$\boxed{\text{1. Connecting } T_1 \text{ to } S_\lambda(\omega)}$

$$\Gamma_{1\downarrow\lambda} = \frac{2\pi}{\hbar}|M_\lambda|^2\rho(\hbar\omega_q) = \frac{1}{\hbar^2}|d_\lambda|^2 S_\lambda(\omega_q) = D_{\lambda,\perp}^2 S_\lambda(\omega_q), \quad d_\lambda = \langle 1|\frac{dH}{d\lambda}|0\rangle = \hbar D_{\lambda,\perp},$$

$$S_\lambda(\omega_q) = 2\pi\hbar(\delta\lambda)^2\rho(\hbar\omega_q)$$

$\boxed{\text{2. Connecting } T_\varphi \text{ to } S_\lambda(\omega)}$

$$\rho = \begin{pmatrix} 1+(|\alpha|^2-1)e^{-\Gamma_1 t} & \alpha\beta^* e^{i\omega t} e^{-\frac{\Gamma_1}{2}t} e^{-\chi_N(t)} \\ \alpha^*\beta e^{-i\omega t} e^{-\frac{\Gamma_1}{2}t} e^{-\chi_N(t)} & |\beta|^2 e^{-\Gamma_1 t} \end{pmatrix}$$

$$S_\lambda(\omega) = \int_{-\infty}^{\infty} d\tau \langle\lambda(\tau)\lambda(0)\rangle e^{-i\omega\tau}$$

$$\varphi(t) = \int_0^t \omega_q dt' = \langle\omega_q\rangle t + \delta\varphi(t), \quad \delta\varphi(t) = \int_0^t \delta\omega_q(t')dt'$$

$D_{\lambda,z}$ is the qubit's longitudinal sensitivity to λ-noise

$$\langle e^{i\delta\varphi(t)}\rangle = \langle 1 + i\delta\varphi(t) - \frac{1}{2!}\delta\varphi^2(t) - \frac{i}{3!}\delta\varphi^3(t) + \frac{1}{4!}\delta\varphi^4(t) + \cdots\rangle$$

$$= \langle 1 - \frac{1}{2!}\delta\varphi^2(t) + \frac{1}{4!}\delta\varphi^4(t) - \frac{1}{6!}\delta\varphi^6(t) + \cdots\rangle \cong \langle 1 - \frac{1}{2}\delta\varphi^2(t)\rangle = 1 - \frac{1}{2}\langle\delta\varphi^2(t)\rangle$$

$$\cong e^{-\frac{1}{2}\langle\delta\varphi^2(t)\rangle}$$

$$\equiv e^{-\chi_N(t)}$$

$$\chi_N(\tau) = \frac{1}{2}\langle\delta\varphi^2(t)\rangle = \frac{1}{2}\langle\left[\int_0^t \delta\omega_q(t')dt'\right]^2\rangle$$

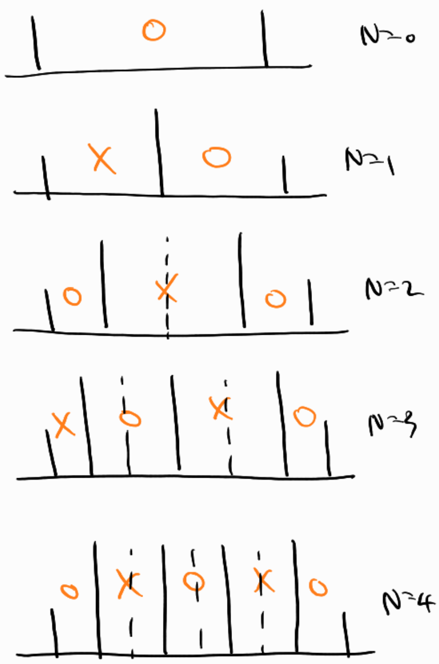
$$\delta\omega_q(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(i\omega)e^{i\omega t}d\omega \qquad \int_0^t \delta\omega_q(t')dt' = \int_{-\infty}^{\infty}\frac{X(i\omega)}{i\omega}(e^{i\omega t}-1)d\omega$$

$$\delta\varphi(t) = \int_{-\infty}^{\infty}\frac{X(i\omega)}{i\omega}(e^{i\omega t}-1)d\omega \qquad = \frac{\tau^2}{2}\int_{-\infty}^{\infty} S(\omega)\underbrace{\boxed{\frac{\sin^2\left(\frac{\omega\tau}{2}\right)}{\left(\frac{\omega\tau}{2}\right)^2}}}_{g_0(\omega,\tau)}d\omega$$

$$\delta\varphi^2(t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{|X(i\omega)|^2}{\omega^2}|1-e^{i\omega t}|^2 d\omega d\omega$$

$$\Rightarrow \chi_N(\tau) = \frac{1}{2}\langle\delta\varphi^2(\tau)\rangle = \frac{1}{2}\int_{-\infty}^{\infty}\langle|X(i\omega)|^2\rangle\frac{|1-e^{i\omega\tau}|^2}{\omega^2}d\omega = \frac{\tau^2}{2}\int_{-\infty}^{\infty}S(\omega)\underbrace{\boxed{\frac{2-2\cos(\omega\tau)}{(\omega\tau)^2}}}_{g_0(\omega,\tau)}d\omega$$

N π-pulse



$$\begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}\begin{pmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{pmatrix} = -i\begin{pmatrix} \rho_{10} & \rho_{11} \\ \rho_{00} & \rho_{01} \end{pmatrix}$$

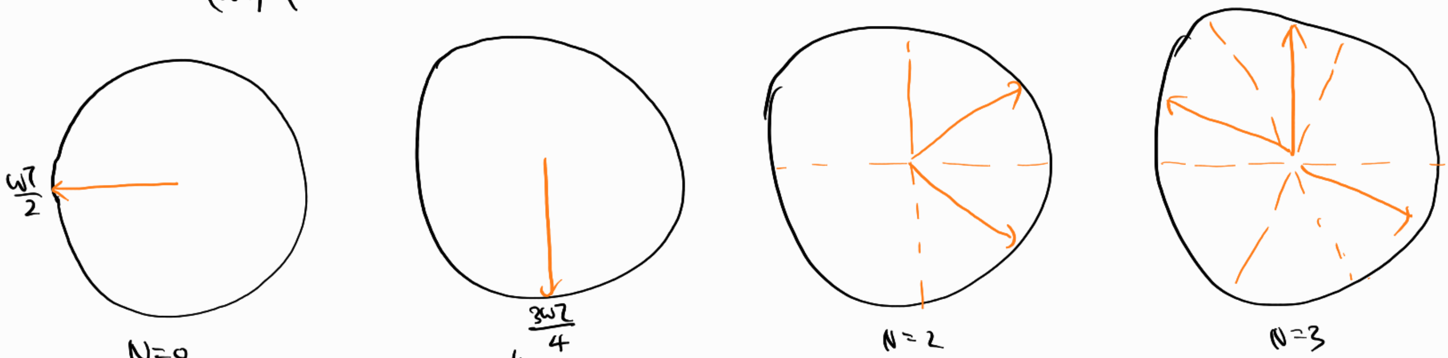
$$\begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}\begin{pmatrix} -i\rho_{10} & -i\rho_{11} \\ -i\rho_{00} & -i\rho_{01} \end{pmatrix} = \begin{pmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{pmatrix}$$

$$\delta\varphi(t) = \delta_0\int_0^{\frac{\tau}{2N}}\delta\omega_g(t')dt' + \delta_1\int_{\frac{\tau}{2N}}^{\frac{\tau}{2N}+\frac{\tau}{N}}\delta\omega_g(t')dt' + \cdots + \delta_{N-1}\int_{\frac{\tau}{2N}+(N-2)\frac{\tau}{N}}^{\frac{\tau}{2N}+(N-1)\frac{\tau}{N}}\delta\omega_g(t')dt' + \delta_N\int_{\frac{\tau}{2N}+(N-1)\frac{\tau}{N}}^{\tau}\delta\omega_g(t')dt'$$

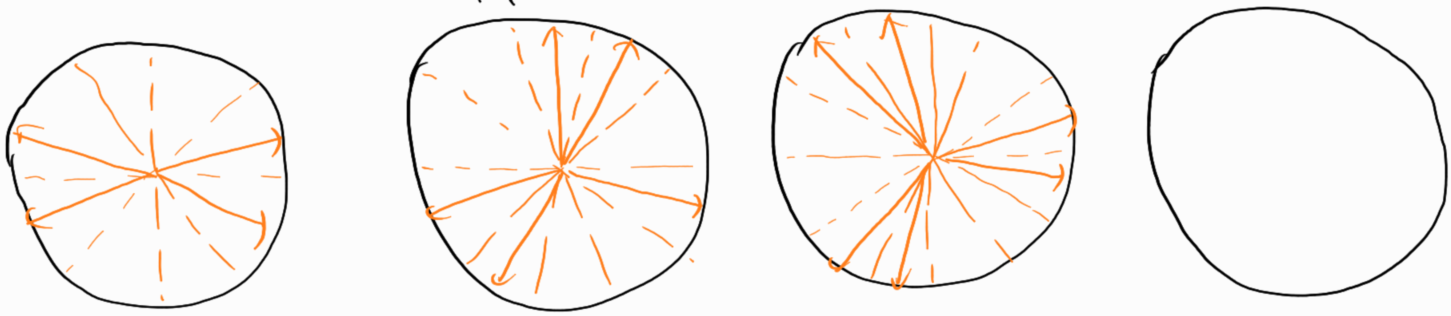
$\delta_N = 1$, $\delta_{N-1} = 0$, $\cdots$

$$\delta\varphi(t) = \int_{-\infty}^{\infty}\frac{X(\omega)}{j\omega}\left[e^{j\omega\tau} - e^{j\omega\tau\frac{2N-1}{2N}} + e^{j\omega\tau\frac{2N-3}{2N}} - e^{j\omega\tau\frac{2N-5}{2N}} + \cdots\right]d\omega$$

$$\Rightarrow g_N(\omega,\tau) = \frac{1}{(\omega\tau)^2}\left(e^{j\omega\tau} - e^{j\omega\tau\frac{2N-1}{2N}} + e^{j\omega\tau\frac{2N-3}{2N}} - e^{j\omega\tau\frac{2N-5}{2N}} + \cdots\right)\left(e^{-j\omega\tau} - e^{-j\omega\tau\frac{2N-1}{2N}} + e^{-j\omega\tau\frac{2N-3}{2N}} - e^{-j\omega\tau\frac{2N-5}{2N}} + \cdots\right)$$



N=0 N=1 N=2 N=3



N=4 N=5 N=6 N=7

N이 짝수이면...

$$0 \sim \frac{\omega\tau}{N} : \frac{\sin^2\left(\frac{\omega\tau}{4N}\right)}{\omega^2}\left(e^{j\frac{\omega\tau}{4N}} + e^{-j\frac{3\omega\tau}{4N}}\right) = \frac{\sin^2\left(\frac{\omega\tau}{4N}\right)}{\omega^2}4\cos^2\left(\frac{3\omega\tau}{4N}\right)\left|e^{j\frac{\omega\tau}{N}}\right|^2$$

$$\Rightarrow \frac{16}{\omega^2}\sin^2\left(\frac{\omega\tau}{4N}\right)\cos^2\left(\frac{3\omega\tau}{4N}\right)\left|1 - e^{j\frac{\omega\tau}{N}} + \cdots + e^{j\frac{N-1}{N}\omega\tau}\right|^2 = \frac{16}{\omega^2}\sin^2\left(\frac{\omega\tau}{4N}\right)\cos^2\left(\frac{3\omega\tau}{4N}\right)\left|\frac{1+(-1)^{N+1}e^{j\omega\tau}}{1+e^{j\frac{\omega\tau}{N}}}\right|^2$$

$$= \frac{16}{\omega^2}\sin^2\left(\frac{\omega\tau}{4N}\right)\cos^2\left(\frac{3\omega\tau}{4N}\right)\frac{\sin^2\left(\frac{\omega\tau}{2}\right)}{\cos^2\left(\frac{\omega\tau}{2N}\right)}$$

3. Gaussian Noise

$S(\omega) = C.$

$$X_0(\tau) = \frac{\tau^2}{2}\int_{-\infty}^{\infty}S(\omega)\frac{\sin^2\left(\frac{\omega\tau}{2}\right)}{\left(\frac{\omega\tau}{2}\right)^2}d\omega = \frac{\tau^2}{2}C\int_{-\infty}^{\infty}\frac{\sin^2\left(\frac{\omega\tau}{2}\right)}{\left(\frac{\omega\tau}{2}\right)^2}d\omega$$