I.15. The area of a triangle is equal to half the area of the parallelogram defined by two sides of the triangle and the angle between them. The latter area is the absolute value of the cross product of the two vectors corresponding to the two sides. Taking all three pairs of the sides and comparing the obtained area, we get

$$\frac{1}{2}AB\sin\gamma = \frac{1}{2}AC\sin\beta = \frac{1}{2}BC\sin\alpha = \text{Area}(\Delta ABC).$$

Dividing through by $\frac{1}{2}ABC$, we obtain

$$\frac{\sin\alpha}{A} = \frac{\sin\beta}{B} = \frac{\sin\gamma}{C}.$$

I.16. Let a, b and c denote the sides of a triangle and let $\vec{\mathbf{a}}$, $\vec{\mathbf{b}}$, and $\vec{\mathbf{c}}$ denote the corresponding vectors directed so that $\vec{\mathbf{a}} = \vec{\mathbf{b}} + \vec{\mathbf{c}}$. Then

$$c^2 = \vec{\mathbf{c}}\cdot\vec{\mathbf{c}} = (\vec{\mathbf{a}} - \vec{\mathbf{b}})\cdot(\vec{\mathbf{a}} - \vec{\mathbf{b}}) = \vec{\mathbf{a}}\cdot\vec{\mathbf{a}} - 2\,\vec{\mathbf{a}}\cdot\vec{\mathbf{b}} + \vec{\mathbf{b}}\cdot\vec{\mathbf{b}} = a^2 - 2ab\cos\gamma + b^2$$

where γ is the angle between $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$.

I.25a. By Stokes' (Greene's) theorem,

$$\oint_C \vec{\mathbf{u}}\cdot d\vec{\mathbf{l}} = \int\int_S \vec{\nabla}\times\vec{\mathbf{u}}\, dA$$

where the second integral is over the area enclosed by the curve C. For this problem,

$$\vec{\nabla}\times\vec{\mathbf{u}} = \frac{\partial(-xy^2)}{\partial x} - \frac{\partial(x^2y)}{\partial y} = -y^2 - x^2.$$

In polar coordinates $x = r\cos\theta$, $y = r\sin\theta$, $dxdy = rdrd\phi$, so $x^2 + y^2 = r^2$ and the integral becomes

$$\int_0^{2\pi}\left(\int_0^1 -r^3dr\right)d\theta = -\frac{\pi}{2}.$$

I.26a. By the divergence theorem,

$$\int\int_S \vec{\mathbf{v}}\cdot d\vec{\mathbf{A}} = \int\int\int_V \vec{\nabla}\cdot\vec{\mathbf{v}}\, dV$$

where the second integral is over the volume enclosed by the surface A. Now

$$\vec{\nabla}\cdot\vec{\mathbf{v}} = \frac{\partial(x^3)}{\partial x} + \frac{\partial(3yz^2)}{\partial y} + \frac{\partial(3y^2z)}{\partial z} = 3x^2 + 3z^2 + 3y^2.$$

In spherical coordinates, $x = r\cos\theta\cos\phi$, $y = r\cos\theta\sin\phi$, $z = r\sin\theta$, $dxdydz = r^2\sin\phi drd\phi d\theta$, so $x^2 + y^2 + z^2 = r^2$ and the integral becomes

$$\int_{-\pi/2}^{\pi/2}\left(\int_0^{2\pi}\left(\int_0^2 3r^4dr\right)d\theta\right)\sin\phi d\phi = \frac{384\pi}{5}.$$

II.2. If $z = r(\cos\theta + \mathrm{i}\sin\theta)$, then, by de Moivre's formula, $z^3 = r^3(\cos(3\theta) + \mathrm{i}\sin(3\theta))$. On the other hand, computing z^3 directly, we get

$$z^3 = r^3(\cos^3\theta + 3\mathrm{i}\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - \mathrm{i}\sin^3\theta).$$

The real and the imaginary parts must match, so we get

$$\begin{aligned}\cos(3\theta) &= \cos^3\theta - 3\cos\theta\sin^2\theta,\\ \sin(3\theta) &= 3\cos^2\theta\sin\theta - \sin^3\theta.\end{aligned}$$

II.5. Since the absolute value $|\cdot|$ is the same as the distance to the origin from a point in the complex plane $\mathbb{C}$, we see that $|z-c|$ is the distance between z and a and $|z-b|$ is the distance between z and b. Therefore, the equation says that the sum of these two distances should be constant, which is the geometric definition of an ellipse with foci a and b.

The main axes are therefore determined by the line connecting a and b, whose equation is

$$\frac{a+b}{2} + (a-b)t \quad : \quad t \in \mathbb{R}$$

and by the perpendicular line through the midpoint $\frac{a+b}{2}$, i.e., by the line

$$\frac{a+b}{2} + \mathrm{i}(b-a)t \quad : \quad t \in \mathbb{R}.$$

The length of the major axis, which goes through the foci, can be determined by looking at the point of intersection of the ellipse itself and its major axis. Denote the distance between this point and the nearest focus by x. Then, from the equation of an ellipse we get $|a-b| + 2x = \alpha$, and that's exactly the length of the major axis, so the major axis is the interval

$$\{\frac{a+b}{2} + \alpha\frac{a-b}{|a-b|}t \quad : \quad t \in \left[-\frac{1}{2}, \frac{1}{2}\right]\}.$$

To determine the length of the minor axis, consider the point of intersection of the ellipse and that axis. The distance between that point and each focus is the same and therefore equal to $\alpha/2$. So we get an isosceles triangle with sides $\alpha/2$, $\alpha/2$, $|b-a|$. Its height is therefore $\frac{1}{2}\sqrt{\alpha^2 - |a-b|^2}$, so the length of the minor axis is $\sqrt{\alpha^2 - |a-b|^2}$ and its equation is

$$\{\frac{a+b}{2} + \mathrm{i}\sqrt{\alpha^2 - |a-b|^2}\frac{a-b}{|a-b|}t \quad : \quad t \in \left[-\frac{1}{2}, \frac{1}{2}\right]\}.$$

II.7a. Since $-1 = e^{\mathrm{i}\pi}$, the 5th roots of -1 are given by the formula

$$e^{\mathrm{i}(\pi + 2\pi k)/5} : \qquad k = 0, 1, 2, 3, 4.$$

b. Since $16 = 2^4$, its 4th roots are

$$2e^{\mathrm{i}\pi k/2} : \qquad k = 0, 1, 2, 3.$$