Problem 1

$$\begin{array}{cccccc} \uparrow & \times & \uparrow & \downarrow & \cdots & \times \\ 0 & 1 & 2 & 3 & & L-1 \end{array}$$

"x" = spin 0

a) $$Z = \sum_{\{\sigma\}} e^{-\beta H}$$

$$= \sum_{\{\sigma\}} e^{\sum_{i=0}^{L-1} \beta J \delta_{\sigma_i \sigma_{i+1}}}$$

$$= \sum_{\{\sigma\}} \prod_{i=0}^{L-1} e^{\beta J \delta_{\sigma_i \sigma_{i+1}}}$$

Let's define the transfer matrix

$$T_{\sigma_i \sigma_{i+1}} = e^{\beta J \delta_{\sigma_i \sigma_{i+1}}}$$

$$= \begin{pmatrix} e^{\beta J} & 1 & 1 \\ 1 & e^{\beta J} & 1 \\ 1 & 1 & e^{\beta J} \end{pmatrix}$$

We then have

$$Z = \sum_{\{\sigma\}} \prod_{i=0}^{L-1} e^{\beta J \delta_{\sigma_i \sigma_{i+1}}}$$

$$= \sum_{\{\sigma\}} \prod_{i=0}^{L-1} T_{\sigma_i \sigma_{i+1}}$$

$$= \sum_{\{\sigma\}} T_{\sigma_0 \sigma_1} T_{\sigma_1 \sigma_2} \cdots T_{\sigma_{L-1} \sigma_0}$$

$$= \sum_{\{\sigma_0\}} (T^N)_{\sigma_0 \sigma_0} \qquad (N = L)$$

$$= \mathrm{Tr}\{T^N\}$$

$$= \mathrm{Tr}\left\{ \begin{pmatrix} e^{\beta J} & 1 & 1 \\ 1 & e^{\beta J} & 1 \\ 1 & 1 & e^{\beta J} \end{pmatrix}^N \right\}$$

We need to diagonalize T. We start by finding its eigenvalues:

$$0 = \det(T - \lambda \mathbb{1})$$

$$= \begin{vmatrix} e^{\beta J}-\lambda & 1 & 1 \\ 1 & e^{\beta J}-\lambda & 1 \\ 1 & 1 & e^{\beta J}-\lambda \end{vmatrix}$$

$$0 = (e^{\beta J}-\lambda)^3 + 2 - 3(e^{\beta J}-\lambda)$$

Get rid of exponents by guessing $\lambda = e^{\beta J} - \lambda_0$ for some λ_0. This gives

$$0 = \lambda_0^3 + 2 - 3\lambda_0$$

Can quickly see that $\lambda_0 = 1$ is a solution. Using polynomial division we find

$$(\lambda_0^3 - 3\lambda_0 + 2) : (\lambda_0 - 1) = \underline{\lambda_0^2 + \lambda_0 - 2}$$

$$\begin{array}{l} \lambda_0^3 - \lambda_0^2 \\ \hline \quad \lambda_0^2 - 3\lambda_0 + 2 \\ \quad \lambda_0^2 - \lambda_0 \\ \hline \qquad -2\lambda_0 + 2 \\ \qquad -2\lambda_0 + 2 \\ \hline \qquad\qquad 0 \end{array}$$

$$\Rightarrow \lambda_0 = \frac{-1 \pm \sqrt{1+8}}{2} = \frac{-1 \pm 3}{2}$$

$$\Rightarrow \lambda_0 = 1 \lor \lambda_0 = -2$$

Thus we have

$\lambda = e^{\beta J} + 2$ with multiplicity 1

$\lambda = e^{\beta J} - 1$ with multiplicity 2

Diagonalizing gives

$$T = RDR^{-1}$$

with $D = \begin{pmatrix} e^{\beta J}+2 & 0 & 0 \\ 0 & e^{\beta J}-1 & 0 \\ 0 & 0 & e^{\beta J}-1 \end{pmatrix}$ and R an invertible matrix with the eigenvectors of T as columns.

Using this we have

$$Z = \mathrm{Tr}\{T^N\}$$

$$= \mathrm{Tr}\{\overbrace{\cancel{R^{-1}} D \cancel{R} \cancel{R^{-1}} D \cancel{R} \cdots \cancel{R^{-1}} D \cancel{R}}^{N \text{ times}}\}$$ ↳ cyclic property

$$= \mathrm{Tr}\{D^N\}$$

$$Z = \left(e^{\beta J}+2\right)^N + 2\left(e^{\beta J}-1\right)^N$$

Average total internal energy:

$$U = \frac{1}{Z}\sum_{\{\sigma\}} H e^{-\beta H}$$

$$= -\frac{1}{Z}\frac{\partial Z}{\partial \beta}$$

$$= -\frac{1}{Z}\left[\frac{\partial}{\partial\beta}\left(e^{\beta J}+2\right)^N + 2\frac{\partial}{\partial\beta}\left(e^{\beta J}-1\right)^N\right]$$

$$= -\frac{1}{Z}\left[N\left(e^{\beta J}+2\right)^{N-1} e^{\beta J}\cdot J + 2N\left(e^{\beta J}-1\right)^{N-1} e^{\beta J}\cdot J\right]$$

$$= -NJe^{\beta J}\frac{\left(e^{\beta J}+2\right)^{N-1} + 2\left(e^{\beta J}-1\right)^{N-1}}{\left(e^{\beta J}+2\right)^N + 2\left(e^{\beta J}-1\right)^N}$$

At very high temperatures, $\beta \to 0$:

$$\lim_{\beta\to 0} U = -NJ\frac{(1+2)^{\cancel{N-1}} + \cancel{2\cdot(1-1)^{N-1}}}{\cancel{(1+2)^N} + \cancel{2\cdot(1-1)^N}}$$

$$= -\frac{1}{3}NJ$$ High T

At very low temperatures, $\beta \to \infty$:

$$\lim_{\beta\to\infty} U = -NJ\lim_{\beta\to\infty} e^{\beta J}\frac{\left(e^{\beta J}\cancel{+2}\right)^{N-1} + 2\left(e^{\beta J}\cancel{-1}\right)^{N-1}}{\left(e^{\beta J}\cancel{+2}\right)^N + 2\left(e^{\beta J}\cancel{-1}\right)^N}$$ negligible as $\beta\to\infty$

$$= -NJ\lim_{\beta\to\infty}\frac{e^{N\beta J}+2e^{N\beta J}}{e^{N\beta J}+2e^{N\beta J}}$$

$$= -NJ$$ Low T

At low T all the spins have the same alignment. This can be seen since it is the only way to get $H = -J\sum_{i=0}^{L-1} \delta_{\sigma_i \sigma_{i+1}}$ to be $-JN$.

At high T the spin alignments are more random, with every spin taking on a random of the three possible values. On average, two neighbouring spins take the same value only one third of the time, resulting in $U = -\frac{1}{3}NJ$.

b) $\langle m \rangle = \frac{1}{Z} \sum_{\{\sigma\}} \frac{1}{N} \sum_{j=0}^{N-1} e^{i\frac{2\pi}{3}\sigma_j} \prod_{i=0}^{N-1} T_{\sigma_i \sigma_{i+1}}$

$$= \frac{1}{NZ} \sum_{j=0}^{N-1} \sum_{\{\sigma\}} e^{i\frac{2\pi}{3}\sigma_j} T_{\sigma_0\sigma_1} \cdots T_{\sigma_{N-1}\sigma_0}$$

$$= \frac{1}{NZ} \sum_{j=0} \sum_{\{\sigma\}} T_{\sigma_0\sigma_1} \cdots T_{\sigma_{j-1}\sigma_j} e^{i\frac{2\pi}{3}\sigma_j} T_{\sigma_j\sigma_{j+1}} \cdots T_{\sigma_{N-1}\sigma_0}$$

Let's define a matrix with the possible m_j-values along the diagonal:

$$M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{i\frac{2\pi}{3}} & 0 \\ 0 & 0 & e^{i\frac{4\pi}{3}} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{i\frac{2\pi}{3}} & 0 \\ 0 & 0 & e^{-i\frac{2\pi}{3}} \end{pmatrix}$$

We then have

$$\langle m \rangle = \frac{1}{NZ} \sum_{j=0}^{N-1} \sum_{\{\sigma\}} T_{\sigma_0\sigma_1} \cdots T_{\sigma_{j-1}\sigma_j} M_{\sigma_j\sigma_j} T_{\sigma_j\sigma_{j+1}} \cdots T_{\sigma_{N-1}\sigma_0}$$

$$= \frac{1}{NZ} \sum_{j=0}^{N-1} \mathrm{Tr}\{T^j M T^{N-j}\}$$

$$= \frac{1}{NZ} \sum_{j=0}^{N-1} \mathrm{Tr}\{M T^N\}$$

$$= \frac{1}{NZ} \sum_{j=0}^{N-1} \mathrm{Tr}\{M R D^N R^{-1}\}$$

Need eigenvectors of T to construct R. Let $\lambda_0 = e^{\beta J} + 2$, $\lambda_1 = \lambda_2 = e^{\beta J} - 1$. We find the eigenvectors by Gaussian elimination:

$$\lambda_0: \quad T - \lambda \mathbb{1} = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} \overset{\text{Wolfram}}{\sim} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \vec{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\lambda_{1,2}: \quad T - \lambda \mathbb{1} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \vec{v}_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \quad \vec{v}_3 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

$$\Rightarrow R = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix}, \quad R^{-1} \overset{\text{Wolfram}}{=} \frac{1}{3} \begin{pmatrix} 1 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}$$

Using this we find

$$T^N = R D^N R^{-1}$$

$$= \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix}\begin{pmatrix} \lambda_0^N & 0 & 0 \\ 0 & \lambda_1^N & 0 \\ 0 & 0 & \lambda_1^N \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}$$

$$= \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix}\begin{pmatrix} \lambda_0^N & \lambda_0^N & \lambda_0^N \\ \lambda_1^N & -2\lambda_1^N & \lambda_1^N \\ \lambda_1^N & \lambda_1^N & -2\lambda_1^N \end{pmatrix}$$

$$= \frac{1}{3}\begin{pmatrix} \lambda_0^N+2\lambda_1^N & \lambda_0^N-\lambda_1^N & \lambda_0^N-\lambda_1^N \\ \lambda_0^N-\lambda_1^N & \lambda_0^N+2\lambda_1^N & \lambda_0^N-\lambda_1^N \\ \lambda_0^N-\lambda_1^N & \lambda_0^N-\lambda_1^N & \lambda_0^N+2\lambda_1^N \end{pmatrix}$$

Furthermore, this gives

$$M T^N = \frac{1}{3}\begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{i\frac{2\pi}{3}} & 0 \\ 0 & 0 & e^{-i\frac{2\pi}{3}} \end{pmatrix}\begin{pmatrix} \lambda_0^N+2\lambda_1^N & \lambda_0^N-\lambda_1^N & \lambda_0^N-\lambda_1^N \\ \lambda_0^N-\lambda_1^N & \lambda_0^N+2\lambda_1^N & \lambda_0^N-\lambda_1^N \\ \lambda_0^N-\lambda_1^N & \lambda_0^N-\lambda_1^N & \lambda_0^N+2\lambda_1^N \end{pmatrix}$$

Wolfram

$$= \frac{1}{3}\begin{pmatrix} \lambda_0^N+2\lambda_1^N & \lambda_0^N-\lambda_1^N & \lambda_0^N-\lambda_1^N \\ e^{i\frac{2\pi}{3}}(\lambda_0^N-\lambda_1^N) & e^{i\frac{2\pi}{3}}(\lambda_0^N+2\lambda_1^N) & e^{i\frac{2\pi}{3}}(\lambda_0^N-\lambda_1^N) \\ e^{-i\frac{2\pi}{3}}(\lambda_0^N-\lambda_1^N) & e^{-i\frac{2\pi}{3}}(\lambda_0^N-\lambda_1^N) & e^{-i\frac{2\pi}{3}}(\lambda_0^N+2\lambda_1^N) \end{pmatrix}$$

Thus we find

$$\langle m \rangle = \frac{1}{NZ}\sum_{j=0}^{N-1} \mathrm{Tr}\{M T^N\}$$

$$= \frac{1}{NZ}\sum_{j=0}^{N} \frac{1}{3}\Big(1+\underbrace{e^{i\frac{2\pi}{3}}+e^{-i\frac{2\pi}{3}}}_{=2\cos(\frac{2\pi}{3}) = -1}\Big)(\lambda_0^N+2\lambda_1^N)$$

$$\langle m \rangle = 0$$

c) We have that

$$\langle m \rangle = \frac{1}{Z} \sum_{\{\sigma\}} \frac{1}{N} \sum_{i=0}^{N-1} m_i \, e^{-\beta H}$$

$$= \frac{1}{N} \sum_{i=0}^{N-1} \frac{1}{Z} \sum_{\{\sigma\}} m_i \, e^{-\beta H}$$

$$= \frac{1}{N} \sum_{i=0}^{N-1} \langle m_i \rangle$$

From the calculation in b) we know the content of this sum is zero, thus

$$\underline{\langle m_i \rangle = 0} \quad \forall i$$

We thus have

$$C(r) = \langle m_0^* m_r \rangle$$

$$= \frac{1}{Z} \sum_{\{\sigma\}} e^{-i\frac{2\pi}{3}\sigma_0} e^{i\frac{2\pi}{3}\sigma_r} \prod_{i=0}^{N-1} T_{\sigma_i \sigma_{i+1}}$$

$$= \frac{1}{Z} \sum_{\{\sigma\}} e^{-i\frac{2\pi}{3}\sigma_0} T_{\sigma_0\sigma_1} \cdots T_{\sigma_{r-1}\sigma_r} e^{i\frac{2\pi}{3}\sigma_r} T_{\sigma_r\sigma_{r+1}} \cdots T_{\sigma_{N-1}\sigma_0}$$

$$= \frac{1}{Z} \sum_{\{\sigma\}} M^\dagger_{\sigma_0\sigma_0} T_{\sigma_0\sigma_1} \cdots T_{\sigma_{r-1}\sigma_r} M_{\sigma_r\sigma_r} T_{\sigma_r\sigma_{r+1}} \cdots T_{\sigma_{N-1}\sigma_0}$$

$$= \frac{1}{Z} \mathrm{Tr}\left\{ M^\dagger T^r M T^{N-r} \right\}$$

From b) we have

$$M T^{N-r} = \frac{1}{3} \begin{pmatrix} \lambda_0^{N-r} + 2\lambda_1^{N-r} & \lambda_0^{N-r} - \lambda_1^{N-r} & \lambda_0^{N-r} - \lambda_1^{N-r} \\ e^{i\frac{2\pi}{3}}(\lambda_0^{N-r} - \lambda_1^{N-r}) & e^{i\frac{2\pi}{3}}(\lambda_0^{N-r} + 2\lambda_1^{N-r}) & e^{i\frac{2\pi}{3}}(\lambda_0^{N-r} - \lambda_1^{N-r}) \\ e^{-i\frac{2\pi}{3}}(\lambda_0^{N-r} - \lambda_1^{N-r}) & e^{-i\frac{2\pi}{3}}(\lambda_0^{N-r} - \lambda_1^{N-r}) & e^{-i\frac{2\pi}{3}}(\lambda_0^{N-r} + 2\lambda_1^{N-r}) \end{pmatrix}$$

and similarly we find

$$M^{\dagger}T^{r} = \frac{1}{3}\begin{pmatrix} \lambda_0^r+2\lambda_1^r & \lambda_0^r-\lambda_1^r & \lambda_0^r-\lambda_1^r \\ e^{-i\frac{2\pi}{3}}(\lambda_0^r-\lambda_1^r) & e^{-i\frac{2\pi}{3}}(\lambda_0^r+2\lambda_1^r) & e^{-i\frac{2\pi}{3}}(\lambda_0^r-\lambda_1^r) \\ e^{i\frac{2\pi}{3}}(\lambda_0^r-\lambda_1^r) & e^{i\frac{2\pi}{3}}(\lambda_0^r-\lambda_1^r) & e^{i\frac{2\pi}{3}}(\lambda_0^r+2\lambda_1^r) \end{pmatrix}$$

Putting the two together gives

$$\mathrm{Tr}\{M^{\dagger}T^{r}MT^{N-r}\} = \frac{1}{9}\underbrace{\mathrm{Tr}}_{\text{Trace, so we only need diagonal elements}}\left\{\begin{pmatrix} \lambda_0^r+2\lambda_1^r & \lambda_0^r-\lambda_1^r & \lambda_0^r-\lambda_1^r \\ e^{-i\frac{2\pi}{3}}(\lambda_0^r-\lambda_1^r) & e^{-i\frac{2\pi}{3}}(\lambda_0^r+2\lambda_1^r) & e^{-i\frac{2\pi}{3}}(\lambda_0^r-\lambda_1^r) \\ e^{i\frac{2\pi}{3}}(\lambda_0^r-\lambda_1^r) & e^{i\frac{2\pi}{3}}(\lambda_0^r-\lambda_1^r) & e^{i\frac{2\pi}{3}}(\lambda_0^r+2\lambda_1^r) \end{pmatrix}\right.$$

$$\left.\cdot\begin{pmatrix} \lambda_0^{N-r}+2\lambda_1^{N-r} & \lambda_0^{N-r}-\lambda_1^{N-r} & \lambda_0^{N-r}-\lambda_1^{N-r} \\ e^{i\frac{2\pi}{3}}(\lambda_0^{N-r}-\lambda_1^{N-r}) & e^{i\frac{2\pi}{3}}(\lambda_0^{N-r}+2\lambda_1^{N-r}) & e^{i\frac{2\pi}{3}}(\lambda_0^{N-r}-\lambda_1^{N-r}) \\ e^{-i\frac{2\pi}{3}}(\lambda_0^{N-r}-\lambda_1^{N-r}) & e^{-i\frac{2\pi}{3}}(\lambda_0^{N-r}-\lambda_1^{N-r}) & e^{-i\frac{2\pi}{3}}(\lambda_0^{N-r}+2\lambda_1^{N-r}) \end{pmatrix}\right\}$$

$$= \frac{1}{9}\Big[(\lambda_0^r+2\lambda_1^r)(\lambda_0^{N-r}+2\lambda_1^{N-r}) + \left(e^{i\frac{2\pi}{3}}+e^{-i\frac{2\pi}{3}}\right)(\lambda_0^r-\lambda_1^r)(\lambda_0^{N-r}-\lambda_1^{N-r})$$
$$+ (\lambda_0^r+2\lambda_1^r)(\lambda_0^{N-r}+2\lambda_1^{N-r}) + \left(e^{-i\frac{2\pi}{3}}+e^{\cancelto{i\frac{2\pi}{3}}{-i\frac{4\pi}{3}}}\right)(\lambda_0^r-\lambda_1^r)(\lambda_0^{N-r}-\lambda_1^{N-r})$$
$$+ (\lambda_0^r+2\lambda_1^r)(\lambda_0^{N-r}+2\lambda_1^{N-r}) + \left(e^{i\frac{2\pi}{3}}+e^{\cancelto{-i\frac{2\pi}{3}}{i\frac{4\pi}{3}}}\right)(\lambda_0^r-\lambda_1^r)(\lambda_0^{N-r}-\lambda_1^{N-r})\Big]$$

$$= \frac{1}{3}\Big[(\lambda_0^r+2\lambda_1^r)(\lambda_0^{N-r}+2\lambda_1^{N-r}) + \underbrace{2\cos\left(\tfrac{2\pi}{3}\right)}_{=-1}(\lambda_0^r-\lambda_1^r)(\lambda_0^{N-r}-\lambda_1^{N-r})\Big]$$

$$= \frac{1}{3}\Big[(\lambda_0^r+2\lambda_1^r)(\lambda_0^{N-r}+2\lambda_1^{N-r}) - (\lambda_0^r-\lambda_1^r)(\lambda_0^{N-r}-\lambda_1^{N-r})\Big]$$

$$= \frac{1}{3}\Big[\cancel{\lambda_0^N} + \cancelto{3}{2}\lambda_0^r\lambda_1^{N-r} + \cancelto{3}{2}\lambda_0^{N-r}\lambda_1^r + \cancelto{3}{4}\lambda_1^N - \cancel{\lambda_0^N} + \cancel{\lambda_0^r\lambda_1^{N-r}} + \cancel{\lambda_0^{N-r}\lambda_1^r} - \cancel{\lambda_1^N}\Big]$$

$$= \underline{\lambda_1^N + \lambda_0^r\lambda_1^{N-r} + \lambda_0^{N-r}\lambda_1^r}$$

Thus we get

$$C(r) = \frac{\lambda_1^N + \lambda_0^r\lambda_1^{N-r} + \lambda_0^{N-r}\lambda_1^r}{Z}$$

$$\underline{\underline{C(r) = \frac{\lambda_1^N + \lambda_0^r\lambda_1^{N-r} + \lambda_0^{N-r}\lambda_1^r}{\lambda_0^N + 2\lambda_1^N}}} \qquad = \frac{\lambda_1^r}{\lambda_0^r}$$

The limit $L \to \infty$ is equivalent to $N \to \infty$. Using $\lambda_0 > \lambda_1$ we find in this limit:

$$\lim_{N\to\infty} C(r) = \frac{\lambda_1^N + \lambda_0^r \lambda_1^{N-r} + \lambda_0^{N-r}\lambda_1^r}{\lambda_0^N + 2\lambda_1^N} \qquad \begin{matrix} \cdot \frac{1}{\lambda_0} \\ \cdot \frac{1}{\lambda_0} \end{matrix}$$

$$= \lim_{N\to\infty} \frac{\left(\frac{\lambda_1}{\lambda_0}\right)^N + \left(\frac{\lambda_1}{\lambda_0}\right)^{N-r} + \left(\frac{\lambda_1}{\lambda_0}\right)^r}{1 + 2\left(\frac{\lambda_1}{\lambda_0}\right)^N}$$

$0 < \frac{\lambda_1}{\lambda_0} < 1$

$$= \left(\frac{\lambda_1}{\lambda_0}\right)^r$$

$$= \underline{\underline{\left(\frac{e^{\beta J} - 1}{e^{\beta J} + 2}\right)^r}} \qquad \text{for } L \to \infty$$

For $T = 0$ we have $\beta \to \infty$, thus $\lambda_0 \approx \lambda_1 \approx e^{\beta J}$, so we get

$$\lim_{\beta\to\infty} C(r) = \lim_{\beta\to\infty} \frac{\lambda_1^N + \lambda_0^r \lambda_1^{N-r} + \lambda_0^{N-r}\lambda_1^r}{\lambda_0^N + 2\lambda_1^N}$$

$$= \lim_{\beta\to\infty} \frac{e^{\beta N J} + \cancel{e^{\beta r J}}\, e^{\beta(N\cancel{-r})J} + e^{\beta(N\cancel{-r})J}\, \cancel{e^{\beta r J}}}{e^{\beta N J} + 2e^{\beta N J}}$$

$$= \lim_{\beta\to\infty} \frac{3e^{\beta N J}}{3e^{\beta N J}}$$

$$= \underline{\underline{1}} \qquad \text{for } T = 0$$