Homework 10 Problem 2

We are given:

$$\begin{pmatrix} b_1 & c_1 & 0 & \cdots & 0 \\ a_2 & b_2 & c_2 & & \vdots \\ 0 & a_3 & b_3 & \ddots & 0 \\ \vdots & & \ddots & \ddots & c_{n-1} \\ 0 & \cdots & 0 & a_n & b_n \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \\ \vdots \\ d_n \end{pmatrix}$$

Now,

$$\begin{pmatrix} 1 & \frac{c_1}{b_1} & 0 & \cdots & 0 \\ a_2 & b_2 & c_2 & & \vdots \\ 0 & a_3 & b_3 & \ddots & 0 \\ \vdots & & \ddots & \ddots & c_{n-1} \\ 0 & \cdots & 0 & a_n & b_n \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} \frac{d_1}{b_1} \\ d_2 \\ d_3 \\ \vdots \\ d_n \end{pmatrix}$$

$$= \begin{pmatrix} 1 & \frac{c_1}{b_1} & 0 & \cdots & 0 \\ 0 & b_2 - a_2\frac{c_1}{b_1} & c_2 & & \vdots \\ 0 & a_3 & b_3 & \ddots & 0 \\ \vdots & & \ddots & \ddots & c_{n-1} \\ 0 & \cdots & 0 & a_n & b_n \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} \frac{d_1}{b_1} \\ d_2 - a_2\frac{d_1}{b_1} \\ d_3 \\ \vdots \\ d_n \end{pmatrix}$$

$$= \begin{pmatrix} 1 & \frac{c_1}{b_1} & 0 & \cdots & 0 \\ 0 & 1 & \frac{c_2}{b_2 - a_2\frac{c_1}{b_1}} & & \vdots \\ 0 & a_3 & b_3 & \ddots & 0 \\ \vdots & & \ddots & \ddots & c_{n-1} \\ 0 & \cdots & 0 & a_n & b_n \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} \frac{d_1}{b_1} \\ \frac{d_2 - a_2\frac{d_1}{b_1}}{b_2 - a_2\frac{c_1}{b_1}} \\ d_3 \\ \vdots \\ d_n \end{pmatrix}$$

$$= \begin{pmatrix} 1 & \frac{c_1}{b_1} & 0 & \cdots & 0 \\ 0 & 1 & \frac{c_2}{b_2 - a_2\frac{c_1}{b_1}} & & \vdots \\ 0 & 0 & b_3 - a_3\frac{c_2}{b_2 - a_2\frac{c_1}{b_1}} & \ddots & 0 \\ \vdots & & \ddots & \ddots & c_{n-1} \\ 0 & \cdots & 0 & a_n & b_n \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} \frac{d_1}{b_1} \\ \frac{d_2 - a_2\frac{d_1}{b_1}}{b_2 - a_2\frac{c_1}{b_1}} \\ d_3 - a_3\frac{d_2 - a_2\frac{d_1}{b_1}}{b_2 - a_2\frac{c_1}{b_1}} \\ \vdots \\ d_n \end{pmatrix}$$

$$
= \begin{pmatrix} 1 & \frac{c_1}{b_1} & 0 & \cdots & 0 \\ 0 & 1 & \frac{c_2}{b_2 - a_2 \frac{c_1}{b_1}} & & \vdots \\ 0 & 0 & 1 & \ddots & 0 \\ \vdots & & \ddots & \ddots & c_{n-1} \\ 0 & \cdots & 0 & a_n & b_n \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} \frac{d_1}{b_1} \\ \frac{d_2 - a_2 \frac{d_1}{b_1}}{b_2 - a_2 \frac{c_1}{b_1}} \\ \frac{d_3 - a_3 \frac{d_2 - a_2 \frac{d_1}{b_1}}{b_2 - a_2 \frac{c_1}{b_1}}}{b_3 - a_3 \frac{c_2}{b_2 - a_2 \frac{c_1}{b_1}}} \\ \vdots \\ d_n \end{pmatrix}
$$

Here c_3 becomes $\frac{c_3}{b_3 - a_3 \frac{c_2}{b_2 - a_2 \frac{c_1}{b_1}}}$. Let changed c_i become c_i', and changed d_i become d_i'. Giving,

$$
= \begin{pmatrix} 1 & c_1' & 0 & \cdots & 0 \\ 0 & 1 & \frac{c_2}{b_2 - a_2 c_1'} & & \vdots \\ 0 & 0 & 1 & \ddots & 0 \\ \vdots & & \ddots & \ddots & c_{n-1} \\ 0 & \cdots & 0 & a_n & b_n \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} d_1' \\ \frac{d_2 - a_2 d_1'}{b_2 - a_2 c_1'} \\ \frac{d_3 - a_3 d_2'}{b_3 - a_3 c_2'} \\ \vdots \\ d_n \end{pmatrix}
$$

Here c_3 becomes $\frac{c_3}{b_3 - a_3 c_2'}$. Continuing we get:

$$
= \begin{pmatrix} 1 & c_1' & 0 & \cdots & 0 \\ 0 & 1 & \frac{c_2}{b_2 - a_2 c_1'} & & \vdots \\ 0 & 0 & 1 & \ddots & 0 \\ \vdots & & \ddots & \ddots & \frac{c_{n-1}}{b_{n-1} - a_{n-1} c_{n-2}'} \\ 0 & \cdots & 0 & a_n & b_n \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ x_n \end{pmatrix} = \begin{pmatrix} d_1' \\ \frac{d_2 - a_2 d_1'}{b_2 - a_2 c_1'} \\ \frac{d_3 - a_3 d_2'}{b_3 - a_3 c_2'} \\ \vdots \\ \frac{d_{n-1} - a_{n-1} d_{n-2}'}{b_{n-1} - a_{n-1} c_{n-2}'} \\ d_n \end{pmatrix}
$$

Finally,

$$
= \begin{pmatrix} 1 & c_1' & 0 & \cdots & 0 \\ 0 & 1 & \frac{c_2}{b_2 - a_2 c_1'} & & \vdots \\ 0 & 0 & 1 & \ddots & 0 \\ \vdots & & \ddots & \ddots & \frac{c_{n-1}}{b_{n-1} - a_{n-1} c_{n-2}'} \\ 0 & \cdots & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ x_n \end{pmatrix} = \begin{pmatrix} d_1' \\ \frac{d_2 - a_2 d_1'}{b_2 - a_2 c_1'} \\ \frac{d_3 - a_3 d_2'}{b_3 - a_3 c_2'} \\ \vdots \\ \frac{d_{n-1} - a_{n-1} d_{n-2}'}{b_{n-1} - a_{n-1} c_{n-2}'} \\ \frac{d_n - a_n d_{n-1}'}{b_n - a_n c_{n-1}'} \end{pmatrix}
$$

So, we have:

$$c'_i = \begin{cases} \frac{c_i}{b_i} & i = 1, \\ \frac{c_i}{b_i - a_i c'_{i-1}} & i = 2, ..., n-1 \end{cases} \qquad d'_i = \begin{cases} \frac{d_i}{b_i} & i = 1, \\ \frac{d_i - a_i d'_{i-1}}{b_i - a_i c'_{i-1}} & i = 2, ..., n \end{cases}$$

Now getting the solutions:

$$x_n = d'_n$$

$$\begin{aligned} x_{n-1} + x_n c'_{n-1} &= d'_{n-1} \\ x_{n-1} &= d'_{n-1} - x_n c'_{n-1} \end{aligned}$$

$$\begin{aligned} x_{n-2} + x_{n-1} c'_{n-2} &= d'_{n-2} \\ x_{n-2} &= d'_{n-2} - x_{n-1} c'_{n-2} \end{aligned}$$

So clearly for the solutions:

$$x_i = \begin{cases} d'_i & i = n, \\ d'_i - x_{i+1} c'_i & i = n-1, ..., 1 \end{cases}$$