Chapter 6
Queueing Models

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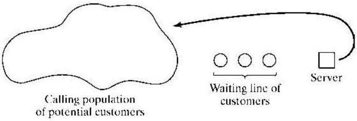
Discrete-Event System Simulation

Purpose

- Simulation is often used in the analysis of queueing models.
- A simple but typical queueing model:



- Queueing models provide the analyst with a powerful tool for designing and evaluating the performance of queueing systems.
- Typical measures of system performance:
  - Server utilization, length of waiting lines, and delays of customers
  - For relatively simple systems, compute mathematically
  - For realistic models of complex systems, simulation is usually required.

Outline

- Discuss some well-known models (not development of queueing theories):
  - General characteristics of queues,
  - Meanings and relationships of important performance measures,
  - Estimation of mean measures of performance.
  - Effect of varying input parameters,
  - Mathematical solution of some basic queueing models.

Characteristics of Queueing Systems

- Key elements of queueing systems:
  - Customer: refers to anything that arrives at a facility and requires service, e.g., people, machines, trucks, emails.
  - Server: refers to any resource that provides the requested service, e.g., repairpersons, retrieval machines, runways at airport.

Calling Population

[Characteristics of Queueing System]

- Calling population: the population of potential customers, may be assumed to be finite or infinite.
  - Finite population model: if arrival rate depends on the number of customers being served and waiting, e.g., model of one corporate jet, if it is being repaired, the repair arrival rate becomes zero.
  - Infinite population model: if arrival rate is not affected by the number of customers being served and waiting, e.g., systems with large population of potential customers.

System Capacity

[Characteristics of Queueing System]

- System Capacity: a limit on the number of customers that may be in the waiting line or system.
  - Limited capacity, e.g., an automatic car wash only has room for *10* cars to wait in line to enter the mechanism.
  - Unlimited capacity, e.g., concert ticket sales with no limit on the number of people allowed to wait to purchase tickets.

Arrival Process

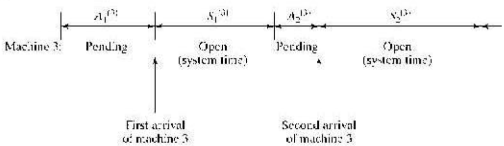
[Characteristics of Queueing System]

- For infinite-population models:
  - In terms of interarrival times of successive customers.
  - Random arrivals: interarrival times usually characterized by a probability distribution.
    - Most important model: Poisson arrival process (with rate λ), where A_n represents the interarrival time between customer $n-1$ and customer n, and is exponentially distributed (with mean $1/\lambda$).
  - Scheduled arrivals: interarrival times can be constant or constant plus or minus a small random amount to represent early or late arrivals.
    - e.g., patients to a physician or scheduled airline flight arrivals to an airport.
  - At least one customer is assumed to always be present, so the server is never idle, e.g., sufficient raw material for a machine.

Arrival Process

[Characteristics of Queueing System]

- For finite-population models:
  - Customer is pending when the customer is outside the queueing system, e.g., machine-repair problem: a machine is “pending” when it is operating, it becomes “not pending” the instant it demands service form the repairman.
  - Runtime of a customer is the length of time from departure from the queueing system until that customer’s next arrival to the queue, e.g., machine-repair problem, machines are customers and a runtime is time to failure.
  - Let $A_1^{(i)}$, $A_2^{(i)}$, ... be the successive runtimes of customer i, and $S_1^{(i)}$, $S_2^{(i)}$ be the corresponding successive system times:



Queue Behavior and Queue Discipline

[Characteristics of Queueing System]

- Queue behavior: the actions of customers while in a queue waiting for service to begin, for example:
  - Balk: leave when they see that the line is too long,
  - Renege: leave after being in the line when its moving too slowly,
  - Jockey: move from one line to a shorter line.
- Queue discipline: the logical ordering of customers in a queue that determines which customer is chosen for service when a server becomes free, for example:
  - First-in-first-out (FIFO)
  - Last-in-first-out (LIFO)
  - Service in random order (SIRO)
  - Shortest processing time first (SPT)
  - Service according to priority (PR).

Service Times and Service Mechanism

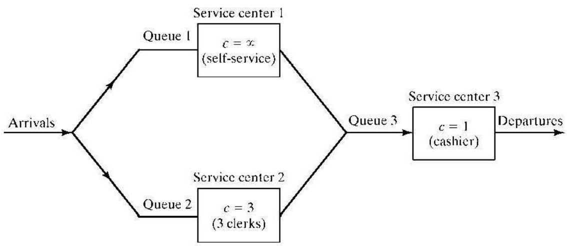
[Characteristics of Queueing System]

- Service times of successive arrivals are denoted by S_1, S_2, S_3.
  - May be constant or random.
  - $\{S_1, S_2, S_3, \ldots\}$ is usually characterized as a sequence of independent and identically distributed random variables, e.g., exponential, Weibull, gamma, lognormal, and truncated normal distribution.
- A queueing system consists of a number of service centers and interconnected queues.
  - Each service center consists of some number of servers, c, working in parallel, upon getting to the head of the line, a customer takes the *1st* available server.

Service Times and Service Mechanism

[Characteristics of Queueing System]

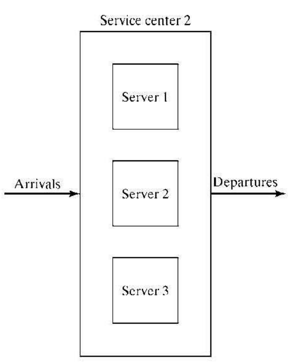
- Example: consider a discount warehouse where customers may:
  - Serve themselves before paying at the cashier:



Service Times and Service Mechanism

[Characteristics of Queueing System]

  - Wait for one of the three clerks:



  - Batch service (a server serving several customers simultaneously), or customer requires several servers simultaneously.

Queueing Notation

[Characteristics of Queueing System]

- A notation system for parallel server queues: *A/B/c/N/K*
  - *A* represents the interarrival-time distribution,
  - *B* represents the service-time distribution,
  - *c* represents the number of parallel servers,
  - *N* represents the system capacity,
  - *K* represents the size of the calling population.

Queueing Notation

[Characteristics of Queueing System]

- Primary performance measures of queueing systems:
  - P_n: steady-state probability of having n customers in system,
  - $P_n(t)$: probability of n customers in system at time t,
  - λ: arrival rate,
  - λ_e: effective arrival rate,
  - μ: service rate of one server,
  - ρ: server utilization,
  - A_n: interarrival time between customers n-1 and n,
  - S_n: service time of the nth arriving customer,
  - W_n: total time spent in system by the nth arriving customer,
  - W_n^Q: total time spent in the waiting line by customer n,
  - $L(t)$: the number of customers in system at time t,
  - $L_Q(t)$: the number of customers in queue at time t,
  - L: long-run time-average number of customers in system,
  - L_Q: long-run time-average number of customers in queue,
  - w: long-run average time spent in system per customer,
  - w_Q: long-run average time spent in queue per customer.

Time-Average Number in System L

[Characteristics of Queueing System]

- Consider a queueing system over a period of time T,
  - Let T_i denote the total time during $[0,T]$ in which the system contained exactly i customers, the time-weighted-average number in a system is defined by:

$$\hat{L} = \frac{1}{T}\sum_{i=0}^{\infty} iT_i = \sum_{i=0}^{\infty} i\left(\frac{T_i}{T}\right)$$

  - Consider the total area under the function is $L(t)$, then,

$$\hat{L} = \frac{1}{T}\sum_{i=0}^{\infty} iT_i = \frac{1}{T}\int_0^T L(t)\,dt$$

  - The long-run time-average # in system, with probability 1:

$$\hat{L} = \frac{1}{T}\int_0^T L(t)\,dt \rightarrow L \quad \text{as} \quad T \rightarrow \infty$$

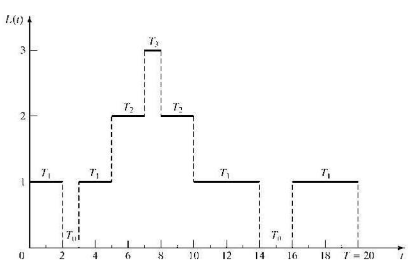
Time-Average Number in System L

[Characteristics of Queueing System]

- The time-weighted-average number in queue is:

$$\hat{L}_Q = \frac{1}{T}\sum_{i=0}^{\infty} iT_i^Q = \frac{1}{T}\int_0^T L_Q(t)\,dt \rightarrow L_Q \quad \text{as} \quad T \rightarrow \infty$$

- *G/G/1/N/K* example: consider the results from the queueing system ($N > 4$, $K > 3$).



$$\hat{L} = [0(3) + 1(12) + 2(4) + 3(1)]/20$$
$$= 23/20 = 1.15 \text{ cusomters}$$

$$L_Q(t) = \begin{cases} 0, & \text{if } L(t) = 0 \\ L(t) - 1, & \text{if } L(t) \geq 1 \end{cases}$$

$$\hat{L}_Q = \frac{0(15) + 1(4) + 2(1)}{20} = 0.3 \text{ customers}$$

Average Time Spent in System Per Customer w

[Characteristics of Queueing System]

- The average time spent in system per customer, called the average system time, is:

$$\hat{w} = \frac{1}{N}\sum_{i=1}^{N} W_i$$

  where $W_1, W_2, \ldots, W_N$ are the individual times that each of the N customers spend in the system during $[0, T]$.
  - For stable systems: $\hat{w} \to w$ as $N \to \infty$
  - If the system under consideration is the queue alone:

$$\hat{w}_Q = \frac{1}{N}\sum_{i=1}^{N} W_i^Q \to w_Q \quad \text{as} \quad N \to \infty$$

  - *G/G/1/N/K* example (cont.): the average system time is

$$\hat{w} = \frac{W_1 + W_2 + \ldots + W_5}{5} = \frac{2 + (8-3) + \ldots + (20-16)}{5} = 4.6 \text{ time units}$$

The Conservation Equation

[Characteristics of Queueing System]

- Conservation equation (a.k.a. Little's law)

$$\hat{L} = \hat{\lambda}\hat{w}$$

Average # in system

Arrival rate

Average System time

$$L = \lambda w \quad \text{as} \quad T \to \infty \text{ and } N \to \infty$$

  - Holds for almost all queueing systems or subsystems (regardless of the number of servers, the queue discipline, or other special circumstances).
  - *G/G/1/N/K* example (cont.): On average, one arrival every *4* time units and each arrival spends *4.6* time units in the system. Hence, at an arbitrary point in time, there is *(1/4)(4.6) = 1.15* customers present on average.

Server Utilization

[Characteristics of Queueing System]

- Definition: the proportion of time that a server is busy.
  - Observed server utilization, $\hat{\rho}$, is defined over a specified time interval [0,T].
  - Long-run server utilization is ρ.
  - For systems with long-run stability: $\hat{\rho} \rightarrow \rho \quad \text{as} \quad T \rightarrow \infty$

Server Utilization

[Characteristics of Queueing System]

- For $G/G/1/\infty/\infty$ queues:
  - Any single-server queueing system with average arrival rate λ customers per time unit, where average service time $E(S) = 1/\mu$ time units, infinite queue capacity and calling population.
  - Conservation equation, $L = \lambda w$, can be applied.
  - For a stable system, the average arrival rate to the server, λ_s, must be identical to λ.
  - The average number of customers in the server is:

$$\hat{L}_s = \frac{1}{T}\int_0^T \left(L(t) - L_Q(t)\right)dt = \frac{T - T_0}{T}$$

Server Utilization

[Characteristics of Queueing System]

- In general, for a single-server queue:

$$\hat{L}_s = \hat{\rho} \to L_s = \rho \text{ as } T \to \infty$$

$$\text{and} \quad \rho = \lambda E(s) = \frac{\lambda}{\mu}$$

  - For a single-server stable queue: $\rho = \frac{\lambda}{\mu} < 1$
  - For an unstable queue ($\lambda > \mu$), long-run server utilization is *1*.

Server Utilization

[Characteristics of Queueing System]

- For *G/G/c/∞/∞* queues:
  - A system with c identical servers in parallel.
  - If an arriving customer finds more than one server idle, the customer chooses a server without favoring any particular server.
  - For systems in statistical equilibrium, the average number of busy servers, L_s, is: L_s, = $\lambda E(s)$ = λ/μ.
  - The long-run average server utilization is:

$$\rho = \frac{L_s}{c} = \frac{\lambda}{c\mu}, \quad \text{where } \lambda < c\mu \text{ for stable systems}$$

Server Utilization and System Performance

[Characteristics of Queueing System]

- System performance varies widely for a given utilization ρ.
  - For example, a *D/D/1* queue where $E(A) = 1/\lambda$ and $E(S) = 1/\mu$, where:

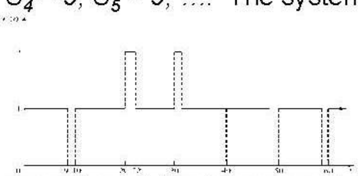
    $$L = \rho = \lambda/\mu, \quad w = E(S) = 1/\mu, \quad L_Q = W_Q = 0.$$

    - By varying λ and μ, server utilization can assume any value between *0* and *1*.
    - Yet there is never any line.
  - In general, variability of interarrival and service times causes lines to fluctuate in length.

Server Utilization and System Performance

[Characteristics of Queueing System]

- Example: A physician who schedules patients every *10* minutes and spends S_i minutes with the i^{th} patient: $S_i = \begin{cases} 9 \text{ minutes with probability } 0.9 \\ 12 \text{ minutes with probability } 0.1 \end{cases}$
  - Arrivals are deterministic, $A_1 = A_2 = \ldots = \lambda^{-1} = 10$.
  - Services are stochastic, $E(S_i) = 9.3$ min and $V(S_0) = 0.81$ min².
  - On average, the physician's utilization = $\rho = \lambda/\mu = 0.93 < 1$.
  - Consider the system is simulated with service times: $S_1 = 9$, $S_2 = 12$, $S_3 = 9$, $S_4 = 9$, $S_5 = 9$, ... The system becomes:



  - The occurrence of a relatively long service time ($S_2 = 12$) causes a waiting line to form temporarily.

Costs in Queueing Problems

[Characteristics of Queueing System]

- Costs can be associated with various aspects of the waiting line or servers:
  - System incurs a cost for each customer in the queue, say at a rate of *$10* per hour per customer.
    - The average cost per customer is:

$$\sum_{j=1}^{N} \frac{\$10 * W_j^Q}{N} = \$10 * \hat{w}_Q$$

W_j^Q is the time customer j spends in queue

    - If $\hat{\lambda}$ customers per hour arrive (on average), the average cost per hour is:

$$\left(\hat{\lambda}\frac{\text{customer}}{\text{hour}}\right)\left(\frac{\$10 * \hat{w}_Q}{\text{customer}}\right) = \$10 * \hat{\lambda}\hat{w}_Q = \$10 * \hat{L}_Q \text{ / hour}$$

  - Server may also impose costs on the system, if a group of *c* parallel servers ($1 \le c \le \infty$) have utilization r, each server imposes a cost of *$5* per hour while busy.
    - The total server cost is: $\$5 * c\rho$.

Steady-State Behavior of Infinite-Population Markovian Models

- Markovian models: exponential-distribution arrival process (mean arrival rate = λ).
- Service times may be exponentially distributed as well (*M*) or arbitrary (*G*).
- A queueing system is in statistical equilibrium if the probability that the system is in a given state is not time dependent:

$$P(L(t) = n) = P_n(t) = P_n.$$

- Mathematical models in this chapter can be used to obtain approximate results even when the model assumptions do not strictly hold (as a rough guide).
- Simulation can be used for more refined analysis (more faithful representation for complex systems).

Steady-State Behavior of Infinite-Population Markovian Models

- For the simple model studied in this chapter, the steady-state parameter, L, the time-average number of customers in the system is:

$$L = \sum_{n=0}^{\infty} nP_n$$

  - Apply Little's equation to the whole system and to the queue alone:

$$w = \frac{L}{\lambda}, \quad w_Q = w - \frac{1}{\mu}$$

$$L_Q = \lambda w_Q$$

- *G/G/c/∞/∞* example: to have a statistical equilibrium, a necessary and sufficient condition is $\lambda/(c\mu) < 1$.

M/G/1 Queues [Steady-State of Markovian Model]

- Single-server queues with Poisson arrivals & unlimited capacity.
- Suppose service times have mean $1/\mu$ and variance σ^2 and $\rho = \lambda/\mu < 1$, the steady-state parameters of *M/G/1* queue:

$$\boxed{\begin{aligned} &\rho = \lambda/\mu, \quad P_0 = 1-\rho \\ &L = \rho + \frac{\rho^2(1+\sigma^2\mu^2)}{2(1-\rho)}, \quad L_Q = \frac{\rho^2(1+\sigma^2\mu^2)}{2(1-\rho)} \\ &w = \frac{1}{\mu} + \frac{\lambda(1/\mu^2+\sigma^2)}{2(1-\rho)}, \quad w_Q = \frac{\lambda(1/\mu^2+\sigma^2)}{2(1-\rho)} \end{aligned}}$$

M/G/1 Queues [Steady-State of Markovian Model]

- No simple expression for the steady-state probabilities $P_0, P_1, \ldots$
- $L - L_Q = \rho$ is the time-average number of customers being served.
- Average length of queue, L_Q, can be rewritten as:

$$L_Q = \frac{\rho^2}{2(1-\rho)} + \frac{\lambda^2\sigma^2}{2(1-\rho)}$$

  - If λ and μ are held constant, L_Q depends on the variability, σ^2, of the service times.

M/G/1 Queues [Steady-State of Markovian Model]

- Example: Two workers competing for a job, Able claims to be faster than Baker on average, but Baker claims to be more consistent,
  - Poisson arrivals at rate $\lambda = 2$ per hour ($1/30$ per minute).
  - Able: $1/\mu = 24$ minutes and $\sigma^2 = 20^2 = 400$ minutes^2:

$$L_Q = \frac{(1/30)^2[24^2 + 400]}{2(1-4/5)} = 2.711 \text{ customers}$$

    - The proportion of arrivals who find Able idle and thus experience no delay is $P_0 = 1-\rho = 1/5 = 20\%$.
  - Baker: $1/\mu = 25$ *minutes* and $\sigma^2 = 2^2 = 4$ minutes^2:

$$L_Q = \frac{(1/30)^2[25^2 + 4]}{2(1-5/6)} = 2.097 \text{ customers}$$

    - The proportion of arrivals who find Baker idle and thus experience no delay is $P_0 = 1-\rho = 1/6 = 16.7\%$.
  - Although working faster on average, Able's greater service variability results in an average queue length about *30%* greater than Baker's.

M/M/1 Queues

[Steady-State of Markovian Model]

- Suppose the service times in an *M/G/1* queue are exponentially distributed with mean $1/\mu$, then the variance is $\sigma^2 = 1/\mu^2$.
  - *M/M/1* queue is a useful approximate model when service times have standard deviation approximately equal to their means.
  - The steady-state parameters:

$$\rho = \lambda/\mu, \quad P_n = (1-\rho)\rho^n$$

$$L = \frac{\lambda}{\mu-\lambda} = \frac{\rho}{1-\rho}, \quad L_Q = \frac{\lambda^2}{\mu(\mu-\lambda)} = \frac{\rho^2}{1-\rho}$$

$$w = \frac{1}{\mu-\lambda} = \frac{1}{\mu(1-\rho)}, \quad w_Q = \frac{\lambda}{\mu(\mu-\lambda)} = \frac{\rho}{\mu(1-\rho)}$$

M/M/1 Queues

[Steady-State of Markovian Model]

- Example: *M/M/1* queue with service rate $\mu=10$ customers per hour.
  - Consider how *L* and *w* increase as arrival rate, λ, increases from *5* to *8.64* by increments of *20%*:

λ	5.0	6.0	7.2	8.64	10.0
ρ	0.500	0.600	0.720	0.864	1.000
L	1.00	1.50	2.57	6.35	∞
w	0.20	0.25	0.36	0.73	∞

  - If $\lambda/\mu \geq 1$, waiting lines tend to continually grow in length.
  - Increase in average system time (*w*) and average number in system (*L*) is highly nonlinear as a function of ρ.

Effect of Utilization and Service Variability

[Steady-State of Markovian Model]

- For almost all queues, if lines are too long, they can be reduced by decreasing server utilization (ρ) or by decreasing the service time variability (σ^2).
- A measure of the variability of a distribution, coefficient of variation (cv):

$$(cv)^2 = \frac{V(X)}{[E(X)]^2}$$

  - The larger cv is, the more variable is the distribution relative to its expected value

Effect of Utilization and Service Variability

[Steady-State of Markovian Model]

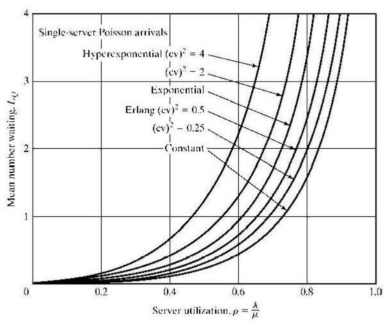
- Consider L_Q for any *M/G/1* queue:

$$L_Q = \frac{\rho^2(1+\sigma^2\mu^2)}{2(1-\rho)}$$

$$= \left(\frac{\rho^2}{1-\rho}\right)\left(\frac{1+(cv)^2}{2}\right)$$

L_Q for *M/M/1* queue

Corrects the *M/M/1* formula to account for a non-exponential service time dist'n



Multiserver Queue [Steady-State of Markovian Model]

- *M/M/c/∞/∞* queue: *c* channels operating in parallel.
  - Each channel has an independent and identical exponential service-time distribution, with mean $1/\mu$.
  - To achieve statistical equilibrium, the offered load (λ/μ) must satisfy $\lambda/\mu < c$, where $\lambda/(c\mu) = \rho$ is the server utilization.
  - Some of the steady-state probabilities:

$$\rho = \lambda / c\mu$$

$$P_0 = \left\{ \left[\sum_{n=0}^{c-1} \frac{(\lambda/\mu)^n}{n!} \right] + \left[\left(\frac{\lambda}{\mu} \right)^c \left(\frac{1}{c!} \right) \left(\frac{c\mu}{c\mu - \lambda} \right) \right] \right\}^{-1}$$

$$L = c\rho + \frac{(c\rho)^{c+1} P_0}{c(c!)(1-\rho)^2} = c\rho + \frac{\rho P(L(\infty) \geq c)}{1-\rho}$$

$$w = \frac{L}{\lambda}$$

Multiserver Queue [Steady-State of Markovian Model]

- Other common multiserver queueing models:
  - *M/G/c/∞*: general service times and c parallel server. The parameters can be approximated from those of the *M/M/c/∞/∞* model.
  - *M/G/∞*: general service times and infinite number of servers, e.g., customer is its own system, service capacity far exceeds service demand.
  - *M/M/C/N/∞*: service times are exponentially distributed at rate m and c servers where the total system capacity is $N \geq c$ customer (when an arrival occurs and the system is full, that arrival is turned away).

Steady-State Behavior of Finite-Population Models

- When the calling population is small, the presence of one or more customers in the system has a strong effect on the distribution of future arrivals.
- Consider a finite-calling population model with *K* customers (*M/M/c/K/K*):
  - The time between the end of one service visit and the next call for service is exponentially distributed, (mean = $1/\lambda$).
  - Service times are also exponentially distributed.
  - *c* parallel servers and system capacity is *K*.

Steady-State Behavior of Finite-Population Models

- Some of the steady-state probabilities:

$$P_0 = \left\{ \sum_{n=0}^{c-1} \binom{K}{n} \left(\frac{\lambda}{\mu}\right)^n + \sum_{n=c}^{K} \frac{K!}{(K-n)!c!c^{n-c}} \left(\frac{\lambda}{\mu}\right)^n \right\}^{-1}$$

$$P_n = \begin{cases} \binom{K}{n} \left(\frac{\lambda}{\mu}\right)^n P_0, & n = 0,1,\ldots,c-1 \\ \frac{K!}{(K-n)!c!c^{n-c}} \left(\frac{\lambda}{\mu}\right)^n, & n = c, c+1, \ldots K \end{cases}$$

$$L = \sum_{n=0}^{K} nP_n, \quad w = L/\lambda_e, \quad \rho = \lambda_e / c\mu$$

where λ_e is the long run effective arrival rate of customers to queue (or enterin B17 iting service)

$$\lambda_e = \sum_{n=0}^{K} (K-n)\lambda P_n$$

Steady-State Behavior of Finite-Population Models

- Example: two workers who are responsible for *10* milling machines.
  - Machines run on the average for *20* minutes, then require an average *5*-minute service period, both times exponentially distributed: $\lambda = 1/20$ and $\mu = 1/5$.
  - All of the performance measures depend on P_0:

$$P_0 = \left\{ \sum_{n=0}^{2-1} \binom{10}{n} \left(\frac{5}{20}\right)^n + \sum_{n=2}^{10} \frac{10!}{(10-n)!2!2^{n-2}} \left(\frac{5}{20}\right)^n \right\}^{-1} = 0.065$$

    - Then, we can obtain the other P_n.
    - Expected number of machines in system:

$$L = \sum_{n=0}^{10} nP_n = 3.17 \text{ machines}$$

    - The average number of running machines:

$$K - L = 10 - 3.17 = 6.83 \text{ machines}$$

Networks of Queues

- Many systems are naturally modeled as networks of single queues: customers departing from one queue may be routed to another.
- The following results assume a stable system with infinite calling population and no limit on system capacity:
  - Provided that no customers are created or destroyed in the queue, then the departure rate out of a queue is the same as the arrival rate into the queue (over the long run).
  - If customers arrive to queue *i* at rate λ_i, and a fraction $0 \le p_{ij} \le 1$ of them are routed to queue j upon departure, then the arrival rate form queue *i* to queue *j* is $\lambda_i p_{ij}$ (over the long run).

Networks of Queues

- The overall arrival rate into queue j:

$$\lambda_j = a_j + \sum_{\text{all}\, i} \lambda_i p_{ij}$$

Arrival rate from outside the network

Sum of arrival rates from other queues in network

- If queue j has $c_j < \infty$ parallel servers, each working at rate μ_j, then the long-run utilization of each server is $\rho_j=\lambda_j/(c\mu_j)$ (where $\rho_j < 1$ for stable queue).
- If arrivals from outside the network form a Poisson process with rate a_j for each queue j, and if there are c_j identical servers delivering exponentially distributed service times with mean $1/\mu_j$, then, in steady state, queue j behaves likes an $M/M/c_j$ queue with arrival rate $\lambda_j = a_j + \sum_{\text{all}\, i} \lambda_i p_{ij}$

Network of Queues

- Discount store example:
  - Suppose customers arrive at the rate 80 per hour and 40% choose self-service. Hence:
    - Arrival rate to service center 1 is $\lambda_1 = 80(0.4) = 32$ per hour
    - Arrival rate to service center 2 is $\lambda_2 = 80(0.6) = 48$ per hour.
  - $c_2 = 3$ clerks and $\mu_2 = 20$ customers per hour.
  - The long-run utilization of the clerks is:

$$\rho_2 = 48/(3*20) = 0.8$$

  - All customers must see the cashier at service center 3, the overall rate to service center 3 is $\lambda_3 = \lambda_1 + \lambda_2 = 80$ per hour.
    - If $\mu_3 = 90$ per hour, then the utilization of the cashier is:

$$\rho_3 = 80/90 = 0.89$$

Summary

- Introduced basic concepts of queueing models.
- Show how simulation, and some times mathematical analysis, can be used to estimate the performance measures of a system.
- Commonly used performance measures: L, L_Q, w, w_Q, ρ, and λ_e.
- When simulating any system that evolves over time, analyst must decide whether to study transient behavior or steady-state behavior.
  - Simple formulas exist for the steady-state behavior of some queues.
- Simple models can be solved mathematically, and can be useful in providing a rough estimate of a performance measure.