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Probability Theory
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Book 1. Lecture Notes

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Contents

Chapter 1 Set Theory

1.1 Set algebra

1.1.1 Terminology

A set is a collection of distinct *elements*.

- If a is an element of the set A, we write this as $a \in A$.
- If a is *not* an element of A, we write this as $a \notin A$.
- The *cardinality* of a set is the number of elements it contains.
- The *empty set* contains no elements, and is denoted by $\emptyset$.

Algebra is the study of *relations* and *operations*.

- The basic relations of set algebra are *set inclusion* and *set equality*.
- The basic operations of set algebra are *complementation*, *union* and *intersection*.

1.1.2 Set relations

Definition 1.1
Let A and B be sets.

(1) If every element of A is also an element of B, we say that A is a *subset* of B.
This is denoted by $A \subseteq B$.

(2) If every element of A is an element of B, and every element of B is an element of A, we say that A and B are *equal*.
This is denoted by $A = B$.

(3) If A is a subset of B, but A is not equal to B, we say that A is a *proper subset* of B.
This is denoted by $A \subset B$.

Example 1.2
Let $A = \{a, b\}$, $B = \{a, b\}$ and $C = \{a, b, c\}$.

- A is a subset of B: $A \subseteq B$,
- A is also equal to B: $A = B$, and
- A is a proper subset of C: $A \subset C$.

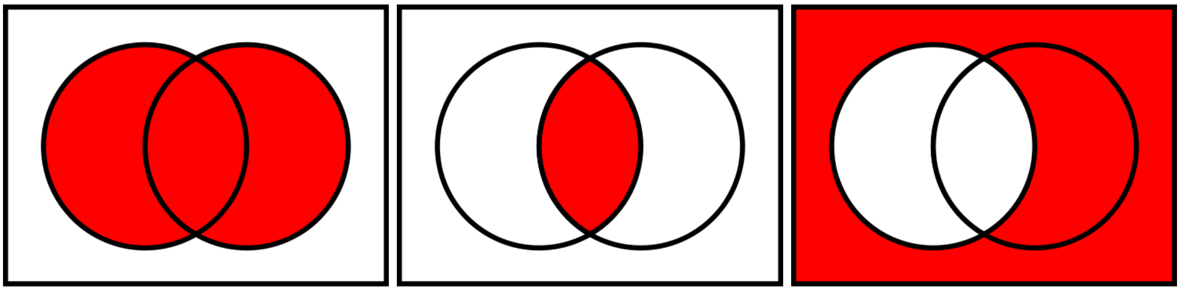
1.1.3 Set operations

Definition 1.3
Let A, B and Ω be sets, with $A, B \subseteq \Omega$.

(1) The *union* of A and B is the set

$$A \cup B = \{a \in \Omega : a \in A \text{ or } a \in B\}.$$

Figure 1.1: The basic set operations.



(a) Union (b) Intersection (c) Complement

Table 1.1: Correspondence with logical operators.

Set Theory		Logic		
Union	$A \cup B$	Disjunction	OR	$\vee$
Intersection	$A \cap B$	Conjunction	AND	$\wedge$
Complement	A^c	Negation	NOT	$\neg$

(2) The *intersection* of A and B is the set

$$A \cap B = \{a \in \Omega : a \in A \text{ and } a \in B\}.$$

(3) The *complement* of A is the set

$$A^c = \{a \in \Omega : a \notin A\}.$$

Example 1.4
Let $\Omega = \{a, b, c, d\}$, $A = \{a, b\}$ and $B = \{b, c\}$. Find $A \cup B$, $A \cap B$ and A^c.

Solution

- $A \cup B = \{a, b, c\}$ is the set of elements in A or B (or both).
- $A \cap B = \{b\}$ is the set of elements in A and B.
- $A^c = \{c, d\}$ is the set of elements not in A.

1.1.4 Set algebra

Definition 1.5

(1) Commutative property.

- $A \cup B = B \cup A$,
- $A \cap B = B \cap A$.

(2) Associative property.

- $(A \cup B) \cup C = A \cup (B \cup C)$,
- $(A \cap B) \cap C = A \cap (B \cap C)$.

(3) Distributive property.

- $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$,
- $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Remark 1.6
A statement such as $A \cup B \cap C$ is ambiguous.

1.2 De Morgan's laws

Union and intersection swap roles under complementation.

Theorem 1.7

(1) $(A \cup B)^c = A^c \cap B^c$.

(2) $(A \cap B)^c = A^c \cup B^c$.

Proof

(1) Let $a \in (A \cup B)^c$. Then $a \notin A$ and $a \notin B$, so $a \in A^c \cap B^c$. Hence $(A \cup B)^c \subseteq A^c \cap B^c$.
Let $a \in A^c \cap B^c$. Then $a \notin A$ and $a \notin B$, so $a \notin A \cup B$. Hence $A^c \cap B^c \subseteq (A \cup B)^c$.
Thus it follows that $(A \cup B)^c = A^c \cap B^c$.

(2) Apply part (1) to the sets A^c and B^c: $(A^c \cup B^c)^c = A \cap B$.
Then take the complement of both sides: $(A \cap B)^c = A^c \cup B^c$.

1.3 Set difference

Definition 1.8
Let A, B and Ω be sets, with $A, B \subseteq \Omega$.

(1) The *set difference* between A and B is the set

$$A \setminus B = \{a \in \Omega : a \in A \text{ and } a \notin B\}.$$

(2) The *symmetric difference* between A and B is the set

$$A \triangle B = (A \setminus B) \cup (B \setminus A).$$

- $A \setminus B$ is the set of points that are in A but not in B.
- $A \triangle B$ is the set of points that are in either A or B, but not both.

Example 1.9
Let $A = \{a, b\}$ and $B = \{b, c\}$. Then

- $A \setminus B = \{a\}$
- $A \triangle B = \{a, c\}$.

1.4 Assignments

Homework 1.1

1. Illustrate the basic set operations using Venn diagrams.
2. State and prove De Morgan's laws.

Chapter 2 Events

2.1 A brief history of probability

Games of chance have been played since antiquity, but the mathematical principles of chance and uncertainty were first established only in the 17th century:

1654	Classical principles	Blaise Pascal (1623–1662)
		Pierre de Fermat (1601–1665)
1657	*De Ratiociniis in Ludo Aleae*	Christiaan Huygens (1629–1695)
1713	*Ars Conjectandi*	Jakob Bernoulli (1654–1705)
1718	*The Doctrine of Chances*	Abraham de Moivre (1667–1754)
1812	*Theorie Analytique des Probabilites*	Pierre de Laplace (1749-1827)
1919	Relative frequency	Richard von Mises (1883–1953)
1933	Modern axiomatic theory	Andrey Kolmogorov (1903–1987)

2.2 Sample spaces

Definition 2.1

(1) Any process of observation or measurement will be called an *experiment* or *trial*.

(2) Any experiment whose outcome is uncertain is called a *random experiment*.

(3) A random experiment has a set of possible *outcomes*.

(4) Each time a random experiment is performed, *exactly one* of its outcomes will occur.

(5) The set of all possible outcomes is called the *sample space* of the experiment, denoted by Ω.

(6) Outcomes are also called *elementary events*, and denoted by $\omega \in \Omega$.

Example 2.2
The sample space of a random experiment is the set of all possible outcomes:

Experiment	Sample space
A coin is tossed once.	$\Omega = \{H, T\}$
A six-sided die is rolled once.	$\Omega = \{1, 2, 3, 4, 5, 6\}$
A coin is tossed repeatedly until a head occurs.	$\Omega = \{1, 2, 3, \ldots\}$
The height of a randomly chosen student is measured:	$\Omega = [0, \infty)$

Exercise 2.3
Think of a situation in which randomness occurs. Can you describe the set of possible outcomes? Can you write it down using mathematical notation?

2.3 Events

Definition 2.4

- An *event* A is a subset of the sample space, Ω.

- If outcome ω occurs, we say that event A *occurs* if and only if $\omega \in A$.
- Two events A and B with $A \cap B = \emptyset$ are called *disjoint* or *mutually exclusive*.
- The empty set $\emptyset$ is called the *impossible event*.
- The sample space Ω is called the *certain event*.

Remark 2.5

- If A occurs and $A \subseteq B$, then B must also occur.
- If A occurs and $A \cap B = \emptyset$, then B does not occur.

Example 2.6
A die is rolled once. The sample space can be represented by $\Omega = \{1, 2, 3, 4, 5, 6\}$. We may be interested in whether or not the following events occur:

Event	Subset
The outcome is the number 1.	$\{1\}$
The outcome is an even number.	$\{2, 4, 6\}$
The outcome is even but does not exceed 3.	$\{2, 4, 6\} \cap \{1, 2, 3\}$
The outcome is not even	$\Omega \setminus \{2, 4, 6\}$

2.4 Families of events

Definition 2.7
Let Ω be any set.

(1) The set of all subsets of Ω is called the *power set* of Ω, which we denote by 2^{Ω}.

(2) An arbitrary set $\mathcal{F}$ of subsets of Ω is called a *family of sets over* Ω.

Let Ω be the sample space of some random experiment. If we are interested in events A and B, we must also be interested in the following.

- Event A *or* event B occurs: this is the event $A \cup B$,
- Event A *and* event B occur: this is the event $A \cap B$,
- Event A does *not* occur: this is the event A^c.

As the basis for investigating random experiments, so that we can consider all events that may be of interest, we must allow only families of sets over Ω that are *closed* under certain set operations.

Definition 2.8
A family of sets $\mathcal{F}$ over Ω is said to be

(1) *closed under complementation* if $A^c \in \mathcal{F}$ for every $A \in \mathcal{F}$,

(2) *closed under pairwise unions* if $A \cup B \in \mathcal{F}$ for every $A, B \in \mathcal{F}$,

Definition 2.9
A family of sets $\mathcal{F}$ over Ω is called a *field of sets* over Ω if

(1) $\Omega \in \mathcal{F}$,

(2) $\mathcal{F}$ is closed under complementation, and

(3) $\mathcal{F}$ is closed under pairwise unions.

Table 2.1: Table of correspondence (Grimmett & Stirzaker 2001).

Notation	Set theory	Probability theory
Ω	Universal set	Sample space
$\omega \in \Omega$	Element of Ω	Elementary event, outcome
$A \subseteq \Omega$	Subset of Ω	Event A
$A \subseteq B$	Inclusion	If A occurs, then B occurs
$A \cup B$	Union	A or B occurs
$A \cap B$	Intersection	A and B occur
A^c	Complement of A	A does not occur
$A \setminus B$	Difference	A occurs, but B does not
$A \triangle B$	Symmetric difference	A or B occurs, but not both
$\emptyset$	Empty set	Impossible event
Ω	Universal set	Certain event

Example 2.10
A six-sided die is rolled once, and the score is observed. A suitable sample space for this experiment is the set $\Omega = \{1, 2, 3, 4, 5, 6\}$. The power set of Ω will always provide a field of sets to work with. However, suppose we are only interested in whether or not the outcome is an even number. In this case, we need only consider the following family of events:

$$\mathcal{F} = \big\{\emptyset, \{1, 3, 5\}, \{2, 4, 6\}, \{1, 2, 3, 4, 5, 6\}\big\}.$$

We can see that $\mathcal{F}$ is a field of sets over Ω, because

(1) it contains the sample space $\{1, 2, 3, 4, 5, 6\}$,

(2) the complement of every set in $\mathcal{F}$ is also contained in $\mathcal{F}$, and

(3) the union of any two sets in $\mathcal{F}$ is also contained in $\mathcal{F}$.

Theorem 2.11 (Properties of fields)
Let $\mathcal{F}$ be a field over Ω. Then

(1) $\emptyset \in \mathcal{F}$,

(2) $\mathcal{F}$ is closed under pairwise intersections,

(3) $\mathcal{F}$ is closed under set differences.

Proof

(1) We know that $\emptyset = \Omega^c$, and that $\Omega \in \mathcal{F}$. Because $\mathcal{F}$ is closed under complementation, it thus follows that $\emptyset \in \mathcal{F}$.

(2) Let $A, B \in \mathcal{F}$. By De Morgan's laws, we have that $A \cap B = (A^c \cup B^c)^c$. Because $\mathcal{F}$ is closed under complementation and pairwise unions, it thus follows that $A \cap B \in \mathcal{F}$.

(3) Let $A, B \in \mathcal{F}$. Set difference can be written as $A \setminus B = A \cap B^c$. Furthermore, by De Morgan's laws we see that $A \cap B^c = (A^c \cup B)^c$. Because $\mathcal{F}$ is closed under complementation and pairwise unions, it thus follows that $A \setminus B \in \mathcal{F}$.

2.5 Assignments

Homework 2.1

1. Identify a sample space, and the subset corresponding to event A, in each of the following scenarios:
   (a) A coin is tossed three times. A is the event that at least two heads are obtained.

Answer:
$\Omega = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$ and $A = \{HHH, HHT, HTH, THH\}$.
Alternatively, if we are only interested in the number of heads, we could take $\Omega = \{0, 1, 2, 3\}$ and $A = \{2, 3\}$.

(b) A game of football is played. A is the event that the match ends in a draw.

Answer: $\Omega = \{(a, b) : a, b = 0, 1, 2, \ldots\}$ and $A = \{(a, b) : a = b\}$ where a and b are the numbers of goals scored by the first and second teams, respectively. Note that this is a (countably) infinite set.
Alternatively, we could take $\Omega = \{W, D, L\}$ and $A = \{D\}$ where W, D, L are respectively the events that the first team wins, draws or loses the game.

(c) A couple have two children. A is the event that both are girls.

Answer: $\Omega = \{GG, GB, BG, BB\}$ and $A = \{GG\}$.
Alternatively, we could take $\Omega = \{0, 1, 2\}$ and $A = \{2\}$.

(d) A shot hits a circular target of radius 10cm. A is the event that the shot hits within 3cm of the centre.

Answer: $\Omega = \{(x, y) : x^2 + y^2 \leq 10^2\}$ and $A = \{(x, y) : x^2 + y^2 \leq 3^2\}$.

2. A family of sets $\mathcal{F}$ over Ω is said to be

- *closed under finite unions* if $A_1 \cup A_2 \cup \ldots \cup A_n \in \mathcal{F}$ whenever $A_1, A_2, \ldots A_n \in \mathcal{F}$, and
- *closed under finite intersections* if $A_1 \cap A_2 \cap \ldots \cap A_n \in \mathcal{F}$ whenever $A_1, A_2, \ldots A_n \in \mathcal{F}$.

If $\mathcal{F}$ is a field of sets over Ω, show that $\mathcal{F}$ is closed under finite unions and finite intersections.

Answer:

- Proof by induction. Suppose that $\mathcal{F}$ is closed under unions of n sets (where $n \geq 2$). Let $A_1, A_2, \ldots, A_{n+1} \in \mathcal{F}$. By the inductive hypothesis, $\cup_{i=1}^{n} A_i \in \mathcal{F}$. Thus $\cup_{i=1}^{n+1} A_i = \big[\cup_{i=1}^{n} A_i\big] \cup A_{n+1} \in \mathcal{F}$, because $\mathcal{F}$ is closed under pairwise unions.
- Let $A_1, A_2, \ldots, A_n \in \mathcal{F}$. Then $\cap_{i=1}^{n} A_i = \big[\cup_{i=1}^{n} A_i^c\big]^c$ (De Morgan's laws). Hence $\cap_{i=1}^{n} A_i \in \mathcal{F}$ because $\mathcal{F}$ is closed under complementation and finite unions.

Chapter 3 Probability

3.1 Probability measures

Probability is defined to be a *function* that assigns numerical value to random events.

Definition 3.1
Let Ω be the sample space of some random experiment, and let $\mathcal{F}$ be a field of sets over Ω. A *probability measure* on $(\Omega, \mathcal{F})$ is a function

$$\begin{aligned} \mathbb{P}: \quad \mathcal{F} &\to [0,1] \\ A &\mapsto \mathbb{P}(A) \end{aligned}$$

such that $\mathbb{P}(\Omega) = 1$, and for any countable collection of pairwise disjoint events $\{A_1, A_2, \ldots\}$,

$$\mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mathbb{P}(A_i).$$

The triple $(\Omega, \mathcal{F}, \mathbb{P})$ is called a *probability space.*

Remark 3.2

- The second property is called *countable additivity.*
- The number $\mathbb{P}(A)$ is called the *probability* of event $A \in \mathcal{F}$.

Example 3.3
Consider a random experiment in which a fair six-sided die is rolled once.

- A suitable sample space for the experiment is $\Omega = \{1, 2, 3, 4, 5, 6\}$.
- A suitable field of events for the experiment is the power set, $\mathcal{F} = \mathcal{P}(\Omega)$.
- Because the die is fair, a suitable probability measure is given by the function

$$\begin{aligned} \mathbb{P}: \mathcal{F} &\to [0,1] \\ A &\mapsto \tfrac{1}{6}|A|, \qquad \text{where } |A| \text{ denotes the cardinality of } A. \end{aligned}$$

Event	Subset	Probability
The outcome is the number 1.	$\{1\}$	$\mathbb{P}(A) = 1/6$
The outcome is an even number.	$\{2, 4, 6\}$	$\mathbb{P}(A) = 3/6$
The outcome is even but does not exceed 3.	$A = \{2, 4, 6\} \cap \{1, 2, 3\}$	$\mathbb{P}(A) = 1/6$
The outcome is not even	$A = \Omega \setminus \{2, 4, 6\}$	$\mathbb{P}(A) = 3/6$

Example 3.4
A fair six-sided die is rolled once. If we are only interested in whether the outcome is an odd or even number, we can take

- Sample space: $\Omega = \{1, 2, 3, 4, 5, 6\}$,
- Events: $\mathcal{F} = \big\{\emptyset, \{1, 3, 5\}, \{2, 4, 6\}, \{1, 2, 3, 4, 5, 6\}\big\}$
- Probability measure: $\mathbb{P}(\emptyset) = 0$, $\mathbb{P}(\{1, 3, 5\}) = 1/2$, $\mathbb{P}(\{2, 4, 6\}) = 1/2$, $\mathbb{P}(\{1, 2, 3, 4, 5, 6\}) = 1$.

3.2 Properties of probability measures

Theorem 3.5 (Properties of probability measures)
Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and let $A, B \in \mathcal{F}$.

(1) Complementarity: $\mathbb{P}(A^c) = 1 - \mathbb{P}(A)$.

(2) $\mathbb{P}(\emptyset) = 0$,

(3) Monotonicity: if $A \subseteq B$ then $\mathbb{P}(A) \leq \mathbb{P}(B)$.

(4) Addition rule: $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$.

Proof

(1) Since $A \cup A^c = \Omega$ is a disjoint union and $\mathbb{P}(\Omega) = 1$, it follows by additivity that

$$1 = \mathbb{P}(\Omega) = \mathbb{P}(A \cup A^c) = \mathbb{P}(A) + \mathbb{P}(A^c).$$

(2) Since $\emptyset = \Omega^c$ and $\mathbb{P}(\Omega) = 1$, it follows by complemenarity that

$$\mathbb{P}(\emptyset) = \mathbb{P}(\Omega^c) = 1 - \mathbb{P}(\Omega) = 1 - 1 = 0.$$

(3) Let $A \subseteq B$ and let us write $B = A \cup (B \setminus A)$.
Since A and $B \setminus A$ are disjoint sets, it follows by additivity that

$$\mathbb{P}(B) = \mathbb{P}\big[A \cup (B \setminus A)\big] = \mathbb{P}(A) + \mathbb{P}(B \setminus A).$$

Hence, because $\mathbb{P}(B \setminus A) \geq 0$, it follows that $\mathbb{P}(B) \geq \mathbb{P}(A)$.

(4) Let us write:

- $A \cup B = (A \setminus B) \cup (B \setminus A) \cup (A \cap B)$
- $A = (A \setminus B) + (A \cap B)$
- $B = (B \setminus A) + (A \cap B)$

These are disjoint unions, so by additivity,

- $\mathbb{P}(A \cup B) = \mathbb{P}(A \setminus B) + \mathbb{P}(B \setminus A) + \mathbb{P}(A \cap B)$
- $\mathbb{P}(A) = \mathbb{P}(A \setminus B) + \mathbb{P}(A \cap B)$
- $\mathbb{P}(B) = \mathbb{P}(B \setminus A) + \mathbb{P}(A \cap B)$

Hence $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$, as required.

3.3 Exercises

Exercise 3.6

1. What does it mean to say that $\mathbb{P}$ is a probability measure over $(\Omega, \mathcal{F})$?

   Answer: Bookwork. The symbols Ω (sample space) and $\mathcal{F}$ (field of events) should be defined before giving the definition of $\mathbb{P}$.

2. Show that $\mathbb{P}(A \cup B) \leq \mathbb{P}(A) + \mathbb{P}(B)$ for any two events A and B.

Answer: First we express A, B and $A \cup B$ as disjoint unions:

$$A = (A \cap B^c) \cup (A \cap B)$$
$$B = (B \cap A^c) \cup (A \cap B)$$
$$A \cup B = (A \cap B^c) \cup (A \cap B) \cup (B \cap A^c)$$

By the additivity property of probability measures,

$$\mathbb{P}(A) = \mathbb{P}(A \cap B^c) + \mathbb{P}(A \cap B)$$
$$\mathbb{P}(B) = \mathbb{P}(B \cap A^c) + \mathbb{P}(A \cap B)$$
$$\mathbb{P}(A \cup B) = \mathbb{P}(A \cap B^c) + \mathbb{P}(A \cap B) + \mathbb{P}(B \cap A^c)$$

From here, it follows that $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$, and because $\mathbb{P}(A \cap B) \geq 0$ for any two events $A, B\mathcal{F}$, we see that $\mathbb{P}(A \cup B) \leq \mathbb{P}(A) + \mathbb{P}(B)$, as required.

3. Let A and B be events such that $\mathbb{P}(A) = 0.4$, $\mathbb{P}(B) = 0.5$ and $\mathbb{P}(A \ \cup \ B) = 0.8$.

   Compute the following probabilities:

   (a) $\mathbb{P}(A \cap B)$.

   Answer: $\mathbb{P}(A \cap B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cup B) = 0.4 + 0.5 - 0.8 = 0.1$.

   (b) $\mathbb{P}(A \cup B^c)$.

   Answer: $\mathbb{P}(A \cup B^c) = 1 - \mathbb{P}(B \setminus A) = 1 - \big[\mathbb{P}(B) - \mathbb{P}(A \cap B)\big] = 1 - 0.4 = 0.6$.

3.4 Assignments

Test 3.1

1. Who is the odd one out?

   A. John

   B. Paul

   C. George

   D. Bingo

Test 3.2

1. Which of the following statements are correct?

   ✓ $1 + 1 = 2$

   ☐ $1 + 1 = 3$

   ☐ $2 + 2 = 3$

   ✓ $2 + 2 = 4$

Homework 3.1

1. Let A and B be random events, with probabilities $\mathbb{P}(A) = 1/2$ and $\mathbb{P}(B) = 3/4$.

   (a) Show that $\dfrac{1}{4} \leq \mathbb{P}(A \cap B) \leq \dfrac{1}{2}$.

Answer: $A \cap B \subseteq A$ and $A \cap B \subseteq B$ means that:

$$\mathbb{P}(A \cap B) \leq \min\left\{\mathbb{P}(A), \mathbb{P}(B)\right\} = \frac{1}{2}.$$

Furthermore, $\mathbb{P}(A \cup B) \leq 1$ means that:

$$\mathbb{P}(A \cap B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cup B) \geq \frac{1}{4}.$$

(b) Show that $\frac{3}{4} \leq \mathbb{P}(A \cup B) \leq 1$.

Answer: $A \subseteq A \cup B$ and $B \subseteq A \cup B$ means that:

$$\mathbb{P}(A \cup B) \geq \max\left[\mathbb{P}(A), \mathbb{P}(B)\right] = \frac{3}{4}.$$

Furthermore, $\mathbb{P}(A \cup B) \leq 1$ means that:

$$\mathbb{P}(A \cup B) \leq \min\{1, \mathbb{P}(A) + \mathbb{P}(B)\} = 1.$$