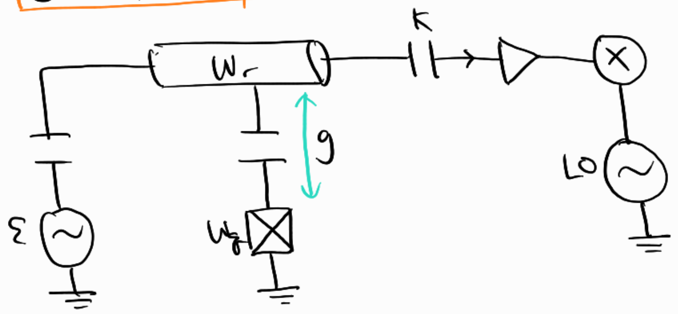
Multi-level Transmon Dispersive Readout

① Hamiltonian



ground state $|g\rangle$, the first excited state $|e\rangle$,
the second excited state $|f\rangle$

ω_g : bare qubit frequency

δ_g : anharmonicity ($\delta_g > 0$, $\delta_g/\omega_g \sim 0.2\,\text{GHz}/6\,\text{GHz} \ll 1$)

ω_r : bare resonator frequency

a : annihilation operator for resonator

g : coupling between qubit and resonator

$\tilde{g}$: coupling " involving levels $|e\rangle$, $|f\rangle$,

$\tilde{g} \simeq \sqrt{2}g$

ε_r : normalized amplitude of the microwave drive

H_κ : resonator energy damping with the rate κ.

H_r : intrinsic qubit relaxation.

double excitation은 무시, coupling은 real. $g = g^*$, $\tilde{g} = \tilde{g}^*$.

$$H = \omega_g |e\rangle\langle e| + (2\omega_g - \delta_g)|f\rangle\langle f| + \omega_r a^\dagger a$$
$$+ g(a^\dagger |g\rangle\langle e| + a|e\rangle\langle g|) + \sqrt{2}g(a^\dagger|e\rangle\langle f| + a|f\rangle\langle e|) + \varepsilon_r a^\dagger e^{-i\omega_d t} + \varepsilon_r^* a e^{i\omega_d t}$$
$$\underbrace{+ H_\kappa + H_r}_{\text{damping term 추가}}$$

driving term, damping term 제외하면

$$H|g,n\rangle = \omega_r n |g,n\rangle + g\sqrt{n}|e,n-1\rangle$$
$$H|e,n\rangle = \omega_g|e,n\rangle + \omega_r n|e,n\rangle + g\sqrt{n+1}|g,n+1\rangle + \sqrt{2}g\sqrt{n}|f,n-1\rangle$$
$$H|f,n\rangle = (2\omega_g - \delta_g)|f,n\rangle + \omega_r n|f,n\rangle + \sqrt{2}g\sqrt{n+1}|e,n+1\rangle$$

$$H|g,n+1\rangle = \omega_r(n+1)|g,n+1\rangle + g\sqrt{n+1}|e,n\rangle$$
$$\Rightarrow H|e,n\rangle = (\omega_r n + \omega_g)|e,n\rangle + g\sqrt{n+1}|g,n+1\rangle + \sqrt{2}g\sqrt{n}|f,n-1\rangle$$
$$H|f,n-1\rangle = (\omega_r(n-1) + 2\omega_g - \delta_g)|f,n-1\rangle + \sqrt{2}g\sqrt{n}|e,n\rangle$$

$$H^{(3\times3)} = \begin{bmatrix} \omega_r(n+1) & g\sqrt{n+1} & 0 \\ g\sqrt{n+1} & \omega_r n + \omega_g & g\sqrt{2n} \\ 0 & g\sqrt{2n} & \omega_r(n-1) + 2\omega_g - \delta_g \end{bmatrix}$$

$$|H - \lambda I| = 0.$$
$$[\omega_r(n+1) - \lambda]\left((\omega_r n + \omega_g - \lambda)(\omega_r(n-1) + 2\omega_g - \delta_g - \lambda) - \tilde{g}^2 n\right)$$
$$- g^2(n+1)(\omega_r(n-1) + 2\omega_g - \delta_g - \lambda)$$

$$|H - \lambda I| = -\lambda^3 + \lambda^2\Big[\underbrace{\omega_r(n+1) + \omega_r n + \omega_g + \omega_r(n-1) + 2\omega_g - \delta_g}_{3\omega_r n + 3\omega_g - \delta_g}\Big]$$

$$-\lambda\Big[\underbrace{(\omega_r n + \omega_g)(\omega_r(n-1) + 2\omega_g - \delta_g) + \omega_r(n+1)(\omega_r n + \omega_g) + \omega_r(n+1)(\omega_r(n-1) + 2\omega_g - \delta_g) - \tilde{g}^2 n - g^2(n+1)}_{\omega_r^2(3n^2-1) + \omega_r\left[\omega_g(n-1) + n(2\omega_g - \delta_g) + \omega_g(n+1) + (n+1)(2\omega_g - \delta_g)\right] + 2\omega_g^2 - \omega_g\delta_g - n(g^2 + \tilde{g}^2) - g^2}\Big]$$

$$\underbrace{\omega_g(n-1) + n(2\omega_g - \delta_g) + \omega_g(n+1) + (n+1)(2\omega_g - \delta_g)}_{2\omega_g n + (2n+1)(2\omega_g - \delta_g) = (6\omega_g - 2\delta_g)n + (2\omega_g - \delta_g)}$$

$$+\omega_r(n+1)(\omega_r n+\omega_q)(\omega_r(n-1)+2\omega_q-\delta_q) - \tilde{g}^2\omega_r n(n+1) - g^2(n+1)(\omega_r(n-1)+2\omega_q-\delta_q) = 0$$

$$\omega_r^2 n(n-1) + \omega_r\left((n-1)\omega_q + n(2\omega_q-\delta_q)\right) + \omega_q(2\omega_q-\delta_q) \qquad -\omega_r(n+1)\left(\tilde{g}^2 n + g^2(n-1)\right) - g^2(n+1)(2\omega_q-\delta_q)$$

$$\omega_r^3(n+1)n(n-1) + \omega_r^2(n+1)\left(n(3\omega_q-\delta_q)-\omega_q\right) + \omega_r(n+1)\left[\omega_q(2\omega_q-\delta_q) - \tilde{g}^2 n - g^2(n-1)\right] - g^2(n+1)(2\omega_q-\delta_q)$$

$$\lambda^3 - \lambda^2\left[3\omega_r n + 3\omega_q - \delta_q\right] + \lambda\left[\omega_r^2(3n^2-1) + \omega_r\left[(6\omega_q-2\delta_q)n + (2\omega_q-\delta_q)\right] + \omega_q(2\omega_q-\delta_q) - n(g^2+\tilde{g}^2) - g^2\right]$$

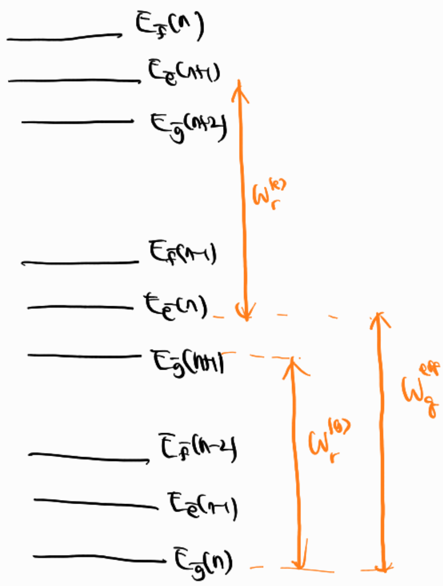
$$-(n+1)\left[\omega_r^3 n(n-1) + \omega_r^2\left[n(3\omega_q-\delta_q)-\omega_q\right] + \omega_r\left[\omega_q(2\omega_q-\delta_q) - \tilde{g}^2 n - g^2(n-1)\right] - g^2(2\omega_q-\delta_q)\right] = 0$$

$|g,n+1\rangle$, $|e,n\rangle$, $|f,n-1\rangle$ 의 coupling은 $\overline{|g,n\rangle}$, $\overline{|e,n\rangle}$, $\overline{|f,n\rangle}$ 에서 $E_{\bar{g}}(n)$, $E_{\bar{e}}(n)$, $E_{\bar{f}}(n)$ 계산.

$$E_{\bar{g}}(n) = n\omega_r - \frac{ng^2}{\omega_q-\omega_r}$$

$$E_{\bar{e}}(n) = \omega_q + n\omega_r + \frac{(n+1)g^2}{\omega_q-\omega_r} - \frac{n\tilde{g}^2}{\omega_q-\omega_r-\delta_q}$$

$$E_{\bar{f}}(n) = 2\omega_q - \delta_q + n\omega_r - \frac{g^2}{\omega_q-\omega_r} + \frac{n\tilde{g}^2}{\omega_q-\omega_r-\delta_q}$$



$$\omega_r^{(g)} = E_{\bar{g}}(n+1) - E_{\bar{g}}(n) = \omega_r - \frac{g^2}{\Delta}$$

$$\omega_r^{(e)} = E_{\bar{e}}(n+1) - E_{\bar{e}}(n) = \omega_r + \frac{g^2}{\Delta} - \frac{\tilde{g}^2}{\Delta-\delta_q}$$

$$\omega_q^{eff} = E_{\bar{e}}(n) - E_{\bar{g}}(n) = \omega_q + \frac{(2n+1)g^2}{\Delta} - \frac{n\tilde{g}^2}{\Delta-\delta_q}$$

let $2\chi \equiv \frac{2g^2}{\Delta} - \frac{\tilde{g}^2}{\Delta-\delta_q}$ $\qquad \omega_q^{eff} = \omega_q + \frac{g^2}{\Delta} + 2n\chi$

Lamb shift $\qquad$ ac Stark shift

$$\boxed{\begin{aligned}\omega_r^{(g)} &= \omega_r - \frac{g^2}{\Delta}\\ \omega_r^{(e)} &= \omega_r + \frac{g^2}{\Delta} - \frac{\tilde{g}^2}{\Delta-\delta_q}\\ \omega_q^{eff} &= \omega_q + \frac{g^2}{\Delta} + 2n\left(\frac{g^2}{\Delta} - \frac{\tilde{g}^2/2}{\Delta-\delta_q}\right)\end{aligned}}$$

notation 변경

$$\overline{|g,n\rangle} = \left(1 - \frac{(n+1)g^2}{2\Delta^2}\right)|g,n+1\rangle + \frac{\sqrt{n(n+1)}\,g\tilde{g}}{\Delta(2\Delta-\delta_q)}|f,n-1\rangle - \frac{\sqrt{n+1}\,g}{\Delta}\left(1 - \frac{3(n+1)g^2}{2\Delta^2} + \frac{n\tilde{g}^2}{\Delta(2\Delta-\delta_q)}\right)|e,n\rangle$$

$$\overline{|e,n\rangle} \cong \frac{\sqrt{n+1}\,g}{\Delta}\left(1 - \frac{3(n+1)g^2}{2\Delta^2} + \frac{n\tilde{g}^2(\Delta-2\delta_q)}{2\Delta(\Delta-\delta_q)^2}\right)|g,n+1\rangle - \frac{\sqrt{n}\,\tilde{g}}{\Delta-\delta_q}|f,n-1\rangle$$
$$+ \left(1 - \frac{(n+1)g^2}{2\Delta^2} - \frac{n\tilde{g}^2}{2(\Delta-\delta_q)^2}\right)|e,n\rangle$$

② Dispersive Regime

First two lines of the Hamiltonian can be approximated

$$H = -\frac{\omega_q^{eff}}{2}\sigma_z + \omega_r^{eff} a^\dagger a - \chi a^\dagger a \sigma_z,$$

$$\omega_r^{eff} = \frac{1}{2}\left(\omega_r^{(g)} + \omega_r^{(e)}\right) = \omega_r - \frac{1}{2}\frac{\tilde{g}^2}{\Delta-\delta_q}$$

$$\omega_q^{eff} = \omega_q + \frac{g^2}{\Delta} + 2\chi$$

$$\chi = \frac{g^2}{\Delta} - \frac{\tilde{g}^2/2}{\Delta-\delta_q} = \left(-\frac{g^2\delta_q}{\Delta(\Delta-\delta_q)}\right) + \frac{g^2-\tilde{g}^2/2}{\Delta-\delta_q}$$

$$\cong -\frac{g^2\delta_q}{\Delta^2}$$

$\tilde{g} \cong g$!

SHO. $\delta_q \to 0$

two level. $\delta_q \to \infty$.

dispersive shift harmonic oscillator with 0.

What is n_{crit}, $\tilde{n}_{crit}$.

$n \ll \min(n_{crit}, \tilde{n}_{crit})$, $n_{crit} = \frac{\Delta^2}{4g^2}$, $\tilde{n}_{crit} = \frac{(\Delta-\delta_q)^2}{4\tilde{g}^2}$

③ Purcell rate

Qubit Decay Rate

$$\Gamma_{e\to g}(n) = \kappa\left|\langle \overline{g,n-1}|a|\overline{e,n}\rangle\right|^2 = \kappa\frac{g^2}{\Delta^2}\left[1 - \frac{3g^2}{\Delta^2} - 6n\frac{g^2}{\Delta^2} + n\frac{\tilde{g}^2(3\Delta-4\delta_q)}{\Delta(\Delta-\delta_q)^2}\right]$$

collapse op $\sqrt{\kappa}\,a$

④ Hamiltonian with drive

$$H = \omega_q|e\rangle\langle e| + (2\omega_q-\delta_q)|f\rangle\langle f| + \omega_r a^\dagger a + g(a^\dagger|g\rangle\langle e| + a|e\rangle\langle g|) + \tilde{g}(a^\dagger|e\rangle\langle f| + a|f\rangle\langle e|)$$
$$+ A(t)(a+a^\dagger)\cos(\omega_d t)$$

$$R = e^{i\omega_d t(|e\rangle\langle e| + 2|f\rangle\langle f| + a^\dagger a)} \qquad i\dot{R}R^\dagger = -\omega_d|e\rangle\langle e| - \omega_d|f\rangle\langle f| - \omega_d a^\dagger a$$

$$\tilde{H} \cong (\omega_q-\omega_d)|e\rangle\langle e| + (2\omega_q-\delta_q-2\omega_d)|f\rangle\langle f| + (\omega_r-\omega_d)a^\dagger a + Rg(a^\dagger|g\rangle\langle e| + a|e\rangle\langle g|)R^\dagger$$
$$+ R\tilde{g}(a^\dagger|e\rangle\langle f| + a|f\rangle\langle e|)R^\dagger + A(t)\frac{a+a^\dagger}{2}$$

$$R = e^{i\omega_d t|e\rangle\langle e|}\, e^{2i\omega_d t|f\rangle\langle f|}\, e^{i\omega_d t a^\dagger a} \qquad R^\dagger = e^{-i\omega_d t|e\rangle\langle e|}\, e^{-2i\omega_d t|f\rangle\langle f|}\, e^{-i\omega_d t a^\dagger a}$$

$$e^{i\omega_d t a^\dagger a}\, a^\dagger\, e^{-i\omega_d t a^\dagger a} = a^\dagger e^{+i\omega_d t} \qquad B = i\omega_d t\begin{bmatrix}0&0&0\\0&1&0\\0&0&1\end{bmatrix} \qquad A = \begin{bmatrix}0&1&0\\0&0&0\\0&0&0\end{bmatrix}$$

$$e^{i\omega_d t a^\dagger a}\, a\, e^{-i\omega_d t a^\dagger a} = a e^{-i\omega_d t}$$

$$[B,A] = i\omega_d t\begin{bmatrix}0&0&0\\0&1&0\\0&0&2\end{bmatrix}\begin{bmatrix}0&1&0\\0&0&0\\0&0&0\end{bmatrix} - i\omega_d t\begin{bmatrix}0&1&0\\0&0&0\\0&0&0\end{bmatrix}\begin{bmatrix}0&0&0\\0&1&0\\0&0&2\end{bmatrix} = -\begin{bmatrix}0&1&0\\0&0&0\\0&0&0\end{bmatrix} = -i\omega_d t|g\rangle\langle e|$$

$$\Rightarrow e^{B}\begin{bmatrix}0&1&0\\0&0&0\\0&0&0\end{bmatrix}e^{-B} = |g\rangle\langle e|e^{-i\omega_d t}$$

$$A = \begin{bmatrix}0&0&0\\1&0&0\\0&0&0\end{bmatrix} \quad [B,A] = i\omega_d t\begin{bmatrix}0&0&0\\0&1&0\\0&0&2\end{bmatrix}\begin{bmatrix}0&0&0\\1&0&0\\0&0&0\end{bmatrix} - i\omega_d t\begin{bmatrix}0&0&0\\1&0&0\\0&0&0\end{bmatrix}\begin{bmatrix}0&0&0\\0&1&0\\0&0&2\end{bmatrix} = i\omega_d t|e\rangle\langle g| \Rightarrow e^{B}\begin{bmatrix}0&0&0\\1&0&0\\0&0&0\end{bmatrix}e^{-B} = |e\rangle\langle g|e^{+i\omega_d t}$$

$$A = |e\rangle\langle f| = \begin{bmatrix}0&0&0\\0&0&1\\0&0&0\end{bmatrix} \quad [B,A] = i\omega_d t\begin{bmatrix}0&0&0\\0&1&0\\0&0&2\end{bmatrix}\begin{bmatrix}0&0&0\\0&0&1\\0&0&0\end{bmatrix} - i\omega_d t\begin{bmatrix}0&0&0\\0&0&1\\0&0&0\end{bmatrix}\begin{bmatrix}0&0&0\\0&1&0\\0&0&2\end{bmatrix} = -i\omega_d t|e\rangle\langle f|,\ e^{B}|e\rangle\langle f|e^{-B} = |e\rangle\langle f|e^{-i\omega_d t}$$

$$A = |f\rangle\langle e|\begin{bmatrix}0&0&0\\0&0&0\\0&1&0\end{bmatrix} \quad [B,A] = i\omega_d t\begin{bmatrix}0&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}0&0&0\\0&0&1\\0&0&0\end{bmatrix} - i\omega_d t\begin{bmatrix}0&0&0\\0&0&1\\0&0&0\end{bmatrix}\begin{bmatrix}0&0&0\\0&1&0\\0&0&1\end{bmatrix} = i\omega_d t|f\rangle\langle e| \quad e^{B}|f\rangle\langle e|e^{-B} = |f\rangle\langle e|e^{i\omega_d t}$$

$$\tilde{H} = (\omega_g-\omega_d)|e\rangle\langle e| + (2\omega_g-\delta_g-2\omega_d)|f\rangle\langle f| + (\omega_r-\omega_d)a^\dagger a + g(a^\dagger|g\rangle\langle e| + a|e\rangle\langle g|)$$
$$+\tilde{g}(a^\dagger|e\rangle\langle f| + a|f\rangle\langle e|) + A(t)\frac{a(1+\cancel{e^{-2i\omega_d t}}) + a^\dagger(1+\cancel{e^{2i\omega_d t}})}{2}$$

$$\boxed{= (\omega_g-\omega_d)|e\rangle\langle e| + (2\omega_g-\delta_g-2\omega_d)|f\rangle\langle f| + (\omega_r-\omega_d)a^\dagger a + g(a^\dagger|g\rangle\langle e| + a|e\rangle\langle g|) + \tilde{g}(a^\dagger|e\rangle\langle f| + a|f\rangle\langle e|) + A(t)\frac{a+a^\dagger}{2}}$$

⑤ Time Evolution

$$\dot{X} = i[\tilde{H},X] + \kappa\left[a^\dagger X a - \tfrac{1}{2}a^\dagger a X - \tfrac{1}{2}X a^\dagger a\right] + \gamma\left(b^\dagger X b - \tfrac{1}{2}b^\dagger b X - \tfrac{1}{2}X b^\dagger b\right)$$

$$\dot{a} = -i(\omega_r-\omega_d)a - ig|g\rangle\langle e| - i\tilde{g}|e\rangle\langle f| - \frac{iA(t)}{2}$$

$$\dot{a}^\dagger = i(\omega_r-\omega_d)a^\dagger + ig|e\rangle\langle g| + i\tilde{g}|f\rangle\langle e| + \frac{iA(t)}{2}$$

$|g\rangle\langle g|$ purcell decay $\gamma + \left(\frac{g}{\Delta}\right)^2\kappa$.

Collapse operator $\quad \gamma,\ 2\gamma \leftarrow$ Fermi's Golden Rule

$\uparrow$ e→g $\quad \uparrow$ f→e.

FFT

Transmon Qubit.

Intrinsic Quality Factor

$$\dot{\rho}_f = -\Gamma_{ef}\rho_f$$
$$\dot{\rho}_e = \Gamma_{ef}\rho_f - \Gamma_{ge}\rho_e$$
$$\dot{\rho}_g = +\Gamma_{ge}\rho_e$$