# #1. Spin Chains 101

$[A,B] = AB - BA.$

① $S_j^\alpha = \sigma_j^\alpha \sigma_{j+1}^\alpha$, $[S_j^\alpha, S_j^\beta] = \sigma_j^\alpha \sigma_{j+1}^\alpha \sigma_j^\beta \sigma_{j+1}^\beta - \sigma_j^\beta \sigma_{j+1}^\beta \sigma_j^\alpha \sigma_{j+1}^\alpha = I^{\otimes j-1} \otimes \sigma^\alpha \sigma^\beta \otimes \sigma^\alpha \sigma^\beta \otimes I^{\otimes N-1-j}$

$- I^{\otimes j-1} \otimes \sigma^\beta \sigma^\alpha \otimes \sigma^\beta \sigma^\alpha \otimes I^{\otimes N-1-j}$

(i) $\alpha = \beta \Rightarrow (\sigma^\alpha)^2 = I \Rightarrow [S_j^\alpha, S_j^\beta] = I^{\otimes N} - I^{\otimes N} = 0$

(ii) $\alpha \neq \beta$. let $\sigma^\alpha \sigma^\beta = \sigma^\gamma$ $(\gamma \neq \alpha, \beta) \Rightarrow \sigma^\alpha \sigma^\beta = \sigma^\gamma$, $\sigma^\beta \sigma^\alpha = -\sigma^\gamma$.

$\Rightarrow [S_j^\alpha, S_j^\beta] = I^{\otimes j-1} \otimes \sigma^\gamma \otimes \sigma^\gamma \otimes I^{\otimes N-1-j} - I^{\otimes j-1} \otimes (-\sigma^\gamma) \otimes (-\sigma^\gamma) \otimes I^{\otimes N-1-j}$

$= 0.$

$\therefore [S_j^\alpha, S_j^\beta] = 0$

② $[S_j^\alpha, S_{j+1}^\beta] = \sigma_j^\alpha \sigma_{j+1}^\alpha \sigma_{j+1}^\beta \sigma_{j+2}^\beta - \sigma_{j+1}^\beta \sigma_{j+2}^\beta \sigma_j^\alpha \sigma_{j+1}^\alpha$

$= I^{\otimes j-1} \otimes \sigma^\alpha \otimes \sigma^\alpha \sigma^\beta \otimes \sigma^\beta \otimes I^{N-2-j} - I^{\otimes j-1} \otimes \sigma^\alpha \otimes \sigma^\beta \sigma^\alpha \otimes \sigma^\beta \otimes I^{N-2-j}$, when $\alpha \neq \beta$.

$= I^{\otimes j-1} \otimes \sigma^\alpha \otimes \underbrace{(\sigma^\alpha \sigma^\beta - \sigma^\beta \sigma^\alpha)}_{[\sigma^\alpha, \sigma^\beta] = 2i\sigma^\gamma} \otimes \sigma^\beta \otimes I^{N-2-j} = 2i I^{\otimes j-1} \otimes \sigma^\alpha \otimes \sigma^\gamma \otimes \sigma^\beta \otimes I^{N-2-j} \neq 0.$

$\therefore [S_j^\alpha, S_{j+1}^\beta] \neq 0$

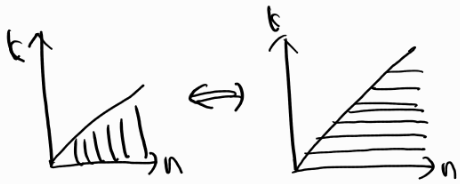
# #2. Schrodinger dynamics

$z(t) = e^{tA} z(0)$. $z'(t) = \frac{d}{dt}(e^{tA}) z(0) = \frac{d}{dt}\left(\sum_{n=0}^\infty \frac{t^n A^n}{n!}\right) z(0) = \left[\sum_{n=0}^\infty \frac{d}{dt}\left(\frac{t^n A^n}{n!}\right)\right] z(0)$

$= \left(\sum_{n=1}^\infty \frac{t^{n-1} A^n}{(n-1)!}\right) z(0) = A\left(\sum_{n=0}^\infty \frac{t^n A^n}{n!}\right) z(0) = A e^{tA} z(0) = A z(t).$

$\therefore z(t) = e^{tA} z(0)$ is a solution to $z'(t) = A z(t)$

# #3. (Foundations: N=2 Case)



(a) ① $AB = BA \Rightarrow e^A e^B = \left(\sum_{n=0}^\infty \frac{A^n}{n!}\right)\left(\sum_{m=0}^\infty \frac{B^m}{m!}\right)$

$e^{A+B} = \sum_{n=0}^\infty \frac{(A+B)^n}{n!} = \sum_{n=0}^\infty \sum_{k=0}^n \frac{{}_nC_k A^k B^{n-k}}{n!} = \sum_{n=0}^\infty \sum_{k=0}^n \frac{1}{n!} \frac{n!}{(n-k)!k!} A^k B^{n-k} \xlongequal{n-k=m} \sum_{k=0}^\infty \sum_{m=0}^\infty \frac{A^k}{k!} \frac{B^m}{m!}$

$= e^A e^B$ $\therefore e^A e^B = e^{A+B}$

② 2×2 matrix

$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ $[A,B] = AB - BA = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = Z.$

$A + B = X.$ $A^2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = 0$ $B^2 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = 0$

$$e^A = 1 + A + \frac{A^2}{2!} + \cdots = I + A, \quad e^B = 1 + B + \frac{B^2}{2!} + \cdots = I + B \Rightarrow e^A e^B = \begin{bmatrix}1&1\\0&1\end{bmatrix}\begin{bmatrix}1&0\\1&1\end{bmatrix} = \begin{bmatrix}2&1\\1&1\end{bmatrix}$$

$$e^{A+B} = e^X = e^{|+X\rangle\langle+X| - |-X\rangle\langle-X|} = e|+X\rangle\langle+X| + \frac{1}{e}|-X\rangle\langle-X| = \begin{bmatrix}\frac{e}{2}&\frac{e}{2}\\\frac{e}{2}&\frac{e}{2}\end{bmatrix} + \begin{bmatrix}\frac{1}{2e}&-\frac{1}{2e}\\-\frac{1}{2e}&\frac{1}{2e}\end{bmatrix} = \begin{bmatrix}\frac{1}{2}(e+\frac{1}{e})&\frac{1}{2}(e-\frac{1}{e})\\\frac{1}{2}(e-\frac{1}{e})&\frac{1}{2}(e+\frac{1}{e})\end{bmatrix}$$

$$\Rightarrow e^{A+B} \neq e^A e^B \qquad \therefore A = \begin{bmatrix}0&1\\0&0\end{bmatrix} \quad B = \begin{bmatrix}0&0\\1&0\end{bmatrix} \Rightarrow e^{A+B} = e^A e^B$$

(b) $$H_s = \frac{1}{2}\left(J_x \sigma_0^x \sigma_1^x + J_y \sigma_0^y \sigma_1^y + J_z \sigma_0^z \sigma_1^z\right)$$

$$\sigma_0^x \sigma_1^x = \begin{bmatrix}0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0\end{bmatrix} \quad \sigma_0^y \sigma_1^y = \begin{bmatrix}0&0&0&-1\\0&0&1&0\\0&1&0&0\\-1&0&0&0\end{bmatrix} \quad \sigma_0^z \sigma_1^z = \begin{bmatrix}1&0&0&0\\0&-1&0&0\\0&0&-1&0\\0&0&0&-1\end{bmatrix}$$

$$B(\theta) = \prod_\alpha e^{-i\frac{t}{2}J_\alpha \sigma_0^\alpha \sigma_1^\alpha} = \begin{bmatrix}e^{-i\theta_z}\cos\gamma & 0 & 0 & -e^{-i\theta_z}\sin\gamma\\ 0 & e^{i\theta_z}\cos\delta & -e^{i\theta_z}\sin\delta & 0\\ 0 & -ie^{i\theta_z}\sin\delta & e^{i\theta_z}\cos\delta & 0\\ -ie^{-i\theta_z}\sin\gamma & 0 & 0 & e^{-i\theta_z}\cos\gamma\end{bmatrix}$$

for $N=2$

$$H_s = \sum_{j=1}^{N-1} H_j, \quad H_j = \frac{1}{2}\left(J_x \sigma_j^x \sigma_{j+1}^x + J_y \sigma_j^y \sigma_{j+1}^y + J_z \sigma_j^z \sigma_{j+1}^z\right)$$

$= H_{even} + H_{odd}$ when $H_{odd} = \sum_{j\,odd} H_j$, $H_{even} = \sum_{j\,even} H_j$

$$e^{-itH_{odd}} = e^{-it\sum_{j\,odd} H_j} = e^{-it\sum_j \frac{1}{2}\left(J_{x,2j+1}\sigma_{2j+1}^x \sigma_{2j+2}^x + J_{y,2j+1}\sigma_{2j+1}^y \sigma_{2j+2}^y + J_{z,2j+1}\sigma_{2j+1}^z \sigma_{2j+2}^z\right)}$$

$$= e^{-it\sum_j \frac{1}{2}J_{x,2j+1} S_{2j+1}^x + \frac{1}{2}J_{y,2j+1} S_{2j+1}^y + \frac{1}{2}J_{z,2j+1} S_{2j+1}^z}$$

$$[S_j^\alpha, S_j^\beta] = 0$$

$$[S_j^\alpha, S_{j+2}^\beta] = \sigma_j^\alpha \sigma_{j+1}^\alpha \sigma_{j+2}^\beta \sigma_{j+3}^\beta - \sigma_{j+2}^\beta \sigma_{j+3}^\beta \sigma_j^\alpha \sigma_{j+1}^\alpha = I^{\otimes j-1} \otimes \sigma^\alpha \otimes \sigma^\alpha \otimes \sigma^\beta \otimes \sigma^\beta \otimes I^{\otimes N-3-j}$$

$$- I^{\otimes j-1} \otimes \sigma^\alpha \otimes \sigma^\alpha \otimes \sigma^\beta \otimes \sigma^\beta \otimes I^{\otimes N-3-j} = 0$$

$\Rightarrow$ for $i, j \in \mathbb{N}$, $[S_{2i+1}^\alpha, S_{2j+1}^\beta] = 0$

$$\Rightarrow e^{-itH_{odd}} = \prod_{j,\alpha\in\{x,y,z\}} e^{-\frac{it}{2}J_{\alpha,2j+1} S_{2j+1}^\alpha} = \prod_j B_{2j-1}(\theta) \quad \text{when } \theta = \frac{t}{2}[J_x\ J_y\ J_z]^T$$

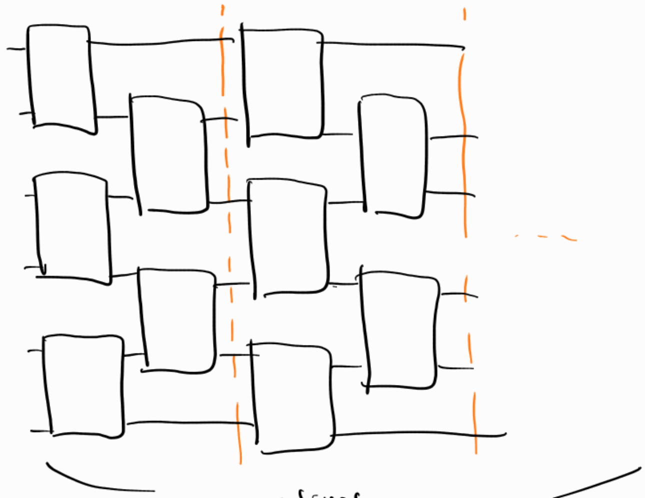
$$e^{-itH_{even}} = e^{-it\sum_{j\,even} H_j} = e^{-it\sum_j \frac{1}{2}\left(J_{x,2j}\sigma_{2j}^x \sigma_{2j+1}^x + J_{y,2j}\sigma_{2j}^y \sigma_{2j+1}^y + J_{z,2j}\sigma_{2j}^z \sigma_{2j+1}^z\right)}$$

$$= e^{-it\sum_j \frac{1}{2}J_{x,2j} S_{2j}^x + \frac{1}{2}J_{y,2j} S_{2j}^y + \frac{1}{2}J_{z,2j} S_{2j}^z}$$

$$[S_j^\alpha, S_j^\beta] = 0, \quad [S_j^\alpha, S_{j+2}^\beta] = 0$$

$$\Rightarrow e^{-itH_{even}} = \prod_j B_{2j}(\theta), \quad \theta = \frac{t}{2}[J_x\ J_y\ J_z]^T$$

(c)



n times

$\eta$ trotter time, 2 gates for each step

$\Rightarrow$ depth 2

$\therefore$ 2.

$$S_j^{\alpha} = \sigma_j^{\alpha}\sigma_{j+1}^{\alpha}$$

$$U(t) \cong \left(\prod_j B_j(\theta/n)\right)^n \qquad \prod_j B_j(\theta/n) = e^{-it\sum_{J=1}^{N-1}\frac{1}{2}\left(J_{xJ} S_J^x + J_{yJ} S_J^y + J_{zJ} S_J^z\right)}$$

$\Rightarrow$ for each qubit (j), qubit undergoes gate J-1 th, Jth.

$\Rightarrow$ depth 2. $\therefore$ 2

# #4. Staggered Magnetization

$O = \sum_x \lambda_x |x\rangle\langle x|$ let $|\psi\rangle = \sum_x \alpha_x |x\rangle$, $P_x = |\langle x|\psi\rangle|^2 = |\alpha_x|^2$

O is Hermitian Matrix. Therefore, O can be represented by spectral decomposition with real eigenvalue $\lambda_x$.

$$\langle\psi|O|\psi\rangle = \langle\psi|\sum_x \lambda_x |x\rangle\langle x|\,|\psi\rangle = \sum_x \lambda_x \langle\psi|x\rangle\langle x|\psi\rangle = \sum_x \lambda_x |\langle x|\psi\rangle|^2 = \sum_x P_x\lambda_x$$

$$\therefore \langle\psi|O|\psi\rangle = \sum_x P_x\lambda_x$$

# #. Test Pass

I passed test from test_uxz() to test_xyzm_evolution().