Revisionist Integral Deferred Correction and the Pseudo-Spectral Method

Bradley Gadd-Speakman

2023

Durham University

Introduction

The goal of this presentation is to be able to demonstrate how revisionist integral deferred correction (RIDC) and spectral methods can be combined in order to numerically solve a PDE with a very high order of accuracy, in both time and space, within an optimised amount of computation time via parallelism.

To demonstrate, we want to solve the following:

The goal of this presentation is to be able to demonstrate how revisionist integral deferred correction (RIDC) and spectral methods can be combined in order to numerically solve a PDE with a very high order of accuracy, in both time and space, within an optimised amount of computation time via parallelism.

To demonstrate, we want to solve the following:

2D Incompressible Vorticity Equation

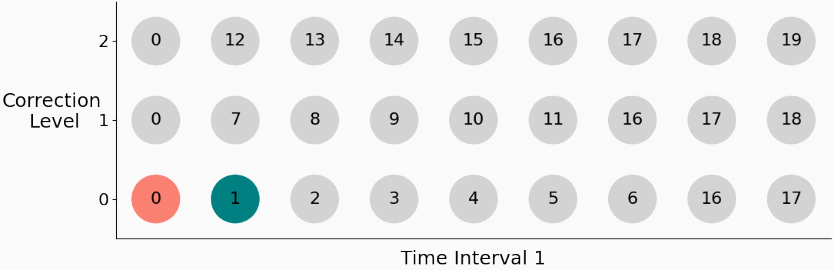
$$\frac{\partial w}{\partial t} = \nu \nabla^2 w - (\boldsymbol{u} \cdot \nabla) w, \quad -\nabla^2 \boldsymbol{u} = \nabla \times \boldsymbol{w}$$

$$\boldsymbol{u}(x, y, t) = \boldsymbol{u}(x + P, y + P, t), \quad \boldsymbol{u}(x, y, t = 0) = \boldsymbol{u}_0.$$

Section 1
RIDC

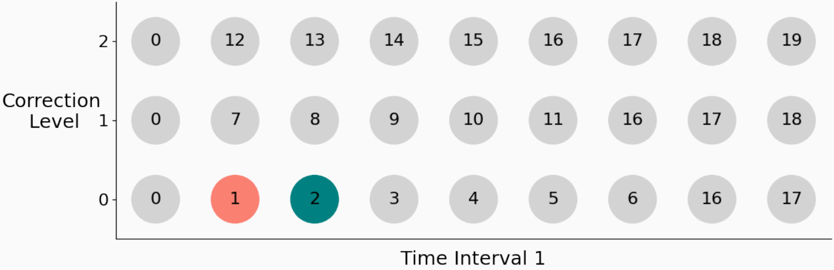
RIDC Recap

RIDC is a complex time integration method which can be nicely visualised using stencil diagrams.



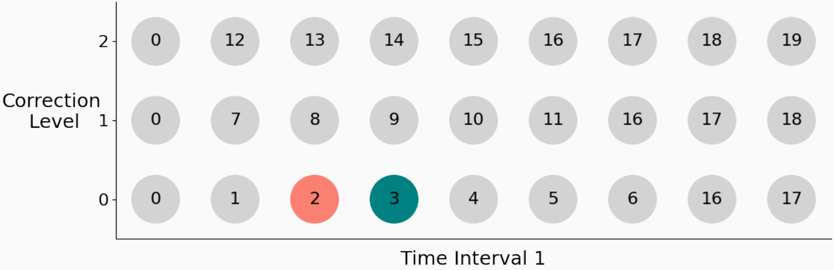
RIDC Recap

RIDC is a complex time integration method which can be nicely visualised using stencil diagrams.



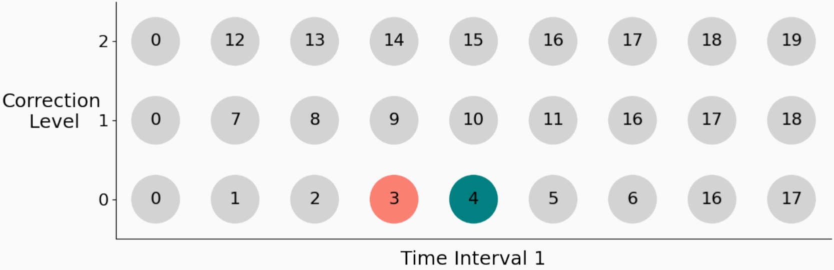
RIDC Recap

RIDC is a complex time integration method which can be nicely visualised using stencil diagrams.



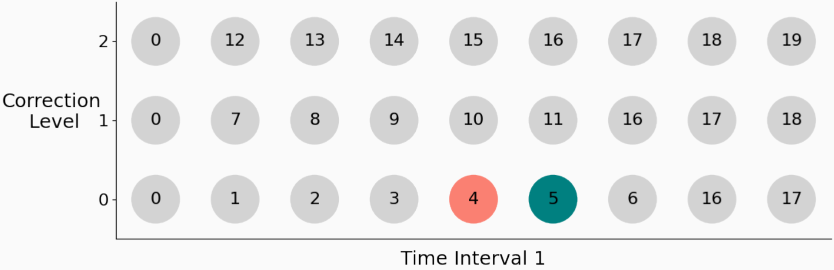
RIDC Recap

RIDC is a complex time integration method which can be nicely visualised using stencil diagrams.



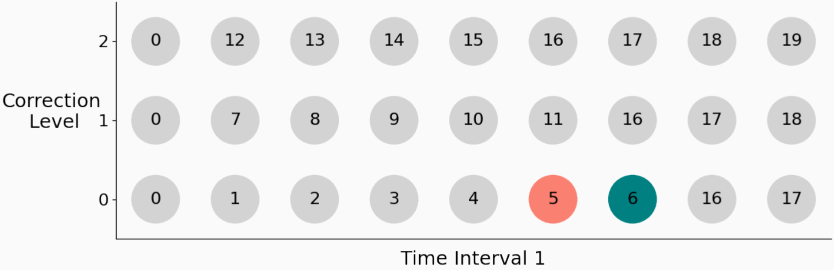
RIDC Recap

RIDC is a complex time integration method which can be nicely visualised using stencil diagrams.



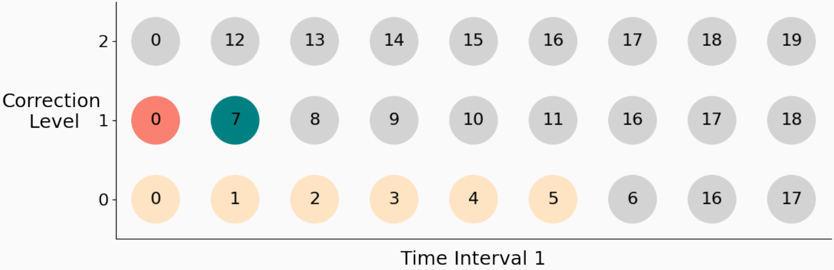
RIDC Recap

RIDC is a complex time integration method which can be nicely visualised using stencil diagrams.



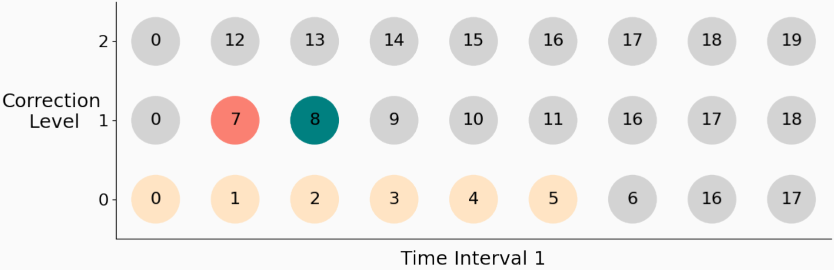
RIDC Recap

RIDC is a complex time integration method which can be nicely visualised using stencil diagrams.



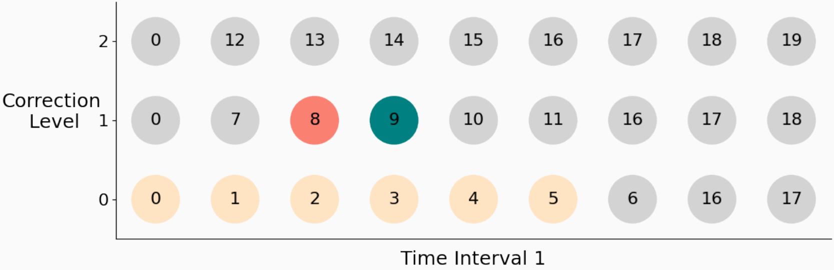
RIDC Recap

RIDC is a complex time integration method which can be nicely visualised using stencil diagrams.



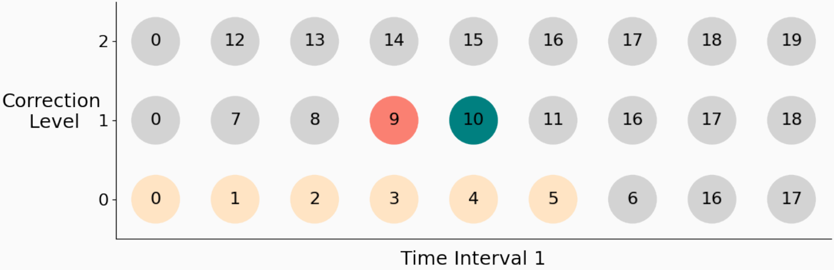
RIDC Recap

RIDC is a complex time integration method which can be nicely visualised using stencil diagrams.



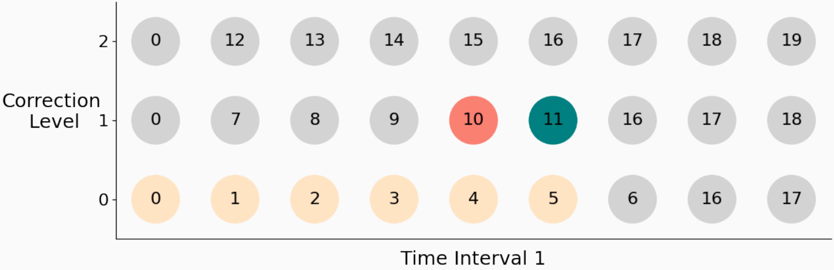
RIDC Recap

RIDC is a complex time integration method which can be nicely visualised using stencil diagrams.



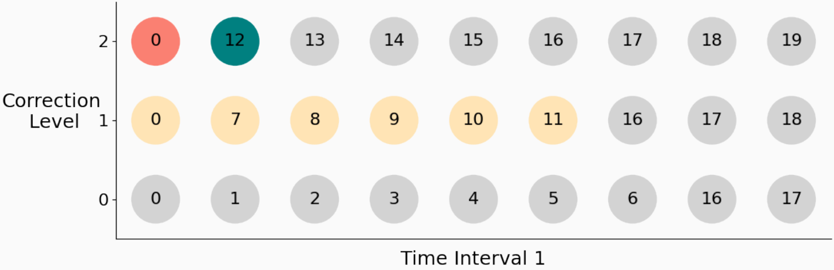
RIDC Recap

RIDC is a complex time integration method which can be nicely visualised using stencil diagrams.



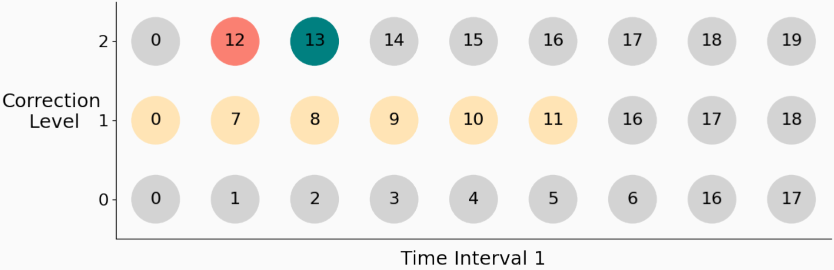
RIDC Recap

RIDC is a complex time integration method which can be nicely visualised using stencil diagrams.



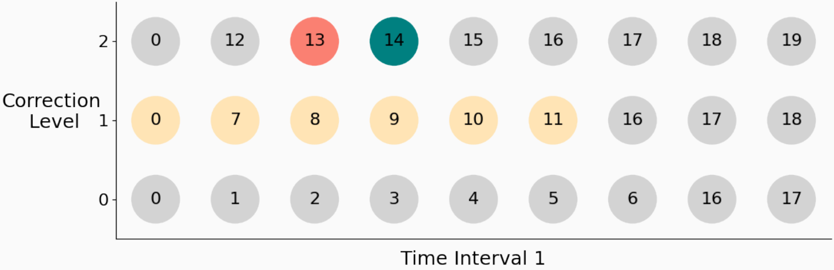
RIDC Recap

RIDC is a complex time integration method which can be nicely visualised using stencil diagrams.



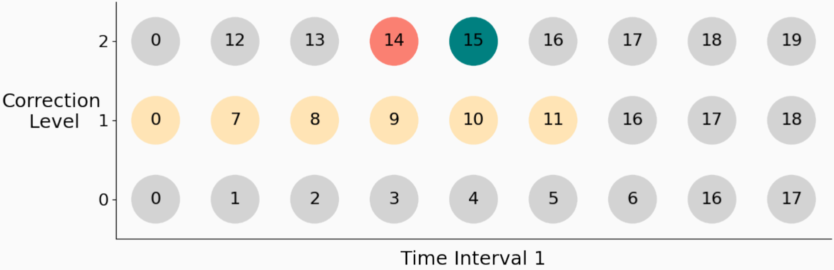
RIDC Recap

RIDC is a complex time integration method which can be nicely visualised using stencil diagrams.



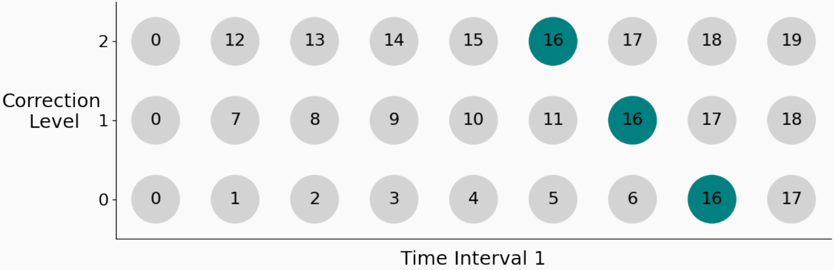
RIDC Recap

RIDC is a complex time integration method which can be nicely visualised using stencil diagrams.



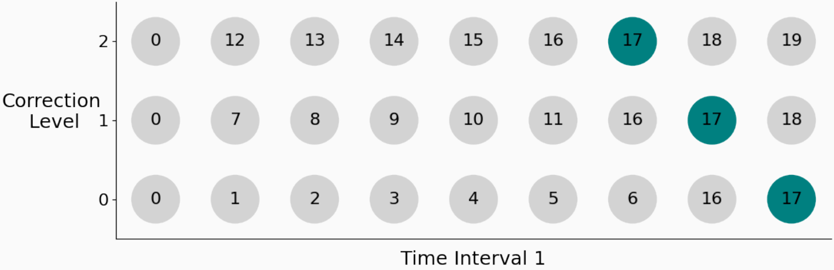
RIDC Recap

RIDC is a complex time integration method which can be nicely visualised using stencil diagrams.



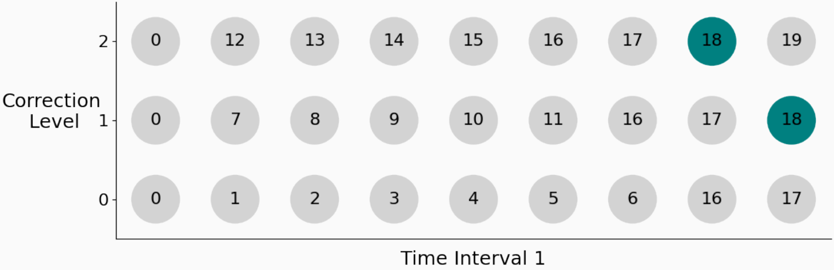
RIDC Recap

RIDC is a complex time integration method which can be nicely visualised using stencil diagrams.



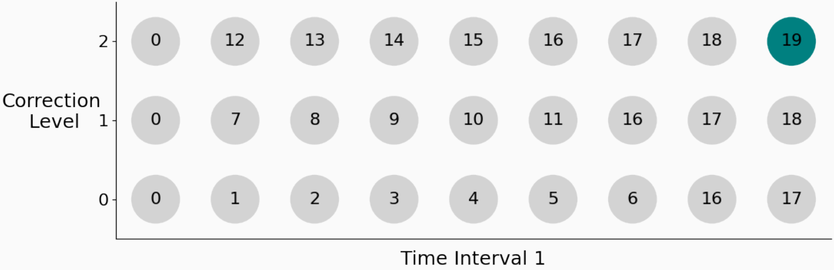
RIDC Recap

RIDC is a complex time integration method which can be nicely visualised using stencil diagrams.



RIDC Recap

RIDC is a complex time integration method which can be nicely visualised using stencil diagrams.



Why Use RIDC?

Benefits of RIDC:

Why Use RIDC?

Benefits of RIDC:

- High order of accuracy.[2]

Why Use RIDC?

Benefits of RIDC:

- High order of accuracy.[2]
- Parallelisable over the correction levels.

Why Use RIDC?

Benefits of RIDC:

- High order of accuracy.[2]
- Parallelisable over the correction levels.
- The final correction values are dropped down to the prediction level at the end of every time interval.

Why Use RIDC?

Benefits of RIDC:

- High order of accuracy.[2]
- Parallelisable over the correction levels.
- The final correction values are dropped down to the prediction level at the end of every time interval.
- Using equidistant nodes means no recalculation of the quadrature weights matrix is required.

Section 2
Pseudo-Spectral Method

Introduction to Spectral Methods

Ansatz

Spectral methods assume a solution of the form

$$u(x) = \sum_n c_n \psi_n(x)$$

where the c_n are coefficients to be found and the ψ_n are the predetermined basis functions.

Some examples:

- If $\psi_n = e^{\frac{2\pi i n x}{P}}$ then $u(x)$ is a Fourier series.
- If ψ_n are the Legendre polynomials then $u(x)$ is a Fourier-Legendre series.
- If ψ_n are non-linear and adaptive basis functions composed of L function compositions then $u(x)$ is a neural network with depth L.

Problems with the Spectral Method

We want to multiply our numerical solution $u(x)$ by the function $v(x)$.

$$\hat{u}(x) = u(x)v(x)$$

$$u(x) = \sum_{n=0}^{N} c_n \psi_n(x), \quad \hat{u}(x) = \sum_{n=0}^{N} \hat{c}_n \psi_n(x)$$

We need to be able to find the coefficents $\hat{c}_n$ in the sum of $\hat{u}(x)$.

$$\hat{c}_n = \langle \hat{u}, \psi_n \rangle = \langle uv, \psi_n \rangle$$

$$\hat{c}_n = \sum_{m=0}^{N} V_{n,m}\, c_m, \quad V_{n,m} = \langle v\psi_m, \psi_n \rangle$$

Finding these coefficients has computational complexity of $O(N^2)$ and the matrix V also needs to be precomputed, adding an extra step to the method.

Pseudo-Spectral Method

We could instead use the pseudo-spectral method which works by discretising the domain and approximating the inner product with a known quadrature.

$$\langle \psi_n, \psi_m \rangle \approx \sum_{k=0}^{N} w_k \psi_n(x_k) \overline{\psi_m(x_k)}, \quad n, m = 0, 1, ..., N$$

We then assume the quadrature can also adequately approximate:

$$\hat{c}_n = \langle \hat{u}, \psi_n \rangle \approx \sum_{k=0}^{N} w_k u(x_k) v(x_k)$$

This removes the need to do any prerequisite calculations, but how can we also reduce the computational complexity?

Fourier Pseudo-Spectral Method

If ψ_n are plane wave basis functions and $w_k = 1$ then the quadrature is given by the discrete Fourier transform.

$$\hat{c}_n = \langle \hat{u}, \psi_n \rangle \approx \sum_{k=0}^{N} u(x_k) v(x_k) e^{-\frac{2\pi i}{N} kn}$$

This is more straightforwardly written using the symbol for the discrete Fourier transform.

$$\hat{\boldsymbol{c}} \approx \mathcal{F}_{\boldsymbol{x}}\{\boldsymbol{u} \cdot \boldsymbol{v}\}$$

By using the fast Fourier transform we can reduce the computational complexity of finding $\hat{\boldsymbol{c}}$ from $O(N^2)$ to $O(NlogN)$.

Applying the Pseudo-Spectral Method

2D Incompressible Vorticity Equation

$$\frac{\partial w}{\partial t} = \nu \nabla^2 w - (\boldsymbol{u} \cdot \nabla) w, \quad -\nabla^2 \boldsymbol{u} = \nabla \times \boldsymbol{w} \tag{1}$$

Applying the Pseudo-Spectral Method

2D Incompressible Vorticity Equation

$$\frac{\partial w}{\partial t} = \nu \nabla^2 w - (\boldsymbol{u} \cdot \nabla) w, \quad -\nabla^2 \boldsymbol{u} = \nabla \times \boldsymbol{w} \tag{1}$$

Spectral Form of Equation (1)

$$\begin{aligned}\frac{\partial \tilde{w}}{\partial t} = &- \mathcal{F}_{\boldsymbol{x}} \left[u_x \mathcal{F}_{\boldsymbol{x}}^{-1}[ik_x \tilde{w}] + u_y \mathcal{F}_{\boldsymbol{x}}^{-1}[ik_y \tilde{w}] \right] \\ &- \nu (k_x^2 + k_y^2) \tilde{w}\end{aligned}$$

Applying the Pseudo-Spectral Method

2D Incompressible Vorticity Equation

$$\frac{\partial w}{\partial t} = \nu \nabla^2 w - (\boldsymbol{u} \cdot \nabla) w, \quad -\nabla^2 \boldsymbol{u} = \nabla \times \boldsymbol{w} \tag{1}$$

Spectral Form of Equation (1)

$$\begin{aligned}\frac{\partial \tilde{w}}{\partial t} = & - \mathcal{F}_{\boldsymbol{x}} \left[u_x \mathcal{F}_{\boldsymbol{x}}^{-1} [i k_x \tilde{w}] + u_y \mathcal{F}_{\boldsymbol{x}}^{-1} [i k_y \tilde{w}] \right] \\ & - \nu (k_x^2 + k_y^2) \tilde{w}\end{aligned}$$

Pseudo-Spectral Form of Equation (1)

After discretising our space into a mesh-grid and changing from continuous to discrete Fourier transforms we arrive at:

$$\begin{aligned}\frac{\partial \tilde{w}}{\partial t} = & - \mathcal{F}_{\boldsymbol{x}} \left\{ \mathcal{F}_{\boldsymbol{x}}^{-1} \{ \tilde{u}_x \} \mathcal{F}_{\boldsymbol{x}}^{-1} \{ i k_x \tilde{w} \} + \mathcal{F}_{\boldsymbol{x}}^{-1} \{ \tilde{u}_y \} \mathcal{F}_{\boldsymbol{x}}^{-1} \{ i k_y \tilde{w} \} \right\} \\ & - \nu (k_x^2 + k_y^2) \tilde{w}\end{aligned} \tag{2}$$

Why Use Spectral Methods?

Benefits of spectral methods:

Why Use Spectral Methods?

Benefits of spectral methods:

- Very high order of accuracy.[3]

Why Use Spectral Methods?

Benefits of spectral methods:

- Very high order of accuracy.[3]
- Not very computationally intensive.

Why Use Spectral Methods?

Benefits of spectral methods:

- Very high order of accuracy.[3]
- Not very computationally intensive.
- Relatively easy to implement.

Section 3
Application

Example Problem

As an example, we can now numerically solve the 2D incompressible vorticity equation given below:

Example Setup

$$\frac{\partial w}{\partial t} = \nu \nabla^2 w - (\boldsymbol{u} \cdot \nabla) w, \quad -\nabla^2 \boldsymbol{u} = \nabla \times \boldsymbol{w}$$

$$\nu = 0.001,\ t \in [0, 80],\ \boldsymbol{x} \in [0, 2\pi]^2$$

$$\boldsymbol{w}_0 = e^{-5((x-0.2\pi)^2+(y-1.1\pi)^2)} - e^{-5((x-0.2\pi)^2+(y-0.9\pi)^2)} + e^{-5((x-1.8\pi)^2+(y-0.9\pi)^2)} - e^{-5((x-1.8\pi)^2+(y-1.1\pi)^2)}.$$

We achieve this by using RIDC to time integrate and obtain an approximate solution to equation (2):

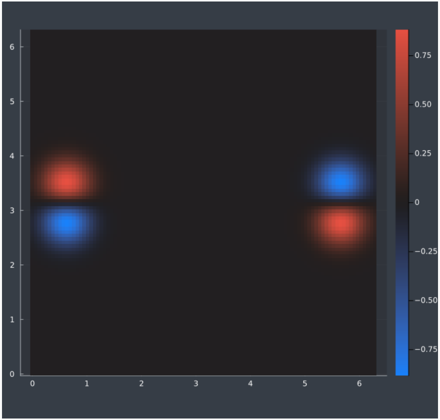
Pseudo-Spectral Form

$$\frac{\partial \tilde{w}}{\partial t} = -\mathcal{F}_{\boldsymbol{x}}\left\{\mathcal{F}_{\boldsymbol{x}}^{-1}\{\tilde{u}_x\}\mathcal{F}_{\boldsymbol{x}}^{-1}\{ik_x\tilde{w}\} + \mathcal{F}_{\boldsymbol{x}}^{-1}\{\tilde{u}_y\}\mathcal{F}_{\boldsymbol{x}}^{-1}\{ik_y\tilde{w}\}\right\} - \nu(k_x^2 + k_y^2)\tilde{w}.$$

Example Solution

An RIDC method using a prediction and a correction level both of order two incorporated 12000 time nodes and 12 time intervals with 1000 nodes each, while the pseudo-spectral method used a 100x100 spatial grid. With this setup, the following time evolution of the vorticity was obtained:

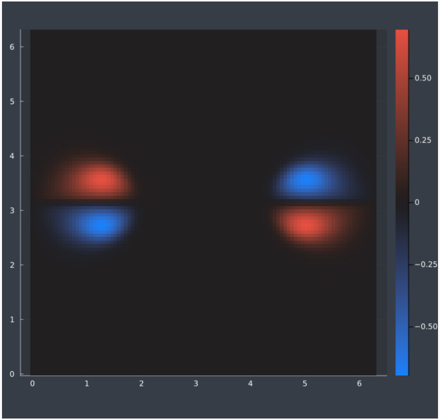
$T = 0$



Example Solution

An RIDC method using a prediction and a correction level both of order two incorporated 12000 time nodes and 12 time intervals with 1000 nodes each, while the pseudo-spectral method used a 100x100 spatial grid. With this setup, the following time evolution of the vorticity was obtained:

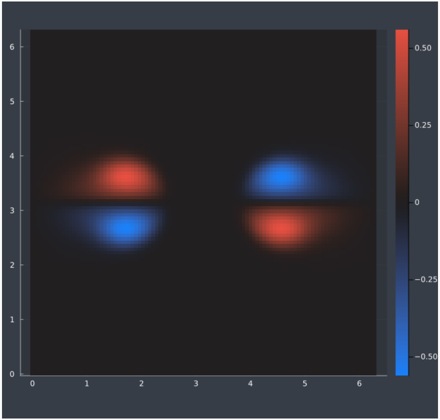
$T = 10$



Example Solution

An RIDC method using a prediction and a correction level both of order two incorporated 12000 time nodes and 12 time intervals with 1000 nodes each, while the pseudo-spectral method used a 100x100 spatial grid. With this setup, the following time evolution of the vorticity was obtained:

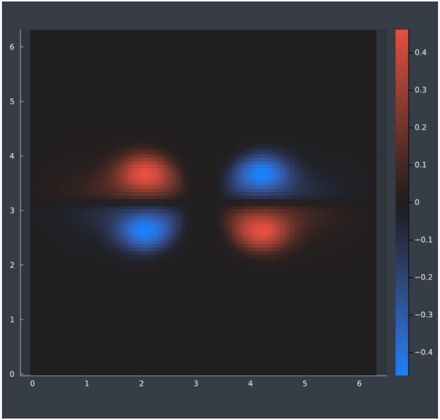
$T = 20$



Example Solution

An RIDC method using a prediction and a correction level both of order two incorporated 12000 time nodes and 12 time intervals with 1000 nodes each, while the pseudo-spectral method used a 100x100 spatial grid. With this setup, the following time evolution of the vorticity was obtained:

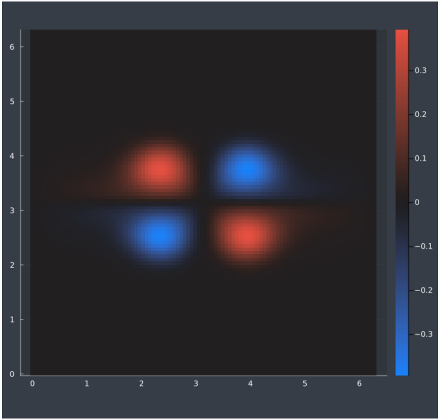
$T = 30$



Example Solution

An RIDC method using a prediction and a correction level both of order two incorporated 12000 time nodes and 12 time intervals with 1000 nodes each, while the pseudo-spectral method used a 100x100 spatial grid. With this setup, the following time evolution of the vorticity was obtained:

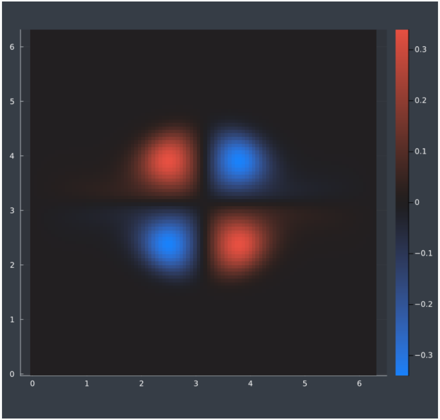
$T = 40$



Example Solution

An RIDC method using a prediction and a correction level both of order two incorporated 12000 time nodes and 12 time intervals with 1000 nodes each, while the pseudo-spectral method used a 100x100 spatial grid. With this setup, the following time evolution of the vorticity was obtained:

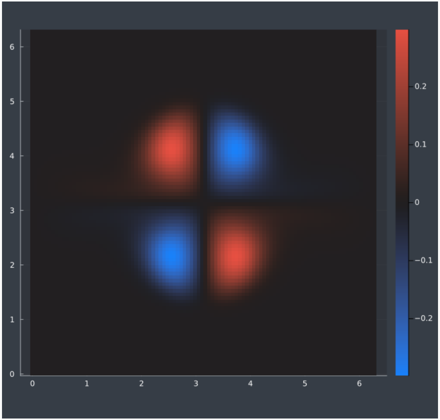
$T = 50$



Example Solution

An RIDC method using a prediction and a correction level both of order two incorporated 12000 time nodes and 12 time intervals with 1000 nodes each, while the pseudo-spectral method used a 100x100 spatial grid. With this setup, the following time evolution of the vorticity was obtained:

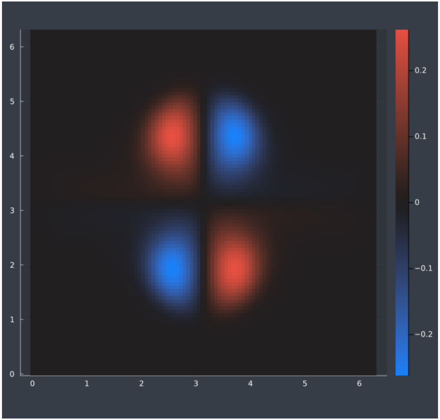
$T = 60$



Example Solution

An RIDC method using a prediction and a correction level both of order two incorporated 12000 time nodes and 12 time intervals with 1000 nodes each, while the pseudo-spectral method used a 100x100 spatial grid. With this setup, the following time evolution of the vorticity was obtained:

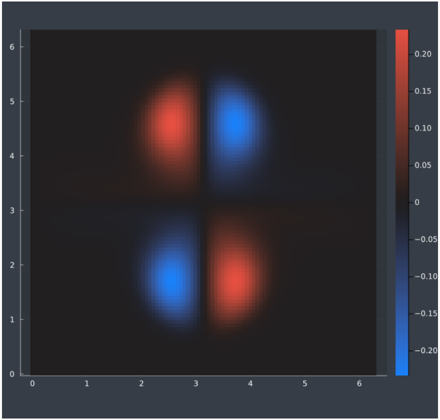
$T = 70$



Example Solution

An RIDC method using a prediction and a correction level both of order two incorporated 12000 time nodes and 12 time intervals with 1000 nodes each, while the pseudo-spectral method used a 100x100 spatial grid. With this setup, the following time evolution of the vorticity was obtained:

$T = 80$



Questions?

References

[1] John Charles Butcher. *Numerical methods for ordinary differential equations*. John Wiley & Sons, 2016.

[2] Andrew J. Christlieb, Colin B. Macdonald, and Benjamin W. Ong. "Parallel High-Order Integrators". In: *SIAM Journal on Scientific Computing* (2010). DOI: 10.1137/09075740X.

[3] Lloyd N Trefethen. *Spectral methods in MATLAB*. SIAM, 2000.