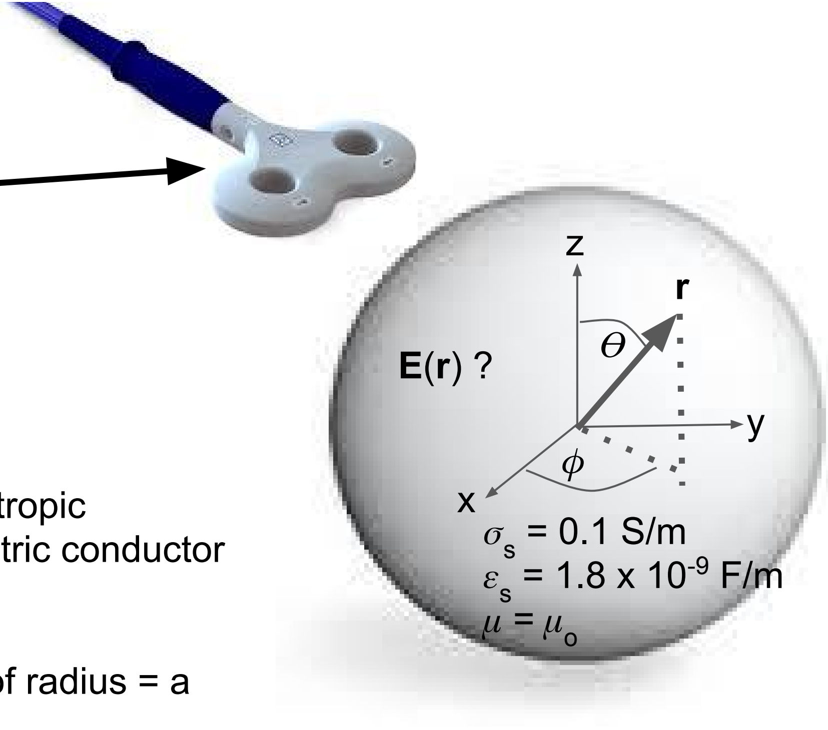
TMS Physics: Quasi-Static E Field within a Spherical Conductor Due to an External Current Source

Known externally applied current **J**(**r**)



σ = 0
$\mu = \mu_o = 4\pi \times 10^{-7}$ H/m
$\varepsilon = \varepsilon_o = 8.85 \times 10^{-12}$ F/m

σ_s = 0.1 S/m
$\varepsilon_s = 1.8 \times 10^{-9}$ F/m
$\mu = \mu_o$

Homogeneous isotropic spherically symmetric conductor

spherical volume of radius = a

See [6] for typical material constants

The Physics

- Time varying current supplied by the TMS unit induces a *primary* electric field in the conductor which produces a distribution of charge via an ohmic current which gives rise to a *secondary* electric field.

- Quasi-static approximation: Typical TMS frequency of 10 KHz so system behaves as though it reaches equilibrium on a time-scale much less than the period of the characteristic temporal frequency. As a consequence any charge build up will reside only at the surface of the conductor as is the case with a conductor in equilibrium. Also since ∇ □**J**(**r**,t) = 0 for a conductor at equilibrium then, at the surface of a conductor with homogeneous isotropic conductivity, ie E_r(**r**,t) = 0 at r = a.

- The vector potential, **A**(**r**,t)**,** is determined by the current within the stimulation coils only. The skin depth for an arbitrary conductor is given by [6]

$$\delta = (2\pi f)^{-1}([\sqrt{1+(\sigma/2\pi f\epsilon)^2}-1]\mu\epsilon/2)^{-1/2}$$

Using the assumed constants the skin depth is 0.8 m at f = 10 MHz. Such a relatively large skin depth (compared to sphere diameter) means that the ohmic current in the conductor is not a significant contributor to **A**(**r**,t). As a consequence the condition ∇ x **B**(**r**,t) = 0 applies within the conducting sphere when calculating **A**(**r**,t).

The Physics

- We want to determine **E**(**r**,t) within the conducting sphere therefore we need to determine the scalar potential **Φ**(**r**,t) and magnetic vector potential **A**(**r**,t). The **E** field is then given by

$$\mathbf{E}(\mathbf{r},t) = -\nabla\Phi(\mathbf{r},t) - \frac{1}{c}\frac{\partial\mathbf{A}(\mathbf{r},t)}{\partial t}$$

secondary electric field

primary electric field

The Math: The Vector Potential

For quasi-static approximation the vector potential is given everywhere by

$$\mathbf{A}(\mathbf{r},t)=\frac{1}{c}\iiint\frac{\mathbf{J}(\mathbf{r},t)}{|\mathbf{r}-\mathbf{r}'|}d\mathbf{r}^3$$

Where the current density is that due to the TMS coils only. Using a vector spherical harmonic expansion [5] of the integrand

$$\mathbf{A}(\mathbf{r},t)=\frac{4\pi}{c}\sum_{l\,jm}\frac{r^l}{2l+1}\mathbf{Y}^l_{jm}(\theta,\phi)\iiint\frac{1}{r'^{l+1}}\mathbf{J}(\mathbf{r}',t)\cdot\mathbf{Y}^{*l}_{jm}(\theta',\phi')r'^2\sin\theta'\,d\theta'\,d\phi'$$

where j = 0, …, ∞ ; l = j, j±1 and m = -j, …, 0, … j. Which can be rewritten as.

$$\mathbf{A}(\mathbf{r},t)=\sum_{l\,jm}\frac{r^l}{2l+1}\mathbf{Y}^l_{jm}(\theta,\phi)A^l_{jm}(t)$$

The Math: The Vector Potential

where

$$A^{l}_{jm}(t) = \frac{4\pi}{c} \iiint \frac{1}{r'^{l+1}} \mathbf{J}(\mathbf{r}', t) \cdot \mathbf{Y}^{*l}_{jm}(\theta', \phi')\, r'^{2} \sin\theta'\ dr' d\theta' d\phi'$$

Since the applied current is assumed to be known then the coefficients given above are known as well. Applying the previously mentioned constraint ∇ x **B**(**r**,t) = ∇ x ∇ x **A**(**r**,t) = 0 one finds that

$$A^{j+1}_{jm}(t) = 0$$

so that

$$\mathbf{A}(\mathbf{r}, t) = \sum_{jm} \left[\frac{r^{j}}{2j+1} \mathbf{Y}^{j}_{jm}(\theta, \phi) A^{j}_{jm}(t) + \frac{r^{j-1}}{2j-1} \mathbf{Y}^{j-1}_{jm}(\theta, \phi) A^{j-1}_{jm}(t) \right]$$

The Math: The Scalar Potential

The scalar potential in quasistatic approximation is given by

$$\Phi(\mathbf{r}, t) = \int \frac{\rho(\mathbf{r}', t)}{|\mathbf{r} - \mathbf{r}'|} d^3\mathbf{r}'$$

Spherical harmonic expansion applicable to interior of the sphere

$$\Phi(\mathbf{r}, t) = 4\pi \iint \rho(r', \theta', \phi', t) \sum_{jm} \frac{1}{2j+1} \frac{r^j}{r'^{j+1}} Y_{jm}(\theta, \phi) Y^*_{jm}(\theta', \phi') r'^2 \sin\theta' \, dr' \, d\theta' \, d\phi'$$

Assuming the charge is distributed on the surface of the sphere only:

$$\rho(r, \theta, \phi, t) = \frac{1}{4\pi r^2} \delta(r - a) \sigma(\theta, \phi, t)$$

The Math: The Scalar Potential

and integrating with respect to *r*' we get

$$\Phi(\mathbf{r},t) = \iint \sigma(\theta',\phi',t)\sum_{jm}\frac{1}{2j+1}\frac{r^j}{a^{j+1}}Y_{jm}(\theta,\phi)Y^*_{jm}(\theta',\phi')\sin\theta'\,d\theta'\,d\phi'$$

which can be rewritten as

$$\Phi(\mathbf{r},t) = \sum_{jm}\frac{r^j}{2j+1}Y_{jm}(\theta,\phi)C_{jm}(t)$$

where

$$C_{jm}(t) = \frac{1}{a^{j+1}}\iint \sigma(\theta',\phi',t)Y^*_{jm}(\theta',\phi')\sin\theta'\,d\theta'\,d\phi'$$

The Math: Applying Boundary Condition

Our job is to find the $C_{jm}(t)$ in terms of the known the $A^{I}_{jm}(t)$. We do this by using the boundary condition on the component of E normal to the surface (the radial component) at *r* = *a*:

$$0 = \left[\nabla \Phi(\mathbf{r}, t) \cdot \hat{\mathbf{r}} + \frac{1}{c} \frac{\partial \mathbf{A}(\mathbf{r}, t)}{\partial t} \cdot \hat{\mathbf{r}} \right]_{r=a}$$

First we calculate the **E** field inside sphere

$$\mathbf{E}(\mathbf{r}, t) = \sum_{jm} C_{jm}(t) \sqrt{\frac{j}{2j+1}}\, r^{j-1}\, \mathbf{Y}^{j-1}_{jm}(\theta, \phi)$$
$$+ \frac{1}{c} \sum_{jm} \left[\frac{\partial A^{j}_{jm}}{\partial t} \frac{r^{j}}{2j+1} \mathbf{Y}^{j}_{jm}(\theta, \phi) + \frac{\partial A^{j-1}_{jm}}{\partial t} \frac{r^{j-1}}{2j-1} \mathbf{Y}^{j-1}_{jm}(\theta, \phi) \right]$$

The Math: Applying Boundary Condition

And then we apply the boundary condition on the radial component of **E** and find

$$C_{jm}(t) = -\frac{1}{c(2j-1)}\sqrt{\frac{2j+1}{j}}\frac{\partial A_{jm}^{j-1}}{\partial t}$$

Putting it all together the **E** field inside the sphere is:

$$\mathbf{E}(\mathbf{r},t) = \frac{1}{c}\sum_{jm}\frac{\partial A_{jm}(t)}{\partial t}\frac{r^j}{2j+1}\,\mathbf{Y}_{jm}^{j}(\theta,\phi)$$

where the particular vector spherical harmonic above is given by

$$\mathbf{Y}_{jm}^{j}(\theta,\phi) = -\mathbf{e}_\theta\frac{m}{\sqrt{j(j+1)}}\,\frac{1}{\sin\theta}\,Y_{jm}(\theta,\phi) - \mathbf{e}_\phi\frac{i}{\sqrt{j(j+1)}}\,\frac{\partial Y_{jm}(\theta,\phi)}{\partial\theta}$$

Discussion

- The electric field has no radial component.
- The magnitude of the electric field decreases as *r* decreases (as we move toward sphere’s center).
- The magnitude of spatially small-scale components of the field fall off faster with respect to *r*. Therefore any focal angular distribution the field magnitude near the surface will blur as *r* decreases.

References

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[2] M. Bencsik, R. Bowtell and R. M. Bowley: Electric fields induced in a spherical volume conductor by temporally varying magnetic field gradients, Phys. Med. Biol., Vol 47, 2002, 557-576.

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[4] R. Plonsey and D. B. Heppner: Considerations of Quasi-stationarity in Electrophysiological Systems, Bulletin of Mathematical Biophysics,, Vol 29, 1967, 657-664.

[5] D. A. Varshalovich, A. N. Moskalev and V. K. Khersonskii, Quantum Theory of Angular Momentum, World Scientific, 208-229, 1988.

References

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