Chapter 1

Soft Tissue Engineering

1.1 Stress & strain in large deformations

1.1.1 Kinematics

- Mapping function

$$\vec{x} = \underline{\varphi}(\vec{X}, t) \tag{1.1}$$

- Displacement function

$$\vec{x} = \vec{X} + \underline{U}(\vec{X}, t) \tag{1.2}$$

- The deformation gradient tensor

$$d\vec{x} = \mathbf{F}(X, t)d\vec{X} \Rightarrow \mathbf{F} = \frac{\partial \vec{x}}{\partial \vec{X}} = \frac{\partial \underline{\varphi}}{\partial \vec{X}} \tag{1.3}$$

- Volume transformation

$$dv = \det(\mathbf{F})dV = JdV \tag{1.4}$$

- Surface transformation: Nanson's Formula

$$\vec{n}ds = J\mathbf{F}^{-\top}\vec{N}dS \tag{1.5}$$

- Polar decomposition

$$\mathbf{F} = \mathbf{RU} = \mathbf{VR} \tag{1.6}$$
$$\Rightarrow \mathbf{U}^2 = \mathbf{F}^\top\mathbf{F} = \mathbf{C} \tag{1.7}$$
$$\Rightarrow \mathbf{V}^2 = \mathbf{F}\mathbf{F}^\top = \mathbf{B} \tag{1.8}$$

1.1.2 Strain

- Nominal strain, Biot strain or Engineering strain

$$\mathbf{e} = \mathbf{U} - \mathbf{I} \tag{1.9}$$

- Green-Lagrange strain

$$\mathbf{E} = \frac{1}{2}(\mathbf{F}^\top\mathbf{F} - \mathbf{I}) = \frac{1}{2}(\mathbf{U}^2 - \mathbf{I}) \tag{1.10}$$

- Euler-Almansi strain

$$\mathbf{A} = \frac{1}{2}(\mathbf{I} - \mathbf{V}^{-2}) \tag{1.11}$$

- True strain or logarithmic strain

$$\boldsymbol{\varepsilon} = \ln(\mathbf{U}) \tag{1.12}$$

1.1.3 Stress

- True stress or Cauchy stress

$$\boldsymbol{\sigma}^\top = \frac{d\vec{f}}{\vec{n}dS} \tag{1.13}$$

- 1PK: first Piola Kirchoff stress (Nominal stress or Engineering stress)

$$\mathbf{P} = \frac{d\vec{f}}{\vec{N}dS_0} \tag{1.14}$$

- 2PK: second Piola Kirchoff stress

$$\mathbf{S}^\top = \frac{\mathbf{F}^{-1}d\vec{f}}{\vec{N}dS_0} \tag{1.15}$$

- Swithcing between stresses:

	$\boldsymbol{\sigma}$	$\mathbf{P}$	$\mathbf{S}$
$\boldsymbol{\sigma}$	$\boldsymbol{\sigma}$	$J^{-1}\mathbf{P}\mathbf{F}^\top$	$J^{-1}\mathbf{F}\mathbf{S}\mathbf{F}^\top$
$\mathbf{P}$	$J\boldsymbol{\sigma}\mathbf{F}^\top$	$\mathbf{P}$	$\mathbf{F}\mathbf{S}$
$\mathbf{S}$	$J\mathbf{F}^{-1}\boldsymbol{\sigma}\mathbf{F}^{-\top}$	$\mathbf{F}^{-1}\mathbf{P}$	$\mathbf{S}$

1.2 Constitutive modeling

The most general form:

$$\boldsymbol{\sigma}(t) = \mathcal{F}_{\tau \leq t}\{\mathbf{F}(\tau)\} \tag{1.16}$$

1.2.1 Hyperelasticity

Strain energy density function Ψ related to 2PK

$$\mathbf{S} = \frac{\partial \Psi}{\partial \mathbf{E}} = 2\frac{\partial \Psi}{\partial \mathbf{C}} \tag{1.17}$$

Strain energy density function related to Cauchy stress

$$\boldsymbol{\sigma} = \frac{1}{J}\frac{\partial \Psi}{\partial \mathbf{F}}\mathbf{F}^\top \tag{1.18}$$

Small strain - linear elasticity (using Hooke's law)

$$\Psi_{\text{elastic}} = \frac{E\varepsilon^2}{2} \tag{1.19}$$

1.2.2 Isotropic hyperelastic materials

Right Cauchy-Green Tensor (with λ_i the PRICIPAL stretches)

$$\mathbf{C} = \begin{bmatrix} \lambda_1^2 & 0 & 0 \\ 0 & \lambda_2^2 & 0 \\ 0 & 0 & \lambda_3^2 \end{bmatrix} \tag{1.20}$$

with invariants:

$$I_1 = \text{tr}(\mathbf{C}) = \lambda_1^2 + \lambda_2^2 + \lambda_3^2 \tag{1.21}$$

$$I_2 = \frac{1}{2}[\text{tr}(\mathbf{C})^2 - \text{tr}(\mathbf{C}^2)] = \lambda_1^2\lambda_2^2 + \lambda_1^2\lambda_3^2 + \lambda_2^2\lambda_3^2 \tag{1.22}$$

$$I_3 = \det(\mathbf{C}) = \lambda_1^2\lambda_2^2\lambda_3^2 = J^2 \tag{1.23}$$

Some forms of isotropic SEDFs

- Generalized Ogden

$$\Psi = \sum_{p=1}^{N} \frac{\mu_p}{\alpha_p}(\lambda_1^{\alpha_p} + \lambda_2^{\alpha_p} + \lambda_3^{\alpha_p} - 3) \tag{1.24}$$

- Neo-Hookean ($N = 1$, $\alpha_1 = 2$)

$$\psi = c_1(I_1 - 3) \tag{1.25}$$

- Mooney-Rivlin ($N = 2$, $\alpha_1 = 2$, $\alpha_2 = -2$)

$$\psi = c_1(I_1 - 3) + c_2(\lambda_1^{-2} + \lambda_2^{-2} + \lambda_3^{-2} - 3) \tag{1.26}$$

1.2.3 The NeoHookean material formulation

$$\boldsymbol{\sigma}_{\mathrm{NH}} = \frac{2}{J} c_1 \mathbf{B} \tag{1.27}$$

1.2.4 Incompressibility

Consider as SEDF

$$\Psi_{\mathrm{tot}} = \Psi(\mathbf{F}) - p(J - 1) \tag{1.28}$$

the 2PK stress becomes

$$S = 2\left(\frac{\partial \Psi}{\partial \mathbf{C}} - p\frac{J}{2}\mathbf{C}^{-\top}\right) \tag{1.29}$$

hence the Cauchy stress takes the following form

$$\boldsymbol{\sigma} = \frac{2}{J} < textbfF \frac{\partial \Psi}{\partial \mathbf{C}} \mathbf{F}^{\top} - p\mathbf{I} \tag{1.30}$$

Using the Neohookean SEDF gives

$$\boldsymbol{\sigma} = 2c_1\mathbf{B} - p\mathbf{I} = -p_h\mathbf{I} \tag{1.31}$$

1.3 Mechanical testing & parameter fitting

1.3.1 Uniaxial tensile testing

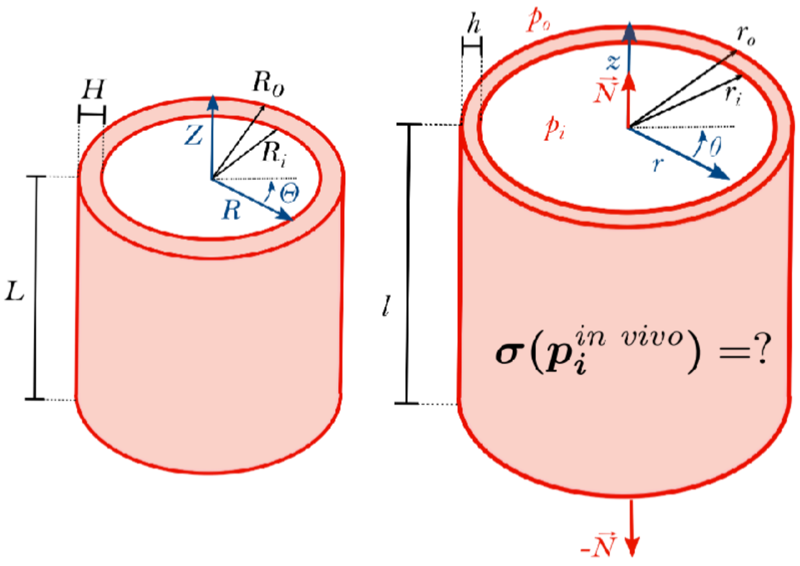
1.3.2 Biaxial tensile testing

1.3.3 Extenson-inflation testing

1.3.4 Strain mapping

1.4 In vivi wall stress estimation

1.4.1 Laplace's law for thin-walled tubes



The deformation gradient tensor is givan as

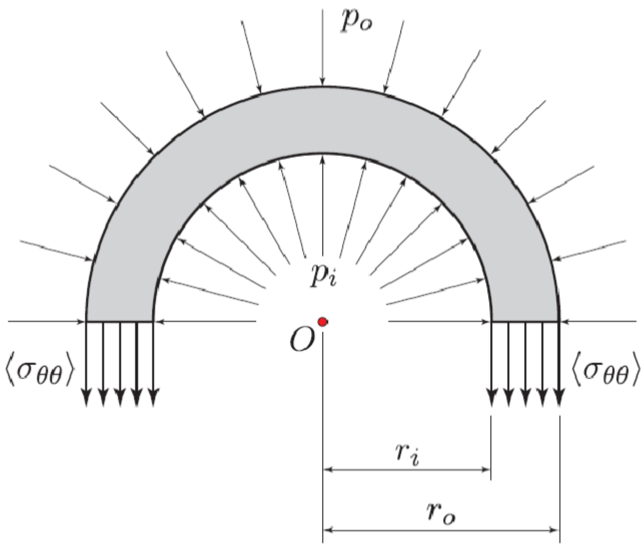
$$\mathbf{F} = \begin{bmatrix} 1/(\lambda_\theta \lambda_z) & 0 & 0 \\ 0 & \lambda_\theta & 0 \\ 0 & 0 & \lambda_z \end{bmatrix} \tag{1.32}$$

with corresponding stresses

$$\langle \sigma_{rr} \rangle = -\frac{P}{2} \tag{1.33}$$

$$\langle \sigma_{\theta\theta} \rangle = \frac{P}{h/r_i} \tag{1.34}$$

$$\langle \sigma_{zz} \rangle = \frac{||\vec{N}||}{\pi(r_0^2 - r_i^2)} \tag{1.35}$$



1.4.2 Analytical analysis fot thick-walled cylinders

Stress equilibrium (Balance of Momentum)

$$\nabla \cdot \boldsymbol{\sigma} + \vec{f} = \rho \frac{D^2 \vec{u}}{Dt^2} \tag{1.36}$$

Stress equilibrium in a thick-walled cylinder

$$P = \int_{r_i}^{r_o} (\sigma_{\theta\theta} - \sigma_{rr}) \frac{dr}{r} \tag{1.37}$$

NeoHookean formulation for the Cauchy Stress in a thick-walled cylinder

$$\sigma_{rr} = 2c_1 \lambda_\theta^{-2} \lambda_z^{-2} - p \tag{1.38}$$

$$\sigma_{\theta\theta} = 2c_1 \lambda_\theta^2 - p \tag{1.39}$$

$$\sigma_{zz} = 2c_1 \lambda_z^2 - p \tag{1.40}$$

with

$$\lambda_\theta = \frac{r}{R} \tag{1.41}$$

$$r = \sqrt{\lambda_{\theta,i}^2 R_i^2 + \frac{R^2 - R_i^2}{\lambda_z}} \tag{1.42}$$

1.4.3 The Finite Element Method

1.5 Other biomechanical applications