1. Given: $F_1 = (V, E_1)$ and $F_2 = (V, E_2)$be any two acyclic graphs(a.k.a forests) on vertex set V and $|E_1| < |E_2|$

To Prove: $\exists e \in E_2 \setminus E_1$ such that $F = (V, E_1 \cup \{e\})$ is also an acyclic graph
Proof: As we are given $\mid E_1 \mid < \mid E_2 \mid$,
As F_1 is acyclic,
We can consider F_1 to have n connected components $C_1, C_2, C_3 ..., C_n$ with $V_1, V_2, V_3 ..., V_n$ vertex sets respectively.
So we divide the given situation into two cases.

Case 1: $\exists$ such e $\in E_2 \setminus E_1$ whose endpoints lie in different connected components. i.e., e = (u, v) then u $\in C_i$ and $v \in C_2$

Claim 1: joining one more edge will also give a acyclic graph in this case.
Proving claim 1: We know that all the connected components of acyclic tree are acyclic subgraphs and joining two connected components with an edge also create a acyclic connected graph with one more edge. We claim that joining two acyclic connected components with no. of vertices in 1st component being a and b in 2nd component and edges a-1 and b-1 and will make a acyclic connected component with edges a+b-1. Also other components are not affected by this, So overall graph is an acyclic graph.

Case 2: For all $(u, v) \in E_2 \setminus E_1, \exists$ e with endpoints u,v such that its endpoints lie in different connected components.
Claim: Case 2 can not exist
Proving the claim by contradiction:
As we had considered that no such edge lie in $E_2 \backslash E_1$ that has endpoints in different connected components of E_1. We can claim that all the edges of E_2 lie in only different components of E_1.
So if all the edges of E_2 lie in only the different components of E_1, We can say that,
$\mid E_1 \mid \geq \mid E_2 \mid$
, this is beacuse if all the edges lie in different connected components then as we know that no. of edges in a connected component is α_i - 1, where α_i are no. of edges in C_i.
and So, $\sum(\alpha_i - 1) \geq \mid E_2 \mid$
therefore, $\mid E_1 \mid \geq \mid E_2 \mid$
This is contradicting the given fact that $\mid E_2 \mid > \mid E_1 \mid$
So, this case can not exist.
Hence, our claim is true that $\exists$ $e \epsilon E_2$ such that $E_1 \cup e$ also forms an acyclic graph with vertex set V.

2. Claim 1: Every connected graph G = (V,E) has a closed walk which traverses every edge in E exactly twice.
Proof: Applying induction on number of edges of the graph:-
let number of edges of the graph G(V,E) be m.
Base Case: m=0
If m = 0 $\Rightarrow$
The graph contains an isolated single vertex. Also, number of edges = 0 this case is true as no edge has to be traversed.
Induction Hypothesis: $m < n$
By assumption under Strong Induction let the property holds when number of edges are less than n.
Induction Step: m =n
Consider u,v $\in$V, such that $\{(u, v)\} \in E$. Consider G'(V,E$\backslash\{(u, v)\}$):

Case 1: G' is connected-
since m= n-1 for G' $\Rightarrow$ by IH there exist a closed walk in G' such that every edge is covered twice. Let this walk be W = $uv_1v_2...u$ and this walk does not contain uv.

For G we can construct a walk R such that

$$R = W - vu = uv_1v_2...uvu$$

Since W traverse through every edge twice except uv and thus after appending the closed walk uvu to W, All edges are traversed exactly twice.

Case 2: G' is disconnected-
The graph G'(V,E') must have two connected components and u,v lies in different connected components. Let these two components be G1(V1,E1') and G2(V2,E2'). since number of edges for G1' and G2' are less than n therefore by IH there exist a closed walk from u in G1' such that every edge is covered twice and there exist a closed walk from in G2' such that every edge is covered twice. Let these walks be W1 and W2 respectively.
For G we can construct a walk R such that

$$R = W1 - W2 - u = uv_1v_2...uvw_1w_2...vu$$

Since W1 and W2 traverse through every edge twice except uv and thus first travelling W1 and then uv once and then travelling W2 and then uv again covers all edges twice.
Therefore by PMI our claim is true.

Claim 2: Every connected graph G = (V;E) has a closed walk of length $2|V| - 2$ which visits every vertex in V at least once.
Proof: Since every connected graph has a subgraph on vertex set V which is a tree.(Spanning Tree). Let on G(V,E) the spanning Tree be T(V,E').
Also, $|E'| = |V| - 1$
. Since T is connected and thus by Claim 1 we can say that T has a closed walk W on T which traverse every vertex exactly twice $\Rightarrow$ Walk visits every vertex at least once.
Also length of this walk $= 2 * |E'| = 2|V| - 2$ as Walk traverse every edge twice.
Therefore, similar closed walk on G visits every vertex at least once and length of the closed walk $= 2|V| - 2$.
Hence Proved.

3. Claim: The edge set of every bipartite regular graph can be partitioned into perfect matchings.
Proof: Applying induction on degree of the bipartite graph.
Let the regular bipartite graph G(V,E) has bipartition V1,V2 and degree of each vertex be d.
Note that edges from V1 are $d * |V1|$ and edges from V2 are $d * |V2|$. These must be equal to $|E| \Rightarrow$

$$d * |V1| = d * |V2| \Rightarrow |V1| = |V2|$$

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Base Case: d =1$\Rightarrow$
Every vertex in V1 is connected to exactly 1 vertex in V2 as every vertex in V1 has degree 1 and due to bipartition its neighbour must lie in V2. Also two vertex can not have same neighbour as degree of vertex in V2 is also 1. Therefore this set of edges is already a matching and thus Base case is true.

Inductive Hypothesis: $d = n - 1$
Let for d = n-1 the set of edges can be partitioned into perfect matchings.

Induction Step: d =n
Consider S be any set such that S⊆V1. Let S_o be the set of edges from S and
N(S) = {v∈ $V2$| v is neighbour of element in S }
Let $N(S)_i$ be the set of edges from elements of N(S).
since all edges from S incidents on N(S)$\Rightarrow$

$$|N(S)_i| >= |S_o|$$

$$\Rightarrow\ d * |N(S)| >= d * |S|$$
$$\Rightarrow\ |N(S)| >= |S|\ \forall\ S \subseteq V1$$

Therefore By Hall's theorem we can say that G has a matching M which matches all vertices of V1. Also since $|V1| = |V2|$, this matching also matches all vertices of V2. Clearly M is a regular matching. E can be partitioned into M and E' such that:
E' = $\{x | x \in E \backslash \{m\} \forall m \in M\}$
Consider G'(V,E'). Degree of vertex of G' is d-1 as M is removed from E. Thus degree is reduced by one. Since d-1 < d, By IH E' can be partitioned into perfect matchings. Let set of these partition be P'.
Consider partition set P = M ∪ P'. Clearly P is the set of partitions of E with each of the partition being a perfect matching. Hence by PMI, claim is true.

4. Let us prove by cases.
Let $G = (V, E), |V| = n$
Case I: n is odd.
The number of perfect matchings in this case is 0 as perfect matchings are only possible for graphs with an even number of vertices.
Case II: n is even.
Proof by construction. Let us select a vertex in V. As the graph is complete, we have $(n-1)$ options to select a vertex to construct a perfect matching. For the next vertex, we have $(n-3)$ options to select a vertex. Continuing this process, for the $(n/2)^{th}$ vertex there is only one option to select a vertex.
∴ The number of perfect matchings of G is given by:
$$(n-1)(n-3)...1 = \frac{n(n-1)...1}{n(n-2)...2} = \frac{n!}{n(n-2)...2} = \frac{n!}{2^{n/2} * (n/2) * (n/2-1)...1} = \frac{n!}{2^{n/2} * (n/2)!} \tag{1}$$
Proof by induction on n:
Base Case: n=2 In this case, there are 2 vertices joined by 1 edge so there is only one perfect matching. The formula arrived at above gives the correct result in this case ($\frac{2!}{2^1 * 1!} = 1$).
Induction Hypothesis: For a complete graph with n vertices, the result is true, i.e. number of perfect matchings is $\frac{n!}{2^{n/2} * (n/2)!}$.
Induction Step: Considering a graph with $n+2$ vertices. Choosing a vertex from these, we have $n+1$ options to pair with this vertex. Now 2 vertices have been chosen and n are left. For the remaining n, the number of perfect matchings is given by $\frac{n!}{2^{n/2} * (n/2)!}$ (According to induction hypothesis). Hence the number of perfect matchings for $n+2$ vertices is given by:
$$(n+1) * \frac{n!}{2^{n/2} * (n/2)!} = \frac{(n+1)!}{2^{n/2} * (n/2)!} = \frac{(n+1)! * (n+2)}{2^{n/2} * (n/2)! * (n/2+1) * 2} = \frac{(n+2)!}{2^{(n+2)/2} * ((n+2)/2)!} \tag{2}$$
∴ The number of perfect matchings in a complete graph with n vertices is $\frac{n!}{2^{n/2} * (n/2)!}$

5. Let the m passengers be seated in a line in any arbitrary order.
Let x_0/x_m be the number of seats between the left/right most passenger and left/right most edge.
Let x_i $(1 \leq i < m)$ denote the number of empty seats between the i^{th} and $(i+1)^{th}$ passenger. (Counting is done from the left here)
According to the given conditions:
$x_0, x_m \geq 0$ and $x_i \geq 2\, \forall\, 1 \leq i < m$. As m seats are occupied out of n, the number of empty seats is $n-m$, i. e. $\Sigma_{i=0}^{m} x_i = n - m$. Now let
$y_i = x_i$ if $i = 0$ or $i = m$
$y_i + 2 = x_i$ if $1 \leq i < m$
Replacing x_i with y_i, we get:
$\Sigma_{i=0}^{m} y_i + \Sigma_{i=1}^{m-1} = n - m$
$\Sigma_{i=0}^{m} y_i = n - 3m + 2$
Now as $y_i \geq 0\, \forall\, 0 \leq i \leq m$, the summation $\Sigma_{i=0}^{m} y_i = n - 3m + 2$ is equivalent to putting $n - 3m + 2$ identical

balls in $m+1$ distinct bins such that bins can be empty.
$\therefore$ as discussed in class, the number of ways to seat the passengers in the given order is given by:

$$C((n-3m+2)+(m+1)-1,(m+1)-1) = C(n-2m+2,m) = \frac{(n-2m+2)!}{m!(n-3m+2)!} \tag{3}$$

This is the number of ways to seat the passengers in an arbitrarily chosen order. Hence this is the number of ways to seat the passengers for all possible orders. As we have m passengers, there are $m!$ possible orders.
$\therefore$ The number of ways to seat the passengers according to the distancing rule is:

$$m! * \frac{(n-2m+2)!}{m!(n-3m+2)!} = \frac{(n-2m+2)!}{(n-3m+2)!} \tag{4}$$