1. Connecting T_1 to $S_\lambda(\omega)$

$$\Gamma_{1\downarrow\lambda} = \frac{2\pi}{\hbar}|M_\lambda|^2 \rho(\hbar\omega_q) = \frac{1}{\hbar^2}|d_\lambda|^2 S_\lambda(\omega_q) = D_{\lambda,\perp}^2 \, S_\lambda(\omega_q), \quad d_\lambda = \langle 1|\frac{dH}{d\lambda}|0\rangle = \hbar D_{\lambda,\perp},$$

$$S_\lambda(\omega_q) = 2\pi\hbar\,(\delta\lambda)^2 \rho(\hbar\omega_q)$$

2 Connecting T_φ to $S_\lambda(\omega)$

$$\rho = \begin{pmatrix} 1+(|\alpha|^2-1)e^{-\Gamma_1 t} & \alpha\beta^* e^{i\omega_q t} e^{-\frac{\Gamma_1}{2}t} e^{-\chi_N(t)} \\ \alpha^*\beta e^{-i\omega_q t} e^{-\frac{\Gamma_1}{2}t} e^{-\chi_N(t)} & |\beta|^2 e^{-\Gamma_1 t} \end{pmatrix}$$

$$S_\lambda(\omega) = \int_{-\infty}^{\infty} d\tau \, \langle \lambda(\tau)\lambda(0)\rangle e^{-i\omega\tau}$$

$$\varphi(t) = \int_0^t \omega_q dt' = \langle\omega_q\rangle t + \delta\varphi(t), \quad \delta\varphi(t) = \int_0^t \delta\omega_q(t')dt'$$

$D_{\lambda,z}$ is the qubit's longitudinal sensitivity to λ-noise

$$\langle e^{i\delta\varphi(t)}\rangle = \langle 1 + i\delta\varphi(t) - \frac{1}{2!}\delta\varphi^2(t) - \frac{i}{3!}\delta\varphi^3(t) + \frac{1}{4!}\delta\varphi^4(t) + \cdots\rangle$$

$$= \langle 1 - \frac{1}{2!}\delta\varphi^2(t) + \frac{1}{4!}\delta\varphi^4(t) - \frac{1}{6!}\delta\varphi^6(t) + \cdots\rangle \cong \langle 1 - \frac{1}{2}\delta\varphi^2(t)\rangle = 1 - \frac{1}{2}\langle\delta^2(t)\rangle$$

$$\cong e^{-\frac{1}{2}\langle\delta\varphi^2(t)\rangle}$$

$$\equiv e^{-\chi_N(t)}$$

$$\chi_N(\tau) = \frac{1}{2}\langle\delta^2(t)\rangle = \frac{1}{2}\left\langle\left[\int_0^t \delta\omega_q(t')dt'\right]^2\right\rangle$$

$$\delta\omega_q(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(i\omega)e^{i\omega t}d\omega \qquad \int_0^t \delta\omega_q(t')dt' = \int_{-\infty}^{\infty}\frac{X(i\omega)}{i\omega}(e^{i\omega t}-1)d\omega$$

$$\delta\varphi(t) = \int_{-\infty}^{\infty}\frac{X(i\omega)}{i\omega}(e^{i\omega t}-1)d\omega \qquad = \frac{\tau^2}{2}\int_{-\infty}^{\infty} S(\omega)\underbrace{\boxed{\frac{\sin^2\left(\frac{\omega\tau}{2}\right)}{\left(\frac{\omega\tau}{2}\right)^2}}}_{g_0(\omega,\tau)}d\omega$$

$$\delta\varphi^2(t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{|X(i\omega)|^2}{\omega^2}|1-e^{i\omega t}|^2 d\omega d\omega$$

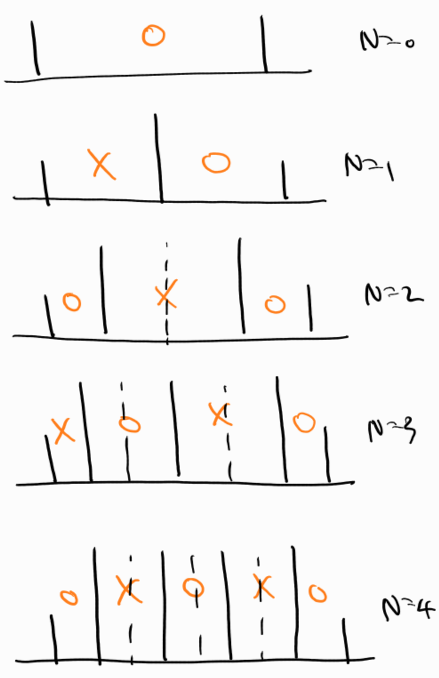
$$\Rightarrow \chi_N(\tau) = \frac{1}{2}\langle\delta\varphi^2(\tau)\rangle = \frac{1}{2}\int_{-\infty}^{\infty}\langle|X(i\omega)|^2\rangle\frac{|1-e^{i\omega\tau}|^2}{\omega^2}d\omega = \frac{\tau^2}{2}\int_{-\infty}^{\infty}S(\omega)\underbrace{\boxed{\frac{2-2\cos(\omega\tau)}{(\omega\tau)^2}}}_{g_0(\omega,\tau)}d\omega$$

$N\pi$ π-pulse

$$\begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}\begin{pmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{pmatrix}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \rho_{10} & \rho_{11} \\ \rho_{00} & \rho_{01} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \rho_{11} & \rho_{10} \\ \rho_{01} & \rho_{00} \end{pmatrix}$$



$$\delta\varphi(t) = \delta_0 \int_0^{\frac{\tau}{2N}} \delta\omega_g(t')dt' + \delta_1 \int_{\frac{\tau}{2N}}^{\frac{\tau}{2N}+\frac{\tau}{N}} \delta\omega_g(t')dt' + \cdots + \delta_{N-1} \int_{\frac{\tau}{2N}+(N-2)\frac{\tau}{N}}^{\frac{\tau}{2N}+(N-1)\frac{\tau}{N}} \delta\omega_g(t')dt' + \delta_N \int_{\frac{\tau}{2N}+(N-1)\frac{\tau}{N}}^{\tau} \delta\omega_g(t')dt'$$

$$\delta_N = 1, \quad \delta_{N-1} = -1, \quad \cdots \quad \delta_0 = (-1)^N$$

$$\delta\varphi(t) = \int_{-\infty}^{\infty} \frac{X(\omega)}{j\omega}\left[e^{j\omega\tau} - 2e^{j\omega\tau\frac{2N-1}{2N}} + 2e^{j\omega\tau\frac{2N-3}{2N}} - e^{j\omega\tau\frac{2N-5}{2N}} + \cdots\right] d\omega$$

$$\Rightarrow g_N(\omega,\tau) = \frac{1}{(\omega\tau)^2}\left| e^{j\omega\tau} + (-1)^{N+1} - 2e^{j\omega\tau}\underbrace{\left(e^{-j\omega\tau\frac{1}{2N}} - e^{-j\omega\tau\frac{3}{2N}} + \cdots + (-1)^N e^{-j\omega\tau\frac{2N-1}{2N}}\right)}_{e^{-j\omega\tau\frac{1}{2N}}\left(1 - e^{-j\omega\tau\frac{1}{N}} + \cdots + (-1)^N e^{-j\omega\tau\frac{N-1}{N}}\right)}\right|^2$$

$$= e^{-j\omega\tau\frac{1}{2N}} \frac{1-(-1)^N e^{-j\omega\tau}}{1+e^{-j\omega\tau\frac{1}{N}}} = e^{-j\omega\tau\frac{1}{2N}} \frac{1+(-1)^{N+1} e^{-j\omega\tau}}{1+e^{-j\omega\tau\frac{1}{N}}}$$

$$N=1: \left| e^{j\omega\tau} - 2e^{\frac{j\omega\tau}{2}} + 1 \right|^2$$

$$= \left(e^{j\omega\tau} - 2e^{j\frac{\omega\tau}{2}} + 1\right)\left(e^{-j\omega\tau} - 2e^{-j\frac{\omega\tau}{2}} + 1\right) = 6 - 2e^{-j\frac{\omega\tau}{2}} + e^{-j\omega\tau} - 2e^{j\frac{\omega\tau}{2}} - 2e^{-j\frac{\omega\tau}{2}} + e^{j\omega\tau} - 2e^{j\frac{\omega\tau}{2}}$$

$$= 6 - 8\cos\left(\frac{\omega\tau}{2}\right) + 2\cos(\omega\tau) = 6 - 8\cos\left(\frac{\omega\tau}{2}\right) + 4\cos^2\left(\frac{\omega\tau}{2}\right) - 2 = 4\left(\cos\left(\frac{\omega\tau}{2}\right) - 1\right)^2 = 16\sin^4\left(\frac{\omega\tau}{4}\right)$$

$$\Rightarrow g_N(\omega,\tau) = \frac{\sin^4\left(\frac{\omega\tau}{4}\right)}{\left(\frac{\omega\tau}{4}\right)^2}$$

N 짝수.
$$e^{j\omega\tau} - 1 - 2je^{j\omega\tau\frac{2N-1}{2N}} \frac{e^{\frac{j\omega\tau}{2}}}{e^{-j\frac{\omega\tau}{2N}}} \frac{\sin\left(\frac{\omega\tau}{2}\right)}{\cos\left(\frac{\omega\tau}{2N}\right)} = 2je^{\frac{j\omega\tau}{2}}\sin\left(\frac{\omega\tau}{2}\right) - 2je^{\frac{j\omega\tau}{2}} \frac{\sin\left(\frac{\omega\tau}{2}\right)}{\cos\left(\frac{\omega\tau}{2N}\right)}$$

$$\Rightarrow g_N(\omega,\tau) = \frac{\sin^2\left(\frac{\omega\tau}{2}\right)}{\left(\frac{\omega\tau}{2}\right)^2}\left(1 - \sec\left(\frac{\omega\tau}{2N}\right)\right)^2$$

3. Gaussian Noise

$S(\omega) = C.$

$$X_0(\tau) = \frac{\tau^2}{2}\int_{-\infty}^{\infty} S(\omega) \frac{\sin^2\left(\frac{\omega\tau}{2}\right)}{\left(\frac{\omega\tau}{2}\right)^2} d\omega = \frac{\tau^2}{2} C \int_{-\infty}^{\infty} \frac{\sin^2\left(\frac{\omega\tau}{2}\right)}{\left(\frac{\omega\tau}{2}\right)^2} d\omega$$