Resonator Drive Pulse

① Hamiltonian $H_r(t)$

$$\hat H_r(t) = \omega_r a^\dagger a + A_0 (a + a^\dagger)\cos(\omega_{rd} t)$$

$$i\frac{\partial|\psi\rangle}{\partial t} = \hat H_r(t)|\psi\rangle \quad \text{let } \tilde H_r = U H_r U^\dagger + i\dot U U^\dagger, \quad U = e^{i\omega_{rd} t a^\dagger a} = \sum_{n=0}^{\infty} e^{i\omega_{rd} n t}|n\rangle\langle n|$$

$$\tilde H_r = \omega_r e^{i\omega_{rd} t a^\dagger a}(a^\dagger a) e^{-i\omega_{rd} t a^\dagger a} + A_0 e^{i\omega_{rd} t a^\dagger a}(a^\dagger + a)\cos(\omega_{rd} t) e^{-i\omega_{rd} t a^\dagger a}$$

$$- \left(\sum_{n=0}^{\infty} n\omega_{rd} e^{i\omega_{rd} n t}|n\rangle\langle n|\right)\sum_{n=0}^{\infty} e^{-i\omega_{rd} n t}|n\rangle\langle n|$$

$$= (\omega_r-\omega_{rd}) a^\dagger a + \left(A_0 \sum_{n=0}^{\infty} e^{i\omega_{rd} n t}|n\rangle\langle n| \sum_{n=1}^{\infty} \sqrt{n}|n-1\rangle\langle n| \sum_{n=0}^{\infty} e^{-i\omega_{rd} n t}|n\rangle\langle n| + A_0 \sum_{n=0}^{\infty} e^{i\omega_{rd} n t}|n\rangle\langle n|\sum_{n=0}^{\infty}\sqrt{n+1}|n+1\rangle\langle n| \sum_{n=0}^{\infty} e^{-i\omega_{rd} n t}|n\rangle\langle n| \right) \cos(\omega_{rd} t)$$

$$\left(A_0 e^{i\omega_{rd} t}\sum_{n=0}^{\infty}\sqrt{n+1}|n+1\rangle\langle n| + A_0 e^{-i\omega_{rd} t}\sum_{n=1}^{\infty}\sqrt{n}|n-1\rangle\langle n|\right)\cos(\omega_{rd}t)$$

$$= (\omega_r-\omega_{rd})a^\dagger a + A_0 \frac{(1+e^{2i\omega_{rd}t})a^\dagger + (1+e^{-2i\omega_{rd}t})a}{2} \simeq (\omega_r-\omega_{rd})a^\dagger a + A_0\frac{a+a^\dagger}{2}$$

② Steady state

$$\dot a = +i[H,a] = +i(\omega_r-\omega_{rd})\underbrace{(a^\dagger a a - a a^\dagger a)}_{-a} + i[a^\dagger, a]\frac{A_0}{2}$$

$$= -i(\omega_r-\omega_{rd})a - \frac{iA_0}{2}$$

$$\dot a = i[H,a] + \gamma\left(a^\dagger a a - \frac12 a^\dagger a a - \frac12 a a^\dagger a\right)$$

$$= -i(\omega_r-\omega_{rd})a - \frac{i}{2}A_0 + \frac{\gamma}{2}(a^\dagger a - a a^\dagger)a = \boxed{\left[-i(\omega_r-\omega_{rd}) - \frac{\gamma}{2}\right]a - \frac{iA_0}{2}}$$

$$\dot a = 0 \ (\text{steady state}) \Rightarrow a|\alpha\rangle = \alpha|\alpha\rangle. \quad \dot a|\alpha\rangle + a|\dot\alpha\rangle = \dot\alpha|\alpha\rangle + \alpha|\dot\alpha\rangle$$

$$\Rightarrow \dot\alpha = \left(-i(\omega_r-\omega_{rd}) - \frac{\gamma}{2}\right)\alpha - \frac{i}{2}A_0 = 0 \Rightarrow \boxed{\alpha_{ss} = -\frac{\frac{i}{2}A_0}{i(\omega_r-\omega_{rd}) + \frac{\gamma}{2}}}$$

$$\boxed{n = |\alpha|^2 = \frac{\left(\frac{A_0}{2}\right)^2}{(\omega_r-\omega_{rd})^2 + \left(\frac{\gamma}{2}\right)^2}}$$

③ $n(t)$ time evolution

$$\dot a = \left[-i(\omega_r-\omega_{rd}) - \frac{\gamma}{2}\right]a - \frac{i}{2}A_0 \Rightarrow a(t) = -\frac{\frac{i}{2}A_0}{i(\omega_r-\omega_{rd})+\frac{\gamma}{2}} + \left[a(0) + \frac{\frac i2 A_0}{i(\omega_r-\omega_{rd})+\frac{\gamma}{2}}\right] e^{[-i(\omega_r-\omega_{rd}) - \frac{\gamma}{2}]t}$$

$$\langle n\rangle = \langle a^\dagger a\rangle, \quad a(t) = a(0) e^{[-i(\omega_r-\omega_{rd})-\frac{\gamma}{2}]t} - \frac{\frac i2 A_0}{i(\omega_r-\omega_{rd})+\frac\gamma2}\left(1 - e^{(-i(\omega_r-\omega_{rd})-\frac\gamma2)t}\right)$$

$$(a^\dagger a)(t) = (a^\dagger a)(0) e^{-\gamma t} + \frac{\left(\frac{A_0}{2}\right)^2}{(\omega_r-\omega_{rd})^2+\left(\frac{\gamma}{2}\right)^2}\left(1 + e^{\gamma t} - 2e^{-\frac{\gamma}{2}t}\cos(\omega_r-\omega_{rd})t\right)$$

$$+ \langle a(0)\rangle \boxed{} + \langle a^\dagger(0)\rangle \boxed{}^*$$

$$\Rightarrow \boxed{n(t) = n(0)e^{-\gamma t} + \frac{\left(\frac{A_0}{2}\right)^2}{(\omega_r-\omega_{rd})^2 + \left(\frac{\gamma}{2}\right)^2}\left(1 - 2e^{-\frac{\gamma}{2}t}\cos(\omega_r-\omega_{rd})t + e^{-\gamma t}\right)}$$

$$\boxed{t \to \infty \Rightarrow n(t) \to \frac{\left(\frac{A_0}{2}\right)^2}{(\omega_r-\omega_{rd})^2 + \left(\frac{\gamma}{2}\right)^2}}$$

④ $A_0 \to A(t)$

$$\tilde{H}_r \cong (\omega_r-\omega_{rd})a^\dagger a + \frac{A_0}{2}(a + a^\dagger)$$

$$\tilde{H}_r \cong (\omega_r-\omega_{rd})a^\dagger a + A(t)\,\frac{a + a^\dagger}{2}$$

보충 필요

$$\dot{a} = i[H, a] + \gamma\left(a^\dagger a a - \tfrac{1}{2}a^\dagger a a - \tfrac{1}{2}a a^\dagger a\right)$$

$$= \left[-i(\omega_r-\omega_{rd}) - \frac{\gamma}{2}\right]a - \frac{A(t)}{2}$$

보충 필요

$$a = (a(0) - B(0))\, e^{\left(-i(\omega_r-\omega_{rd}) - \frac{\gamma}{2}\right)t} + B(t),$$ steady state는 $B(t)$.

⑤ Lab Frame

$$\tilde{H} = UHU^\dagger + i\dot{U}U^\dagger$$ a frame $|\tilde{\psi}\rangle = U|\psi\rangle$ $|\psi\rangle = U^\dagger|\tilde{\psi}\rangle$

$$i\frac{\partial|\psi\rangle}{\partial t} = H|\psi\rangle \longrightarrow i\frac{\partial|\tilde{\psi}\rangle}{\partial t} = \tilde{H}|\tilde{\psi}\rangle$$

$$\Rightarrow iU\frac{\partial|\psi\rangle}{\partial t} + i\dot{U}|\psi\rangle = \tilde{H}U|\psi\rangle$$

$$\Rightarrow UH|\psi\rangle + i\dot{U}|\psi\rangle = \tilde{H}U|\psi\rangle$$

$$\tilde{H} = UHU^\dagger + i\dot{U}U^\dagger$$

$$\tilde{H} - i\dot{U}U^\dagger = UHU^\dagger \qquad H = U^\dagger\tilde{H}U - iU^\dagger\dot{U}, \quad U = e^{i\omega_{rd}t\, a^\dagger a}$$

$$\dot{a} = i[H, a] + \gamma\left(a^\dagger a a - \tfrac{1}{2}a^\dagger a a - \tfrac{1}{2}a a^\dagger a\right)$$

at lab frame, $a(t) = U^\dagger\tilde{a}(t)U$, $(a^\dagger a)(t) = U^\dagger\tilde{a}^\dagger(t)\tilde{a}(t)U$, $\langle\psi(0)|U^\dagger|\tilde{a}\,a(t)|U|\psi(0)\rangle$

$$\boxed{n(t) = n(0)e^{-\gamma t} + \frac{\left(\frac{A_0}{2}\right)^2}{(\omega_r-\omega_{rd})^2 + \left(\frac{\gamma}{2}\right)^2}\left(1 - 2e^{-\frac{\gamma}{2}t}\cos(\omega_r-\omega_{rd})t + e^{-\gamma t}\right),}$$ 동일.

⑥ Coherent State

과연 위의 steady state가 coherent state일까.

$$\tilde{H}_r = (\omega_r-\omega_{rd})a^\dagger a + \frac{A_0}{2}(a + a^\dagger)$$

let eigenstate $|h\rangle = \sum_{n=0}^{\infty} C_n|n\rangle$

$$\tilde{H}_r|h\rangle = \sum_{n=0}^{\infty}\left[(\omega_r-\omega_{rd})nC_n + \frac{A_0}{2}\left(\sqrt{n}\,C_{n-1} + \sqrt{n+1}\,C_{n+1}\right)\right]|n\rangle = h\sum_{n=0}^{\infty}C_n|n\rangle$$

점화식을 푸는 것은 굉장히 힘들지만... let $|h\rangle = |\alpha\rangle = e^{-\frac{|\alpha|^2}{2}}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle$, $\alpha = \dfrac{-\frac{i}{2}A_0}{i(\omega_r-\omega_{rd}) + \cancel{\frac{\gamma}{2}}}$

$\Rightarrow |\alpha\rangle$는 eigenstate이다.

$$\dot{\rho} = -i[H,\rho] + \gamma\left(a\rho a^\dagger - \tfrac{1}{2}a^\dagger a\rho - \tfrac{1}{2}\rho a^\dagger a\right)$$ at rotating frame

$$= -i(\omega_r - \omega_{rd})(a^\dagger a\rho - \rho a^\dagger a) - i\frac{A_0}{2}(a\rho - \rho a + a^\dagger\rho - \rho a^\dagger) + \gamma\left(a\rho a^\dagger - \tfrac{1}{2}a^\dagger a\rho - \tfrac{1}{2}\rho a^\dagger a\right)$$

$$= \left[-i(\omega_r - \omega_{rd}) - \frac{\gamma}{2}\right]a^\dagger a\rho + \left[i(\omega_r - \omega_{rd}) - \frac{\gamma}{2}\right]\rho a^\dagger a + \gamma a\rho a^\dagger - i\frac{A_0}{2}(a\rho + a^\dagger\rho - \rho a - \rho a^\dagger) = 0$$

at Steady State

$\rho = |\alpha\rangle\langle\alpha|$ and $\quad a\rho a^\dagger = a|\alpha\rangle\langle\alpha|a^\dagger = |\alpha|^2|\alpha\rangle\langle\alpha|$

$a^\dagger a\rho = a^\dagger a|\alpha\rangle\langle\alpha| = \alpha\, a^\dagger|\alpha\rangle\langle\alpha| = \alpha a^\dagger\rho$

$\rho a^\dagger a = |\alpha\rangle\langle\alpha|a^\dagger a = \alpha^*|\alpha\rangle\langle\alpha|a = \alpha^*\rho a$

$a\rho = a|\alpha\rangle\langle\alpha| = \alpha|\alpha\rangle\langle\alpha| = \alpha\rho$

$\rho a^\dagger = |\alpha\rangle\langle\alpha|a^\dagger = \alpha^*|\alpha\rangle\langle\alpha| = \alpha^*\rho$

$$\alpha = -\frac{\frac{i}{2}A_0}{i(\omega_r - \omega_{rd}) + \frac{\gamma}{2}} = \frac{\frac{i}{2}A_0\left(i(\omega_r - \omega_{rd}) - \frac{\gamma}{2}\right)}{(\omega_r - \omega_{rd})^2 + \left(\frac{\gamma}{2}\right)^2}$$

$\gamma|\alpha|^2 - \frac{-A_0}{i2}(\alpha + \alpha^*)$

$= 0$ [illegible] $\quad \rho(t \to \infty) = |\alpha\rangle\langle\alpha|$