Integrating an Infinite Power Tower

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1 Introduction

For this paper, we will look at the following integral:

$$\int_1^{e^{\frac{1}{e}}} x^{x^{x^{\cdots}}} dx$$

Where $x^{x^{x^{\cdots}}} = \lim_{n\to\infty} {}^n x$ (Tetrations) Before we can even start however, we need a more rigorous way to define the function in question.

2 A More Rigorous Definition of an Infinite Power Tower

start by defining:

$$y = x^{x^{x^{\cdots}}}$$

Using recursive substitution, we can make the following relationship from this:

$$y = x^y$$

$$x = y^{\frac{1}{y}}, y \neq 0$$

From this, we can see the infinite power tower is the inverse of $x^{\frac{1}{x}}$. From this point on, all mentions of an infinite power tower are to be defined as the inverse of $x^{\frac{1}{x}}$. An important note is that the infinite power tower has a restricted domain and range. The function has a domain of [e^{-e},$e^{\frac{1}{e}}$], and a range of [$\frac{1}{e}$,e].

3 Integrating with respect to an inverse

It can be graphically shown for any function f(x) that is invertable, with arbitrary bounds of integration, a and b:

$$\int_a^b f(x)dx + \int_{f(a)}^{f(b)} f^{-1}(y)dy = f(b)b - f(a)a$$

if $f(x)$ is continuous for all values x, where x $\in$ [a,b], and $f^{-1}(x)$ is continuous for all values x, where x $\in$ [f(a),f(b)]. Since $\int\limits_{f(a)}^{f(b)} f^{-1}(y)dy = \int\limits_{f(a)}^{f(b)} f^{-1}(x)dx$,

$$\int_a^b f(x)dx + \int_{f(a)}^{f(b)} f^{-1}(x)dx = f(b)b - f(a)a$$

$$\int_{f(a)}^{f(b)} f^{-1}(x)dx = f(b)b - f(a)a - \int_a^b f(x)dx$$

4 Applying inverse integration to infinite power towers

Start by defining f(x) to be $x^{\frac{1}{x}}$. If we plug in 1 and e as the bounds of integration, we can see:

$$\int_{f(1)}^{f(e)} x^{x^{x^{\cdots}}} = f(e)e - f(1)1 - \int_1^e x^{\frac{1}{x}}dx$$

Plugging in f(1), and f(e), we get the following:

$$\int_1^{e^{\frac{1}{e}}} x^{x^{x^{\cdots}}} = (e^{\frac{1}{e}})e - 1 - \int_1^e x^{\frac{1}{x}}dx$$

5 Solving the integral of $x^{\frac{1}{x}}$

We will solve this integral using the Taylor series expansion of e^x:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

If we rewrite $x^{\frac{1}{x}}$ in terms of base e, we can see:

$$x^{\frac{1}{x}} = e^{\frac{ln(x)}{x}} = \sum_{n=0}^{\infty} \frac{\left(\frac{ln(x)}{x}\right)^n}{n!}$$

Now, we can rewrite our integral with this infinite sum (Note: we will call the integral in question I_1)

$$I_1 = \int_1^e \sum_{n=0}^{\infty} \frac{ln(x)^n}{x^n n!}dx$$

Luckily, this infinite sum has absolute convergence on the interval from 1 to e, and therefore we can switch the order of summation and integration (Fubini-Tonelli Theorem).

$$I_1 = \sum_{n=0}^{\infty} \int_1^e \frac{ln(x)^n}{x^n n!}dx$$

Although it will not make sense now, we will separate the first two terms of this series from the rest of the series, as such:

$$I_1 = \int_1^e dx + \int_1^e \frac{ln(x)}{x} dx + \sum_{n=2}^{\infty} \left(\int_1^e \frac{ln(x)^n}{x^n n!} dx \right)$$

To simplify notation, we will call the sum of these first two terms C, and also we will factor out the n! term (it's constant with respect to x).

$$I_1 = \sum_{n=2}^{\infty} \left(\frac{1}{n!} \int_1^e \frac{ln(x)^n}{x^n} dx \right) + C$$

Now, we will do a series of u-substitutions: Set $u = ln(x)$, $x = e^u$,$dx = e^u du$

$$I_1 = \sum_{n=2}^{\infty} \left(\frac{1}{n!} \int_0^1 u^n e^{u(-n+1)} du \right) + C$$

Set $v = u(n-1)$,$u = \frac{v}{n-1}$,$du = \frac{1}{n-1} dv$

$$I_1 = \sum_{n=2}^{\infty} \left(\frac{1}{n!} \int_0^{(n-1)} \frac{\left(\frac{v}{(n-1)}\right)^n e^{-v}}{n-1} dv \right) + C$$

Simplify, and we get

$$I_1 = \sum_{n=2}^{\infty} \left(\frac{1}{n!} \int_0^{(n-1)} \frac{v^n e^{-v}}{(n-1)^{(n+1)}} dv \right) + C$$

Factoring the n terms once again, we get:

$$I_1 = \sum_{n=2}^{\infty} \left(\frac{1}{(n-1)^{(n+1)} n!} \int_0^{(n-1)} v^n e^{-v} dv \right) + C$$

The lower incomplete gamma function is defined as follows:

$$\gamma(s, x) = \int_0^x t^{s-1} e^{-t} dt$$

We can then apply this to the remaining integral in our summation:

$$I_1 = \sum_{n=2}^{\infty} \left(\frac{\gamma((n+1), (n-1))}{(n-1)^{(n+1)} n!} \right) + C$$

We can clean up the index of summation with another quick u-sub: Set: $k = (n-1)$

$$I_1 = \sum_{k=1}^{\infty} \left(\frac{\gamma((k+2), (k))}{k^{(k+2)} (k+1)!} \right) + C$$

Remember that C is defined as

$$C = \int_1^e dx + \int_1^e \frac{ln(x)}{x} dx$$

The first integral is trivial, and thus I won't bother showing work.

$$C = (e-1) + \int_1^e \frac{ln(x)}{x} dx$$

The second integral can be easily conquered with a quick u-substitution. Set $u = ln(x)$,$du = \frac{1}{x}dx$

$$C = (e-1) + \int_0^1 udu$$

$$C = (e-1) + \frac{1}{2}$$

$$C = e - \frac{1}{2}$$

Plugging this back in, we see that:

$$I_1 = \int_1^e x^{\frac{1}{x}} dx = e - \frac{1}{2} + \sum_{k=1}^{\infty} \left(\frac{\gamma((k+2),(k))}{k^{(k+2)}(k+1)!} \right)$$

6 Final Touches!

From earlier, we know:

$$\int_1^{e^{\frac{1}{e}}} x^{x^{x^{\cdot^{\cdot^{\cdot}}}}} = (e^{\frac{1}{e}})e - 1 - \int_1^e x^{\frac{1}{x}}$$

So plugging in our result for I_1, we see:

$$\int_1^{e^{\frac{1}{e}}} x^{x^{x^{\cdot^{\cdot^{\cdot}}}}} = (e^{\frac{1}{e}})e - 1 - \left(e - \frac{1}{2} + \sum_{k=1}^{\infty} \left(\frac{\gamma((k+2),(k))}{k^{(k+2)}(k+1)!} \right) \right)$$

Distributing the negative sign, and combining like terms, we get a final answer of:

$$\int_1^{e^{\frac{1}{e}}} x^{x^{x^{\cdot^{\cdot^{\cdot}}}}} = (e^{\frac{1}{e}})e - e - \frac{1}{2} - \sum_{k=1}^{\infty} \left(\frac{\gamma((k+2),(k))}{k^{(k+2)}(k+1)!} \right)$$

Since $(k+1)! = \Gamma(k+2)$, we can replace it as such:

$$\int_1^{e^{\frac{1}{e}}} x^{x^{x^{\cdot^{\cdot^{\cdot}}}}} = (e^{\frac{1}{e}})e - e - \frac{1}{2} - \sum_{k=1}^{\infty} \left(\frac{\gamma((k+2),(k))}{k^{(k+2)}\Gamma(k+2)} \right)$$

Q.E.D.