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# *Mathematical Investigations IV*

## *Trigonometry — Beyond the Right Triangles*
## *Ambiguous Triangles*

On the previous two worksheets we worked with the laws of sines and cosines. Given enough information about a triangle, they allowed us to solve for all the unknown sides and angles of a triangle. We also saw, in the last problem on each sheet, that sometimes you can give information that cannot possibly form a triangle. But what is “enough” information? And when does the information given fail to form a unique triangle?

1) Consider the ASA theorem from geometry. What does it state?

2) Explain why AAS is also a congruence theorem about triangles. How can it be derived from other congruence theorems?

If you know two of the angles of a triangle, you really already know the third. That means you know the triangle “up to similarity.” Then once you specify the length of one of the sides, you know exactly which triangle it is. Thus, any two angles plus any one side is “enough information” and can never lead to a case where no triangle is possible.

3) What other triangle congruence theorems are there in geometry? List some of them here.

SAS gives enough information to pinpoint a triangle, so two sides and the included angle is “enough” information, and can’t lead to a case where no triangle is possible. SSS also gives enough information, but *can* lead to a case where no triangle exists.

4) What conditions must the sides $a$, $b$, and $c$ of a triangle satisfy to ensure the existence of a triangle?

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So if we have a congruence theorem, we have enough information to solve for the triangle. We'll see shortly that if we have information that *doesn't* fit into a congruence theorem, other things can happen.

5) Which law (sines or cosines) would you use to solve the triangle if the given information:

a) Fits the ASA theorem?

b) Fits the AAS theorem?

c) Fits the SAS theorem?

d) Fits the SSS theorem?

It's never enough to have just two pieces of information—two angles or two sides or one angle and one side can never pinpoint the triangle. There are always lots of triangles that will work. Three pieces is usually enough, and we have seen that AAS, ASA, SAS, and SSS lead to situations where the laws of sines and cosines solve the rest of the triangle for you. What are the other combinations?

6) Explain why AAA can't be used to solve triangles.

The last possibility is SSA—two sides, but the angle you know is *not* between them. In problem 7 of the law of sines packet, we had this case, but we were lucky to be able to solve it. Problem 8 of that packet had one where the triangle was impossible. Let's try those triangles again here, this time with the law of *cosines*.

7) In $\Delta KLM$, $\angle K = 82°$, $m = 12$, and $k = 15$. Solve for $l$ using the *law of cosines*. You should get a quadratic equation, with one positive and one negative solution. Is the positive solution you found the same as the length $l$ you found previously?

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8) In $\Delta CDE$, $c = 15$, $e = 25$, and $m\angle C = 85°$. What happens when you try to solve for $d$ using the law of cosines?

Using the law of cosines to solve for an unknown side always leads to a quadratic equation. Quadratic equations always have two solutions (well, almost always). Of course, sides can't be negative, so if you have one positive and one negative solution you can ignore the negative one. In problem 8) you have two *non-real* solutions, so you had no real triangles. But what if you have *two* positive solutions?

9) In $\Delta XYZ$, $x = 20$, $y = 14$, and $m\angle Y = 35°$. Use the law of cosines to find side $z$. You should get two answers. Do both lead to triangles that work?

10) Solve the rest of both triangles in problem 9). What do you notice about the two possibilities for angle $X$?

11) What do you notice about $\sin(X)$ for the two possibilities for angle $X$?

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12) For the triangle in problem 9) you could also try the law of sines. First compute $\sin(X)$. does the value match the one found in problem 11)?

13) Now when you use your calculator to find angle $X$ knowing its sine, why do you only get one answer, even though there are two possible triangles?

We have just run into the ***ambiguous case***. That is where we should have enough information to solve the triangle, but there are actually two triangles that are possible. This can only happen when we have SSA information. *Sometimes* SSA gives you one triangle. Other times, none, and still other times two.

It's *usually* easiest to decide whether there is zero, one, or two triangles by trying to solve them and finding how many solutions there are. Law of cosines leads to a quadratic, and the $\pm$ in the quadratic formula can lead to two answers. If one is negative it can be ignored. This always happens if the side *opposite* the given angle is larger than the other given side. This can also happen if the sides and angles work out exactly so that the triangle is a right triangle.

14) In $\Delta ABC$, suppose angle $A$, side $a$, and side $c$ are given.

a) What is the relationship between these three parts of the triangle so that there is a right angle at $C$? What does this imply about the length of $a$?

b) What happens if $a$ is shorter than this length?

c) *Challenge:* If the relationship in problem 14a) is true, what happens when you solve for side $b$ using the law of cosines?

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Complete the summary table by stating the relationship between sin($A$), $a$ and $c$:

## In $\Delta ABC$, given angle $A$ and sides $a$ and $c$.

| No (0) Triangle Possible | ONE (1) Triangle Possible | TWO (2) triangles possible |
|---|---|---|
| | | |
| | | |

15) In the case of two triangles, how are the two possible measures for angle $C$ related?

16) Since this is true, the two possible angle measure will always have the same sine.

a) Why does using $\sin^{-1}$ on your calculator only find one of the angles?

b) Which one does it find?

c) How will you find the other possibility?

Then you can continue to solve for *both* triangles, using the law of sines.
OR you can use the law of cosines and the quadratic equation will help you find both triangles.

Now let's PRACTICE!

For each situation below, decide how many triangles there are and solve for all possibilities.

17) $\Delta FGH$, $m\angle F = 18^\circ$, $f = 8$, and $g = 12$.

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18) $\Delta PQR$, $\angle P = 54°$, $\angle Q = 39°$, and $p = 15$.

19) $\Delta XYZ$, $\angle X = 64°$, $x = 30$, $y = 13$.

<u>Note:</u> In the problem below and in future problems, a designation such as "S55°E" is used. In this case, first draw a line due South. From this line, move 55° towards the East. (This will be 35° below the horizontal.) Similarly, a designation of N25°W means that one should first draw a line due North. From there, move 25° towards the West (left).

20) A weather forecaster in the tower at DuPage Airport spots a tornado at S55°E and this same tornado is spotted from Willis Tower at S35°W. The distance between the two observers is approximately 38 miles. How far is the tornado from each observer and how far south of the line connecting the two is the tornado? Note: the Willis Tower is S86ºE of the DuPage airport.