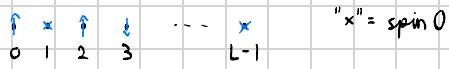


Problem 1



"x" = spin 0

$$\begin{aligned}
 a) Z &= \sum_{\{\sigma\}} e^{-\beta H} \\
 &= \sum_{\{\sigma\}} e^{\sum_{i=0}^{L-1} \beta J \delta_{\sigma_i \sigma_{i+1}}} \\
 &= \sum_{\{\sigma\}} \prod_{i=0}^{L-1} e^{\beta J \delta_{\sigma_i \sigma_{i+1}}}
 \end{aligned}$$

Let's define the transfer matrix

$$\begin{aligned}
 T_{\sigma_i \sigma_{i+1}} &= e^{\beta J \delta_{\sigma_i \sigma_{i+1}}} \\
 &= \begin{pmatrix} e^{\beta J} & 1 & 1 \\ 1 & e^{\beta J} & 1 \\ 1 & 1 & e^{\beta J} \end{pmatrix}
 \end{aligned}$$

We then have

$$\begin{aligned}
 Z &= \sum_{\{\sigma\}} \prod_{i=0}^{L-1} e^{\beta J \delta_{\sigma_i \sigma_{i+1}}} \\
 &= \sum_{\{\sigma\}} \prod_{i=0}^{L-1} T_{\sigma_i \sigma_{i+1}} \\
 &= \sum_{\{\sigma\}} T_{\sigma_0 \sigma_1} T_{\sigma_1 \sigma_2} \cdots T_{\sigma_{L-1} \sigma_0} \\
 &= \sum_{\{\sigma\}} (T^N)_{\sigma_0 \sigma_0} \quad (N=L) \\
 &= \text{Tr}\{T^N\} \\
 &= \text{Tr}\left\{\left(\begin{pmatrix} e^{\beta J} & 1 & 1 \\ 1 & e^{\beta J} & 1 \\ 1 & 1 & e^{\beta J} \end{pmatrix}\right)^N\right\}
 \end{aligned}$$

We need to diagonalize T. We start by finding its eigenvalues:

$$0 = \det(T - \lambda I)$$

$$= \begin{vmatrix} e^{\beta J} - \lambda & 1 & 1 \\ 1 & e^{\beta J} - \lambda & 1 \\ 1 & 1 & e^{\beta J} - \lambda \end{vmatrix}$$

$$0 = (e^{\beta J} - \lambda)^3 + 2 - 3(e^{\beta J} - \lambda)$$

Get rid of exponents by guessing $\lambda = e^{\beta J} - \lambda_0$ for some λ_0 . This gives

$$0 = \lambda_0^3 + 2 - 3\lambda_0$$

Can quickly see that $\lambda_0 = 1$ is a solution. Using polynomial division we find

$$\begin{aligned} (\lambda_0^3 - 3\lambda_0 + 2) : (\lambda_0 - 1) &= \underline{\lambda_0^2 + \lambda_0 - 2} \\ \underline{\lambda_0^3 - \lambda_0^2} & \\ \underline{\lambda_0^2 - 3\lambda_0 + 2} & \\ \underline{\lambda_0^2 - \lambda_0} & \\ \underline{-2\lambda_0 + 2} & \\ \underline{-2\lambda_0 + 2} & \\ 0 & \end{aligned} \Rightarrow \lambda_0 = \frac{-1 \pm \sqrt{1+8}}{2} = \frac{-1 \pm 3}{2} \Rightarrow \lambda_0 = 1 \vee \lambda_0 = -2$$

Thus we have

$$\lambda = e^{\beta J} + 2 \text{ with multiplicity 1}$$

$$\lambda = e^{\beta J} - 1 \text{ with multiplicity 2}$$

Diagonalizing gives

$$T = RDR^{-1}$$

with $D = \begin{pmatrix} e^{\beta J} + 2 & 0 & 0 \\ 0 & e^{\beta J} - 1 & 0 \\ 0 & 0 & e^{\beta J} - 1 \end{pmatrix}$ and R an invertible matrix with the eigenvectors

of T as columns.

Using this we have

$$Z = \text{Tr} \{ T^N \}$$

N times

$$= \text{Tr} \{ \underbrace{R^T D R R^T D R \dots R^T D R}_\text{cyclic property} \}$$

$$= \text{Tr} \{ D^N \}$$

$$\underline{\underline{Z = (e^{\beta J} + 2)^N + 2(e^{\beta J} - 1)^N}}$$

Average total internal energy:

$$\begin{aligned} U &= \frac{1}{Z} \sum_{E \in S} E e^{-\beta E} \\ &= -\frac{1}{Z} \frac{\partial Z}{\partial \beta} \\ &= -\frac{1}{Z} \left[\frac{\partial}{\partial \beta} (e^{\beta J} + 2)^N + 2 \frac{\partial}{\partial \beta} (e^{\beta J} - 1)^N \right] \\ &= -\frac{1}{Z} \left[N (e^{\beta J} + 2)^{N-1} e^{\beta J} \cdot J + 2 N (e^{\beta J} - 1)^{N-1} e^{\beta J} \cdot J \right] \\ &= -N J e^{\beta J} \frac{(e^{\beta J} + 2)^{N-1} + 2(e^{\beta J} - 1)^{N-1}}{(e^{\beta J} + 2)^N + 2(e^{\beta J} - 1)^N} \end{aligned}$$

At very high temperatures, $\beta \rightarrow 0$:

$$\lim_{\beta \rightarrow 0} U = -N J \frac{(1+2)^{N-1} + 2(1-1)^{N-1}}{(1+2)^N + 2(1-1)^N}$$

$$= -\frac{1}{3} N J \quad \text{High T}$$

At very low temperatures, $\beta \rightarrow \infty$:

$$\begin{aligned} \lim_{\beta \rightarrow \infty} U &= -N J \lim_{\beta \rightarrow \infty} e^{\beta J} \frac{(e^{\beta J} + 2)^{N-1} + 2(e^{\beta J} - 1)^{N-1}}{(e^{\beta J} + 2)^N + 2(e^{\beta J} - 1)^N} \quad \text{negligible as } \beta \rightarrow \infty \\ &= -N J \lim_{\beta \rightarrow \infty} \frac{e^{N\beta J} + 2e^{N\beta J}}{e^{N\beta J} + 2e^{N\beta J}} \end{aligned}$$

$$= -NJ \quad \text{Low } T$$

At low T all the spins have the same alignment. This can be seen since it is the only way to get $H = -J \sum_{i=0}^{L-1} \delta_{\sigma_i \sigma_{i+1}}$ to be $-JN$.

At high T the spin alignments are more random, with every spin taking on a random of the three possible values. On average, two neighbouring spins take the same value only one third of the time, resulting in $U = -\frac{1}{3}NJ$.

$$\begin{aligned}
 b) \langle m \rangle &= \frac{1}{N^2} \sum_{\sigma_0} \frac{1}{N} \sum_{i=0}^{N-1} e^{i \frac{2\pi}{3} \sigma_i} \prod_{i=0}^{N-1} T_{\sigma_i \sigma_{i+1}} \\
 &= \frac{1}{N^2} \sum_{i=0}^{N-1} \sum_{\sigma_0} e^{i \frac{2\pi}{3} \sigma_i} T_{\sigma_0 \sigma_1} \dots T_{\sigma_{N-1} \sigma_0} \\
 &= \frac{1}{N^2} \sum_{i=0}^{N-1} \sum_{\sigma_0} T_{\sigma_0 \sigma_1} \dots T_{\sigma_{i-1} \sigma_i} e^{i \frac{2\pi}{3} \sigma_i} T_{\sigma_i \sigma_{i+1}} \dots T_{\sigma_{N-1} \sigma_0}
 \end{aligned}$$

Let's define a matrix with the possible m_i -values along the diagonal:

$$M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{i \frac{2\pi}{3}} & 0 \\ 0 & 0 & e^{i \frac{4\pi}{3}} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{i \frac{2\pi}{3}} & 0 \\ 0 & 0 & e^{-i \frac{2\pi}{3}} \end{pmatrix}$$

We then have

$$\begin{aligned}
 \langle m \rangle &= \frac{1}{N^2} \sum_{i=0}^{N-1} \sum_{\sigma_0} T_{\sigma_0 \sigma_1} \dots T_{\sigma_{i-1} \sigma_i} M_{\sigma_i \sigma_i} T_{\sigma_i \sigma_{i+1}} \dots T_{\sigma_{N-1} \sigma_0} \\
 &= \frac{1}{N^2} \sum_{i=0}^{N-1} \text{Tr} \{ T^i M T^{N-i} \} \\
 &= \frac{1}{N^2} \sum_{i=0}^{N-1} \text{Tr} \{ M T^N \} \\
 &= \frac{1}{N^2} \sum_{i=0}^{N-1} \text{Tr} \{ M R D^N R^{-1} \}
 \end{aligned}$$

Need eigenvectors of T to construct R . Let $\lambda_0 = e^{\frac{2\pi i}{3}} + 1$, $\lambda_1 = \lambda_2 = e^{\frac{2\pi i}{3}} - 1$. We find the eigenvectors by Gaussian elimination:

$$\lambda_0: T - \lambda_0 \mathbb{1} = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} \xrightarrow{\text{Wolfram}} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \vec{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\lambda_{1,2}: T - \lambda_1 \mathbb{1} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \vec{v}_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \vec{v}_3 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

$$\Rightarrow R = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix}, R^{-1} = \frac{1}{3} \begin{pmatrix} 1 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}$$

Using this we find

$$T^N = R D^N R^{-1}$$

$$= \frac{1}{3} \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} \lambda_0^N & 0 & 0 \\ 0 & \lambda_1^N & 0 \\ 0 & 0 & \lambda_1^N \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}$$

$$= \frac{1}{3} \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} \lambda_0^N & \lambda_0^N & \lambda_0^N \\ \lambda_1^N & -2\lambda_1^N & \lambda_1^N \\ \lambda_1^N & \lambda_1^N & -2\lambda_1^N \end{pmatrix}$$

$$= \frac{1}{3} \begin{pmatrix} \lambda_0^N + 2\lambda_1^N & \lambda_0^N - \lambda_1^N & \lambda_0^N - \lambda_1^N \\ \lambda_0^N - \lambda_1^N & \lambda_0^N + 2\lambda_1^N & \lambda_0^N - \lambda_1^N \\ \lambda_0^N - \lambda_1^N & \lambda_0^N - \lambda_1^N & \lambda_0^N + 2\lambda_1^N \end{pmatrix}$$

Furthermore, this gives

$$MT^N = \frac{1}{3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{i\frac{2\pi}{3}} & 0 \\ 0 & 0 & e^{-i\frac{2\pi}{3}} \end{pmatrix} \begin{pmatrix} \lambda_0^N + 2\lambda_1^N & \lambda_0^N - \lambda_1^N & \lambda_0^N - \lambda_1^N \\ \lambda_0^N - \lambda_1^N & \lambda_0^N + 2\lambda_1^N & \lambda_0^N - \lambda_1^N \\ \lambda_0^N - \lambda_1^N & \lambda_0^N - \lambda_1^N & \lambda_0^N + 2\lambda_1^N \end{pmatrix}$$

Wolfram

$$= \frac{1}{3} \begin{pmatrix} \lambda_0^N + 2\lambda_1^N & \lambda_0^N - \lambda_1^N & \lambda_0^N - \lambda_1^N \\ e^{i\frac{2\pi}{3}}(\lambda_0^N - \lambda_1^N) & e^{-i\frac{2\pi}{3}}(\lambda_0^N + 2\lambda_1^N) & e^{i\frac{2\pi}{3}}(\lambda_0^N - \lambda_1^N) \\ e^{-i\frac{2\pi}{3}}(\lambda_0^N - \lambda_1^N) & e^{i\frac{2\pi}{3}}(\lambda_0^N - \lambda_1^N) & e^{-i\frac{2\pi}{3}}(\lambda_0^N + 2\lambda_1^N) \end{pmatrix}$$

Thus we find

$$\begin{aligned} \langle m \rangle &= \frac{1}{Nz} \sum_{i=0}^{N-1} \text{Tr} \{ MT^N \} \\ &= \frac{1}{Nz} \sum_{i=0}^N \frac{1}{3} \left(1 + \underbrace{e^{i\frac{2\pi}{3}} + e^{-i\frac{2\pi}{3}}}_{= 2 \cos(\frac{2\pi}{3})} \right) (\lambda_0^N + 2\lambda_1^N) \\ \langle m \rangle &= 0 \end{aligned}$$

c) We have that

$$\begin{aligned}\langle m \rangle &= \frac{1}{Z} \sum_{\sigma_0} \frac{1}{N} \sum_{i=0}^{N-1} m_i e^{-\beta H} \\ &= \frac{1}{N} \sum_{i=0}^{N-1} \frac{1}{Z} \sum_{\sigma_0} m_i e^{-\beta H} \\ &= \frac{1}{N} \sum_{i=0}^{N-1} \langle m_i \rangle\end{aligned}$$

From the calculation in b) we know the content of this sum is zero, thus

$$\langle m_i \rangle = 0 \quad \forall i$$

We thus have

$$C(r) = \langle m_0^* m_r \rangle$$

$$\begin{aligned}&= \frac{1}{Z} \sum_{\sigma_0} e^{-i \frac{2\pi}{3} \sigma_0} e^{i \frac{2\pi}{3} \sigma_r} \prod_{i=0}^{N-1} T_{\sigma_i \sigma_{i+1}} \\ &= \frac{1}{Z} \sum_{\sigma_0} e^{-i \frac{2\pi}{3} \sigma_0} T_{\sigma_0 \sigma_1} \dots T_{\sigma_{r-1} \sigma_r} e^{i \frac{2\pi}{3} \sigma_r} T_{\sigma_r \sigma_{r+1}} \dots T_{\sigma_{N-1} \sigma_0} \\ &= \frac{1}{Z} \sum_{\sigma_0} M_{\sigma_0 \sigma_1}^\dagger T_{\sigma_0 \sigma_1} \dots T_{\sigma_{r-1} \sigma_r} M_{\sigma_r \sigma_{r+1}} T_{\sigma_r \sigma_{r+1}} \dots T_{\sigma_{N-1} \sigma_0} \\ &= \frac{1}{Z} \text{Tr} \{ M^\dagger T^r M T^{N-r} \}\end{aligned}$$

From b) we have

$$M T^{N-r} = \frac{1}{3} \begin{pmatrix} \lambda_0^{N-r} + 2\lambda_1^{N-r} & \lambda_0^{N-r} - \lambda_1^{N-r} & \lambda_0^{N-r} - \lambda_1^{N-r} \\ e^{i \frac{2\pi}{3} (\lambda_0^{N-r} - \lambda_1^{N-r})} & e^{i \frac{2\pi}{3} (\lambda_0^{N-r} + 2\lambda_1^{N-r})} & e^{i \frac{2\pi}{3} (\lambda_0^{N-r} - \lambda_1^{N-r})} \\ e^{-i \frac{2\pi}{3} (\lambda_0^{N-r} - \lambda_1^{N-r})} & e^{-i \frac{2\pi}{3} (\lambda_0^{N-r} - \lambda_1^{N-r})} & e^{-i \frac{2\pi}{3} (\lambda_0^{N-r} + 2\lambda_1^{N-r})} \end{pmatrix}$$

and similarly we find

$$M^T T^r = \frac{1}{3} \begin{pmatrix} \lambda_o^r + 2\lambda_i^r & \lambda_o^r - \lambda_i^r & \lambda_o^r - \lambda_i^r \\ e^{i\frac{2\pi}{3}}(\lambda_o^r - \lambda_i^r) & e^{-i\frac{2\pi}{3}}(\lambda_o^r + 2\lambda_i^r) & e^{-i\frac{2\pi}{3}}(\lambda_o^r - \lambda_i^r) \\ e^{i\frac{2\pi}{3}}(\lambda_o^r - \lambda_i^r) & e^{i\frac{2\pi}{3}}(\lambda_o^r - \lambda_i^r) & e^{i\frac{2\pi}{3}}(\lambda_o^r + 2\lambda_i^r) \end{pmatrix}$$

Putting the two together gives

$$\text{Tr} \left\{ M^T T^r M T^{N-r} \right\} = \frac{1}{9} \text{Tr} \left\{ \begin{pmatrix} \lambda_o^r + 2\lambda_i^r & \lambda_o^r - \lambda_i^r & \lambda_o^r - \lambda_i^r \\ e^{i\frac{2\pi}{3}}(\lambda_o^r - \lambda_i^r) & e^{-i\frac{2\pi}{3}}(\lambda_o^r + 2\lambda_i^r) & e^{-i\frac{2\pi}{3}}(\lambda_o^r - \lambda_i^r) \\ e^{i\frac{2\pi}{3}}(\lambda_o^r - \lambda_i^r) & e^{i\frac{2\pi}{3}}(\lambda_o^r - \lambda_i^r) & e^{i\frac{2\pi}{3}}(\lambda_o^r + 2\lambda_i^r) \end{pmatrix} \cdot \begin{pmatrix} \lambda_o^{N-r} + 2\lambda_i^{N-r} & \lambda_o^{N-r} - \lambda_i^{N-r} & \lambda_o^{N-r} - \lambda_i^{N-r} \\ e^{i\frac{2\pi}{3}}(\lambda_o^{N-r} - \lambda_i^{N-r}) & e^{-i\frac{2\pi}{3}}(\lambda_o^{N-r} + 2\lambda_i^{N-r}) & e^{-i\frac{2\pi}{3}}(\lambda_o^{N-r} - \lambda_i^{N-r}) \\ e^{-i\frac{2\pi}{3}}(\lambda_o^{N-r} - \lambda_i^{N-r}) & e^{-i\frac{2\pi}{3}}(\lambda_o^{N-r} - \lambda_i^{N-r}) & e^{i\frac{2\pi}{3}}(\lambda_o^{N-r} + 2\lambda_i^{N-r}) \end{pmatrix} \right\}$$

Trace, so we only need diagonal elements

$$= \frac{1}{9} \left[(\lambda_o^r + 2\lambda_i^r)(\lambda_o^{N-r} + 2\lambda_i^{N-r}) + (e^{i\frac{2\pi}{3}} + e^{-i\frac{2\pi}{3}})(\lambda_o^r - \lambda_i^r)(\lambda_o^{N-r} - \lambda_i^{N-r}) \right. \\ \left. + (\lambda_o^r + 2\lambda_i^r)(\lambda_o^{N-r} + 2\lambda_i^{N-r}) + (e^{-i\frac{2\pi}{3}} + e^{i\frac{2\pi}{3}})(\lambda_o^r - \lambda_i^r)(\lambda_o^{N-r} - \lambda_i^{N-r}) \right. \\ \left. + (\lambda_o^r + 2\lambda_i^r)(\lambda_o^{N-r} + 2\lambda_i^{N-r}) + (e^{i\frac{2\pi}{3}} + e^{i\frac{2\pi}{3}})(\lambda_o^r - \lambda_i^r)(\lambda_o^{N-r} - \lambda_i^{N-r}) \right]$$

$$= \frac{1}{3} \left[(\lambda_o^r + 2\lambda_i^r)(\lambda_o^{N-r} + 2\lambda_i^{N-r}) + \underbrace{2 \cos\left(\frac{2\pi}{3}\right)}_{=-1} (\lambda_o^r - \lambda_i^r)(\lambda_o^{N-r} - \lambda_i^{N-r}) \right]$$

$$= \frac{1}{3} \left[(\lambda_o^r + 2\lambda_i^r)(\lambda_o^{N-r} + 2\lambda_i^{N-r}) - (\lambda_o^r - \lambda_i^r)(\lambda_o^{N-r} - \lambda_i^{N-r}) \right]$$

$$= \frac{1}{3} \left[\cancel{\lambda_1^N} + \cancel{2\lambda_o^r \lambda_i^{N-r}} + \cancel{2\lambda_o^{N-r} \lambda_i^r} + \cancel{4\lambda_o^N \lambda_i^N} - \cancel{\lambda_o^N} + \cancel{\lambda_o^r \lambda_i^{N-r}} + \cancel{\lambda_o^{N-r} \lambda_i^r} - \cancel{\lambda_i^N} \right]$$

$$= \underline{\lambda_1^N + \lambda_o^r \lambda_i^{N-r} + \lambda_o^{N-r} \lambda_i^r}$$

Thus we get

$$C(r) = \frac{\lambda_1^N + \lambda_o^r \lambda_i^{N-r} + \lambda_o^{N-r} \lambda_i^r}{z}$$

$$C(r) = \frac{\lambda_1^N + \lambda_o^r \lambda_i^{N-r} + \lambda_o^{N-r} \lambda_i^r}{\lambda_o^N + 2\lambda_1^N} = \underline{\underline{\frac{\lambda_1^r}{\lambda_o^r}}}$$

The limit $L \rightarrow \infty$ is equivalent to $N \rightarrow \infty$. Using $\lambda_0 > \lambda_1$, we find in this limit:

$$\lim_{N \rightarrow \infty} C(r) = \frac{\lambda_1^N + \lambda_0 \lambda_1^{N-r} + \lambda_0^r \lambda_1^N}{\lambda_0^N + 2\lambda_1^N} \cdot \frac{\frac{1}{\lambda_0}}{\frac{1}{\lambda_0}}$$

$$= \lim_{N \rightarrow \infty} \frac{\left(\frac{\lambda_1}{\lambda_0}\right)^N + \left(\frac{\lambda_1}{\lambda_0}\right)^{N-r} + \left(\frac{\lambda_1}{\lambda_0}\right)^r}{1 + 2\left(\frac{\lambda_1}{\lambda_0}\right)^N}$$

$$0 < \frac{\lambda_1}{\lambda_0} < 1 \\ = \left(\frac{\lambda_1}{\lambda_0}\right)^r$$

$$= \underline{\underline{\left(\frac{e^{\beta r} - 1}{e^{\beta r} + 2}\right)^r}} \quad \text{for } L \rightarrow \infty$$

For $T=0$ we have $\beta \rightarrow \infty$, thus $\lambda_0 \approx \lambda_1 \approx e^{\beta r}$, so we get

$$\lim_{\beta \rightarrow \infty} C(r) = \lim_{\beta \rightarrow \infty} \frac{\lambda_1^N + \lambda_0 \lambda_1^{N-r} + \lambda_0^r \lambda_1^N}{\lambda_0^N + 2\lambda_1^N}$$

$$= \lim_{\beta \rightarrow \infty} \frac{e^{\beta N r} + e^{\beta r} e^{\beta(N-r)} + e^{\beta(N-r)} e^{\beta r}}{e^{\beta N r} + 2e^{\beta N r}}$$

$$= \lim_{\beta \rightarrow \infty} \frac{3e^{\beta N r}}{3e^{\beta N r}}$$

$$= \underline{\underline{1}} \quad \text{for } T=0$$