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PREFACE

Theoretically, the unlimited amplification of electrical signals, i.e. the detection of arbitrarily small signals, has been implemented by the vacuum tube which may be considered to be the first active element of electronics. However, it soon became evident that these small signals are absorbed by disturbances of various origins. In the early years of radio reception, atmospheric and power line interferences were observed; cosmic noise sources have been discovered subsequently. However, in spite of perfect screening against external disturbances, random small voltage and current fluctuations in circuits have been experienced. Investigations have shown that these fluctuations have two origins.

1. Current is carried by elementary charges having uneven energy distribution. This has the effect that the number of charges per unit time passing through the vacuum space of an electron tube or the p–n junction of a solid state device is not constant.
2. A random brownian motion by elementary charges results in collision between these charges and the atoms of the solid body forming the circuit. Each collision has the effect of changing the direction and magnitude of an elementary current filament.

Both phenomena are outcomes of fundamental laws of nature so the signal fluctuations caused by these phenomena even theoretically cannot be eliminated. The term "electronic noise", in short "noise", will be applied to describe these phenomena. This book deals exclusively with noise; and external disturbances will not be covered.

During the evolution of radio communication, with increasing annoyance, noise became a limiting factor. A thorough investigation of noise was therefore launched leading to unexpected, useful results. It turned out that a knowledge

of noise properties may contribute to our understanding of the physical background of active device operation. Very soon, in addition to theoretical investigations, the evolution of noise-measuring methods of increasing sensitivity and accuracy gained in importance.

Theoretical investigations have shown that noise may be considered as a series of random stochastic events. Moreover, measurements have proved that in most cases the useful signals to be transmitted also have stochastic properties. It is thus appropriate to investigate electronic devices, circuits, and systems by applying stochastic driving signals. The response of a system driven by a stochastic signal will include the responses to sinusoidal and step signals, as treated by classical electrical theorems.

The methods of mathematical statistics and probability theory, taking shape during this period, have turned out to offer useful means of analyzing random signals. However, these methods in themselves seemed to be insufficient, and several additional considerations were needed to analyze and understand basically the physical origins of noises appearing in electronic devices and circuits. The practicing engineer may have problems in trying to form a unified picture based on theoretical physical, mathematical, and his own practical considerations.

This book's main task is to help the engineer in his composite model-forming process. The extensive literature available may provide a suitable background, but a definite starting point and guidance is needed for the correct evaluation of highly specialized references. The definite statement of basic relations is only possible if preceded by the formation of a unified system comprising basic background knowledge and the physical contents of the mathematical formulas.

The main relations refer to noise power. With due caution, these may be extended to voltages and currents, and in the case of linear circuits, linear superposition may be applied to these voltages and currents. Noise equivalent circuits may be introduced, leading to well-proven circuit design methods. These are then followed by noise parameters and noise equivalent circuits of active devices.

Noise measurements usually require a detector having some kind of nonlinear characteristic. This requires, to some extent, the analysis of responses resulting from noise driven, nonlinear networks. The possible methods of noise measurement will also be investigated, and the theoretical limitations of noise measurements will be treated.

This relatively short book cannot go into details of all the existing problems. For more detailed investigations, the reader is referred to the references at the end of each Chapter.

András Ambrózy

CHAPTER

ONE

INTRODUCTION

As an introduction, a few important basic terms will be defined by the analysis of a simple example, the shot-effect noise. However, this analysis cannot yet be complete as information treated in later Chapters is not yet known. We shall come back to the unanswered questions in the appropriate Chapters.

1-1 BASIC TERMS

Figure 1-1 shows the energy diagram of a vacuum-tube cathode and of a p–n junction. In order that an electron can pass from the cathode into the vacuum or from the n layer into the p layer, it must possess sufficient energy to pass through the energy gap. The energies of electrons are randomly distributed, resulting in a fluctuation of charges passing over per time unit, and thus in a fluctuation of current. In a loudspeaker, the sound of the amplified current fluctuation is similar to the dropping of shots, explaining the term shot noise.

In order to calculate this fluctuation, let us investigate an interval of time T, and let us assume that during this interval the number of electrons passing through the energy gap will be N. Let us further assume that at the beginning of this time interval, the number of electrons within the cathode (or within the n layer) will also be N, and for the sake of our mental experiment, let us distinguish between these electrons in some way. After the time interval T, these electrons will be used up.

Let us now divide the time interval investigated into intervals Δt of equal lengths. As shown in Fig. 1-2, the number of electrons passing through the energy gap during intervals $\Delta t_1, \Delta t_2, \ldots, \Delta t_i$ will be $n_1, n_2, \ldots, n_i$. We note that $n_1, n_2, \ldots, n_i$ is not a monotonous series, members of different subscripts may be equal to each other. The following question should be answered: what

is the probability of the electron bearing number 17, for example, passing through the energy gap in that time interval which is lined up in Fig. 1-2?

This event would certainly take place during time interval T, so the probability of it occurring during Δt is $\Delta t/T$. The situation is different if the probability of a definite electron combination passing (e.g., electrons 1, 17, and 23) has to be calculated (see Fig. 1-1). During this time interval, only the designated combination is allowed to pass over, excluding all those electrons which do not belong to the combination. The probability of not passing will be $1-\Delta t/T$ for each single electron. The movements of electrons are independent of each other. The joint probability of occurrence of independent events is given by the

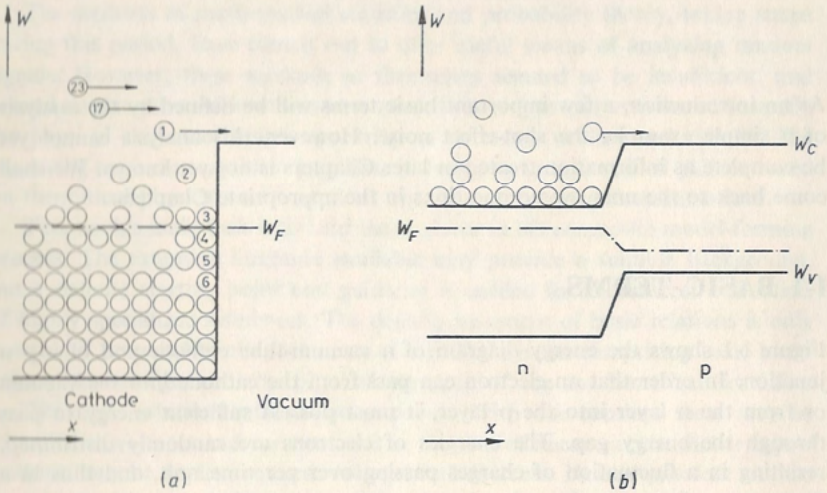

Figure 1-1 *(a)* Energy gap in a vacuum tube. *(b)* Energy gap in a p–n junction. (W, electron energy; W_C, bottom of the conduction band; W_F, Fermi level, W_V, top of the valence band).

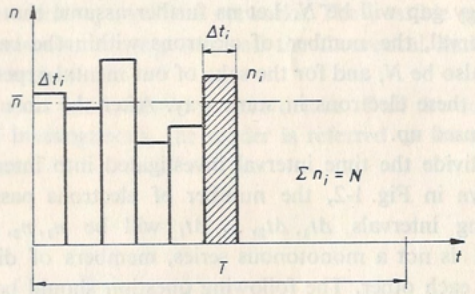

Figure 1-2 Model of shot-noise origin.

product of the single probabilities of occurrences, thus the passing probability of a particular combination is

$$\left(\frac{\Delta t}{T}\right)^{n_i}\left(1-\frac{\Delta t}{T}\right)^{N-n_i}$$

However, single electrons may not actually be distinguished. What may be stated is only: what is the probability of the joint passing of n_i electrons during an interval Δt? From a group of N electrons there are $\binom{N}{n_i}$ ways to select n_i electrons. Thus the probability of the above joint passing will be stated by the well-known formula giving the probabilities of events occurring according to the binomial distribution. Thus

$$f(n_i) = \binom{N}{n_i}\left(\frac{\Delta t}{T}\right)^{n_i}\left(1-\frac{\Delta t}{T}\right)^{N-n_i} \tag{1-1}$$

This distribution type is characteristic of cases in which alternatives are governed by chance. (In our case the alternatives are given by "pass occurs"—"pass does not occur".) However, it is rather inconvenient to apply the binomial distribution, especially for very large numbers of N. A good approximation is achieved by allowing both N and T to increase beyond any limit, i.e. $T\to\infty$, $N\to\infty$, and thus $\Delta t/T\to 0$, but

$$N\frac{\Delta t}{T} = \lambda \tag{1-2}$$

where λ is a finite and constant number. It is immediately evident from Fig. 1-2 that Eq. (1-2) gives the average number of electrons passing during time interval Δt. Taking into account Eq. (1-2) in rewriting Eq. (1-1), we have

$$f(n) = \binom{N}{n}\left(\frac{\lambda}{N}\right)^{n}\left(1-\frac{\lambda}{N}\right)^{N-n}$$

$$= \frac{\lambda^n}{n!}\,\frac{N(N-1)\ldots(N-n+1)}{N^n}\left(1-\frac{\lambda}{N}\right)^{-n}\left(1-\frac{\lambda}{N}\right)^{N} \tag{1-3}$$

The second factor in the right-hand four factor product approaches 1 if $N\to\infty$. Thus

$$\lim_{N\to\infty}\left[1\left(1-\frac{1}{N}\right)\left(1-\frac{2}{N}\right)\ldots\left(1-\frac{n-1}{N}\right)\right] = 1$$

The third factor is

$$\lim_{N\to\infty}\left[\left(1-\frac{\lambda}{N}\right)^{-n}\right] = 1$$

and the fourth factor is

$$\lim_{N\to\infty}\left[\left(1-\frac{\lambda}{N}\right)^{N}\right] = e^{-\lambda}$$

From these expressions, we have

$$\lim_{\substack{N\to\infty \\ \Delta t/T\to 0}} f(n) = \frac{\lambda^n}{n!} e^{-\lambda} \tag{1-4}$$

which is the Poisson distribution as a limiting case of the binomial distribution. The probability of the passing of n electrons during time interval dt is $f(n)\,dt$. In spite of the limiting process $N\to\infty$, the distribution is not continuous and is interpreted for only positive integers of n. A fractional number of electrons cannot be interpreted physically.

This approximation of the binomial distribution can only be applied for cases of $\Delta t/T \ll 1$. This condition is not met for every fluctuation phenomenon!

If the fluctuation of n may be described by the Poisson distribution, it may be shown that the mean value of n is

$$M(n) \equiv \bar{n} = \lambda \tag{1-5}$$

where the mean value is denoted by $M(\ )$ (see Chapter 2). On the other hand, the variance of n, i.e. the mean square value of the fluctuation, is

$$D^2(n) \equiv \overline{(n-\bar{n})^2} = \lambda \tag{1-6}$$

Let us now calculate the current carried by the electrons. The instantaneous value of the current* in a time interval Δt is

$$i = \frac{nq}{\Delta t} \tag{1-7}$$

where $|q| = 1.6\times 10^{-19}$ As, the elementary charge. The mean value of the current is

$$I = \bar{i} = \frac{\bar{n}q}{\Delta t} \tag{1-8}$$

The mean square value of the current fluctuation, which is directly proportional to the noise power, is given by

$$\overline{i^2} = \overline{(i-I)^2} = D^2(i) = \left(\frac{q}{\Delta t}\right)^2 D^2(n) = \left(\frac{q}{\Delta t}\right)^2 \bar{n} \tag{1-9}$$

Expressing $\bar{n}$ from Eq. (1-8) and substituting into Eq. (1-9) we have

$$\overline{i^2} = \frac{q}{\Delta t} I \tag{1-10}$$

* If the DC component is also taken into account in the instantaneous value, the symbol i is used. The DC mean value is denoted by upper case letters, the stochastic signal of zero mean value is denoted by lower case letters.

Thus the mean square value of the current fluctuation is directly proportional to the DC mean value and to the charge carrying the current, and is inversely proportional to the density of time interval divisions. The latter may be demonstrated in Fig. 1-3. The right-hand side shows the same fluctuation phenomenon as the left-hand side, but the intervals Δt_1 have been united in groups of five, thus smoothing out the short-lasting fluctuations.

Δt may not be chosen arbitrarily because this would not allow a unique definition of $\overline{i^2}$. On the other hand, should we consider the stochastic time function shown in Figs. 1-2 and 1-3, respectively, as a series of pulses and should we pass these pulses through a linear network, then the duration of the shortest pulse that could be transmitted through this network would be of the order of the reciprocal bandwidth. Thus the mean square current fluctuation is a function of the bandwidth:

$$\boxed{\overline{i^2} = 2qI\Delta f} \tag{1-11}$$

The conversion from Eq. (1-10) to Eq. (1-11) is not explained here, but we shall come back to this at the end of Chapter 3.

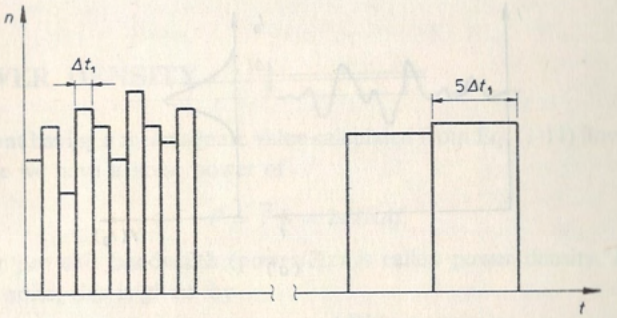

Figure 1-3 Changing of time division.

The frequency is not contained in Eq. (1-11), thus the frequency band of width Δf may be selected anywhere. This infers that $\overline{i^2}\to\infty$ would result in the case of $\Delta f\to\infty$, which is evidently impossible. In Chapter 3, it will be shown that the finite time needed for the passage (e.g., the electron transit time in a vacuum tube) results in a frequency-dependent factor in Eq. (1-11) of less than one.

The shot noise is only an example of fluctuation phenomena in electronic devices. In order to achieve a unified treatment a few general terms are now introduced.

1-2 AMPLITUDE DENSITY

The instantaneous value of the current flowing through an electronic device may be given in the form

$$i(t) = I + i(t) \tag{1-12}$$

which is similar to that used for periodic signals superimposed on direct current. Unfortunately, this similarity is of no use here as the exact function $i(t)$ is not known. On the other hand, the probability of n electrons participating in the current flow may be given using Eq. (1-4)

$$f(n) = \frac{\lambda^n}{n!} e^{-\lambda}$$

In practical cases, n is a very large number. When utilizing current-measuring instruments, we can not distinguish single electrons, only finite current ranges. The probability of $i(t)$ falling between I_0 and $I_0 + \Delta I$ if $\Delta I \to 0$ (see Fig. 1-4*a*) is by

$$\mathscr{P}[I_0 \leqq i(t) < I_0 + \Delta I] = f(I_0)\,\Delta I \tag{1-13}$$

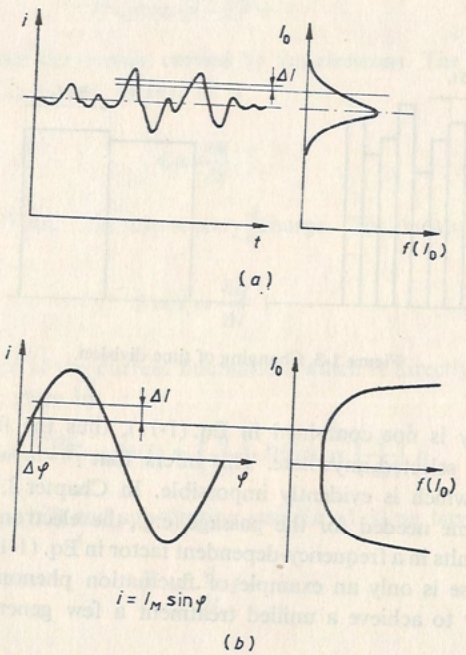

Figure 1-4 Time function and amplitude density function. *(a)* Stochastic signal. *(b)* Sine-wave signal.

The function $f(I_0)$ is called the amplitude density function, its dimension being A^{-1}. In a similar way, the amplitude density function $f(U_0)$ may also be derived, its dimension being V^{-1}.

In case of shot-effect noise, $f(I_0)$ is also a Poisson function as i is proportional to n. Evidently other kinds of distributions would be valid for other types of noise. The distribution types frequently occurring in the theory of electronic noise will be surveyed in Chapter 2.

Signals with known time behavior—i.e. deterministic signals—also have amplitude density functions. For instance, the function of the sinusoidal signal is shown in Fig. 1-4*b*. It is easily shown that

$$f(I_0)_{\sin}\,\Delta I = \frac{2\Delta\varphi}{2\pi} \tag{1-14}$$

and

$$\lim_{\Delta i \to 0} f(I_0)_{\sin} = \frac{1}{\pi}\frac{d\varphi}{di} = \frac{1}{\pi}\frac{d}{di}\left(\arcsin\frac{i}{I_{\max}}\right) = \frac{1}{\pi I_{\max}}\frac{1}{\sqrt{1-(I_0/I_{\max})^2}} \tag{1-15}$$

A knowledge of amplitude density functions pertaining to deterministic signals is especially important for the investigation of nonlinear networks (e.g., demodulators).

1-3 POWER DENSITY

If the current having a mean square value calculated from Eq. (1-11) flows through a resistance we have a noise power of

$$P = \overline{i^2}R = 2qIR\Delta f \tag{1-16}$$

The power per unit bandwidth (power/Hz) is called power density. In case of shot-effect noise, this is given by

$$p = 2qIR\left(\frac{\mathrm{W}}{\mathrm{Hz}}\right) \tag{1-17}$$

If the bandwidth of the measuring instrument is limited to the low-frequency range, then the power density of the shot-effect noise is independent of frequency (see Fig. 1-5*a*). This means that the power spectrum is uniform, which is true in many practical cases. This kind of noise is called white noise, taking the analogy from the mixed nature of white light. However, in the general case (see Fig. 1-5*b*), we have

$$p = p(f) \quad \text{or} \quad p = p(\omega) \tag{1-18}$$

and the total noise is given by the integral

$$P = \int_0^\infty p(f)\,df \tag{1-19}$$

In practical calculations, frequently $\overline{i^2}$ or $\overline{u^2}$ are calculated. In most cases, it would be inconvenient to convert these quantities into powers. This is why the spectral density of the squared fluctuations is written in the form

$$S = S(f) \quad \text{or} \quad S = S(\omega) \tag{1-20}$$

(dimensions: A^2/Hz, V^2/Hz, $A^2/rad\ s^{-1}$, $V^2/rad\ s^{-1}$)

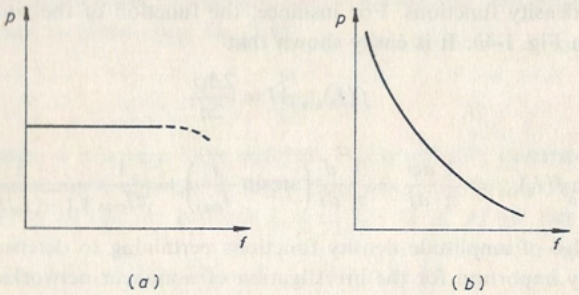

Figure 1-5 *(a)* Frequency-independent power spectrum (white noise). *(b)* Frequency dependent power spectrum ($1/f$ noise).

Due to further considerations of convenience, the root mean square noise current and noise voltage

$$i_{rms} = \sqrt{\overline{i^2}} \qquad u_{rms} = \sqrt{\overline{u^2}} \tag{1-21}$$

are used. It should be stressed, however, that for stochastic signals, only spectra of the type given in Eqs. (1-18) and (1-20) may be interpreted. From these spectra, no conclusions regarding the frequency spectrum of the stochastically changing current or voltage may be drawn. On the other hand, the power spectrum may be easily determined from the so-called autocorrelation function dealt with in Chapter 3, which reveals an internal property of the stochastic signal.

PROBLEMS

1-1 Let the number of electrons passing during the time interval Δt be $\lambda = 10$. Let us compare the passing probabilities of 12 and 10 electrons!

Answer: From Eq. (1-4) we have

$$\frac{f(12)}{f(10)} = \frac{\lambda^{n_2} n_1!}{\lambda^{n_1} n_2!} = \frac{10^{12}}{10^{10}} \times \frac{1}{11 \times 12} = 0.76$$

Let us repeat this by taking $n = \lambda = 100$, $n_2 = 120$, and $n_1 = 100$! For such large numbers, the values of factorials may be computed from the Stirling formula.

$$n! \cong n^n e^{-n} \sqrt{2\pi n} \quad \text{if} \quad n \to \infty$$

$$f(100) = \left(\frac{\lambda}{\bar{n}}\right)^{n_1} \frac{e^{n_1}}{\sqrt{2\pi n_1}} e^{-\lambda} = \frac{e^{n_1}}{\sqrt{2\pi n_1}} e^{-\lambda}$$

$$(\lambda \equiv \bar{n})$$

$$f(120) = \left(\frac{\lambda}{n_2}\right)^{n_2} \frac{e^{n_2}}{\sqrt{2\pi n_2}} e^{-\lambda}$$

$$\frac{f(120)}{f(100)} = \left(\frac{1}{1.2}\right)^{120} e^{20} \sqrt{\frac{1}{1.2}} = 0.134$$

It can be seen that by increasing n, which according to Eq. (1-8) may be attained by increasing I or Δt, the probability of a given relative difference, i.e. the relative fluctuation, gets smaller.

1-2 The power density of the so-called $1/f$ noise may be written in the form $p(f) = p_0 f_0 / f$. Let $p_0 = 10^{-6}$ W/Hz and $f_0 = 1$ kHz. What is the noise power between 1 Hz and 10 kHz?

Answer:

$$P = p_0 f_0 \int_{f_1}^{f_2} \frac{df}{f} = p_0 f_0 \ln \frac{f_2}{f_1} = 9.21 \text{ mW}$$

Note that for $f_1 \to 0$, $P \to \infty$. A peculiarity of the $1/f$ noise is the fact that it follows the above frequency dependence even at frequencies several orders less than 1 Hz. At the same time, its power is finite, which may be explained by energetic reasoning. No satisfactory explanation for this phenomenon has yet been found.

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CHAPTER

TWO

DISTRIBUTION AND DENSITY FUNCTIONS OF INSTANTANEOUS VALUES

As mentioned in Chapter 1, the exact time function of a stochastic signal is unknown. However, the occurrence probabilities of given instantaneous values may be computed utilizing the physical background knowledge of the random process. The probability terms applied in our investigations will be treated without striving for complete coverage, and occasionally limited to our special needs. For thorough investigations, publications dealing with probability theory should be consulted.[4,5,11,12,14,15] However, care will be taken in making our relations suitable for describing the random variables of physical processes.

2-1 BASIC TERMS OF PROBABILITY THEORY

In a general sense, a random trial is characterized by the fact that its outcome is not determined uniquely by the conditions taken into account. The possible outcomes of a trial are called the elementary events. In a mathematical sense, a possible event is a subset of the set Ω comprising the elementary events forming the event space.

In probability theory, the Kolmogorov set of axioms is generally accepted. According to this set of axioms, the $P(A)$ function defined on the subsets of the event space Ω is called probability if the following axioms are met for this function:

$$0 \leqq \mathscr{P}(A) \leqq 1 \tag{2-1}$$

$$\mathscr{P}(\Omega) = 1 \tag{2-2}$$

$$\mathscr{P}\left(\sum_k A_k\right) = \sum_k P(A_k) \tag{2-3}$$

if $A_1, A_2, \ldots, A_k$ is a finite or infinite series of mutually exclusive events.

The random variable is a real function defined over the set Ω comprising the elementary events. The real number characterizing the elementary event ω will generally be denoted by ξ, and in some cases, the event ω will also be indicated, e.g., $\xi(\omega)$. In most cases, the numerical result of the trial is given by this random variable.

The functional relationship between the random variable and the event is not necessarily mutually unique. Taking again the example of Fig. 1-2, the i-th event means the passing of n_i electrons: in this case, the random variable is n_i itself. But if two vacuum tubes are connected in parallel, and their overall current is investigated, the new random variable will be $n_i' = n_{i1} + n_{i2}$, and several combinations of the elementary events within the vacuum tube will pertain to a given n_i'.

2-2 DISTRIBUTIONS

Now the term "stochastic process" may be defined more precisely. By a stochastic process is meant a single parameter assembly of random variables $\xi(t)$ where parameter t covers a time-instant set continuously. For instance, the previously mentioned n_i' is a function of the time as a parameter and is made up of a random variable defined over two event spaces. Such a stochastic process may be formed by the number of calls per unit time in a telephone exchange. A more detailed investigation of stochastic processes will be presented in Chapter 3. We will now consider the distribution of random variables, together with the parameters and characteristic types of distribution.

The distribution function $F(x)$ of a random variable ξ represents the probability of ξ having a value of less than x. Thus

$$F(x) = \mathscr{P}(\xi < x) \tag{2-4}$$

The properties of the distribution function $F(x)$ are as follows:

1. Monotonically increasing, i.e.

$$F(x_2) \geqq F(x_1), \quad \text{if} \quad x_2 > x_1 \tag{2-5}$$

2. $$\lim_{x \to -\infty} F(x) = 0 \tag{2-6}$$

3. $$\lim_{x \to \infty} F(x) = 1 \tag{2-7}$$

4. At all places, continuous from the left, i.e.

$$\lim_{x \to x_0 - 0} F(x) = F(x_0) \tag{2-8}$$

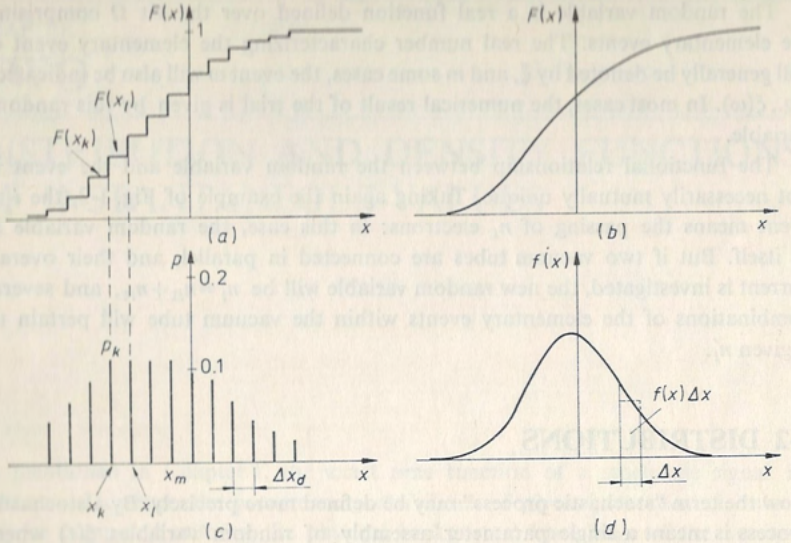

Figure 2-1 *(a)* Discrete distribution. *(b)* Continuous distribution. *(c)* Probabilities of discrete distribution *(d)* Density function of continuous distribution.

As shown in Fig. 2-1,

$$\mathscr{P}(x_k \leqq \xi < x_l) = F(x_k) - F(x_l) \tag{2-9}$$

Two basic types of a distribution function—discrete and continuous—may be distinguished. In the case of a discrete distribution, the possible members of a random variable ξ are members of a finite or infinite series. If the probability of the possible values $x_1, x_2, \ldots, x_i$ is $p_1, p_2, \ldots, p_i$, respectively, i.e.

$$P_i = \mathscr{P}(\xi = x_i) \qquad i = 1, 2, \ldots \tag{2-10}$$

then the distribution function is

$$F(x) = \mathscr{P}(\xi < x) = \sum_{x_i < x} p_i \tag{2-11}$$

Comparing this with Eq. (1-7) we see that if $x \to \infty$ the sum of probabilities p_i tends toward one. The distribution is said to be continuous if a function $f(x) \geqq 0$ exists for which the condition

$$F(x_2) - F(x_1) = \mathscr{P}(x_1 \leqq \xi < x_2) = \int_{x_1}^{x_2} f(x)\, dx \tag{2-12}$$

is met in any left-bounded interval (x_1, x_2).

Figures 2-1*a* to 2-1*d* show a discrete and a continuous distribution, as well as the pertaining discrete probabilities p_i and the density function $f(x)$:

$$F(x_j) - F(x_i) = p_i \tag{2-13a}$$

and

$$\frac{dF(x)}{dx} = f(x) \tag{2-13b}$$

where p_i can be calculated by taking the difference in two consecutive steps of the staircase distribution function, and $f(x)$ may be calculated by differentiating $F(x)$.

The density function of a discrete distribution may not be defined. However, the discrete probabilities of a series of measurements comprising a large number of data may be approximated by a continuous function. In a case where the "spectrum lines" representing the probabilities p_i are equidistant and relatively densely spaced

$$p_i = f(x)\, \Delta x_d \qquad f(x) = \frac{p_i}{\Delta x_d} \tag{2-14}$$

where Δx_d is the distance between consecutive spectrum lines of the discrete distribution. The probability for unity Δx is just $f(x)$, which is therefore called the probability density function, or in short, the density function. Its dimension is $1/\Delta x$ where Δx may be an increment of any variable, e.g., current, voltage, resistance, time, etc.

It is clear that if step heights are decreased beyond all limits the discrete distribution will turn into a continuous distribution, and a continuous distribution may be converted into a discrete distribution by a staircase approximation.

Both conversion types will be utilized below. One of these conversions has already been applied in Sec. 1-2, where the discrete probabilities $p_i = f(n)$ of the Poisson distribution, defined only for integers, have been substituted by a continuous function because of the large values of n.

2-3 EXPECTED VALUE AND STANDARD DEVIATION

The expected value is the most important characteristic of a random variable distribution. The expected value is that number around which the algebraic mean—the so-called empirical mean—of the independent observations related to the random variable in question will fluctuate. If the random variable ξ has a discrete distribution and its values $x_1, x_2, \ldots, x_i$ have the probabilities $p_1, p_2, \ldots, p_i$ then the expected value will be the sum of the series

$$\sum_{i=1}^{\infty} x_i p_i \qquad p_i = \lim_{N \to \infty} \frac{n_i}{N} \tag{2-15}$$

on the condition that this series is absolutely convergent. The expected value of the random variable ξ will be denoted by $M(\xi)$. This notation is derived from the initial of the word "mean", and is an operational instruction for the random variable, in parentheses.

From Eqs. (2-14) and (2-15), the expected value of a continuous distribution is

$$M(\xi) = \int_{-\infty}^{\infty} xf(x)\,dx \tag{2-16}$$

This has to meet the condition of absolute integrability, i.e.

$$\int_{-\infty}^{\infty} |x|\,f(x)\,dx < \infty \tag{2-17}$$

Otherwise the expected value is not defined. The formulas needed for determining the expected value are very similar to those applied to the calculation of plane figure moments referring to a specific axis. This is why $M(\xi)$ is usually called the first-order moment of the distribution (Fig. 2-2).

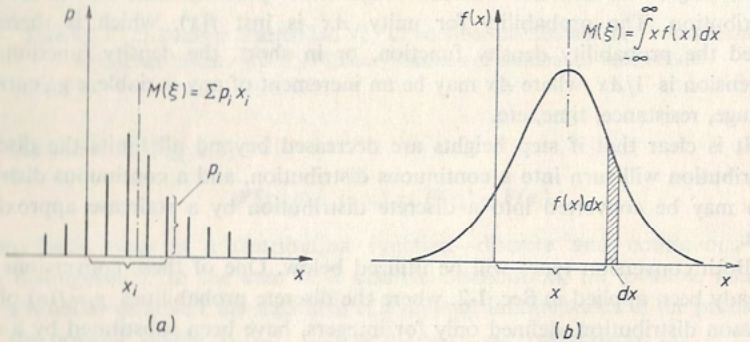

Figure 2-2 First-order partial moments for calculating the expected value. *(a)* Discrete distribution. *(b)* Continuous distribution.

A few typical properties of the expected value will now be presented:

$$M(c\xi) = cM(\xi) \tag{2-18}$$

where c is a constant. The expected value of the sum of two or more random variables is

$$M(\xi_1+\xi_2+\ldots+\xi_n) = M(\xi_1)+M(\xi_2)+\ldots+M(\xi_n) \tag{2-19}$$

It follows from Eq. (2-19) that for $\xi_1=\xi$ and $\xi_2=c$

$$M(\xi+c) = M(\xi)+c \tag{2-20}$$

If ξ_1 and ξ_2 are statistically independent random variables,

$$M(\xi_1\xi_2) = M(\xi_1)M(\xi_2) \tag{2-21}$$

The condition for independence is that the joint distribution, i.e. the probability of the simultaneous occurrence of two events pertaining to random variables ξ_1 and ξ_2 given by

$$\mathscr{P}(\xi_1 < x_1,\ \xi_2 < x_2) = G(x_1, x_2) \tag{2-22}$$

should have the density function

$$g(x_1, x_2) = f_1(x_1)f_2(x_2) \tag{2-23}$$

i.e. it should be the product of the density functions of the two variables.

The random variable fluctuates around the mean value $M(\xi)$. However, we cannot yet tell anything about the amplitude of the fluctuation. The instantaneous deviation from the expected value may be either positive or negative, thus the deviations will cancel each other out. This requires then that the measure of fluctuation should be independent of the sign. Let us therefore introduce the positive square root of the mean square value of fluctuation, the so-called standard deviation

$$D(\xi) = \sqrt{M\{[\xi-M(\xi)]^2\}} \tag{2-24}$$

This means that the standard deviation of the random variable ξ is the positive square root taken from the expected value of the squared random variable $\xi-M(\xi)$. Here, the operational instruction $D(\ )$ refers to the calculation of the dispersion of the random variable in parentheses.

The application of Eq. (2-24) is somewhat inconvenient. Let us perform the indicated operations

$$D^2(\xi) = M\{[\xi-M(\xi)]^2\} = M[\xi^2-2\xi M(\xi)+M^2(\xi)] \tag{2-25}$$

The factor $M(\xi)$ in the central right-hand term is constant, so by applying Eq. (2.18), we have

$$D^2(\xi) = M(\xi^2)-2M^2(\xi)+M^2(\xi) = M(\xi^2)-M^2(\xi) \tag{2-26}$$

This means that the variance may be expressed in another way too, namely by the difference between the expected value of the squared random variable and the squared expected value. However, care should be taken when applying Eq. (2-26) as in some cases it may be much less accurate than Eq. (2-24). For example, let us assume that the expected value $M(\xi)$ has been determined with an error ε. This means that according to Eq. (2-24), we have

$$\begin{aligned} D^2(\xi) &= M\{[\xi-M(\xi)-\varepsilon]^2\} \\ &= M[\xi^2-2\xi M(\xi)+M^2(\xi)-2\varepsilon\xi+2\varepsilon M(\xi)+\varepsilon^2] \\ &= M(\xi^2)-M^2(\xi)+\varepsilon^2 \end{aligned} \tag{2-27}$$

The terms $-2\varepsilon\xi+2\varepsilon M(\xi)$ are canceled because $M(\xi)=M[M(\xi)]$ as the expected value of a constant number is that number itself. Let us now repeat the calculation by applying Eq. (2-26).

$$D^2(\xi) = M(\xi^2)-[M(\xi+\varepsilon)]^2$$
$$= M(\xi^2)-M^2(\xi)-2\varepsilon M(\xi)-\varepsilon^2 \qquad (2\text{-}27a)$$

where Eq. (2-20) has been utilized to calculate $M(\xi+\varepsilon)$. In the latter case, the error of $D^2(\xi)$ is substantially higher as $2\varepsilon M(\xi)\gg\varepsilon^2$ if $\varepsilon\ll M(\xi)$.

Knowledge of the density function permits the calculation of any distribution variance. According to Eq. (2-24), the variance is the expected value of the squared fluctuations, so for a discrete distribution, it follows from Eq. (2-15) that

$$D^2(\xi) = \sum_i [x_i-M(\xi)]^2 p_i \qquad (2\text{-}28)$$

and for a continuous distribution, we have from (2-17) that

$$D^2(\xi) = \int_{-\infty}^{\infty} [x-M(x)]^2 f(x)\,dx \qquad (2\text{-}29)$$

The latter relationship may be converted in a similar way to Eq. (2-26). Thus

$$D^2(\xi) = \int_{-\infty}^{\infty} x^2 f(x)\,dx - M^2(\xi) = \mu_2-\mu_1^2 \qquad (2\text{-}30)$$

where $\mu_1=M(\xi)$ is the first-order moment of the distribution. It can be seen that the integral of $x^2f(x)dx$ is the second-order moment of the density function taken for $x=0$, denoted by μ_2. Equation (2-30) gives the so-called central second-order moment.

Let us present two typical properties of the variance

$$D^2(a\xi+b) = M\{[a\xi+b-M(a\xi+b)]^2\}$$
$$= M\{[a\xi+b-aM(\xi)-b]^2\} = a^2D^2(\xi) \qquad (2\text{-}31)$$

According to Eq. (2-24), the standard deviation is thus multiplied by the factor $|a|$, but is not changed by the additive term.

Another important theorem is that the variance of mutually independent random variables $\xi_1, \xi_2, \ldots, \xi_n$ is given by

$$D^2(\xi_1+\xi_2+\ldots+\xi_n) = D^2(\xi_1)+D^2(\xi_2)+\ldots+D^2(\xi_n) \qquad (2\text{-}32)$$

which may be verified as follows:

$$D^2(\xi_1+\xi_2+\ldots+\xi_n) = M\{[\xi_1+\xi_2+\ldots+\xi_n-M(\xi_1+\xi_2+\ldots+\xi_n)]^2\}$$
$$= M\{[\xi_1-M(\xi_1)+\xi_2-M(\xi_2)+\ldots+\xi_n-M(\xi_n)]^2\} \qquad (2\text{-}33)$$

In the course of squaring, we obtain the squares and double products of terms that have the form $\xi_i-M(\xi_i)$. The squared terms add up to just $D^2(\xi)$, and the double product terms cancel because

$$M\{[\xi_i-M(\xi_i)][\xi_k-M(\xi_k)]\} = M[\xi_i-M(\xi_i)]\,M[\xi_k-M(\xi_k)] = 0 \qquad (2\text{-}34)$$

The first equality follows from the independence requirement. Provisionally denoting the random variables within the brackets as η_1, η_2, it follows from Eq. (2-21) that $M(\eta_1,\eta_2)=M(\eta_1)M(\eta_2)$. However, the last two factors disappear separately because $M(\xi_i)=M[M(\xi_i)]$.

If the variances of independent random variables $\xi_1, \xi_2, \ldots, \xi_n$ are equal

$$D^2(\xi_1+\xi_2+\ldots+\xi_n) = nD^2(\xi) \qquad (2\text{-}35)$$

The presence of additive constants, i.e. the differences between the density functions of the random variables, have no influence on the end result.

As in the case of the first-order and second-order moments, the higher-order moments of distributions may also be defined by

$$\mu_n = \int_{-\infty}^{\infty} x^n f(x)\,dx \qquad (2\text{-}36)$$

The investigation of higher-order moments may yield valuable information about the density function. For example, in Fig. 2-3 symmetrical and asymmetrical density functions having all zero first-order moments and variances not differing significantly are compared. However, for the asymmetrical function $\mu_3\neq 0$, the third-order moment is characteristic of the skewness. By comparing the

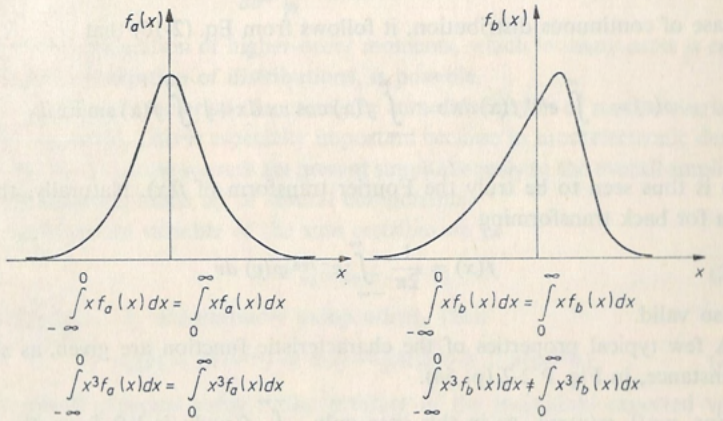

Figure 2-3 Comparison of third-order moments pertaining to distributions with zero expected values. The origin of the asymmetrical density function $f_b(x)$ is chosen for $M(\xi)=0$.

second-order moment μ_2 and the fourth-order moment μ_4, the kurtosis of the density function may be derived. For instance, if the density function has a long tail, i.e. it is not zero even at points that are separated from the mean value by several multiples of the standard deviation, then the fourth-order moment is significantly increased.

2-4 CHARACTERISTIC FUNCTION, SUM DISTRIBUTIONS

The introduction of the characteristic function is justified partly because of the simplified calculation of higher-order moments and partly because of the investigation of so-called sum distributions. It should be mentioned in advance that the characteristic function is the Fourier transform of the density function, and this transform will be used several times. It cannot always be calculated in closed form, necessitating in these cases computer-aided methods.[18]

The characteristic function of a random variable ξ is defined for all real numbers as

$$\varphi(v) = M(e^{j\xi v}) = M(\cos \xi v) + jM(\sin \xi v) \tag{2-37}$$

In noise analysis, ξ generally depends on time; $\xi = \xi(t)$. Naturally, the characteristic function may also be defined for time independent random variables.

Taking into account the instruction for calculating the expected value, we have the following Fourier series for discrete values [see Eq. (2-15)]:

$$\varphi(v) = \sum_i e^{jx_i v} p_i = \sum_i p_i \cos x_i v + j \sum_i p_i \sin x_i v \tag{2-38}$$

In case of continuous distribution, it follows from Eq. (2-16) that

$$\varphi(v) = \int_{-\infty}^{\infty} e^{jxv} f(x)\, dx = \int_{-\infty}^{\infty} f(x) \cos vx\, dx + j \int_{-\infty}^{\infty} f(x) \sin vx\, dx \tag{2-39}$$

$\varphi(v)$ is thus seen to be truly the Fourier transform of $f(x)$. Naturally, the formula for back transforming

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-jvx} \varphi(v)\, dv \tag{2-40}$$

is also valid.

A few typical properties of the characteristic function are given, as shown, for instance, in Fig. P 2-2 (p. 54).

For $v=0$, $\varphi(v)=1$, as in this case only $\int_{-\infty}^{+\infty} f(x)dx$ is left from Eq. (2-39) which is equal to unity according to Eqs. (2-6) and (2-7). It may also be proven that in the case of $v \neq 0$, $|\varphi(v)| \leqq 1$.

Generally $\varphi(v)$ is complex. However, if $f(x)$ is an even function, the second integral on the right-hand side of Eq. (2-39) disappears, so $\varphi(v)$ becomes real. If it is not required for $f(x)$ to be even, then

$$\varphi(-v) = \varphi^*(v) \tag{2-41}$$

This is proved by noting that $M[\cos(-\xi v)] = M(\cos \xi v)$ and $M[\sin(-\xi v)] = M[-\sin(\xi v)]$. (The asterisk denotes now and in the following the conjugated value of a complex number.)

Let us now calculate the derivatives of $\varphi(v)$ with respect to v.

$$\frac{d\varphi}{dv} = j \int_{-\infty}^{\infty} e^{jvx} x f(x)\, dx$$

$$\frac{d^2\varphi}{dv^2} = -\int_{-\infty}^{\infty} e^{jvx} x^2 f(x)\, dx$$

$$\frac{d^m\varphi}{dv^m} = j^m \int_{-\infty}^{\infty} e^{jvx} x^m f(x)\, dx \tag{2-42}$$

Substituting $v=0$, we have

$$\left.\frac{d\varphi}{dv}\right|_0 = jM(\xi) = j\mu_1$$

$$\left.\frac{d^2\varphi}{dv^2}\right|_0 = -\mu_2$$

$$\left.\frac{d^m\varphi}{dv^m}\right|_0 = j^m \mu_m \tag{2-43}$$

Thus an easy calculation of higher-order moments, which in many cases is essential for the investigation of distributions, is possible.

Utilizing the characteristic function, the sum distribution of random variables may be calculated. This is especially important because in most electronic devices or circuits, several noise sources are present simultaneously so the overall amplitude density function is made up of several components.

Let the random variable of the sum distribution be

$$\eta = \xi_1 + \xi_2 + \ldots + \xi_n \tag{2-44}$$

where $\xi_1, \xi_2, \ldots, \xi_n$ are mutually independent. Then

$$\varphi_\eta(v) = M(e^{jv\eta}) = M(e^{jv\xi_1}) M(e^{jv\xi_2}) \ldots M(e^{jv\xi_n}) \tag{2-45}$$

as the overall expected value is the product of the individual expected values because of the condition of mutual independence. This means that the Fourier transform of the sum distribution density function is the product of the Fourier transforms corresponding to the individual density functions.

Let us, for the time being, restrict our considerations to the two-term sum

$$\eta = \xi_1 + \xi_2 \tag{2-46}$$

and let the density functions of ξ_1, ξ_2 and η be $f_1(x_1)$, $f_2(x_2)$ and $f(y)$, respectively. Using the Fourier transform to transform back the product of the characteristic functions

$$\varphi_\eta(v) = \varphi_1(v)\varphi_2(v) \tag{2-47}$$

we have the well-known convolution integral

$$f(y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-jvy}\varphi_1(v)\varphi_2(v)\,dv = \int_{-\infty}^{\infty} f_1(x_1)f_2(y-x_1)\,dx_1 \tag{2-48}$$

This can be derived using another reasoning too. The joint probability of independent events is given by the product of their probabilities, so $f_1(x_1)$ and $f_2(x_2)$ are to be multiplied, but the condition of $x_2 = y - x$, for any value of y, should also be met. Covering all possible values of x_1, the density function of y is calculated by the addition of the partial products, i.e. by integration in the case of a continuous density function. A few examples of the convolution of typical density functions representing electronic noise are given by Arend.[1]

A simple example will now be given to explain the density function of a sum distribution. Let us consider the throwing of two dice, and suppose that our interest in the outcome extends to our knowing the sum of the numbers appearing on the dice. What are the probabilities of the possible sum values? The probability of any number appearing when throwing a single die is 1/6. The probability that a six will appear with both dice is 1/36 as we have independent events. The probability of obtaining 2 and 12 as a sum will be only 1/36, but, for instance, the number five may be cast in four different ways: i.e. 1+4, 2+3, 3+2, 4+1. Expressing Eq. (2-48) for a discrete distribution, in our case we have

$$p(5) = \sum_{i=1}^{4} p_i p_{(5-i)} = 4 \times \frac{1}{6} \times \frac{1}{6} = \frac{1}{9} \tag{2-49}$$

Figure 2.4*a* shows the discrete distributions for throwing dice, and the corresponding continuous, so-called uniform distribution is shown in Fig. 2.4*b*. Given two random variables having uniform distributions, the density function of their sum is the convolution of two rectangular density functions: this convolution is triangular.

Before considering the central limit theorem, the reader is asked to calculate the sum density function of three random variables having equal uniform distributions (e.g., the tossing of three dice). Furthermore, by increasing the number of members giving the sum, how will the overall density function be modified?

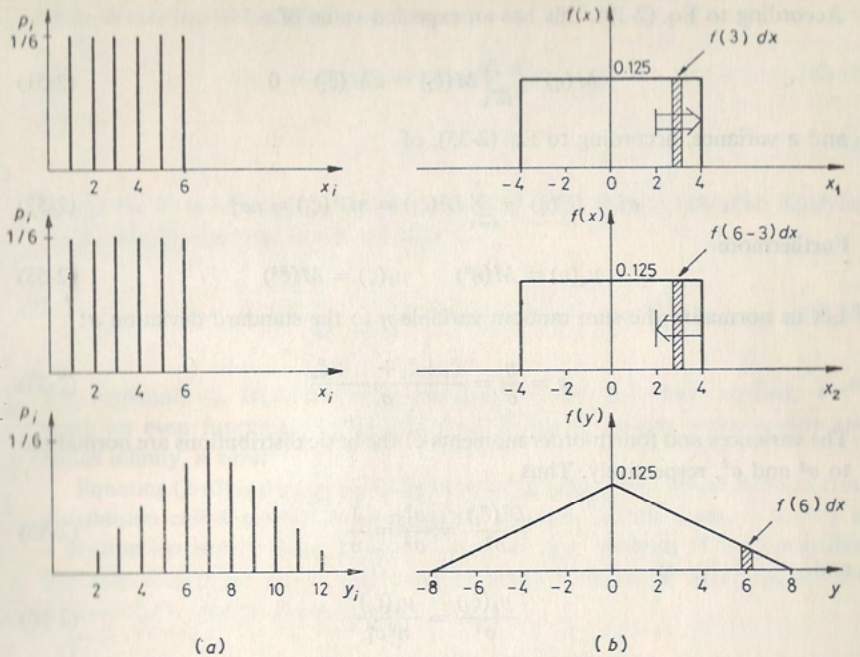

Figure 2-4 Discrete and continuous sum distributions. *(a)* Probabilities of dice casting and sum distribution. *(b)* Convolution of uniform distributions.

2-5 THE CENTRAL LIMIT THEOREM

The noise appearing in devices and circuits may frequently be considered as a superposition of many random phenomena with equal distributions. An important question is: can the density function of the sum distribution be determined in this case, especially when the number of components is high and unknown?[6,16] According to the central limit theorem of mathematical statistics, the distribution of a sum made up of a large number of independent random variables follows normal (gaussian) distribution.

Let $\xi_1, \xi_2, \ldots, \xi_i, \ldots, \xi_n$ be mutually independent random variables with zero expected values and σ_i standard deviations; furthermore $\mu_3 = 0$ skewness.* Let the sum random variable be defined by

$$\eta = \sum_{i=1}^{n} \xi_i \tag{2-50}$$

* This requirement simplifies the derivation. However, the central limit theorem may also be derived with more general starting-point conditions.

According to Eq. (2-19), this has an expected value of

$$M(\eta) = \sum_{i=1}^{n} M(\xi_i) = nM(\xi_i) = 0 \tag{2-51}$$

and a variance, according to Eq. (2-35), of

$$\sigma^2 = D^2(\eta) = \sum_{i=1}^{n} D^2(\xi_i) = nD^2(\xi_i) = n\sigma_i^2 \tag{2-52}$$

Furthermore

$$\mu_4(\eta) = M(\eta^4) \qquad \mu_4(\xi) = M(\xi^4) \tag{2-53}$$

Let us normalize the sum random variable η to the standard deviation σ:

$$z = \frac{\eta}{\sigma} = \frac{\xi_1+\xi_2+\ldots+\xi_n}{\sigma} \tag{2-54}$$

The variances and fourth-order moments of the basic distributions are normalized to σ^2 and σ^4, respectively. Thus

$$\frac{D^2(\xi_i)}{\sigma^2} = \frac{\sigma_i^2}{\sigma^2} = \frac{1}{n} \tag{2-55}$$

and

$$\frac{\mu_4(\xi_i)}{\sigma^4} = \frac{\mu_4(\xi_i)}{n^2\sigma_i^4} \tag{2-56}$$

Using Eq. (2-45), let us express the characteristic function of the sum distribution as

$$\varphi(v) = M(e^{jzv}) = \prod_{i=1}^{n} M(e^{j(v/\sigma)\xi_i}) = \left[\varphi_i\left(\frac{v}{\sigma}\right)\right]^n \tag{2-57}$$

The function $\varphi_i\left(\frac{v}{\sigma}\right)$ may be expanded to

$$\varphi_i\left(\frac{v}{\sigma}\right) = M\left[1+j\left(\frac{v}{\sigma}\right)\xi_i - \frac{1}{2!}\left(\frac{v}{\sigma}\right)^2\xi_i^2 - \frac{j}{3!}\left(\frac{v}{\sigma}\right)^3\xi_i^3+\ldots\right]$$

$$= 1+j\left(\frac{v}{\sigma}\right)M(\xi_i) - \frac{1}{2!}\left(\frac{v}{\sigma}\right)^2 D^2(\xi_i) - \frac{j}{3!}\left(\frac{v}{\sigma}\right)^3\mu_3(\xi_i)+\ldots \tag{2-58}$$

Taking into account that $M(\xi_i)=0$ and $\mu_3(\xi_i)=0$, and utilizing Eqs. (2-55) and (2-56), we have

$$\varphi_i\left(\frac{v}{\sigma}\right) = 1-\frac{1}{2}\frac{v^2}{n}+\frac{1}{4!}\frac{v^4}{n^2}-+\ldots \tag{2-59}$$

In Eq. (2-57), we have the product of functions φ_i. Let us therefore take the logarithm of φ_i and apply the approximation $\ln(1+y)\cong y$. If $n\to\infty$ and $y\to 0$

$$\ln\varphi_i\left(\frac{v}{\sigma}\right) \cong \ln\left(1-\frac{v^2}{2n}\right) \cong -\frac{v^2}{2n} \tag{2-60}$$

Then, according to Eq. (2-57), we have

$$\ln\varphi(v) = n\ln\varphi_i\left(\frac{v}{\sigma}\right) = -\frac{v^2}{2} \tag{2-61}$$

and so

$$\varphi_v = e^{-v^2/2} \tag{2-62}$$

This is the Fourier transform of the sum distribution density function. Applying the inverse Fourier transform, we have

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-v^2/2}\cos vx\,dv = \frac{1}{\sqrt{2\pi}}e^{-x^2/2} \tag{2-63}$$

(see Appendix B). Here the following simplification has been applied: $e^{-v^2/2}$ being an even function, the integral of $e^{-v^2/2}\sin vx$, taken between plus and minus infinity, is zero.

Equation (2-63) is the density function of an important and frequently occurring distribution called normal or gaussian distribution. In this form, it applies to a distribution with a mean value of $M(z)=0$ and, because of the normalization according to Eq. (2-54), with unity standard deviation. If $M(z)=\mu_1\neq 0$ and $D^2(z)=\sigma^2\neq 1$, then in general form,

$$\boxed{f(x) = \frac{1}{\sigma\sqrt{2\pi}}e^{-(x-\mu_1)^2/2\sigma^2}} \tag{2-64}$$

It is worthwhile mentioning that for $n\to\infty$, i.e. when the density function of a sum made up of many random variables with equal distributions has to be determined, the result will always be a normal distribution; the only requirement to be met is that the basic distributions should not be skew.

The random variable ξ in the fluctuation phenomena is a function of time t. In this case, the density function gives the probability of individual instantaneous values. For instance, if an infinite number of equal-amplitude, incoherent sine waves is superimposed (see Fig. 1.4*b* for the peaked amplitude density function), we again have instantaneous values with normal distribution.* It is therefore not surprising that a stochastic signal occupies a broad frequency range. It should be stressed, however, that a "white" power spectrum in itself does not necessarily imply instantaneous values of normal distribution.

We have seen that meeting a single moderate requirement is sufficient always to arrive, in a limiting case, at the normal distribution. It follows from this fact

* This may be derived, for instance, from Eq. (30) in Chapter 7 of the book by Miller,[10] after expansion and neglections.

that the distribution of a sum made up of two random variables with equal expected values and normal distributions is also normal. This fact will be utilized in calculating the instantaneous values and powers of noises originating from different sources.

2-6 NORMAL DISTRIBUTION

Normal distribution is of special importance, so a separate Section has been assigned to present its properties. Its density function is given, in the general case, by Eq. (2-64):

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu_1)^2/2\sigma^2}$$

and the distribution function, according to Eq. (2-9), by

$$F(x) = \int_{-\infty}^{x} f(u)\,du = \Phi\left(\frac{x-\mu_1}{\sigma}\right) \tag{2-65}$$

The latter may not be expressed in closed form, but is tabulated in many books (see, for instance, Jahnke and Emde[9]) with parameters $\mu_1=0$ and $\sigma=1$. For $\mu_1=0$, the first few terms of its Taylor series are given by

$$\Phi\left(\frac{x}{\sigma}\right) = \frac{1}{2}+\frac{1}{\sqrt{2\pi}}\left[\left(\frac{x}{\sigma}\right)-\frac{1}{3!}\left(\frac{x}{\sigma}\right)^3+\frac{1\times 3}{5!}\left(\frac{x}{\sigma}\right)^5-+\ldots\right] \tag{2-66}$$

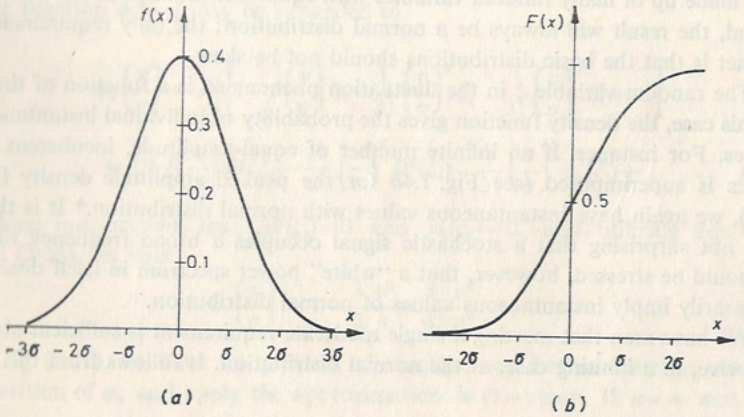

Figure 2-5 Normal (gaussian) distribution. *(a)* Density function. *(b)* Distribution function.

The density function and the distribution function are shown in Fig. 2-5. The question frequently asked is what is the probability of the instantaneous value of a signal with normal distribution falling between the limits $x=\pm k\sigma$? These probabilities for a few normalized values are shown in Table 2-1.

Table 2-1 Tabulated values of the error function

$\frac{x}{\sigma}$	$\Phi\left(\frac{x}{\sigma}\right)-\Phi\left(-\frac{x}{\sigma}\right)$
1	0.683
2	0.954
3	0.9972
4	0.99994

The expected value, i.e. the first-order moment of a normal distribution symmetrical with respect to $x=0$, is zero. However, let us calculate the expected value of the half-distribution pertaining to positive x values according to Eq. (2-16)

$$M(\xi) = \int_0^\infty xf(x)\,dx \tag{2-67}$$

$$= \frac{1}{\sigma\sqrt{2\pi}}\int_0^\infty xe^{-x^2/2\sigma^2}\,dx = \frac{\sigma}{\sqrt{2\pi}} \cong 0.4\sigma \tag{2-68}$$

$$x \geqq 0$$

This will be characteristic for a half-wave detection of a normal distribution signal. For full-wave detection, we have

$$M(|\xi|) = 2\int_0^\infty xf(x)\,dx = \sqrt{\frac{2}{\pi}}\,\sigma \cong 0.8\sigma \tag{2-69}$$

Now let us calculate the higher-order moments by multiple differentiation of the characteristic function. Let $\varphi(v)$ now be the Fourier transform of a normal distribution with zero mean value and arbitrary standard deviation [see Appendix B]. Thus

$$\varphi(v) = \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty}\cos vx e^{-x^2/2\sigma^2}\,dx = e^{-\sigma^2v^2/2} \tag{2-70}$$

The derivatives and the moments are

$$\frac{d\varphi}{dv} = -\sigma^2 v e^{-\sigma^2 v^2/2}\big|_{v=0} = 0 = j\mu_1 \tag{2-71}$$

$$\frac{d^2\varphi}{dv^2} = \sigma^2 e^{-\sigma^2 v^2/2}(\sigma^2 v^2 - 1)_{v=0} = -\sigma^2 = -\mu_2 \tag{2-72}$$

$$\frac{d^3\varphi}{dv^3} = \sigma^4 v e^{-\sigma^2 v^2/2}(3-\sigma^2 v^2)_{v=0} = 0 = -j\mu_3 \tag{2-73}$$

$$\frac{d^4\varphi}{dv^4} = \sigma^4 e^{-\sigma^2 v^2/2}(3-6\sigma^2 v^2+\sigma^4 v^4)_{v=0} = 3\sigma^4 = \mu_4 \tag{2-74}$$

An important property of a normal distribution is that

$$\frac{\mu_4}{\mu_2^2} = 3 \tag{2-75}$$

Naturally, Eqs. (2-19) and (2-32) are valid as before if $\xi_1, \xi_2, \ldots, \xi_n$ are independent random variables with normal distribution. Since $\eta = \xi_1 + \xi_2 + \ldots + \xi_n$,

$$M(\eta) = M(\xi_1) + M(\xi_2) + \ldots + M(\xi_n)$$

$$D^2(\eta) = D^2(\xi_1) + D^2(\xi_2) + \ldots + D^2(\xi_n)$$

In this case, according to Sec. 2-4, we may be sure that η also has a normal distribution.

2-7 DISCRETE DISTRIBUTIONS

In probability theory, there are many types of discrete distributions. We are interested in only two of these: the previously mentioned binomial distribution and Poisson distribution [see Eqs. (1-1) and (1-4)].

Let us consider two sets of events that may have only two, mutually exclusive outcomes. This is called a simple alternative. An example of this is the emission or nonemission of an electron from the vacuum-tube cathode. Another example is related to two electrodes with the possibility of either of them being the target for an emitted electron. The noise originating from the random distribution of current fluctuation may also be treated using this model.

Let $\mathscr{P}(A) = p_1$ be the probability of the event in question, i.e. that electrode A is hit by the electron. The probability of the event not occurring is $p_2 = 1 - p_1$. Thus in the first case the value of the random variable is $\xi = x_1 = 1$, and in the second case, $\xi = x_2 = 0$. The expected value of ξ is then

$$M(\xi) = \sum_{i=1}^{2} p_i x_i = 1 \cdot p_1 + 0 \cdot p_2 = p_1 \tag{2-76}$$

and its variance is

$$D^2(\xi) = M(\xi^2) - M^2(\xi) = \sum_{i=1}^{2} p_i x_i^2 - M^2(\xi)$$

$$= 1 \cdot p_1 + 0 \cdot p_2 - p_1^2 = p_1(1-p_1) = p_1 p_2 \tag{2-77}$$

Let us repeat the above experiment N times. What is the probability of the event occurring just n times, and not occurring $N-n$ times? If the single experiments are independent from each other, then the overall probability is given by the product of the single probabilities, i.e.

$$\prod_{n} p_1 = p_1^n \quad \text{and} \quad \prod_{N-n} p_2 = p_2^{N-n} \tag{2-78}$$

However, as there is no restriction regarding the sequence of outcomes, n events from a total of N may occur in $\binom{N}{n}$ different ways. Thus

$$\boxed{\mathscr{P}(\eta = n) = \binom{N}{n} p_1^n p_2^{N-n}} \tag{2-79}$$

which are probabilities according to the binomial distribution. Here the random variable η may be defined according to

$$\eta = \xi_1 + \xi_2 + \ldots + \xi_N \tag{2-80}$$

where $\xi_i = 1$ or 0.

Because of the requirement for independent alternatives, we have from Eq. (2-19)

$$M(\eta) = NM(\xi) = Np_1 \tag{2-81}$$

and from Eq. (2-32)

$$D^2(\eta) = ND^2(\xi) = Np_1 p_2 \tag{2-82}$$

If $p_1 \leqq 1$, $p_2 = 1 - p_1 \cong 1$ and

$$D^2(\eta) = M(\eta) \tag{2-83}$$

If, on the other hand, $p_1 = 1/2$, $p_2 = 1/2$ and

$$D^2(\eta) = \frac{M(\eta)}{2} = \frac{N}{4} \tag{2-84}$$

Figure 2-6 shows the probabilities according to the binomial distribution for $N = 10$ and $p_1 = 0.1$, as well as for $N = 10$ and $p_1 = 0.5$.

If single experiments have several outcomes instead of simple alternatives, their probability is given by the polynomial distribution. However, as in actually utilized electronic devices the number of independent electron- (hole-) collecting electrodes is normally not more than two, the polynomial distribution is seldom used.

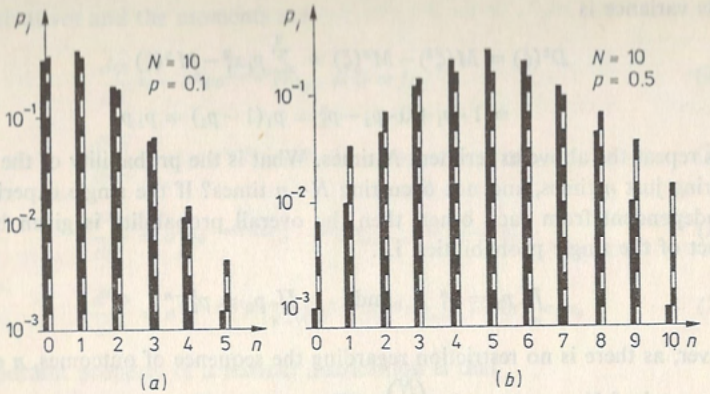

Figure 2-6 Ten-term binomial distribution. *(a)* $p=0.1$. *(b)* $p=0.5$.
(For comparison, the dashed columns show the Poisson distributions for $\lambda=1$ and $\lambda=5$. The logarithmic scale of p is compressed in the vertical direction.)

Equations (1-2) to (1-4) have shown that if $N\to\infty$ and $p\to 0$ so that $Np=\lambda>0$, then the binomial distribution goes to the Poisson distribution given by

$$\boxed{\mathscr{P}(\eta=n)=\frac{\lambda^n}{n!}e^{-\lambda}=f(n)} \tag{2-85}$$

Thus if the above conditions are met, then the difference between the two distributions will be arbitrarily small for all n values. Figure 2-6 shows also the probabilities according to the Poisson distribution. It is noted that for $N=10$, $p=0.1$ the approximation is fair, but for $N=10$, $p=0.5$ the approximation is not applicable, especially at the extremes of the distribution. Thus if the analysis of a specific problem leads to a binomial distribution then, before a Poisson approximation is carried out, its justification should be determined. [For instance, a simple calculation and comparison can be made for $\mathscr{P}(\eta=0)$.] As an example, the intervals Δt shown in Fig. 1-1 may have been refined arbitrarily, so the condition $p\to 0$ could be met. On the other hand, the probability of current distribution between two electrodes is given ($0<p_1<1$ and $0<p_2=1-p_1<1$, respectively), so the refinement of the time intervals, or the meeting of the condition $N\to\infty$ in itself will not allow the Poisson approximation.

The moments of the Poisson distribution may be calculated from the characteristic function

$$\varphi(v)=\sum_{n=0}^{\infty}e^{jvn}\frac{\lambda^n}{n!}e^{-\lambda}=e^{-\lambda}\sum_{n=0}^{\infty}\frac{(\lambda e^{jv})^n}{n!}=e^{-\lambda}e^{\lambda e^{jv}}=e^{\lambda(e^{jv}-1)} \tag{2-86}$$

where $e^x=\sum_0^{\infty}x^n/n!$ has been used. The first derivative is

$$\left.\frac{d\varphi}{dv}\right|_{v=0}=j\lambda e^{jv}e^{\lambda(e^{jv}-1)}\big|_{v=0}=j\lambda=jM(\eta) \tag{2-87}$$

and yields j times the expected value. The second derivative is

$$\frac{d^2\varphi}{dv^2}=-\lambda^2e^{2jv}e^{\lambda(e^{jv}-1)}-\lambda e^{jv}e^{\lambda(e^{jv}-1)}\big|_{v=0}=-\lambda^2-\lambda=-\mu_2 \tag{2-88}$$

Utilizing Eq. (2-30), we have

$$D^2(\eta)=\mu_2-\mu_1^2=\lambda \tag{2-89}$$

The expected value and the variance have already been utilized in Chapter 1 to calculate the shot noise.

In these cases, $M(\eta)>0$ as the Poisson distribution is defined only for positive integers. The probabilities of the distribution are shown in Fig. 2-7. It is seen that the "envelope" which has only an illustrative role for discrete random variables is not symmetrical with respect to $M(\eta)$, so generally $\mu_3(\eta)\neq 0$. In spite of this, the normal distribution may be used as a good approximation for high enough values of n and λ, if n is near λ (i.e. not for arbitrarily high n values).

For large n values, the Stirling formula*

$$n!=n^ne^{-n}\sqrt{2\pi n} \tag{2-90}$$

may be applied. Let us substitute this into Eq. (2-85) and let us take the natural logarithm of Eq. (2-85)

$$\ln\mathscr{P}=-n\ln\frac{n}{\lambda}+n-\lambda-\frac{1}{2}\ln n-\frac{1}{2}\ln 2\pi \tag{2-91}$$

Let us make use of the identity

$$n\equiv\lambda\left(1+\frac{n-\lambda}{\lambda}\right)=\lambda\left(1+\frac{\Delta n}{\lambda}\right)=\lambda(1+\alpha) \tag{2-92}$$

and take into account that if $n\to\lambda$, then $\alpha\ll 1$ and

$$\ln(1+\alpha)=\alpha-\frac{1}{2}\alpha^2+\ldots \tag{2-93}$$

Equation (2-91) may thus be rewritten as

$$\ln\mathscr{P}=-\lambda(1+\alpha)\left(\alpha-\frac{\alpha^2}{2}+\ldots\right)+\lambda\alpha-\frac{1}{2}\ln\lambda-\frac{1}{2}\left(\alpha-\frac{\alpha^2}{2}+\ldots\right)-\frac{1}{2}\ln 2\pi$$

$$=-\frac{1}{2}\lambda\alpha^2+\frac{1}{2}\lambda\alpha^3-\frac{1}{2}\alpha+\frac{1}{4}\alpha^2-\frac{1}{2}\ln 2\pi\lambda+-\ldots \tag{2-94}$$

* A more accurate approximation of $n!$ is available.[17]

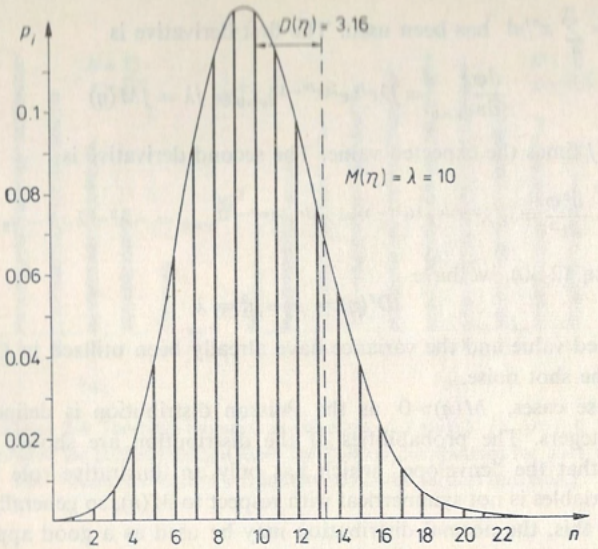

Figure 2-7 Probabilities of Poisson distribution.

or

$$\ln \mathscr{P} = -\frac{1}{2}\frac{\Delta n^2}{\lambda} + \frac{1}{2}\frac{\Delta n^2}{\lambda^2}\left(\Delta n + \frac{1}{2}\right) - \frac{1}{2}\frac{\Delta n}{\lambda} - \frac{1}{2}\ln 2\pi\lambda + \dots \tag{2-95}$$

As $\Delta n \ll \lambda$ but $\Delta n = 0, 1, 2, \dots$, the two middle terms may be neglected compared with the two outer terms. Thus

$$\mathscr{P}(\eta = n) = \frac{1}{\sqrt{2\pi\lambda}} e^{-\Delta n^2/2\lambda} = \frac{1}{D(\eta)\sqrt{2\pi}} e^{-\Delta n^2/2D^2(\eta)} \tag{2-96}$$

because according to Eq. (2-89) $D(\eta) = \sqrt{\lambda}$.

Considering continuous distribution, we arrive at the density function of the gaussian distribution. It may thus be concluded that if $D(\eta)$ is sufficiently small compared with $M(\eta)$, i.e. the relative standard deviation is

$$\frac{D(\eta)}{M(\eta)} = \frac{\sqrt{\lambda}}{\lambda} = \frac{1}{\sqrt{\lambda}} \ll 1 \tag{2-97}$$

the Poisson distribution may be approximated by the normal distribution, at least in the vicinity of $M(\eta)$.* The condition for this is a sufficiently high average current and thus a sufficiently high λ. With the exception of rare cases, the shot noise of electronic devices will therefore be regarded as having normal distribution.

* The relative error may be high far from $M(\eta)$. (See Hubbard.[8])

The binomial distribution that may not be approximated by Poisson distribution has still to be investigated ($N \to \infty$, but p does not go to zero). This is really a sum distribution made up of alternative event pairs; for large numbers, the validity of the central limit theorem may be expected. However, a condition of this—the symmetrical basic distribution—is not generally met, as may be seen from Eq. (2-76). Therefore, the distribution in the vicinity of $n = Np_1 = M(\eta)$ will be investigated separately. Here

$$\mathscr{P}(\eta = Np_1) = \mathscr{P}_0 = \binom{N}{Np_1} p_1^{Np_1} p_2^{N-Np_1} = \frac{N!\,p_2^N}{(Np_1)!(N-Np_1)!}\left(\frac{p_1}{1-p_1}\right)^{Np_1} \tag{2-98}$$

Let us determine the relative probability of $\eta = Np_1 + k$ ($k \ll Np_1$). For this, $(Np_1)!$ has to be increased to $(Np_1+k)!$ which means evidently a multiplication by $(Np_1+1)(Np_1+2)\dots(Np_1+k)$. Similarly, $(N-Np_1)!$ has to be divided by $(N-Np_1)(N-Np_1-1)\dots[N-Np_1-(k-1)]$. Thus

$$\frac{\mathscr{P}_k}{\mathscr{P}_0} = \frac{\mathscr{P}(\eta = Np_1+k)}{\mathscr{P}(\eta = Np_1)}$$

$$= \frac{(N-Np_1)(N-Np_1-1)\dots[N-Np_1-(k-1)]}{(Np_1+1)(Np_1+2)\dots(Np_1+k)}\left(\frac{p_1}{1-p_1}\right)^k \tag{2-99}$$

A common factor in the numerator is $(N-Np_1)^k$, and in the denominator is $(Np_1)^k$. Their quotient multiplied by the last factor in Eq. (2-99) gives unity. Let us now take the natural logarithm of the remaining equation below

$$\frac{\mathscr{P}_k}{\mathscr{P}_0} = \frac{[1-1/(N-Np_1)][1-2/(N-Np_1)]\dots[1-(k-1)/(N-Np_1)]}{(1+1/Np_1)(1+2/Np_1)\dots(1+k/Np_1)}$$

Its natural logarithm is given by

$$\ln \mathscr{P}_k = \ln \mathscr{P}_0 + \sum_{i=0}^{k-1} \ln\left(1 - \frac{i}{N-Np_1}\right) - \sum_{i=1}^{k} \ln\left(1 + \frac{i}{Np_1}\right) \tag{2-100}$$

Because of the requirement that $k \ll Np_1$, the approximation $\ln(1+x) \cong x$ may be used. Thus

$$\ln \mathscr{P}_k = \ln \mathscr{P}_0 - \sum_{i=0}^{k-1} \frac{i}{N-Np_1} - \sum_{i=1}^{k} \frac{i}{Np_1} \tag{2-101}$$

Both summations will result in sums of algebraic series. Thus

$$\ln \mathscr{P}_k = \ln \mathscr{P}_0 - \frac{k}{2N}\left(\frac{k-1}{1-p_1} + \frac{k+1}{p_1}\right) = \ln \mathscr{P}_0 - \frac{k(k+1-2p_1)}{2Np_1(1-p_1)} \tag{2-102}$$

Note that according to Eq. (2-82), $2Np_1(1-p_1) = 2D^2(\eta)$ and the numerator has an approximate value of k^2. For $p_1 = 0.5$, i.e. for symmetrical distribution, this value is accurate. So

$$\mathscr{P}_k = \mathscr{P}_0 e^{-k^2/2D^2(\eta)} \tag{2-103}$$

$\mathscr{P}_0$ may be calculated from Eq. (2-98) by utilizing the Stirling formula. Thus

$$N! = N^N e^{-N} \sqrt{2\pi N} \tag{2-104}$$

$$(Np_1)! = N^{Np_1} p_1^{Np_1} e^{-Np_1} \sqrt{2\pi N p_1} \tag{2-105}$$

$$(N-Np_1)! = N^{N(1-p_1)}(1-p_1)^{N(1-p_1)} e^{-N(1-p_1)} \sqrt{2\pi N(1-p_1)} \tag{2-106}$$

After simplification, we have

$$\mathscr{P}_0 = \frac{1}{D(\eta)\sqrt{2\pi}} \tag{2-107}$$

which is the expected coefficient of the well-approximating gaussian distribution.

It may be concluded from this derivation that for sufficiently high $M(\eta)$, which is nearly always the case for charge carriers in electronic devices, the binomial distribution may be substituted by a normal distribution.

2-8 CONTINUOUS DISTRIBUTIONS

The processes within electronic devices are in connection with elementary charge carriers and should strictly speaking be characterized by discrete distributions only. However, due to the large number of charge carriers, in many cases the approximation given in Eq. (2-14) may be applied with the change over to continuous distribution. In this category, the uniform distribution is the most simple.

The continuous random variable ξ has a uniform distribution if in the interval (a, b) the density function is

$$f(x) = \frac{1}{a-b} \qquad a < x < b \tag{2-108}$$

and outside this interval is $f(x)=0$. It should again be borne in mind that the density function describing the physical process is generally not dimensionless. In Eq. (2-108), x may have any physical interpretation. In our case, it is voltage or current, so $[f(x)]=\mathrm{V}^{-1}$ or A^{-1}.

Let us confine ourselves to the uniform distribution with zero expected value as shown in Fig. 2-4*b*, with limits $-a$ and $+a$. The density function is then

$$f(x) = \frac{1}{2a} \qquad -a < x < a \tag{2-109}$$

which meets the condition of Eq. (2-12)

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

According to Eq. (2-30), the variance is given by

$$D^2(\xi) = \int_{-a}^{a} x^2 \frac{1}{2a}\,dx = \frac{x^3}{6a}\bigg|_{-a}^{a} = \frac{a^2}{3} \tag{2-110}$$

and the standard deviation by

$$D(\xi) = \frac{a}{\sqrt{3}} \tag{2-111}$$

Note that the dimension of the standard deviation is in this case again equal to the dimension of the random variable.

The fourth-order moment is

$$\mu_4 = \int_{-a}^{a} x^4 \frac{1}{2a}\,dx = \frac{a^4}{5} \tag{2-112}$$

According to Sec. 2-3, the kurtosis of the uniform distribution will be characterized by

$$\frac{\mu_4}{\mu_2^2} = \frac{\mu_4}{D^4(\xi)} = \frac{9}{5} < 3 \tag{2-113}$$

The next three types of continuous distribution will be derived from the normal distribution (χ^2, average absolute deviation, Rayleigh distribution). All three distributions are mainly applied to methods of measuring stochastic signals.

For instance, in the case of quadratic detection, the instantaneous value of the voltage or current to be measured is squared (by using a nonlinear network with infinite speed and zero time constant), and the signal thus generated is averaged or integrated. Thus the distribution of squared instantaneous values having normal distribution should be known, as should the k-term sum distribution, derived from the former distribution (if k-independent values are taken during averaging).

Let us consider as a starting point, a normal distribution of zero expected value and standard deviation σ. The probability of $\xi^2 < y$ is given by

$$F(y) = \mathscr{P}(\xi^2 < y) = \mathscr{P}(-\sqrt{y} < \xi < \sqrt{y}) \tag{2-114}$$

As the distribution of ξ is normal and symmetrical with respect to $x=0$, we have

$$F(y) = \frac{2}{\sigma\sqrt{2\pi}} \int_0^{\sqrt{y}} e^{-u^2/2\sigma^2}\,du = 2\left[\Phi\left(\frac{\sqrt{y}}{\sigma}\right) - \Phi(0)\right] \tag{2-115}$$

The density function is given by

$$f(y) = \frac{dF(y)}{dy} = \frac{d}{d\sqrt{y}} 2\Phi\left(\frac{\sqrt{y}}{\sigma}\right) \frac{d\sqrt{y}}{dy} = \frac{e^{-y/2\sigma^2}}{\sigma\sqrt{2\pi y}} \tag{2-116}$$

This is shown in Fig. 2-8, and may only be defined for $y>0$. This is called a chi-square distribution with one degree of freedom (χ^2 distribution).

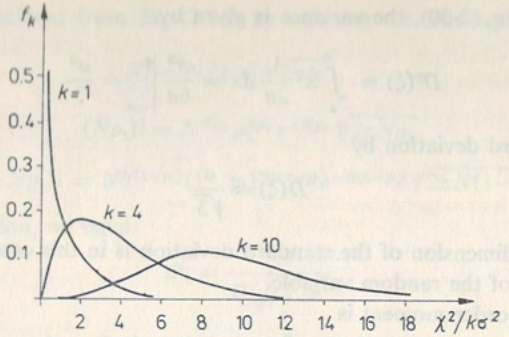

Figure 2-8 χ^2 distributions for squared sums having different number of terms.

Consider now a random variable that is not the square of a single instantaneous value ξ but the sum of squared random variables $\xi_1, \xi_2, \ldots, \xi_k$ having the same normal distribution

$$\chi^2 = \xi_1^2 + \xi_2^2 + \ldots + \xi_k^2 \tag{2-117}$$

We then have a χ^2 distribution of k degrees of freedom, having a density function

$$\boxed{f_k(y) = \frac{y^{(k/2)-1} e^{-y/2\sigma^2}}{2^{k/2}\Gamma(k/2)\sigma^k} = \frac{x^{k-2} e^{-x^2/2\sigma^2}}{2^{k/2}\Gamma(k/2)\sigma^k} = f_k(x^2)} \tag{2-118}$$

where f_k is given both as a function of the variable x pertaining to the original normal distribution and as a function of the variable $y = x^2$ of the χ^2 distribution. The relation may be derived either by convolution[11] or by raising to the appropriate power the characteristic function of Eq. (2-116) [see Eq. (2-45)] and transforming it back. $\Gamma(v)$ is the generalization of function $v!$ for an arbitrary nonintegral number, defined by the integral[3]

$$\Gamma(v) = \int_0^\infty x^{v-1} e^{-x}\, dx$$

In the form given in Eq. (2-118), $f_k(x)$ and $f_k(y)$ are suitable for describing a physical variable of any dimension; naturally $[x] = [\sigma]$.

Let us calculate the expected value and standard deviation of the χ^2 distribution. As $\xi_1, \xi_2, \ldots, \xi_k$ are independent, utilizing Eqs. (2-19) and (2-32) we have

$$M(\chi^2) = M(\xi_1^2) + M(\xi_2^2) + \ldots + M(\xi_k^2) = kM(\xi^2) \tag{2-119}$$

$$D^2(\chi^2) = kD^2(\xi^2) = k[M(\xi^4) - M^2(\xi^2)] \tag{2-120}$$

At the right-hand sides of these equations there are the second- and fourth-order moments of the basic distribution. Due to the requirement that ξ should have a normal distribution

$$D^2(\chi^2) = k(3\sigma^4 - \sigma^4) = 2k\sigma^4 \tag{2-121}$$

Furthermore

$$M(\chi^2) = k\sigma^2 \tag{2-122}$$

The relative standard deviation of the χ^2 distribution is

$$\frac{D(\chi^2)}{M(\chi^2)} = \sqrt{\frac{2}{k}} \tag{2-123}$$

Figure 2-8 shows functions pertaining to different k parameters. If k is a large number, the χ^2 distribution also goes to the normal distribution as do the sum distributions in general.

In the case of full-wave, piecewise linear detection, the distribution of the detected output signal follows the distribution of $|\xi|$. If before detection ξ had a normal distribution, the density function of $|\xi|$ will have the form shown in Fig. 2-9

$$f(x) = \frac{1}{\sigma}\sqrt{\frac{2}{\pi}}\, e^{-x^2/2\sigma^2} \qquad 0 < x < \infty \tag{2-124}$$

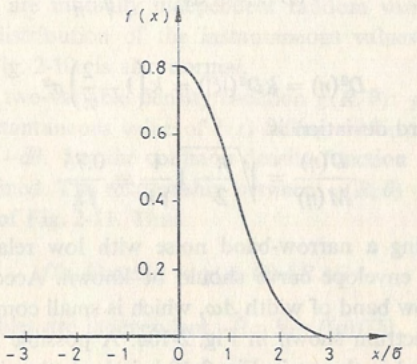

Figure 2-9 Distribution of absolute value

The expected value [see Eq. (2-69)] is

$$M(|\xi|) = \frac{1}{\sigma}\sqrt{\frac{2}{\pi}} \int_0^\infty x e^{-x^2/2\sigma^2}\, dx = \sqrt{\frac{2}{\pi}}\,\sigma$$

and the variance is

$$D^2(|\xi|) = \mu_2 - \mu_1^2 = \frac{1}{\sigma}\sqrt{\frac{2}{\pi}} \int_0^\infty x^2 e^{-x^2/2\sigma^2}\, dx - \frac{2}{\pi}\sigma^2 = \left(1 - \frac{2}{\pi}\right)\sigma^2 \tag{2-125}$$

Naturally, during the course of measurement, the output signal of a linear detector is also averaged, so the distribution

$$\eta = |\xi_1| + |\xi_2| + \ldots + |\xi_k| \tag{2-126}$$

of the k-term sum formed by the absolute values of the independent random variables having normal distribution is again needed. For a small number of terms, the density function[7] is

$$f(\eta) = k^{3/2} 2^{-(k+1)/2} \pi^{(k-1)/2} \sum_{i=1}^{k-1} \binom{k}{i} e^{-\eta^2 k^3/8i(k-i)} G_{i-1}\left(\frac{k\eta}{2}\right) G_{k-i-1}\left(\frac{k\eta}{2}\right) \tag{2-127}$$

where $G_i(x)$ may be calculated from the following recursive formula

$$G_i(x) = \int_0^x e^{-t^2/2i(i+1)} G_{i-1}(t)\, dt$$

$$G_0(x) = 1$$

However, actual accuracy requirements call for a longer averaging time, i.e. for a higher number of terms. The sum distribution of Eq. (2-126) may then be approximated, according to Eq. (2-69), by a normal distribution having an expected value of

$$M(\eta) = kM(|\xi|) = k\sqrt{\frac{2}{\pi}} \tag{2-128}$$

and a variance of

$$D^2(\eta) = kD^2(|\xi|) = k\left(1 - \frac{2}{\pi}\right)\sigma^2 \tag{2-129}$$

The relative standard deviation is

$$\frac{D(\eta)}{M(\eta)} = \sqrt{\frac{\pi}{2} - 1}\,\frac{1}{\sqrt{k}} \cong \frac{0.75}{\sqrt{k}} \tag{2-130}$$

Finally, for detecting a narrow-band noise with low relative bandwidth, the distribution of the envelope curve should be known. According to Fig. 2-10*b*, let us select a narrow band of width $\Delta\omega$, which is small compared with ω_0, from the white-noise spectrum shown in Fig. 2-10*a*. A possible time function of the signal thus generated is shown in Fig. 2-10*d*: indeed, the signal with centre frequency ω_0 and bandwidth $\Delta\omega$ may be considered as a carrier of frequency ω_0, stochastically modulated in amplitude and phase, and having an instantaneous value of

$$\xi(t) = \varrho(t) \cos[\omega_0 t + \vartheta(t)] \tag{2-131}$$

where $\xi(t)$, $\varrho(t)$ and $\vartheta(t)$ are stochastic time functions; $\varrho(t)$ and $\vartheta(t)$ are the instantaneous values of the envelope and the phase angle, respectively, $(0 < \varrho < \infty$, $0 < \vartheta < 2\pi)$. $\xi(t)$ may also be written in the form

$$\xi(t) = \alpha(t) \cos \omega_0 t + \beta(t) \sin \omega_0 t \tag{2-132}$$

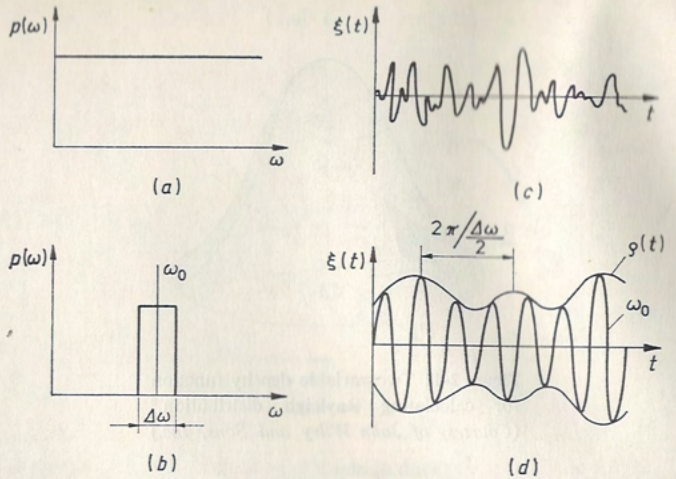

Figure 2-10 White-noise characteristics. *(a)* Power spectrum without band limiting. *(b)* Power spectrum with band limiting. *(c)* Time function without band limiting. *(d)* Time function and envelope function with band limiting.

where $\alpha(t)$ and $\beta(t)$ are mutually independent random variables with normal distributions if the distribution of the instantaneous values pertaining to the wide-band noise of Fig. 2-10*c* is also normal.

Let us define the two-variable density function $g(R, \theta)$: $g(R, \theta)dR\,d\theta$ is the probability of the instantaneous value of $\xi(t)$ falling within the intervals R and $R + dr$, and θ and $\theta + d\theta$. Let the common density function $f(a, b)$ of $\alpha(t)$ and $\beta(t)$ be similarly defined. The relationship between $g(R, \theta)$ and $f(a, b)$ is calculated with the aid of Fig. 2-11. Thus

$$f(a, b)\Delta a \Delta b = g(R, \Theta)\Delta R \Delta\Theta \tag{2-133}$$

As the random variables are independent, $f(a, b) = f(a)f(b)$, so

$$f(a)f(b) = \left(\frac{1}{\sigma\sqrt{2\pi}}\right)^2 e^{-(a^2+b^2)/2\sigma^2} = \frac{1}{2\pi\sigma^2} e^{-R^2/2\sigma^2} \tag{2-134}$$

For the coordinate transformation $a, b \to R, \Theta$, we have the following relation between the infinitely small surface elements:

$$dR d\Theta = \begin{bmatrix} \dfrac{dR}{da} & \dfrac{dR}{db} \\ \dfrac{d\Theta}{da} & \dfrac{d\Theta}{db} \end{bmatrix} da\, db = |J|\, da\, db \tag{2-135}$$

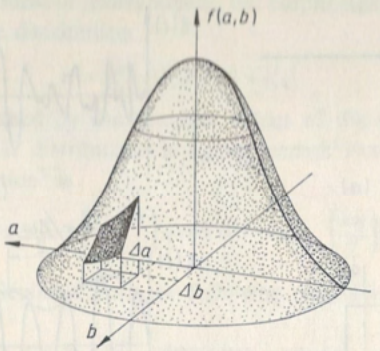

Figure 2-11 Two-variable density function for calculating Rayleigh distribution.[2] (*Courtesy of John Wiley and Sons, Inc.*)

where $|J|$ is the absolute value of the Jacobi determinant. In this case,

$$R = \sqrt{a^2+b^2}$$

$$\Theta = \arctan \frac{b}{a} \tag{2-136}$$

$$|J| = \frac{1}{R}$$

so

$$g(R, \Theta) = \frac{f(a, b)}{|J|} = \frac{R}{2\pi\sigma^2} e^{-R^2/2\sigma^2} \tag{2-137}$$

As the distribution of Θ between 0 and 2π is uniform and independent of R, Θ is easily eliminated. Thus

$$f(R) = \int_0^{2\pi} g(R, \Theta)\, d\Theta = 2\pi g(R, \Theta) \tag{2-138}$$

From this equation, the single variable density function of the envelope curve Rayleigh distribution is

$$\boxed{f(R) = \frac{R}{\sigma^2} e^{-R^2/2\sigma^2} \qquad R \geqq 0} \tag{2-139}$$

Integrating the distribution function of Eq. (2-139) between 0 and R, we have

$$F(R) = \mathscr{P}(\varrho < R) = 1 - e^{-R^2/2\sigma^2} \tag{2-140}$$

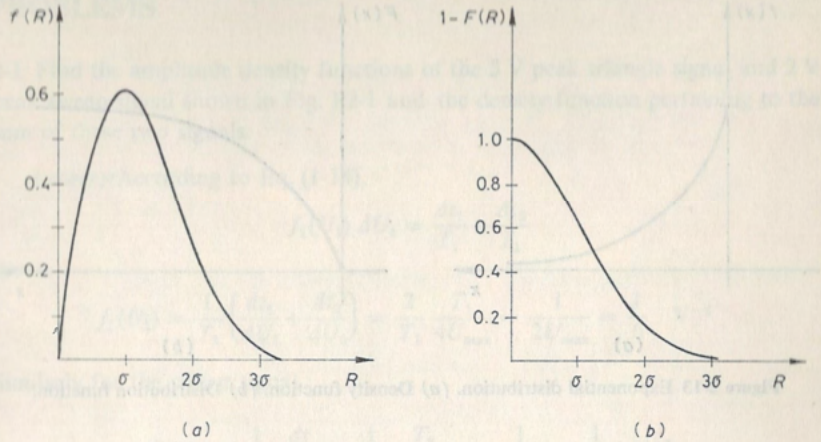

Figure 2-12 *(a)* Density function of Rayleigh distribution. *(b)* Probability of $R > k$.

Thus the probability of an envelope amplitude higher than R is given by $e^{-R^2/2\sigma^2}$ (see Fig. 2-12*b*). The expected value of the distribution is

$$M(\varrho) = \int_0^\infty \frac{R^2}{\sigma^2} e^{-R^2/2\sigma^2}\, dR = \sqrt{\frac{\pi}{2}}\,\sigma \cong 1.25\sigma \tag{2-141}$$

and the variance is

$$D^2(\varrho) = \mu_2 - \mu_1^2 = \int_0^\infty \frac{R^3}{\sigma^2} e^{-R^2/2\sigma^2}\, dR - \mu_1^2 = 2\sigma^2 - \frac{\pi}{2}\sigma^2 = 0.43\sigma^2 \tag{2-142}$$

Again taking into account the averaging process following the envelope detector, the relative standard deviation of the sum variable formed by the independent, random variables $\varrho_1, \varrho_2, \ldots, \varrho_k$, as in Eqs. (2-123) and (2-130), is

$$\frac{D(\sum_k \varrho)}{M(\sum_k \varrho)} = \frac{\sqrt{0.43}}{1.25}\frac{1}{\sqrt{k}} = \frac{0.525}{\sqrt{k}} \tag{2-143}$$

The Rayleigh distribution of Eq. (2-139) may also be derived in a different way.[13]

Another continuous distribution of importance is the exponential distribution that is utilized to describe decomposition, collision, and relaxation processes in atoms and solid state structures.

A continuous random variable ξ has an exponential distribution if the density function has the form

$$f(x) = \begin{cases} 0 & \text{if} \quad x \leqq 0 \\ \lambda e^{-\lambda x} & \text{if} \quad x > 0 \end{cases} \tag{2-144}$$

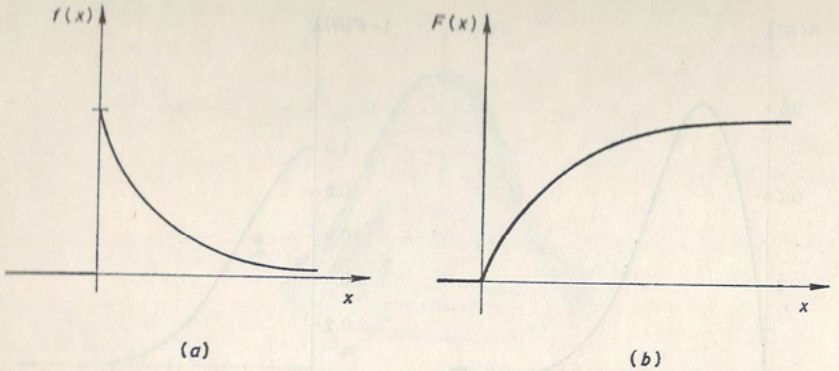

Figure 2-13 Exponential distribution. *(a)* Density function. *(b)* Distribution function.

where λ may be an arbitrary positive number called distribution parameter. The distribution function of ξ, shown in Fig. 2-13*b*, is given by

$$F(x) = \begin{cases} 0 & \text{if} \quad x \leqq 0 \\ 1-e^{-\lambda x} & \text{if} \quad x > 0 \end{cases} \tag{2-145}$$

The first- and second-order moments and the variance of a random variable of exponential distribution and of distribution parameter λ are

$$M(\xi) = \int_0^\infty x\lambda e^{-\lambda x}\,dx = \frac{1}{\lambda} \tag{2-146}$$

$$M(\xi^2) = \int_0^\infty x^2\lambda e^{-\lambda x}\,dx = \frac{2}{\lambda^2} \tag{2-147}$$

$$D^2(\xi) = M(\xi^2) - M^2(\xi) = \frac{1}{\lambda^2} \tag{2-148}$$

Let us assume that the random variable ξ stands for a time interval and has an exponential distribution. If any time instant x is selected and if the random time interval is not finished at this time instant, the whole process may be regarded as though it had started at the instant x. This property makes the exponential distribution suitable for describing the statistic behavior of elementary particles.

PROBLEMS

2-1 Find the amplitude density functions of the 3 V peak triangle signal and 2 V peak sweep signal shown in Fig. P2-1 and the density function pertaining to the sum of these two signals.

Answer: According to Eq. (1-14),

$$f_1(U_1)\,\Delta U_1 = \frac{\Delta t_1}{T_1} + \frac{\Delta t_2}{T_1}$$

thus

$$f_1(U_1) = \frac{1}{T_1}\left(\frac{dt_1}{dU_1} + \frac{dt_2}{dU_1}\right) = \frac{2}{T_1}\frac{T_1}{4U_{\max}} = \frac{1}{2U_{\max}} = \frac{1}{6} \quad \mathrm{V}^{-1}$$

Similarly for the sweep signal

$$f_2(U_2) = \frac{1}{T_2}\frac{dt}{dU_2} = \frac{1}{T_2}\frac{T_2}{2U_{\max}} = \frac{1}{2U_{\max}} = \frac{1}{4} \quad \mathrm{V}^{-1}$$

The convolution of the two density functions yields the density function of $U = U_1 + U_2$:

$$f(U) = \begin{cases} \int_{-\infty}^{\infty} f_1(U_1) f_2(U-U_1)\,dU_1 = 0 & U < -5\,\mathrm{V} \quad U > +5\,\mathrm{V} \\ 0.042\left(\frac{1}{\mathrm{V}^2}\right)(U+5\,\mathrm{V}) & -5\,\mathrm{V} < U < -1\,\mathrm{V} \\ 0.167\left(\frac{1}{\mathrm{V}}\right) & -1\,\mathrm{V} < U < 1\,\mathrm{V} \\ -0.042\left(\frac{1}{\mathrm{V}^2}\right)(U-5\,\mathrm{V}) & 1\,\mathrm{V} < U < 5\,\mathrm{V} \end{cases}$$

2-2 Random numbers of normal distribution are required for a computer model of a noise problem. However, only random numbers of uniform distribution, falling in the interval $-1 < \xi < +1$ are available (see Fig. P2-2*a*). What method may be applied to approximate the normal distribution?

Answer: According to the central limit theorem, the sum of an infinite number of terms has a normal distribution if $\sigma_1 = \sigma_2 = \ldots \sigma_n$. Let us investigate this for a finite number of terms, and let the number of terms be, in this case, 6! The basic distribution being symmetrical, the sum distribution is also symmetrical, with an expected value of $M(\eta) = 0$ and a variance of

$$D^2(\eta) = 6D^2(\xi) = 6 \times \frac{1}{3} = 2 = \sigma^2$$

This is explained by Eq. (2-111), according to which the variance of a uniform distribution between ± 1 is 1/3.

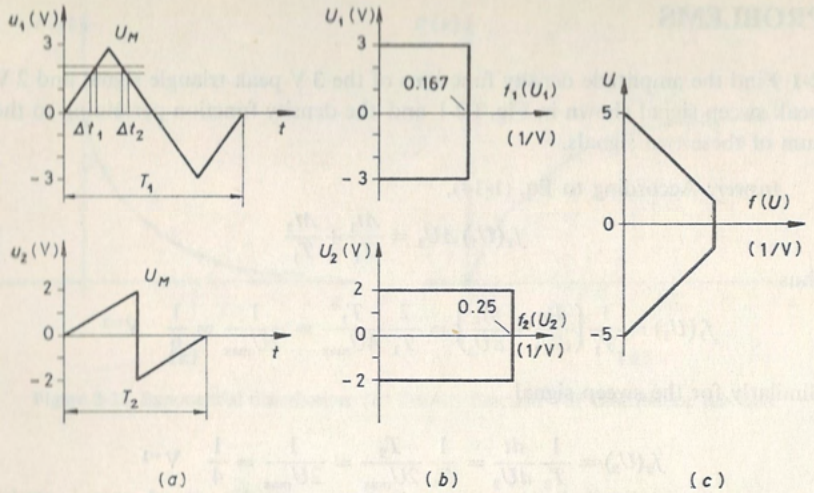

Figure P2-1 Noncoherent triangular signal and sweep signal *(a)* Time functions. *(b)* Amplitude density functions. *(c)* Density function corresponding to the sum of the two signals.

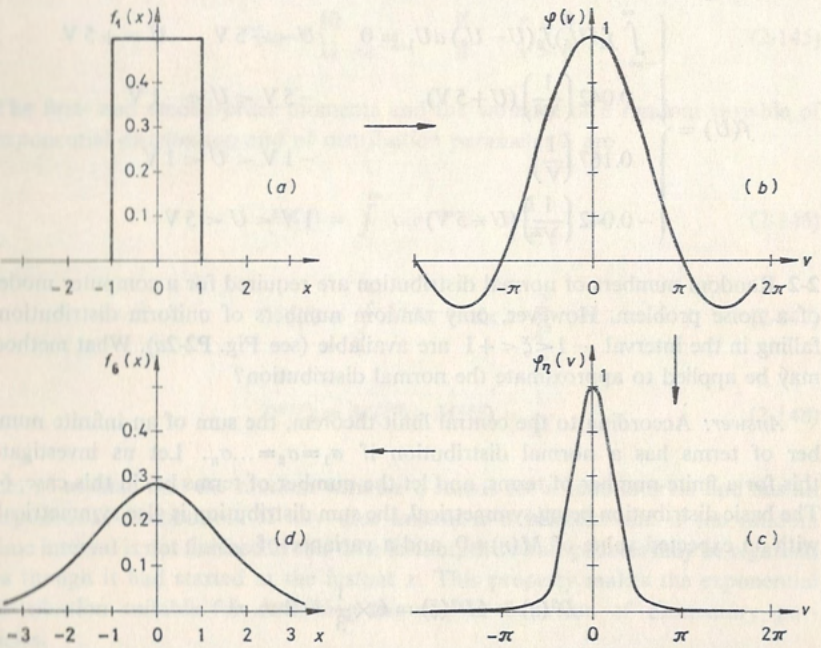

Figure P2-2 *(a)* Even distribution. *(b)* Characteristic function of even distribution. *(c)* Characteristic function on the sixth power. *(d)* Sum distribution.

The density function $f(x)=1/2$ $(-1<\xi<1)$ is symmetrical, so its characteristic function is given, according to Eq. (2-39) (see Fig. P2-2*b*), by

$$\varphi(v)=\int_{-1}^{1}\cos vx\,\frac{1}{2}\,dx=\frac{1}{2}\,\frac{\sin vx}{v}\bigg|_{-1}^{1}=\frac{\sin v}{v}$$

The characteristic function of an n-term sum, according to Eq. (2-45), is (see Fig. P2-2*c*)

$$\varphi_\eta(v)=\varphi^n(v)=\left(\frac{\sin v}{v}\right)^n$$

The density function of the sum is determined by the inverse Fourier transform (see Fig. P2-2*d* and Appendix B)

$$f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\cos vx\left(\frac{\sin v}{v}\right)^n dv=\frac{1}{\pi}\int_{0}^{\infty}\cos vx\left(\frac{\sin v}{v}\right)^n dv$$

$$=\frac{n}{2^n}\sum_{k=0}^{(x+n)/2}\frac{(-1)^k(n+x-2k)^{n-1}}{k!(n-k)!}$$

For $n=6$ and $-6\leqq x<-4$, only the polynomial term pertaining to $k=0$ is left. Thus

$$f(x)=\frac{n}{2^n}\,\frac{1}{n!}(n+x)^{n-1}\qquad -6\leqq x<-4$$

Further sections of the function are

$$f(x)=\begin{cases}\dfrac{n}{2^n}\left[\dfrac{1}{n!}(n+x)^{n-1}-\dfrac{1}{(n-1)!}(n-2+x)^{n-1}\right] & \text{if}\quad -4<x<-2\\[2ex] \dfrac{n}{2^n}\left[\dfrac{1}{n!}(n+x)^{n-1}-\dfrac{1}{(n-1)!}(n-2+x)^{n-1}+\dfrac{1}{2(n-2)!}(n-4+x)^{n-1}\right] & \text{if}\quad -2\leqq x<0\end{cases}$$

The polynomial need not be expressed for $x>0$ as $f(x)=f(-x)$. In the following table, the density function thus determined is compared with the density function of a normal distribution with a standard deviation of $\sigma=\sqrt{2}$. For ease of evaluation, both functions are multiplied by the reciprocal value of $1/\sigma\sqrt{2\pi}=1/\sqrt{2}\sqrt{2\pi}$.

Table 2-2

$\pm x$	$2\sqrt{\pi}f_6(x)$	$e^{-x^2/4}$
0	0.9748	1
1	0.7763	0.7788
2	0.3840	0.3679
3	0.1094	0.1054
4	0.0148	0.0183
5	0.00046	0.00193
6	0	0.000123

According to the required accuracy, the number of terms may be increased or decreased. A sum distribution with unity standard deviation may be realized using the definition

$$\eta = \sqrt{\frac{3}{n}} \sum_{i=1}^{n} \xi_i$$

2-3 A p–n junction carries a DC current of 1 μA, and the shot noise is mesaured in a bandwidth of 10 MHz. May the difference between the instantaneous value and the mean value of the current be regarded as having a normal distribution?

Answer: According to Eq. (1-11), which has been derived from the Poisson distribution, the square root taken from the mean square value of the fluctuation, i.e. the standard deviation, is

$$\sqrt{\overline{i^2}} = \sqrt{2qI\Delta f} = \sqrt{3.2\times10^{-19}\times10^{-6}\times10^{7}} = 1.8\times10^{-9}\ \mathrm{A}$$

As $\sqrt{\overline{i^2}} \ll I$, the Poisson distribution may by substituted by a normal distribution according to Eqs. (2-90) to (2-96). This may not be justified for devices operating at extremely low current and high bandwidth, e.g., for optoelectronic devices.

2-4 The narrow-band noise of normal distribution, shown in Fig. 2-10*d* is used to feed two detectors—an absolute-value detector and an envelope detector—and the detected signals are integrated by a high time-constant *RC* network (Fig. P2-3). Will the two output voltages be different?

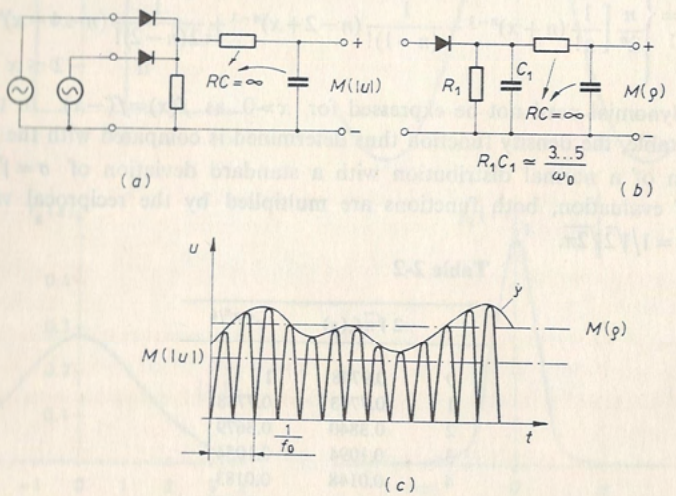

Figure P2-3 *(a)* Absolute-value detector. *(b)* Envelope detector. *(c)* Absolute value of narrow-band noise voltage.

Answer: Let the root mean square value of the input noise voltage be

$$U_{\mathrm{rms}} = \sqrt{\overline{u^2}} = D(u) = \sigma$$

Then, according to Eq. (2-69), the DC output voltage of the absolute-value detector will be

$$M(|u|) = \sqrt{\frac{2}{\pi}}\, U_{\mathrm{rms}}$$

On the other hand, the DC output voltage of the envelope detector, according to Eq. (2-141), will correspond to the expected value of the Rayleigh distribution

$$M(\varrho) = \sqrt{\frac{\pi}{2}}\, U_{\mathrm{rms}} = \frac{\pi}{2} M(|u|)$$

or

$$M(|u|) = \frac{2}{\pi} M(\varrho)$$

Taking into account the approximate sinusoidal waveform of the narrow-band noise, the output of the absolute-value detector will be a series of half sine waves, with amplitudes corresponding to the envelope function. The mean value of this half sine-wave series is $2/\pi$ times the peak value.

2-5 Show that the density function χ^2 of the distribution (2-118) meets the requirement

$$\int_{-\infty}^{\infty} f(y)\, dy = 1$$

following from the definition of the density function, e.g. for $k=4$.

Answer: For integers, $\Gamma(\alpha+1)=\alpha!$, $\Gamma(2)=1$, moreover $f_k(y)=0$ for $y<0$, so with partial integration, we have

$$\int_0^{\infty} f_4(y)\, dy = \frac{1}{4\sigma^4}\int_0^{\infty} e^{-y/2\sigma^2}\, dy = \frac{1}{4\sigma^4}\left[-2\sigma^2 y e^{-y/2\sigma^2} - 4\sigma^4 e^{-y/2\sigma^2}\right]_0^{\infty} = 1.$$

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CHAPTER

THREE

STOCHASTIC SIGNALS IN THE TIME AND FREQUENCY DOMAIN

In Chapter 2, the stochastic process has been defined as a single parameter assembly of random variables with the variable covering a time instant set. Thus $\xi=\xi(t)$ may be regarded as the discrete or continuous function of time. In our case, the specific value of ξ may be voltage or current.

Normally, the time function of voltage or current may be expressed by a simple and unique mathematical formula. Time functions of precise time dependence are called deterministic functions. In the case of these functions, a few parameters such as amplitude, frequency and phase are sufficient to describe either the past or the future of the signal. Evidently, the "past" or the "future" is restricted to the time interval of the signal; this time interval has to be defined for each specific case. On the other hand, the past knowledge of a stochastic signal does not allow the precise determination of the next instantaneous value; instead its probability may be ascertained. In other words, an ideal stochastic signal has no "memory". However, there is always some kind of time constant in physically realizable devices and circuits (e.g., the transit time of the charge carrier in a vacuum or solid state device). Owing to this effect, the consecutive instantaneous values of the noise appearing at the output of a device or circuit, assuming infinitely small time intervals between these instantaneous values, are not completely independent from each other. This internal property—autocorrelation—of real stochastic signals will be frequently utilized in this Chapter. However, the time and frequency domain description of deterministic and stochastic signals will first be presented systematically.

3-1 DETERMINISTIC AND STOCHASTIC SIGNALS[1,2,3,4,7,9,10,11]

Figure 3-1 shows all possible types of signals. Both main groups may be divided further into classes and subclasses. Wherever possible, the usual description in the time, frequency, and complex frequency domains will be presented, as well as the transformations between these domains.

The simplest deterministic signal is the periodic signal with a time function comprising sine and cosine terms, but only with integral multiples of ω_0 in the arguments of this function. The discrete version of the Fourier transform (the Fourier series) is applied for the passing over from the time domain to the frequency domain.

The "almost periodical" signal is an extension of the periodical signal comprising harmonic components. It may be regarded as a superposition of discrete and fixed-frequency sine and cosine terms. However, the quotients of these discrete frequencies are neither integers nor rational fractions, i.e.

$$\frac{\omega_j}{\omega_k} = \text{irrational number}$$

meaning that the signal waveform is repeated only after an infinite time. Such a signal is, for instance, the superposition of the triangular wave and the sweep signal in Problem 2-1, provided that their fundamental frequencies are not harmonically related, according to the above condition. Separately, both signals may be represented in the frequency domain by a discrete spectrum. The connection between the time domain and the frequency domain is given by the Fourier integral, resulting in a spectrum comprising a series of Dirac-delta functions.

Transients, which may be described by a starting function $x(t)=0$ if $t<0$, may also be classified as deterministic functions. As a transient may not repeat itself, i.e. it has an infinite period, the Fourier transform may be used to change over from the time to the frequency domain. Unfortunately, the Fourier transform in itself is not necessarily convergent; the conditions for convergence would have to be investigated for each case. The Laplace transform is much more suitable for the investigation of deterministic transient signals. In spite of this, the Fourier transform will be applied in this book as, with due caution, it may be equally useful for nonperiodic, deterministic signals and for stochastic signals.

Stochastic signals are comparable to the almost periodical signals, with the difference that the frequency spectrum of stochastic signals is, at least piecewise, continuous. The main problem is that neither the amplitude nor the phase can be predicted for an arbitrary time instant, so the amplitude and phase of the Fourier transform are unknown. Stochastic signals should therefore be described by other methods.

A stochastic process may be defined by a function $\xi(t)$ in the same way as a deterministic process. The function values may be real or complex numbers, or

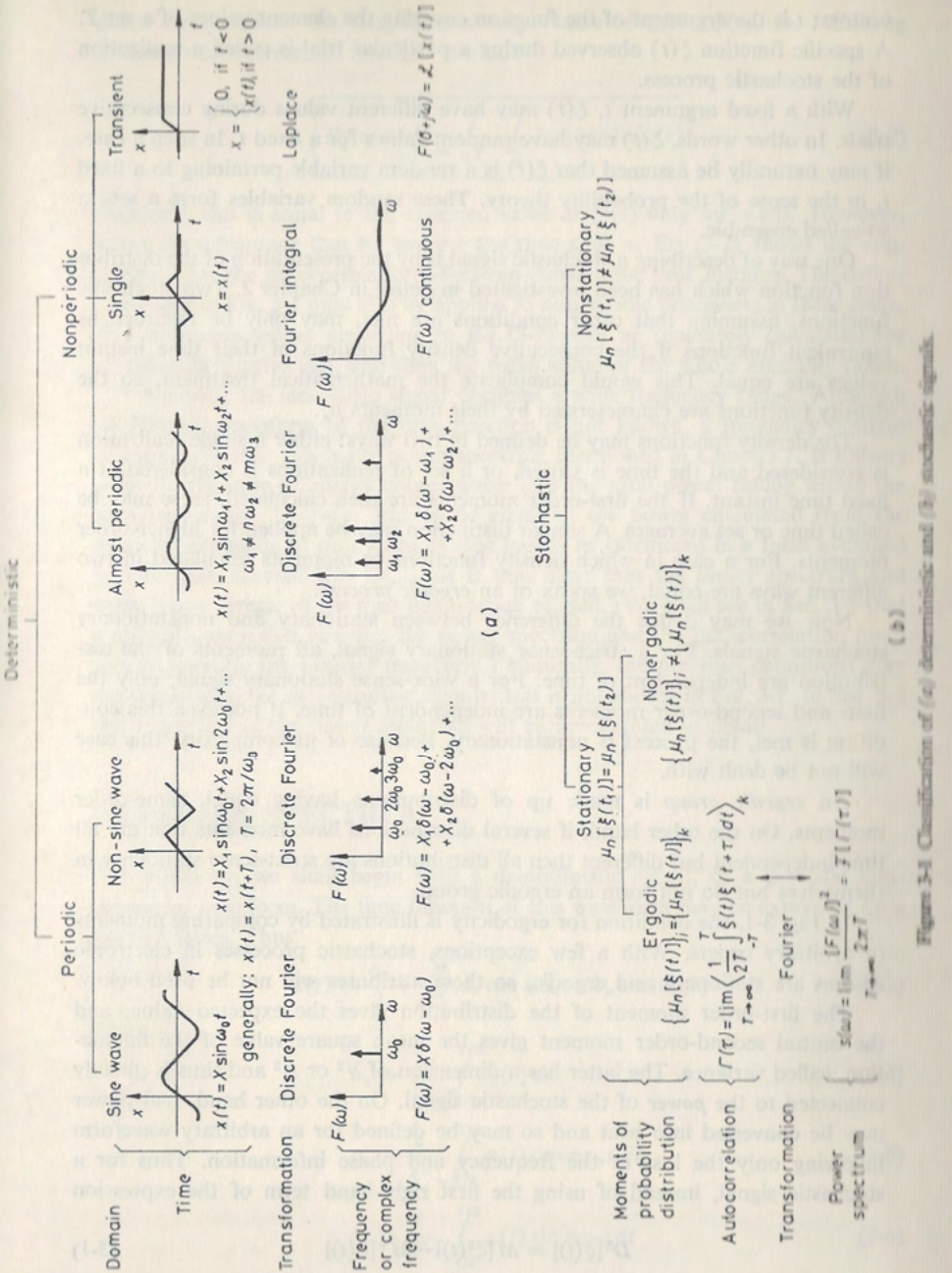

Figure 3-1 Classification of (a) deterministic and (b) stochastic signals

vectors; t is the argument of the function covering the element values of a set T. A specific function $\xi(t)$ observed during a particular trial is called a realization of the stochastic process.

With a fixed argument t, $\xi(t)$ may have different values during consecutive trials. In other words, $\xi(t)$ may have random values for a fixed t. In such a case, it may naturally be assumed that $\xi(t)$ is a random variable pertaining to a fixed t, in the sense of the probability theory. These random variables form a set—a so-called ensemble.

One way of describing a stochastic signal is by the presentation of the distribution function which has been investigated in detail in Chapter 2. Two stochastic functions, assuming that other conditions are met, may only be regarded as equivalent functions if the consecutive density functions of their time instant values are equal. This would complicate the mathematical treatment, so the density functions are characterized by their moments μ_n.

The density functions may be defined in two ways: either a single realization is considered and the time is varied, or a set of realizations is considered at a fixed time instant. If the first-order moments are then calculated, these may be called time or set averages. A similar distinction may be applied for higher-order moments. For a case in which density functions or moments calculated in two different ways are equal, we speak of an *ergodic process*.

Now we may define the difference between stationary and nonstationary stochastic signals. For a strict-sense stationary signal, all moments of the distribution are independent of time. For a wide-sense stationary signal, only the first- and second-order moments are independent of time. If not even this condition is met, the process is nonstationary. Because of its complexity, this case will not be dealt with.

An *ergodic group* is made up of distributions having equal, same-order moments. On the other hand, if several distributions have moments that are all time-independent but different then all distributions are strict-sense stationary in themselves but do not form an ergodic group.

In Fig. 3-1, the condition for ergodicity is illustrated by comparing moments of arbitrary orders. With a few exceptions, stochastic processes in electronic devices are stationary and ergodic, so these attributes will not be used below.

The first-order moment of the distribution gives the expected value, and the central second-order moment gives the mean square value of the fluctuation, called variance. The latter has a dimension of V^2 or A^2 and thus is directly connected to the *power* of the stochastic signal. On the other hand, real power may be converted into heat and so may be defined for an arbitrary waveform involving only the loss of the frequency and phase information. Thus for a stochastic signal, instead of using the first right-hand term of the expression

$$D^2[\xi(t)] = M[\xi^2(t)] - M^2[\xi(t)] \tag{3-1}$$

which really means the long-term average value of $\xi(t)\times\xi(t)$, the following so-called autocorrelation function is used:

$$\boxed{\Gamma_\xi(\tau) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \xi(t)\xi(t+\tau)\,dt} \tag{3-2}$$

Evidently, this is equal to the expected value of $\xi^2(t)$ only for $\tau=0$. However, it has the advantage that by varying the time shift τ, Eq. (3-2) shows the connection—i.e. the autocorrelation—between consecutive time instants. The definition given in Fig. 3-1 may also be applied to nonergodic processes; in these cases, the time mean values according to Eq. (3-2) may be separately calculated for all elementary processes, and then the mean value of the whole assembly taken.

Generally, the stochastic signal occupies a wide frequency range. Although the Fourier transform of the time function is not known, a frequency domain description exists; this is the power spectrum mentioned in Chapter 1. It follows from the theorem of energy conservation that the total power falling within the frequency range $0<f<\infty$ has to be equal to the power calculated from the mean square value of time instants that has to be calculated in a form averaged for the time interval $T\gg 1/f_{\min}$. It is thus clear that the power spectrum and some characteristic of the time instants are related. (We shall see in Sec. 3-3 that a mutual connection between the power spectrum and the autocorrelation function is given by the Fourier transform.) Naturally, the previous definitions may also be applied for deterministic signals; this is not shown in Fig. 3-1. In Sec. 3-2 the properties of the power spectrum will be investigated in detail.

3-2 POWER SPECTRUM

For simplicity, we shall begin with a deterministic signal, or a periodic nonsinusoidal waveform. The time function of this waveform may be expressed by a Fourier series. Thus

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\omega_0 t + b_n \sin n\omega_0 t) \tag{3-3}$$

where

$$a_0 = \frac{2}{T} \int_{-T/2}^{T/2} x(t)\,dt \tag{3-4}$$

$$a_n = \frac{2}{T} \int_{-T/2}^{T/2} x(t) \cos n\omega_0 t\,dt \tag{3-5}$$

$$b_n = \frac{2}{T} \int_{-T/2}^{T/2} x(t) \sin n\omega_0 t\,dt \tag{3-6}$$

and

$$T = \frac{2\pi}{\omega_0} \tag{3-7}$$

Utilizing the Euler relation, we have

$$\cos n\omega_0 t = \frac{e^{jn\omega_0 t} + e^{-jn\omega_0 t}}{2} \tag{3-8}$$

$$\sin n\omega_0 t = \frac{e^{jn\omega_0 t} - e^{-jn\omega_0 t}}{2j} \tag{3-9}$$

and Eq. (3-3) transforms to

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(\frac{a_n + jb_n}{2} e^{-jn\omega_0 t} + \frac{a_n - jb_n}{2} e^{jn\omega_0 t} \right) \tag{3-10}$$

Let us now introduce the complex coefficients

$$c_n^- = \frac{a_n + jb_n}{2} \tag{3-11}$$

and

$$c_n^+ = \frac{a_n - jb_n}{2} \tag{3-12}$$

Thus $x(t)$ is made up of a constant term plus the sum of vectors rotating with angular velocities $n\omega_0$ and $-n\omega_0$. The absolute value and phase of these vectors are given by coefficients c_n^- and c_n^+. In the following, let $a_0 = 0$. The introduction of the seemingly unjustified negative angular frequency $-n\omega_0$ results in significant mathematical simplifications. Note that

$$c_n^- = (c_n^+)^* \tag{3-13}$$

i.e. the corresponding coefficient pair members are conjugates of each other. Let us substitute $-n\omega_0$ instead of $n\omega_0$ into the defining formulas (3-5) and (3-6) for a_n and b_n, respectively; then b_n changes sign while a_n does not change. Thus in the range $-\infty < n < \infty$, we may apply the universal coefficient

$$c_n = \frac{a_n - jb_n}{2} = \frac{1}{T} \int_{-T/2}^{T/2} x(t) e^{-jn\omega_0 t}\, dt \tag{3-14}$$

instead of the separate coefficients c_n^- and c_n^+. Thus $x(t)$ may be written in the simplified form

$$x(t) = \sum_{-\infty}^{\infty} c_n e^{jn\omega_0 t} \tag{3-15}$$

Expressing c_n as given in Eq. (3-14), Eq. (3-15) will comprise a vector pertaining to increasing time.

Let us now calculate the mean square value of $x(t)$. To this end, the multiterm right-hand side expression in Eq. (3-15) has to be multiplied by its own conjugated value and then has to be averaged in time. Thus

$$M[x^2(t)] = \frac{1}{T} \int_{-T/2}^{T/2} (c_1 c_1^* + c_1 e^{j\omega_0 t} c_2^* e^{-j2\omega_0 t} + \ldots + c_2 e^{j2\omega_0 t} c_1^* e^{-j\omega_0 t} + c_2 c_2^* + \ldots)\, dt$$

$$= \frac{1}{T} \int_{-T/2}^{T/2} \sum_n \sum_m c_n c_m^* e^{j(n-m)\omega_0 t}\, dt \tag{3-16}$$

The period $T = 2\pi/\omega_0$ of the lowest frequency component is chosen as average time. In the exponent of the exponential term, we find n and m, and t as factors of a single product, so the sequence of operations involving these factors may be inverted. Thus

$$M[x^2(t)] = \frac{1}{T} \sum_n \sum_m \left(c_n c_m^* \int_{-T/2}^{T/2} e^{j(n-m)\omega_0 t}\, dt \right) \tag{3-17}$$

For $n \neq m$, we have

$$\int_{-T/2}^{T/2} e^{j(n-m)(2\pi t/T)}\, dt = 0$$

so finally

$$M[x^2(t)] = \sum_{n=-\infty}^{\infty} c_n c_n^* = \sum_{n=-\infty}^{\infty} |c_n|^2 \tag{3-18}$$

However, according to Eq. (3-13), the coefficients c_n^+ and c_n^- pertaining to the same number n are conjugates of each other, thus their absolute values are equal. It follows therefore that $|c_n|^2$ is an even function, which provides substantial computational benefits.

Figure 3-2*a* shows the complex coefficients c_n and the spectrum distribution of the mean square value pertaining to an arbitrarily chosen deterministic function. Figure 3-2*b* cannot be regarded truly as a power spectrum as the dimension of coefficients c_n is V or A, thus the dimension of $|c_n|^2$ is V^2 or A^2. Actually, we may call this a power spectrum only in cases where this voltage appears across, or this current flows through, a given resistance R. Unless this leads to ambiguity, the coefficients $|c_n|^2$ will be called the terms of the power spectrum. However, it should be noted that in calculating the overall power of the signal $x(t) = i(t)$, Eq. (3-18) has to be multiplied by the value of resistance R. Thus

$$P = R \sum_{n=-\infty}^{\infty} |c_n|^2 \text{ (W)} \tag{3-19}$$

Equation (3-19) expresses a simple theorem: the power of a periodic nonsinusoidal wave may be calculated by taking into account all Fourier coefficients.

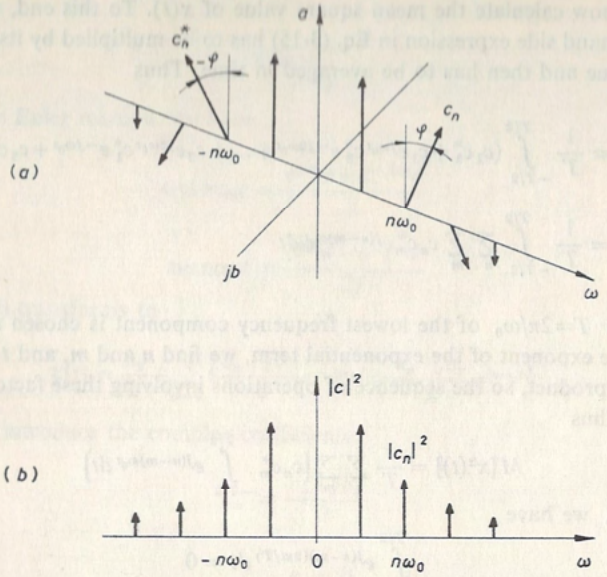

Figure 3-2 Periodic nonsinusoidal signal. *(a)* Complex Fourier coefficients as a function of discrete $n\omega_0$ frequencies. *(b)* Double-sided power spectrum.

The use of negative frequencies is not convenient from the measurement standpoint. As $|c_n|^2$ is an even function and according to our condition $c_0=a_0=0$, the following relation also holds:

$$M[x^2(t)] = 2\sum_{n=1}^{\infty}|c_n|^2 = \sum_{n=1}^{\infty}|C_n|^2 \tag{3-20}$$

where $|C_n|^2=2|c_n|^2$. In the following, lower case letters will be used to denote members of the two-sided power spectrum, while upper case letters denote members of the single-sided power spectrum which is only interpreted for positive frequencies.

Let us now investigate a deterministic signal with a continuous Fourier spectrum (see Fig. 3-4*a*). For this case, the discrete spectrum made up of coefficients c_n is changed into a continuous spectrum $F(\omega)$, using the following relation which is analogous to Eq. (3-14):[5,6,8]

$$F(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt \tag{3-21}$$

As Eq. (3-21) is an improper integral and $|e^{-j\omega t}|=1$, the following conditions have to be met for a finite value of $x(t)$:

1. $x(t)$ should be a continuous function capable of differentiation.
2. The integral of the absolute value of $x(t)$ should be finite, i.e.

$$\int_{-\infty}^{\infty} |x(t)|\,dt = \text{finite} \tag{3-22}$$

This means that for $t\to\pm\infty$, $x(t)\to 0$. As this last condition would substantially reduce the applicable function types—and, as will be shown, the stationary stochastic processes would also be excluded—let, for the time being, the finite value of Eq. (3-21) be secured by the condition that the function $x(t)$ may differ from zero only in the interval between $-T$ and $+T$ (see Fig. 3-3). Thus

$$x(t) = \begin{cases} x_T(t) & \text{if } -T<t<T \\ 0 & \text{otherwise} \end{cases} \tag{3-23}$$

Equation (3-21) may then be written as

$$F_T(\omega) = \int_{-\infty}^{\infty} x_T(t)e^{-j\omega t}\,dt = \int_{-T}^{T} x(t)e^{-j\omega t}\,dt \tag{3-24}$$

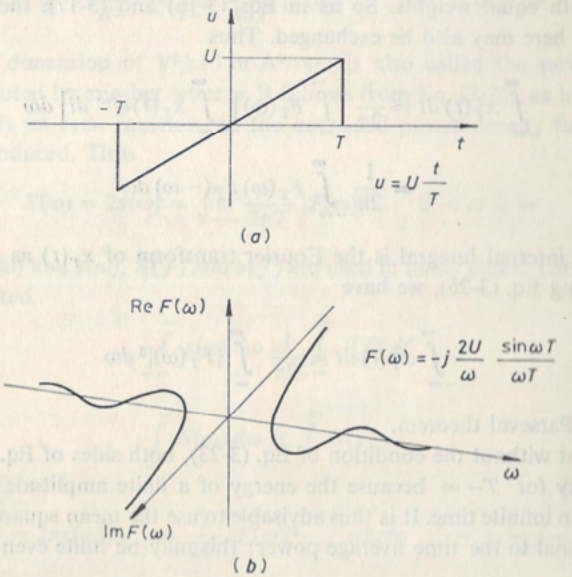

Figure 3-3 Nonperiodic signal truncated in time. *(a)* Time function. *(b)* Fourier spectrum.

Due to the requirement that $x(t)$ be real, Eq. (3-24) may be divided into two parts:

$$F_T(\omega) = \int_{-\infty}^{\infty} x_T(t)\cos\omega t\,dt - j\int_{-\infty}^{\infty} x_T(t)\sin\omega t\,dt \tag{3-25}$$

Writing $(-\omega)$ instead of ω, the first right-hand side term is unchanged while the second term changes sign. Thus

$$F_T(-\omega) = F_T^*(\omega) \tag{3-26}$$

Knowledge of $F_T(\omega)$ allows the determination of $x_T(t)$ by inverse Fourier transform

$$x_T(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F_T(\omega)e^{j\omega t}\,d\omega \tag{3-27}$$

Let us now calculate the time integral of $x_T^2(t)$, which is proportional to the signal energy. According to condition (3-22) we have to arrive at a finite result

$$\int_{-\infty}^{\infty} x_T^2(t)\,dt = \int_{-\infty}^{\infty} x_T(t)x_T(t)\,dt = \int_{-\infty}^{\infty} x_T(t)\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} F_T(\omega)e^{j\omega t}\,d\omega\right]dt \tag{3-28}$$

where Eq. (3-27) has also been used. Note that in the double integral of Eq. (3-28), $x_T(t)$ and $F_T(\omega)$ are symmetrically placed, and $e^{j\omega t}$ comprises both integral variables with equal weights. So as in Eqs. (3-16) and (3-17), the sequence of integrations here may also be exchanged. Thus

$$\int_{-\infty}^{\infty} x_T^2(t)\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} F_T(\omega)\left[\int_{-\infty}^{\infty} x_T(t)e^{j\omega t}\,dt\right]d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} F_T(\omega)F_T(-\omega)\,d\omega \tag{3-29}$$

because the internal integral is the Fourier transform of $x_T(t)$ as a function of $-\omega$. Utilizing Eq. (3-26), we have

$$\int_{-\infty}^{\infty} x_T^2(t)\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |F_T(\omega)|^2\,d\omega \tag{3-30}$$

This is the Parseval theorem.

Note that without the condition of Eq. (3-23), both sides of Eq. (3-30) would go to infinity for $T\to\infty$ because the energy of a finite amplitude real signal is infinite for an infinite time. It is thus advisable to use the mean square value which is proportional to the time average power; this may be finite even in this case:

$$M[x_T^2(t)] = \frac{1}{2T}\int_{-T}^{T} x_T^2(t)\,dt = \frac{1}{4\pi T}\int_{-\infty}^{\infty} |F_T(\omega)|^2\,d\omega \tag{3-31}$$

Let us remove the restriction imposed by Eq. (3-23) and now we only require that $M[x^2(t)]$ should go to a finite limiting value for $T\to\infty$. Then

$$M[x^2(t)] = \lim_{T\to\infty}\frac{1}{4\pi T}\int_{-\infty}^{\infty} |F(\omega)|^2\,d\omega \tag{3-32}$$

If $x(t)$ has current or voltage dimension, the average power may be defined as

$$P = R^{\pm 1}M[x^2(t)] \tag{3-33}$$

where $[R]=\mathrm{V/A}$.

The power pertaining to one radian per second which is the *power density* is given by

$$p(\omega) = \frac{dP}{d\omega} = R^{\pm 1}\lim_{T\to\infty}\frac{1}{4\pi T}|F(\omega)|^2 \tag{3-34}$$

which now may be accurately defined with the aid of Eq. (3-34). Do not forget, however, that this definition applies to a two-sided power spectrum, i.e.

$$P = \int_{-\infty}^{\infty} p(\omega)\,d\omega \tag{3-35}$$

With some degree of inaccuracy, the quantity

$$s(\omega) = \frac{p(\omega)}{R^{\pm 1}} = \lim_{T\to\infty}\frac{1}{4\pi T}|F(\omega)|^2 \qquad -\infty < \omega < \infty \tag{3-36}$$

which has a dimension of $\mathrm{V^2/rad}$ or $\mathrm{A^2/rad}$, is also called the power density, although denoted by another letter, s. It follows from Eq. (3-26), as in Eq. (3-20), that $|F(\omega)|^2$ is an even function, so the one-sided power density function may again be introduced. Thus

$$S(\omega) = 2s(\omega) = \lim_{T\to\infty}\frac{1}{2\pi T}|F(\omega)|^2 \qquad 0 < \omega < \infty \tag{3-37}$$

Instead of $S(\omega)$ and $s(\omega)$, $S(f)$ and $s(f)$ are used in many cases. These terms are evidently related.

$$\int_{-\infty}^{\infty} s(\omega)\,d\omega \equiv \int_{-\infty}^{\infty} s(f)\,df \tag{3-38a}$$

and

$$\int_{0}^{\infty} S(\omega)\,d\omega \equiv \int_{0}^{\infty} S(f)\,df \tag{3-38b}$$

As $d\omega = 2\pi df$, we have

$$s(f) = 2\pi s(\omega) = \lim_{T\to\infty}\frac{1}{2T}|F(\omega)|^2 \quad \text{where} \quad -\infty < f < \infty \tag{3-39}$$

and

$$S(f) = 2\pi S(\omega) = \lim_{T\to\infty}\frac{1}{T}|F(\omega)|^2 \quad \text{where} \quad 0 < f < \infty \tag{3-40}$$

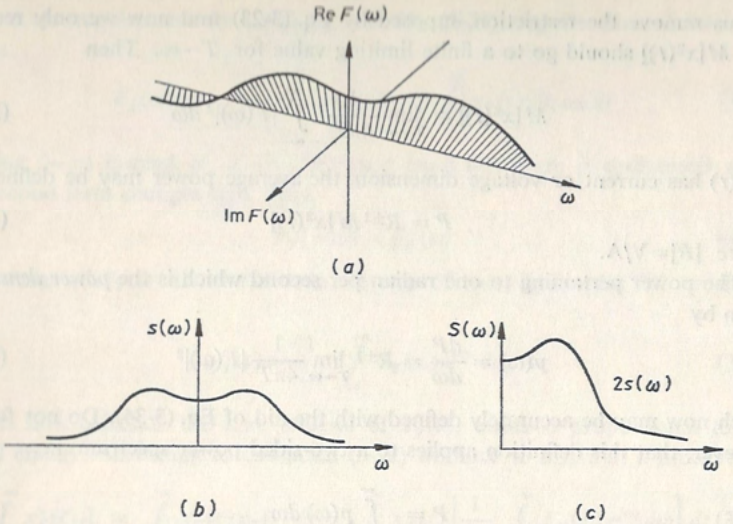

Figure 3-4 Nonperiodic, deterministic signal. *(a)* Continuous Fourier spectrum. *(b)* Double-sided power density function. *(c)* Single-sided power density function

Figure 3-4 shows a continuous Fourier spectrum, as well as a two-sided and a one-sided power density function. However, it is important to note that while $F(\omega)$ is a complex function with a real variable, so for each ω it defines a pair of real–imaginary values; $s(\omega)$ is a real function with one variable. When calculating $s(\omega)$, the phase information is lost, so $F(\omega)$ may not be determined in a unique sense from $s(\omega)$.

The mean square value may also be defined for stochastic processes, so utilizing Eq. (3-33) the average power of a stochastic signal may also be determined. It is also true that this power is not related to a single frequency as all random processes involve several components of different frequencies. Thus the power density spectrum may also be applied for a stochastic signal, i.e.

$$M[\xi^2(t)] = \int_{-\infty}^{\infty} s(\omega)\, d\omega \tag{3-41}$$

$s(\omega)$ may also comprise Dirac pulses. However, the chief problem is not the impossibility of computing the Fourier spectrum back from this, but that $F(\omega)$ may not exist at all. [It is not finite, so the condition of Eq. (3-22) is not met. Simultaneously, the Laplace transform may exist.] For instance, consider a stationary stochastic signal of infinite time duration. The requirement of infinite time duration will make Eq. (3-24) an improper integral, and the stationary property will

not allow the condition of Eq. (3-22) to be met. This is explained by the fact that when only $\xi(t) \not\equiv 0$ then

$$\int_{-\infty}^{\infty} |\xi(t)|\, dt \to \infty \tag{3-42}$$

However, $s(\omega)$ may be defined even under these circumstances. The stationarity condition will render $M[\xi^2(t)] = \mu_2[\xi(t)]$ a finite, time-independent constant, so the right-hand side of Eq. (3-41) will also be finite. Naturally, this will not exclude the possibility that very high amplitude, but very narrow pulses (Dirac deltas), appear in the spectrum $s(\omega)$.

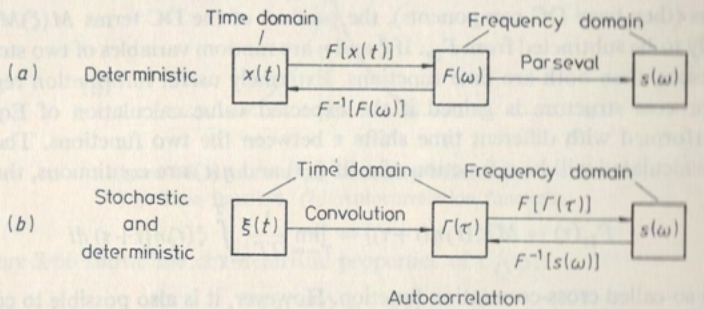

Figure 3-5 Characterizations of *(a)* deterministic, and *(b)* stochastic signals, with permissible transformations.

Figure 3-5*a* shows the time- and frequency-domain descriptions of the deterministic signals, together with the allowed transforms. The Fourier transform of a stochastic signal may not exist even if the time function $\xi(t)$ is known, and this may prevent the calculation of $s(\omega)$. This necessitates the introduction of a new type of description, the autocorrelation function (see Fig. 3-5*b*) which may equally be defined both for stochastic and for deterministic signals.

3-3 CORRELATION, AUTOCORRELATION

In our previous calculations, the independence of the random variables has been generally assumed. However, this may not always be the case, so the measure of dependence should somehow be defined. If the two random variables in question are ξ and η, then their relation may be expressed as

$$C_{\xi\eta} = M\{[\xi - M(\xi)][\eta - M(\eta)]\} \tag{3-43}$$

In Eq. (2-34) it has been utilized that for independent random variables $C_{\xi\eta}=0$. Performing the prescribed operations, we have

$$C_{\xi\eta} = M(\xi\eta)-2M(\xi)M(\eta)+M(\xi)M(\eta)$$
$$= M(\xi\eta)-M(\xi)M(\eta) \tag{3-44}$$

If $\eta=\xi$, we once more have the much-used relation [see Eq. (2-26)]

$$C_{\xi\xi} = D^2(\xi) = M(\xi^2)-M^2(\xi)$$

Let us now introduce the relation*

$$\Gamma_{\xi\eta} = M(\xi\eta) = C_{\xi\eta}+M(\xi)M(\eta) \tag{3-45}$$

and let us restrict ourselves to calculating $\Gamma_{\xi\eta}$. If ξ and η have nonzero expected values (they have DC components), the product of the DC terms $M(\xi)M(\eta)$ has simply to be subtracted from $\Gamma_{\xi\eta}$. If ξ and η are random variables of two stochastic processes, then both are time functions. Extremely useful information regarding the process structure is gained if the expected value calculation of Eq. (3-45) is performed with different time shifts τ between the two functions. The result thus calculated will be a function of τ. If $\xi(t)$ and $\eta(t)$ are continuous, then

$$\Gamma_{\xi\eta}(\tau) = M[\xi(t)\eta(t+\tau)] = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\xi(t)\eta(t+\tau)\,dt \tag{3-46}$$

is the so-called cross-correlation function. However, it is also possible to compare the value at time instant t of a single stochastic process with the value measured earlier by a time interval τ, and to take the average for the whole time scale. We thus arrive to the autocorrelation function

$$\Gamma_{\xi}(\tau) = M[\xi(t)\xi(t+\tau)] = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\xi(t)\xi(t+\tau)\,dt \tag{3-47}$$

Naturally it is necessary to choose $T\gg\tau$. It is clear that if there is no correlation between $\xi(t)$ and $\xi(t+\tau)$, not even for infinitely small τ, the sequentially following instantaneous values are totally independent then $\Gamma_{\xi}(\tau)\equiv 0$. On the other hand, the longer lasting the effect of the instantaneous values the slower $\Gamma_{\xi}(\tau)$ tends to zero for $t\to\infty$. The autocorrelation function, being of greater significance, will be treated in more detail.

If the stochastic process is stationary, $\Gamma_{\xi}(\tau)$ is independent of time, so the integration interval between $-T$ and $+T$ may be chosen at any point. Equation (3-47) is valid for stationary ergodic processes. For nonergodic processes, the reasoning of Sec. 3-1 requires the additional averaging of the autocorrelation functions pertaining to partial processes due to various sources.

* In the literature, there is no unified terminology for $\Gamma_{\xi\eta}$ and $C_{\xi\eta}$. $\Gamma_{\xi\eta}$ is usually called correlation and $C_{\xi\eta}$ is usually called covariance.

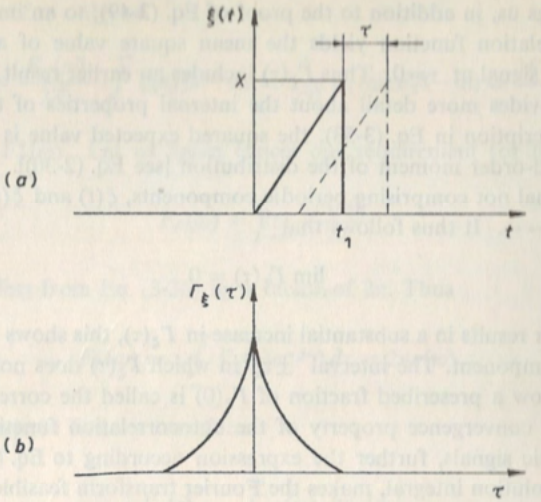

Figure 3-6 Asymmetrical stationary triangular wave.
(a) Time function. *(b)* Autocorrelation function.

Figure 3-6*b* shows the characteristic properties of $\Gamma_{\xi}(\tau)$:

$$\Gamma_{\xi}(-\tau) = \Gamma_{\xi}(\tau) \tag{3-48}$$

$$|\Gamma_{\xi}(\tau)| \leqq \Gamma_{\xi}(0) \tag{3-49}$$

Equation (3-48) can simply be proved. Let us write $\vartheta\to\tau-t$ instead of t in Eq. (3-47), then $d\vartheta=dt$ and

$$\Gamma_{\xi}(-\tau) = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\xi(\vartheta-\tau)\xi(\vartheta)\,d\vartheta = \Gamma_{\xi}(\tau) \tag{3-50}$$

as for a stationary process, Γ_{ξ} is dependent only on the time shift τ and not on t or ϑ. Γ_{ξ} is always an even function independent of the shape of $\xi(t)$. The second relationship, Eq. (3-49), may be proved as follows. Let us calculate the expected value

$$M\{[\xi(t)\pm\xi(t+\tau)]^2\} = M\{[\xi^2(t)\pm 2\xi(t)\xi(t+\tau)+\xi^2(t+\tau)]\} \geqq 0 \tag{3-51}$$

This can only be zero or positive, ξ being real. However, for a stationary process $M[\xi^2(t)]=M[\xi^2(t+\tau)]$ so

$$M[\xi^2(t)] \geqq M|\xi(t)\xi(t+\tau)|. \tag{3-52}$$

It can be seen that the right-hand side equals $|\Gamma_{\xi}(\tau)|$ according to Eq. (3-47), and the left side is

$$\Gamma_{\xi}(0) = M[\xi^2(t)] = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\xi^2(t)\,dt = \int_{-\infty}^{\infty}x^2 f(x)\,dx \tag{3-53}$$

This brings us, in addition to the proof of Eq. (3-49), to an important result: the autocorrelation function yields the mean square value of a stochastic or deterministic signal at $\tau=0$. Thus $\Gamma_\xi(\tau)$ includes an earlier result as a particular case and provides more detail about the internal properties of the signal. For complete description in Eq. (3-53), the squared expected value is also expressed by the second-order moment of the distribution [see Eq. (2-30)].

For a signal not comprising periodic components, $\xi(t)$ and $\xi(t+\tau)$ are independent if $\tau\to\infty$. It thus follows that

$$\lim_{\tau\to\infty}\Gamma_\xi(\tau)=0 \tag{3-54}$$

If increasing τ results in a substantial increase in $\Gamma_\xi(\tau)$, this shows the presence of a periodic component. The interval $\pm\tau_k$ in which $\Gamma_\xi(\tau)$ does not decrease permanently below a prescribed fraction of $\Gamma_\xi(0)$ is called the correlation interval.

This zero convergence property of the autocorrelation function in the case of nonperiodic signals, further the expression according to Eq. (3-47), which is really a convolution integral, makes the Fourier transform feasible. As a starting point let us reconsider a stochastic or deterministic signal, limited in time, again defined by Eq. (3-23). Then, requiring that $\tau\ll T$, we have

$$\Gamma_\xi(\tau)=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\xi(t)\xi(t+\tau)\,dt=\frac{1}{2T}\int_{-\infty}^{\infty}\xi_T(t)\xi_T(t+\tau)\,dt \tag{3-55}$$

and the Fourier transform of $\Gamma_\xi(\tau)$ is given by

$$F_\Gamma(\omega)=\int_{-\infty}^{\infty}e^{-j\omega\tau}\left[\frac{1}{2T}\int_{-\infty}^{\infty}\xi_T(t)\xi_T(t+\tau)\,dt\right]d\tau \tag{3-56}$$

Let us extend this expression by $e^{j\omega t}$, and let us include $e^{-j\omega\tau}$ in the inside integral. (It is allowed as this is a constant factor for the inside integral.) Thus

$$F_\Gamma(\omega)=\int_{-\infty}^{\infty}\frac{1}{2T}\int_{-\infty}^{\infty}\xi_T(t)e^{j\omega t}\xi_T(t+\tau)e^{-j\omega(t+\tau)}\,dt\,d\tau \tag{3-57}$$

Again t and $t+\tau$ are symmetrically placed variables, so the sequence of integrations may be exchanged. Thus

$$F_\Gamma(\omega)=\int_{-\infty}^{\infty}\frac{1}{2T}\left[\int_{-\infty}^{\infty}\xi_T(t+\tau)e^{-j\omega(t+\tau)}\,d\tau\right]\xi_T(t)e^{j\omega t}\,dt \tag{3-58}$$

By introducing a new variable $t+\tau=\vartheta$ into the inside integral $d\tau=d\vartheta$, thus the expression

$$\int_{-\infty}^{\infty}\xi_T(\vartheta)e^{-j\omega\vartheta}\,d\vartheta=F_T(\omega) \tag{3-59}$$

is the function of ω only, so

$$F_\Gamma(\omega)=\frac{F_T(\omega)}{2T}\int_{-\infty}^{\infty}\xi_T(t)e^{j\omega t}\,dt=\frac{1}{2T}F_T(\omega)F_T(-\omega)=\frac{|F_T(\omega)|^2}{2T} \tag{3-60}$$

as $F_T(-\omega)=F_T^*(\omega)$. Let us again cancel our requirement for finite T. Thus

$$F_\Gamma(\omega)=\lim_{T\to\infty}\frac{|F(\omega)|^2}{2T} \tag{3-61}$$

which only differs from Eq. (3-36) by a factor of 2π. Thus

$$F_\Gamma(\omega)=\int_{-\infty}^{\infty}\Gamma_\xi(\tau)e^{-j\omega\tau}\,d\tau=2\pi s(\omega) \tag{3-62}$$

or, using the inverse Fourier transform

$$\boxed{\Gamma_\xi(\tau)=\int_{-\infty}^{\infty}s(\omega)e^{j\omega\tau}\,d\omega} \tag{3-63}$$

It should be recalled that for a nonperiodic stochastic time function, ω is a continuous variable. The above pair of equations, giving correspondence between the autocorrelation function and the power density, has been introduced by Wiener[15] and Khintchine.[8] In the literature, several forms of these equations are used and below the most frequently used expressions are summarized.

$\Gamma_\xi(\tau)$ is a real, even function. Thus

$$\boxed{\begin{aligned}F_\Gamma(\omega)&=\int_{-\infty}^{\infty}\Gamma_\xi(\tau)\cos\omega\tau\,d\tau=2\int_{0}^{\infty}\Gamma_\xi(\tau)\cos\omega\tau\,d\tau\\&=2\pi s(\omega)\quad\text{if}\quad-\infty<\omega<\infty\end{aligned}} \tag{3-64}$$

$$F_\Gamma(\omega)=\begin{cases}\pi S(\omega) & \text{if}\quad 0<\omega<\infty & (3\text{-}65)\\ s(f) & \text{if}\quad -\infty<f<\infty & (3\text{-}66)\\ \frac{1}{2}S(f) & \text{if}\quad 0<f<\infty & (3\text{-}67)\end{cases}$$

Knowing one of the power spectrum expressions, it follows from Eq. (3-63) that

$$\Gamma_\xi(\tau)=\int_{-\infty}^{\infty}s(\omega)\cos\omega\tau\,d\omega=\int_{0}^{\infty}S(\omega)\cos\omega\tau\,d\omega \tag{3-68}$$

$$\Gamma_\xi(\tau)=\int_{-\infty}^{\infty}s(f)\cos 2\pi f\tau\,df=\int_{0}^{\infty}S(f)\cos 2\pi f\tau\,df \tag{3-69}$$

Substituting $\tau=0$ into Eq. (3-68), we have

$$\Gamma_\xi(0) = \int_{-\infty}^{\infty} s(\omega)\, d\omega = M[\xi^2(t)] \tag{3-70}$$

which is in accordance with Eqs. (3-41) and (3-53).

After the introduction of the autocorrelation function concept, we may now define the gaussian process. Generally, a stochastic process is termed a gaussian process if an ensemble density function follows a multi-dimensional gaussian distribution at any fixed time instant. Most of the stochastic functions in electronics are gaussian, and they have the important property that the distribution type is unchanged after a linear transformation.

In the following, let us assume only two random variables: $\xi_1=\xi_1(t)$ and $\xi_2=\xi_2(t+\tau)$. Their values may be x_1 and x_2. Let the expected value of x_1 and x_2 be zero, their variance σ_x^2, and let them be within the two-variable, normal distribution $f(x_1, x_2)$ defined by Eqs. (2-22) and (2-23). Then

$$\sigma_x^2 = M[\xi^2(t)] = M[\xi^2(t+\tau)] = \int_{-\infty}^{\infty} x^2 f(x)\, dx \tag{3-71}$$

$$\Gamma_\xi(\tau) = M[\xi(t)\xi(t+\tau)] = \iint_{-\infty}^{\infty} x_1 x_2 f(x_1, x_2)\, dx_1\, dx_2 \tag{3-72}$$

The process may be described as gaussian if

$$f(x_1, x_2) = \frac{1}{2\pi\sigma_x^2\sqrt{1-\varrho}}\, e^{(x_1^2-2\varrho x_1x_2+x_2^2)/2\sigma_x^2(1-\varrho^2)} \tag{3-73}$$

where

$$\varrho=\varrho_x(\tau) = \frac{\Gamma_\xi(\tau)}{\sigma_x^2} \tag{3-74}$$

This means that the common density function of the instantaneous values always has a normal distribution independent of τ; its parameter ϱ depends on τ.

3-4 TYPICAL AUTOCORRELATION FUNCTIONS AND THEIR POWER SPECTRA

This Section is primarily intended to present illustrative examples in order to appreciate the applications of the autocorrelation function. Also it is proposed to present a few typical properties of electronic noise sources.

In our first problem, let us calculate the autocorrelation function and power spectrum of the most simple deterministic signal

$$u(t) = U = \text{const.} \qquad -\infty < t < \infty \tag{3-75}$$

The autocorrelation function is

$$\Gamma(\tau) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} u(t)u(t+\tau)\, dt = U^2 \tag{3-76}$$

The result is thus the mean square value, as in this case there is a unit correspondence between instantaneous values separated by arbitrary time intervals. The power spectrum is calculated from Eq. (3-64). Thus

$$s(\omega) = \frac{1}{\pi}\int_0^{\infty} U^2 \cos\omega\tau\, d\tau = \frac{U^2}{\pi}\pi\delta(\omega) \tag{3-77}$$

where $\delta(\omega)$ is the Dirac pulse having the properties

$$\delta(\omega) = \begin{cases} 0 & \text{if} \quad \omega \neq 0 \\ \infty & \text{if} \quad \omega = 0 \end{cases}$$

$$\int_{-\varepsilon}^{\varepsilon} \delta(\omega)\, d\omega = 1 \quad \text{for any} \quad \varepsilon \quad (\varepsilon > 0); \qquad \delta(-\omega) = \delta(\omega) \tag{3-78}$$

As a control, let us transform $s(\omega)$ back. Thus

$$\Gamma(\tau) = U^2 \int_{-\infty}^{\infty} \delta(\omega)\cos\omega\tau\, d\omega = \lim_{\varepsilon\to 0} U^2 \int_{-\varepsilon}^{\varepsilon} \frac{\cos\omega\tau}{2\varepsilon}\, d\omega$$

$$= \lim_{\varepsilon\to 0}\left(U^2 \frac{\sin\omega\tau}{2\varepsilon\tau}\right)_{-\varepsilon}^{\varepsilon} = \lim_{\varepsilon\to 0}\left(U^2 \frac{\sin\varepsilon\tau}{\varepsilon\tau}\right) = U^2 \tag{3-79}$$

This means that a constant autocorrelation function has a power spectrum of the form $s(\omega)=\text{const.}\,\delta(\omega)$.

In the next problem, the signal

$$u(t) = U\sin(\omega_0 t+\varphi) \tag{3-80}$$

will be investigated. Thus

$$\Gamma(\tau) = \lim_{T\to\infty} \frac{U^2}{2T} \int_{-T}^{T} \sin(\omega_0 t+\varphi)\sin(\omega_0 t+\omega_0\tau+\varphi)\, dt \tag{3-81}$$

After elementary trigonometric transformations, we have

$$\Gamma(\tau) = \frac{U^2}{2}\cos\omega_0\tau \tag{3-82}$$

independent of the starting phase φ. This means that Eq. (3-82) is valid for both sine and cosine signals, and meets the conditions of Eqs. (3-48) and (3-49). How-

ever, the phase information is lost. The power spectrum, as in Eq. (3-77), is

$$s(\omega) = \frac{U^2}{2\pi}\int_0^\infty \cos\omega_0\tau\cos\omega\tau\,d\tau$$

$$= \frac{U^2}{2\pi}\left[\int_0^\infty \frac{\cos(\omega-\omega_0)\tau}{2}\,d\tau + \int_0^\infty \frac{\cos(\omega+\omega_0)\tau}{2}\,d\tau\right]$$

$$= \frac{U^2}{2}\left[\frac{\delta(\omega-\omega_0)}{2} + \frac{\delta(\omega+\omega_0)}{2}\right] \tag{3-83}$$

This is made up of two half-area Dirac pulses[2,8] placed symmetrical to the origin, disregarding the coefficient $U^2/2$. The single-sided power spectrum is

$$S(\omega) = \frac{U^2}{2}\delta(\omega-\omega_0) \quad \text{if} \quad 0 < \omega < \infty \tag{3-84}$$

clearly illustrating that the power of a single-frequency deterministic signal is concentrated in a single, well-defined spectrum line.

In the previous two cases, the autocorrelation function and then the power spectrum may easily be determined using known time functions. However, the situation may be the reverse for stochastic signals. In this case $\Gamma(\tau)$ has to be calculated from $S(\omega)$, but the original time function may not be reconstructed because of the lack of phase information.

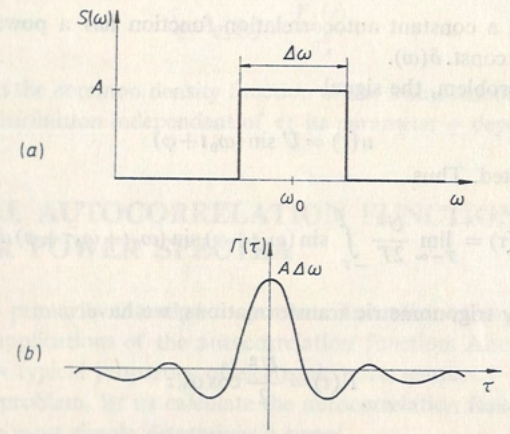

Figure 3-7 Band-limited white noise. *(a)* Power spectrum. *(b)* Autocorrelation function.

Let us now investigate a band-limited white noise having a single-sided power spectrum shown in Fig. 3-7*a* and given by

$$S(\omega) = \begin{cases} A & \text{if} \quad \omega_0 - \dfrac{\Delta\omega}{2} < \omega < \omega_0 + \dfrac{\Delta\omega}{2} \\ 0 & \text{elsewhere} \end{cases} \tag{3-85}$$

The autocorrelation function is calculated from Eq. (3-68). Thus

$$\Gamma_\xi(\tau) = \int_{\omega_0-\Delta\omega/2}^{\omega_0+\Delta\omega/2} A\cos\omega\tau\,d\omega = A\Delta\omega\cos\omega_0\tau\,\frac{\sin\tau\Delta\omega/2}{\tau\Delta\omega/2} \tag{3-86}$$

(see Eig. 3-7*b*). Let us investigate two limiting cases.

1. Let

$$\tau\Delta\omega \ll 1 \tag{3-87}$$

resulting in

$$\Gamma_\xi(\tau) \cong A\Delta\omega\cos\omega_0\tau \tag{3-88}$$

which closely resembles Eq. (3-82). This suggests that for low bandwidths and short correlation times, even a stochastic signal may be approximated by a sine (cosine) signal of ω_0 angular frequency. In Sec. 2-8 the Rayleigh distribution has been considered and it has been shown that a narrow-band noise may be regarded as a carrier modulated stochastically in both amplitude and phase. It is seen from Eq. (3-86) that the carrier amplitude can be regarded as constant as long as $\Delta\omega$ is small. Figure 3-8 shows that the constant amplitude carrier is well approximated even at $\Delta\omega/2\omega_0 = 0.1$. In network theory, it is a general rule that the smaller the bandwidth the slower the random phase changes, i.e. the stochastic signal becomes more and more deterministic. In Chapter 5, the noise parameters

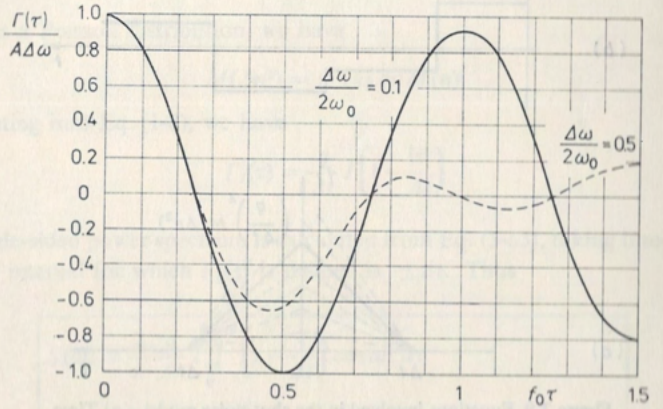

Figure 3-8 Autocorrelation function of narrow-band white noise.[9] *(Courtesy of Verlag Technik, Berlin.)*

of the linear networks are calculated using this simplifying condition, thus making use of proven methods in classical circuit theory.

2. Let

$$\frac{\Delta\omega}{2} = \omega_0 \tag{3-89}$$

Then, we have

$$\Gamma_\xi(\tau) = \int_0^{\Delta\omega} A\cos\omega\tau\, d\omega = A\Delta\omega\frac{\sin\tau\Delta\omega}{\tau\Delta\omega} \tag{3-90}$$

The higher the fixed value of $\Delta\omega$ the faster $|\Gamma_\xi(\tau)|$ will tend to zero. For

$$\lim_{\Delta\omega\to\infty}\Gamma_\xi(\tau) = A\pi\delta(\tau). \tag{3-91}$$

Thus the autocorrelation time interval of an ideal white noise having infinite bandwidth is zero. This means that there is absolutely no correlation between instantaneous values separated by an infinitely small time (see Figs. 3-1*a* and *c*).

Now we can come back to the calculation of the shot-noise power spectrum not detailed in Chapter 1. To this end the model shown in Fig. 1-2 is further

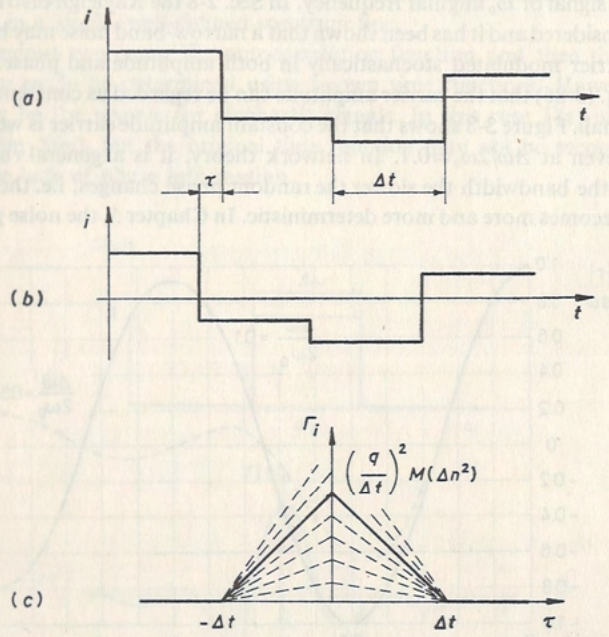

Figure 3-9 Functions involved in the shot-noise model. *(a)* Time function of the current fluctuation $i(t)$. *(b)* Function $i(t-\tau)$. *(c)* Autocorrelation function.

developed. Figure 3-9*a* illustrates the current fluctuation $i(t)$, i.e. the deviation from the mean value, based on the relation

$$i(t) = \frac{q}{\Delta t}[n(t)-\bar{n}] = \frac{q}{\Delta t}\Delta n(t) \tag{3-92}$$

which is a combination of Eqs. (1-7) and (1-8). This has the autocorrelation function

$$\Gamma_i(\tau) = M[i(t)i(t+\tau)] = \lim_{T\to\infty}\frac{1}{2T}\left(\frac{q}{\Delta t}\right)^2\int_{-T}^{T}\Delta n(t)\Delta n(t+\tau)\,dt \tag{3-93}$$

according to Eq. (3-47). In order to calculate this function, $i(t)$ has to be shifted either to the left or to the right (see Fig. 3-9*b*), and the product of the instantaneous values has to be averaged for a long time. As the values

$$\Delta n(t),\ \Delta n(t+\Delta t),\ \Delta n(t+2\Delta t),\ \ldots$$

are independent of each other, the autocorrelation interval may not be longer than $\pm\Delta t$. According to Fig. 3-9*c*, the time function made up of rectangular current pulses has a convolution resulting in an isosceles triangle. The height of this triangle is different for each pulse, but the average height for a long average time is calculated from Eq. (3-93), by substituting $\tau=0$, to be

$$(q/\Delta t)^2 M(\Delta n^2),$$

i.e.

$$\Gamma_i(\tau) = \begin{cases}\left(\dfrac{q}{\Delta t}\right)^2 M(\Delta n^2)\left(1-\dfrac{|\tau|}{\Delta t}\right) & \text{if} \quad |\tau| \leqq \Delta t \\ 0 & \text{elsewhere}\end{cases} \tag{3-94}$$

As n has a Poisson distribution, we have

$$M(\Delta n^2) = D^2(n) = M(n) \tag{3-95}$$

Substituting into Eq. (1-8), we have

$$\Gamma_i(\tau) = \frac{q}{\Delta t}I\left(1-\frac{|\tau|}{\Delta t}\right) \tag{3-96}$$

The single-sided power spectrum is calculated from Eq. (3-65), taking into account that the interval for which $\Gamma_i(\tau)$ is defined is $\pm\Delta t$. Thus

$$\boxed{\begin{aligned} S(\omega) &= \frac{2}{\pi}\frac{q}{\Delta t}I\int_0^{\Delta t}\left(1-\frac{\tau}{\Delta t}\right)\cos\omega\tau\,d\tau \\ &= \frac{2}{\pi}\frac{q}{\Delta t}I\frac{1}{\omega^2\Delta t}(1-\cos\omega\Delta t) = \frac{1}{\pi}qI\frac{\sin^2(\omega\Delta t/2)}{(\omega\Delta t/2)^2}\end{aligned}} \tag{3-97}$$

For $\omega\Delta t \ll 2\pi$ the ratio is approximately unity, so

$$S(\omega) = \frac{1}{\pi} qI \tag{3-98}$$

and from Eq. (3-40)

$$S(f) = 2\pi S(\omega) = 2qI \tag{3-99}$$

The mean square value of the fluctuation in a frequency range Δf, according to Eq. (3-41), is

$$M[i^2(t)]_{\Delta f} = \int_f^{f+\Delta f} S(f)\,df = 2qI\Delta f$$

which is the well-known formula for shot noise [see Eq. (1-11)].

The above calculation shows also the limitations of the simple relationship of Eq. (1-11): the intervals Δt may not be decreased arbitrarily. This is explained by the fact that if the charge carriers moving within the electronic device need a finite time interval to overcome a hindrance then the events within this time interval may not be regarded as independent from each other. In many cases, the investigated phenomenon may be modeled not using rectangular pulses as shown in Fig. 3-9 but using pulses of another shape. In these cases, the multiplying factor of $S(\omega)$ also differs from $(\sin x/x)^2$.

In the above example, the investigated time interval has been divided into arbitrary parts, and the pulses have been obtained by this arbitrary subdivision. However, there are types of electronic noise that originally have pulse waveforms (burst noise and some processes connected with voltage breakdown). The time function of these processes is illustrated by the so-called random telegraph signal shown in Fig. 3-10.

For simplicity, let us investigate a symmetrical random telegraph signal, the values $+U$ and $-U$ having equal probabilities of 0.5. Let the changeovers follow a Poisson distribution; this means that if there are ν changeovers during unit time, then the probability of having $0, 1, 2, \ldots, n$ changeovers during a time interval τ will be

$$P(\eta = n) = \frac{(\nu\tau)^n}{n!} e^{-\nu\tau} \tag{3-100}$$

It should be noted that in this case the time intervals between changeovers follow an exponential distribution with a density function according to Eq. (2-144).

As the process is assumed to be stationary, the product $\xi(t)\xi(t+\tau)$ may only be $+U^2$ or $-U^2$, depending on whether the number of changeovers during a time-interval τ is $0, 2, 4, \ldots$ or $1, 3, 5, \ldots$. Thus after averaging we have

$$\begin{aligned} M[\xi(t)\xi(t+\tau)] = {} & U^2 \times (\text{the probability of even changeovers}) - \\ & - U^2 \times (\text{the probability of odd changeovers}) \end{aligned} \tag{3-101}$$

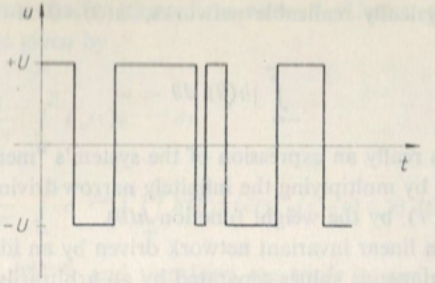

Figure 3-10 Random telegraph signal.

Combining Eqs. (3-100) and (3-101) gives

$$M[\xi(t)\xi(t+\tau)] = U^2 e^{-\nu\tau}\left[1 + \frac{(\nu\tau)^2}{2!} + \frac{(\nu\tau)^4}{4!} + \ldots\right] - U^2 e^{-\nu\tau}\left[\frac{\nu\tau}{1!} + \frac{(\nu\tau)^3}{3!} + \ldots\right] \tag{3-102}$$

Rearrangement of this expression yields

$$\Gamma_\xi(\tau) = U^2 e^{-2\nu\tau} \tag{3-103}$$

Note that $\Gamma_\xi(\tau)$ is always even, therefore the correct end result is

$$\Gamma_\xi(\tau) = U^2 e^{-2\nu|\tau|} \tag{3-104}$$

According to Appendix B, the Fourier transform of this expression is

$$s(\omega) = \frac{4U^2\nu}{4\nu^2 + \omega^2} \tag{3-105}$$

This is called a Lorentz spectrum.

3-5 RESPONSE OF A LINEAR NETWORK DRIVEN BY A STOCHASTIC SIGNAL[1,2,3,7,9,10,12]

For small signals, electronic devices behave as linear networks. Several noise sources may be present in a single device such as a transistor, and between the noise sources and output terminals there are usually resistive and reactive network elements of the equivalent circuit. This network has an effect on the output autocorrelation function and power spectrum.

A linear time-invariant network may be characterized by its weight function $h(\vartheta)$. If the input driving signal is $x(t)$, the output response is

$$y(t) = \int_{-\infty}^{\infty} h(\vartheta)x(t-\vartheta)\,d\vartheta \tag{3-106}$$

In the case of physically realizable networks, $h(\vartheta)=0$ if $\vartheta<0$. The stability criterion is

$$\int_{-\infty}^{\infty} |h(\vartheta)|\, d\vartheta < \infty \tag{3-107}$$

Equation (3-106) is really an expression of the system's "memory": the response function is formed by multiplying the infinitely narrow driving elements falling in the interval $(t-\vartheta, t)$ by the weight function $h(\vartheta)$.

Let us assume a linear invariant network driven by an ideal white noise with independent instantaneous values separated by an arbitrarily short time interval $[\Gamma_\xi(\tau)=0$ if $|\tau|>0]$. However, the response function will include not only the driving signal value pertaining to the given instant t but also earlier signal values, although with decreased weight, unless the weight function $h(\vartheta)$ shows a very fast convergence to zero with increasing ϑ. This means that the autocorrelation functions of the driving and response waveforms will be different.

Let us assume an input driving signal $\xi(t)$ which is stationary and ergodic with an autocorrelation function $\Gamma_\xi(\tau)$, and let us determine the autocorrelation function $\Gamma_\eta(\tau)$ of the response $\eta(t)$. The product $\eta(t)\eta(t+\tau)$ is first calculated. Thus

$$\eta(t)\eta(t+\tau) = \int_{-\infty}^{\infty} h(\vartheta)\xi(t-\vartheta)\, d\vartheta \int_{-\infty}^{\infty} h(\Theta)\xi(t+\tau-\Theta)\, d\Theta$$

$$= \iint_{-\infty}^{\infty} h(\vartheta)h(\Theta)\xi(t-\vartheta)\xi(t+\tau-\Theta)\, d\Theta\, d\vartheta \tag{3-108}$$

Here two kinds of integration variables ϑ and Θ have been introduced for formal reasons only. According to Eq. (3-2)

$$\Gamma_\eta(\tau) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \eta(t)\eta(t+\tau)\, dt \tag{3-109}$$

Thus Eq. (3-108) should be averaged in time. As t, ϑ and Θ are independent of each other, the sequence of integrations may again be changed. Let us first calculate Eq. (3-109). Thus

$$\Gamma_\eta(\tau) = \lim_{T\to\infty} \iint_{-\infty}^{\infty} h(\vartheta)h(\Theta)\left[\frac{1}{2T}\int_{-T}^{T} \xi(t-\vartheta)\xi(t+\tau-\Theta)\, dt\right] d\Theta\, d\vartheta$$

$$= \iint_{-\infty}^{\infty} h(\vartheta)h(\Theta)\Gamma_\xi(\tau-\Theta+\vartheta)\, d\Theta\, d\vartheta \tag{3-110}$$

The application of this expression is rather awkward. However, the following relationships are known:

$$h(t) \xleftrightarrow{\text{Fourier transform}} H(\omega)$$

$$\Gamma(\tau) \longleftrightarrow s(\omega) \tag{3-111}$$

where $H(\omega)$ is the transfer function of the network. Utilizing these, the Fourier transform of $\Gamma_\eta(\tau)$ is given by

$$s_\eta(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Gamma_\eta(\tau)e^{-j\omega t}\, d\tau$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-j\omega\tau}\left[\iint_{-\infty}^{\infty} h(\vartheta)h(\Theta)\Gamma_\xi(\tau-\Theta+\vartheta)\, d\Theta\, d\vartheta\right] d\tau \tag{3-112}$$

Let us multiply by $e^{j\omega(\vartheta-\Theta)}$ and transform the triple integral into a product of three integrals (the independence of the three variables allows this). Thus

$$s_\eta(\omega) = \int_{-\infty}^{\infty} h(\vartheta)e^{j\omega\vartheta}\, d\vartheta \int_{-\infty}^{\infty} h(\Theta)e^{-j\omega\Theta}\, d\Theta \frac{1}{2\pi}\int_{-\infty}^{\infty} \Gamma_\xi(\tau-\Theta-\vartheta)e^{-j\omega(\tau-\Theta+\vartheta)}\, d\tau \tag{3-113}$$

As $h(\vartheta)$ is real, the first two Fourier transforms form a conjugate pair [similar to Eq. (3-26)], so their product is the square of the absolute value of the transfer function. The third integral being just $s_\xi(\omega)$ we have

$$\boxed{s_\eta(\omega) = |H(\omega)|^2 s_\xi(\omega)} \tag{3-114}$$

This relation, comprising only frequency domain parameters, is simple and instructive. It is normally used instead of Eq. (3-110) for the calculation of the autocorrelation functions, according to the scheme

$$\begin{array}{ccccc} \Gamma_\xi(\tau) & \xrightarrow{\mathscr{F}} & s_\xi(\omega)|H(\omega)|^2 & \xrightarrow{\mathscr{F}^{-1}} & \Gamma_\eta(\tau) \\ & & \mathscr{F}\uparrow & & \\ & & h(\vartheta) & & \end{array} \tag{3-115}$$

The transformation $h(\vartheta)\to H(\omega)$ is frequently not performed separately as the transfer function of the network is normally known.

As an example, let us calculate the autocorrelation function of the output signal for a low-pass filter with a transfer function

$$H(\omega) = \frac{1}{1+j(\omega/\omega_a)} \tag{3-116}$$

when the input is driven by white noise. Let the single-sided power spectrum of the white noise again be

$$S_\xi(\omega) = A \tag{3-117}$$

At the filter output we have

$$S_\eta(\omega) = \frac{A}{1+(\omega/\omega_a)^2} \tag{3-118}$$

from which, according to Eq. (3-68) (see Appendix B), it follows that

$$\Gamma_\eta(\tau) = \int_0^\infty \frac{A\cos\omega\tau}{1+(\omega/\omega_a)^2}\,d\omega = A\omega_a\frac{\pi}{2}e^{-|\omega_a\tau|} \tag{3-119}$$

The power spectra and the autocorrelation functions are shown in Fig. 3-11.

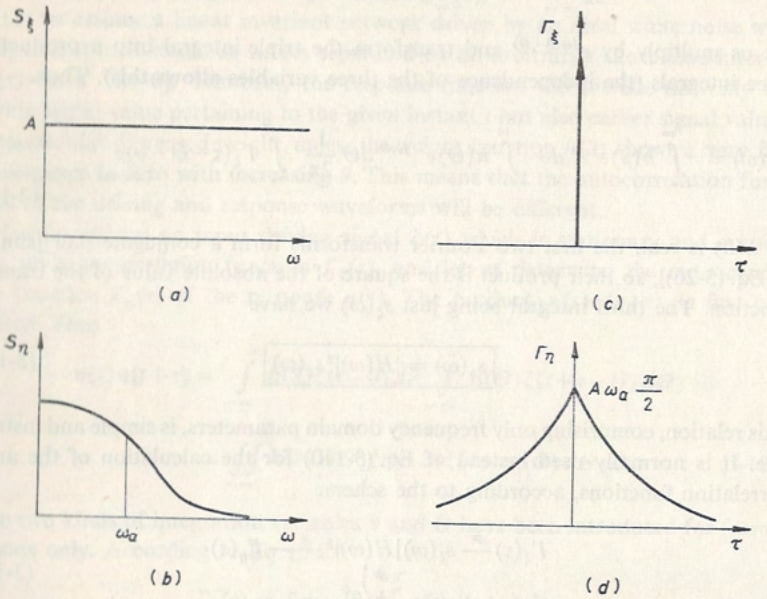

Figure 3-11 White-noise characteristics. *(a)* Power spectrum, without band-limiting. *(b)* Power spectrum, with band-limiting. *(c)* Autocorrelation function, without band-limiting. *(d)* Autocorrelation function, with band-limiting.

In the case of an input signal with normal distribution, the output signal will also have normal distribution. A band-limiting network driven by an input with anormal distribution will provide a response that has a distribution closer to normal because, according to Eq. (3-106), $y(t)$ is a function of not a single instantaneous value but of an infinite number of instantaneous values weighted by $h(\vartheta)$. Regarding this series of instantaneous values as members of a sum distribution, a normal distribution should be attained as a consequence of the central limit theorem. The actual situation is much more complicated: the instantaneous values of the input signal (roughly inside the autocorrelation interval τ_k) are not independent from each other, so the central limit theorem may not be

applied. As the problem at present has no general solution,[13,14] the only qualitative statement possible is that the higher the time constant of the band-limiting network compared with the autocorrelation interval of the input signal the better the output signal distribution will approximate the normal distribution.

PROBLEM

3-1 Calculate the autocorrelation function and power spectrum of the triangnlar pulse shown in Fig. 3-6.

Answer:

$$\xi(t) = X\frac{t}{t_1} \quad \text{if} \quad 0 < t < t_1$$

$$\xi(t-\tau) = X\frac{t-\tau}{t_1} \quad \text{if} \quad \tau < t < t_1+\tau$$

The autocorrelation function (see Fig. 3-6*b*) is

$$\Gamma_\xi(\tau) = \Gamma_\xi(-\tau) = M\left[X\frac{t}{t_1}X\frac{t-\tau}{t_1}\right] = \frac{X^2}{3}\left[1-\frac{3}{2}\frac{|\tau|}{t_1}+\frac{1}{2}\frac{|\tau|^3}{t_1^3}\right] \qquad 0 < |\tau| < t_1$$

The power spectrum, according to Eq. (3-64), is

$$S(\omega) = \frac{1}{\pi}\frac{X^2}{3}\int_0^{t_1}\left[1-\frac{3}{2}\frac{\tau}{t_1}+\frac{1}{2}\left(\frac{\tau}{t_1}\right)^3\right]\cos\omega\tau\,d\tau$$

Partial integration and trigonometrical rearrangements provide the result

$$S(\omega) = \frac{t_1}{2\pi}\frac{X^2}{(\omega t_1)^4}[(\omega t_1)^2+2(1-\cos\omega t_1-\omega t_1\sin\omega t_1)]$$

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CHAPTER

FOUR

THE PHYSICAL ASPECTS OF NOISE

The utilization of the concepts in probability theory described in previous Chapters results in models that provide a good approximation of most of the noise phenomena in electronic devices. However, these models have been constructed by neglecting several parameters, and while this is permitted for a general description a more rigorous treatment is necessary for the analysis of physical aspects. The relation $\overline{i^2}=2qI\Delta f$ for shot noise given in Chapter 1 [see Eq. (1-11)] has been further developed in Chapter 3. From Eq. (3-97) we have

$$\overline{i^2} = 2qI\frac{\sin^2(\omega\Delta t/2)}{(\omega\Delta t/2)^2}\Delta f \tag{4-1}$$

The multiplying factor $(\sin x/x)^2 \leqq 1$ means a finite upper-frequency limit that is evidently connected with the geometrical layout and physical operation of the noise source. However, expression (4-1) is only valid for the current fluctuation model shown in Fig. 1-2. In an actual device, the stochastic processes are made up of elementary pulses having different shapes.

4-1 SHOT NOISE

Shot noise is generated when the charge carriers supplying the current pass through an energy gap. The energy gaps within electronic devices are always brought about by some heterogeneous structure: in a vacuum tube by the cathode metal and the vacuum, while in a solid state device by two layers with different conduction types or two different solid materials.[118] On the other hand, the dimensions of the vacuum space or of the solid state device depletion layers cannot be arbitrarily decreased, both for technological and theoretical reasons. The transit process

of the charge carriers is concluded by the passage through the energy gap and the finite space involving this gap, thus requiring a finite time interval. Therefore our previous models assuming Dirac pulses are merely approximate ones.

Figure 4-1 shows the cross section of a vacuum diode with plane electrodes. The electron with a charge q has zero initial velocity at the cathode and travels to the anode with constant acceleration. The transit time (time of flight) is

$$t_f = \frac{2d}{v_{\max}} \tag{4-2}$$

where $v_{\max}$ is the maximum velocity at the anode. If, in the meantime, no further electron is emitted from the cathode, the external circuit has also to pass a charge q during time interval t_f. However, the time dependence of the current flowing in the external circuit is not yet known.

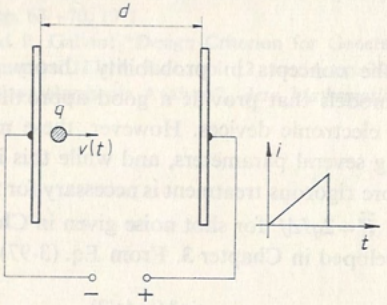


Figure 4-1 Movement of an electron in a vacuum between planar electrodes.

When moving along a path length dx, the additional energy gained by the electron is

$$dW = F\,dx = qE\,dx = q\frac{U_{ac}}{d}dx \tag{4-3}$$

This additional energy is supplied by the power supply:

$$dW = U_{ac}\,i(t)\,dt \tag{4-4}$$

where $i(t)$ is the instantaneous value of the current flowing in the external circuit. Using Eqs. (4-3) and (4-4), we have

$$i(t) = \frac{q}{d}\frac{dx}{dt} = \frac{q}{d}v(t) \tag{4-5}$$

In the case of constant acceleration, $v(t)$ increases linearly up to $v_{\max}$:

$$i(t) = \frac{q}{d}v_{\max}\left(\frac{t}{t_f}\right) = \frac{2q}{t_f}\left(\frac{t}{t_f}\right) \tag{4-6}$$

where Eq. (4-2) has been utilized to eliminate d. Let the number of electrons emitted simultaneously be n. This number is a random variable with Poisson distribution having an expected value of $M(n)$. The mean value of the current is

$$I = M[ni(t)] = M(n)\frac{2q}{t_f^2}\int_0^{t_f}\left(\frac{t}{t_f}\right)dt = M(n)\frac{q}{t_f} \tag{4-7}$$

which is directly self-evident. The autocorrelation function of the current pulses $ni(t)$ is

$$\Gamma_i = M[ni(t)\,ni(t+\tau)] \tag{4-8}$$

As $M[ni(t)]\neq 0$, let us convert expression (4-8) according to Eq. (3-45). Thus

$$\Gamma_i = M[\Delta ni(t)\,\Delta ni(t+\tau)] + I^2 = \Gamma_i + I^2 \tag{4-9}$$

where $\Delta n = n - M(n)$, and Γ_i is the autocorrelation function of the fluctuation component. In the following, the investigation of Γ_i will be sufficient:

$$\Gamma_i = M(\Delta n^2)\frac{4q^2}{t_f^2}M\left[\frac{t}{t_f}\frac{t+\tau}{t_f}\right] = \frac{4qI}{t_f}\left[1 - \frac{3}{2}\frac{|\tau|}{t_f} + \frac{1}{2}\frac{|\tau|^3}{t_f^3}\right] \tag{4-10}$$

where the relation (2-89), $D^2(n) = M(\Delta n^2) = M(n)$, which pertains to the Poisson distribution, has been used. Equation (4-7) and the example given in Problem 3-1 have also been utilized.

The power spectrum[95] is given by

$$s(\omega) = \frac{1}{2\pi}qI\frac{4}{(\omega t_f)^4}[(\omega t_f)^2 + 2(1 - \cos\omega t_f - \omega t_f \sin\omega t_f)] \tag{4-11}$$

which is difficult to evaluate in this form. Expanding the angular functions for $\omega t_f \ll 1$ and simplifying, we have

$$s(\omega) = \frac{1}{2\pi}qI\left[1 - \frac{(\omega t_f)^2}{18} + \ldots\right] \tag{4-12}$$

Expressing now the single-sided density function $S(f)$ and taking into account Eqs. (3-64) to (3-67), according to which $S(f) = 4\pi s(\omega)$,

$$\overline{i^2} = S(f)\,\Delta f = 2qI\left[1 - \frac{(\omega t_f)^2}{18} + \ldots\right]\Delta f = 2qIF_{fp}^2\,\Delta f \tag{4-13}$$

where $F_{fp} \leqq 1$ which is a factor depending on the transit angle ωt_f for a planar diode.

For other geometrical configurations and field distributions both the form of the current pulses and the power spectrum will be modified. For a cylindrical diode, we have instead of Eq. (4-3)

$$dW = F\,dx = qE(r)\,dr = \frac{qU_{ac}}{r \ln \varrho_a}\,dr \tag{4-14}$$

$$\varrho_a = \frac{r_a}{r_c}$$

thus, substituting Eq. (4-4) and $dr/dt = v(t)$,

$$i(t) = \frac{q}{\ln \varrho_a}\,\frac{v(t)}{r(t)} \tag{4-15}$$

The function $i(t)$ may be plotted utilizing the relations

$$v(r) = v_{\max}\sqrt{\frac{\ln \varrho}{\ln \varrho_a}} \qquad \varrho = \frac{r}{r_c} \tag{4-16}$$

$$t(r) = \int_{r_c}^{r} \frac{dr}{v(r)} = \frac{2r_c}{v_{\max}}\sqrt{\ln \varrho_a}\int_0^{\sqrt{\ln \varrho}} e^{y^2}\,dy \tag{4-17}$$

where

$$\ln \varrho = \ln \frac{r}{r_c} = y^2 \qquad dr = 2r_c y e^{y^2}$$

In Fig. 4-2, $i(t)/I$ is plotted for three diodes having different electrode configurations.[35,105] If $r_a/r_c \to \infty$ then $i(t)$ will approximate the Dirac pulse, so the Fourier transform of the autocorrelation function would result, according

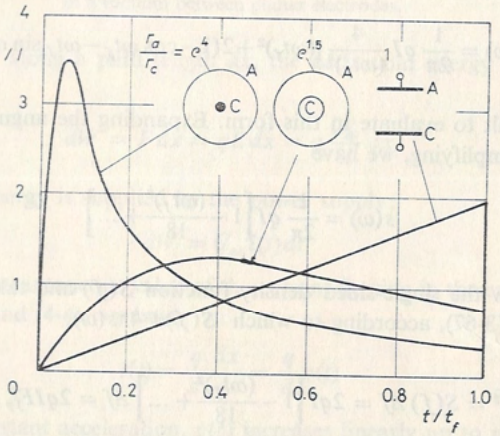


Figure 4-2 Shape of current pulse generated in the external circuit for different electrode configurations. (*Courtesy of Siemens.*)

to Eq. (3-91), in a white power spectrum. However, for practically realizable structures, taking into account the factor F_{fc}^2 depending on the transit time, we have, according to Spenke,[103]

$$F_{fc}^2 = \left[\frac{1}{z_a}\int_0^{z_a} \cos \omega t_f \frac{\psi(\sqrt{z})}{\psi(\sqrt{z_a})}\,dz\right]^2 + \left[\frac{1}{z_a}\int_0^{z_a} \sin \omega t_f \frac{\psi(\sqrt{z})}{\psi(\sqrt{z_a})}\,dz\right]^2 \tag{4-18}$$

where $z = \ln \varrho$ and

$$\psi(\sqrt{z}) = \int_0^{u=\sqrt{z}} e^{u^2}\,du \tag{4-19}$$

Figure 4-3 shows the factors F_f in Eqs. (4-11) and (4-18). The three curves nearly coincide in the interval $0 < \omega t_f < \pi/2$, so the two-term approximation given in Eq. (4-13) may be applied for F_f in all three cases. The transit time is also dependent on the structure. If $r_a/r_c \to \infty$ then $(t_f)_c \to \frac{1}{2}(t_f)_p$ (assuming the same anode voltages and electrode distances).[87]

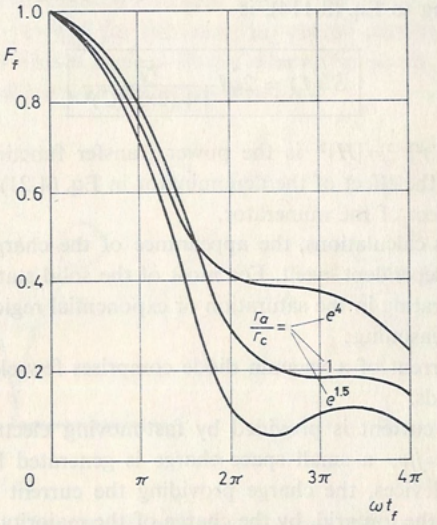


Figure 4-3 Factor F_f depending on the transit angle for different electrode configurations. (*Courtesy of Siemens.*)

In the depletion layers of a solid state device p–n junction, the velocity of the charge carriers is nearly constant. If the external circuit is a short circuit (i.e. it does not comprise delaying elements) then the current pulse due to the passing

of a single charge carrier will be rectangular. The power spectrum, according to Eq. (3-97), is

$$S(f) = 2qI\left(\frac{\sin \omega t_f/2}{\omega t_f/2}\right)^2 = 2qI\left[1-\frac{(\omega t_f)^2}{12}+\dots\right] \tag{4-20}$$

At low frequencies, $S(f)$ is constant and independent of the pulse shape. If $f=2/\pi t_f$, then the power density will be 5 to 10 percent less compared with the low-frequency value, according to the different F_f values. This means that the spectrum of the shot noise may be regarded as white in a wide frequency range.

The frequency dependence of the shot-noise power density is affected by other factors in addition to the finite transit time. This is explained by the fact that usually parasitic elements, such as electrode capacitance, lead resistance, and inductance, are present between the noise source and the external circuit, and these form a low-pass network. According to Sec. 3-5, this modifies both the power spectrum and the autocorrelation function. For instance, if the noise current does not flow directly into the external circuit but through an RC circuit made up of the series resistance R and electrode capacitance C, then the power spectrum, according to Eq. (3-114), is

$$\boxed{S'(f) = 2qI\frac{F_f^2}{1+(2\pi fRC)^2}} \tag{4-21}$$

where $[1+(2\pi fRC)^2]^{-1}=|H|^2$ is the power transfer function of the network. In practical cases, the effect of the denominator in Eq. (4-21) is frequently much higher than the effect of the numerator.

In our previous calculations, the appearance of the charge carriers has been regarded as an independent event. For most of the solid state devices and for a vacuum diode operating in the saturation or exponential region, this is explained by the following reasoning:

1. The starting current of a vacuum diode comprises few electrons with nearly independent fields.
2. The saturation current is provided by fast-moving electrons. According to the relation $\varrho=j/v$, a small space charge is generated by a given current.
3. In solid state devices, the charge providing the current is balanced by the fixed charge of the ion grid, by the charge of the majority carriers and by the surface charges. As a consequence of charge neutrality, there is no space charge.

However, in the vicinity of the cathode in a vacuum tube operating in the space charge region, the electrons are slowed down and form a space charge cloud. This affects the electrons that are subsequently emitted. The space charge cloud smoothes the fluctuations of the cathode current, so

$$\overline{i^2} = 2qIF_s^2\,\Delta f \tag{4-22}$$

Here $F_s<1$ is the noise attenuation factor characteristic for the space charge operating mode.[119] Thus

$$F_s^2 = \frac{9(1-\pi/4)kT_c}{q(U_a-U_{\min})} \tag{4-23}$$

where $k=1.38\times10^{-23}$ J/K is the Boltzmann constant, T_c is the cathode temperature, U_a is the anode voltage (in the case of a tube with grids, the effective voltage in the plane of the control grid), and $U_{\min}$ is the potential minimum in the vicinity of the cathode.

4-2 PARTITION NOISE

In many cases, the charge carriers emitted within an electronic device follow random paths. Let us assume that according to the general arrangement shown in Fig. 4-4*a* a constant number of charge carriers are emitted per unit time from electrode 0, and these charge carriers reach either electrode 1 or electrode 2. Let the probability of reaching electrode 1 be p_1 and the probability of reaching electrode 2 be $p_2=1-p_1$. (In this case, no charge carriers will return to the emitting electrode.) This is again a simple alternative, so the model presented in Sec. 1-1 may be utilized with slight modification.

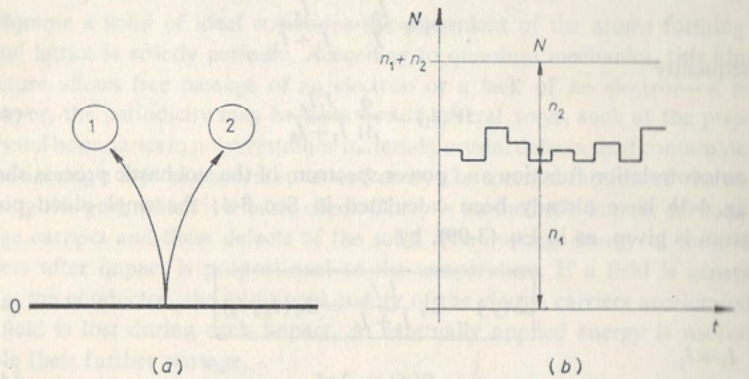


Figure 4-4 *(a)* Current partition between two collector electrodes. *(b)* Model for the fluctuation of current components.

Suppose that the number of evenly emitted charges during time Δt is Nq, where N is a constant, i.e. it is not a random variable. The average value of the total current is then

$$I = \bar{i} = \frac{Nq}{\Delta t} \tag{4-24}$$

which is randomly distributed in two parts

$$i_1 = \frac{n_1 q}{\Delta t} \quad \text{and} \quad i_2 = \frac{n_2 q}{\Delta t} \tag{4-25}$$

where i_1 and i_2 are the instantaneous values of the currents provided by n_1 and n_2 electrons, respectively. n_1 and n_2 have binomial distributions, so according to Eqs. (2-81) and (2-82), we have

$$M(n_1) = Np_1 \tag{4-26}$$

and

$$D^2(n_1) = Np_1 p_2 = D^2(n_2) \tag{4-27}$$

The expression $D^2(n_1)=D^2(n_2)$ is not only a formal consequence of Eq. (4-27). N being constant in any interval Δt, the fluctuation in n_1 creates a fluctuation in n_2 of equal magnitude but of opposite sense (see Fig. 4-4*b*). Thus

$$D^2(i_1) = D^2(i_2) = \left(\frac{q}{\Delta t}\right)^2 Np_1(1-p_1) \tag{4-28}$$

Combining Eqs. (4-25) and (4-26), we have

$$I_1 = M(i_1) = \frac{q}{\Delta t} M(n_1) = \frac{q}{\Delta t} Np_1 \tag{4-29}$$

Furthermore, it is easily shown that

$$1-p_1 = \frac{I_2}{I} = \frac{I_2}{I_1+I_2} \tag{4-30}$$

Consequently

$$D^2(i_1) = \frac{q}{\Delta t}\frac{I_1 I_2}{I_1+I_2} \tag{4-31}$$

The autocorrelation function and power spectrum of the stochastic process shown in Fig. 4-4*b* have already been calculated in Sec. 3-4; the single-sided power spectrum is given, as in Eq. (3-99), by

$$\boxed{S(f) = 2q\frac{I_1 I_2}{I_1+I_2} = 2q(I_1 \| I_2)} \tag{4-32}$$

For $I_1 \ll I_2$,

$$S(f) \cong 2qI_1 \tag{4-33}$$

Thus, the analogy between the partition noise and shot noise is clearly seen. If the condition of Eq. (4-33) is met, the instantaneous values follow a Poisson distribution; otherwise, we have a binomial distribution. In practical cases, N is normally so high that both may be approximated by a normal distribution (see Sec. 2-7).

It is well known that within an electronic device the emission of electrons is not even and this fluctuation brings about the shot noise. Taking into account

this fluctuation too, Eq. (4-32) may be rewritten in the following form, referred to electrode 1:[20,129]

$$S_{i1}(f) = \left(\frac{I_1}{I}\right)^2 2qI + 2q\frac{I_1 I_2}{I} \tag{4-34}$$

This means that now $S_{i1}(f) \neq S_{i2}(f)$; the latter may be calculated in a similar way to Eq. (4-34) by incorporating I_2 into the numerator of the first term.

4-3 THERMAL NOISE

In homogeneous conductors or semiconductors, there are no energy pedestals, but in spite of this, noise is generated even without current flow. This is due to the thermal motion of charge carriers and to the nonideal structure of the conductor or semiconductor. (This latter prevents the achievement of infinite conductance.) It may be proven generally[142] that there is direct proportionality between the conductance of solids and the root mean square value of the noise current generated within these solids. Our corpuscular kinetic model is far from being general. Nevertheless, it is used because it allows direct conclusions concerning the microphysical processes within the solids.[36] This is evident from the following model of current conduction which will be needed to establish our subsequent results.

Assume a solid of ideal structure: the placement of the atoms forming the crystal lattice is strictly periodic. According to quantum mechanics, this kind of structure allows free passage of an electron or a lack of an electron—a hole.[9] However, the periodicity may be disturbed in several ways, such as the presence of crystal boundaries in polycrystalline materials, crystal defects, and contaminating atoms in single crystal materials, as well as by the thermal motion of the atoms forming the grid, which is a basic effect. There is an inelastic impact between the charge carriers and these defects of the solid. The average energy of the charge carriers after impact is proportional to the temperature. If a field is generated within the conductor, the additional energy of the charge carriers accelerated by this field is lost during each impact, so externally applied energy is needed to enable their further passage.

Let us assume that in the instant $t=0$ the number of charge carriers that suffer impacts is C, and let us investigate their subsequent paths. The impacts being independent events, the change in the number of charge carriers that have not yet suffered impacts between t and $t+\Delta t$ is

$$dC(t) = \frac{C(t)\,dt}{\Theta} \tag{4-35}$$

where Θ is the average time between impacts (the change is in a decreasing sense).

Solving this differential equation, the density function of the time distribution between two impacts may be determined. Thus

$$f(t) = \frac{1}{\Theta} e^{-t/\Theta} \qquad t > 0 \tag{4-36}$$

This is in agreement with the density function (2-144) which has the first- and second-order moments $M(t)=\Theta$ and $M(t^2)=2\Theta^2$, respectively [see Eq. (2-147)]. Let the field strength in a prism-shaped solid be E_x. The path of one of the electrons may have the shape shown in Fig. 4-5 because of the impacts. The projection of these path elements in direction x is given by

$$x_i = \frac{a_x}{2} t_i^2 \tag{4-37}$$

where t_i is the time interval from the $(i-1)$-th impact to the i-th impact and the acceleration of the electron is

$$a_x = -\frac{q}{m^*} E_x \tag{4-38}$$

where m^* is the effective mass of the electron in the crystal. Taking into account that according to Eq. (2-147) $M(t^2)/2=\Theta^2$, the distance covered during a large number of impacts K will be given by

$$x_K = \frac{a}{2} \sum_1^K t_i^2 = \frac{a}{2} KM(t^2) = aK\Theta^2 \tag{4-39}$$

and the time needed to cover this distance is

$$t_K = \sum_1^K t_i = KM(t) = K\Theta \tag{4-40}$$

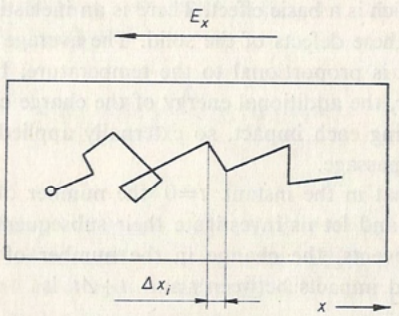


Figure 4-5 Path of an electron moving in an electric field within a solid. The relatively slow movement in the x-direction is superimposed on a random thermal movement.

From these expressions, the average drift velocity is

$$\bar{v}_d = \frac{x_K}{t_K} = a\Theta = -\frac{q}{m^*} E_x \Theta \tag{4-41}$$

If the number of electrons moving with a velocity v_d within a unit volume is n_0 then the current passing through a unit area is

$$J_x = (-q) n_0 \overline{v_d} = \frac{q^2}{m^*} n_0 \Theta E_x \tag{4-42}$$

Utilizing the definition of conductivity, we have

$$\gamma = \frac{J_x}{E_x} = \frac{q^2}{m^*} n_0 \Theta \tag{4-43}$$

so γ is directly proportional to the average time interval Θ between two impacts. The order of Θ is 10^{-14} to 10^{-13} s, and the average free distance is between 10 and 30 nm.

During the calculation of Eq. (4-43), our model has been basically classic, in spite of the fact that the results of quantum mechanics have been taken into account through the introduction of m^*. The same result can be achieved using more accurate quantum mechanical calculations.[64]

For the case of $E=0$, there is no macroscopically measurable current flow within the solid. However, there is a random thermal motion of the charge carriers which has a velocity of v which is much higher than the practically occurring values of v_d. The direction of the charge carrier motion is changed during each impact so there are microscopic current changes both in the volume in question and in the external circuit connected to it. The average value of the velocity is not changed significantly by impact as the sites of impact are at the same temperature.

Let the number of charge carriers having equal polarity again be n_0 within the unit volume of the solid with cross section A, as shown in Fig. 4-6. However,

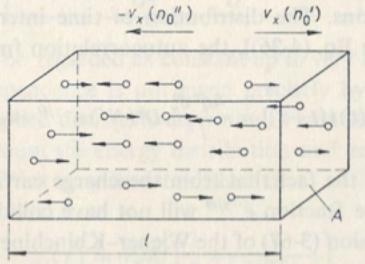


Figure 4-6 The x-direction component of the thermal velocity of charge carriers.

the velocity components in the x directions should now be different: $+v_x$ for n_0' charge carriers, and $-v_x$ for $n_0''=n_0-n_0'$ charge carriers. In this case, the current flowing in one direction is

$$i' = AJ' = An_0'qv_x = \frac{N'}{l}qv_x \tag{4-44}$$

and the current flowing in the opposite direction is given by

$$i'' = An_0''q(-v_x) = -\frac{N-N'}{l}qv_x \tag{4-45}$$

where $N=Aln_0$, $N'=Aln_0'$, $N''=Aln_0''$ is the number of charge carriers within the entire volume. The difference between the two currents is

$$i(t) = i'-i'' = \frac{qv_x}{l}(2N'-N) = \frac{2qv_x}{l}\left[N'(t)-\frac{N}{2}\right] \tag{4-46}$$

The sign of Eq. (4-46) depends on the type of charge carriers and the instantaneous value of N'. The sign may be disregarded below as only the mean square value of i will be needed; our calculation will thus be valid both for electrons and holes.

If the two kinds of charge carriers are simultaneously present, N' should be calculated by taking into account the holes of velocity $+v_x$ and electrons of velocity $-v_x$, and N'' should be calculated by taking into account electrons and holes moving in opposite directions.

N' and N'' are random variables pertaining to mutually exclusive events, their expected value being $N/2$ and their distribution being binomial. According to Eq. (2-84) we have

$$D^2(N') = \frac{M(N')}{2} = \frac{N'}{4} \tag{4-47}$$

The current fluctuation $i(t)$ is the consequence of random collisions of carriers and lattice imperfections. The distribution of time intervals between collisions being exponential [see Eq. (4-36)], the autocorrelation function of $i(t)$ is

$$\Gamma_i(\tau) = M[i(t)i(t+\tau)] = \frac{4q^2\overline{v_x^2}}{l^2}D^2(N')e^{-\tau/\Theta} = \frac{q^2\overline{v_x^2}}{l}n_0Ae^{-\tau/\Theta} \tag{4-48}$$

which is explained by the fact that from the charge carriers moving together in time instant t only the fraction $e^{-\tau/\Theta}$ will not have collided after the time interval τ. From the expression (3-67) of the Wiener–Khinchine theorem (Appendix B), we have

$$S_i(f) = 4\int_0^\infty \Gamma_i(\tau)\cos\omega\tau\,d\tau = \frac{4q^2\overline{v_x^2}}{l}n_0A\frac{\Theta}{1+\omega^2\Theta^2} \tag{4-49}$$

The power density thus has a Lorentz spectrum. However, this has no significance for $\omega<10^{12}$ rad/s as $\omega\Theta\ll1$.

$\overline{v_x^2}$ may be expressed utilizing the equipartition theorem

$$\frac{1}{2}m^*\overline{v_x^2} = \frac{1}{2}kT \tag{4-50}$$

where $k=1.38\times10^{-23}$ J/K, the Boltzmann constant. After substituting Eqs. (4-50) and (4-43), the power density of the noise current flowing through the two-pole is

$$\boxed{S_i(f) = 4kT\gamma\frac{A}{l} = 4kTG} \tag{4-51}$$

where G is the conductance of the solid shown in Fig. 4-6. This end result is the well-known Nyquist formula.

This calculation has been based on an important simplification: namely that $|v_x|$ has been regarded as constant for all charge carriers. However, actually

$$v_x = v\cos\varphi \tag{4-52}$$

where φ is the angle with the x-axis; v follows the Fermi–Dirac statistics in metals, and the Maxwell–Boltzmann statistics in semiconductors. However, for isotropic velocity distribution, Eq. (4-50) is derived again after detailed calculation. It may further be shown that for $\omega\ll1/\Theta$, the end result is independent of the distribution of v.[8] Several other derivations of the Nyquist formula may be found in the literature.[41,126]

Expression (4-51) shows that the power density of the thermal-noise current is proportional to the conductance G and absolute temperature T of the two-pole. The power density of the noise voltage across the two-pole terminals is

$$\boxed{S_u(f) = \frac{S_i(f)}{G^2} = \frac{4kT}{G} = 4kTR} \tag{4-53}$$

$S_i(f)$, and $S_u(f)$ may be regarded as constant up to very high frequencies. However, the frequency dependence is not given precisely by Eq. (4-49) because of the approximations applied during its derivation. From the quantum mechanical model, taking into account the energy distribution and zero-point energy of the solid-state electrons,[33,142] it may be established that

$$S_i(f) = 4Ghf\left(\frac{1}{2}+\frac{1}{e^{hf/kT}-1}\right) \tag{4-54}$$

which is identical to Eq. (4-51) for $hf\ll kT$. At room temperature, $hf=kT$ if f

6×10^{12} Hz. Expanding Eq. (4-54) for small hf/kT values we have the approximation

$$S_i(f) = 4kTG\left[1+\frac{1}{12}\left(\frac{hf}{kT}\right)^2+\dots\right] \tag{4-55}$$

At extremely low temperatures the increase in $S_i(f)$ should be taken into account even in the microwave region. Increasing further the frequency f towards the optical regions, $S_i(f)$ is increased since the optical radiation is inherently quantized and the flow of quanta is random like the passover of charge carriers causing shot noise. This kind of noise is called *quantum noise*.

The reactances in parallel or in series with the resistive noise source have an effect on the overall power spectrum in this case too (see Sec. 3-5). However, an ideally reactive circuit element will not generate any noise as real power can neither be transmitted to nor received from its environment.

4-4 AVALANCHE NOISE

Avalanche noise may be generated between conductors in contact with each other at microscopic points, or in a p–n junction of a semiconductor with reverse bias. The breakdown mechanism of a p–n junction is fairly well known. The low-voltage breakdown below approximately 6 V is caused by the Zener (tunnel) effect while the high-voltage breakdown is the result of an avalanche-type charge carrier multiplication.[84] Avalanche noise is only generated in the latter case.[29]

There is an extremely high field strength in a p–n junction biased near breakdown. The minority carriers providing the reverse current may therefore have a high acceleration, capable of generating one or more new charge carrier pairs by ionizing on impact the neutral atoms forming the crystal lattice. Thus the plasma, characteristic of breakdown, is formed.

Figure 4-7 shows the cross section of the depletion layer of width w which is the region of high field strength in the p–n junction. The entrance of holes is at

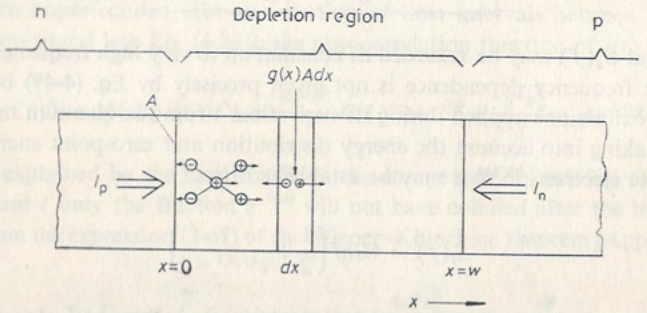


Figure 4-7 Processes in the depletion layer.

$x=0$, while that for electrons is at $x=w$. Heating or electromagnetic radiation has the effect of generating in the volume Adx further charge carrier pairs of number $g(x)Adx$ [here $g(x)$ is the number of pairs generated at point x in a unit volume and during a unit time interval]. The primary charge carriers may then generate secondary carriers during impacts, so the total current is

$$I = I_p(0)M(0)+I_n(w)M(w)+qA\int_0^w g(x)M(x)\,dx \tag{4-56}$$

where I_p and I_n are the hole and electron currents, respectively, flowing in at the ends. $M(0)$, $M(w)$, and $M(x)$ are the multiplication factors[83] pertaining to the charge carriers entering at the ends 0 and w, or generated at point x, respectively.

Let the current increase in volume Adx be $dI(x)$, which makes the following contribution to the squared fluctuation of the total current:

$$d[\overline{i^2}(x)] = 2qM^2(x)\,dI(x)\,\Delta f \tag{4-57}$$

in the external circuit, the squared fluctuation generated by current element $dI(x)$ will have a weighting factor $M^2(x)$ due the multiplication.

The mean square value of the total fluctuation should be calculated carefully as both $dI(x)$ and $M(x)$ are random variables. For the case where $M(x)=M$ independent of the distance and the primary (not multiplicated) current has a Poisson distribution then, according to the variance theory,[20,130] we have

$$\overline{i^2} = 2qI_0\overline{M^2}\,\Delta f \tag{4-58}$$

where

$$I_0 = I_p(0)+I_n(w)+qA\int_0^w g(x)\,dx \tag{4-59}$$

It has been shown by Tager[116,117] using detailed analysis of the multiplication factor M that $\overline{M^2}=\overline{M}^3$; so finally

$$S_i(f) = 2qI_0\overline{M}^3 \tag{4-60}$$

Our model may further be refined by taking into account the fact that the probability of impacts generating new charge carrier pairs is a function of the field strength, which is dependent on the distance, and is not the same for electrons and holes. If the number of new charge carriers generated in unit distance by an electron is α and by a hole is β and $\beta=k\alpha$, then according to McIntyre,[88]

$$S_i(f) = 2qI_p(0)M^3\left[1+\frac{1-k}{k}\left(\frac{M-1}{M}\right)^2\right] \tag{4-61}$$

if the primary current is mainly due to holes and

$$S_i(f) = 2qI_n(w)M^3\left[1-(1-k)\left(\frac{M-1}{M}\right)^2\right] \tag{4-62a}$$

or

$$S_i(f) = 2qI_n(w)M^2\left[kM+(1-k)\left(2-\frac{1}{M}\right)\right] \tag{4-62b}$$

if the primary current is mainly due to electrons. Note that the latter case is more favorable, i.e. it generates less noise than the former because for silicon $k \ll 1$. This fact is actually utilized in opto-electronic diodes.[16,141]

The transition range 2–14 V between tunnel breakdown and pure avalanche breakdown has only been recently investigated.[75] It has been found that

$$S_i(f) = 2qI_0[1+f(M^*)] \tag{4-63}$$

where $M^* = (I/I_0)-1$, and $f(M^*)$ is a fairly complicated function of the averaged ionized rates. At such low voltages, the primary charge carriers are generated by the high field emission at the junction, and by these, relatively few new pairs are generated due to the short ionization distance available. The earlier theories were refined and generalized recently.[135,136]

It has not previously been taken into account that a finite time is needed for the charge carriers to pass through the depletion layer. This has the effect of making the multiplication factor time-dependent. Thus

$$M(x, \omega) = \frac{M(x)}{1+j\omega M\tau} \tag{4-64}$$

where τ is the so-called intrinsic response time,[91,98] its order of magnitude being 10^{-11} to 10^{-10} s.

However, the upper frequency limit is generally not determined by $M\tau$ but by the capacitance and series resistance of the p–n junction. As surprisingly high amplitude signals may be generated here, it is wise to apply a graphical analysis.[42] As shown by Fig. 4-8*a*, the junction biased up to breakdown is connected to the voltage source U_0 via current-limiting resistance R_L. Figure 4-8*b* shows the equivalent circuit of the junction. U_b is the breakdown voltage extrapolated for zero current, r_s is the resistance in series with the junction, and C is the total capacitance of the junction.

Figure 4-8*c* shows the graphic analysis of the equivalent circuit. After connecting the voltage source, capacitance C is charged to the source voltage U_0 with a time constant $R_L C$. In spite of $U_0 > U_b$, breakdown will only take place if the acceleratable primary charge carrier is within the small volume of the microplasma to be formed. Because of thermal reasons, for example, this will take place after randomly changing time intervals—the earlier, on average, the higher U_0.

The avalanche-like breakdown will have the effect of inserting resistance r_s, causing a discharge of capacitance C by a time constant $(R_L \| r_s)C$ to the voltage corresponding to the stationary working point P. If the primary current, again due to the random generation process, then decreases to zero by an order of time needed to pass the depletion layer the breakdown will be terminated, as shown by one cycle illustrated in Fig. 4-8*c*. Figure 4-8*d* shows the voltage–time function across the p–n junction. Both intervals t_1 and t_2 reveal a stochastic change.

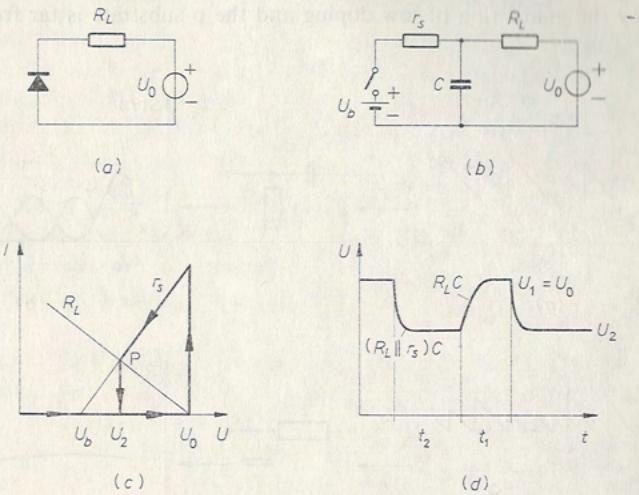


Figure 4-8 *(a)* Reverse biased p–n junction. *(b)* Equivalent circuit of the junction. *(c)* Graphical analysis of the cycle. *(d)* Voltage waveform. (*Courtesy of the American Institute of Physics and the Author.*)

The circuit shown in Fig. 4-8 and the stochastic process introduced by this circuit may also be used, under controlled conditions, as a calibrating noise source. Two possibilities for realizing this are shown in Fig. 4-9. In both cases, current generator feed is used instead of voltage source U_0 and resistance R_L. Breakdown is initiated by the field emission. In the case shown in Fig. 4-9*a*, the probability of field emission is given by the abruptly changing function $g_1(u)$, so the breakdowns occur by nearly equal voltages. Differentiating the stochastic triangular wave thus obtained, the power spectrum of the obtained pulses will be slightly wavy (see Fig. 4-9*b*). On the other hand, in the case shown in Fig. 4-9*c*, $g_2(u)$ is much less abrupt in making the distribution of the triangular wave cover a much wider interval. After integration, the fairly flat power spectrum (see Fig. 4-9*d*) is obtained.[43]

Low-voltage varactor diodes are applied as high-frequency noise sources.[40,104,110,111]

During our previous investigations, it has been implicitly assumed that the breakdown takes place simultaneously over the complete cross section of the p–n junction (uniform junction). However, this is only true if the junction is formed by a special guard ring as shown in Fig. 4-10*a*. In this diode, breakdown may only take place perpendicularly to the surface, in a well-defined cross section *D*, between layers p and n+. Reaching the state of the breakdown, the junction formed by the guard ring of low doping and the p substrate is far from breakdown.

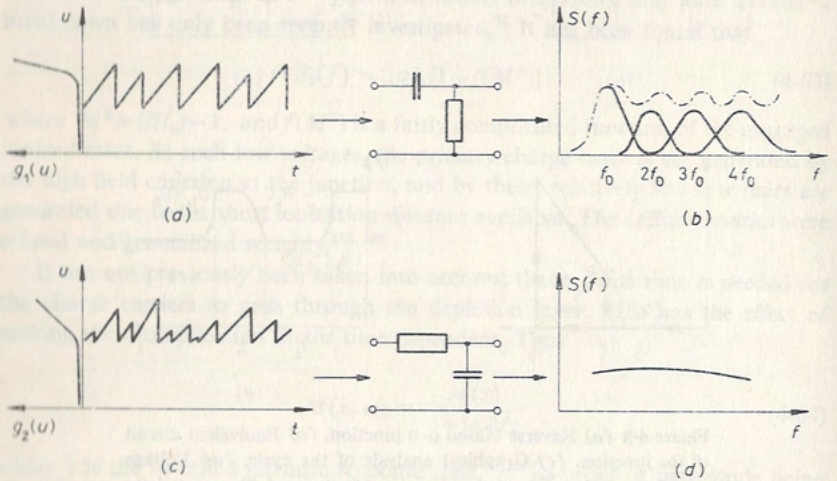


Figure 4-9 *(a)* Sawtooth signal with almost constant repetition rate, for a field emission probability function $g_1(u)$. *(b)* Power spectrum of signal *a*. *(c)* Waveform for a different type of function $g_2(u)$. *(d)* Power spectrum of signal *c*. (*Courtesy of the Institute of Electrical and Electronics Engineers.*)

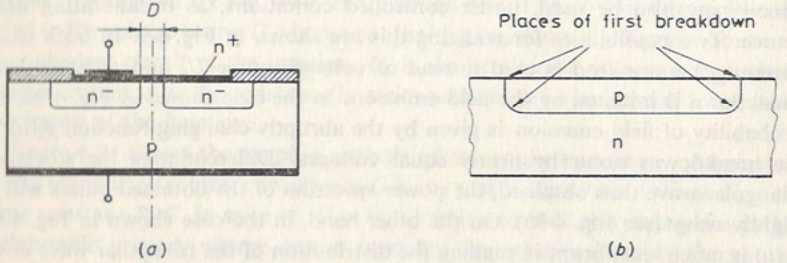


Figure 4-10 *(a)* Diode with guard ring for realization of the uniform junction.[43] *(b)* Planar diode. (*Courtesy of the Institute of Electrical and Electronics Engineers.*)

Planar diodes (Fig. 4-10*b*) may comprise two regions that may break down earlier than the planar junction. First, the breakdown voltage at curvatures is always less than at planar regions.[115,144] Second, the perimeter where the junction reaches the surface is very delicate. If, in addition, there is some imperfect crystal structure at these sensitive points (dislocation, precipitation, or local fluctuation in the concentration of the doping), low-volume regions with sufficient field strength to generate a local avalanche—*microplasma*—may be formed.

Microplasmas have the following characteristics:

1. Generally small diameter (1–2 μm)
2. Low current intensity (of the order of 10 μA)
3. Relatively high series resistance (around 10 kΩ)
4. High noise due to stochastic in–out switching[22]
5. Occasionally weak light emission[28]

In the diodes, either a single microplasma may be generated before the complete breakdown of the junction or a multitude of microplasmas. Most of the commercially available reference diodes with more than 6-V breakdown voltage show this undesirable effect, so it is not feasible to use them with too low an operating current.

It is fairly difficult to define the parameters of the stochastic process describing the switch-on and switch-off of the microplasma. More or less general results have only been published recently.[34] It has been found that the probability of switch-on may be written in the form

$$P_{01} = [\mu i(t)+\nu][u(t)-U_b]\,dt \tag{4-65}$$

where $i(t)$ is the reverse current instantaneous value, $u(t)$ is the reverse voltage instantaneous value, U_b is the breakdown voltage extrapolated for zero current, and μ and ν are values characterizing the current-dependent and current-independent changeover frequencies, respectively. Let us introduce the auxiliary variable

$$z = (\mu I_0+\nu)\tau_c U_0 \tag{4-66}$$

here I_0 is the stationary reverse current, τ_c is the time constant formed by the junction capacitance and the series resistance [$\tau_c \gg \tau$, defined in Eq. (4-64)], and U_0 is the supply voltage. Utilizing this auxiliary variable, the amplitude distribution of the stochastic signal generated by a *single* microplasma is

$$F\left(\frac{u}{U_0}\right) = 1-\frac{\Gamma[z(1-u/U_0)]}{\Gamma(z)} \tag{4-67}$$

Here $\Gamma(x)$ is the gamma function defined in connection with Eq. (2-118).

4-5 GENERATION–RECOMBINATION NOISE

The resistance R of a homogeneous semiconductor block of length L and cross section A is

$$R = \frac{L}{Aq(\mu_n n + \mu_p p)} = \frac{L^2}{q(\mu_n N + \mu_p P)} \tag{4-68}$$

where n and p are the densities of the electrons and holes, respectively, μ is their mobility, and N and P are the number of electrons and holes, respectively, in the *total* volume. These numbers are not constant (they may fluctuate because of the generation and recombination) so $N = N_0 + \Delta N(t)$ and $P = P_0 + \Delta P(t)$. The fluctuation is random, so the resistance will also fluctuate stochastically. Thus

$$\Delta R(t) = -\frac{L^2}{q}\frac{\mu_n \Delta N(t) + \mu_p \Delta P(t)}{\mu_n N_0 + \mu_p P_0} = -R\frac{b\Delta N + \Delta P}{bN_0 + P_0} \tag{4-69}$$

where $b = \mu_n/\mu_p$. If now a DC current I flows through the semiconductor block then the noise voltage will be

$$\Delta U(t) = -IR\frac{b\Delta N(t) + \Delta P(t)}{bN_0 + P_0} \tag{4-70}$$

For simplicity, let us investigate an intrinsic semiconductor with only pairwise recombination and generation possibilities. In this case

$$\Delta U(t) = -\frac{IR}{bN_0 + P_0}(b+1)\,\Delta N(t) \tag{4-71}$$

and expressing the mean square value of the fluctuation by the square of the flowing DC current, we have

$$M(\Delta I^2) = M\left(\frac{\Delta U^2}{R^2}\right) = I^2\frac{(b+1)^2}{(bN_0 + P_0)^2}M(\Delta N^2) \tag{4-72}$$

It may be shown generally[133] that

$$M(\Delta N^2) = \frac{g(N_0)}{r'(N_0) - g'(N_0)} \tag{4-73}$$

where g is the generation rate, and g' and r' are the generation and recombination rate derivatives, respectively, in respect to N. For the intrinsic semiconductor in our example, $M(\Delta N^2) = N_0 P_0/(N_0 + P_0)$.

Naturally, the autocorrelation function of $\Delta N(t)$ is again needed to calculate the power spectrum. The process of generation–recombination is described by a differential equation of the type given by Eq. (4-35), so it is not surprising to find that

$$\Gamma_{\Delta N}(t') = M(\Delta N^2)e^{-t'/\tau} \tag{4-74}$$

where τ is a time constant characteristic for the generation–recombination process. The power spectrum is of the Lorentz type:

$$S_{\Delta N}(\omega) = 4M(\Delta N^2)\frac{\tau}{1 + \omega^2\tau^2} \tag{4-75}$$

For a nonintrinsic-type semiconductor, the relationship $\Delta P = \Delta N$ does not hold, but these two quantities are not independent either. In this case, all components and the correlations between them have to be taken into account in the power spectrum of the current fluctuation.[132] Thus

$$S_i(f) = \left(\frac{I}{bN_0 + P_0}\right)^2 [b^2 S_{nn}(f) + 2bS_{np}(f) + S_{pp}(f)] \tag{4-76}$$

where S_{nn} and S_{pp} are the power density spectra of the electron number fluctuation and the hole number fluctuation, respectively, and S_{np} is the so-called cross power density spectrum [for its definition see Eq. (5-15)].

In semiconductors operating at not too low temperatures and having the usual level of doping, the generation–recombination noise has no significance. This is explained by the fact that in this case the conductance is determined by the density of the majority charge carriers, and this is constant being practically equal to the density of the built-in doping atoms. On the other hand, the investigation of low-concentration contaminants by evaluating the generation–recombination noise is most interesting.

During the early 1950s when the first semiconductor devices and the more-or-less pure basic materials had been prepared, the generation–recombination phenomenon and the noise due to this phenomenon had been thoroughly investigated. Interest has fallen in more recent years with the witnessing of other, stronger noise sources which suppress the former kind of noise. However, these stronger noises have been reduced by the developing technology, resulting in a renewed interest in the generation–recombination noise.[62, 131, 146] Another possibility justifying this interest is to reveal extremely low levels of contaminants or structural imperfections by careful analysis of the generation–recombination noise.

Let us take into account in the kinetics of the generation–recombination process some impurities—so-called traps. In the semiconductor under test, the density of electron traps is N_t, in addition to the completely ionized donor and acceptor impurities (see Fig. 4-11).[129] Let the density of the free electrons be n, and the density of the electrons captured by the traps be n_t. The condition of balance may be formulated in the form

$$\text{Free electron} + \text{empty trap} \rightleftharpoons \text{filled trap}$$

It is assumed this time that the traps do not comprise holes or electrons from the valency band. Our results may, of course, be generalized for these cases.

Following any perturbation of the free and captured electron distribution, the system will come back to balance according to the differential equation

$$\frac{dn}{dt} = -vs(N_t - n_t)n + v_0 e^{(-W_n/kT)} n_t \tag{4-77}$$

and

$$\frac{dn_t}{dt} = -\frac{dn}{dt} \tag{4-78}$$

where v = thermal velocity of the free electrons

s = effective cross-section of the traps

$v_0 e^{-W_n/kT}$ = re-emission rate of the captured electrons.

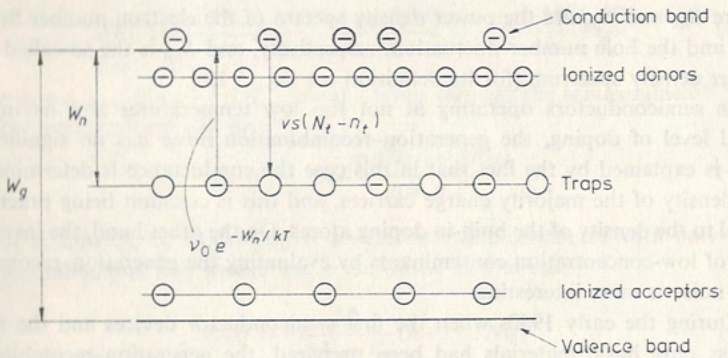


Figure 4-11 Illustration of a generation–recombination process involving traps.[129]

In the first member of Eq. (4-77), the well-known rule of the recombination is comprised: the rate is proportional to the density of unfilled traps and free electrons. The second member is introduced physically by the lattice oscillations which have the effect of freeing the captured electron from the trap. The rate v_0 has an order of 10^{12} s^{-1}.

Introducing the time constants τ_c and τ_r we have

$$\frac{1}{\tau_c} = vs(N_t - n_t) \tag{4-79a}$$

and

$$\frac{1}{\tau_r} = v_0 e^{-W_n/kT} \tag{4-79b}$$

Solving the differential equation (4-77) will give the following time function for the perturbation $\Delta n(t)$ with initial value Δn_0:

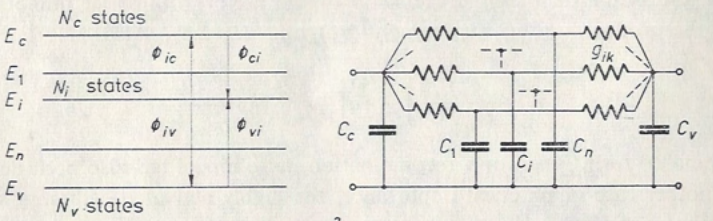
$$\Delta n(t) = \Delta n_0 e^{-t/\tau} \tag{4-80}$$

if $1/\tau = 1/\tau_c + 1/\tau_r$. Equation (4-74) may be utilized again for the characterization of the power spectrum of the process.

According to Eq. (4-79*b*), τ_r has a strong temperature dependence, allowing the possibility of investigating material structure by analyzing the generation–recombination noise. By measuring the time constant at several temperatures—generally attainable by cooling—and plotting this time constant as a function of $1/T$, the activation energy level W_n of the trap may be determined. On the other hand, the amplitude of the generation–recombination noise yields the density of the traps.

For more than one trap level, the time constants may not be clearly recognized in the overall noise spectrum, so the above method is only feasible with definitely separable trap levels so that the effect of the traps may be considered to be independent within the various temperature ranges. The general theory of multilevel generation–recombination fluctuation although known[133,143] is difficult to utilize in evaluating practical measurements due to mathematical complications.

The resemblance of the differential equation (4-77) with the differential equations applying to electrical networks has resulted in descriptions of the generation–recombination phenomenon by equivalent networks[25,103] and in the calculation of the generation–recombination noise by equivalent networks.[25] Figure 4-12 shows the energy diagram, the equivalent network, and the relationships between parameters applying to a semiconductor with multilevel traps. In these relations, N_i is the number of available states within the i-th level or band, f_i^0 is the equilibrium value of the probability of occupancy, and Φ_{ik}^0 [s^{-1}] is the equilibrium value of the probability of transition. Taking the conductance and valence bands as single levels, the number of total levels in the model is two more than the number of generation–recombination levels. The capacitances are proportional to the maximum number of electrons or holes within the level (band), and the conductances are proportional to the communication between these levels. The den-



$$C_i = \frac{q^2}{kT} N_i f_i^0 (1 - f_i^0)$$

$$g_{ik} = \frac{q^2}{kT} \Phi_{ik}^0$$

Figure 4-12 Energy diagram and equivalent circuit of a multilevel generation–recombination process.

sity of charges giving rise to the conductance will only fluctuate if an electron enters or leaves the conductance band, or if a hole enters or leaves the valence band. Thus the transitions between generation–recombination levels are unimportant, so the internal nodes of the equivalent network are not connected by conductances. If it is assumed that $C_c, C_v \gg C_1, C_2, \ldots, C_n$, i.e. the maximum number of electrons or holes within the conductance and valence bands is high with respect to this number in the generation–recombination centers, then closed-form relationships for expressing the power spectra needed for evaluating Eq. (4-76) may be derived (with allowable approximations).[7]

To this end, the impedance matrix of the equivalent circuit shown in Fig. 4-12 has to be calculated as

$$S_{nn}(f) = N_0^2\left(\frac{q}{kT}\right)^2 4kT\,\mathrm{Re}\,Z_{11}$$

$$S_{np}(f) = -N_0 P_0\left(\frac{q}{kT}\right)^2 4kT\,\mathrm{Re}\,Z_{12} \tag{4-81}$$

$$S_{pp}(f) = P_0^2\left(\frac{q}{kT}\right)^2 4kT\,\mathrm{Re}\,Z_{22}$$

Omitting details of the calculation, we have

$$\mathrm{Re}\,Z_{11} = \frac{C_v/C_c}{C_c+C_v}\frac{\tau_0}{1+\omega^2\tau_0^2} \tag{4-82}$$

$$\mathrm{Re}\,Z_{22} = \frac{C_c/C_v}{C_c+C_v}\frac{\tau_0}{1+\omega^2\tau_0^2} \tag{4-83}$$

$$\mathrm{Re}\,Z_{12} \cong -\frac{1}{C_c+C_v}\frac{1}{(1+\omega^2\tau_0^2)\prod_1^n(1+\omega^2\tau_i^2)}\sum_{i=1}^{n}\left[\frac{G_i}{G}(\tau_0+\tau_i)\prod_{k=1}^{n}{}'(1+\omega^2\tau_k^2)\right] \tag{4-84}$$

where the prime denotes the exclusion of the factor with serial number $k=i$ from the product, and $G_i = g_{ic}g_{iv}/(g_{ic}+g_{iv})$ and $G=\Sigma G_i$. Also

$$\tau_0 = \frac{C_c C_v}{C_c+C_v}\left[\sum_{i=1}^{n} G_i\right]^{-1} \qquad \tau_i = \frac{C_i}{g_{ic}+g_{iv}} \tag{4-85}$$

A closed-form generation–recombination noise model has also been derived,[7] for another case of practical importance, the highly activated cadmium sulfide-type photoconductor.

In most kinds of solid state devices, the generation–recombination noise does not have much significance as it is usually suppressed by other kinds of noise sources. The junction field-effect transistor is usually regarded as producing the lowest noise level. However, if this device is cooled below room temperature in order to reduce the thermal noise to meet special requirements, the generation–recombination noise may become predominant. This has been investigated in recent publications.[27,48,134] These investigations have been aimed, for instance, at proving the structural damage produced by neutron or electron irradiation,[71,140] or at demonstrating that the generation–recombination centers introduced by channel dopants are dependent on the dopant suppliers, in spite of the fact that these are so-called chemically pure elements.[47]

4-6 FLICKER NOISE

In thin metallic or semiconductor layers, in carbon resistors, carbon microphones, vacuum tubes, solid state devices, etc., another noise source in addition to those previously investigated is also present. The power spectrum of this noise source may be expressed in the general form

$$\boxed{S(f) = \mathrm{const}\,\frac{I^\beta}{f^\alpha}} \tag{4-86}$$

where I = DC current flowing through the device
f = frequency of measurement
$\beta \cong 2$
$\alpha \cong 1$

The current dependence may be easily interpreted. It will be shown that in the case of $\beta=2$, this kind of noise may be caused by the fluctuation in the conductance or resistance. On the other hand, the exponent $\alpha \cong 1$ may not presently be explained by a universal physical model, although it may be present even at a frequency of 10^{-6} Hz.[21] Furthermore, $\alpha=1$ leads also to mathematical complications: considering Problem 1-2, it is at once seen that an infinite noise power results if the lower integration limit tends to zero, and this is evidently impossible.

It is extremely difficult to give a short summary of flicker noise; this topic has been extensively dealt with in many cases. Therefore, our discussions will be restricted to investigations of homogeneous structures not comprising p–n junctions (in addition to presenting general characteristics). Also, results published since 1968 will only be presented.

In order to establish a model for conductance fluctuation, let us investigate a conductor of length l and cross section A, comprising electrons of number n_0 and mobility μ_n, per unit volume. The conductivity is then

$$\gamma = q\mu_n n_0 \tag{4-87}$$

and

$$R = \frac{l}{\gamma A} = \frac{l}{Aq\mu_n n_0} = \frac{l^2}{q\mu_n N} \tag{4-88}$$

where $N = n_0 Al$ is the number of charge carriers within the whole volume. If N changes by dN then

$$\frac{dR}{M(R)} = -\frac{dN}{M(N)} \tag{4-89}$$

and

$$(dR)^2 = M^2(R)\left[\frac{dN}{M(N)}\right]^2 \tag{4-90}$$

Its expected value may be calculated from that of $(dN)^2$. Thus

$$M(dN)^2 = M[N - M(N)]^2 = CM(N) \tag{4-91}$$

where it has been assumed that the mean square value of the fluctuation is somehow proportional to the total number of charge carriers.

From the above, we have

$$M(dR)^2 = M^2(R)\frac{C}{M(N)} = M^2(R)\frac{C}{n_0 V} \tag{4-92}$$

that is the smaller the volume in question the higher will be the fluctuation of the resistance. This explains why this additional noise is primarily generated in thin layers and granular materials, as well as in weakly doped semiconductors (N_0 is small).

After investigating several materials and structures, Hooge[49] ventured a generalization according to which the squared relative fluctuation of a two-pole of resistance R is given by

$$\frac{\overline{\Delta R^2}}{R^2} = \frac{\alpha}{N}\frac{\Delta f}{f} \tag{4-93}$$

where N is the total number of free charge carriers within the resistance block and α is a universal constant (2×10^{-3}). Information gathered since this publication is contradictory, and even Hooge has admitted in a publication in 1976 and comprising 87 references that "his formula may not be adapted for all inhomogeneous cases".[51]

If a current I flows through a resistance, the mean square value of the voltage fluctuation across the resistance is

$$M(u^2) = I^2 M(dR)^2 \tag{4-94}$$

i.e. it is proportional to the current squared. However, if the voltage connected across the resistance is constant then, according to a similar calculation, the mean square fluctuation of the current flowing through the resistance is

$$M(i^2) = U^2 M(dG)^2 \tag{4-95}$$

It is thus seen from Eq. (4-94) that the noise voltage is always generated by the current flowing through the resistance. This is why the $1/f$ noise or excess noise is frequently called current noise. However, it should be emphasized that the current is not noisy but the noise originates from the fluctuation (modulation) noise of the conductance, this being only unfolded by the current flow. (On the other hand, the generation–recombination noise is also quadratically dependent on the current, so this might also be called current noise.) The term "current noise" will therefore not be used in this book.

Of course, it should be proved that the additional noise of a resistance is actually generated by the resistance fluctuation, and that the current flow is only needed to make this fluctuation perceptible according to Eq. (4-94). It could also be possible that the current flow is already responsible for generating the fluctuation, that is the additional noise might be a result of some non-equilibrium process. However, in this case the exponent of the current in Eq. (4-94) would have to be different from two.

The validity of Eq. (4-94) has been investigated in several ways. According to one method adopted the resistance under test is considered as the series and parallel resultant of several components. This decomposition has been realized macroscopically by composing the resistance R with the aid of 1, 4, 9, 16, ... parallel and series elements.[30] The microscopic analogy of this method is the investigation of integrated circuit resistors[55] by altering the length, width, and depth. Both publications have proved the validity of Eq. (4-94).

According to another method, the exciting current I has been changed. Sutcliffe[112, 113] showed that it is relevant whether a DC or an RF current flow is applied. Jones and Francis[60] have utilized simultaneous DC and RF current flow and found a correlation factor of 0.96 ± 0.05 between the noise spectra corresponding to the respective current flows. This proves that the noise is independent of the excitation.

Other RF investigations[80, 81] have given rise to the possibility that the nonlinearity[46, 114] and/or temperature coefficient[59] may also contribute to the generation of flicker noise. On the other hand, the noise components in the frequency ranges $f\pm\Delta f$, $2f\pm\Delta f$, or $3f\pm\Delta f$ have been proved to be uncorrelated,[61] thus speaking against the nonlinearity effect. The controversy about RF investigations[13, 88] will probably last for some time.

Voss and Clarke[137, 138] have performed a basic experiment to prove that the resistance fluctuates even under conditions of thermal equilibrium, so establishing that the fluctuation is not caused by the current flow. They first investigated small volume samples, and measured $\Delta R/R$ in the usual way. Thereafter, they measured thermal noise *without current flow* which has a long time mean square value of

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t U^2\,dt' = 4kTR(f_2 - f_1) \tag{4-96}$$

if the measurement is performed between the frequencies f_1 and f_2. If the fluctuation of R has a $1/f$ spectrum, the same spectrum should also be present at the

left-hand side of the equation by choosing correct measurement intervals. The experiment proved the fluctuation of R. Other workers have obtained similar results.[10]

It is much more difficult to explain the $1/f$ frequency dependence. If the current fluctuation is explained by some kind of generation–recombination process then, according to Eqs. (4-74) and (4-75), an exponential autocorrelation function and a Lorentz power spectrum should result. Both comprise the time constant τ which is regarded as constant.

There exists a continuous distribution of the time constant which results, with good approximation, in a power spectrum of type $1/f$ in both an upper- and lower-bound frequency range. According to McWorther[85] and van der Ziel,[128] this may be conceived by the following approach.

Assume that the direct sources of the flicker noise are also generation–recombination centers which, however, are driven by slow processes caused, for example by oxide contaminations on the semiconductor surface. An exchange of charges between the semiconductor surface and the contaminant may only take place by means of the tunnel effect (see Fig. 4-13). The greater the distance y of the contaminant from the surface the more will be the average time between transitions. Thus

$$\tau = \tau_1 e^{\delta y} \tag{4-97}$$

where τ_1 is a time constant characterizing the tunnel passage, and δ is a constant of the order of $10^8\ \mathrm{cm}^{-1}$.

Let us divide the surface into elements of area ΔA in order to determine the distribution of time constants τ. In this case, a dominant trap within the oxide at a distance y may be assigned for each surface element. Should there be additional traps beyond these ones, these may be neglected because, according to Eq. (4-97), their time constant will increase abruptly.

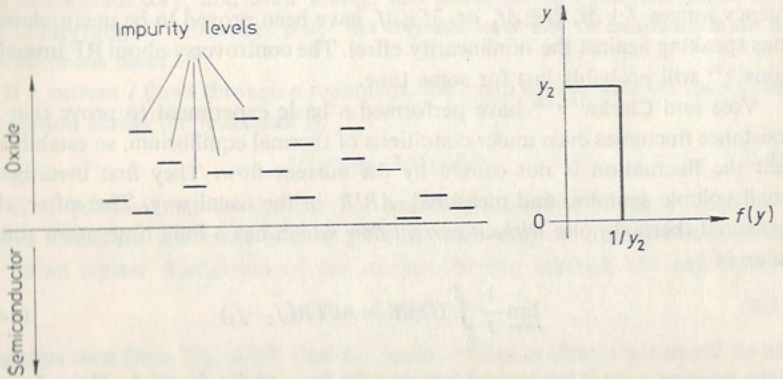


Figure 4-13 Traps in the oxide covering the surface of a semiconductor and their distribution.

Let the distribution of the traps as a function of y be uniform if $0<y<y_2$ (see Fig. 4-13), i.e. let the density function be of the form

$$f(y) = \frac{1}{y_2} \tag{4-98}$$

From this expression, the density function of the time constant, by applying the transformation formula

$$g(\tau) = \frac{dy}{d\tau} f(y) \tag{4-99}$$

will be given, by the expression

$$g(\tau) = \begin{cases} \dfrac{1}{\delta y_2}\dfrac{1}{\tau} = \dfrac{1}{\ln \tau_2/\tau_1}\dfrac{1}{\tau} & \text{if} \quad \tau_1 < \tau < \tau_2 \\ 0 & \text{elsewhere} \end{cases} \tag{4-100}$$

Here $\tau_2 = \tau_1 e^{\delta y_2}$. It is interesting to note that by assuming the tunnel effect the reciprocal-type density function, which is quite unusual, is derived in a natural way.

The overall noise generated by the generation–recombination centers pertaining to the surface elements is determined by integrating with respect to the time constants τ. Thus

$$S(f) = 4\overline{a^2} \int_{\tau_1}^{\tau_2} \frac{\tau}{1+\omega^2\tau^2} \frac{d\tau/\tau}{\ln \tau_2/\tau_1} = \frac{4\overline{a^2}}{\omega \ln \tau_2/\tau_1} \int_{\omega\tau_1}^{\omega\tau_2} \frac{dx}{1+x^2}$$

$$= \frac{4\overline{a^2}}{\omega \ln \tau_2/\tau_1} (\arctan \omega\tau_2 - \arctan \omega\tau_1) \tag{4-101}$$

In the range $1/\tau_2 \ll \omega < 1/\tau_1$,

$$S(f) \cong \frac{\overline{a^2}}{f \ln \tau_2/\tau_1} \tag{4-102}$$

which is the typical $1/f$-type spectrum.

The tunnel effect may not only take place in the above model but also at the grain boundaries of resistances with granular structure, or between contact surfaces. These structures typically reveal flicker noise.

There is a high probability that the passage of electrons through the isolating layer between two metallic layers or conducting granules is made possible by the tunnel effect. This isolator is, however, seldom perfect as it may comprise traps that may be occupied by electrons (see Fig. 4-14). The height and width of the potential barrier thus formed, and so the charge quantity passed by the tunnel effect, are modulated by the occupancy of these traps. However, the occupation or depletion of the trap may only take place by the tunnel effect. Thus the distribution of time intervals between transitions here also corresponds to Eq.

(4-100).[73] According to another theorem, the modulation of the potential barrier is affected by the thermal noise of the isolator loss resistance.[66]

As the spectrum of the flicker noise is omnipresent in differing physical conditions, an attempt has been made to describe the process by purely mathematical means. Halford[44] has found a class of fluctuation phenomena meeting the following boundary conditions: only finite time constants, finite energies, and finite amplitudes are permitted. Even under these conditions, a $1/f$ spectrum of arbitrary width may be present; however, this has a lower boundary region of $\geqq 0$ exponent, and an upper boundary region of $\leqq -2$ exponent. Two other publications have utilized even more extended mathematical abstractions.[77,96]

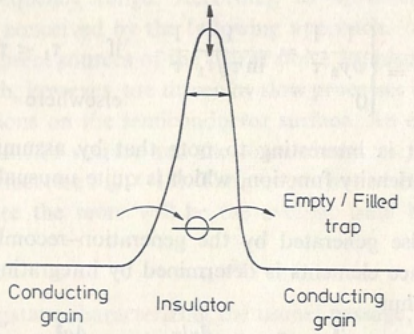


Figure 4-14 Modulation of a barrier formed in a thin insulation layer due to the varying occupancy of the trap within the insulation.

In order to discover the physical process resulting in flicker noise, numerous experiments have been carried out and several models have been created. Of more general significance are the models based on temperature fluctuation,[65,138] on mobility fluctuation,[50] and on the assumption of slow surface conditions.[85] Regarding particular structures, reference is made in the literature to metallic thin layers,[23,24,53,82,139] thick film resistances,[26,99,106,124,145] contacts,[54,121,122,123,125] and granulated structures.[1,5,6,17,45,90]

Careful investigations have revealed another peculiar characteristic of the flicker noise: its power has a substantially higher variance than the power of white noise in the same bandwidth. These investigations have been carried out on carbon resistors,[17,18,19,31,74,86,94] photoconductors,[31,52] semiconductor blocks and p–n junctions,[12,18,19,94] and operational amplifiers.[109] More or less significant deviations have been ascertained from the theoretical relative variances given by Eq. (8-6). In some papers, the discrepancy is explained by the nonstationary nature of the process;[17,31,74,118] in other sources, it is proved that the process is stationary[51,86,107,109] and the discrepancy is explained by the strong effect of small

perturbations.[12,86] It has also been investigated whether the frequency response of the measurement apparatus may be responsible for the discrepancy.[14,99,108] Recently the role of temperature fluctuations has also been considered.[97]

By investigating only the statistical properties of the process, the following explanations may be suggested:

1. The amplitude distribution is not gaussian.
2. The samples may not be regarded as uncorrelated.
3. The process is not stationary.

The first assumption has been disproved by many experimental results. The third assumption can neither be proved nor disproved definitely. It thus seems that a careful investigation of the second assumption may be useful.

In order to avoid analytical difficulties, a process lacking a $1/f$-type spectrum will be investigated. Fuller[37] has shown that the variance of the autocorrelation function pertaining to a process with finite length is strongly dependent on the internal characteristics of that process. For instance, consider a gaussian process of time interval T and autocorrelation function

$$\Gamma(\tau) = e^{-\lambda|\tau|} \tag{4-103}$$

The variance in this autocorrelation function is

$$\sigma_G^2(\tau) \cong \begin{cases} \dfrac{1}{2\lambda T}\left[1-\dfrac{\tau}{2T}+(1+2\lambda\tau)e^{(-2\lambda\tau)}\right] & \text{if} \quad 0 \leqq \tau \leqq 2T \\ 0 & \text{elsewhere} \end{cases} \tag{4-104}$$

provided that $\lambda T \gg 1$.

Substituting $\tau=0$ we get the variance of the noise power which is given here by $\sigma_G^2(0)=1/\lambda T$. However, there may be a process that is also stationary and ergodic, but nevertheless not gaussian, in spite of the fact that it has a normal amplitude distribution. This process is illustrated by the Gauss–Poisson square wave (see Fig. 4-15). A Poisson square wave is a signal that has a level changing according to a Poisson distribution in a discrete way along the time axis; accordingly, the distribution of time intervals between changeovers is exponential. The variance of the autocorrelation function at $\tau=0$ is

$$\sigma_{GP}^2(0) = \frac{2}{\lambda T} = 2\sigma_G^2(0) \tag{4-105}$$

The above results were first applied to high-variance noises by the author utilizing computer simulation for experimental verification.[2] It can be seen that the variance of a non-gaussian process is much higher compared with that of a pure gaussian process.

High variances have always been observed experimentally for $1/f$-type noises, so the above considerations have had to be further developed for this case too.

The analytical approximation may not be applied to this case as the autocorrelation function of the Poisson process is a simple exponential expression, while that of the flicker noise is an integral exponential or an even more complicated function, depending on the response of the band-limiting filter.[30] However, computer simulation can easily be applied. The random numbers with uniform distribution R_i are transformed by the transformation

$$t_i = \tau_1 \left(\frac{\tau_2}{\tau_1}\right)^{R_i} \tag{4-106}$$

into random numbers with reciprocal distribution as given by Eq. (4-100) [τ_1 and τ_2 correspond to Eq. (4-100)]. By simulating the stochastic square-wave signal according to Fig. 4-15 with such parameters and by determining the variance of the powers (mean square values), we arrived at the result[3]

$$\sigma_{GF}^2 = (5\ldots6)\sigma_G^2 \tag{4-107}$$

which explains the increased variance.

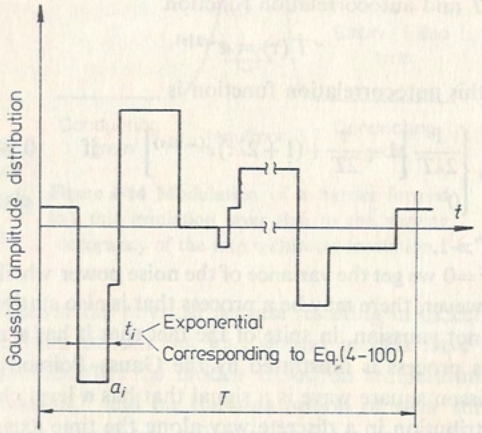


Figure 4-15 Square wave with gaussian amplitude distribution and Poisson or 1/*f*-type changeover distribution.

Even more interesting is the distribution of the mean square values which, for a gaussian process and a high number of results, should also be gaussian. On the other hand, Fig. 4-16 shows a skew distribution as the result of the simulation. For comparison, the dotted line shows the distribution of the mean square values of a gaussian process having the same instantaneous values.[4] The skew distribution determined for the high-variance process should be compared with the histogram evaluating the experiments of Greenstein and Brophy.[39]

Experience shows that high-variance noise is generated if the fluctuations are the result of a series of slow effects, each effect being substantial in itself. These effects are higher than the shot-effect pulses by several orders of magnitude. Based on this observation, Fig. 4-17 shows a new physical model of a resistor generating additional noise comprising pulse noise too. Conductance modulation is given by a series of elementary, although not necessarily equal, conductances

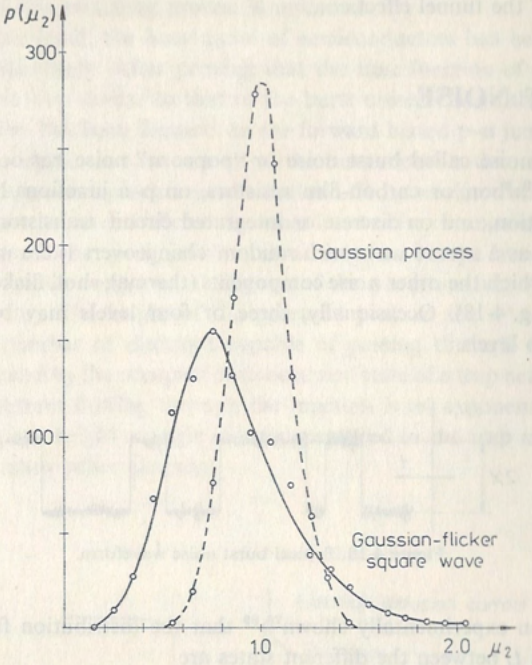


Figure 4-16 Mean square distributions.

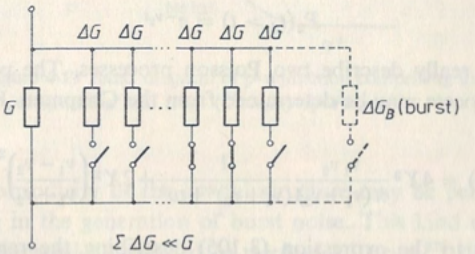


Figure 4-17 Model of resistance with additional noise, comprising a possible burst component.

that are switched in and out at random; more than one switch may change its position at a given instant.

If the ΔG values are equal then their instantaneous sum has a binomial distribution.[127] This tends to a normal distribution for a sufficiently high number of branches, thus giving a fluctuation in the overall resistance value according to the normal distribution. The time distribution of the change of states may be a distribution with a $1/f$ spectrum if the switches are driven by the above mechanism of the tunnel effect.

4-7 BURST NOISE

A pulse-type noise called burst noise or "popcorn" noise has occasionally been observed on carbon or carbon-film resistors, on p–n junctions biased into cut-off or conduction, and on discrete or integrated circuit transistors.[11,22,38,72] This noise appears as a square wave with random changeovers (i.e. random telegraph signal) onto which the other noise components (thermal, shot, flicker) are superimposed (see Fig. 4-18). Occasionally, three or four levels may be distinguished instead of two levels.

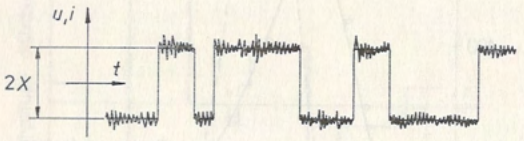


Figure 4-18 Typical burst noise waveform.

It has been experimentally shown[78,89] that the distribution functions of the time intervals t_d between the different states are

$$P_1(t_d > t) = e^{-\nu_1 t}$$
$$P_2(t_d > t) = e^{-\nu_2 t} \tag{4-108}$$

These functions really describe two Poisson processes. The power spectrum of the stochastic process may be determined from the Chapman–Kolmogorov equation[93]

$$s(\omega) = 4X^2 \frac{\nu_1 \nu_2}{\pi(\nu_1+\nu_2)} \frac{1}{(\nu_1+\nu_2)^2+\omega^2} + 2X^2 \left(\frac{\nu_1-\nu_2}{\nu_1+\nu_2}\right)^2 \delta(\omega) \tag{4-109}$$

which is similar to the expression (3-105) describing the random symmetrical telegraph signal. The term comprising the Dirac pulse represents the DC component due to the asymmetry.

Burst noise is much less frequently observed than flicker noise, appearing generally in passive or active devices of imperfect structure. Its analysis led to several theories.[92]

Publications on burst noise in resistors are relatively rare[11,63,70,86] and are rather descriptive in nature. According to our present knowledge, the overall conductance of a resistor revealing burst noise is determined by some two-state mechanism, e.g., a parallel resistor bridge that is switched on and off. The physical background of this switching process is unknown.

On the other hand, the burst noise of semiconductors has been investigated much more extensively. After proving that the time function of one avalanche-noise variant is very similar to that of the burst noise[56,67] and thus may be separated, attention has been focused on the forward biased p–n junctions.[15,57,68,69] All conclusions agree that the source of the burst noise is due to the imperfect nature of the semiconductor's crystal structure. Some workers[76,79] claim that the presence of dislocations is sufficient to cause burst noise; in another publication,[58] a precipitation in the p–n junction is assumed to be the burst-noise source. Figure 4-19 shows the energy–band diagram of the p semiconductor–metal–n semiconductor system. The height of the potential peak at the n^+–metal junction, and thus the number of electrons capable of passing through the junction, is strongly influenced by the occupied or unoccupied state of a trap near the precipitation. As the current flowing through the junction is an exponential function of the potential peak height a single electron captured in the trap may control the movement of many other electrons.

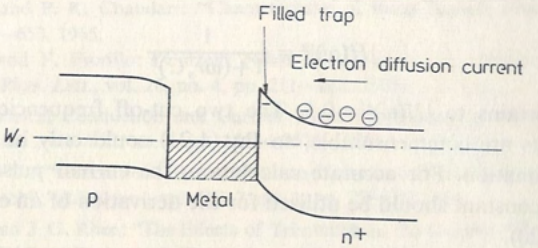


Figure 4-19 Band diagram of p–n junction comprising metal precipitation and trap. (*Courtesy of Pergamon Press.*)

The perfect periodicity of the crystal structure may be perturbed by further factors, resulting in the generation of burst noise. This kind of burst noise has been observed on samples subjected to mechanical stress,[101] dopant diffusion of high concentration,[79] thermal shock,[15] or UV irradiation.[100] The dislocation may also generate burst noise due to the mechanical stress produced by it.[32]

Burst noise was generated in a high percentage of analog integrated circuits produced in the late 1960s—a phase of rapid expansion but immature technology. This high incidence has lately been reduced to a fifth of its former value[102] using a technology that has substantially reduced the formation of metal precipitation.

PROBLEM

4-1 Assume a reverse biased semiconductor diode with a depletion layer having a width of 5 μm. The minority charge carriers pass through this depletion layer at a nearly constant velocity of 10^5 m/s. The capacity of the biased junction is 10 pF, the series diode resistance is 2 Ω. Assume an external loading circuit which is short circuited for AC. What is the frequency at which the power spectrum of the shot-noise current flowing through this circuit is decreased to half of its low-frequency value?

Answer: According to Eq. (4-20), the decrease in the power density depending on the transit time is

$$F_f^2 = \left(\frac{\sin \omega t_f/2}{\omega t_j/2}\right)^2$$

$\omega t_f/2 \cong 1.4$ pertainsto $F_f^2 = 0.5$. As $t_f = 5\times10^{-6}\,\mathrm{m}/10^5\,\mathrm{m\,s^{-1}} = 5\times10^{-11}$ s, and $f = 8.9$ GHz. Neglecting the effect of the transit time, the impedance formed by the diode capacity and the series resistance also has the effect of decreasing the power density, according to Eq. (3-114). Thus

$$|H(\omega)|^2 = \frac{1}{1+(\omega r_s C)^2}$$

$f = 8$ GHz pertains to $|H(\omega)|^2 = 0.5$. The two cut-off frequencies due to two different effects are commensurable, so Eq. (4-21) could only be applicable for gross approximation. For accurate calculation, the current pulses distorted by the RC time constant should be utilized for the derivation of an expression similar to Eq. (4-20).

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CHAPTER

FIVE

NOISE PARAMETERS OF LINEAR NETWORKS

The noise levels due to physical processes treated in the previous Chapter are relatively low. At these levels, the electronic devices and passive network elements with good approximation may be regarded as linear. The classic mathematical methods for establishing linear network models and equivalent circuits are well known so it seems justified to apply these methods also to noisy networks.

5-1 ONE-PORTS

In the course of our investigations of the physical aspects of noise, our end results have been expressed in terms of voltage squared, current squared, or power density. The small-signal equivalent circuit of the component or device in question is made up of resistive and reactive components, so a noise equivalent circuit may be devised by including additional noise current or noise voltage generators.

Figure 5-1 shows the noise equivalent circuit of a two-pole having conductance G. If our investigations are restricted to a frequency band of width Δf and $hf \ll kT$, the mean square value of the thermal noise current is given, according to Eqs. (3-41) and (4-51), by

$$\overline{i^2} = \int_f^{f+\Delta f} S_i(f)\,df = 4kTG\Delta f \tag{5-1}$$

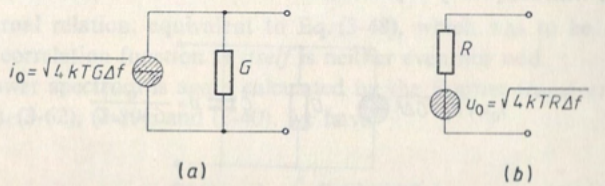

Figure 5-1 Real conductance noise equivalent circuits. *(a)* Norton. *(b)* Thévenin.