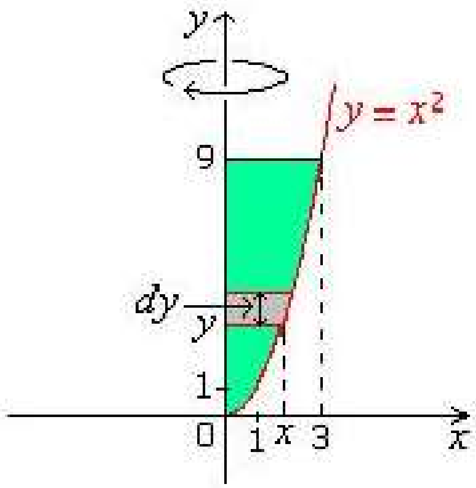
$$v = \pi \int_1^4 \left(\sqrt{x}\right)^2 dx - \pi \int_1^4 \left(\frac{1}{2}\right)^2 dx$$

$$= \pi \int_1^4 \left(x^2 - \frac{1}{4}\right) dx$$

$$= \pi \left[\frac{x^3}{3} - \frac{1}{4}x\right]_1^4 = \frac{225\pi}{32} = 22.09 cubic\, units$$

b)Compute the volume of the solid generated by revolving the plane region bounded by$y = x^2$, y = 9, and x = 0 about the y-axis.

Solution



$$v = \pi \int_{y=a}^{y=b} x^2 dy$$

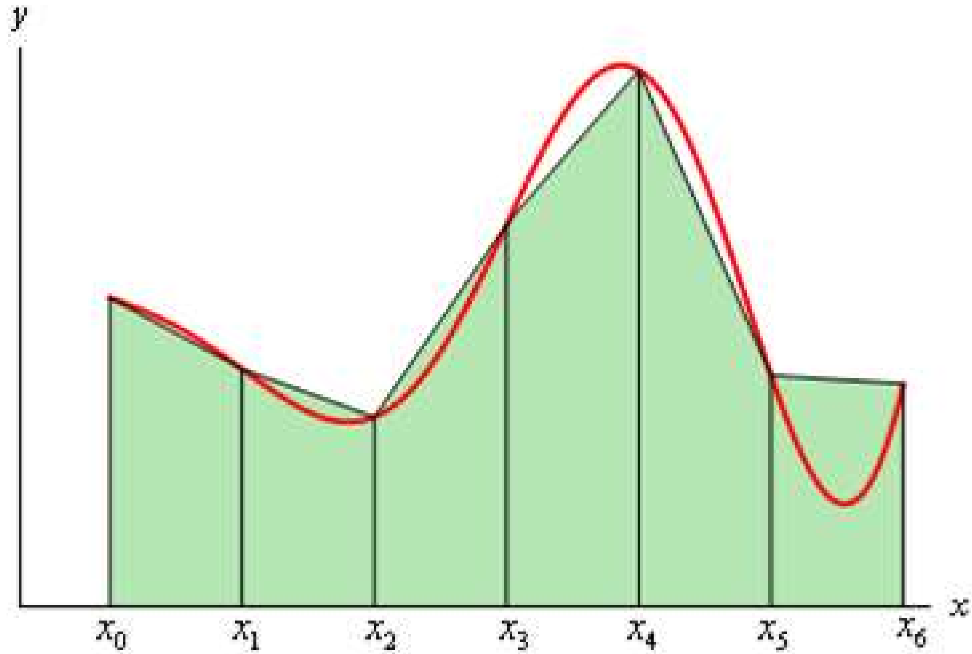
$$= \pi \int_0^9 \left(\sqrt{y}\right)^2 dy$$

$$= \pi \left[\frac{y^2}{2}\right]_0^9 = \pi \left[\frac{9^2}{2} - 0\right] = 127.23 cubic\, units$$

9 NUMERICAL ANALYSIS

A) Trapezoidal rule:

Trapezoidal rule defines area under a curve as the summation of individual trapezium areas that constitutes a given space under a curve.



Area of trapezium =1/2(sum of parallel sides*perpendicular distance between them)

Area of trapezium 1= $\frac{1}{2}h\left(y_0 + y_1\right)$

Area of trapezium 2= $\frac{1}{2}h\left(y_1 + y_2\right)$

Area of trapezium 3= $\frac{1}{2}h\left(y_2 + y_3\right)$

Area of trapezium 4= $\frac{1}{2}h\left(y_3 + y_4\right)$

Area of trapezium 5= $\frac{1}{2}h\left(y_{n-1} + y_n\right)$

Summing up the above areas,

$$A = \frac{h}{2}\left[\left(y_0 + y_n\right) + 2\left(y_1 + y_3 + y_4 + ... + y_{n-1}\right)\right]$$

which is the trapezoidal rule of finding area under a curve.

The equal spacing h is given by $h = \frac{b-a}{n}$

B) Simpson's rule:

$$A = \frac{h}{3}\left[(y_0 + 4y_1 + y_2) + (y_2 + 4y_3 + y_4) + (y_4 + 4y_5 + y_6) + ...\right]$$

$$A = \frac{h}{3}\left[(y_0 + y_n) + 2\left(y_2 + y_4 + y_6 + ...\right) + 4\left(y_3 + y_5 + y_7 + ...\right)\right]$$

The equal spacing h is given by $h = \frac{b-a}{n}$

Examples:

1. Evaluate the integral $I = \int_0^1 \frac{dx}{1+x}$ using,

(i) composite trapezoidal rule

(ii) composite Simpson's rule

using 2, 4 and 8 equal sub intervals.

case 1:n=2

- composite trapezoidal rule

$h = \frac{b-a}{n} = \frac{1-0}{2} = 0.5$

x	0	0.5	1.0
$y = \frac{1}{1+x}$	1	0.667	0.5

$$A = \frac{0.5}{2}\left[(1 + 0.5) + 2\left(0.667\right)\right]$$

$$= 0.70833$$

- composite Simpson rule

$$A = \frac{0.5}{3}\left[1 + (2 \times 0.667) + 0.5\right]$$

$$= 0.69444$$

case 2: n=4

- composite trapezoidal rule

$h = \frac{b-a}{n} = \frac{1-0}{4} = 0.25$

x	0	0.25	0.5	0.75	1.0
$y = \frac{1}{1+x}$	1	0.800	0.667	0.5714	0.5

$$A = \frac{0.25}{2}\left[(1+0.5) + 0.800 + 0.667 + 0.5714\right]$$

$$= 0.697024$$

- composite Simpsons rule

$$A = \frac{0.25}{3}\left[(1+0.5) + 4\left(0.800 + 0.5714\right) + 2\left(0.667\right)\right]$$

$$= 0.693254$$

case 3: n=8

- composite trapezoidal rule

$h = \frac{b-a}{n} = \frac{1-0}{8} = 0.125$

x	$y = \frac{1}{1+x}$
0	1
0.125	0.8889
0.25	0.800
0.375	0.7273
0.5	0.667
0.625	0.6154
0.750	0.5714
0.875	0.5333
1.0	0.5

$$A = \frac{0.125}{2}\left[(1+0.5)+0.8889+0.800+0.7273+0.667+0.6154+0.5714+0.5333\right]$$

$$= 0.694122$$

- composite Simpsons rule

$$A = \frac{0.125}{3}\left[(1+0.5)+4\left(0.8889+0.7273+0.6154+0.5333\right)+2\left(0.800+0.667+0.5714\right)\right]$$

$$= 0.693155$$

Exercise:

Compute the following integrals using both trapezoidal and Simpsons rules of sub-intervals 4,6 and 8.

a) $\int_0^1 \frac{1}{1+x^2}dx$

b) $\int_0^1 \frac{x}{1+x^2}dx$

10 ERROR ANALYSIS

(a) Trapezoidal Error Bound

The absolute value of error made when $\int_a^b f(x)dx$is approximated by the trapezoidal approximation with n subdivisions is given by:

$$\xi = \frac{k\left(b-a\right)^3}{12n^2}$$

where k is any number such that $|f''(x)| \leq k$ for all x in the closed interval, $[a, b]$.

(b) Simpson's Rule Error bound

The absolute value of the error made when $\int_a^b f(x)dx$ is approximated by Simpsons rule with n subdivisions is given by:

$$\xi = \frac{m\left(b-a\right)^5}{180n^4}$$

where m is any number such that $|f^4(x)| \leq m$ for all x in the closed interval, $[a, b]$.

Example:

1. Estimate the absolute value of the maximum error that can occur when approximating $\int_0^1 \sqrt{1+x^3}dx$by

(a) The trapezoidal approximation with n=4

(b) Simpson's rule with n=4

Solution

(a)The trapezoidal approximation

$$\xi = \frac{k\left(b-a\right)^3}{12n^2}$$

$$f'(x) = \frac{1}{2}\left(1+x^3\right)^{-\frac{1}{2}} \cdot 3x^2 = \frac{3x^2}{2\sqrt{1+x^3}}$$

$$f''(x) = \frac{3}{4}x\left(1+x^3\right)^{-\frac{3}{2}}\left(x^3+4\right)$$

By choosing the maximum value of each term over the interval [0,1], we find;

$$f''(x) = \frac{3}{4}\cdot 1\,(1+1)^{-\frac{3}{2}}\,(1+4) = \frac{15}{4}$$

Hence we can take $k = \frac{15}{4}$, so that the maximum error can be at most

$$\xi = \frac{\frac{15}{4}\,(1-0)^3}{12\cdot 4^2} \simeq 0.02$$

(b) Simpsons rule

$$\xi = \frac{m\,(b-a)^5}{180n^4}$$

$$f^4(x) = \frac{9}{16}\left(1+x^3\right)^{-\frac{7}{2}}\left(x^8+56x^5-80x^2\right)$$

This is on the interval [0,1], we have

$$m = \frac{513}{16}$$

Therefore the maximum error will be

$$\xi = \frac{\frac{513}{16}\,(1-0)^5}{180\cdot 4^4} \simeq 0.007$$

2. How many subdivisions n, should be taken to ensure that the absolute value of the error in the approximation of $\int_1^2 \frac{1}{x^2}$is at most $\xi = 0.001$when using

(a) Trapezoidal approximation

(b) Simpsons approximation

Solution

$$f(x) = x^{-2}$$

$$f'(x) = -2x^{-3}$$

$$f''(x) = 6x^{-4}$$

$$f'''(x) = -24x^{-5}$$

$$f^4(x) = 120x^{-6}$$

$$f^5(x) = -720x^{-7}$$

Both $f^4(x)$and $f^5(x)$are decreasing functions because their derivatives are negative in this interval .

Hence each achieves its maximum on [1,2] at the left end point, x=1

$$k = |f''(x)| = |f''(1)| = 6$$

$$m = \left|f^4(x)\right| = \left|f^4(1)\right| = 120$$

(a) Using the trapezoidal approximation with n subdivisions, the absolute value of the error is at most;

$$\xi \simeq \frac{k\left(b-a\right)^3}{12n^2}$$

$$0.001 \simeq \frac{6\left(2-1\right)^3}{12n^2}$$

$$n = 22.36 = 23$$

(b) Using Simpsons rule approximation with n subdivisions, the absolute value of the error is at most;

$$\xi \simeq \frac{m\left(b-a\right)^{5}}{180n^{4}}$$

$$0.001 \simeq \frac{120\left(2-1\right)^{5}}{180n^{4}}$$

$$n = 6$$