21-242 HW 1

Instructor: Wesley Pegden

Due September 3, 2025

Question 1. Referencing the axioms of a vector space, for each of the following, prove or disprove the statement. These proofs should be complete but need not be long.

(a) $V = \left\{ \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \mid v_1, v_2 \in \mathbb{R}, v_1 + v_2 = 1 \right\}$ is an $\mathbb{R}$-vector space when endowed with the usual operations.

(b) Define the operation $\oplus$ on $\mathbb{R}^+ \times \mathbb{R}^+$ by $a \oplus b = a \cdot b$ (standard multiplication) and $\otimes$ on $\mathbb{R} \times \mathbb{R}^+$ by $a \otimes b = b^a$ (exponentiation). (Note that both operations indeed always return positive real numbers). Then $V = \mathbb{R}^+$ is an $\mathbb{R}$-vector space with $\oplus$ as the addition operation on V and $\otimes$ as the scalar multiplication. Note, $\mathbb{R}^+$ is the set of positive real numbers.

(c) With the same operations as above, $V = \mathbb{Q}^+$ is a $\mathbb{Q}$-vector space.

(d) Let V be the set of polynomials of degree at most 3 with coefficients in $\mathbb{Q}$. Then V is a $\mathbb{Q}$ vector space when endowed with the standard operations of polynomial addition and scalar multiplication.

Question 2. In class we proved that if V is a vector space than for any $\vec{v} \in V$ we have $0\vec{v} = \vec{0}$. Use this to prove that $(-1) \cdot \vec{v} = -\vec{v}$.

Question 3. Prove for an $\mathbb{F}$-vector space V that for any $\vec{v} \in V$ and $c \in \mathbb{F}$, if $c\vec{v} = \vec{0}$ then either $c = 0$ or $\vec{v} = \vec{0}$.

Question 4. (a) Consider $\mathbb{R}$ as a vector space over $\mathbb{R}$ with the usual arithmetic operations. Prove that any two vectors are linearly dependent (which just means: not linearly independent).

(b) Consider $\mathbb{R}$ as a vector space over $\mathbb{Q}$ with the usual arithmetic operations. (So, the vectors are elements of $\mathbb{R}$, and $\mathbb{Q}$ is the field). Give an example of two linearly independent vectors; prove they are independent.

(c) **Challenge, not to be graded** Prove that for $\mathbb{R}$ as a vector space over $\mathbb{Q}$, there is no finite list $\mathcal{L}$ of vectors for which $\mathrm{span}(\mathcal{L}) = \mathbb{R}$.

Question 5. Prove that if $\mathcal{L}$ is a list of vectors in the $\mathbb{F}$-vector space V, and $\vec{w}_1, \ldots, \vec{w}_k \in \mathrm{span}(\mathcal{L})$, then $\mathrm{span}(\vec{w_1}, \ldots, \vec{w_k}) \subseteq \mathrm{span}(\mathcal{L})$.

Question 6. Fix a postive integer k and define a vector space V_k over $\mathbb{R}$ as follows:

Vectors are lists $(a_{-k}, \ldots, a_{-1}, a_0, a_1, \ldots, a_k)$ of real numbers such that $a_j = j \cdot a_{-j}$ for each $j = 1, \ldots, k$.

1. Prove that V_k indeed a vector space, with the usual operations of point-wise addition and scalar multiplication.

2. Give two different bases for V_k (with proofs they are bases).