Report Electromagnetism I

FDTD Simulation of Lossless Transmission Lines

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Abstract
Short review of the report.

1 The update functions

The update functions are given as:

$$\tilde{I}_{n+\frac{1}{2}}^{m+\frac{1}{2}} = \tilde{I}_{n+\frac{1}{2}}^{m-\frac{1}{2}} + \alpha \left(V_n^m - V_{n+1}^m \right), \qquad V_n^{m+1} = V_n^m + \alpha \left(\tilde{I}_{n-\frac{1}{2}}^{m+\frac{1}{2}} - \tilde{I}_{n+\frac{1}{2}}^{m+\frac{1}{2}} \right), \tag{1}$$

where

$$\alpha \triangleq \frac{v\Delta t}{\Delta z} \tag{2}$$

is the dimensionless Courant factor and

$$\tilde{I}_{n+\frac{1}{2}}^{m+\frac{1}{2}} = I_{n+\frac{1}{2}}^{m+\frac{1}{2}} R_c \tag{3}$$

is the rescaled current.
At the boundaries the update function for V takes another form.

- At $z = 0$
  The voltage update function is given as:

  $$V_0^{m+1} = V_0^m + \frac{2\Delta t}{C\Delta z} \left(I_g^{m+\frac{1}{2}} - I_{\frac{1}{2}}^{m+\frac{1}{2}} \right), \tag{4}$$

  with

  $$I_g^{m+\frac{1}{2}} = \frac{E_g^{m+\frac{1}{2}}}{R_g} - \frac{V_0^m + V_0^{m+1}}{2R_g}. \tag{5}$$

  Subsituting (4) in (3) and ussing (1), the two relations $v = \frac{1}{\sqrt{LC}}$ and $R_c = \sqrt{\frac{L}{C}}$ and the new defined constant $\tilde{R}_g = \frac{R_c}{R_g}$ yields, after some rearrangements:

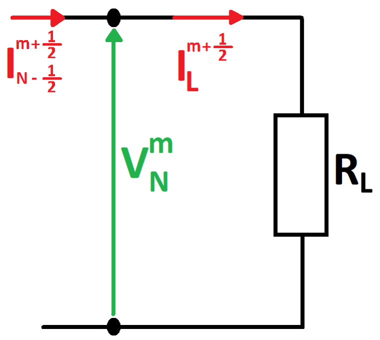
  $$V_0^{m+1} = C_1 V_0^m + C_2 \left(E_g^{m+\frac{1}{2}} \tilde{R}_g - \tilde{I}_{\frac{1}{2}}^{m+\frac{1}{2}} \right) \tag{6}$$

  where

  $$C_1 = \frac{1 - \alpha\tilde{R}_g}{1 + \alpha\tilde{R}_g}, \tag{7}$$
  $$C_2 = \frac{2\alpha}{1 + \alpha\tilde{R}_g}, \tag{8}$$

  are two dimensionless constants.

- At $z = l$



The voltage update function becomes:

$$V_N^{m+1} = V_N^m + \frac{2\Delta t}{C\Delta z}\left(I_{N-\frac{1}{2}}^{m+\frac{1}{2}} - I_L^{m+\frac{1}{2}}\right) \tag{9}$$

Kirchoff's voltage law in discretized form states that

$$I_L^{m+\frac{1}{2}} = \frac{V_N^{m+\frac{1}{2}}}{R_L} \tag{10}$$

$$= \frac{V_N^m + V_N^{m+1}}{2R_L} \tag{11}$$

Subsituting (10) in (8) and using the same relations as for $z = 0$ and the new defined constant $\tilde{R}_L = \frac{R_c}{R_L}$ yields, after some rearrangements:

$$V_N^{m+1} = C_3 V_N^m + C_4 \tilde{I}_{N-\frac{1}{2}}^{m+\frac{1}{2}}, \tag{12}$$

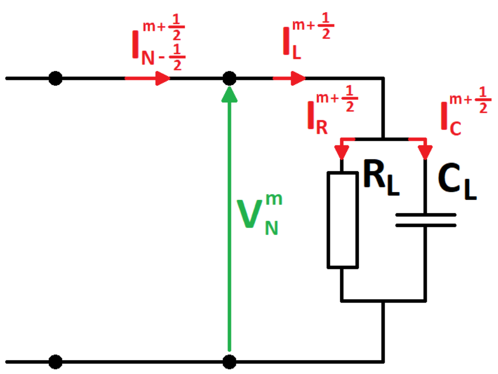
where

$$C_3 = \frac{1 - \alpha\tilde{R}_L}{1 + \alpha\tilde{R}_L}, \tag{13}$$

$$C_4 = \frac{2\alpha}{1 + \alpha\tilde{R}_L}, \tag{14}$$

are two dimensionless constants.

Now if there would be a capacitor added to the load the following situation will occur



Kirchoff's current law states that

$$I_L^{m+\frac{1}{2}} = I_R^{m+\frac{1}{2}} + I_C^{m+\frac{1}{2}}. \tag{15}$$

Using Kirchoff's voltage law at the resistor gives

$$
\begin{aligned}
I_R^{m+\frac{1}{2}} &= \frac{V_N^{m+\frac{1}{2}}}{R_L} && (16)\\
&= \frac{V_N^m + V_N^{m+1}}{2R_L} && (17)
\end{aligned}
$$

The relation between the current and the voltage at the capitor is given as

$$
\hat{i} = -C\frac{\partial \hat{v}}{\partial t}. \tag{18}
$$

For a first order FDM this turns into

$$
I_R^{m+\frac{1}{2}} = -C_L\frac{V_N^{m+1} - V_N^m}{\Delta t}. \tag{19}
$$

Substituting (16) and (18) into (14) gives

$$
I_L^{m+\frac{1}{2}} = \left(\frac{1}{2R_L} - \frac{C_L}{\Delta t}\right)V_N^{m+1} + \left(\frac{1}{2R_L} + \frac{C_L}{\Delta t}\right)V_N^m \tag{20}
$$

and can be rewritten as

$$
\tilde{I}_L^{m+\frac{1}{2}} = \frac{\tilde{Z}_1}{2}V_N^{m+1} + \frac{\tilde{Z}_2}{2}V_N^m \tag{21}
$$

with

$$
\begin{aligned}
\tilde{Z}_1 &= R_C\left(\frac{1}{R_L} - 2\frac{C_L}{\Delta t}\right) && (22)\\
\tilde{Z}_2 &= R_C\left(\frac{1}{R_L} + 2\frac{C_L}{\Delta t}\right) && (23)
\end{aligned}
$$

Substituting (20) into the adjusted voltage update function at $z = l$ yields

$$
V_N^{m+1} = C_3'V_N^m + C_4'\tilde{I}_{N-\frac{1}{2}}^{m+\frac{1}{2}}, \tag{24}
$$

where

$$
\begin{aligned}
C_3' &= \frac{1 - \alpha\tilde{Z}_2}{1 + \alpha\tilde{Z}_1}, && (25)\\
C_4' &= \frac{2\alpha}{1 + \alpha\tilde{Z}_1}. && (26)
\end{aligned}
$$

Notice when $C_L = 0$ then $C_3' = C_3$ and $C_4' = C_4$.