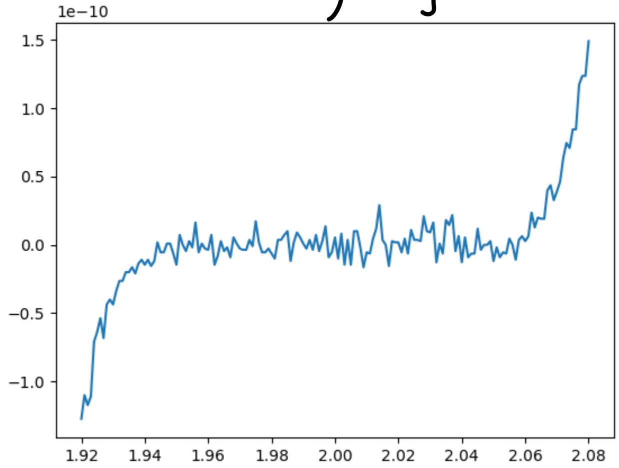
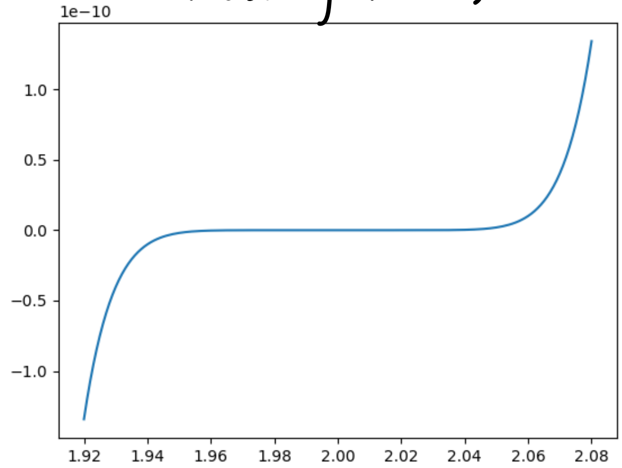
1) i. Plotting Taylor



ii. Plotting $(x-2)^9$



iii. Plot 1 very clearly shows a significant amount of noise. In this case, Plot 2 is the more accurate plot. This noise is likely due to the many subtractions from the taylor polynomial. For each term, there is a pairing term of similar magnitude that could very easily lose precision. For example: $x^9 \simeq 512$, $18x^8 \simeq 2304x$, etc.

2) i. $\sqrt{x+1}-1$ for $x \simeq 0$

do $\left(\sqrt{x+1}-1\right)\left(\dfrac{\sqrt{x+1}+1}{\sqrt{x+1}+1}\right) = \dfrac{x+\cancel{1}-\cancel{1}}{\sqrt{x+1}+1} = \boxed{\dfrac{x}{\sqrt{x+1}+1}}$

This way, there is only addition and therefore no loss of significant figures.

ii. $\sin(x)-\sin(y)$ for $x \simeq y$

assume that we know $h = x-y$ to full precision, and

$\sin(x)-\sin(y) = \boxed{2\cos\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)}$

by the sum to product identities

If we know $h = x-y$ to full precision, then this transformation allows us to avoid cancellations by subtraction.

2) iii. $\frac{1-\cos(x)}{\sin(x)}$ for $x \simeq 0$

Start w/ conjugate:

$\sin^2 x + \cos^2 x = 1$

$$\frac{1-\cos(x)}{\sin(x)} \cdot \frac{1+\cos(x)}{1+\cos(x)} = \frac{1-\cos^2(x)}{\sin(x)(1+\cos(x))} = \frac{\sin^2(x)}{\sin(x)(1+\cos(x))} = \boxed{\frac{\sin(x)}{1+\cos(x)}}$$

This algorithm would allow us to avoid cancellation for $x \approx 0$. Of course, this would create an issue at $x \simeq \pi$, but we can use our original procedure for that case.

$\rightarrow 1+\cos(\sim 0) = 1+(\sim 1) = \sim 2$ ✓

3) For degree 2, find $f'(x)$ and $f''(x)$

$$f(x) = (1+x+x^3)\cos(x) \quad , \quad f(0) = 1$$

$$f'(x) = (1+3x^2)\cos(x) - (1+x+x^3)\sin(x) \quad , \quad f'(0) = 1$$

$$f''(x) = 6x\cos(x) - (1+3x^2)\sin(x) - (1+3x^2)\sin(x) - (1+x+x^3)\cos(x) \quad , \quad f''(0) = -1$$

So, 2^{nd} order Talyor polynomial is

$$P_2(x) = 1 + x - \frac{x^2}{2}$$

a) $P_2(0.5) = 1 + \frac{1}{2} - \frac{1}{8} = \frac{11}{8} = \boxed{1.375}$

Find $\max\{f'''(x)\}$ on $[0, 0.5]$

$$= \max\{(5-3x^2)\cos(x) + (1-5x+x^3)\sin(x) - 12x\sin(x) - (2+6x^2)\cos(x)\}$$

gives a safe upper bound of $M = 4$.

Error Formula is:

$$|f(0.5) - P_2(0.5)| \leq \frac{4}{3!}(0.5)^3$$

$$\Rightarrow |f(0.5) - P_2(0.5)| \leq \frac{1}{12}$$

Actual Error is:

$$|f(0.5) - P_2(0.5)| = |1.426 - 1.375|$$

$$= 0.0511 < \frac{1}{12}$$ ✓

3) b) Rewrite $f'''(x) = (3-9x^2)\cos(x) + (1-17x+x^3)\sin(x)$

when $|1-17x+x^3| - |3-9x^2| \geq 0$

use $1-17x+x^3$ as max

otherwise $3-9x^2$

since $\sin(x)$, $\cos(x)$ have range $[-1, 1]$

assuming interval $0 < x < \infty$

So for $[0, 0.212] \cup [1.699, 10.572]$

use

$$|f(x) - P_2(x)| \leq \frac{|3-9x^2|}{3!}(x)^3$$

and for $(0.212, 1.699) \cup (10.572, \infty)$

use

$$|f(x) - P_2(x)| \leq \frac{|x^3 - 17x + 1|}{3!}x^3$$

c) $\int_0^1 P_2(x)\,dx = x + x^2/2 - x^3/6 \Big]_0^1$

$= (1 + \frac{1}{2} - \frac{1}{6}) - (0) = \boxed{\frac{8}{6}}$

d) since $|f(x) - P_2(x)| \leq \frac{|x^3 - 17x + 1|}{3!}x^3$

for $x = 1$, integrate

$$\int_0^1 \frac{|x^6 - 17x + 1|}{3!}\,dx = \frac{1}{6}\left(\left|\frac{1}{7}x^7 - \frac{17}{2}x^2 + x\right|\right)\Big]_0^1$$

$$= \frac{1}{6}\left(\left|\frac{1}{7} - \frac{17}{2} + 1\right|\right) = \frac{103}{84}$$

so $\int_0^1 |f(x) - P_2(x)|\,dx \leq \boxed{\frac{103}{84}}$

4) a) we have $x^2 - 56x + 1 = 0$

set up $\frac{56 \pm \sqrt{56^2 - 4}}{2} = \frac{56 \pm 55.964}{2}$

we get $x_1 = 55.982$

$x_2 = 0.018$

error is: $\frac{|x - x_n|}{|x|}$

for x_1: $\frac{|55.9821... - 55.9821|}{|55.9821...|} = \boxed{2.45 \cdot 10^{-6}}$

for x_2: $\frac{|0.017862... - 0.0181|}{|0.017862...|} = \boxed{7.68 \cdot 10^{-3}}$

b) convert $ax^2 + bx + c = 0$

to $a(x^2 + \frac{b}{a}x + \frac{c}{a}) = 0$

so $x^2 + \frac{b}{a}x + \frac{c}{a} = 0$

and $(x - r_1)(x - r_2) = 0$

$\Rightarrow x^2 - (r_1 + r_2)x + r_1 r_2 = 0$

$x^2 + \frac{b}{a}x + \frac{c}{a}$

$\Rightarrow \frac{b}{a} = -r_1 - r_2, \quad \frac{c}{a} = r_1 r_2$

to calculate the "bad" root, use the "good" root:

$r_1 = \frac{c}{r_2 a}$

$\Rightarrow \frac{1}{55.982} = 0.017862...$

so now error is $\frac{|0.01786... - 0.01786...|}{|0.017861|}$

about same as first root $\longrightarrow = \boxed{2.49 \cdot 10^{-6}}$

5) a) $|\Delta y| = |\Delta x_1 - \Delta x_2|$

$$= |\tilde{x}_1 - x_1 - (\tilde{x}_2 - x_2)|$$

$$= |(\tilde{x}_1 - \tilde{x}_2) - (x_1 - x_2)|$$

$$\frac{|\Delta y|}{|y|} = \frac{|\Delta x_1 - \Delta x_2|}{|\hat{y} - (\Delta x_1 - \Delta x_2)|}$$

Therefore, the relative error will be large when $\hat{y} \approx \Delta x_1 - \Delta x_2 = \Delta y$

b. $\cos(x + \delta) - \cos(x)$

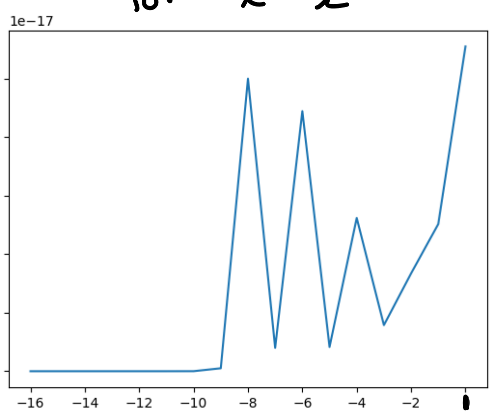
$$= -2 \sin\left(\frac{2x + \delta}{2}\right) \sin\left(\frac{\delta}{2}\right)$$

by the sum to product identities

...plot in python...

error for $x = \pi$

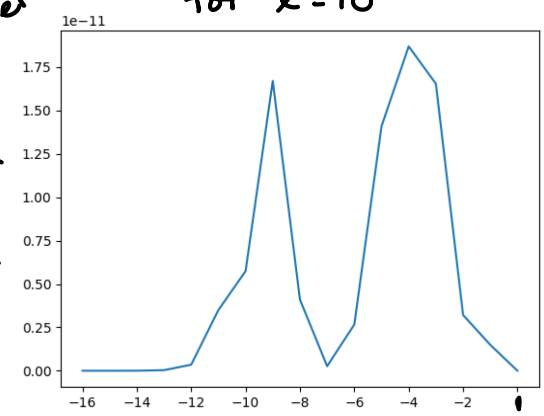
(10^{-17})



$\log(\delta)$

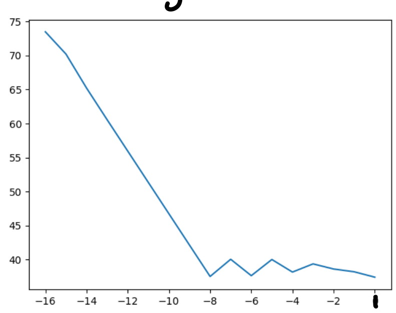
error for $x = 10^6$

(10^{-11})

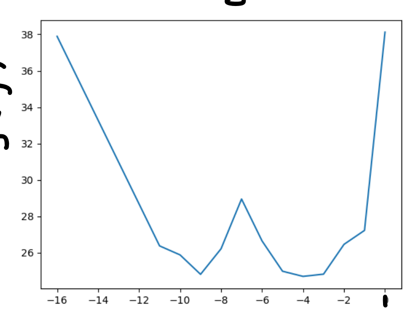


$\log(\delta)$

$\log(y)$



$\log(y)$



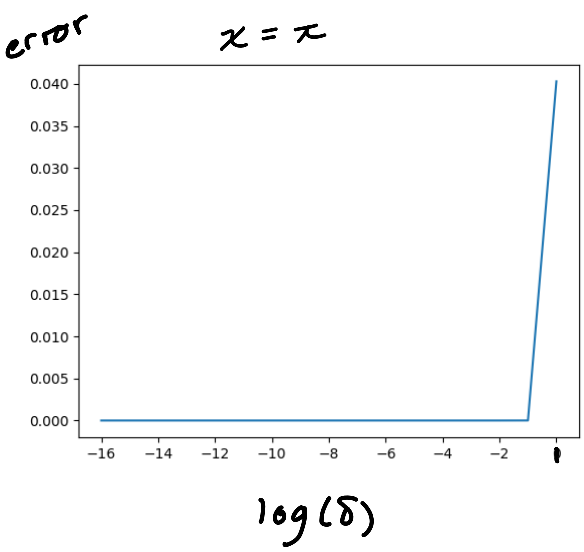
5) c. Since $f(x+\delta) - f(x)$

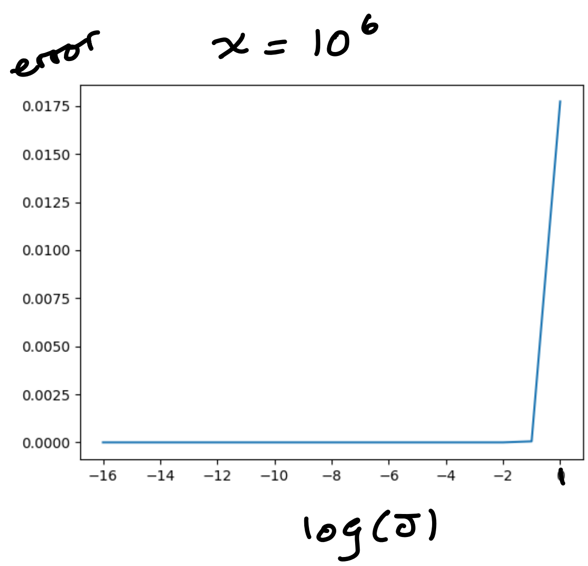
$$= \delta f'(x) + \frac{\delta^2}{2} f''(\xi)$$

$$= -\delta \sin(x) - \frac{\delta^2}{2}\cos(\xi)$$

$$= -\delta\left(\sin(x) + \frac{\delta}{2}\cos(x)\right)$$

This algorithm will avoid 50% of opposite-sign scenarios, unfortunately, in QII, QIV of the unit circle, sin and cos will have opposing signs and encounter some cancellation, so we should expect a worse error than in (b). Shown below is the difference between the manipulated expression in part b.





What I neglected before, is that at $x = 0$, we lose the sin term, causing our error to be substantially larger at this point. As such, the manipulated equation $-2\sin\left(\frac{2x+\delta}{2}\right)\sin\left(\frac{\delta}{2}\right)$ from part b forms a better algorithm.