On a pair of commutative contractions (1963) by Tsuyoshi Andô

Yeetang Kwok and Bradley Hsu

March 2022

Setup

Let T be a contraction a Hilbert space D. A unitary operator $\boldsymbol{U}$ is called a *unitary dilation* of T if $\boldsymbol{U}$ acts on a Hilbert space R containing D as a subspace, and

$$T^n f = PU^n f \qquad (f \in D)$$

Where **P** is the orthogonal projection from R onto D. Note that there exists a unitary dilation of any contraction.

Theorem

Let $T_1,\ T_2$ be a pair of commutative contractions. Then there exists a pair of commutative unitary operators $\boldsymbol{U_1},\ \boldsymbol{U_2}$ on a Hilbert space R containing D as a subspace such that

$$T_1^{n_1} T_2^{n_2} f = PU_1^{n_1} U_2^{n_2} f \qquad (f \in D; n_1, n_2 = 1, 2, ...)$$

Where **P** is the orthogonal projection from R onto D.

Reduction of the problem

First, replace *"unitary"* by *"isometric"* as the unitary operators in question can be readily obtained. Second, if $\boldsymbol{U_1}$, $\boldsymbol{U_2}$ are isometries on R $\supset$D such that

$$(1) T_i f = PU_1 f \qquad (f \in D; i = 1, 2)$$

and

$$(2) U_i(R \ominus D) \subset R \ominus D \qquad (i = 1, 2)$$

then the theorem is necessarily satisfied. Thus it suffices to prove the following proposition instead of the theorem.

Reduction of the problem

For any pair of commutative contractions T_1, T_2 there exists a pair of commutative isometries $\boldsymbol{U_1}$, $\boldsymbol{U_2}$ with the properties above.

Proof setup

Say R is the orthogonal sum of countably many copies of D where, for elements $f_n \in D$, the sequence

$$\phi = \{f_n\}_0^\infty \in R$$

(I think this assumes R is infinite dimensional)

Thus inner products/norms are:

$$||\phi||^2 = \sum_0^\infty ||f_n||^2$$

Proof setup

In other words - say

$$a_d = \{f_0 \dots f_n\} \in D$$

Then you can lift a_d into R by tacking on a bunch of zeros at the end, i.e.

$$a_r = \{a_d, 0, \dots\} = \{f_1 \dots f_n, 0, \dots\} \in R$$

Proof setup

Say R is the orthogonal sum of countably many copies of D where, for elements $f_n \in D$, the sequence

$$\phi = \{f_n\}_0^\infty \in R$$

(I think this assumes R is infinite dimensional)

Thus inner products/norms are:

$$||\phi||^2 = \sum_0^\infty ||f_n||^2$$

Proof setup

Now define operators $V_i, i \in (1,2)$ as $g = V_i f$ iff

$$\{g_0, g_1, g_2, \dots, g_n\} = \{T_i f_0, Z_i f_0, 0, \dots f_{n-2}\}$$

Where Z_i is defined as

$$Z_i = (I - T_i^* T_i)^{1/2}$$

And

$$||Z_i f||^2 = ||f||^2 - ||T_i f||^2, (f \in D; i \in (1,2))$$

Proof setup

Now consider norm V

$$||V_i f||^2 = ||T_1 f_0||^2 + ||Z_i f_0||^2 + \cdots + ||f_n||^2 \dots$$

We have

$$||Z_i f||^2 = ||f||^2 - ||T_i f||^2, (f \in D; i \in (1, 2))$$

So

$$||V_i f||^2 = ||f||^2$$

And V_i are isometries (and therefore unitary).

Similarly, V satisfies (1) and (2) but for V_i instead of U_i

Algebraic Operations

So we have

$$||Z_i f||^2 = ||f||^2 - ||T_i f||^2, (f \in D; i \in (1, 2))$$

Plug in $T_i f$ for f and we have

$||Z_2 T_1 f||^2 + ||Z_1 f||^2 = ||T_1 f||^2 - ||T_2 T_1 f||^2 + ||f||^2 - ||T_1 f||^2$
$= ||f||^2 - ||T_2 T_1 f||^2$

Similarly, $||Z_1 T_2 f||^2 + ||Z_2 f||^2 = ||f||^2 - ||T_1 T_2 f||$

And T_1, T_2 are commutative, so we have

$$||Z_1 T_2 f||^2 + ||Z_2 f||^2 = ||Z_2 T_1 f||^2 + ||Z_1 f||^2$$

Algebraic Operations

$$||Z_1 T_2 f||^2 + ||Z_2 f||^2 = ||Z_2 T_1 f||^2 + ||Z_1 f||^2$$

Construct spaces M_1 and M_2

$$M_1 = \{Z_2 T_1 f, 0, Z_1 f, 0\}$$
$$M_2 = \{Z_1 T_2 f, 0, Z_2 f, 0\}$$

Then there's an isometry W between M_1 and M_2.

Now consider G, that's the orthogonal sum of four copies of D. We can extend W to a unitary operator on G.

Algebraic Operations

$$||Z_1 T_2 f||^2 + ||Z_2 f||^2 = ||Z_2 T_1 f||^2 + ||Z_1 f||^2$$

Construct spaces M_1 and M_2

$$M_1 = \{Z_2 T_1 f, 0, Z_1 f, 0\}$$

$$M_2 = \{Z_1 T_2 f, 0, Z_2 f, 0\}$$

Then there's an isometry W between M_1 and M_2.

Now consider G, that's the orthogonal sum of four copies of D. We can extend W to a unitary operator on G.

Algebraic Operations

Let $\boldsymbol{W}$ be the operator on R defined as follows: $g_n = \boldsymbol{W} f_n$ iff
$g_0 = f_0$ and
$\{g_{4n-3}, g_{4n-2}, g_{4n-1}, g_{4n}\} = W\{f_{4n-3}, f_{4n-2}, f_{4n-1}, f_{4n}\}$

Then the unitarity of $\boldsymbol{W}$ follows from the unitarity of W on G, and both $\boldsymbol{W}$ and $\boldsymbol{W}^*$ have the second property.

Finally the isometries $\boldsymbol{U_1}$ and $\boldsymbol{U_2}$ in question are defined by

$$\boldsymbol{U_1 = WV_1} \text{ and } \boldsymbol{U_2 = V_2 W^*}$$

Note that $\boldsymbol{U_1}$ and $\boldsymbol{U_2}$ are also isometries that satisfy (1) and (2) because $\boldsymbol{W}$ and $\boldsymbol{V}$ satisfy (1) and (2)

Algebraic Operations

Now we just have to prove the commutativity of U_1 and U_2

$\{g_n\} \equiv U_1 U_2 \{f_n\} \equiv \mathrm{W} \mathrm{V}_1 V_2 W^* \{\mathrm{f}_n\}$

$\{h_n\} \equiv U_2 U_1 \{f_n\} \equiv \mathrm{V}_2 W W^* V_1 \{\mathrm{f}_n\} \equiv V_2 V_1 \{f_n\}$

using the definitions of W and U_i's to show that

$$
\begin{gathered}
g_0 = T_1 T_2 f_0 \\
\{g_1, g_2, g_3, g_4\} = W\{Z_1 T_2 f_0, 0, Z_2 f_0, 0\} \\
g_n = f_{n-4} \\
h_0 = T_2 T_1 f_0 \\
\{h_1, h_2, h_3, h_4\} = \{Z_2 T_1 f_0, 0, Z_1 f_0, 0\} \\
h_n = f_{n-4} \qquad\qquad (n > 4)
\end{gathered}
$$

Algebraic Operations

Since $T_1 T_2 = T_2 T_1$

$$W\{Z_1 T_2 f_0, 0, Z_2 f_0, 0\} = \{Z_2 T_1 f_0, 0, Z_1 f_0, 0\}$$

by the definition of W, it follows that $\boldsymbol{U_1 U_2\{f_n\} = U_2 U_1\{f_n\}}$