How to write a mathematical description of a LCU qubisation embedded in a QPE algorithm.

1. LCU Block-Encoding of the Hamiltonian

A Hermitian operator H is written as a emphlinear combination of unitaries

$$H \;=\; \sum_{i=0}^{L-1} \omega_i \, U_i, \qquad \omega_i \in \mathbb{R}, \tag{1}$$

where Ua are unitary matrices with coefficents. The one-norm of the LCU, $\lambda = \sum_{i=0} |\omega_i|$ For the one-qubit toy model we take

$$U_0 = I, \qquad U_1 = X, \qquad U_2 = Z, \quad H = 1.5\, I + 0.5\, X - 0.5\, Z.$$

2. Normalisation Factor

$$\lambda \;=\; \sum_{i=0}^{L-1} |\omega_i| \;=\; 1.5 + 0.5 + 0.5 \;=\; 2.5. \tag{2}$$

Hence

$$\frac{H}{\lambda} \;=\; \sum_{i=0}^{L-1} \frac{|\omega_i|}{\lambda}\, s_i\, U_i, \qquad s_i = \operatorname{sgn}(\omega_i) \in \{\pm 1\}.$$

3. Ancilla Preparation (PREP)

Let $m = \lceil \log_2 L \rceil$ (here $m = 2$). Prepare

$$|\chi\rangle \;=\; \sum_{i=0}^{L-1} \sqrt{\tfrac{|\omega_i|}{\lambda}}\; |i\rangle \;=\; \sqrt{0.6}\, |00\rangle + \sqrt{0.2}\, |01\rangle + \sqrt{0.2}\, |10\rangle, \tag{3}$$

via a unitary

$$\mathsf{PREP} : \; |0^{\otimes m}\rangle \mapsto |\chi\rangle.$$

4. SELECT Operator

$$\mathsf{SELECT} \;=\; \sum_{i=0}^{L-1} |i\rangle\!\langle i| \;\otimes\; s_i\, U_i \;=\; |00\rangle\!\langle 00| \otimes I \;+\; |01\rangle\!\langle 01| \otimes X \;+\; |10\rangle\!\langle 10| \otimes (-Z). \tag{4}$$

5. Block-Encoding Unitary

$$U \;=\; (\mathsf{PREP}^\dagger \otimes I)\; \mathsf{SELECT}\; (\mathsf{PREP} \otimes I). \tag{5}$$

Projecting the ancilla onto $|0^{\otimes m}\rangle$ returns

$$\big(\langle 0^{\otimes m}| \otimes I\big)\, U\, \big(|0^{\otimes m}\rangle \otimes I\big) \;=\; \frac{H}{\lambda}.$$

6. Walk Operator (Qubitisation)

Define the reflection

$$R = 2|0^{\otimes m}\rangle\langle 0^{\otimes m}| - I_{2^m},$$

and the *walk operator*

$$W \;=\; RU. \tag{6}$$

For every eigenvector $|\psi_j\rangle$ of H with eigenvalue E_j,

$$W\big(|0^{\otimes m}\rangle \otimes |\psi_j\rangle\big) \;=\; e^{\pm i\theta_j}\,\big(|0^{\otimes m}\rangle \otimes |\psi_j\rangle\big), \qquad \cos\theta_j = \frac{E_j}{\lambda}. \tag{7}$$

7. Using QPE

Because W is unitary and its phases θ_j directly encode the eigen-energies, standard Quantum Phase Estimation on W yields

$$E_j \;=\; \lambda\,\cos\theta_j.$$

8. Walkoperator in QPE

The walk operator is the essential bridge between a block-encoded Hamiltonian and the phase spectrum required by Quantum Phase Estimation. Its theoretical justification follows directly from the qubitisation results in Low & Chuang, *npj Quantum Information* **3**, 13 (2017).