COL703 Logic For CS SEM I 2023-24

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1 Question 1

1.1 Insertion Rule :

$$\dfrac{\begin{array}{cc} \phi & \psi \\ \vdots & \vdots \\ \psi, & \phi \end{array}}{\phi \leftrightarrow \psi} \leftrightarrow \mathrm{I}$$

1.2 Elimination Rules

:

1.2.1 Elimination 1

:

$$\frac{\phi, \quad \phi \leftrightarrow \psi}{\psi} \leftrightarrow \mathrm{E}_1$$

1.2.2 Elimination 2

:

$$\frac{\psi, \quad \phi \leftrightarrow \psi}{\phi} \leftrightarrow \mathrm{E}_2$$

1.3 Insertion Rule Derivation :

1.	ϕ	Assumption
2.	$\cdot$	R1
3.	ψ	R2
4.	ψ	Assumption
5.	$\cdot$	R3
6.	ϕ	R4
7.	$\phi \rightarrow \psi$	$\rightarrow_I (1-3)$
8.	$\psi \rightarrow \phi$	$\rightarrow_I (4-6)$
9.	$(\phi \rightarrow \psi) \wedge (\psi \rightarrow \phi)$	$\wedge_I(7,8)$

1.4 Elimination Rule Derivation 1 :

1.	$(\phi \rightarrow \psi) \wedge (\phi \rightarrow \psi)$	Premise
2.	ϕ	Premise
3.	$\phi \rightarrow \psi$	$\wedge_{E1}(1)$
4.	ψ	MP(2,3)

1.5 Elimination Rule Derivation 2 :

1.	$(\phi \rightarrow \psi) \wedge (\phi \rightarrow \psi)$	Premise
2.	ψ	Premise
3.	$\psi \rightarrow \phi$	$\wedge_{E2}(1)$
4.	ϕ	MP(2,3)

2 Question 2

2.1 Part a

$(p \to r) \land (q \to r) \vdash p \land q \to r$

1.	$(p \to r) \land (q \to r)$	Premise
2.	$p \to r$	$\land_{E1}(1)$
3.	$q \to r$	$\land_{E2}(1)$
4.	$p \land q$	Assumption
5.	p	$\land_{E1}(4)$
6.	$p \to r$	copy 2
7.	r	MP 5,6
8.	$p \land q \to r$	$\to_I (4-7)$

2.2 Part b

$p \to q \land r \vdash (p \to q) \land (p \to r)$

1.	$p \to q \land r$	Premise
2.	p	Assumption
3.	$q \land r$	M.P 2,1
4.	q	$\land_{E1}(3)$
5.	$p \to q$	$\to_I (2-4)$
6.	p	Assumption
7.	$q \land r$	M.P 6,1
8.	r	$\land_{E2}(7)$
9.	$p \to r$	$\to_I (6-8)$
10.	$(p \to r) \land (q \to r$	$\land_I(5, 9)$

3 Question 3

Let Φ be the set of well-formed formulas created by the set of atoms and $\{\leftrightarrow, \neg\}$.

- L(ϕ) = Number of literals present in the formula. It could be also defined as creating a parse tree and counting number of leaves (It would only give count of literals. Not necessary that leaves would be literals). Here $\phi \epsilon \Phi$

- Formulas with L(ϕ) = 1

  1. Observation 1 : Possible cases are either they are negation of atom or atom itself. In both cases we will observe that number of True values are odd.
  2. Observation 2 : Another observation we will make is that for any truthtable, we will have 2^n, $n \geq 0$ possible valuations. So total possible valuations are always even. Hence number of True values and number of False values are both even or both odd.
  3. Observation 3 : formula made by only connective as $\neg \leftrightarrow$ literal (defined according to Huth and Ryan). This is because negation is unary operator. And for the other side as the second connective in set is binary we will always need two literals atleast.
  4. Unary Operator $\neg$ applied on formula, if number of True values are even, then $\neg(\phi)$ will also have even True values. This is because by observation 2, False values in original formula would also be even. By negating all false values, they will become True in new formula. As false values are even, True values would also be even.

- Claim : For a formula ϕ with $L(\phi) > 1$, number of True values will be even if $\phi \epsilon \Phi$.

  1. We will prove the claim using Induction over length of formula by the notion of length defined above.
  2. Base case : $L(\phi) = 2$; For this case we can consider the possible structures of formula itself.
  3. $\begin{cases} p\leftrightarrow \text{ p : Computing True values for it, we have 2 Trues.} \\ p\leftrightarrow \text{ q : Here q is independent of p. Total True values are 2.} \\ p\leftrightarrow \neg p \text{ : Total True values are 0.} \end{cases}$
  4. Induction Hypothesis : For any formula of length $L(\phi) \leq k$, where $k > 2$, we assume that it contains even number of True values.
  5. Induction Statement :
  6. – We will calculate True values for any formula ϕ with $L(\phi) = k + 1$.
     – The structure of ϕ could be either $\neg\phi_1$ or $\phi_1 \leftrightarrow \phi_2$
     – Case 1 : $\phi = \neg\phi_1$
     – By Observation 4, we can say that number of True values will be yet Even.
     – Case 2 : $\phi = \phi_1 \leftrightarrow \phi_2$
     – Case 2.1 : This is possible that atmost one of them either ϕ_1 or ϕ_2 is of length 1. Consider ϕ_1 of Length 1 WLOG.
     – Consider the Truth table for this. Let the number of possible valuations in ϕ_1 be n.
     – By observation 3, we know that ϕ_2 is literal. This means it contains 1 True value and 1 False value.

- To create the Truth table for ϕ, Every True value of ϕ_2 will occur $n \times m$, where m = number of True values in ϕ_2.
- As n is even, Number of True values that would occur in Truth Table of ϕ_2 will be even.
- Now its remains to show that if two formula with even true values are connected by $\leftrightarrow$ then they will have even number of true values.
- Assume the following :
  Number of True values occurring in $\phi_1 = a_1$
  Number of False values occurring in $\phi_1 = b_1$
  Number of True values occurring in $\phi_2 = a_2$
  Number of True values occurring in $\phi_2 = b_2$
- Now if both these are combined then consider the True False pairs created using the True values of ϕ_1 be x. And number of True False pairs created using True values of ϕ_2 be y.
- So number of True values remained for pairing with True values are : In ϕ_1 It would be $a_1 - x$ and In ϕ_2 It would be $a_2 - y$.
- They will be equal giving that x−y = even.
- Total number of valuations with True True pairs and False False pairs is Total valuations $-$ $(x + y)$.
- As Total valuations is always even, If $(x+y)$ would be even, we are done.
- As we know that $x - y$ is even, this means that x and y both are even or both are odd.
- In both possible cases, $(x + y)$ is even.
- Hence Proved the Claim

7. By this claim, We can state that as we are not able to create boolean funcitons of all the type, the set is not adequate. This is because this property is not satisfied by adequate set.

4 Question 4

4.1 Part a

$\neg r \to (p \vee q), r \wedge (\neg q) \vdash r \to q$

p	q	r	1	2	3
T	T	T	T	F	T
T	T	F	T	F	T
T	F	T	T	T	F
T	F	F	T	F	T
F	T	T	T	F	T
F	T	F	T	F	T
F	F	T	T	T	F
F	F	F	F	F	T

Figure 1: Truth Table

- 1 in Table : $\neg r \to (p \wedge q)$
- 2 in Table : $r \wedge (\neg q)$
- 3 in Table : $r \to q$
- It is visible from line no. 4, p = T, q = F, r = T is satisfying Formula 1 and 2 (giving truth values for them). While Formula 3 is not true for that.

4.2 Part b

$p \to (q \to r) \vdash p \to (r \to q)$

p	q	r	1	2
T	T	T	T	T
T	T	F	F	T
T	F	T	T	F
T	F	F	T	T
F	T	T	T	T
F	T	F	T	T
F	F	T	T	T
F	F	F	T	T

Figure 2: Truth Table

- 1 in Table : $p \to (q \to r)$
- 2 in Table : $p \to (r \to q)$
- It is visible from line no. 4, that p = T, q = F, r = T. And for this Formula 1 is satisfying (giving true value). But for Formula 2, we are getting false.

5 Question 5

5.1 Part a

Set of all formulas $\Phi = \{\alpha_0, \alpha_1, ..\}$
Given : X is a FSS
Construction :

- $X_0 = X$
- $X_{i+1} = \begin{cases} \text{X}_i \cup \{\alpha_i\} \text{ if } X_i \cup \alpha_i \text{ is FSS} \\ \text{X}_i \text{ else} \end{cases}$
- Y $= \cup_{i \geq 0} X_i$

1. Claim 1 : Y is FSS
   - We will prove this by contradiction.
   - Assume Y is not a FSS.
   - By the definition of FSS, if Y is not a FSS, $\exists$ a Z $\subseteq_{fin}$ Y, which is not satisfiable.
   - Let Z $= \alpha_{i_0}, \alpha_{i_1} ...., \alpha_{i_k}$
   - Here $i_0 \leq i_1 ... \leq i_k$
   - As $Z \subseteq Y$, and $\alpha_{i_k} \epsilon Z$, we can say that $X_{i_k+1} = X_{i_k} \cup \{\alpha_{i_k}\}$. Because α_{i_k} could be present in Y only if it is added $i_k + 1$ step.
   - As $Z \subseteq Y$, and max index formula present in Z is α_{ik}, we can say that $Z \subseteq X_{i_k+1}$. This is because $i_0 \leq i_1 .. \leq i_k$. If $\alpha_{i_j} \epsilon Z$, we know that α_{i_j} is added at $i_j + 1$ step. Also, $X_0 \subseteq X_1 ... \subseteq X_{i_k+1}$
   - We know that if α_{i_k} is added at $i_k + 1$ step, then $X_{i_k} \cup \{\alpha_{i_k}\}$ is FSS. As it is FSS, every finite subset is satisfiable. Hence Z should be satisfiable.
   - This is Contradiction.
   - Hence Y is FSS.
2. Claim 2 : Y is Maximal FSS
   - Suppose Y is not Maximal FSS.
   - Then $Y \cup \{\alpha_i\}$ for some $\alpha_i \epsilon \Phi / Y$, is FSS.
   - If α_i does not belong to Y, it must not be added at i^{th} step.
   - This meant that $X_i \cup \{\alpha_i\}$ was not FSS.
   - If $X_i \cup \{\alpha\}$ is not FSS, $\exists Z \subseteq_{fin} X_i \cup \{\alpha_i\}$ is not FSS.
   - As $Z \subseteq Y \cup \{\alpha_i\}$, and Z is not FSS. This contradicts that Y is FSS.
3. Hence Y is maximal FSS.

5.2 Part b

1. Claim 1 : α,$\neg\alpha$ is not FSS.
   - We know by LEM, $\alpha \vee \neg\alpha$ is valid. Hence $\neg(\alpha \vee \neg\alpha)$ is not satisfiable. As $\neg(\alpha \vee \neg\alpha) = (\alpha \wedge \neg\alpha)$,
   - $(\alpha \wedge \neg\alpha)$ is not satisfiable.

2. Claim 2 : At least one of α or $\neg\alpha$ is in X.
    - We will proof this by contradiction.
    - Suppose neither of them belongs to X.
    - As X is Maximal FSS, $\exists Y, Z \subseteq_{fin} X$, where $Y \cup \{\alpha\}$ and $Z \cup \{\neg\alpha\}$ is not satisfiable.
    - Let Y = β_1,β_2,...,β_m and Z = γ_1,...γ_n.
    - As $Z \cup \{\neg\alpha\}$ is not satisfiable $\implies \gamma 1 \wedge \gamma 2... \wedge \neg\alpha$ is not satisfiable.
    - Let $\beta = \beta_1 \wedge ...\beta_m$
    - Let $\gamma = \gamma_1 \wedge ...\gamma_n$
    - And similarly, $\beta 1 \wedge \beta 2... \wedge \alpha$ is not satisfiable.
    - As we know that satisfiability is equivalent to consistency.
    - $\vdash \neg(\alpha \wedge \beta)$ and $\vdash \neg(\neg\alpha \wedge \gamma)$
    - As $\vdash \neg(\alpha \wedge \beta) = \vdash \neg\alpha \vee \neg\beta = \vdash \alpha \rightarrow \neg\beta$
    - Similarly, $\vdash \neg(\neg\alpha \wedge \gamma) = \vdash \alpha \vee \neg\gamma = \vdash \neg\alpha \rightarrow \neg\gamma$
    - Sub-Claim : $\alpha \rightarrow \neg\beta, \neg\alpha \rightarrow \neg\gamma \vdash \neg\beta \rightarrow \gamma$
    - Deduction Theorem, $\alpha \rightarrow \neg\beta, \neg\alpha \rightarrow \neg\gamma, \neg\beta \vdash \gamma$
    - M.T $\alpha \rightarrow \neg\beta, \neg\alpha \rightarrow \neg\gamma, \neg\beta \vdash \neg\alpha$
    - M.P $\alpha \rightarrow \neg\beta, \neg\alpha \rightarrow \neg\gamma, \neg\beta, \neg\alpha \vdash \neg\gamma$
    - Hence Sub Claim Proved.
    - As $(\neg\beta \rightarrow \neg\gamma) = (\neg\beta \vee \neg\gamma) = \neg(\beta \wedge \gamma)$
    - As $(Y \cup Z)$ contains both α and $\neg\alpha$. And by Claim 1 proven above, it is not satisfiable,
    - Hence $(Y \cup Z)$ is not FSS.
    - This is contradiction.
    - Hence, Atleast one of them is in X.
3. Both these properties, prove the claim that $\alpha \epsilon X \leftrightarrow \alpha \not\epsilon X$

5.3 Part c

1. ”=>”
    - Suppose $\alpha \vee \beta \epsilon X$, X is Maximal FSS.
    - By the previous part (b), we know that atleast of the α or $\neg\alpha$ is in X.
    - Suppose α is in X, then it proves the side.
    - Suppose $\neg\alpha$ is in X.
    - Given that $\alpha \vee \beta$ is in X, which is equivalent to $\neg\alpha \rightarrow \beta$.
    - We will prove that if β is not in X, then X is not a Maximal FSS.
    - This could be done by proving that $X \cup \{\beta\}$ is FSS.
    - Consider $Y = (X \cup \{\beta\})$
    - We will prove Y as FSS
    - Now Suppose Y is not a FSS
    - Then by Satisfiability Lemma, $\exists Z \subseteq_{fin} Y$ which is not satisfiable

- It is easy to see that Z will contain β otherwise $Z \subseteq_{fin} X$, As X is FSS, Z would be satisfiable. Hence Z must contain β.
- Let Z = $\{\beta, \gamma_0, .., \gamma_m\}$
- We can also write that $\neg(\beta \wedge \gamma)$ is valid. As Z is not satisfiable.
- As $\neg\alpha$ and $\neg\alpha \rightarrow \beta$ are in X they are satisfiable.
- We know that $\{\gamma_0, ..., \gamma_m, \neg\alpha, \neg\alpha \rightarrow \beta\}$ is satisfiable. As it is finite subset of X.
- We will show $\gamma_0, ..., \gamma_m, \neg\alpha, \neg\alpha \rightarrow \beta \vdash \beta \wedge \gamma$
- Here $\gamma = \gamma_0, \gamma_1, .., \gamma_m$
- By showing this we can say that $\beta \wedge \gamma$ is satisfaible (As Hilbert System is complete So Satisfiability of Premises imples Satisfiablity of Provable Formulas), we can say that $\neg(\beta \wedge \gamma)$ is not valid, which will lead to contradiction.
- Proof of $\gamma_0, ..., \gamma_m, \neg\alpha, \neg\alpha \rightarrow \beta \vdash \beta \wedge \gamma$:
- M.P β
- (Premise) γ
- $(\wedge_i)$ $\beta \wedge \gamma$
- As we know that all premises are satisfiable, there will exist some valuations for which $\beta \wedge \gamma$ will be satisfied, Hence it is satisfiable.
- So we reached to contradiction.
- Hence Y is FSS.
- Hence Proved.

2. ” <= ”
    - Given that α is in X
    - We will prove that if $\alpha \vee \beta$ is not in X then X is not maximal FSS
    - We will prove this by proving that $X \cup \{\alpha \vee \beta\}$ is FSS
    - Let $Y = X \cup \{\alpha \vee \beta\}$
    - Suppose Y is not FSS
    - Then $\exists Z \subseteq_{fin} Y$ not satisfiable and Z must have $\alpha \vee \beta$ by the argument same as used in provinig the other side
    - Now, Suppose $Z = \{\alpha \vee \beta, \gamma\}$, where $\gamma = \gamma_0, \gamma_1, .., \gamma_m$
    - As Z is not satisfiable, we can say that $\neg((\alpha \vee \beta) \wedge \gamma)$ is valid
    - Now as α and γ is in X, $\{\alpha, \gamma\}$ is satisfaible as it is a finite subset of X.
    - It is easy to see that $\alpha, \gamma \vdash (\alpha \vee \beta) \wedge \gamma$
    - As ”or” insertion and ”and” insertion can be proven in Hilbert System.
    - We know that as Left side (premises) are satisfaible
    - As Hilbert System is complete. We have $\alpha, \gamma \models (\alpha \vee \beta) \wedge \gamma$
    - This shows that $(\alpha \vee \beta) \wedge \gamma$ is satisfaible, Hence $\neg((\alpha \vee \beta) \wedge \gamma)$ is not valid.
    - This is a contradiction.
    - Hence Y is FSS
3. This proves the claim.

5.4 Part d

1. Let us assume that v_α is set of all possible valuations for which α has a valuation of True.

2. Here $\alpha \epsilon X$

3. For any $Z \subseteq X$, we will define $v_Z = \cap_{\alpha \epsilon Z}(v_\alpha)$

4. We will define $v^* = \cap_{\alpha \epsilon X} v_\alpha$

5. This is observable that v_* is non empty. Otherwise If it is empty, then $\exists Z \epsilon X$ which is empty (By Konnigs Lemma).

6. And as X is FSS, Z must be satisfiable but if v_Z is empty then there exist no valuation such that v_Z could be True.

7. This contradicts our Assumption, Hence v^* is not empty.

8. Now we will prove that $\exists v_x \epsilon v^*$ such that $v_x \models \alpha \leftrightarrow \alpha \epsilon X$

9. " <= "
    - As v^* is intersection over all possible valuations $\forall \alpha \epsilon X$, It must be a subset of $v_\alpha \forall \alpha$
    - So $v_x \models \alpha$
    - Hence Proved

10. " => "
    - Suppose $\alpha \not\epsilon X$
    - By the part b, if $\alpha \not\epsilon X$ then $\neg \alpha \epsilon X$
    - As $\neg \alpha \epsilon X$, By " <= " direction we can say that $v_x \models \neg \alpha$
    - But we know that $v_x \models \alpha$, which is a contradiction.
    - Hence $\alpha \epsilon X$

11. This proves the claim

5.5 Part e

1. By part d, we know that $\alpha \epsilon X \leftrightarrow v_x \models X$, if X is maximal FSS, $\forall \alpha$

2. For any FSS, Y we can construct a maximal FSS X_0.

3. So $Y \subseteq X$

4. As $v_x \models \alpha \ \forall \alpha \epsilon X_0$.

5. So,$v_x \models \alpha \ \forall \alpha \epsilon Y$, as $Y \subseteq X_0$.

6. Let Y' = Any finite subset of Y

7. So, $v_x \models \wedge_{\alpha \epsilon Y'} \alpha$

8. As all the finite subsets of Y are satisfiable by only valuation v_x.

9. Y is simultaneously satisfiable

5.6 Part f

1. (<=):
   - If $Y \subseteq_{fin} X$, and $Y \models \alpha$.
   - $Y \cup Z \models \alpha, \forall Z$
   - Let $Z = X/Y$
   - $X \models \alpha$
   - As $v \models X$ then $v \models Y$.
   - As $Y \models \alpha$.
   - By logical sequence, $v \models \alpha$
2. (=>)
   - Sub-Claim : For all $Z \subseteq \Phi$ and all $\beta \epsilon \Phi$, $Z \models \beta$ iff $Z \cup \{\neg\beta\}$ is not satisfiable.
   - Sub-Claim Proof : If Z is not satisfiable, then we are done. If Z is satisfiable and $Z \cup \{\neg\beta\}$ is also satisfiable. As $Z \models \beta$, then $Z \cup Y \models \beta$ for $Z \cup Y$ begin satisfiable. As $Z \cup \{\neg\beta\}$ is satisfiable, $Z \cup \{\neg\beta\} \models \beta$. This is contradiction. Hence sub claim proven.
   - Suppose $X \models \alpha$.
   - By sub claim $X \cup \{\neg\alpha\}$ is not satisfiable. As proven in class, if X is satisfiable then all its finite subset are satisfiable by konigs lemma. As $X \cup \{\neg\alpha\}$ is not satisfiable, $\exists Y \subseteq_{fin} X$ not satisfiable. As Y could be rewritten as $(Y/\{\neg\alpha\}) \cup \{\alpha\}$, which is not satisfiable, by sub claim, we can say that $(Y/\{\neg\alpha\}) \models \alpha$, as $(Y/\{\neg\alpha\}) \subseteq_{fin} X$, this proves the claim.
3. Hence Proved.