CRAMER'S RULE FOR A 3×3 SYSTEM (WITH THREE VARIABLES)

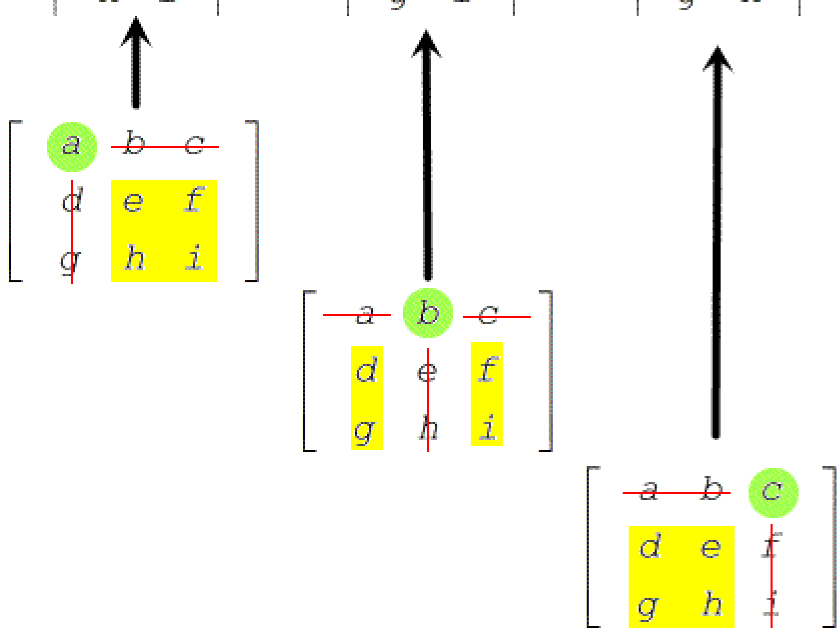
In our previous lesson, we studied how to use Cramer's Rule with two variables. Our goal here is to expand the application of Cramer's Rule to three variables usually in terms of *x*, *y*, and *z*. I will go over five (5) worked examples to help you get familiar with this concept.
To do well on this topic, you need to have an idea on how to find the determinant of a 3✕3 matrix. So, this is what we are going to do first.
Formula to Find the Determinant of a 3✕3 Matrix

- Given a 3✕3 matrix

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

- Its determinant can be calculated using the following formula.

$$|A| = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a \cdot \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \cdot \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \cdot \begin{vmatrix} d & e \\ g & h \end{vmatrix}$$



Let's do a quick example of this.

Find the determinant of matrix A

$$A = \begin{bmatrix} 6 & 2 & -4 \\ 5 & 6 & -2 \\ 5 & 2 & -3 \end{bmatrix}$$

Solution: Make sure that you follow the formula on how to find the determinant of a 3×3 matrix carefully, as shown above. More so, don't rush when you perform the required arithmetic operations in every step. This is where common errors usually occur, but it can be prevented. When you do it right, your solution should be similar to the one below.

$$|A| = \begin{vmatrix} 6 & 2 & -4 \\ 5 & 6 & -2 \\ 5 & 2 & -3 \end{vmatrix} = 6 \cdot \begin{vmatrix} 6 & -2 \\ 2 & -3 \end{vmatrix} - (2) \cdot \begin{vmatrix} 5 & -2 \\ 5 & -3 \end{vmatrix} + (-4) \cdot \begin{vmatrix} 5 & 6 \\ 5 & 2 \end{vmatrix}$$

$$= 6\left[-18 - (-4)\right] - 2\left[-15 - (-10)\right] - 4\left[10 - 30\right]$$

$$= 6(-14) - 2(-5) - 4(-20)$$

$$= -84 + 10 + 80$$

$$= 6$$

Now, it's time to go over the procedure on how to use Cramer's Rule in a linear system involving three variables.

Cramer's Rules for Systems of Linear Equations with Three Variables

- Given a linear system

x-column z-column

$$
\begin{aligned}
a_1x + b_1y + c_1z &= d_1 \\
a_2x + b_2y + c_2z &= d_2 \\
a_3x + b_3y + c_3z &= d_3
\end{aligned}
$$

y-column constant-column

- Labeling each of the four matrices

coefficient matrix:

$$D = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$$

X – matrix:

$$D_x = \begin{bmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{bmatrix}$$

Y – matrix:

$$D_y = \begin{bmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{bmatrix}$$

Z – matrix:

$$D_z = \begin{bmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{bmatrix}$$

- To solve for *x*:

$$x = \frac{|D_x|}{|D|} = \frac{\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}$$

- To solve for *y*:

$$y = \frac{|D_y|}{|D|} = \frac{\begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}$$

- To solve for z:

$$z = \frac{|D_z|}{|D|} = \frac{\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}$$

Things to observe from the setup above:
1) The coefficients of variables x, y, and z make use of subscripted a, b, and c, respectively. While the constant terms use subscripted d.
2) The denominators to find the values of x, y, and z are all the same which is the determinant of the coefficient matrix (coefficients coming from the columns of x, y, and z).
3) To solve for x, the coefficients of the x-column is replaced by the constant column (in red).
4) To solve for y, the coefficients of the y-column is replaced by the constant column (in red).
5) In the same manner, to solve for z, the coefficients of the z-column is replaced by the constant column (in red).

Examples of How to Solve Systems of Linear Equations with Three Variables using Cramer's Rule

Example 1: Solve the system with three variables by Cramer's Rule.

$$\begin{cases} x + 2y + 3z = -5 \\ 3x + y - 3z = 4 \\ -3x + 4y + 7z = -7 \end{cases}$$

From the given system of linear equations, solve for the values of x, y, z.
Use the guide above to correctly setup these special matrices.

- **coefficient matrix**

$$D = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & -3 \\ -3 & 4 & 7 \end{bmatrix}$$

- **X – matrix**

$$D_x = \begin{bmatrix} -5 & 2 & 3 \\ 4 & 1 & -3 \\ -7 & 4 & 7 \end{bmatrix}$$

- **Y – matrix**

$$D_y = \begin{bmatrix} 1 & -5 & 3 \\ 3 & 4 & -3 \\ -3 & -7 & 7 \end{bmatrix}$$

- **Z – matrix**

$$D_z = \begin{bmatrix} 1 & 2 & -5 \\ 3 & 1 & 4 \\ -3 & 4 & -7 \end{bmatrix}$$

solving for the determinant of each matrix.

The values of the determinants are listed below.

Determinants of each matrix:

$$|D| = 40$$

$$|D_x| = -40$$

$$|D_y| = 40$$

$$|D_z| = -80$$

The final answers or solutions are easily computed or calculated once all the required determinants are found.

Solved values for x, y, and z.

$$x = \frac{|D_x|}{|D|} = \frac{-40}{40} = -1$$

$$y = \frac{|D_y|}{|D|} = \frac{40}{40} = 1$$

$$z = \frac{|D_z|}{|D|} = \frac{-80}{40} = -2$$

The final answer written in point notation is {x, y, z} = { - 1,1, - 2}

Example 2: Solve the system with three variables by Cramer's Rule.

$$\begin{cases} -2x - y - 3z = 3 \\ 2x - 3y + z = -13 \\ 2x \qquad - 3z = -11 \end{cases}$$

Consider the coefficient matrix as the "primary" matrix because the other three matrices are derived from it. For instance, the x matrix is just the "primary" matrix with the x-column replaced by the constant column (in red). You can observe that the same pattern is applied in constructing the other matrices: y and z

- **coefficient matrix**

$$D = \begin{bmatrix} -2 & -1 & -3 \\ 2 & -3 & 1 \\ 2 & 0 & -3 \end{bmatrix}$$

- **X – matrix**

$$D_x = \begin{bmatrix} 3 & -1 & -3 \\ -13 & -3 & 1 \\ -11 & 0 & -3 \end{bmatrix}$$

- **Y – matrix**

$$D_y = \begin{bmatrix} -2 & 3 & -3 \\ 2 & -13 & 1 \\ 2 & -11 & -3 \end{bmatrix}$$

- **Z – matrix**

$$D_z = \begin{bmatrix} -2 & -1 & 3 \\ 2 & -3 & -13 \\ 2 & 0 & -11 \end{bmatrix}$$

After solving the determinant of each matrix, Determinants of each matrix are

$$|D| = -44$$

$$|D_x| = 176$$

$$|D_y| = -88$$

$$|D_z| = -44$$

The values for x, y and z are calculated as follows. Notice that x is obtained by taking the determinant of the x-matrix divided by the determinant of the coefficient matrix. This rule holds for the rest.

Solved values for x, y, and z are

$$x = \frac{|D_x|}{|D|} = \frac{176}{-44} = -4$$

$$y = \frac{|D_y|}{|D|} = \frac{-88}{-44} = 2$$

$$z = \frac{|D_z|}{|D|} = \frac{-44}{-44} = 1$$

Our final answer is {x, y, z} = { - 4, 2, 1}

Example 3: Solve the system with three variables by Cramer's Rule.

$$\begin{cases} \quad -y - 2z = -8 \\ x \qquad + 3z = 2 \\ 7x + y + z = 0 \end{cases}$$

This problem is much easier than the first two examples because of the presence of zero entries in the x, y, and constant columns.
In fact, as you increase the number of zeroes in a square matrix, the work done to find its determinant is greatly reduced.

Here are the matrices extracted from the system of linear equations.

- **coefficient matrix**

$$D = \begin{bmatrix} 0 & -1 & -2 \\ 1 & 0 & 3 \\ 7 & 1 & 1 \end{bmatrix}$$

- **X – matrix**

$$D_x = \begin{bmatrix} -8 & -1 & -2 \\ 2 & 0 & 3 \\ 0 & 1 & 1 \end{bmatrix}$$

- **Y – matrix**

$$D_y = \begin{bmatrix} 0 & -8 & -2 \\ 1 & 2 & 3 \\ 7 & 0 & 1 \end{bmatrix}$$

- **Z – matrix**

$$D_z = \begin{bmatrix} 0 & -1 & -8 \\ 1 & 0 & 2 \\ 7 & 1 & 0 \end{bmatrix}$$

Solving for their determinants, Determinants of each matrix:

$$|D| = -22$$

$$|D_x| = 22$$

$$|D_y| = -132$$

$$|D_z| = -22$$

This leads us to easily set up and calculate the final answers.

Solved values for x, y and z are:

$$x = \frac{|D_x|}{|D|} = \frac{22}{-22} = -1$$

$$y = \frac{|D_y|}{|D|} = \frac{-132}{-22} = 6$$

$$z = \frac{|D_z|}{|D|} = \frac{-22}{-22} = 1$$

The final answer is {x, y, z} = { - 1, 6, 1}

Example 4: Solve the system with three variables by Cramer's Rule

$$\begin{cases} -2x + y + z = 4 \\ -4x + 2y - z = 8 \\ -6x - 3y + z = 0 \end{cases}$$

Write down the four special matrices.

- **coefficient matrix**

$$D = \begin{bmatrix} -2 & 1 & 1 \\ -4 & 2 & -1 \\ -6 & -3 & 1 \end{bmatrix}$$

- **X – matrix**

$$D_x = \begin{bmatrix} 4 & 1 & 1 \\ 8 & 2 & -1 \\ 0 & -3 & 1 \end{bmatrix}$$

- **Y – matrix**

$$D_y = \begin{bmatrix} -2 & 4 & 1 \\ -4 & 8 & -1 \\ -6 & 0 & 1 \end{bmatrix}$$

- **Z – matrix**

$$D_z = \begin{bmatrix} -2 & 1 & 4 \\ -4 & 2 & 8 \\ -6 & -3 & 0 \end{bmatrix}$$

Evaluate each matrix to find its determinant.

These are the determinants of each matrix:

$$|D| = 36$$

$$|D_x| = -36$$

$$|D_y| = 72$$

$$|D_z| = 0$$

Use the Cramer's Rule to get the following solutions.

Solved values for x, y and z

$$x = \frac{|D_x|}{|D|} = \frac{-36}{36} = -1$$

$$y = \frac{|D_y|}{|D|} = \frac{72}{36} = 2$$

$$z = \frac{|D_z|}{|D|} = \frac{0}{36} = 0$$

The final answer is {x, y, z} { - 1, 2, 0}

Example 5: Solve the system with three variables by Cramer's Rule

$$\begin{cases} x - 8y + z = 4 \\ -x + 2y + z = 2 \\ x - y + 2z = -1 \end{cases}$$

Construct the four special matrices.

- **coefficient matrix**

$$D = \begin{bmatrix} 1 & -8 & 1 \\ -1 & 2 & 1 \\ 1 & -1 & 2 \end{bmatrix}$$

- **X – matrix**

$$D_x = \begin{bmatrix} 4 & -8 & 1 \\ 2 & 2 & 1 \\ -1 & -1 & 2 \end{bmatrix}$$

- **Y – matrix**

$$D_y = \begin{bmatrix} 1 & 4 & 1 \\ -1 & 2 & 1 \\ 1 & -1 & 2 \end{bmatrix}$$

- **Z – matrix**

$$D_z = \begin{bmatrix} 1 & -8 & 4 \\ -1 & 2 & 2 \\ 1 & -1 & -1 \end{bmatrix}$$

Find the determinant of each square matrix.

Determinants of each matrix

$$|D| = -20$$

$$|D_x| = 60$$

$$|D_y| = 16$$

$$|D_z| = -12$$

Solve for x, y and z using the given formula.
Solved values for x, y and z

$$x = \frac{|D_x|}{|D|} = \frac{60}{-20} = -3$$

$$y = \frac{|D_y|}{|D|} = \frac{16}{-20} = \frac{-4}{5}$$

$$z = \frac{|D_z|}{|D|} = \frac{-12}{-20} = \frac{3}{5}$$

The final answer in point form {x, y, z} = {- 3, - 4/53/5}