1. Representation of Qubit Hamiltonian

$$H = -\frac{\omega_q}{2}\sigma_z + \frac{\varepsilon}{2}\sigma_x + \frac{A_{rf}}{2}\cos(\omega_{rf}t+\phi)\,\sigma_x$$

$$= -\frac{\sqrt{\omega_q^2+\varepsilon^2}}{2}\left(\underbrace{\frac{\omega_q}{\sqrt{\omega_q^2+\varepsilon^2}}}_{\cos\theta}\sigma_z - \underbrace{\frac{\varepsilon}{\sqrt{\omega_q^2+\varepsilon^2}}}_{\sin\theta}\sigma_x\right) + \frac{A_{rf}}{2}\sigma_x\cos(\omega_{rf}t+\phi)$$

$$\begin{bmatrix} -\cos\theta & -\sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \Rightarrow \begin{vmatrix} -\cos\theta-\lambda & -\sin\theta \\ -\sin\theta & \cos\theta-\lambda \end{vmatrix} = \lambda^2 - \cos^2\theta - \sin^2\theta = \lambda^2 - 1 = 0 \Rightarrow \lambda = \pm 1$$

$$\lambda = 1 \quad \begin{bmatrix} -\cos\theta-1 & -\sin\theta \\ -\sin\theta & \cos\theta-1 \end{bmatrix} \Rightarrow \frac{1}{\sqrt{2+2\cos\theta}}\begin{bmatrix} -\sin\theta \\ \cos\theta+1 \end{bmatrix} = \frac{1}{2\cos(\frac{\theta}{2})}\begin{bmatrix} -\sin\theta \\ \cos\theta+1 \end{bmatrix}$$

$$\lambda = -1 \quad \begin{bmatrix} -\cos\theta+1 & -\sin\theta \\ -\sin\theta & \cos\theta+1 \end{bmatrix} \Rightarrow \frac{1}{\sqrt{2-2\cos\theta}}\begin{bmatrix} -\sin\theta \\ \cos\theta-1 \end{bmatrix} = \frac{1}{2\sin(\frac{\theta}{2})}\begin{bmatrix} -\sin\theta \\ \cos\theta-1 \end{bmatrix}$$

$$R = e^{-i\frac{\omega_{rf}t}{2}\underbrace{(\cos\theta\sigma_z - \sin\theta\sigma_x)}_{A}} \qquad R = e^{-i\frac{\omega_{rf}t}{2}A} = 1 - i\frac{\omega_{rf}}{2}tA + \frac{1}{2!}\left(-i\frac{\omega_{rf}}{2}tA\right)^2 + \cdots$$

$$\dot{R} = -i\frac{\omega_{rf}}{2}A\left(1 - i\frac{\omega_{rf}}{2}tA + \cdots\right) = -i\frac{\omega_{rf}}{2}A\,e^{-i\frac{\omega_{rf}}{2}tA}$$

$$\Rightarrow i\dot{R}R^\dagger = \frac{\omega_{rf}}{2}(\cos\theta\,\sigma_z - \sin\theta\,\sigma_x)$$

$$e^{-i\frac{\omega_{rf}}{2}t(\cos\theta\sigma_z - \sin\theta\sigma_x)}\,\sigma_x\,e^{i\frac{\omega_{rf}}{2}t(\cos\theta\sigma_z - \sin\theta\sigma_x)}$$

$$= \sigma_x + \left(-i\frac{\omega_{rf}t}{2}\right)2i\cos\theta\,\sigma_y + \frac{1}{2!}\left(-i\frac{\omega_{rf}t}{2}\right)^2(2i)^2(-\cos^2\theta\,\sigma_x - \sin\theta\cos\theta\,\sigma_z)$$

$$+\frac{1}{3!}\left(-i\frac{\omega_{rf}t}{2}\right)^3(2i)^3(-\cos\theta\,\sigma_y) + \frac{1}{4!}\left(-i\frac{\omega_{rf}t}{2}\right)^4(2i)^4(\cos^2\theta\,\sigma_x + \sin\theta\cos\theta\,\sigma_z)$$

$$= \sigma_x + \cos\theta\,\sigma_y\left(\omega_{rf}t - \frac{1}{3!}(\omega_{rf}t)^3 + \cdots\right)$$

$$-\cos^2\theta\,\sigma_x - \sin\theta\cos\theta\,\sigma_z + (\cos^2\theta\,\sigma_x + \sin\theta\cos\theta\,\sigma_z)\left(1 - \frac{1}{2!}(\omega_{rf}t)^2 + \cdots\right)$$

$$= \sin^2\theta\,\sigma_x - \sin\theta\cos\theta\,\sigma_z + \cos\theta\,\sigma_y\sin(\omega_{rf}t)$$

$$+ (\cos^2\theta\,\sigma_x + \sin\theta\cos\theta\,\sigma_z)\cos(\omega_{rf}t)$$

$$e^{B}Ae^{-B} = A + [B,A] + \frac{1}{2!}[B,[B,A]] + \cdots$$

$$[\cos\theta\sigma_z - \sin\theta\sigma_x, \sigma_x] = 2i\cos\theta\,\sigma_y$$

$$[\cos\theta\sigma_z - \sin\theta\sigma_x, \cos\theta\sigma_y] = (2i)^2\cos^2\theta\,\sigma_x - \sin\theta\cos\theta\,\sigma_z$$

$$[\cos\theta\sigma_z - \sin\theta\sigma_x, -\cos^2\theta\sigma_x - \sin\theta\cos\theta\sigma_z]$$

$$= (2i)^3(\cos^3\theta\,\sigma_y - \sin^2\theta\cos\theta\,\sigma_y) = (2i)^3(-\cos\theta\,\sigma_y)$$

$$\vdots$$

$$\tilde{H} = RHR^\dagger + i\dot{R}R^\dagger = -\frac{\sqrt{\omega_q^2+\varepsilon^2}-\omega_{rf}}{2}(\cos\theta\sigma_z - \sin\theta\sigma_x) + \frac{A_{rf}}{2}\frac{e^{i(\omega_{rf}t+\phi)}+e^{-i(\omega_{rf}t+\phi)}}{2}\,\frac{e^{i\omega_{rf}t}-e^{-i\omega_{rf}t}}{2i}\cos\theta\,\sigma_y$$

$$+\frac{A_{rf}}{2}\frac{e^{i(\omega_{rf}t+\phi)}+e^{-i(\omega_{rf}t+\phi)}}{2}\,\frac{e^{i\omega_{rf}t}+e^{-i\omega_{rf}t}}{2}(\cos^2\theta\,\sigma_x + \sin\theta\cos\theta\,\sigma_z)$$

$$= -\frac{\sqrt{\omega_q^2+\varepsilon^2}-\omega_{rf}}{2}(\cos\theta\sigma_z - \sin\theta\sigma_x) + \frac{A_{rf}}{2}\cos\theta\,\frac{1}{2}\Big[-\sin\phi\,\sigma_y + \cos\phi\,(\sin\theta\sigma_z + \cos\theta\sigma_x)\Big]$$

2. General Perturbation expansion of Hamiltonian

Hamiltonian absence of perturbation : $H = -\frac{1}{2}\vec{H}_0(\lambda_0)\cdot\vec{\sigma}$

corresponds to the static values of flux, charge ... etc

perturbation $\delta\lambda$.

$$H = -\frac{1}{2}\left[\vec{H}_0(\lambda_0) + \frac{\partial \vec{H}_0}{\partial\lambda}\delta\lambda + \frac{\partial^2 \vec{H}_0}{\partial\lambda^2}\frac{\delta\lambda^2}{2} + \cdots\right]\cdot\vec{\sigma}$$

$$= -\frac{1}{2}\vec{H}_0(\lambda_0)\cdot\vec{\sigma} - \frac{1}{2}\left(\frac{\partial \vec{H}_0}{\partial\lambda}\delta\lambda + \frac{\partial^2 \vec{H}_0}{\partial\lambda^2}\frac{\delta\lambda^2}{2}\right)\cdot\vec{\sigma}$$

3. Bloch-Redfield Approach

relaxation process characterized by two rates $\begin{cases}\Gamma_1 = \frac{1}{T_1} \\ \Gamma_2 = \frac{1}{T_2} = \frac{\Gamma_1}{2} + \Gamma_\varphi\end{cases}$

example ①

$$H = -\frac{\omega_q}{2}\sigma_z \qquad L = \sqrt{\Gamma}\sigma_x$$

$$[H,\rho] = -\frac{\omega_q}{2}[\sigma_z,\rho] = -\frac{\omega_q}{2}(\sigma_z\rho - \rho\sigma_z)$$

$$\dot\rho = i\frac{\omega_q}{2}(\sigma_z\rho - \rho\sigma_z) + \Gamma(\sigma_x\rho\sigma_x - \rho)$$

$$\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}\rho_{00} & \rho_{01}\\ \rho_{10} & \rho_{11}\end{bmatrix} - \begin{bmatrix}\rho_{00} & \rho_{01}\\ \rho_{10} & \rho_{11}\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix} = \begin{bmatrix}\rho_{00} & \rho_{01}\\ -\rho_{10} & -\rho_{11}\end{bmatrix} - \begin{bmatrix}\rho_{00} & -\rho_{01}\\ \rho_{10} & -\rho_{11}\end{bmatrix} = \begin{bmatrix}0 & 2\rho_{01}\\ -2\rho_{10} & 0\end{bmatrix}$$

$$\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}\rho_{00} & \rho_{01}\\ \rho_{10} & \rho_{11}\end{bmatrix}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix} = \begin{bmatrix}\rho_{10} & \rho_{11}\\ \rho_{00} & \rho_{01}\end{bmatrix}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix} = \begin{bmatrix}\rho_{11} & \rho_{10}\\ \rho_{01} & \rho_{00}\end{bmatrix}$$

$$\begin{bmatrix}\dot\rho_{00} & \dot\rho_{01}\\ \dot\rho_{10} & \dot\rho_{11}\end{bmatrix} = i\omega_q\begin{bmatrix}0 & \rho_{01}\\ -\rho_{10} & 0\end{bmatrix} + \Gamma\begin{bmatrix}\rho_{11}-\rho_{00} & \rho_{10}-\rho_{01}\\ \rho_{01}-\rho_{10} & \rho_{00}-\rho_{11}\end{bmatrix}$$

$$\dot\rho_{00} = -\Gamma(\rho_{00}-\rho_{11})$$
$$\dot\rho_{11} = \Gamma(\rho_{00}-\rho_{11})$$
$$\dot\rho_{01} = (i\omega_q - \Gamma)\rho_{01} + \Gamma\rho_{10}$$
$$\dot\rho_{10} = (-i\omega_q - \Gamma)\rho_{10} + \Gamma\rho_{01}$$

Initial condition $\rho = \begin{bmatrix}\alpha\\ \beta\end{bmatrix}(\alpha^* \ \beta^*) = \begin{bmatrix}|\alpha|^2 & \alpha\beta^*\\ \alpha^*\beta & |\beta|^2\end{bmatrix}$

$$\rho_{00} + \rho_{11} = |\alpha|^2 + |\beta|^2 = 1$$

$$\rho_{11} = 1 - \rho_{00}, \quad \rho_{00} - \rho_{11} = \rho_{00} - 1 + \rho_{00} = 2\rho_{00} - 1 = 1 - 2\rho_{11}$$

$$\dot\rho_{00} = -2\Gamma\rho_{00} + \Gamma = -2\Gamma\left(\rho_{00} - \tfrac{1}{2}\right) \Rightarrow \rho_{00} = \frac{1}{2} + \left(|\alpha|^2 - \tfrac{1}{2}\right)e^{-2\Gamma t}$$

$$\dot\rho_{11} = -2\Gamma\rho_{11} + \Gamma = -2\Gamma\left(\rho_{11} - \tfrac{1}{2}\right) \Rightarrow \rho_{11} = \frac{1}{2} + \left(|\beta|^2 - \tfrac{1}{2}\right)e^{-2\Gamma t}$$

$$(\rho_{01} + \rho_{10})' = i\omega_q(\rho_{01} - \rho_{10}) \qquad A' = i\omega_q B$$

$$(\rho_{01} - \rho_{10})' = i\omega_q(\rho_{01} + \rho_{10}) - 2\Gamma(\rho_{01} - \rho_{10}) \qquad B' = i\omega_q A - 2\Gamma B$$

$$B'' = -2\Gamma B' - \omega_q^2 B, \quad B'' + 2\Gamma B' + \omega_q^2 B = 0$$

$$\Rightarrow -\Gamma \pm \sqrt{\Gamma^2 - \omega_q^2} = -\Gamma \pm i\sqrt{\omega_q^2 - \Gamma^2}$$

$$B = e^{-\Gamma t}\left(c_1 e^{i\omega t} + c_2 e^{-i\omega t}\right)$$

$$A = \frac{1}{i\omega_g}\left(\beta' + 2\Gamma\beta\right) = \frac{1}{i\omega_g}\left((+\Gamma + i\Delta\omega)C_1 e^{(\Gamma+i\Delta\omega)t} + (+\Gamma - i\Delta\omega)C_2 e^{(\Gamma - i\Delta\omega)t}\right)$$

$$\frac{\Gamma + i\Delta\omega}{i\omega_g}C_1 + \frac{\Gamma - i\Delta\omega}{i\omega_g}C_2 = \alpha\beta^* + \alpha^*\beta \,, \quad \frac{\Delta\omega}{\omega_g}(C_1 - C_2) - i\frac{\Gamma}{\omega_g}(C_1 + C_2) = \alpha\beta^* + \alpha^*\beta = 2\,\mathrm{Re}(\alpha\beta^*)$$

$$C_1 + C_2 = \alpha\beta^* - \alpha^*\beta$$

$$C_1 + C_2 = \alpha\beta^* - \alpha^*\beta = 2i\,\mathrm{Im}(\alpha\beta^*)$$

$$\Delta\omega(C_1 - C_2) + 2\Gamma\,\mathrm{Im}(\alpha\beta^*) = 2\omega_g\,\mathrm{Re}(\alpha\beta^*)$$

$$\rho_{01} = \alpha\beta^* e^{i\Delta\omega t} e^{-\Gamma t}$$

$$\rho_{10} = \alpha\beta^* e^{-i\Delta\omega t} e^{-\Gamma t}$$

$$\boxed{\Gamma_1 = 2\Gamma \quad \Gamma_2 = \frac{\Gamma_1}{2} = \Gamma \qquad \rho = \begin{pmatrix} \frac{1}{2} + \left(|\alpha|^2 - \frac{1}{2}\right)e^{-\Gamma_1 t} & \alpha\beta^* e^{i\Delta\omega t} e^{-\frac{\Gamma_1}{2}t} \\ \alpha\beta^* e^{-i\Delta\omega t} e^{-\frac{\Gamma_1}{2}t} & \frac{1}{2} + \left(|\beta|^2 - \frac{1}{2}\right)e^{-\Gamma_1 t} \end{pmatrix}}$$

example ②

$$H = -\frac{\omega_g}{2}\sigma_z \qquad L = \sqrt{\Gamma}\,\sigma_z$$

$$[H,\rho] = -\frac{\omega_g}{2}(\sigma_z\rho - \rho\sigma_z) \qquad \dot\rho = i\frac{\omega_g}{2}(\sigma_z\rho - \rho\sigma_z) + \Gamma(\sigma_z\rho\sigma_z - \rho)$$

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} \rho_{00} & \rho_{01} \\ -\rho_{10} & -\rho_{11} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} \rho_{00} & -\rho_{01} \\ -\rho_{10} & \rho_{11} \end{bmatrix}$$

$$\begin{bmatrix} \dot\rho_{00} & \dot\rho_{01} \\ \dot\rho_{10} & \dot\rho_{11} \end{bmatrix} = i\omega_g\begin{bmatrix} 0 & \rho_{01} \\ -\rho_{10} & 0 \end{bmatrix} - 2\Gamma\begin{bmatrix} 0 & \rho_{01} \\ \rho_{10} & 0 \end{bmatrix}$$

$$\rho_{00} = |\alpha|^2 \qquad \rho_{11} = |\beta|^2$$

$$\dot\rho_{01} = (i\omega_g - 2\Gamma)\rho_{01}$$

$$\dot\rho_{10} = (-i\omega_g - 2\Gamma)\rho_{10}$$

$$\boxed{\Gamma_1 = 0 \quad \Gamma_2 = \Gamma_\varphi = 2\Gamma \qquad \rho = \begin{pmatrix} |\alpha|^2 & \alpha\beta^* e^{i\Delta\omega t} e^{-\Gamma_\varphi t} \\ \alpha^*\beta\, e^{-i\Delta\omega t} e^{-\Gamma_\varphi t} & |\beta|^2 \end{pmatrix}}$$

Bloch – Redfield Approach

$$\boxed{\Gamma_1 = \frac{1}{T_1} \quad \Gamma_2 = \frac{1}{T_2} = \frac{\Gamma_1}{2} + \Gamma_\varphi \qquad \rho_{BR} = \begin{pmatrix} 1 + (|\alpha|^2 - 1)e^{-\Gamma_1 t} & \alpha\beta^* e^{i\Delta\omega t} e^{-\Gamma_2 t} \\ \alpha^*\beta\, e^{-i\Delta\omega t} e^{-\Gamma_2 t} & |\beta|^2 e^{-\Gamma_1 t} \end{pmatrix}}$$

dephasing is induced by broadband fluctuation.

Noise spectra is Lorentzian.

(Bloch - Redfield approach can only describe Lorentzian noise)

4. Decoupling