Time Independent Qubit – Resonator Hamiltonian (Quantum Rabi Model)

① Hamiltonian without drive

$$H = -\frac{\omega_q}{2}\sigma_z \otimes I + \omega_r I \otimes a^\dagger a + g\sigma_x \otimes (a + a^\dagger)$$

RWA

$$H = -\frac{\omega_q}{2}\sigma_z \otimes I + \omega_r I \otimes a^\dagger a + g(\sigma_+ \otimes a + \sigma_- \otimes a^\dagger)$$

$\hat{P}_e = |0\rangle\langle 0| + |1\rangle\langle 1| = I \qquad [H, P_e] = 0$

$\hat{N}_e = |1\rangle\langle 1| + a^\dagger a \qquad [H, N_e] = 0$

$\hat{H} = \hat{H}_1 + \hat{H}_{11}$, $\quad \hat{H}_1 = \omega_r \hat{N}_e + \left(\frac{\omega_q}{2} - \omega_r\right)\hat{P}_e \qquad \hat{H}_2 = (\omega_r - \omega_q)|0\rangle\langle 0| + g(\sigma_+ a + \sigma_- a^\dagger)$

$\omega_r a^\dagger a + \omega_r |1\rangle\langle 1| + \frac{\omega_q}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|) - \omega_r(|0\rangle\langle 0| + |1\rangle\langle 1|)$

$= \omega_r a^\dagger a - \omega_r |0\rangle\langle 0| + \frac{\omega_q}{2}|0\rangle\langle 0| + \frac{\omega_q}{2}|1\rangle\langle 1|$

$[H_1, H_2] = 0$

$$H|0\rangle|n+1\rangle = -\frac{\omega_q}{2}|0\rangle|n+1\rangle + (n+1)\omega_r|0\rangle|n+1\rangle + \sqrt{n+1}\,g|1\rangle|n\rangle$$

$$= \left(\omega_r(n+1) - \frac{\omega_q}{2}\right)|0\rangle|n+1\rangle + g\sqrt{n+1}|1\rangle|n\rangle$$

$$H|1\rangle|n\rangle = \left(\omega_r n + \frac{\omega_q}{2}\right)|1\rangle|n\rangle + g\sqrt{n+1}|0\rangle|n+1\rangle$$

$$H(\alpha|0,n+1\rangle + \beta|1,n\rangle) = \left[\alpha\left(\omega_r(n+1) - \frac{\omega_q}{2}\right) + \beta g\sqrt{n+1}\right]|0,n+1\rangle + \left[\alpha g\sqrt{n+1} + \beta\left(\omega_r n + \frac{\omega_q}{2}\right)\right]|1,n\rangle$$

$$= h(\alpha|0,n+1\rangle + \beta|1,n\rangle)$$

$$\Rightarrow \frac{\alpha\left(\omega_r(n+1) - \frac{\omega_q}{2}\right) + \beta g\sqrt{n+1}}{\alpha} = \frac{\beta\left(\omega_r n + \frac{\omega_q}{2}\right) + \alpha g\sqrt{n+1}}{\beta}$$

$\Delta = \omega_r - \omega_q$

$\Delta + \left(\frac{t^2-1}{t}\right)g\sqrt{n+1} = 0$, $\quad t = \frac{\beta}{\alpha}$

$$\Rightarrow t^2 g\sqrt{n+1} - t\Delta - g\sqrt{n+1} = 0.$$

eigen state: $\frac{1}{\sqrt{t_1^2+1}}|0,n+1\rangle + \frac{t_1}{\sqrt{t_1^2+1}}|1,n\rangle$,

$\frac{1}{\sqrt{t_2^2+1}}|0,n+1\rangle + \frac{t_2}{\sqrt{t_2^2+1}}|1,n\rangle$

$$\Rightarrow t = \frac{\Delta \pm \sqrt{\Delta^2 + 4g^2(n+1)}}{2g\sqrt{n+1}} = \frac{\beta}{\alpha}$$

$$= \frac{1 \pm \sqrt{1 + \tan^2(\Phi_n)}}{\tan(\Phi_n)}, \quad \tan(\Phi_n) = \frac{2g\sqrt{n+1}}{\Delta}$$

$$t^2+1 = \frac{1 + (1 + \tan^2(\Phi_n)) \pm 2\sqrt{1+\tan^2\Phi_n}}{\tan^2(\Phi_n)} + 1 = \frac{\sec^2\Phi_n \pm 2\sec\Phi_n + 1 + \tan^2\Phi_n}{\tan^2\Phi_n} = \frac{2(\sec^2\Phi_n \pm \sec\Phi_n)}{\tan^2\Phi_n}$$

$$\sin\left(\frac{\Phi_n}{2}\right) = \frac{\sqrt{1 - \cos\Phi_n}}{\sqrt{2}} \qquad \cos\left(\frac{\Phi_n}{2}\right) = \frac{\sqrt{1 + \cos\Phi_n}}{\sqrt{2}}$$

$$\frac{1}{\sqrt{t^2+1}} = \frac{1}{\sqrt{2}}\frac{\tan\Phi_n}{\sqrt{\sec\Phi_n}\sqrt{\sec\Phi_n \pm 1}} = \frac{1}{\sqrt{2}}\frac{\sin\Phi_n}{\sqrt{1 \pm \cos\Phi_n}} = \sqrt{\frac{1 \mp \cos\Phi_n}{2}} = \sin\left(\frac{\Phi_n}{2}\right) \text{ or } \cos\left(\frac{\Phi_n}{2}\right)$$

$$\frac{t}{\sqrt{t^2+1}} = \sqrt{\frac{1 \mp \cos\Phi_n}{2}}\frac{\cos\Phi_n \pm 1}{\sin\Phi_n} = \frac{1}{\sqrt{2}}\frac{\cos\Phi_n \pm 1}{\sqrt{1 \pm \cos\Phi_n}} = \cos\left(\frac{\Phi_n}{2}\right) \text{ or } -\sin\left(\frac{\Phi_n}{2}\right)$$

$$|n,+\rangle = \cos\left(\frac{\theta_n}{2}\right)|1,n\rangle + \sin\left(\frac{\theta_n}{2}\right)|0,n+1\rangle \qquad E_\pm(n) = \left(n+\frac{1}{2}\right)\omega_g \pm \sqrt{(\omega_g-\omega_r)^2+4g^2(n+1)}$$

$$|n,-\rangle = -\sin\left(\frac{\theta_n}{2}\right)|1,n\rangle + \cos\left(\frac{\theta_n}{2}\right)|0,n+1\rangle$$

$\boxed{\text{② Dispersive Regime}}$

$$H = H_0 + H_1, \quad i\frac{d}{dt}|\psi_{sp}(t)\rangle = (H_0+H_1)|\psi_{sp}(t)\rangle, \quad |\psi_{Ip}(t)\rangle = U_0^{-1}|\psi_{sp}(t)\rangle$$

$$i\frac{d}{dt}|\psi_{Ip}(t)\rangle = H_{Ip}|\psi_{Ip}(t)\rangle \quad \text{when} \quad H_{Ip}(t) = U_0^\dagger H U_0 - iU_0^\dagger \frac{dU_0}{dt}$$

$$U_0 = e^{-iH_0 t} \quad U_0^\dagger = e^{iH_0 t} \qquad H_0 = \sum \lambda|\lambda\rangle\langle\lambda|, \; U_0 = e^{-i\lambda t}|\lambda\rangle\langle\lambda|, \; \frac{dU_0}{dt} = -i\lambda e^{-i\lambda t}|\lambda\rangle\langle\lambda|$$

$$H_{Ip} = e^{iH_0 t}(H_0+H_1)e^{-iH_0 t} - H_0 = e^{iH_0 t}H_1 e^{-iH_0 t}$$

$$[H_0,H_1] = it\left[-\frac{\omega_g}{2}\sigma_z + \omega_r a^\dagger a,\; g(a\sigma_+ + a^\dagger\sigma_-)\right] = -\frac{\omega_g}{2}g(a\sigma_z\sigma_+ + a^\dagger\sigma_z\sigma_-) + \omega_r g(a^\dagger a a\sigma_+ + a^\dagger a a^\dagger\sigma_-)$$

$$+\frac{\omega_g}{2}g(a\sigma_+\sigma_z + a^\dagger\sigma_-\sigma_z) - \omega_r g(a a^\dagger a\sigma_+ + a^\dagger a^\dagger a\sigma_-)$$

$$\sigma_+\sigma_z = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \qquad \sigma_z\sigma_+ = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}$$

$$\sigma_-\sigma_z = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} \qquad \sigma_z\sigma_- = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$[H_0,H_1] = \omega_g g(a\sigma_+ - a^\dagger\sigma_-) + \omega_r g(-a\sigma_+ + a^\dagger\sigma_-) \qquad \Delta = \omega_r - \omega_g$$

$$= it\Delta g(a^\dagger\sigma_- - a\sigma_+)$$

$$[H_0,[H_0,H_1]] = -t^2\Delta^2 g(+a^\dagger\sigma_- + a\sigma_+) \quad [H_0,[H_0,[H_0,H_1]]] = -it^3\Delta^3 g(+a^\dagger\sigma_- - a\sigma_+) \cdots$$

$$\Rightarrow g(a^\dagger\sigma_- e^{i\Delta t} + a\sigma_+ e^{-i\Delta t})$$

$$H_{Ip}(t) = g\left(a^\dagger\sigma_- e^{i(\omega_r-\omega_g)t} + a\sigma_+ e^{-i(\omega_r-\omega_g)t}\right)$$

$$|\psi_{Ip}(t)\rangle = \hat{\tau}\left[\exp\left(-i\int_0^t H_{Ip}(t')dt'\right)\right]|\psi_{Ip}(0)\rangle$$

$$= \left[I - i\int_0^t H_{Ip}(t')dt' - \frac{1}{2}\left(\int_0^t H_{Ip}(t')dt'\right)^2 + \cdots\right]|\psi_{Ip}(0)\rangle$$

$$g\int_0^t a^\dagger\sigma_- e^{i\Delta t'} + a\sigma_+ e^{-i\Delta t'}\,dt' = \frac{g}{i\Delta}\left[(e^{i\Delta t}-1)a^\dagger\sigma_- - (e^{-i\Delta t}-1)a\sigma_+\right]$$

$$\Rightarrow \hat{\tau}(\quad) = I - \frac{g}{\Delta}\left[(e^{i\Delta t}-1)a^\dagger\sigma_- - (e^{-i\Delta t}-1)a\sigma_+\right] + \frac{ig^2 t}{\Delta}[a\sigma_+, a^\dagger\sigma_-]$$

$\langle a\sigma_+\rangle \cong \langle a^\dagger\sigma_- a\sigma_+\rangle^{\frac{1}{2}}$ is not too large and $\left|\frac{g}{\Delta}\langle A^\dagger A\rangle^{\frac{1}{2}}\right| \ll 1$, then second term is dropped

$$\hat{\tau}\left[\exp\left(-i\int_0^t dt'\hat{H}_{Ip}(t')\right)\right] \cong \hat{I} - it\hat{H}_{eff} \qquad [a\sigma_+, a^\dagger\sigma_-] = aa^\dagger\sigma_+\sigma_- - a^\dagger a\sigma_-\sigma_+$$

$$\sigma_+\sigma_- = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \quad \sigma_-\sigma_+ = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$= \sigma_+\sigma_- + (aa^\dagger - (aa^\dagger - a^\dagger a))\sigma_+\sigma_- - a^\dagger a\,\sigma_-\sigma_+ = \sigma_+\sigma_- - \sigma_z a^\dagger a$$

$$H_{eff} = \underbrace{\frac{g^2}{\omega_q-\omega_r}}_{\chi}\left(\sigma_+\sigma_- - \sigma_z a^\dagger a\right)$$

$\boxed{\text{③ Hamiltonian with drive}}$ i) first try

$$H = -\frac{\omega_q}{2}\sigma_z\otimes I + \omega_r I\otimes a^\dagger a + g(\sigma_+ a + \sigma_- a^\dagger) + A(t)(a+a^\dagger)\cos\omega_{rd}t$$

$$\tilde{H}_r = RHR^\dagger + i\dot{R}R^\dagger, \quad R = e^{i\omega_{rd}tH_0}, \quad H_0 = -\frac{\omega_q}{2}\sigma_z + \omega_r a^\dagger a + g(\sigma_+ a + \sigma_- a^\dagger)$$

$$\tan\Phi_n = \frac{2g\sqrt{n+1}}{\Delta}$$

$$|n,+\rangle = \cos\left(\frac{\Phi_n}{2}\right)|1,n\rangle + \sin\left(\frac{\Phi_n}{2}\right)|0,n+1\rangle \qquad E_\pm(n) = \left(n+\tfrac{1}{2}\right)\omega_q \pm \sqrt{(\omega_r-\omega_q)^2 + 4g^2(n+1)}$$

$$|n,-\rangle = -\sin\left(\frac{\Phi_n}{2}\right)|1,n\rangle + \cos\left(\frac{\Phi_n}{2}\right)|0,n+1\rangle$$

$$\sin\left(\frac{\Phi_n}{2}\right) = \frac{1}{\sqrt{2}}\sqrt{1 - \frac{\Delta}{\sqrt{\Delta^2+4g^2(n+1)}}}$$

$$\cos\left(\frac{\Phi_n}{2}\right) = \frac{1}{\sqrt{2}}\sqrt{1 + \frac{\Delta}{\sqrt{\Delta^2+4g^2(n+1)}}}$$

$$e^{i\omega_{rd}tH_0} = e^{i\omega_{rd}t\sum_{n=0}^{\infty} E_+(n)|n,+\rangle\langle n,+| + E_-(n)|n,-\rangle\langle n,-|}$$

$$= \sum_{n=0}^{\infty} e^{i\omega_{rd}t(n+\frac{1}{2})\omega_q}\left(e^{+i\omega_{rd}t\sqrt{\Delta^2+4g^2(n+1)}}|n,+\rangle\langle n,+| + e^{-i\omega_{rd}t\sqrt{\Delta^2+4g^2(n+1)}}|n,-\rangle\langle n,-|\right)$$

$$= \sum_{n=0}^{\infty} e^{i(n+\frac{1}{2})\omega_{rd}\omega_q t}\Big[e^{i\omega_{rd}\sqrt{\Delta^2+4g^2(n+1)}\,t}\Big(\cos^2\left(\tfrac{\Phi_n}{2}\right)|1,n\rangle\langle 1,n| + \sin\left(\tfrac{\Phi_n}{2}\right)\cos\left(\tfrac{\Phi_n}{2}\right)|1,n\rangle\langle 0,n+1| + \sin\left(\tfrac{\Phi_n}{2}\right)\cos\left(\tfrac{\Phi_n}{2}\right)|0,n+1\rangle\langle 1,n|$$
$$+ \sin^2\left(\tfrac{\Phi_n}{2}\right)|0,n+1\rangle\langle 0,n+1|\Big) + e^{-i\omega_{rd}\sqrt{\Delta^2+4g^2(n+1)}\,t}\Big(\sin^2\left(\tfrac{\Phi_n}{2}\right)|1,n\rangle\langle 1,n| - \sin\left(\tfrac{\Phi_n}{2}\right)\cos\left(\tfrac{\Phi_n}{2}\right)|1,n\rangle\langle 0,n+1| - \sin\left(\tfrac{\Phi_n}{2}\right)\cos\left(\tfrac{\Phi_n}{2}\right)|0,n+1\rangle\langle 1,n|$$
$$+ \cos^2\left(\tfrac{\Phi_n}{2}\right)|0,n+1\rangle\langle 0,n+1|\Big)\Big]$$

$$e^{i\alpha_n t}\frac{1+\cos\Phi_n}{2} + e^{-i\alpha_n t}\frac{1-\cos\Phi_n}{2} = \cos\alpha_n t + i\sin\alpha_n t\cos\Phi_n$$

$$e^{i\alpha_n t}\frac{\sin\Phi_n}{2} - e^{-i\alpha_n t}\frac{\sin\Phi_n}{2} = i\sin\alpha_n t\sin\Phi_n$$

$$e^{i\alpha_n t}\frac{1-\cos\Phi_n}{2} + e^{-i\alpha_n t}\frac{1+\cos\Phi_n}{2} = \cos\alpha_n t - i\sin\alpha_n t\cos\Phi_n$$

$$\alpha_n = \omega_{rd}\sqrt{\Delta^2+4g^2(n+1)}$$
$$\Phi_n = \tan^{-1}\left(\frac{2g\sqrt{n+1}}{\Delta}\right)$$

$$\rightarrow e^{i\omega_{rd}tH_0} = \sum_{n=0}^{\infty} e^{i(n+\frac{1}{2})\omega_{rd}\omega_q t}\Big[(\cos\alpha_n t + i\sin\alpha_n t\cos\Phi_n)|1,n\rangle\langle 1,n| + (\cos\alpha_n t - i\sin\alpha_n t\cos\Phi_n)|0,n+1\rangle\langle 0,n+1|$$
$$+ i\sin\alpha_n t\sin\Phi_n\left(|1,n\rangle\langle 0,n+1| + |0,n+1\rangle\langle 1,n|\right)\Big]$$

$$e^{-i\omega_{rd}tH_0} = \sum_{n=0}^{\infty} e^{-i(n+\frac{1}{2})\omega_{rd}\omega_q t}\Big[(\cos\alpha_n t - i\sin\alpha_n t\cos\Phi_n)|1,n\rangle\langle 1,n| + (\cos\alpha_n t + i\sin\alpha_n t\cos\Phi_n)|0,n+1\rangle\langle 0,n+1|$$
$$- i\sin\alpha_n t\sin\Phi_n\left(|1,n\rangle\langle 0,n+1| + |0,n+1\rangle\langle 1,n|\right)\Big]$$

$$\tilde{H} = RHR^\dagger + i\dot{R}R^\dagger$$

$$\dot{R} = \sum_{n=0}^{\infty} i(n+\tfrac{1}{2})\omega_q\omega_{rd}\, e^{i(n+\frac{1}{2})\omega_{rd}\omega_q t}\Big[(\cos\alpha_n t + i\sin\alpha_n t\cos\Phi_n)|1,n\rangle\langle 1,n| + (\cos\alpha_n t - i\sin\alpha_n t\cos\Phi_n)|0,n+1\rangle\langle 0,n+1|$$
$$+ i\sin\alpha_n t\sin\Phi_n\left(|1,n\rangle\langle 0,n+1| + |0,n+1\rangle\langle 1,n|\right)\Big]$$
$$+ \sqrt{\Delta^2+4g^2(n+1)}\,\omega_{rd}\, e^{i(n+\frac{1}{2})\omega_q\omega_{rd} t}\Big[(-\sin\alpha_n t + i\cos\alpha_n t\cos\Phi_n)|1,n\rangle\langle 1,n| + (-\sin\alpha_n t - i\cos\alpha_n t\cos\Phi_n)|0,n+1\rangle\langle 0,n+1|$$
$$+ i\cos\alpha_n t\sin\Phi_n\left(|1,n\rangle\langle 0,n+1| + |0,n+1\rangle\langle 1,n|\right)\Big]$$

let $\Omega(n) = \sqrt{\Delta^2+4g^2(n+1)}$,

$$R = \sum_{n=0}^{\infty}\left(e^{i\omega_d E_+(n)}\cos^2\left(\tfrac{\Phi_n}{2}\right)+e^{i\omega_d E_-(n)}\sin^2\left(\tfrac{\Phi_n}{2}\right)\right)|1,n\rangle\langle 1,n| + \left(e^{i\omega_d E_+(n)}\sin^2\left(\tfrac{\Phi_n}{2}\right)+e^{i\omega_d E_-(n)}\cos^2\left(\tfrac{\Phi_n}{2}\right)\right)|0,n+1\rangle\langle 0,n+1| + \left(e^{i\omega_d E_+(n)}+e^{i\omega_d E_-(n)}\right)\sin\left(\tfrac{\Phi_n}{2}\right)\cos\left(\tfrac{\Phi_n}{2}\right)\left(|0,n+1\rangle\langle 1,n|+|1,n\rangle\langle 0,n+1|\right)$$

$$\dot{R} = \sum_{n=0}^{\infty}\left[i\omega_d E_+(n)\, e^{i\omega_d E_+(n)}\cos^2\left(\tfrac{\Phi_n}{2}\right) + i\omega_d E_-(n)\, e^{i\omega_d E_-(n)}\sin^2\left(\tfrac{\Phi_n}{2}\right)\right]|1,n\rangle\langle 1,n|$$

$$+\left[i\omega_d E_+(n)\, e^{i\omega_d E_+(n)}\sin^2\left(\tfrac{\Phi_n}{2}\right) + i\omega_d E_-(n)\, e^{i\omega_d E_-(n)}\cos^2\left(\tfrac{\Phi_n}{2}\right)\right]|0,n+1\rangle\langle 0,n+1|$$

$$+\left[i\omega_d E_+(n)\, e^{i\omega_d E_+(n)} + i\omega_d E_-(n)\, e^{i\omega_d E_-(n)}\right]\sin\left(\tfrac{\Phi_n}{2}\right)\cos\left(\tfrac{\Phi_n}{2}\right)\left[|0,n+1\rangle\langle 1,n|+|1,n\rangle\langle 0,n+1|\right]$$

$$i\dot{R}R^{\dagger} = -\omega_d\sum_{n=0}^{\infty}\left[E_+(n)e^{i\omega_d E_+(n)t}\cos^2\left(\tfrac{\Phi_n}{2}\right)+E_-(n)e^{i\omega_d E_-(n)t}\sin^2\left(\tfrac{\Phi_n}{2}\right)\right]\left[e^{-i\omega_d E_+(n)t}\cos^2\left(\tfrac{\Phi_n}{2}\right)+e^{-i\omega_d E_-(n)t}\sin^2\left(\tfrac{\Phi_n}{2}\right)\right]|1,n\rangle\langle 1,n|$$

계산이 복잡 너무 Pause (ii) second try

let $R = e^{i\omega_d a^\dagger a t}$ $\quad I\otimes a^\dagger a = \sum_{n=0}^{\infty} n\left(|0,n\rangle\langle 0,n| + |1,n\rangle\langle 1,n|\right)$

$$R = \sum_{n=0}^{\infty} e^{i\omega_d n t}|0,n\rangle\langle 0,n| + e^{i\omega_d n t}|1,n\rangle\langle 1,n| = I\otimes\sum_{n=0}^{\infty} e^{i\omega_d n t}|n\rangle\langle n|$$

$$\dot{R} = i\omega_d\, I\otimes\sum_{n=0}^{\infty} n e^{i\omega_d n t}|n\rangle\langle n| \quad , \quad i\dot{R}R^{\dagger} = -\omega_d\, I\otimes a^\dagger a$$

$$\tilde{H} = \left(I\otimes\sum_{n=0}^{\infty}e^{i\omega_d n t}|n\rangle\langle n|\right)\left[-\frac{\omega_q}{2}\sigma_z\otimes\sum_{n=0}^{\infty}|n\rangle\langle n| + \omega_r I\otimes\sum_{n=0}^{\infty}n|n\rangle\langle n| + g\sigma_+\otimes\sum_{n=1}^{\infty}\sqrt{n}\,|n-1\rangle\langle n| + g\sigma_-\otimes\sum_{n=0}^{\infty}\sqrt{n+1}\,|n+1\rangle\langle n|\right.$$

$$\left.+A(t)(a+a^\dagger)\cos(\omega_d t)\right]\left(I\otimes\sum_{n=0}^{\infty}e^{-i\omega_d n t}|n\rangle\langle n|\right) - \omega_d I\otimes a^\dagger a$$

$$= -\frac{\omega_q}{2}\sigma_z\otimes I + (\omega_r-\omega_d)\, I\otimes a^\dagger a + \underbrace{g\left(\sigma_+ a e^{-i\omega_d t} + \sigma_- a^\dagger e^{i\omega_d t}\right)}_{(X)} + A(t)\left(a^\dagger e^{i\omega_d t} + a e^{-i\omega_d t}\right)\frac{e^{i\omega_d t}+e^{-i\omega_d t}}{2}$$

(iii) third try

$$R = e^{i\omega_d\left(-\frac{1}{2}\sigma_z + a^\dagger a\right)t} = e^{-i\frac{\omega_d}{2}\sigma_z t}\, e^{i\omega_d a^\dagger a t}$$

$$= \left(e^{-i\frac{\omega_d}{2}t}|0\rangle\langle 0| + e^{i\frac{\omega_d}{2}t}|1\rangle\langle 1|\right)\otimes\sum_{n=0}^{\infty}e^{i\omega_d n t}|n\rangle\langle n|$$

$$= \sum_{n=0}^{\infty}e^{i\omega_d(n-\frac{1}{2})t}|0\rangle\langle 0|\otimes|n\rangle\langle n| + \sum_{n=0}^{\infty}e^{i\omega_d(n+\frac{1}{2})t}|1\rangle\langle 1|\otimes|n\rangle\langle n|$$

$$\dot{R} = i\omega_d\sum_{n=0}^{\infty}\left(n-\tfrac{1}{2}\right)e^{i\omega_d(n-\frac{1}{2})t}|0\rangle\langle 0|\otimes|n\rangle\langle n| + i\omega_d\sum_{n=0}^{\infty}\left(n+\tfrac{1}{2}\right)e^{i\omega_d(n+\frac{1}{2})t}|1\rangle\langle 1|\otimes|n\rangle\langle n|$$

$$R^{\dagger} = \left(e^{i\frac{\omega_d}{2}t}|0\rangle\langle 0| + e^{-i\frac{\omega_d}{2}t}|1\rangle\langle 1|\right)\otimes\sum_{n=0}^{\infty}e^{-i\omega_d n t}|n\rangle\langle n|$$

$$i\dot{R}R^{\dagger} = -\omega_d|0\rangle\langle 0|\sum_{n=0}^{\infty}\left(n-\tfrac{1}{2}\right)|n\rangle\langle n| - \omega_d|1\rangle\langle 1|\sum_{n=0}^{\infty}\left(n+\tfrac{1}{2}\right)|n\rangle\langle n|$$

$$= -\omega_d a^\dagger a + \frac{\omega_d\sigma_z}{2}$$

$$\tilde{H} = RHR^{\dagger} + i\dot{R}R^{\dagger} = -\frac{\omega_q-\omega_d}{2}\sigma_z + (\omega_r-\omega_d)a^\dagger a + gR\left(\sigma_- a^\dagger + \sigma_+ a\right)R^{\dagger} + A(t)\left(ae^{-i\omega_d t} + a^\dagger e^{i\omega_d t}\right)\frac{e^{i\omega_d t}+e^{-i\omega_d t}}{2}$$

$$\begin{bmatrix} e^{-i\frac{\omega_d}{2}t} & 0 \\ 0 & e^{i\frac{\omega_d}{2}t}\end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0\end{bmatrix}\begin{bmatrix} e^{i\frac{\omega_d}{2}t} & 0 \\ 0 & e^{-i\frac{\omega_d}{2}t}\end{bmatrix} = \begin{bmatrix} 0 & e^{-i\frac{\omega_d}{2}t} \\ 0 & 0\end{bmatrix}\begin{bmatrix} e^{-i\frac{\omega_d}{2}t} & 0 \\ 0 & e^{-i\frac{\omega_d}{2}t}\end{bmatrix} = \sigma_- e^{-i\omega_d t}$$

$$\begin{bmatrix} e^{-i\frac{\omega_d}{2}t} & 0 \\ 0 & e^{i\frac{\omega_d}{2}t}\end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0\end{bmatrix}\begin{bmatrix} e^{i\frac{\omega_d}{2}t} & 0 \\ 0 & e^{-i\frac{\omega_d}{2}t}\end{bmatrix} = \begin{bmatrix} 0 & 0 \\ e^{i\frac{\omega_d}{2}t} & 0\end{bmatrix}\begin{bmatrix} e^{i\frac{\omega_d}{2}t} & 0 \\ 0 & e^{-i\frac{\omega_d}{2}t}\end{bmatrix} = \sigma_+ e^{i\omega_d t}$$

$$e^{-i\frac{\omega_d}{2}\sigma_z t}\sigma_+ e^{i\frac{\omega_d}{2}\sigma_z t} = \sigma_+ e^{i\omega_d t}, \quad e^{-i\frac{\omega_d}{2}\sigma_z t}\sigma_- e^{i\frac{\omega_d}{2}\sigma_z t} = \sigma_- e^{-i\omega_d t}$$

$$e^{i\omega_d a^\dagger a t}\, a\, e^{-i\omega_d a^\dagger a t} = a e^{-i\omega_d t} \qquad e^{i\omega_d a^\dagger a t}\, a^\dagger e^{-i\omega_d a^\dagger a t} = a^\dagger e^{i\omega_d t}$$

$$\Rightarrow \tilde{H} = RHR^\dagger + i\dot{R}R^\dagger = -\frac{\omega_q-\omega_{rd}}{2}\sigma_z + (\omega_r-\omega_{rd})a^\dagger a + g(\sigma_- a^\dagger + \sigma_+ a) + A(t)\frac{a(1+e^{-2i\omega_{rd}t}) + a^\dagger(1+e^{2i\omega_{rd}t})}{2}$$

$$\underset{RWA}{\simeq} -\frac{\omega_q-\omega_{rd}}{2}\sigma_z + (\omega_r-\omega_{rd})a^\dagger a + g(\sigma_- a^\dagger + \sigma_+ a) + A(t)\frac{a+a^\dagger}{2}$$

④ Evolution with Pulse

$$\dot{a} = i[\tilde{H}, a] + \kappa\left(a^\dagger a a - \frac{1}{2}a^\dagger a a - \frac{1}{2}a a^\dagger a\right)$$

$$[\tilde{H}, a] = (\omega_r-\omega_{rd})(a^\dagger a a - a a^\dagger a) + g\sigma_-(a^\dagger a - a a^\dagger) + \cancel{g\sigma_+(aa - aa)} + \frac{A(t)}{2}(a^\dagger a - a a^\dagger)$$

$$= -(\omega_r-\omega_{rd})a - g\sigma_- - \frac{A(t)}{2}$$

$$\dot{a} = \left[-i(\omega_r-\omega_{rd}) - \frac{\kappa}{2}\right]a - \frac{iA(t)}{2} - ig\sigma_-$$

$$[\sigma_z,\sigma_-] = \begin{bmatrix}1&0\\0&-1\end{bmatrix}\begin{bmatrix}0&1\\0&0\end{bmatrix} - \begin{bmatrix}0&1\\0&0\end{bmatrix}\begin{bmatrix}1&0\\0&-1\end{bmatrix} = \begin{bmatrix}0&1\\0&0\end{bmatrix} - \begin{bmatrix}0&-1\\0&0\end{bmatrix} = 2\sigma_-$$

$$[\sigma_+,\sigma_-] = \begin{bmatrix}0&0\\1&0\end{bmatrix}\begin{bmatrix}0&1\\0&0\end{bmatrix} - \begin{bmatrix}0&1\\0&0\end{bmatrix}\begin{bmatrix}0&0\\1&0\end{bmatrix} = \begin{bmatrix}0&0\\0&1\end{bmatrix} - \begin{bmatrix}1&0\\0&0\end{bmatrix} = -\sigma_z$$

$$[\tilde{H}, a^\dagger] = (\omega_r-\omega_d)(a^\dagger a a^\dagger - a^\dagger a^\dagger a) + g\sigma_+(a a^\dagger - a^\dagger a) + \frac{A(t)}{2}$$

$$= (\omega_r-\omega_d)a^\dagger + g\sigma_+ + \frac{A(t)}{2}$$

$$\kappa\left(a^\dagger a^\dagger a - \frac{1}{2}a^\dagger a a^\dagger - \frac{1}{2}a^\dagger a^\dagger a\right) = \frac{\kappa}{2}(a^\dagger a^\dagger a - a^\dagger a a^\dagger) = -\frac{\kappa}{2}a^\dagger$$

$$\dot{a}^\dagger = \left[i(\omega_r-\omega_{rd}) - \frac{\kappa}{2}\right]a^\dagger + \frac{iA(t)}{2} + ig\sigma_+$$

$$[\sigma_z,\sigma_+] = \begin{bmatrix}1&0\\0&-1\end{bmatrix}\begin{bmatrix}0&0\\1&0\end{bmatrix} - \begin{bmatrix}0&0\\1&0\end{bmatrix}\begin{bmatrix}1&0\\0&-1\end{bmatrix} = \begin{bmatrix}0&0\\-1&0\end{bmatrix} - \begin{bmatrix}0&0\\1&0\end{bmatrix} = -2\sigma_+$$

$[\sigma_z,\sigma_-] = 2\sigma_-$ $\quad [\sigma_z,\sigma_+] = -2\sigma_+$ $\quad [\sigma_+,\sigma_-] = -\sigma_z$ $\qquad \sigma_-^2 = \sigma_+^2 = 0$

$$[\tilde{H},\sigma_-] = -(\omega_q-\omega_{rd})\sigma_- - g\sigma_z a$$

$$\dot{\sigma}_- = -i(\omega_q-\omega_{rd})\sigma_- - ig\sigma_z a + \overbrace{\kappa\left(a^\dagger\sigma_- a - \frac{1}{2}a^\dagger a\sigma_- - \frac{1}{2}\sigma_- a^\dagger a\right)}^{=0} + \gamma\left(\sigma_+\sigma_-\sigma_- - \frac{1}{2}\sigma_+\sigma_-\sigma_- - \frac{1}{2}\sigma_-\sigma_+\sigma_-\right)$$

$$\dot{\sigma}_- = -i(\omega_q-\omega_{rd})\sigma_- - ig\sigma_z a - \frac{\gamma}{2}\sigma_+$$

$$[\tilde{H},\sigma_+] = +(\omega_q-\omega_{rd})\sigma_+ + g\sigma_z a^\dagger$$

$$\dot{\sigma}_+ = i(\omega_q-\omega_{rd})\sigma_+ + ig\sigma_z a^\dagger - \frac{\gamma}{2}\sigma_+$$

$$[\tilde{H},\sigma_z] = -2g\sigma_- a^\dagger + 2g\sigma_+ a$$

$$\dot{\sigma}_z = i(-2g\sigma_- a^\dagger + 2g\sigma_+ a)$$

$$[\sigma_-,\sigma_z] = 0$$

$$\dot{a} = \left[-i(\omega_r-\omega_{rd}) - \frac{\kappa}{2}\right]a - \frac{iA(t)}{2} - ig\sigma_-$$

$$\dot{a}^\dagger = \left[i(\omega_r-\omega_{rd}) - \frac{\kappa}{2}\right]a^\dagger + \frac{iA(t)}{2} + ig\sigma_+$$

$$\dot{\sigma}_- = \left[-i(\omega_q-\omega_{rd}) - \frac{\gamma}{2}\right]\sigma_- - ig\sigma_z a$$

$$\dot{\sigma}_+ = \left[i(\omega_q-\omega_{rd})\sigma_+ - \frac{\gamma}{2}\right]\sigma_+ + ig\sigma_z a^\dagger$$

$$\dot{\sigma}_z = -2ig\sigma_- a^\dagger + 2ig\sigma_+ a$$

where κ: resonator damping

γ: qubit damping

⑤ Approximation

$|\omega_q-\omega_{rd}| \gg g,\quad |\omega_r-\omega_{rd}| \gg g$

$\Rightarrow \sigma_- \simeq e^{-i(\omega_q-\omega_{rd})t} \qquad \sigma_+ \simeq e^{i(\omega_q-\omega_{rd})t}$, σ_-와 σ_+는 fluctuation으로 작용

$a, a^\dagger$는 resonator에서의 움직임과 같다

(i) $A(t) = A_0$

$$\dot{a} \simeq \left[-i(\omega_r-\omega_{rd}) - \frac{\gamma}{2}\right]a - \frac{iA_0}{2}$$

$$\dot{a}^\dagger \simeq \left[i(\omega_r-\omega_{rd}) - \frac{\gamma}{2}\right]a^\dagger + \frac{iA_0}{2}$$

$\Rightarrow$

$$\alpha(t) \simeq -\frac{\frac{i}{2}A_0}{i(\omega_r-\omega_{rd})+\frac{\gamma}{2}} + \left[\alpha(0) + \frac{\frac{i}{2}A_0}{i(\omega_r-\omega_{rd})+\frac{\gamma}{2}}\right] e^{\left[-i(\omega_r-\omega_{rd})-\frac{\gamma}{2}\right]t}$$

$$\alpha^*(t) \simeq +\frac{\frac{i}{2}A_0}{-i(\omega_r-\omega_{rd})+\frac{\gamma}{2}} + \left[\alpha^*(0) - \frac{\frac{i}{2}A_0}{-i(\omega_r-\omega_{rd})+\frac{\gamma}{2}}\right] e^{\left[i(\omega_r-\omega_{rd})-\frac{\gamma}{2}\right]t}$$

$$\sigma_-(t) \simeq e^{\left[-i(\omega_q-\omega_{rd})-\frac{\gamma}{2}\right]t}$$

$$\sigma_+(t) \simeq e^{\left[i(\omega_q-\omega_{rd})-\frac{\gamma}{2}\right]t}$$

$$\dot{\sigma}_z \simeq -2ig\sigma_- a^\dagger + 2ig\sigma_+ a$$

$$= -2ig\frac{\frac{i}{2}A_0}{-i(\omega_r-\omega_{rd})+\frac{\gamma}{2}} e^{-i(\omega_q-\omega_{rd})t}$$

$$-2ig\left[\alpha^*(0) - \frac{\frac{i}{2}A_0}{-i(\omega_r-\omega_{rd})+\frac{\gamma}{2}}\right] e^{-\frac{\gamma}{2}t}$$

$$-2ig\frac{\frac{i}{2}A_0}{i(\omega_r-\omega_{rd})+\frac{\gamma}{2}} e^{i(\omega_q-\omega_{rd})t}$$

$$+2ig\left[\alpha(0) + \frac{\frac{i}{2}A_0}{i(\omega_r-\omega_{rd})+\frac{\gamma}{2}}\right] e^{-\frac{\gamma}{2}t}$$

fluctuate term

decay term

(ii) $A(t) = 0$

$$\alpha(t) \simeq \alpha(0)\, e^{(-i(\omega_r-\omega_{rd})-\frac{\gamma}{2})t}$$

$$\alpha^*(t) \simeq \alpha^*(0)\, e^{(i(\omega_r-\omega_{rd})-\frac{\gamma}{2})t}$$

$$\sigma_-(t) \simeq e^{\left[-i(\omega_q-\omega_{rd})-\frac{\gamma}{2}\right]t}$$

$$\sigma_+(t) \simeq e^{\left[i(\omega_q-\omega_{rd})-\frac{\gamma}{2}\right]t}$$

$\sigma_z(t)$'s evolution 은 exponential decay.

σ_-의 peak 위치 / AC Stark shift

drive 세기 약하게 줄여가며 2차원 여러..

α : np.fft.fft (non-Hermitian 이므로 ..)

peak 2개 나오는 건 beating ...?