Lec1. Introduction to QuTiP

Composite Systems

$$\sigma_z \otimes 1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \qquad |1\rangle \otimes |0\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$1 \otimes \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & -1 \end{bmatrix} \qquad |0\rangle \otimes |1\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

Coupled 2-qubit Hamiltonian

$$\hat{H} = \frac{\varepsilon_1}{2} \sigma_z^{(1)} + \frac{\varepsilon_2}{2} \sigma_z^{(2)} + g\, \sigma_x^{(1)} \sigma_x^{(2)}$$

$$= \frac{\varepsilon_1}{2} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{\varepsilon_2}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + g \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$= \frac{\varepsilon_1}{2} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{bmatrix} + \frac{\varepsilon_2}{2} \begin{bmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & -1 \end{bmatrix} + g \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix} = \begin{bmatrix} \frac{\varepsilon_1+\varepsilon_2}{2} & & & g \\ & \frac{\varepsilon_1-\varepsilon_2}{2} & g & \\ & g & \frac{-\varepsilon_1+\varepsilon_2}{2} & \\ g & & & \frac{-\varepsilon_1-\varepsilon_2}{2} \end{bmatrix}$$

Jaynes-Cummings Hamiltonian

$$\hat{H} = \omega_c a^\dagger a - \frac{\omega_a}{2} \sigma_z + g(a \sigma_+ + a^\dagger \sigma_-)$$

$$a^\dagger |n\rangle = \sqrt{n+1}\,|n+1\rangle \qquad \sigma_+ = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

$$a|n\rangle = \sqrt{n}\,|n-1\rangle \qquad \sigma_- = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$a^\dagger a |n\rangle = n|n\rangle,$$

$$a^\dagger a = \sum_{n=0}^{\infty} n |n\rangle\langle n|$$

$$\hat{H} = \omega_c \begin{bmatrix} 0 & & & \\ & 1 & & \\ & & 2 & \\ & & & \ddots \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \frac{\omega_a}{2} \begin{bmatrix} 1 & & \\ & 1 & \\ & & \ddots \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + g \left(\begin{bmatrix} 0 & & & \\ \sqrt{1} & 0 & & \\ & \sqrt{2} & \ddots & \\ & & \sqrt{3} & \ddots \end{bmatrix} \otimes \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & \sqrt{1} & & \\ & 0 & \sqrt{2} & \\ & & \ddots & \ddots \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right)$$

$$\hat{H} = \omega_c \left(\sum_{n=0}^{\infty} n |n\rangle\langle n| \right) \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \frac{\omega_a}{2} I \otimes \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + g \left(\left(\sum_{n=1}^{\infty} \sqrt{n} |n-1\rangle\langle n| \right) \otimes \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + \left(\sum_{n=0}^{\infty} \sqrt{n+1} |n+1\rangle\langle n| \right) \otimes \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right)$$

$$\hat{H}|n,0\rangle = \omega_c n |n,0\rangle - \frac{\omega_a}{2} |n,0\rangle + g\sqrt{n}\,|n-1,1\rangle = \left(n\omega_c - \frac{\omega_a}{2}\right)|n,0\rangle + g\sqrt{n}\,|n-1,1\rangle$$

$$\hat{H}|n,1\rangle = \omega_c n |n,1\rangle + \frac{\omega_a}{2} |n,1\rangle + g\sqrt{n+1}\,|n+1,0\rangle = \left(n\omega_c + \frac{\omega_a}{2}\right)|n,1\rangle + g\sqrt{n+1}\,|n+1,0\rangle$$

$$\hat{H}|n+1,0\rangle = \left((n+1)\omega_c - \frac{\omega_a}{2}\right)|n+1,0\rangle + g\sqrt{n+1}\,|n,1\rangle$$

$$\Rightarrow \hat{H} \begin{bmatrix} |n,1\rangle \\ |n+1,0\rangle \end{bmatrix} = \begin{bmatrix} n\omega_c + \frac{\omega_a}{2} & g\sqrt{n+1} \\ g\sqrt{n+1} & (n+1)\omega_c - \frac{\omega_a}{2} \end{bmatrix} \begin{bmatrix} |n,1\rangle \\ |n+1,0\rangle \end{bmatrix} \Rightarrow \left(n\omega_c + \frac{\omega_a}{2} - \lambda\right)\left((n+1)\omega_c - \frac{\omega_a}{2} - \lambda\right) - g^2(n+1) = 0$$

$$\lambda^2 - (2n+1)\omega_c \lambda + \left(n\omega_c + \frac{\omega_a}{2}\right)\left((n+1)\omega_c - \frac{\omega_a}{2}\right) - g^2(n+1) = 0$$

$$\lambda = \frac{(2n+1)\omega_c \pm \sqrt{(2n+1)^2\omega_c^2 - 4\left(n\omega_c + \frac{\omega_a}{2}\right)\left((n+1)\omega_c - \frac{\omega_a}{2}\right) + 4g^2(n+1)}}{2} = \frac{(2n+1)\omega_c \pm \sqrt{(\omega_c - \omega_a)^2 + 4g^2(n+1)}}{2} = \left(n+\frac{1}{2}\right)\omega_c \pm \sqrt{\left(\frac{\omega_c - \omega_a}{2}\right)^2 + g^2(n+1)}$$

eigen energy

(S-picture): [illegible] H-picture, Interaction picture 에서도 [illegible].

Unitary Evolution

Mesolve : Master equation solve

$$i\hbar \frac{\partial|\psi\rangle}{\partial t} = \hat{H}|\psi\rangle \Rightarrow \frac{\partial|\psi\rangle}{\partial t} = -\frac{i}{\hbar}\hat{H}|\psi\rangle \quad |\psi\rangle = |\psi(0)\rangle \cdot e^{-\frac{i\hat{H}t}{\hbar}}$$

$$\langle A\rangle = tr(\rho A)$$

$$\hat{H} = \frac{\omega_a}{2}\sigma_x \Rightarrow A(t) = U^\dagger A U = e^{iHt} A e^{-iHt} = e^{\frac{i\omega_a t}{2}\sigma_x} A e^{-\frac{i\omega_a t}{2}\sigma_x} \quad B = \frac{i\omega_a t}{2}\sigma_x$$

$$e^B A e^{-B} = A + [B,A] + \frac{1}{2!}[B,[B,A]] + \frac{1}{3!}[B,[B,[B,A]]] + \cdots$$

(i) $A = \sigma_x$. $[B,A] = 0 \Rightarrow A(t) = A$. $\langle A\rangle = \langle A(t)\rangle = [0\ 1]\begin{bmatrix} 0 & 1 \\ 1 & 0\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix} = 0$

(ii) $A = \sigma_y$ $[B,A] = \frac{i\omega_a t}{2}[\sigma_x, \sigma_y] = -\omega_a t\,\sigma_z$

$$[B,[B,A]] = \frac{i\omega_a t}{2}(-\omega_a t)[\sigma_x, \sigma_z] = (\omega_a t)^2\sigma_y \quad \cdots$$

$$e^B A e^{-B} = \sigma_y - (\omega_a t)\sigma_z - \frac{(\omega_a t)^2}{2!}\sigma_y + \cdots = \cos(\omega_a t)\sigma_y - \sin(\omega_a t)\sigma_z$$

(iii) $A = \sigma_z$. $[B,A] = \frac{i\omega_a t}{2}[\sigma_x, \sigma_z] = \omega_a t\,\sigma_y$, $[B,[B,A]] = -(\omega_a t)^2\sigma_z$, $\cdots$

$$e^B A e^{-B} = \sigma_z + (\omega_a t)\sigma_y - \frac{(\omega_a t)^2}{2!}\sigma_z - \frac{(\omega_a t)^3}{3!}\sigma_y + \cdots = \cos(\omega_a t)\sigma_z + \sin(\omega_a t)\sigma_y$$

$$\langle\sigma_y(0)\rangle = [0\ 1]\begin{bmatrix}0 & -i\\ i & 0\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix} = [0\ 1]\begin{bmatrix}-i\\0\end{bmatrix} = 0 \quad \langle\sigma_z(0)\rangle = [0\ 1]\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix} = -1$$

$$\Rightarrow \langle\sigma_y(t)\rangle = \sin(\omega_a t) \qquad \langle\sigma_z(t)\rangle = -\cos(\omega_a t)$$

Master Equation

$$\dot{\rho} = -\frac{i}{\hbar}[H,\rho] + \gamma\left(L\rho L^\dagger - \frac{1}{2}(L^\dagger L\rho + \rho L^\dagger L)\right)$$

Homework 1

problem 1 - Numerical representation of the transmon qubit

Hamiltonian of a charge qubit. $\hat{H} = 4E_c(\hat{n} - n_g)^2 - E_J\cos\hat{\varphi}$

0 1
1 0 1
1 0 1
1 0 1
1 0

$$e^{i\hat{\varphi}}|n\rangle = |n-1\rangle \qquad \cos\hat{\varphi} = \frac{e^{i\hat{\varphi}} + e^{-i\hat{\varphi}}}{2} = \Sigma\frac{|n+1\rangle\langle n| + |n\rangle\langle n+1|}{2}$$

$$e^{-i\hat{\varphi}} = \Sigma|n+1\rangle\langle n|, \quad \hat{H} = 4E_c\left(\sum_n (n-n_g)|n\rangle\langle n|\right)^2 - \frac{E_J}{2}\sum_n\left(|n+1\rangle\langle n| + |n\rangle\langle n+1|\right)$$

$$= 4E_c\sum_n\left[(n-n_g)^2|n\rangle\langle n|\right] - \frac{E_J}{2}\sum_n\left(|n+1\rangle\langle n| + |n\rangle\langle n+1|\right)$$

Problem 2 - Rabi Oscillation.

Rotating frame $R(t)$ $|\psi_R(t)\rangle = R(t)|\psi_s(t)\rangle \Rightarrow i\hbar\frac{\partial}{\partial t}|\psi_R(t)\rangle = i\hbar\left(\dot{R}|\psi_s(t)\rangle + R\frac{\partial}{\partial t}|\psi_s(t)\rangle\right)$

$$\frac{\partial}{\partial t}|\psi_s(t)\rangle = -\frac{i}{\hbar}\hat{H}(t)|\psi_s(t)\rangle, \quad i\hbar\frac{\partial}{\partial t}|\psi_R(t)\rangle = i\hbar\dot{R}R^\dagger|\psi_R(t)\rangle + RHR^\dagger|\psi_R(t)\rangle$$

$$\Rightarrow i\hbar\frac{\partial}{\partial t}|\psi_R(t)\rangle = \underbrace{\left(RHR^\dagger + i\hbar\dot{R}R^\dagger\right)}_{H_R}|\psi_R(t)\rangle$$

H_R time-independent가 되게 할 수 있다.

$$R(t) = e^{i\omega_d t |1\rangle\langle 1|} = 1 + i\omega_d t |1\rangle\langle 1| + \frac{(i\omega_d t)^2}{2!} |1\rangle\langle 1|1\rangle\langle 1| + \cdots$$

$$\dot R(t) = i\omega_d |1\rangle\langle 1| + (i\omega_d)^2 t |1\rangle\langle 1| + \frac{(i\omega_d)^3}{2!} t^2 |1\rangle\langle 1| + \cdots = (i\omega_d e^{i\omega_d t}) |1\rangle\langle 1|$$

$$e^{+B} A e^{-B} = A + [B,A] + \frac{1}{2!}[B,[B,A]] + \cdots$$

$A = H$, $B = i\omega_d t |1\rangle\langle 1|$.

$$[B,A] = (i\omega_d t)\left[\begin{bmatrix}0&0\\0&1\end{bmatrix}\left(-\frac{\omega_0}{2}\begin{bmatrix}1&0\\0&-1\end{bmatrix} + \Omega_d\cos(\omega_d t+\phi)\begin{bmatrix}0&1\\1&0\end{bmatrix}\right) - \text{반대}\right]$$

$$\begin{bmatrix}0&0\\0&1\end{bmatrix}\begin{bmatrix}1&0\\0&-1\end{bmatrix} = \begin{bmatrix}0&0\\0&-1\end{bmatrix} \qquad \begin{bmatrix}1&0\\0&-1\end{bmatrix}\begin{bmatrix}0&0\\0&1\end{bmatrix} = \begin{bmatrix}0&0\\0&-1\end{bmatrix}$$

$$\begin{bmatrix}0&0\\0&1\end{bmatrix}\begin{bmatrix}0&1\\1&0\end{bmatrix} = \begin{bmatrix}0&0\\1&0\end{bmatrix} \qquad \begin{bmatrix}0&1\\1&0\end{bmatrix}\begin{bmatrix}0&0\\0&1\end{bmatrix} = \begin{bmatrix}0&1\\0&0\end{bmatrix}$$

$$[B,A] = (i\omega_d t)\Omega_d\cos(\omega_d t+\phi)\begin{bmatrix}0&-1\\1&0\end{bmatrix}$$

$$[B,[B,A]] = \begin{bmatrix}0&0\\0&1\end{bmatrix}\begin{bmatrix}0&-1\\1&0\end{bmatrix} - \begin{bmatrix}0&-1\\1&0\end{bmatrix}\begin{bmatrix}0&0\\0&1\end{bmatrix} = \begin{bmatrix}0&0\\1&0\end{bmatrix} - \begin{bmatrix}0&-1\\0&0\end{bmatrix} = \begin{bmatrix}0&1\\1&0\end{bmatrix} \quad \text{(상수생략)}$$

$$RHR^\dagger = -\frac{\omega_0}{2}\hat\sigma_z + \Omega_d\cos(\omega_d t+\phi)\left[\hat\sigma_x + \omega_d t\,\hat\sigma_y - \frac{(\omega_d t)^2}{2!}\hat\sigma_x - \frac{(\omega_d t)^3}{3!}\hat\sigma_y + \cdots\right]$$

$$= -\frac{\omega_0}{2}\hat\sigma_z + \Omega_d\cos(\omega_d t+\phi)\left((\cos\omega_d t)\hat\sigma_x + (\sin\omega_d t)\hat\sigma_y\right)$$

$$\left.\begin{aligned}\cos(\omega_d t+\phi)\cos(\omega_d t) &= \frac{1}{2}\left(\cos(2\omega_d t+\phi) + \cos\phi\right)\\ \cos(\omega_d t+\phi)\sin(\omega_d t) &= \frac{1}{2}\left(\sin(2\omega_d t+\phi) - \sin\phi\right)\end{aligned}\right) \quad RHR^\dagger \simeq -\frac{\omega_0}{2}\hat\sigma_z + \frac{\Omega_d}{2}\cos\phi\,\hat\sigma_x - \frac{\Omega_d}{2}\sin\phi\,\hat\sigma_y$$

$$\dot R R^\dagger = i\omega_d e^{i\omega_d t}|1\rangle\langle 1|\left(|0\rangle\langle 0| + e^{-i\omega_d t}|1\rangle\langle 1|\right) = i\omega_d |1\rangle\langle 1|$$

$$H_R = RHR^\dagger + i\hbar\dot R R^\dagger = -\frac{\omega_0}{2}\hat\sigma_z + \frac{\Omega_d}{2}\cos\phi\,\hat\sigma_x - \frac{\Omega_d}{2}\sin\phi\,\hat\sigma_y - \omega_d|1\rangle\langle 1|$$

$\phi = 0$, let $R = e^{-\frac{i\omega_d t}{2}\hat\sigma_z}$, $H_R = \frac{\omega_d - \omega_0}{2}\hat\sigma_z + \frac{\Omega_d}{2}\hat\sigma_x$