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Homework 9 Problem 1b

We are given:

$$dX_t = \mu(1 - c\ln X_t)X_t dt + \sigma X_t dW_t$$

Now, for stock price, S:

$$dS = \mu(1 - c\ln S)S dt + \sigma S dW_t \tag{1}$$

From the slides: Option price C is a function of $S(t)$ and t, i.e., $C = C(S(t), t) = C(x, t)\Big|_{x=S(t)}$

From **Ito's Lemma** we know that if $X(t)$ is a solution to dX and $F(X(t), t)$:

$$dF = \left[\frac{\partial F}{\partial t} + f\frac{\partial F}{\partial x} + \frac{1}{2}g^2\frac{\partial^2 F}{\partial x^2}\right]dt + g\frac{\partial F}{\partial x}dW$$

In our case, $f = \mu(1 - c\ln S)S$, $g = \sigma S$:

$$dC = \left[\frac{\partial C}{\partial t} + \mu(1 - c\ln S)\frac{\partial C}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 C}{\partial S^2}\right]dt + \sigma S\frac{\partial C}{\partial S}dW$$

Finally, using **Merton's trick** and *from the slides*: consider a portfolio of value π that eliminates risk (i.e. removes randomness):

$$\begin{aligned}
\pi &= \alpha C + \beta S \\
d\pi &= \alpha dC + \beta dS \\
&= \alpha\left[\frac{\partial C}{\partial t} + \mu(1 - c\ln S)\frac{\partial C}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 C}{\partial S^2}\right]dt + \alpha\sigma S\frac{\partial C}{\partial S}dW \\
&+ \beta\mu(1 - c\ln S)S dt + \beta\sigma S dW_t
\end{aligned}$$

Now, if $\beta = -\alpha\frac{\partial C}{\partial S}$:

$$\begin{aligned}
d\pi &= \alpha\left[\frac{\partial C}{\partial t} + \mu(1 - c\ln S)\frac{\partial C}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 C}{\partial S^2}\right]dt + \alpha\sigma S\frac{\partial C}{\partial S}dW \\
&- \alpha\frac{\partial C}{\partial S}\mu(1 - c\ln S)S dt - \alpha\frac{\partial C}{\partial S}\sigma S dW_t \\
&= \alpha\left[\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 C}{\partial S^2}\right]dt
\end{aligned}$$

Now that there is no more randomness in $d\pi$; π must grow with riskless rate, r, (i.e. $\pi(t) = \pi(0)e^{rt}$):

$$\begin{aligned}
d\pi &= \pi r dt \\
&= \alpha \left(C - \frac{\partial C}{\partial S} S \right) r dt \\
\alpha \left[\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} \right] dt &= \alpha \left(C - \frac{\partial C}{\partial S} S \right) r dt \\
\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} &= \left(C - \frac{\partial C}{\partial S} S \right) r \\
\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} &= rC - rS \frac{\partial C}{\partial S} \\
\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + rS \frac{\partial C}{\partial S} &= rCs
\end{aligned}$$

The result is the same as the Black-Scholes(-Merton) equation. This is because the option price is independent of the value of μ (mean) - it depends on the volatility (σ).