Proofs

Brahmasta Jalu Damara

1 Proofs

Theorem 2.21.

Proof. First consider cyclotomic polynomial $\zeta^7 - 1 = 0$ and its solution,

$$\zeta = e^{i2\pi/7} = cos(\frac{2\pi}{7}) + isin(\frac{2\pi}{7}).$$

Using idea from [3] a polynomials will be constructed with a product function,,

$$\prod_{j=0}^{6}(x + \zeta^j y + \zeta^{2j} z + \zeta^{3j} t + \zeta^{4j} u + \zeta^{5j} v + \zeta^{6j} w) = 0 \tag{1.1}$$

with any $y, z, t, u, v, w \in \mathbb{F}$. Thus we got,

$$(x + y + z + t + u + v + w), j = 0$$

$$(x + (-1)^{2/7}y + (-1)^{4/7}z + (-1)^{6/7}t - (-1)^{1/7}u - (-1)^{3/7}v - (-1)^{5/7}w), j = 1$$
$$(x + (-1)^{4/7}y + (-1)^{1/7}z - (-1)^{5/7}t + (-1)^{2/7}u + (-1)^{6/7}v - (-1)^{3/7}w), j = 2$$
$$(x + (-1)^{6/7}y - (-1)^{5/7}z + (-1)^{4/7}t - (-1)^{3/7}u + (-1)^{2/7}v - (-1)^{1/7}w), j = 3$$
$$(x - (-1)^{1/7}y + (-1)^{2/7}z - (-1)^{3/7}t + (-1)^{4/7}u - (-1)^{5/7}v + (-1)^{6/7}w), j = 4$$
$$(x - (-1)^{3/7}y + (-1)^{6/7}z + (-1)^{2/7}t - (-1)^{5/7}u - (-1)^{1/7}v + (-1)^{4/7}w), j = 5$$
$$(x - (-1)^{5/7}y - (-1)^{3/7}z - (-1)^{1/7}t + (-1)^{6/7}u + (-1)^{4/7}v + (-1)^{2/7}w), j = 6$$

Using Maple software, we got a result a polynomials as result of the product function as follows,

$$x^7 + px^5 + qx^4 + rx^3 + sx^2 + \alpha x + \beta$$

with

$$p = -7yw - 7tu - 7zv$$
$$q = 14tvw + 14yzu + 7y^2v + 7zw^2 + 7u^2w + 7 + 7yt^2 + 7z^2t + 7uv^2$$
$$r = -21z^2uw - 21yu^2v - 21zt^2w - 21ytv^2 - 21y^2zt - 21uvw^2 + 14t^2u^2 + 14y^2w^2 + 14z^2v^2$$
$$-7tw^3 - 7v^3w + 7yzvw + 7ytuw - 7t^3v - 7yz^3 - 7ztuv - 7y^3u - 7zu^3$$
$$s = 35yuv^2w + 35yz^2tw + 35yzt^2v + 35y^2tuv + 35ztuw^2 + 35zu^2vw - 21y^3vw - 21yzw^3$$
$$-21yt^3u - 21z^3tv - 21zuv^3 - 21tu^3w + 14y^3z^2 - 14t^2uvw - 14yz^2uv - 14yztu^2 - 14ytvw^2$$
$$-14y^2zuw - 14ztv^2w + 14y^2u^3 + 14yv^4 + 14z^3u^2 + 14t^3w^2 + 14t^2v^3 + 14v^2w^3 - 7y^2zv^2$$
$$-7y^2t^2w - 7yu^2w^2 - 7z^2t^2u - 7z^2vw^2 - 7tu^2v^2 + 7y^4t + 7yv^4 + 7z^4w + 7zt^4 + 7u^4v + 7uw^4$$
$$\alpha = 14ztu^4 + 14yz^4v - 21yz^2t^3 + 14ytw^4 + 14y^4uw - 21y^3t^2v - 21y^3tu^2 - 21y^2uv^3 - 21y^2z^3w$$
$$-7y^2z^2u^2 - 21yv^3w^2 - 21zt^3v^2 - 21z^3tw^2 - 21z^2u^3v + 14zv^4w - 21zu^2w^3 + 14t^4uv - 21t^2uw^3$$
$$-7t^2v^2w^2 - 21u^3v^2w - 105yztuvw + 35tu^2vw^2 + 7tuv^3w + 7ztvw^3 - 14zuv^2w^2 - 14y^2z^2tv$$
$$+35y^2zt^2u + 35y^2zvw^2 - 14y^2tuw^2 + 35y^2tv^2w - 14y^2u^2vw + 7y^3ztw + 7y^3zuv + 7yz^3tu$$

$$+35yz^2uw^2 - 14yz^2v^2w + 7yzu^3w + 35yzu^2v^2 - 14yzt^2w^2 + 7yztv^3 + 7yt^3vw + 35yt^2u^2w$$
$$-14yt^2uv^2 + 7ytu^3v + 7yuvw^3 + 35z^2t^2vw - 14z^2tu^2w + 35z^2tuv^2 + 7z^3uvw + 7zt^3uw$$
$$-14zt^2u^2v - 7yu^5 - 7y^5z + 7y^4v^2 - 7y^3w^3 + 7y^2t^4 - 7z^5u + 7z^4t^2 - 7z^3v^3 + 7z^2w^4 - 7tv^5 - 7t^5w$$
$$-7t^3u^3 + 7u^4w^2 + 7u^2v^4 - 7vw^5$$
$$\beta = +y^7 + z^7 + t^7 + u^7 + v^7 + w^7 + 7y^3tuvw - 14y^2zt^2vw + 35y^2ztu^2w - 14y^2ztuv^2$$
$$+35y^2z^2uvw + 35yz^2tu^2v + 7yz^3tvw + 7yzt^3uv + 7yztuw^3 + 35yztv^2w^2 - 14yzu^2vw^2$$
$$+7yzuv^3w - 14ytu^2v^2w + 35yt^2uvw^2 - 14z^2tuvw^2 + 14yzt^4w - 21yzt^2u^3 - 21y^3z^2tu$$
$$-21y^3zuw^2 - 21y^3zv^2w + 14y^4ztv - 7y^2z^2tw^2 - 21y^2zu^3v - 21y^2t^3uw - 7y^2t^2u^2v - 21y^2tvw^3$$
$$-7y^2uv^2w^2 - 21yz^3u^2w - 21yz^3uv^2 - 7yz^2t^2v^2 - 21yz^2vw^3 - 21yt^2v^3w - 21ytu^3w^2 + 14ytuv^4$$
$$+14yu^4vw + 7z^2t^4v + 14z^2t^3u^2 + 14z^2t^2w^3 - 7ztw^5 - 7z^5vw + 7z^4tv^2 + 7z^4u^2v - 7z^3uw^3$$
$$-7z^3t^3w - 7z^3tu^3 + 14z^3v^2w^2 + 14z^2u^3w^2 + 7z^2uv^4 - 7zv^3w^3 - 7zt^5u + 7zt^2v^4 - 7zu^5w$$
$$+7zu^4v^2 + 14y^2t^3v^2 + 7y^2tu^4 - 7y^5tw - 7y^5uv + 7y^4z^2w + 7y^4zu^2 + 7y^4t^2u + 7y^4vw^2 - 7y^3u^3w$$
$$+14y^3u^2v^2 - 7y^3z^3v - 7y^3zt^3 + 14y^3t^2w^2 - 7y^3tv^3 + 7y^2v^4w + 14y^2u^2w^3 - 7yzv^5 - 7yz^5t$$
$$+7yz^4w^2 + 7yz^2u^4 + 7yv^2w^4 - 7yt^5v + 7yt^4u^2 - 7yt^3w^3 - 7yuw^5 - 7yu^3v^3 + 7ztu^3vw$$
$$+35zt^2uv^2w - 21z^3t^2uv + 14z^4tuw - 21z^2tv^3w - 7z^2u^2v^2w - 21zt^3vw^2 - 7zt^2u^2w^2$$
$$-21ztu^2v^3 + 14zuvw^4 - 21t^3u^2vw - 21tuv^2w^3 + 7t^4uw^2 + 7t^4v^2w - 7t^3uv^3 + 7t^2vw^4 + 7t^2u^4w$$
$$+14t^2u^3v^2 + 7tv^4w^2 - 7tu^5v + 7tu^2w^4 - 7u^3vw^3 + 14u^2v^3w^2 - 7uv^5w + 7y^2z^4u + 14y^2z^3t^2$$
$$+14y^2z^2v^3 + 7y^2zw.$$

Has a solution by radical which is

$$x_j = -\zeta^j y - \zeta^{2j} z - \zeta^{3j} t - \zeta^{4j} u - \zeta^{5j} v - \zeta^{6j} w, j = 0, 1, 2, 3, 4, 5, 6$$

$\square$