Travelling pulses in neural fields with continuous and discontinuous neuronal activation

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1. Preliminaries

In this section we introduce the notations and definitions, and formulate the theorems required in the subsequent sections.

Let $\mathfrak{B}$ be a real Banach space with the norm $\|\cdot\|_{\mathfrak{B}}$. For any open bounded subset D of $\mathfrak{B}$, we denote by ∂D and $\overline{D}$ the boundary and the closure of D in $\mathfrak{B}$, respectively. We denote by $\deg(\Phi, D, \mathfrak{b}_0)$ and $\mathrm{ind}(\Phi, D)$ the degree and the topological index of an arbitrary operator $\Phi : \overline{D} \to \mathfrak{B}$, respectively (if they are well-defined).

Lemma 1.1. *(see [6]) Let Λ be a compact subset of $\mathbb{R}$, and an operator $T : \Lambda \times \overline{D} \to \mathfrak{B}$ be continuous with respect to both variables and the set $\{T(\Lambda, \overline{D})\}$ be a pre-compact set in $\mathfrak{B}$. If $\lambda_n \to \lambda_0$ and $T(\lambda_n, \mathfrak{b}_n) = \mathfrak{b}_n$, then any limit point of $\{\mathfrak{b}_n\}$ is a solution of the equation $T(\lambda_0, \mathfrak{b}) = \mathfrak{b}$.*

Definition 1.2. *(see [4]) The family $\{\mathrm{h}_\lambda\}$, $(\lambda \in [0, 1])$ of operators $\mathrm{h}_\lambda : \overline{D} \to \mathfrak{B}$ is called homotopy if $\mathrm{h}_\lambda(\mathfrak{b})$ is continuous with respect to $(\lambda, \mathfrak{b})$ on $[0, 1] \times \overline{D}$.*

Lemma 1.3. *(see [4]) Let $\{\mathrm{h}_\lambda\}$ be a homotopy of operators $\mathrm{h}_\lambda : \overline{D} \to \mathfrak{B}$ and $\mathrm{h}_\lambda - I$ be compact for each $\lambda \in [0, 1]$. If for any $\mathfrak{b} \in \partial D$ and $\lambda \in [0, 1]$, $\mathrm{h}_\lambda \mathfrak{b} \neq \mathfrak{b}_0$, then $\deg(\mathrm{h}_\lambda, D, \mathfrak{b}_0)$ is independent of λ.*

Definition 1.4. *(see [3]) Let $\mathfrak{D} \subset \mathfrak{B}$ is an absolute neighborhood retract (see, e.g. [3]), $D \subset \mathfrak{D}$ be open and bounded, and $\psi : D \to \mathfrak{D}$ be a continuous mapping. If the fixed point set of ψ is compact in $\mathfrak{B}$, then ψ is called an admissible mapping.*

Lemma 1.5. *(see [3]) Let* $\psi : D \to \mathfrak{D}$ *be an admissible compact mapping and* $\phi : \mathfrak{D} \to \mathfrak{D}'$ *be a homeomorphism. Then* $\phi \circ \psi \circ \phi^{-1} : \phi(D) \to \mathfrak{D}'$ *is also an admissible compact mapping and*

$$\mathrm{ind}(\psi, D) = \mathrm{ind}(\phi \circ \psi \circ \phi^{-1}, \phi(D)).$$

We define $\widetilde{C}^1(\mathbb{R}, \mathbb{R})$ to be the space of all functions $\zeta : \mathbb{R} \to \mathbb{R}$, continuous together with their first derivatives and satisfying the condition $\lim\limits_{x \to -\infty} \zeta(x) = -\lim\limits_{x \to \infty} \zeta(x) = +0$. The space $\widetilde{C}^1(\mathbb{R}, \mathbb{R})$ is obviously a Banach space with respect to the norm $\|\zeta\|_{\widetilde{C}^1(\mathbb{R},\mathbb{R})} = \max\limits_{x \in \mathbb{R}} |\zeta(x)| + \max\limits_{x \in \mathbb{R}} |\zeta'(x)|$.

2. Main results

2.1. Problem setting

We consider the "classical" Amari neural field equation equipped with a recovery mechanism (such as e.g. spike frequency adaptation, synaptic depression, or some other effect that limits the excitation of the neural network) modeled by a slow, local negative feedback component v added to the synaptic spatial coupling (see e.g. [7]), and described by

$$\begin{gathered} \partial_t u(t,x) = -u(t,x) + \int\limits_{\mathbb{R}} \omega(x-y) f_\beta(u(t,y))dy - v(t,x), \\ \frac{1}{\epsilon}\partial_t v(t,x) = u(t,x) - \sigma v(t,x), \\ t \geq 0,\ x \in \mathbb{R},\ \sigma > 0,\ 0 < \epsilon \ll 1. \end{gathered} \tag{1}$$

Here the steepness of the firing rate function f_β is parameterised by $\beta \in [0, \infty)$.

We impose the following assumptions on the functions involved in (1):

(**A1**) The connectivity kernel $\omega \in C^1(\mathbb{R}, \mathbb{R}) \bigcap L(\mathbb{R}, \mu, \mathbb{R})$ is non-negative.

(**A2**) For $\beta = 0$, the firing rate function is represented by the Heaviside-type function

$$f_0(u) = \begin{cases} 0, & u \leq h, \\ 1, & u > h, \end{cases}$$

where $h > 0$.

(**A3**) For each $\beta > 0$, $f_\beta : \mathbb{R} \to [0, 1]$ is non-decreasing and continuous. Moreover, as $\beta \to \widehat{\beta}$:

(i) $f_\beta \to f_{\widehat{\beta}}$ uniformly on $\mathbb{R}$ for $\widehat{\beta} \in (0, \infty)$;

(ii) for any $\varepsilon > 0$, $f_\beta \to f_{\widehat{\beta}}$ uniformly on $\mathbb{R} \setminus B_R(h, \varepsilon)$ for $\widehat{\beta} = 0$.

We introduce now the definition of "travelling pulse"-solutions to the equation (1) (or simply travelling pulses in the neural field (1)).

Definition 2.1. *Let $h > 0$ be fixed. We say that $u \in C([0,\infty), \widetilde{C}^1(\mathbb{R},\mathbb{R}))$ is a travelling pulse with the speed $c \in \mathbb{R}$, if $u(t,x) = U(x - ct)$, where the travelling pulse profile $U \in \widetilde{C}^1(\mathbb{R},\mathbb{R})$ satisfies the following properties:*

(**B1**) *$U(x) > h$ on $(0, a) \subset \mathbb{R}$;*

(**B2**) *$U(x) < h$ on $\mathbb{R} \setminus [0, a]$.*

If $U \in \widetilde{C}^1(\mathbb{R},\mathbb{R})$ also satisfies the property

(**B3**) *$U'(0) \neq 0$ and $U'(a) \neq 0$,*

then the travelling pulse is called a regular travelling pulse.

Remark 2.2. *In Definition 2.1 and further on, the uniqueness of the pair $(0, a > 0)$ is understood modulo translation due to the translation invariance property of $U \in \widetilde{C}^1(\mathbb{R},\mathbb{R})$.*

The rest part of this section is devoted to the problem of existence of regular travelling pulses in neural fields with continuous and discontinuous firing rate functions and continuous dependence of these solutions on the parameter β.

2.2. Existence and continuous dependence on firing rate functions of travelling pulse profiles

In this subsection we prove the main two theorems providing conditions for existence and continuous dependence on firing rate function of travelling pulse profile.

If the travelling pulse solution to (1) exists, the function U that determines the profile of the travelling pulse $u(t,x) = U(x - ct)$ satisfies the following system of equations

$$
\begin{gathered}
-cU'(x) = -U(x) + \int\limits_R \omega(x-y) f_\beta(U(y))dy - V(x), \\
-cV'(x) = \epsilon U(x) - \epsilon\sigma V(x), \\
x \in \mathbb{R},
\end{gathered}
\tag{2}
$$

which we conveniently rewrite in the matrix form

$$\begin{pmatrix} U'(x) \\ V'(x) \end{pmatrix} = A \begin{pmatrix} U(x) \\ V(x) \end{pmatrix} - \frac{1}{c} \begin{pmatrix} (\mathcal{F}_\beta U)(x) \\ 0 \end{pmatrix}, \quad x \in \mathbb{R}, \tag{3}$$

where

$$A = \begin{pmatrix} 1/c & 1/c \\ -\epsilon/c & \epsilon\sigma/c \end{pmatrix}, \tag{4}$$

$$(\mathcal{F}_\beta U)(x) = \int\limits_R \omega(x-y) f_\beta(U(y)) dy, \quad x \in \mathbb{R}. \tag{5}$$

For the travelling pulse speed $c < 0$, the eigenvalues $\lambda_{1,2}$,

$$\lambda_1 = \frac{1 + \epsilon\sigma - \sqrt{1 - \epsilon\sigma - 4\epsilon}}{2c}, \ \lambda_2 = \frac{1 + \epsilon\sigma + \sqrt{1 - \epsilon\sigma - 4\epsilon}}{2c} \tag{6}$$

of the matrix A are negative, so there exists a unique solution of the system

$$\begin{pmatrix} \xi'(x) \\ \eta'(x) \end{pmatrix} = A \begin{pmatrix} \xi(x) \\ \eta(x) \end{pmatrix} - \frac{1}{c} \begin{pmatrix} \xi(x) \\ 0 \end{pmatrix}, \ \xi, \eta \in C(\mathbb{R}, \mathbb{R})), \ x \in \mathbb{R}.$$

in $C(\mathbb{R}, \mathbb{R}) \times C(\mathbb{R}, \mathbb{R})$, which can be found as follows (see e.g. [1]):

$$\begin{pmatrix} \xi(x) \\ \eta(x) \end{pmatrix} = \frac{1}{c} \int\limits_{-\infty}^{x} \exp\Big(A(x-y)\Big) \begin{pmatrix} \xi(y) \\ 0 \end{pmatrix} dy, \quad x \in R. \tag{7}$$

In the studies of travelling pulses, the decay σ of the negative feedback in the neural medium is often neglected (see e.g. [7]). Our approach allows to omit this restriction.

The equation (7) defines a bounded linear operator $\mathcal{L} : C(\mathbb{R}, \mathbb{R}) \to C(\mathbb{R}, \mathbb{R})$

$$(\mathcal{L}\xi)(x) = \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, -\frac{1}{c} \int\limits_{-\infty}^{x} \exp\Big(A(x-y)\Big) \begin{pmatrix} \xi(y) \\ 0 \end{pmatrix} dy, \right) \quad x \in R. \tag{8}$$

The system (2) can thus be reduced to the following operator equation

$$U = \mathcal{H}_\beta U, \tag{9}$$

$$\mathcal{H}_\beta = \mathcal{L}\mathcal{F}_\beta, \tag{10}$$

where the mappings $\mathcal{F}_\beta$ (for each $\beta \in [0, \infty)$) and $\mathcal{L}$ are defined by (5) and (8), respectively.

Below we prove the theorem on continuous dependence of travelling pulse profiles (if they exist) on the parameter β. In order to do that, we demonstrate that the assumptions (**A1**) – (**A3**) provide collective compactness of the operators $\mathcal{H}_\beta$, $\beta \in [0, \infty)$, defined by (10) as acting from some subset of $\widetilde{C}^1(\mathbb{R}, \mathbb{R})$ to $\widetilde{C}^1(\mathbb{R}, \mathbb{R})$.

Theorem 2.3. *Under the assumptions* (**A1**) – (**A3**)*, if for some $h > 0$ and $\widehat{U} \in \widetilde{C}^1(\mathbb{R}, R)$ satisfying* (**B1**) – (**B3**) *there exists* $\mathrm{r} > 0$ *such that for any $\beta \in (0, 1]$, there exist travelling pulse profiles $U_\beta \in B_{\widetilde{C}^1(\mathbb{R},\mathbb{R})}(\widehat{U}, \mathrm{r})$ satisfying* (9)*, then there exists a travelling pulse profile that corresponds to $\beta = 0$ and belongs to $\overline{\{U_\beta\}}$. Moreover, if for $\beta = 0$, the travelling pulse profile, say U_0, is unique then $\|U_\beta - U_0\|_{\widetilde{C}^1(\mathbb{R},\mathbb{R})} \to 0$, as $\beta \to 0$.*

Proof. We first decompose the mapping $\mathcal{F}_\beta$ as follows

$$\mathcal{F}_\beta = \mathcal{W}\mathcal{N}_\beta,$$

where

$$(\mathcal{W}g)(x) = \int\limits_R \omega(x - y)g(y)dy, \quad x \in R, \tag{11}$$

$$(\mathcal{N}_\beta g)(x) = f_\beta(g(y))dy, \quad x \in R. \tag{12}$$

Due to the assumptions (**A2**) and (**A3**), the operator $\mathcal{N}_\beta$ translates any function from $\widetilde{C}^1(\mathbb{R}, \mathbb{R})$ to $L(\mathbb{R}, \mu, \mathbb{R})$. We show that for some $\mathrm{r}_0 > 0$, the operator $\mathcal{N}_\beta : B_{\widetilde{C}^1(\mathbb{R},\mathbb{R})}(\widehat{U}, \mathrm{r}) \to L(\mathbb{R}, \mu, \mathbb{R})$ defined by (12) is continuous at any $\widehat{\beta} \in [0, \infty)$ uniformly on $B_{\widetilde{C}^1(\mathbb{R},\mathbb{R})}(\widehat{U}, \mathrm{r}_0)$. For $\widehat{\beta} \in [0, \infty)$, and $U \in B_{\widetilde{C}^1(\mathbb{R},\mathbb{R})}(\widehat{U}, \mathrm{r}_0)$, we estimate the norm $\|N_\beta U - N_{\widehat{\beta}} U\|_{L(\mathbb{R},\mu,\mathbb{R})}$, as $\beta \to \widehat{\beta}$. If $\widehat{\beta} \in (0, \infty)$, the assumption (**A3**) implies that

$$\int\limits_R |f_\beta(U(x)) - f_{\widehat{\beta}}(U(x))|dx \to 0, \;\; \beta \to \widehat{\beta},$$

uniformly with respect to $U \in B_{\widetilde{C}^1(\mathbb{R},\mathbb{R})}(\widehat{U}, \mathrm{r}_0)$. Consider now the more involved case $\widehat{\beta} = 0$.

$$\int_{\mathbb{R}} |f_\beta(U(x)) - f_0(U(x))|dx =$$

$$= \int_{R\setminus B_R(h,\mathrm{r}_0)} |f_\beta(U(x)) - f_0(U(x))|dx + \int_{B_R(h,\mathrm{r}_0)} |f_\beta(U(x)) - f_0(U(x))|dx. \tag{13}$$

For all $x \in R \setminus B_R(h, \mathrm{r}_0)$ and any $U \in B_{\widetilde{C}^1(\mathbb{R},\mathbb{R})}(\widehat{U}, \mathrm{r}_0)$, the value $U(x)$ belongs to $R \setminus B_R(h, \mathrm{r}_0)$. It follows from (**A3**) that the first summand on the right-hand side of (13) converges to 0 uniformly on $B_{\widetilde{C}^1(\mathbb{R},\mathbb{R})}(\widehat{U}, \mathrm{r}_0)$, as $\beta \to 0$. Next,

$$\int_{B_R(h,\mathrm{r}_0)} |f_\beta(U(x)) - f_0(U(x))|dx < \frac{1}{c_0} \int_{-\|\widehat{U}\|_{\widetilde{C}^1(\mathbb{R},\mathbb{R})}}^{\|\widehat{U}\|_{\widetilde{C}^1(\mathbb{R},\mathbb{R})}} |f_\beta(s) - f_0(s)|ds,$$

where $0 < \mathrm{const} < |U'(x)|$ for all $x \in R\setminus B_R(h, \mathrm{r}_0)$ and any $U \in B_{\widetilde{C}^1(\mathbb{R},\mathbb{R})}(\widehat{U}, \mathrm{r}_0)$ (Without loss of generality we assume here that $\mathrm{r}_0 < \min\limits_{x \in B_R(h,\mathrm{r}_0)} |\widehat{U}'(x)|$). Finally, we notice that assumption (**A3**) guarantees convergence to 0 of the expression on the right-hand side of the latter inequality, as $\beta \to 0$.

Thus, the operator $\mathcal{N}_\beta$ is a bounded continuous mapping from $B_{\widetilde{C}^1(\mathbb{R},\mathbb{R})}(\widehat{U}, \mathrm{r}_0)$ to $L(\mathbb{R}, \mu, \mathbb{R})$. We notice that the operator $\mathcal{W}$ defined by (11) is a linear and continuous mapping from $L(\mathbb{R}, \mu, \mathbb{R})$ to $C(\mathbb{R}, \mathbb{R})$ provided that assumption (**A1**) holds true. Next, the operator $\mathcal{L}$ given by (8) can be considered as a linear continuous mapping from $C(\mathbb{R}, \mathbb{R})$ to $C^1(\mathbb{R}, \mathbb{R})$. We now demonstrate that if $\epsilon\sigma - \tau < 0$, $c < 0$, and $\xi \in C(\mathbb{R}, \mathbb{R})$, $\xi(x) \geq 0$, for all $x \in \mathbb{R}$, then $\mathcal{L}\xi \in \widetilde{C}^1(\mathbb{R}, \mathbb{R})$. Using eigenvalue decomposition of A in (4) we obtain

$$A = P \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} P^{-1},$$

where $\lambda_{1,2}$ are given by (6) and the columns of P are the corresponding eigenvectors of A. Let $P = (p_{ij})$, $i, j = 1, 2$, then $\mathcal{L}\xi$ can be rewritten as

$$\begin{gathered}(\mathcal{L}\xi)(x) = -\frac{1}{c} \int_{-\infty}^{x} k(x - y; c)\xi(y)dy, \\ k(x; c) = l_2 \exp(\lambda_2(c)x) - l_1 \exp(\lambda_1(c)x),\end{gathered} \tag{14}$$

where

$$l_1 = -\frac{p_{11}p_{22}}{p_{11}p_{22} - p_{12}p_{21}} \quad \text{and} \quad l_2 = -\frac{p_{12}p_{21}}{p_{11}p_{22} - p_{12}p_{21}}.$$

In a general form,

$$p_{1j} = \frac{(\epsilon\sigma - 1) - (-1)^j \sqrt{(\epsilon\sigma - 1)^2 - 4\epsilon}}{2\epsilon}, \quad p_{2j} = 1, \quad j = 1, 2.$$

Thus, we have $0 < l_1 < l_2$, so $k(|x|; c)$ is a typical example of a "wizard hat" function (see Fig. 1).

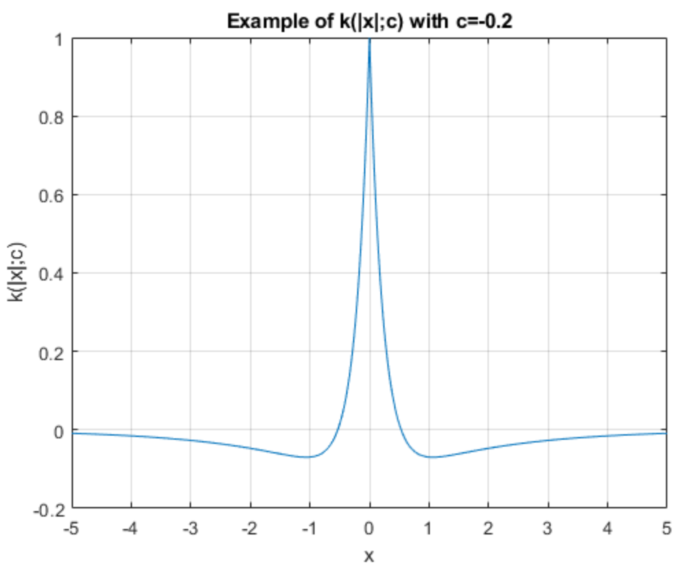

Figure 1: Example of the kernel $k(|x|; c)$ with $c = -0.2$, $\epsilon = 0.1$, $\sigma = 0$.

Now, for each non-negative function $\xi \in C(\mathbb{R}, \mathbb{R})$, it is easy to observe the asymptotics

$$\lim_{x \to -\infty} (\mathcal{L}\xi)(x) = -\lim_{x \to \infty} (\mathcal{L}\xi)(x) = +0.$$

We finally notice, that due to the assumptions (**A1**) – (**A3**), the function $\mathcal{W}\mathcal{H}_\beta g$ is non-negative for any $g \in C(\mathbb{R}, \mathbb{R})$, $\beta \in [0, \infty)$.

Thus, for any $\beta \in [0, \infty)$, it holds true that $\mathcal{H}_\beta : B_{\widetilde{C}^1(\mathbb{R},\mathbb{R})}(\widehat{U}, \mathrm{r}_0) \to \widetilde{C}^1(\mathbb{R}, \mathbb{R})$ and

$$\|\mathcal{H}_\beta U_i - \mathcal{H}_{\widehat{\beta}} U_0\|_{C^1(\mathbb{R},\mathbb{R})} \to 0, \; \beta \to \widehat{\beta},$$

where $U_0 \in B_{\widetilde{C}^1(\mathbb{R},\mathbb{R})}(\widehat{U}, \mathrm{r}_0)$, $\|U_i - U_0\|_{\widetilde{C}^1(\mathbb{R},\mathbb{R})} \to 0$.

We now prove that the operators $\mathcal{H}_\beta : B_{\widetilde{C}^1(\mathbb{R},\mathbb{R})}(\widehat{U}, \mathrm{r}_0) \to \widetilde{C}^1(\mathbb{R},\mathbb{R})$ ($\beta \in [0,\infty)$) are collectively compact.

We notice that due to (**A1**) – (**A3**) we have $\mathcal{WN}_\beta : B_{\widetilde{C}^1(\mathbb{R},\mathbb{R})}(\widehat{U}, \mathrm{r}_0) \to C(\mathbb{R},\mathbb{R})$. Choose arbitrary $\varepsilon > 0$. For the given $\varepsilon > 0$ there exist $\mathrm{r}_\varepsilon > 0$ and $\beta_\varepsilon > 0$ such that

$$\left| \int\limits_R \omega(x-y) f_\beta(U(y)) dy \right| < \varepsilon$$

for any $U \in B_{\widetilde{C}^1(\mathbb{R},\mathbb{R})}(\widehat{U}, \mathrm{r}_0)$, $x \in R \setminus B_R(0, \mathrm{r}_\varepsilon)$, and $\beta \in [0, \beta_\varepsilon)$.

The assumptions (**A1**) – (**A3**) imply that the set of functions

$$\Big\{ \int\limits_R \omega(x-y) f_\beta(U(y)) dy, \ \ U \in B_{\widetilde{C}^1(\mathbb{R},\mathbb{R})}(\widehat{U}, \mathrm{r}_0), \ x \in B_R(0, r - \varepsilon) \Big\}$$

is bounded and equicontinuous. According to Arzela–Ascoli theorem (see e.g [2], Section IV.6.7, Theorem 7), this set possesses an ε-net, say $\{\varsigma_1, \dots, \varsigma_{\mathrm{N}}\}$, $\mathrm{N} < \infty$, $\varsigma_i \in C(B_R(0, r-\varepsilon), \mathbb{R})$, $i = 1, \dots, \mathrm{N}$. For the set of functions

$$\Big\{ \int\limits_R \omega(x-y) f_\beta(U(y)) dy, \ \ U \in B_{\widetilde{C}^1(\mathbb{R},\mathbb{R})}(\widehat{U}, \mathrm{r}_0), \ x \in R \setminus B_R(0, r - \varepsilon) \Big\}$$

we construct an ε-net as follows. Each element of the set $\{\varsigma_1, \dots, \varsigma_{\mathrm{N}}\}$ we continuously connect to zero, so the set of all such continuations forms a finite ε-net for the set

$$\Big\{ \int\limits_R \omega(x-y) f_\beta(U(y)) dy, \ \ U \in B_{\widetilde{C}^1(\mathbb{R},\mathbb{R})}(\widehat{U}, \mathrm{r}_0), \ x \in R \Big\}.$$

Thus, the mappings (compositions) $\mathcal{WN}_\beta : B_{\widetilde{C}^1(\mathbb{R},\mathbb{R})}(\widehat{U}, \mathrm{r}_0) \to C(\mathbb{R},\mathbb{R})$ for $\beta \geq 0$ are collectively compact. As it was shown that the operator $\mathcal{L}$ given by (8) is a linear continuous operator from $C(\mathbb{R},\mathbb{R})$ to $\widetilde{C}^1(\mathbb{R},\mathbb{R})$, we get the collective compactness of the operators $\mathcal{H}_\beta : B_{\widetilde{C}^1(\mathbb{R},\mathbb{R})}(\widehat{U}, \mathrm{r}_0) \to \widetilde{C}^1(\mathbb{R},\mathbb{R})$.

The properties proven allow us to apply Lemma 1.1 to the operator $\mathcal{H}_\beta : B_{\widetilde{C}^1(\mathbb{R},\mathbb{R})}(\widehat{U}, \mathrm{r}_0) \to \widetilde{C}^1(\mathbb{R},\mathbb{R})$ and complete the proof. □

We are now ready to formulate the theorem that guarantees existence of travelling pulse profiles for $\beta \in (0,\infty)$ based on

Theorem 2.4. *(Existence) Let the conditions of Theorem 2.3 be satisfied and the constant* r *be taken from Theorem 2.3. Assume that there exists a travelling pulse profile* $\widehat{U} \in \widetilde{C}^1(\mathbb{R},\mathbb{R})$ *that satisfies* (9) *at* $\beta = 0$ *and conditions* (**B1**) – (**B3**), *and which is unique in* $B_{\widetilde{C}^1(\mathbb{R},\mathbb{R})}(\widehat{U}, \mathrm{r}_1)$ *(for some* $\mathrm{r}_1 < \mathrm{r}$*). Let also*

$$\deg(I - \mathcal{H}_0, B_{\widetilde{C}^1(\mathbb{R},\mathbb{R})}(\widehat{U}, \mathrm{r}_1), 0) \neq 0, \tag{15}$$

where the operator $\mathcal{H}_0 : B_{\widetilde{C}^1(\mathbb{R},\mathbb{R})}(\widehat{U}, \mathrm{r}) \to \widetilde{C}^1(\mathbb{R},\mathbb{R})$ *is defined by* (10). *Then for any* $\beta \in (0,1]$, *there exists a travelling pulse profile* $U_\beta \in B_{\widetilde{C}^1(\mathbb{R},\mathbb{R})}(\widehat{U}, \mathrm{r}_2)$ *satisfying* (9).

Proof. We prove that the family $\{\mathrm{h}_\beta\},\ \beta \in [0,1]$,

$$\mathrm{h}_\beta = I - \mathcal{H}_\beta \tag{16}$$

is homotopy. Continuity of $h_{(\cdot)}(\cdot)$ on $[0,1] \times B_{\widetilde{C}^1(\mathbb{R},\mathbb{R})}(\widehat{U}, \mathrm{r})$ was demonstrated in the proof of Theorem 2.3. It remains to prove that $h_\beta(u) \neq 0$ for any $\beta \in [0,1]$ and $u \in \partial B_{\widetilde{C}^1(\mathbb{R},\mathbb{R})}(\widehat{U}, \mathrm{r}_1)$.

Collective compactness of $\mathcal{H}_\beta : B_{\widetilde{C}^1(\mathbb{R},\mathbb{R})}(\widehat{U}, \mathrm{r}) \to \widetilde{C}^1(\mathbb{R},\mathbb{R})$ $(\beta \in [0,\infty))$, shown in the proof of Theorem 2.3, implies the following two possibilities for any sequence $\{U_{\beta_n}\} \subset B_{\widetilde{C}^1(\mathbb{R},\mathbb{R})}(\widehat{U}, \mathrm{r})$ $(\beta_n \to 0)$ of travelling pulse profiles satisfying (9):

1) $\|U_{\beta_n} - \widehat{U}\|_{\widetilde{C}^1(\mathbb{R},\mathbb{R})} \to 0$, as $\beta_n \to 0$;

2) for some $\widehat{n}$ and any $n > \widehat{n}$, $\|U_{\beta_n} - \widehat{U}\|_{\widetilde{C}^1(\mathbb{R},\mathbb{R})} > \mathrm{r}_1$ (without loss of generality we assume here $\beta_{\widehat{n}} > 1$).

This proves that $(I - \mathcal{H}_\beta)(U) \neq 0$ for any $U \in \partial B_{\widetilde{C}^1(\mathbb{R},\mathbb{R})}(\widehat{U}, \mathrm{r}_1)$ and $\beta \in [0,1]$.

Finally, we apply Lemma 1.5 to the homotopy (16), thus, proving the existence of travelling pulse profiles satisfying (9) for any $\beta \in (0,1]$. □

2.3. Existence and continuous dependence on firing rate functions of travelling pulses

In this subsection we derive conditions providing the validity of the previous section theorems for the travelling pulse profiles, and interpreting these results for the travelling pulses obtain the main result of the present research.

In the case of the Heaviside firing rate function, if a traveling pulse exists, its profile is given as

$$U(x;a,c) = -\frac{1}{c}\int\limits_{-\infty}^{x} k(x-y;c)(W(y-a)-W(y))dy,$$
$$W(y) = \int\limits_{0}^{y} \omega(\xi)d\xi,$$

or, equivalently,

$$\begin{gathered} U(x;a,c) = \frac{1}{c}\int\limits_{0}^{\infty} k(y;c)(W(y-x)-W(y-x+a))dy, \\ W(y) = \int\limits_{0}^{y} \omega(\xi)d\xi, \end{gathered} \tag{17}$$

where (a,c) are such that

$$\begin{cases} U(0;a,c) = \frac{1}{c}\int\limits_{0}^{\infty} k(y;c)(W(y)-W(y+a))dy = h, \\ U(a;a,c) = \frac{1}{c}\int\limits_{0}^{\infty} k(y;c)(W(y-a)-W(y))dy = h. \end{cases} \tag{18}$$

The necessary conditions for the existence of travelling pulse solutions of the speed c to (1) are naturally understood here as the correspondence between the pair $(c,h) \in (-\infty,0)\times(0,+\infty)$ and the pair $(\theta,\alpha) \in \mathbb{R}^2$ such that $U(\theta;\alpha-\theta,c) = h$ and $U(\alpha;\alpha-\theta,c) = h$, which determines the pulse width $\alpha-\theta$. We note that the travelling pulse is regular if

$$\begin{cases} \int\limits_{0}^{\infty} k(y;c)(\omega(y)-\omega(y+a))dy \neq 0, \\ \int\limits_{0}^{\infty} k(y;c)(\omega(y-a)-\omega(y))dy \neq 0. \end{cases} \tag{19}$$

Lemma 2.5. *Let the following condition be satisfied:*

$$\begin{cases} \int\limits_{0}^{\infty} k(y;c)\omega(y-a)dy \neq 0, \\ \int\limits_{0}^{\infty} k(y;c)\omega(y+a)dy \neq 0. \end{cases} \tag{20}$$

Then there exists such $\varepsilon > 0$ that the travelling pulse profile $U = U(\cdot; a, c)$ defined by (18) *is a unique solution to* (2) *in $B_{\widetilde{C}^1(\mathbb{R},\mathbb{R})}(U, \varepsilon)$ for $\beta = 0$.*

Proof. Assessing the derivatives of the expressions $U(\theta; \alpha - \theta, c) = h$ and $U(\alpha; \alpha - \theta, c) = h$ with respect to the parameters θ and α at the point $(\theta, \alpha) = (0, a)$, we obtain

$$\begin{cases} \int\limits_0^\infty k(y; c)\omega(y - a)dy = 0, \\ \int\limits_0^\infty k(y; c)\omega(y + a)dy = 0. \end{cases}$$

Thus, the condition (20) guarantees uniqueness of the solution $U = U(\cdot; a, c)$ in $B_{\widetilde{C}^1(\mathbb{R},\mathbb{R})}(U, \varepsilon)$ for some $\varepsilon > 0$. □

We express (17) in the operator form

$$U = \mathcal{H}_0 U.$$

According to Lemma 2.5, the fixed point $U = U(\cdot; a, c)$ of the operator $\mathcal{H}_0$ is unique in $B_{\widetilde{C}^1(\mathbb{R},\mathbb{R})}(U, \varepsilon)$. Thus, $\mathcal{H}_0$ maps $\overline{B_{\widetilde{C}^1(\mathbb{R},\mathbb{R})}(U, \varepsilon)}$ into some manifold $\mathcal{M} \subset \widetilde{C}^1(\mathbb{R}, \mathbb{R})$, $\mathcal{M} = \Big\{ \upsilon = \frac{1}{c} \int\limits_0^\infty k(y; c)(W(y - x + \theta) - W(y - x + \alpha))dy,\ (\theta, \alpha) \in \mathrm{M} \subset R^2 \Big\}$, where compact set M is chosen in a such way that it contains the points (θ_ν, α_ν) for all $\nu \in \overline{B_{\widetilde{C}^1(\mathbb{R},\mathbb{R})}(U, \varepsilon)}$. We define the mapping $\phi : \mathrm{M} \to \mathcal{M}$ as

$$\begin{gathered} \phi(\theta, \alpha) = \upsilon(x), \\ \upsilon(x) = \tfrac{1}{c} \int\limits_0^\infty k(y; c)(W(y - x + \theta) - W(y - x + \alpha))dy,\ x \in \mathbb{R}. \end{gathered} \tag{21}$$

Lemma 4.2.2. The mapping $\phi : \mathrm{M} \to \mathcal{M}$ defined by (21) is a homeomorphism, and the set $\mathcal{M}$ is an absolute neighborhood retract provided that any of the relations in (20) holds true.

Proof. First, we note that $\phi : \mathrm{M} \to \mathcal{M}$ is surjective by definition. We prove the injectivity of $\phi : \mathrm{M} \to \mathcal{M}$ using the following expressions for the Frechet derivatives of ϕ:

$$\phi'_\theta(\theta, \alpha) = \tfrac{1}{c} \int\limits_0^\infty k(y; c)\omega(y - \cdot + \theta)dy$$

$$\phi'_\alpha(\theta, \alpha) = -\tfrac{1}{c} \int\limits_0^\infty k(y; c)\omega(y - \cdot + \alpha)dy$$

Assuming $\phi'_\theta(0,a)=0$, we get $\int\limits_0^\infty k(y;c)\omega(y-x+\theta)dy=0$, for all $x\in\Omega$, which contradicts with (20). We, thus, have $\phi'_\theta(0,a)\neq 0$. In the same manner we obtain $\phi'_\alpha(0,a)\neq 0$, so the mapping $\phi:\mathrm{M}\to\mathcal{M}$ is a homeomorphism. By properties of homeomorphism, the fact that M is an absolute neighborhood retract implies that $\mathcal{M}=\phi(\mathrm{M})$ is an absolute neighborhood retract as well. $\square$

We now define

$$F=\mathcal{H}_0|_{\mathcal{M}\bigcap\overline{B_{\widetilde{C}^1(\mathbb{R},\mathbb{R})}(U,\varepsilon)}},$$
$$F:\mathcal{M}\bigcap\overline{B_{\widetilde{C}^1(\mathbb{R},\mathbb{R})}(U,\varepsilon)}\to\mathcal{M}.$$

The mapping $F:\mathcal{M}\bigcap\overline{B_{\widetilde{C}^1(\mathbb{R},\mathbb{R})}(U,\varepsilon)}\to\mathcal{M}$ is compact and admissible by its definition. By properties of the topological fixed point index, it holds true that

$$\mathrm{ind}(\mathcal{H}_0,B_{\widetilde{C}^1(\mathbb{R},\mathbb{R})}(U,\varepsilon))=\mathrm{ind}(F,\mathcal{M}\bigcap B_{\widetilde{C}^1(\mathbb{R},\mathbb{R})}(U,\varepsilon)).$$

We apply Lemma 1.3 and obtain

$$\mathrm{ind}(F,\mathcal{M}\bigcap B_{\widetilde{C}^1(\mathbb{R},\mathbb{R})}(U,\varepsilon))=\mathrm{ind}(\phi^{-1}\circ F\circ\phi,\phi^{-1}(F(\mathcal{M}\bigcap B_{\widetilde{C}^1(\mathbb{R},\mathbb{R})}(U,\varepsilon))).$$

Lemma 4.2.3. There exists $\delta>0$ such that $\Psi=\phi^{-1}\circ F\circ\phi$ maps $\overline{B_{\mathbb{R}^2}((0,a)\delta)}$ to M.

Proof. Let $\upsilon(x)=\frac{1}{c}\int\limits_0^\infty k(y;c)(W(y-x+\theta)-W(y-x+\alpha))dy$, $(\theta,\alpha)\in\mathrm{M}$. As the assumption (**A1**) is fulfilled, for any $\varepsilon>0$, one can find $\delta>0$ such that $\|u-U\|_{\widetilde{C}^1(\mathbb{R},\mathbb{R})}<\varepsilon$ for all $(\theta,\alpha)\in\overline{B_{R^2}((0,a),\delta)}$, where U is given by (17). The latter estimate implies that

$$\overline{B_{R^2}((0,a),\delta)}\subset\phi^{-1}(\mathcal{M}\bigcap B_{\widetilde{C}^1(\mathbb{R},\mathbb{R})}(U,\varepsilon)),$$

from where we conclude that

$$\mathcal{M}_\delta\subset F(\mathcal{M}\bigcap B_{\widetilde{C}^1(\mathbb{R},\mathbb{R})}(U,\varepsilon))$$
$$\mathcal{M}_\delta=\Big\{\upsilon(x)=\frac{1}{c}\int\limits_0^\infty k(y;c)(W(y-\cdot+\theta)-W(y-\cdot+\alpha))dy,$$
$$(\theta,\alpha)\in\overline{B_{R^2}((0,a),\delta)}\Big\}.$$

Thus, we obtain

$$\phi^{-1}(\mathcal{M}_\delta)=\overline{B_{R^2}((0,a),\delta)}\subset\phi^{-1}(F(\mathcal{M}\bigcap B_{\widetilde{C}^1(\mathbb{R},\mathbb{R})}(U,\varepsilon))),$$

and complete the proof. $\square$

Due to the fact that U is an isolated fixed point of F and topological invariance property of the index, $(0, a)$ is an isolated fixed point of Ψ. Thus, we have

$$\Psi(0,a) = \big(\Psi_1(0,a), \Psi_2(0,a)\big),$$
$$\Psi(0,a) = \tfrac{1}{c}\int_0^\infty k(y;c)(W(y-\Psi_1(a,b)) - W(y-\Psi_2(a,b)+\alpha))dy.$$

The topological index of the mapping $\Psi = \big(\Psi_1(\theta,\alpha), \Psi_2(\theta,\alpha)\big)$ for $(\theta,\alpha) = (0,a)$ can be found as

$$\operatorname{ind}(\Psi, \phi^{-1}(F(\mathcal{M} \bigcap B_{\widetilde{C}^1(\mathbb{R},\mathbb{R})}(U,\varepsilon))) =$$
$$= \operatorname{sgn}\left(\det\begin{pmatrix} (\Psi_1)'_\theta(0,a) - 1 & (\Psi_1)'_\alpha(0,a) \\ (\Psi_2)'_\theta(0,a) & (\Psi_2)'_\alpha(0,a) - 1 \end{pmatrix}\right).$$

We use the following relations for

$$(U(\theta;\alpha-\theta,c))'_\theta = 0,\ (U(\theta;\alpha-\theta,c))'_\alpha = 0,$$
$$(U(\alpha;\alpha-\theta,c))'_\theta = 0,\ (U(\alpha;\alpha-\theta,c))'_\alpha = 0 \text{ at } (\theta,\alpha) = (0,a)$$

and obtain

$$(\Psi_1)'_\theta(0,a) = \int_0^\infty k(y;c)\frac{\omega(y)}{\omega(y)-\omega(y+a)}dy;$$
$$(\Psi_1)'_\alpha(0,a) = \int_0^\infty k(y;c)\frac{\omega(y+a)}{\omega(y+a)-\omega(y)}dy;$$
$$(\Psi_2)'_\theta(0,a) = \int_0^\infty k(y;c)\frac{\omega(y+a)}{\omega(y-a)-\omega(y)}dy;$$
$$(\Psi_2)'_\alpha(0,a) = \int_0^\infty k(y;c)\frac{\omega(y)}{\omega(y)-\omega(y-a)}dy.$$

Thus, $\deg(I - \mathcal{H}_0, B_{\widetilde{C}^1(\mathbb{R},\mathbb{R})}(U,\varepsilon), 0) \neq 0$ if

$$(\Psi_1)'_\theta(0,a)(\Psi_2)'_\alpha(0,a) - (\Psi_1)'_\alpha(0,a)(\Psi_2)'_\theta(0,a) - (\Psi_1)'_\theta(0,a) - (\Psi_2)'_\alpha(0,a) + 1 \neq 0. \tag{22}$$

The results above together with Theorem 2.3 and Theorem 2.4 imply the main result of the present research.

Theorem 2.6. *Let the assumptions* (**A1**) – (**A3**) *be satisfied. Let the condition* (18) *be fulfilled and the inequalities* (19)*,* (20)*,* (22) *hold true. Then, for each* $\beta \in [0, \infty)$*, there exists regular travelling pulse solution* $u_\beta \in \widetilde{C}^1(\mathbb{R}, \mathbb{R}))$ *of the speed* $c < 0$ *to* (1)*. Moreover,*

$$\|u_\beta - u_0\|_{C([0,\infty),\widetilde{C}^1(\mathbb{R},\mathbb{R}))} \to 0, \;\; as \; \beta \to 0, \tag{23}$$

where $u_0 \in \widetilde{C}^1(\mathbb{R}, \mathbb{R}))$ *is the travelling pulse (of the width* a*) corresponding to* $\beta = 0$ *with the profile defined by* (17)*.*

We note that Theorem 2.3 only provides the convergence of the travelling pulse profiles $\|U_\beta - U_0\|_{\widetilde{C}^1(\mathbb{R},\mathbb{R})} \to 0$, as $\beta \to 0$. However, Definition 2.1 and the metric in $C([0, \infty), \widetilde{C}^1(\mathbb{R}, \mathbb{R}))$ imply the convergence (23).

Finally, we summarize the algorithm of application of the paper results. Given the characteristics of the neural field (namely, the excitatory connections defined by ω, the neuronal activation threshold h, and the negative feedback strength ϵ), one uses the condition (18) to verify the existence of the travelling pulse of the width a and the speed c. As soon as the inequality (18) holds true (i.e. the travelling pulse is regular), one verifies the inequalities (20) and (22). If the inequalities take place, then for any sufficiently steep firing rate function approximating the Heaviside firing rate function with the threshold h, there exists the corresponding travelling pulse of the same speed that approaches (in the sense of the relation (23)) the travelling pulse corresponding to the Heaviside firing rate function.

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