1. Recall that we defined an intersecting family to be a collection of subsets of a given set S such that no two sets in the collection are disjoint. We also proved that when S is a finite set with $|S| = n$, the size of the largets intersecting family of subsets of S is 2^{n-1}. What if we want an intersecting family in which every set has the same given size, say k (a.k.a. a k-uniform intersecting family)? Let us find an answer to this question. Observe that when $k > n/2$, the answer is trivial, so let us assume $k \leq n/2$

Problem 1.1
Prove that there exists a k-uniform intersecting family containing $C(n-1, k-1)$ sets ($C(n-1, k-1)$ denotes "$(n-1)$ choose $(k-1)$").

Proof:
Consider the $n-1$ elements of Set S.
Choose any k-1 sets of cardinality k-1
So we can chose such $k-1$ subsets by $({}^{n-1}C_{k-1})$ ways.
Now put the remaining one element in all the chosen subsets
Hence, all the elements are of cardinality k and intersecting as they will always have atleast one element in common which is the element we added seperately.

Problem 1.2
Suppose $A \subseteq S$ and $|A| = k$. Imagine that the elements of S are to be assigned to n distinct places on the circumference of a circle. How many ways are there to do so in such a way that elements of A appear consecutively?

Claim : no. of ways in which Set S elements can be arranged on circumference of circle when all positions are identical is $M \times N =$ k!(n-k)!
Proof:
A is a subset of S of cardinality | k |
No. of ways such a way that elements of A appear consecutively on a circumference of circle when positions are identical
Consider all the elements of A together and remaining $n-k$ elements.
N = No. of ways elements of A can be arranged is k!
M = No. of ways other elements can be arranged is (n-k)!
So no. of ways in which Set S elements can be arranged on circumference of circle when all positions are identical is $M \times N =$ k!(n-k)!
Using this claim,
Now when positions are distinct So there are n possible ways for every cyclic order, So Total ways when positions are distinct is n(k!)(n-k)!.

Problem 1.3
Suppose $\mathcal{F}$ is a family of subsets of S, each of size k, and $|\mathcal{F}| > C(n-1, k-1)$. Prove that the elements of S can be arranged on the circle in such a way that the elements of more than k of the sets in $\mathcal{F}$ appear consecutively. (Hint: Double counting + pigeon-hole.)

Proof:
So now we will prove the given statement by contradiction.
Suppose We take F such that all the sets in F have atmost k consecutive for a given cyclic order.
Counting the no. of pairs (S,C) where S = no . of elements in F and C are no. of cyclic orders. We get a equation by this approach
So we will count no. of pairs in two ways.
first, for a particular element of F, we can have k!(n-k)! cyclic orders where positions are identical, So total pairs are

$$| F | (k!)(n-k)!$$

.
Secondly, for a particular cyclic order as we assumed there will k consecutive sets can appear at most.

So this gives us the equation:

$$k(n-1)! \geq \mid F \mid (n-k)!k!$$

So,

$$(n-1)!/((k-1)!(n-k)!) \geq \mid F \mid$$

,
As we are given that

$$\mid F \mid >^{n\ -1} C_{k-1}$$

, So this contradicts our assumption.
Hence, We can say that if

$$\mid F \mid >^{n\ -1} C_{k-1}$$

then elements of S can be arranged on the circle in such a way that the elements of more than k of the sets appear consecutively.

Problem 1.4
Hence argue that if

$$\mid F \mid C(n-1, k-1)$$

, then F cannot be a k-uniform intersecting family.

Proof: Proving the given statement using contrapositive statement, WE give claim 1.
Claim 1: If F is k-uniform intersecting family then $C(n-1, k-1) \geq \mid F \mid$

Proving Claim 1: Let us consider the Set S whose subsets are elements of F. And let the $\mid S \mid$ = n. There are (n-1)! cyclic orders exist for this set. For each set we can make different intervals and some of them may disjoint. Let us consider a interval of size k lying in F of this cyclic order.
Let the interval be $A_1 = (a_1, a_2, .., a_k)$.
So other intervals which be in F from this cyclic order must have at least one element common from A_1.
Sub-Claim: All the intervals that are intersecting A_1, from which some will be disjoint.

Let us consider a interval A_2 which is intersecting from A_1, So it must be separating a_i and a_{i+1} for some a_i where a_i is an element of A_1. The two intervals which are separating a_i and a_{i+1} are disjoint.

As there will $2(k-1)$ intervals intersecting A_1 in which k-1 pairs will be disjoint, So at most k-1 intervals intersecting with A_1 and each other can exist. So at most k intervals from a cyclic order can exist including A_1.

As we have proved that for a cyclic order there can exist at most k such intervals.

Consider the no. of pairs of (S,C) where S are elements of F and C are cyclic orders.
Using counting in two ways, we will calculate the at most no. of elements in F.
1st way: For a particular interval S_1, there will be k!(n-k)! cyclic orders will exist where S_1 will exist by the claim of Question 1.2. And so the no. of pairs will be $\mid F \mid$(n-k)!k!.
2nd way: For a particular cyclic order, there will be at most k intervals exist which would be uniform and intersecting. So total no. of pairs at most will be k(n-1)! as there will (n-1)! cyclic orders.
Forming the equation, we get, k(n-1)! $\geq(\mid F \mid)$(n-k)!k!.
So $\frac{(n-1)!}{(k)!(n-k)!} \geq \mid F \mid$
As proving the contrapositive statement is equivalent to prove the given statement, Hence proved the given statement.

2. Consider the poset $(2^S, \subseteq)$, where $S = \{1, \ldots, n\}$ for some $n \in \mathbb{N}$. A non-empty chain $\{A_1, A_2, \ldots, A_k\}$ of this poset, where $A_1 \subseteq A_2 \subseteq \cdots \subseteq A_k$, is said to be a *symmetric chain* if $|A_1| + |A_k| = n$ and $|A_{i+1}| = |A_i| + 1$ for each $i = 1, \ldots, k-1$.

Problem 2.1
Prove that the set 2^S can be partitioned into symmetric chains. (Hint: Induction on n.)

Proof: We will prove this by applying induction on n.
Base Case: n =1
poset $2^S = \{\phi, 1\}$, clearly for this set one symmetric chain is $\{\phi, 1\}$. clearly $|A_1| = 0 \ and |A_2| = 1$ thus it satisfies condition of a symmetric chain and partitions the set.

Induction Hypothesis:
Let for n = k poset $2^S \ where \ S \ = \ \{1...., k+1\}$ can be partitioned into symmetric chains.

Induction Step:
for n = k+1, we can partition the poset 2^S into two sets st :-
S1 = $\{x | x \ \epsilon \ 2^S \quad \& \quad x \ does \ not \ contain \ k+1\}$
S2 = $\{x | x \ \epsilon \ 2^S \quad \& \quad xcontain \ k+1\}$
also S1 $\cup$ S2 = 2^S and both are disjoint
Since the n of S1 = k we can partition S1 into symmetric chains by Induction hypothesis but $|A_1|+|A_d| = k$.
Consider function G on symmetric chains of S1 defined as:
function G: P1 $\rightarrow$ P2 st if C1 ϵ P1 then G(C1)= $\{x | \ if \ y \epsilon \ C1 \ then \ x \ = \ y \cup \{k+1\}\}$ where P1 and P2 are partitions of set S1 and S2 into symmetric chains respectively
also G(C1) covers all elements of S2 is constructed by adding element {k+1} in S1. clearly S2 can be partitioned into symmetric chains as set of G(C1) for all C1 in partition of S1 but $|A_1| + |A_d| = k+2$ for this partition
We can construct partition of poset 2^S as follows:-
Take a chain C1 from partition of set S1 into symmetric chains and chain C2 = G(C1)

$$C1 = \{A_1, A_2.....A_d\}$$

$$C2 = \{A_1 \cup \{k+1\}, A_2 \cup \{k+1\}.....A_d \cup \{k+1\}\}$$

case 1 (if $|C1| \neq 1$)
consider element a as A_1 of C1 which is the smallest size element of C1.
Now consider chain C3 as C1\a, since smallest element of C3 = A_2 of C1 and $|A_2| = |A_1| + 1$, thus C3 is a symmetric chain of set 2^S as $|A_1| + |A_d| = |a| + 1 + |A_d| = k+1$

$$C3 = \{A_2.....A_d\}$$

Also, consider C4 as C2 $\cup$ a, since smallest element of C4 = a and $|a| = |A_1| - 1$ where A_1 is the smallest element of C2, thus C4 is a symmetric chain of set 2^S as $|A_1| + |A_d| = |a| + |A_d| = k+2-1 = k+1$ and $\{a\}$ is a subset of A_1 of C2 as a = $A_1 \backslash \{k+1\}$

$$C4 = \{A_1, A_1 \cup \{k+1\}, A_2 \cup \{k+1\}.....A_d \cup \{k+1\}\}$$

case 2 (if $|C1| = 1$)
let a be the only element in C1. Clearly $|a| = k/2$. also let b be the only element in C2. Thus $|b| = k/2+1$. Consider chain C3 such that it has only two elements a and b. since a is a subset of b as a = $b \backslash \{k+1\}$ and $|A_1| + |A_d| = |a| + |b| = k/2 \ + \ k/2 \ + \ 1 \ = \ k+1$. Thus C3 is a symmetric chain of set 2^S.

Thus for each symmetric chain C1 of set S1 their exist symmetric chains of set 2^S such that elements in C1 $\cup$ G(C1) = elements in the constructed partition of 2^S into symmetric chain, also set of C1$\cup$G(C2) covers all elements in 2^S and each does not intersects thus 2^S can always be partitioned into symmetric chains.

Problem 2.2
Using the above result, find the size of the largest antichain in 2^S as a function of n, and prove your answer.

Proof: Claim: The number of chains in the smallest partition is ${}^nC_{\lfloor n/2 \rfloor}$ and a symmetric chain partitioning is the smallest.
It needs to be shown that each chain has an element of size $\lfloor n/2 \rfloor$.
Consider a chain of size k. If $\mid A_1 \mid = t$ then $\mid A_k \mid = n - t$. Now as $A_1 \subseteq A_k$, $n - t \geq t$, i. e. $t \leq \lfloor n/2 \rfloor$.
$\therefore$ $t \leq \lfloor n/2 \rfloor \leq n - t$.
Hence, for a certain value of t, $\exists$ a set of size $\lfloor n/2 \rfloor$ in all the symmetric chains in the partition.
Therefore for a given set having $[\frac{n}{2}]$ elements, Their exist a unique chain which contains this set.
Also by Dilworth theorem the length of the largest antichain is equal to the minimum number of chain in which you can partition the given set.
Since each chain contains an element of size $[\frac{n}{2}]$, Thus the number of symmetric chains in the partition equals ${}^nC_{[\frac{n}{2}]}$ and thus the length of the largest antichain is ${}^nC_{[\frac{n}{2}]}$.