Time-Independent Dispersive Qubit-Resonator Hamiltonian (Rotating Frame, Dispersive Limit)

① Hamiltonian

$$H = \chi \sigma_z a^\dagger a = \chi\left(|0\rangle\langle 0| - |1\rangle\langle 1|\right) \otimes \left(\sum_{n=0}^{\infty} n|n\rangle\langle n|\right)$$

$|\psi\rangle = |q\rangle \otimes |r\rangle$, $\quad H|\psi\rangle = \chi\sigma_z|q\rangle \otimes a^\dagger a|r\rangle = E|q\rangle\otimes|r\rangle$

$a^\dagger a|r\rangle = \square|r\rangle$ 만족하는 eigenstate : $|n\rangle$ $\quad a^\dagger a|n\rangle = n|n\rangle$

$\sigma_z|q\rangle = \square|q\rangle$ 만족하는 eigenstate : $|0\rangle, |1\rangle$. $\quad \sigma_z|0\rangle = |0\rangle$, $\quad \sigma_z|1\rangle = -|1\rangle$

$\rightarrow H = \chi\sigma_z a^\dagger a$ 의 eigenstate : $|0n\rangle, |1n\rangle$, $n = 0, 1, \dots$

$H|0n\rangle = \chi n|0n\rangle$, $\quad H|1n\rangle = -\chi n|1n\rangle$

② ρ_q 와 $\langle X\rangle, \langle Y\rangle, \langle Z\rangle$

$$\rho(t) = e^{-iHt}\rho(0)e^{+iHt} = \left(\sum e^{-i\chi nt}|0n\rangle\langle 0n| + \sum e^{i\chi nt}|1n\rangle\langle 1n|\right)\rho(0)\left(\sum e^{i\chi nt}|0n\rangle\langle 0n| + \sum e^{-i\chi nt}|1n\rangle\langle 1n|\right)$$

$$= \left[\sum\left(e^{-i\chi nt}|0\rangle\langle 0| + e^{i\chi nt}|1\rangle\langle 1|\right)\otimes|n\rangle\langle n|\right]\rho(0)\left[\sum\left(e^{i\chi nt}|0\rangle\langle 0| + e^{-i\chi nt}|1\rangle\langle 1|\right)\otimes|n\rangle\langle n|\right]$$

$$\rho_q = \sum_{n=0}^{\infty}\langle n|\rho(t)|n\rangle = \sum\left(e^{-i\chi nt}|0\rangle\langle 0| + e^{i\chi nt}|1\rangle\langle 1|\right)\langle n|\rho(0)|n\rangle\left(e^{i\chi nt}|0\rangle\langle 0| + e^{-i\chi nt}|1\rangle\langle 1|\right)$$

let $\rho(0) = \tau\otimes\sigma$, $\quad \tau = \begin{bmatrix}\rho_{00} & \rho_{01}\\ \rho_{10} & \rho_{11}\end{bmatrix}$

$\langle 0|\rho_q|0\rangle = \tau_{00}$ $\qquad \langle 1|\rho_q|1\rangle = \tau_{11}$

$\langle 0|\rho_q|1\rangle = \tau_{01}\sum_{n=0}^{\infty} e^{-2i\chi nt}\sigma_{nn}$ $\qquad \langle 1|\rho_q|0\rangle = \tau_{10}\sum_{n=0}^{\infty} e^{2i\chi nt}\sigma_{nn}$

(i) $\tau = |+\rangle\langle +|$, $\quad \sigma = |\alpha\rangle\langle\alpha|$

$$\langle 0|\rho_q|1\rangle = \frac{e^{-|\alpha|^2}}{2}\sum_{n=0}^{\infty}\frac{1}{n!}\left(|\alpha|e^{-i\chi t}\right)^{2n} \qquad \langle 1|\rho_q|0\rangle = \frac{e^{-|\alpha|^2}}{2}\sum_{n=0}^{\infty}\frac{1}{n!}\left(|\alpha|e^{i\chi t}\right)^{2n}$$

$$= \frac{e^{-|\alpha|^2}}{2}e^{|\alpha|^2 e^{-2i\chi t}} \qquad\qquad = \frac{1}{2}e^{|\alpha|^2\left[e^{2i\chi t}-1\right]}$$

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$$\langle Z\rangle = 0, \quad \langle X\rangle = \frac{e^{|\alpha|^2\left[e^{2i\chi t}-1\right]} + e^{|\alpha|^2\left[e^{-2i\chi t}-1\right]}}{2}, \quad \langle Y\rangle = \frac{i e^{|\alpha|^2\left[e^{-2i\chi t}-1\right]} - i e^{|\alpha|^2\left[e^{2i\chi t}-1\right]}}{2}$$

$$e^{2i\chi t} = \cos(2\chi t) + i\sin(2\chi t)$$

$$e^{|\alpha|^2 e^{2i\chi t}} = e^{|\alpha|^2\cos(2\chi t)}e^{i|\alpha|^2\sin(2\chi t)} = e^{|\alpha|^2\cos(2\chi t)}\left(\cos\left(|\alpha|^2\sin(2\chi t)\right) + i\sin\left(|\alpha|^2\sin(2\chi t)\right)\right)$$

$$e^{|\alpha|^2 e^{-2i\chi t}} = e^{|\alpha|^2\cos(2\chi t)}e^{-i|\alpha|^2\sin(2\chi t)} = e^{|\alpha|^2\cos(2\chi t)}\left(\quad〃\quad - i\quad〃\quad\right)$$

$$\boxed{\begin{aligned}\langle X\rangle &= e^{|\alpha|^2(\cos(2\chi t)-1)}\cos\left(|\alpha|^2\sin(2\chi t)\right) \qquad \langle Z\rangle = 0.\\ \langle Y\rangle &= e^{|\alpha|^2(\cos(2\chi t)-1)}\sin\left(|\alpha|^2\sin(2\chi t)\right)\end{aligned}}$$

$\alpha = 1.$ $\cos(\mathrm{sh}(2\chi t)) > 0$ 이므로, $\langle X \rangle \geq 0$

③ ρ_R 과 $\alpha(t)$

$$\rho_R = \sum_{mn} e^{-i\chi(m-n)t} \tau_{00}\, \sigma_{mn} |m\rangle\langle n| + e^{i\chi(m-n)t} \tau_{11}\, \sigma_{mn} |m\rangle\langle n|$$

$\tau = |0\rangle\langle 0|$ 일 때 $\rho_R = \sum_{mn} e^{-i\chi(m-n)t} \langle m|\alpha\rangle\langle\alpha|n\rangle |m\rangle\langle n| = e^{-|\alpha|^2} \sum_{mn} e^{-i\chi(m-n)t} \frac{(\alpha)^m}{\sqrt{m!}} \frac{(\alpha^*)^n}{\sqrt{n!}} |m\rangle\langle n|$

$$= \boxed{\sum_{mn} e^{-|\alpha|^2} \frac{(\alpha)^m (\alpha^*)^n}{\sqrt{m!n!}} e^{-i\chi(m-n)t} |m\rangle\langle n|}$$

$\tau = |1\rangle\langle 1|$ 일 때 $\rho_R = \boxed{\sum_{mn} e^{-|\alpha|^2} \frac{(\alpha)^m (\alpha^*)^n}{\sqrt{m!n!}} e^{i\chi(m-n)t} |m\rangle\langle n|}$

$\tau = \frac{|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| + |1\rangle\langle 1|}{2}$ 일 때 $\boxed{\rho_R = \sum_{mn} e^{-|\alpha|^2} \frac{(\alpha)^m (\alpha^*)^n}{\sqrt{m!n!}} \cos(\chi(m-n)t) |m\rangle\langle n|}$

$$a = \sum_{n=0}^{\infty} \sqrt{n+1} |n\rangle\langle n+1|$$

$$\rho a = \sum_{mn} e^{-|\alpha|^2} \frac{(\alpha)^m (\alpha^*)^n}{\sqrt{m!n!}} f(t) \sqrt{n+1} |m\rangle\langle n+1|$$

$\tau = |0\rangle\langle 0|$ 일 때 $\alpha = Tr(\rho a) = \alpha e^{-i\chi t}$

$\tau = |1\rangle\langle 1|$ 일 때 $\alpha = Tr(\rho a) = \alpha e^{i\chi t}$

$\tau = |+\rangle\langle +|$ 일 때 $\alpha = Tr(\rho a) = \alpha \cos(\chi t)$