Homework 8 Problem 1a

$$
\begin{aligned}
dX &= f(X,t)dt + g(X,t)dW \\
F(X,t) &= (1+t)^2\cos(X)
\end{aligned}
$$

In the slides, we are given:

$$
dF = \left[\frac{\partial F}{\partial t} + \frac{\partial F}{\partial X}f + \frac{1}{2}\frac{\partial^2 F}{\partial X^2}g^2\right]dt + g\frac{\partial F}{\partial X}dW
$$

For geometric brownian motion: $f = \mu X(t)$ and $g = \sigma X(t)$ - where $X(t)$ is geometric Brownian motion (from Homework 7).

To make things easier:

$$
\begin{aligned}
\frac{\partial F}{\partial t} &= 2(1+t)\cos(X) \\
\frac{\partial F}{\partial X} &= -(1+t)^2\sin(X) \\
\frac{\partial^2 F}{\partial X^2} &= -(1+t)^2\cos(X)
\end{aligned}
$$

Also:

$$
\begin{aligned}
\cos(X) &= \frac{F(X,t)}{(1+t)^2} \\
\sin^2(X) &= 1 - \frac{F(X,t)^2}{(1+t)^4} \\
\sin(X) &= \frac{\sqrt{(1+t)^4 - F(X,t)^2}}{(1+t)^2}
\end{aligned}
$$

Substituting values:

$$
\begin{aligned}
dF &= \left[2(1+t)\cos(X) - (1+t)^2\sin(X)f - \frac{(1+t)^2}{2}\cos(X)g^2\right]dt - \left[g(1+t)^2\sin(X)\right]dW \\
&= \left[2\frac{(1+t)^2}{(1+t)}\cos(X) - (1+t)^2\sin(X)f - \frac{F}{2}g^2\right]dt - \left[g(1+t)^2\sin(X)\right]dW \\
&= \left[\frac{2F}{1+t} - (1+t)^2\sin(X)\mu X - \frac{F}{2}(\sigma X)^2\right]dt - \left[\sigma X(1+t)^2\sin(X)\right]dW \\
&= \left[\frac{2F}{1+t} - \mu(1+t)^2\frac{\sqrt{(1+t)^4 - F(X,t)^2}}{(1+t)^2}\cos^{-1}\left(\frac{F(X,t)}{(1+t)^2}\right) - \frac{F}{2}\left(\sigma\cos^{-1}\left(\frac{F(X,t)}{(1+t)^2}\right)\right)^2\right]dt \\
&- \left[\sigma\cos^{-1}\left(\frac{F(X,t)}{(1+t)^2}\right)(1+t)^2\frac{\sqrt{(1+t)^4 - F(X,t)^2}}{(1+t)^2}\right]dW \\
&= \left[\frac{2F}{1+t} - \mu\sqrt{(1+t)^4 - F(X,t)^2}\cos^{-1}\left(\frac{F(X,t)}{(1+t)^2}\right) - \frac{F}{2}\left(\sigma\cos^{-1}\left(\frac{F(X,t)}{(1+t)^2}\right)\right)^2\right]dt \\
&- \left[\sigma\cos^{-1}\left(\frac{F(X,t)}{(1+t)^2}\right)\sqrt{(1+t)^4 - F(X,t)^2}\right]dW
\end{aligned}
$$