SUBSTITUTION

1) Integral with exponentials $\int \sin^2 x \cos^2 x \, dx$

$$\int e^u \, du = e^u + c$$

$$\int a^u \, du = \frac{a^u}{\ln a} + C$$

$$\int \frac{u'}{u} \, du = \int \frac{1}{u} \, du = \ln|u| + c$$

$$\frac{d}{du}(\ln|u|) = \frac{u'}{u} = \frac{1}{u}$$

$$\int \cancel{du} \frac{d}{\cancel{du}} \ln|u| = \int \frac{1}{u} \, du$$

$$\ln|u| + c = \int \frac{1}{u} \, du$$

Evaluate

i) $\int x^2 e^{x^3} \, dx$

ii) $\int x \, 2^{(x^2+1)} \, dx$

iii) $\int \sqrt{x} \, e^{3\sqrt{x}} \, dx$

iv) $\int \frac{\cos 3x}{\sin 3x} \, dx$

$\boxed{= \int \cot 3x \, dx}$

v) $\int \frac{6x}{3x^2+4} \, dx$

$u = 3x^2 + 4$

$du = 6x \, dx$

v) $\displaystyle\int \frac{\cos\theta}{2+\sin\theta}\,d\theta$ vi) $\displaystyle\int \tan x\,dx$

$u = 2+\sin\theta,\quad du = \cos\theta\,d\theta$

vii) $\displaystyle\int \frac{dx}{\sqrt{x}\,(1+\sqrt{x})}$ viii) $\displaystyle\int \frac{e^{\tan^{-1}x}}{1+x^2}\,dx$

ix) $\displaystyle\int 2^{\sin x}\cos x\,dx$

Solution

i.) $\displaystyle\int x^2 e^{x^3}\,dx$

Let $u = x^3,\quad du = 3x^2\,dx$

$$\frac{du}{3} = x^2\,dx$$

$$\int e^u\,\frac{du}{3} = \frac{1}{3}\int e^u\,du = \frac{1}{3}e^u + c$$

$$= \frac{1}{3}e^{x^3} + c$$

option II

let $u = x^3$ $\quad \dfrac{du}{dx} = 3x^2$ $\quad du = 3x^2\,dx$

$$\frac{du}{3x^2} = dx$$

$$\int x^2 e^{x^3}\,dx = \int \cancel{x^2}\,e^u\,\frac{du}{3\cancel{x^2}} = \int \frac{e^u}{3}\,du$$

$$= \frac{1}{3}\int e^u\,du = \frac{1}{3}e^u + c$$

$$= \frac{1}{3}e^{x^3} + c$$

ii) $\displaystyle\int x\,2^{(x^2+1)}\,dx$

Let $u = x^2+1$ $\qquad \dfrac{du}{dx} = 2x$ $\Big|$ $du = 2x\,dx$

$$\frac{du}{2x} = dx$$

$$\int x\,2^{(x^2+1)}\,dx = \int \underbrace{x\,2^u}\,\frac{du}{2x}$$

$$\frac{(\cancel{x})(2^u)}{(2)(\cancel{x})}$$

$$= \int \frac{2^u}{2}\,du$$

$$= \frac{1}{2}\int 2^u\,du$$

$$= \frac{1}{2}\,\frac{2^u}{\ln 2} + c$$

$$= \frac{2^{x^2+1}}{2\ln 2} + c$$

$$\frac{d}{du}\,a^u = a^u \ln a$$

$$\int d\,(a^u) = \int a^u \ln a\,du$$

$$a^u = \ln a \int a^u\,du$$

$$\boxed{\frac{a^u}{\ln a} = \int a^u\,du}$$

$$\int 2^{\sin x} \underbrace{\cos x \, dx}$$

$$u = \sin x \qquad du = \underbrace{\cos x \, dx} \qquad \frac{du}{\cos x} = dx$$

$$= \int 2^u \, du = \frac{2^u}{\ln 2} + c = \frac{2^{\sin x}}{\ln 2} + c$$

$$\int \frac{e^{\tan^{-1} x}}{1+x^2} \, dx$$

$$\tan\theta = \frac{o}{a}$$

$$\theta = \tan^{-1}\left(\frac{o}{a}\right)$$

Let $u = \tan^{-1} x$

$$x = \tan u$$

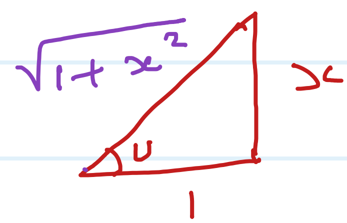
$$\frac{d}{dx}(\qquad d/dx \qquad u' \, \mathrm{Tng}' \, u$$

$$1 = \frac{dx}{dx} = u' \sec^2 u$$

$$1 = \frac{du}{dx} \sec^2 u$$

$$\frac{1}{\sec^2 u} = \frac{du}{dx}$$

$$x = \tan u = \frac{x}{1}$$



$$\sec u = \frac{1}{\cos u} = \frac{H}{A}$$

$$\sec u = \frac{\sqrt{1+x^2}}{1}$$

$$\sec^2 u = 1 + x^2$$

$$\frac{1}{1+x^2} = \frac{du}{dx}$$

$$\frac{dx}{1+x^2} = du$$

$$\int \frac{e^{\tan^{-1}x}}{1+x^2}\,dx = \int \frac{dx}{1+x^2}\, e^{\tan^{-1}x}$$

$$= \int e^u\,du = e^u + c$$

$$= e^{\tan^{-1}x} + c$$

$$\int \frac{\cos 3x}{\sin 3x}\,dx$$

let $u = \sin 3x$

$du = 3\underbrace{\cos 3x\,dx}$

$\frac{du}{3} = \cos 3x\,dx$

$$\int \frac{1}{u}\frac{du}{3} = \frac{1}{3}\int \frac{1}{u}\,du$$

$$= \frac{1}{3}\ln|u| + c = \frac{1}{3}\ln|\sin 3x| + c$$

$$\int \frac{dx}{\underbrace{\sqrt{x}}(1+\sqrt{x})}$$

let $u = 1+\sqrt{x} = 1 + x^{1/2}$

$du = \frac{1}{2}x^{1/2-1}\,dx$

$= \frac{1}{2}x^{-1/2}\,dx$

$du = \frac{1}{2x^{1/2}}\,dx$

$du = \frac{1}{2\sqrt{x}}\,dx$

$2\sqrt{x}\,du = dx$

$$\int \frac{dx}{\sqrt{x}\,(1+\sqrt{x})} = \int \frac{2\cancel{\sqrt{x}}\,du}{\cancel{\sqrt{x}}\quad u} = 2\int \frac{du}{u}$$

$$= 2\int \frac{1}{u}\,du$$

$$= 2\ln|u| + c$$

$$= 2\ln|1+\sqrt{x}| + c$$

Option II

$$u = \sqrt{x} = x^{1/2} \quad / \quad du = \tfrac{1}{2}x^{1/2-1}\,dx = \tfrac{1}{2}x^{-1/2}\,dx$$

$$= \frac{1}{2x^{1/2}}\,dx$$

$$du = \frac{1}{2\sqrt{x}}\,dx$$

$$\int \frac{dx}{\sqrt{x}\,(1+\sqrt{x})} = \int \frac{2\cancel{\sqrt{x}}\,du}{\cancel{\sqrt{x}}\,(1+u)} \qquad 2\sqrt{x}\,du = dx$$

$$= 2\int \frac{du}{1+u} = 2\ln|1+u| + c$$

$$\int \frac{1\,du}{u} = \ln|u|$$

$$= 2\ln|1+\sqrt{x}| + c$$

$$\int \frac{1}{u}\,du = \int \frac{1}{a+u}\,du = \ln|a+u| + c$$

a ↑ Constant

Integrals with Powers of Sin & Cosine

Even power : $\cos^2 x = \frac{1}{2}(1+\cos x)$

$\sin^2 x = \frac{1}{2}(1-\cos x)$

Then integrate

odd power :

$\sin^2 x + \cos^2 x = 1$

Then do Substitution.

✓ i) $\int \sin^2 x\, dx$ iii) $\int \sin^4 x\, dx$ vi) $\int \sin^6 x\, dx$

ii) $\int \cos^2 x\, dx$ iv) $\int \cos^4 x\, dx$ vii) $\int \cos^6 x\, dx$

viii) $\int \sin^3 dx$ x) $\int \sin^5 dx$

ix) $\int \cos^3 dx$ xi) $\int \cos^7 dx$

1) $\int \sin^2 x\, dx = \int \frac{1}{2}(1-\cos 2x)\, dx$

$= \frac{1}{2}\int (1-\cos 2x)\, dx$

$\int \text{Trig}\, u = \dfrac{\int \text{Trig}\, u}{u'}$

$= \frac{1}{2}\left[x - \frac{\sin 2x}{2}\right] + c$

$= \frac{1}{2}\left(x - \frac{\sin 2x}{2}\right) + c$

ii) $\int \cos^4 x \, dx$

$$= \int \left(\cos^2 x\right)^2 dx$$

$$\left(\cos^2 x\right)^2 = \left[\frac{1}{2}(1+\cos 2x)\right]^2$$

$(AB)^2 = A^2 B^2$

$(2\times 3)^2 = 2^2 \times 3^2$

$$= \frac{1}{4}(1+\cos 2x)^2$$

$(a-b)^2 = a^2 - 2ab + b^2$

$$= \frac{1}{4}\left(1 + 2\cos 2x + \underbrace{\cos^2 2x}\right)$$

$$\cos^2 A = \frac{1}{2}(1+\cos 2A)$$

$2x$

$$\cos^2 2x = \frac{1}{2}(1+\cos 4x).$$

$$= \frac{1}{4}\left(1 + 2\cos 2x + \left[\frac{1}{2}(1+\cos 4x)\right]\right)$$

$\frac{2}{8} + \frac{1}{8} = \frac{3}{8}$

$$= \frac{1}{4}\left(1 + 2\cos 2x + \frac{1}{2} + \frac{1}{2}\cos 4x\right)$$

$$= \underbrace{\frac{1}{4}} + \frac{1}{2}\cos 2x + \underbrace{\frac{1}{8}} + \frac{1}{8}\cos 4x$$

$$\cos^4 x = \left(\cos^2 x\right)^2 = \frac{3}{8} + \frac{1}{2}\cos 2x + \frac{1}{8}\cos 4x$$

$$\int \cos^4 x \, dx = \int \left(\frac{3}{8} + \frac{1}{2}\cos 2x + \frac{1}{8}\cos 4x\right) dx$$

$$= \frac{3}{8}x + \frac{1}{2}\,\frac{\sin 2x}{2} + \frac{1}{8}\,\frac{(-\sin 4x)}{4} + c$$

$$= \frac{3}{8}x + \frac{1}{4}\sin 2x - \frac{1}{32}\sin 4x + c$$

$$\int \cos^3 dx = \int \cos^2 x \; \underbrace{\cos x\,dx}_{du}$$

$u = \sin x$

$du = \cos x\,dx$

$\sin^2 x + \cos^2 x = 1$

$\cos^2 x = 1 - \sin^2 x$

$$\int (1 - \sin^2 x)\cos x\,dx$$

$$= \int (1 - u^2)\,du = u - u^3/3 + c$$

$$= \sin x - \frac{\sin^3 x}{3} + c$$

$$\int \sin^5 dx = \int \underline{\sin^4 x} \cdot \underbrace{\sin x\,dx}_{du}$$

$u = \cos x$

$du = -\sin x\,dx$

$-du = \sin x\,dx$

$\sin^2 x + \cos^2 x = 1$

$\sin^4 x = (\sin^2 x)^2$

$= (1 - \cos^2 x)^2$

$$= \int (1 - \cos^2 x)^2 \sin x\,dx$$

$$= -\int (1 - u^2)^2\,du$$

$(a-b)^2 = a^2 - 2ab + b^2$

$a = 1 \quad b = u^2$

$b^2 = u^4$

$$= -\int (1 - 2u^2 + u^4)\,du$$

$$= -u + \frac{2}{3}u^3 - \frac{u^5}{5} + C$$

$$= -\cos x + \frac{2}{3}\cos^3 x - \frac{\cos^5 x}{5} + c$$

Products of sine and cosine

i) All odd

ii) All even

iii) Mixture.

I

i) $\int \sin^3 x \cos^5 x \, dx$ ✓

ii) $\int \overbrace{\sin x}^{du} \cos^3 x \, dx$

iii) $\int \underline{\cos^7 x} \, \underbrace{\sin^5 x} \, dx$

$du = \sin x \, dx$

iv) $\int \sin^5 x \overbrace{\cos x \, dx}^{du}$

iii) $\int \sin^3 x \cos^3 x \, dx$

iv) $\int \sin^5 x \cos^5 x \, dx$

II

i) $\int \sin^2 x \cos^3 x \, dx$

ii) $\int \cos^4 x \sin^3 x \, dx$

$du = \sin x \, dx$

iii) $\int \cos^2 x \sin^3 x \, dx$

$du = \sin x \, dx$

iv) $\int \sin^3 x \cos^4 x \, dx$

v) $\int \cos^6 x \, \underline{\sin^5 x \, dx}$

$du = \sin x \, dx$

vi) $\int \cos^4 x \underbrace{\sin x \, dx}_{du}$

III

i) $\int \sin^2 x \cos^2 x \, dx$

ii) $\int \cos^2 x \sin^4 x \, dx$

iii) $\int \sin^2 x \cos^4 x \, dx$

iv) $\int \sin^4 x \cos^4 x \, dx$

v) $\int \sin^6 x \cos^4 x \, dx$

vi) $\int \cos^6 x \sin^2 x \, dx$

All odds

I i) $\int \sin^3 x \cos^5 x \, dx$

$s^2 s^1 \qquad c^4 c^1$

$du = \sin x \, dx = \checkmark$

$du = \cos x \, dx$

$$\int \sin^2 x \cos^5 x \underbrace{\sin x \, dx}$$

$\sin^2 x + \cos^2 x = 1$

let $u = \cos x$ $\quad du = -\sin x \, dx$

$-du = \sin x \, dx$

$$\int (1 - \cos^2 x) \cos^5 x \sin x \, dx$$

$$\int (1 - u^2) u^5 \, du = \int -(u^5 - u^7) \, du$$

$$= -\frac{u^6}{6} + \frac{u^8}{8} + c$$

$$= -\frac{\cos^6 x}{6} + \frac{\cos^8 x}{8} + c$$

iii) $\int \sin^3 x \cos^3 x \, dx$

$$\int \sin^3 x \cos^2 x \underbrace{\cos x \, dx}_{du}$$

$u = \sin x$

$du = \cos x \, dx$

$$\int \sin^3 x (1 - \sin^2 x) \cos x \, dx$$

$$\int u^3 (1 - u^2) \, du$$

$$\int (u^3 - u^5) \, du = \ldots$$

i) $\int \sin^2 x \cos^3 x \, dx$

$$\int \sin^2 x \cos^2 x \underbrace{\cos x \, dx}_{du} \qquad u = \sin x$$

$$\int \sin^2 x \,(1 - \sin^2 x) \cos x \, dx$$

$$\int u^2 (1 - u^2)\, du = \int (u^2 - u^4)\, du$$

$$= u^3/3 - u^5/5 + C$$

$$= \frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + C$$

Even Powers

i) $\int \sin^2 x \cos^2 x \, dx$

$$= \int \tfrac{1}{2}(1 - \cos 2x)\, \tfrac{1}{2}(1 + \cos 2x)\, dx$$

$$= 1/4 \int (1 - \cos 2x)(1 + \cos 2x)\, dx$$

$$= 1/4 \int (1 + \underbrace{\cos 2x} - \underbrace{\cos 2x} - \cos^2 2x)\, dx$$

$$= 1/4 \int (1 - \cos^2 2x)\, dx$$

$$= 1/4 \int \left[1 - \left(1/2\,(1 + \cos 4x)\right)\right] dx$$

$$= \frac{1}{4} \int \left(1 - \frac{1}{2} - \frac{1}{2} \cos 4x\right) dx$$

$$= \frac{1}{4} \int \left(\frac{1}{2} - \frac{1}{2} \cos 4x\right) dx$$

$$= \frac{1}{4} \left(\frac{x}{2} - \frac{1}{2(4)} \sin 4x \right) + c$$

$$= \frac{x}{8} - \frac{1}{32} \sin 4x + c$$

ii) $\int \underline{\cos^2 x}\ \underline{\sin^4 x}\, dx$

$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$

$\cos^2 x = \frac{1}{2}(1 + \cos 2x)$

$$= \int \frac{1}{2}(1 + \cos 2x)(\sin^2 x)^2\, dx$$

$$= \int \frac{1}{2}(1 + \cos 2x)\left(\frac{1}{2}(1 - \cos 2x)\right]^2 dx$$

$$= \int \frac{1}{2}(1 + \cos 2x)\, \frac{1}{4}(1 - \cos 2x)^2\, dx$$

$(a-b)^2 = a^2 - 2ab + b^2$

$$= \frac{1}{8} \int (1 + \cos 2x)(1 - 2\cos 2x + \cos^2 2x)\, dx$$

$$= \frac{1}{8} \int \left[1 - 2\cos 2x + \cos^2 2x + \cos 2x - 2\cos^2 2x + \cos^3 2x\right] dx$$

$$= \frac{1}{8} \int \left(\left[1 - \cos 2x - \cos^2 2x\right] + \left[\cos^3 2x\right] \right) dx$$

$$= \frac{1}{8} \int \left[(1 - \cos 2x - (\tfrac{1}{2}(1 - \cos 4x)) \right] dx$$

$$+ \tfrac{1}{8} \int \cos^3 2x \, dx$$

$$\frac{1}{8} \int \left[(1 - \cos 2x - (\tfrac{1}{2}(1 - \cos 4x)) \right] dx$$

$$= \frac{1}{8} \int \left(1 - \cos 2x - \frac{1}{2} + \frac{1}{2} \cos 4x \right) dx$$

$$= \frac{1}{8} \int \left(\frac{1}{2} - \cos 2x + \tfrac{1}{2} \cos 4x \right) dx$$

$$= \frac{1}{8} \left(\tfrac{x}{2} - \frac{\sin 2x}{2} + \frac{\sin 4x}{8} \right) + c$$

$$= \frac{x}{16} - \frac{\sin 2x}{16} + \frac{\sin 4x}{64} + c \quad \text{—— (i)}$$

$$\tfrac{1}{8} \int \cos^3 2x \, dx = \frac{1}{8} \int (\cos^2 2x)(\cos 2x) \, dx$$

$$= \frac{1}{8} \int (1 - \sin^2 2x) \cos 2x \, dx$$

Let $u = \sin 2x$

$du = 2 \cos 2x \, dx$

$\frac{du}{2} = \cos 2x \, dx$

$$= \frac{1}{8} \int (1 - u^2) \frac{du}{2} = \tfrac{1}{16} \int (1 - u^2) \, du$$

$$= \frac{1}{16}\left(u - \frac{u^3}{3}\right) + c$$

$$= \frac{1}{16}\sin 2x + \frac{\sin^3 2x}{48} + c$$

Solution

$$\frac{x}{16} - \frac{\sin 2x}{16} + \frac{\sin^u 4x}{64} + c = \frac{1}{16}\sin 2x + \frac{\sin^3 2x}{48} + c$$