3. Let X be the random variable the number of fours observed when two dice are rolled together once. Show that X is a discrete random variable.
4. The pmf of a discrete random variable X is given by $P(X = x) = kx$ for $x = 1, 2, 3, 4, 5, 6$ Find the value of the constant k, $P(X < 4)$ and $P(3 \leq \mathrm{X} < 6)$
5. A fair coin is flip until a head appears. Let N represent the number of tosses required to realize a head. Find the pmf of N
6. A discrete random variable Y has a pmf given by $P(Y = y) = c\left(\frac{3}{4}\right)^x$ for $y = 0, 1, 2, ...$ Find the value of the constant c and $P(\mathrm{X} < 3)$
7. Verify that $\mathrm{f(x)} = \dfrac{2x}{k(k+1)}$ for $x = 0, 1, 2, ...k$ can serve as a pmf of a random variable X.
8. For each of the following determine c so that the function can serve as a pmf of a random variable X.
   a) $\mathrm{f(x)} = cx$ for $x = 1, 2, 3, 4, 5$
   b) $\mathrm{f(x)} = cx^2$ for $x = 0, 1, 2, ...k$
   c) $\mathrm{f(x)} = c\left(\frac{1}{6}\right)^x$ for $x = 0, 1, 2, 3.....$
   d) $\mathrm{f(x)} = c2^{-x}$ for $x =$ for $x = 0, 1, 2, ...$
9. A coin is loaded so that heads is three times as likely as the tails. For 3 independent tosses of the coin find the pmf of the total number of heads realized and the probability of realizing at most 2 heads.

1.3 Continuous Random Variables and Probability Density Function

A **continuous** random variable can assume any value in an interval on the real line or in a collection of intervals. The sample space is uncountable. For instance, suppose an experiment involves observing the arrival of cars at a certain period of time along a highway on a particular day. Let T denote the time that lapses before the 1st arrival, the T is a continuous random variable that assumes values in the interval $[0, \infty)$

Definition: A random variable *X* is *continuous* if there exists a nonnegative function f so that, for every interval *B*, $P(X \in B) = \int_B \mathrm{f(x)\ dx}$. The function $\mathrm{f = f(x)}$ is called the *probability density function* of *X*.

Definition: Let X be a continuous random variable that assumes values in the interval $(-\infty, \infty)$, The f(x) is said to be a probability density function (pdf) of X if it satisfies the following conditions

$$\mathrm{f(x)} \geq 0 \ \text{ for all x }, \ p(a \leq x \leq b) = \int_a^b \mathrm{f(x)\ dx} = 1 \quad \text{and} \quad \int_{-\infty}^{\infty} \mathrm{f(x)\ dx} = 1$$

The support of a continuous random variable is the smallest interval containing all values of x where f(x) >= 0.

Remark A crucial property is that, for any real number x, we have $\mathrm{P(X = x) = 0}$ (implying there is no difference between $\mathrm{P(X \leq x)}$ and $\mathrm{P(X < x)}$); that is it is not possible to talk about the probability of the random variable assuming a particular value. Instead, we talk about the probability of the random variable assuming a value within a given interval. The probability of the random variable assuming a value within some given interval from $x = a$ to $x = b$ is defined to be the area under the graph of the probability density function between $x = a$ and $x = b$.

Example 1

Let X be a continuous random variable. Show that the function

$$f(x) = \begin{cases} \frac{1}{2}x, & 0 \le x \le 2 \\ 0, & elsewhere \end{cases}$$ is a pdf of X hence compute $P(0 \le X < 1)$ and $P(-1 < X < 1)$

Solution

$f(x) \ge 0$ for all x in the interval $0 \le x \le 2$ and $\int_0^2 \frac{1}{2}x\,dx = \frac{1}{4}\left[x^2\right]_0^2 = 1$. Therefore f(x)is indeed a pdf of X.

Now $P(0 \le X < 1) = \int_0^1 \frac{1}{2}x\,dx = \frac{1}{4}\left[x^2\right]_0^1 = \frac{1}{4}$ and

$P(-1 < X < 1) = P(-1 < X < 0) + P(0 < X < 1) = 0 + \frac{1}{4} = \frac{1}{4}$

Example 2

The time X, in hours, between computer failures is a continuous random variable with density

$$f(x) = \begin{cases} \lambda e^{-0.01x} & \text{for } x > 0 \\ 0, & elsewhere \end{cases}$$ Find λ hence compute $P(50 \le X < 150)$ and $P(X < 100)$

Solution

$f(x) \ge 0$ for all x in $0 \le x < \infty$ Thus $1 = \lambda \int_0^\infty e^{-0.01x}dx = -100\lambda\left[e^{-0.01x}\right]_0^\infty = 100\lambda \Rightarrow \lambda = 0.01$..

Now $P(50 \le X < 150) = 0.01\int_{50}^{150} e^{-0.01x}dx = \left[-e^{-0.01x}\right]_{50}^{150} = e^{-0.5} - e^{-1.5} \approx 0.3834005$ and

$P(X < 100) = 0.01\int_0^{100} e^{-0.01x}dx = \left[-e^{-0.01x}\right]_0^{100} = 1 - e^{-1} \approx 0.6321206$

Example 3

A continuous random variable X has a probability density function given by

$$f(x) = \begin{cases} 0.25, & 0 < x < 2 \\ 0.5x + c, & 2 < x < 3 \\ 0, & elsewhere \end{cases}$$ Find c hence compute $P(1 \le X < 2.5)$.

Solution

$\int_{\text{all x}} f(x)dx = 1 = \int_0^2 \frac{1}{4}dx + \int_2^3 \left(\frac{1}{2}x + c\right)dx = \frac{1}{2} + \frac{5}{4} + c \Rightarrow c = -\frac{3}{4}$

$P(1 \le X < 2.5) = P(1 \le X < 2) + P(2 \le X < 2.5) = \int_1^2 \frac{1}{4}dx + \int_2^{2.5}\left(\frac{1}{2}x - \frac{3}{4}\right)dx = \frac{1}{4} + \left[\frac{x^2 - 3x}{4}\right]_2^{2.5} = \frac{1}{4} + \frac{3}{16} = \frac{7}{16}$

Exercise

1) Suppose that the random variable X has p.d.f. given by $f(x) = \begin{cases} cx, & 0 \le x \le 1 \\ 0, & elsewhere \end{cases}$ Find the value of the constant c hence determine m so that $P(X \le m) = \frac{1}{2}$
2) Let X be a continuous random variable with pdf $f(x) = \begin{cases} \frac{x}{5} + k, & 0 \le x \le 3 \\ 0, & elsewhere \end{cases}$ Find the value of the constant k hence compute $P(1 < X < 3)$
3) A continuous random variable Y has the pdf given by $f(y) = \begin{cases} k(1+y), & 4 \le x \le 7 \\ 0, & elsewhere \end{cases}$ Find the value of the constant k hence compute $P(Y < 5)$ and $P(5 < Y < 6)$