Homework 3 Problem 1

We know,

$$FV_m = P(1+y)^m \tag{1}$$

$$P = \frac{F}{(1+y)^n} + \sum_{i=1}^{n} \frac{C}{(1+y)^i} \tag{2}$$

$$MD = \frac{1}{P}\left(\frac{nF}{(1+y)^n} + \sum_{i=1}^{n} \frac{iC}{(1+y)^i}\right) \tag{3}$$

where P is the price of the bond, m is the horizon (equal to D below), y is the interest rate, C is coupon payment, n is time to maturity, and F is the par value.
For two bonds we know that:

$$\begin{aligned} FV &= FV_1 + FV_2 \\ &= P_1(1+y)^D + P_2(1+y)^D \end{aligned} \tag{4}$$

Plugging (2) into (4):

$$\begin{aligned} FV &= \left(\frac{F_1}{(1+y)^n} + \sum_{i=1}^{n} \frac{C_1}{(1+y)^i}\right)(1+y)^D + \left(\frac{F_2}{(1+y)^n} + \sum_{i=1}^{n} \frac{C_2}{(1+y)^i}\right)(1+y)^D \\ &= F_1(1+y)^{D-n_1} + \sum_{i=1}^{n_1} C_1(1+y)^{D-i} + F_2(1+y)^{D-n_2} + \sum_{i=1}^{n_2} C_2(1+y)^{D-i} \end{aligned}$$

The conditions for immunization according to the book are:

1. $FV = L$ at horizon m
2. $\frac{\partial FV}{\partial y} = 0$
3. FV convex around y

At the horizon:

$$\frac{\partial FV}{\partial y} = \frac{\partial FV_1}{\partial y} + \frac{\partial FV_2}{\partial y} = 0$$

$$\begin{aligned}
&\implies (D - n_1)F_1(1+y)^{D-n_1-1} + \sum_{i=1}^{n_1}(D-i)C_1(1+y)^{D-i-1} \\
&+ (D - n_2)F_2(1+y)^{D-n_2-1} + \sum_{i=1}^{n_2}(D-i)C_2(1+y)^{D-i-1} = 0
\end{aligned}$$

$$\begin{aligned}
&\implies \frac{DF_1 - n_1F_1}{(1+y)^{n_1}} + \sum_{i=1}^{n_1}\frac{DC_1 - iC_1}{(1+y)^i} \\
&+ \frac{DF_2 - n_2F_2}{(1+y)^{n_2}} + \sum_{i=1}^{n_2}\frac{DC_2 - iC_2}{(1+y)^i} = 0
\end{aligned}$$

$$\begin{aligned}
&\implies -\left(\frac{n_1F_1}{(1+y)^{n_1}} + \sum_{i=1}^{n_1}\frac{iC_1}{(1+y)^i}\right) + D\left(\frac{n_1F_1}{(1+y)^{n_1}} + \sum_{i=1}^{n_1}\frac{iC_1}{(1+y)^i}\right) \\
&- \left(\frac{n_2F_2}{(1+y)^{n_2}} + \sum_{i=1}^{n_2}\frac{iC_2}{(1+y)^i}\right) + D\left(\frac{n_2F_2}{(1+y)^{n_2}} + \sum_{i=1}^{n_2}\frac{iC_2}{(1+y)^i}\right) = 0 \qquad (5)
\end{aligned}$$

From (2) and (3):

$$MD \cdot P = \left(\frac{nF}{(1+y)^n} + \sum_{i=1}^{n}\frac{iC}{(1+y)^i}\right)$$

Plugging into (5) and also using (2)

$$\begin{aligned}
&- D_1P_1 + DP_1 - D_2P_2 + DP_2 = 0 \\
&\implies D_1P_1 + D_2P_2 = D(P_1 + P_2) \\
&\implies \frac{P_1}{P_1+P_2}D_1 + \frac{P_2}{P_1+P_2}D_2 = D
\end{aligned}$$

Also we know that,

$$\begin{aligned}
\frac{P_1 + P_2}{P_1 + P_2} &= 1 \\
\frac{P_1}{P_1+P_2} + \frac{P_2}{P_1+P_2} &= 1
\end{aligned}$$

Let $\omega_1 = \frac{P_1}{P_1+P_2}$ and $\omega_2 = \frac{P_2}{P_1+P_2}$. Thus:

$$\begin{aligned}
\omega_1D_1 + \omega_2D_2 &= D \\
\omega_1 + \omega_2 &= 1
\end{aligned}$$