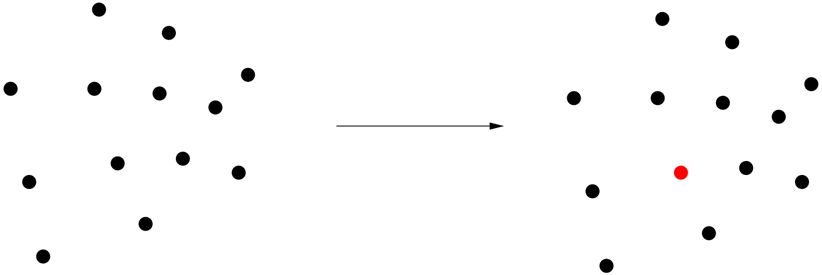
Leader Election in Swarms of Deterministic Robots

Franck Petit

LiP6, UPMC Paris 6

Problem



Question

Given a swarm of n robots, what are the minimal geometric conditions to be able to deterministically agree on a single robot?

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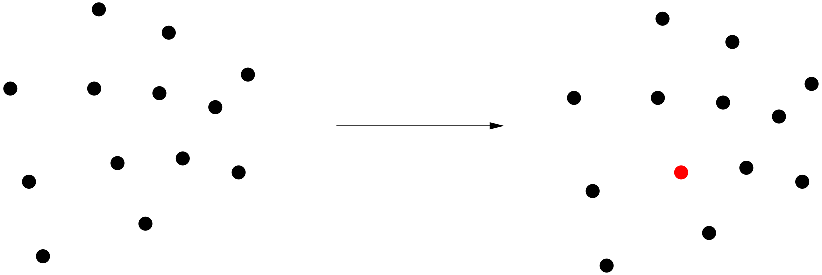

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Previous Works

SoD and Chirality

[Flocchini et al., 1999]

A solution exists for any n.

SoD and No Chirality

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A solution exists if n is odd.

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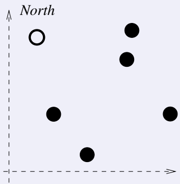

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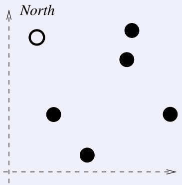

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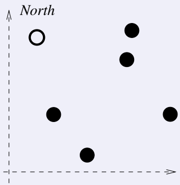

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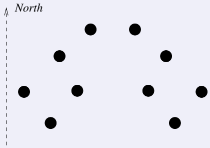

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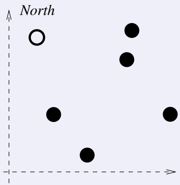

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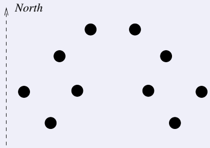

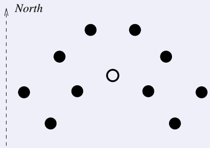

Previous Works

No SoD

[Prencipe 2002]

Impossible in general.

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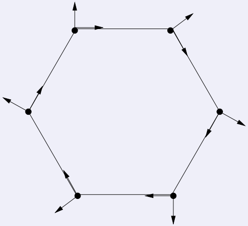

In such a configuration, it is not possible to break the symmetry.

Leader Election With No Sense of Direction

Question

Assuming no sense of direction (with or without chirality), what are the geometric conditions to be able to deterministically agree on a single robot?

To answer to this question, we need tools from the theory on Combinatoric on Words, specifically Lyndon Words.

Leader Election With No Sense of Direction

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To answer to this question, we need tools from the theory on Combinatoric on Words, specifically Lyndon Words.

Lyndon Words

Definition (Word)

Let $A = \{a_0, a_1, \ldots, a_n\}$ be an alphabet. A word is a (possibly empty) sequence of letters in A.

$$A = \{a, b, c, d\}$$

$$abcc \quad a \quad \epsilon \quad dddddddd \equiv d^8$$

Lyndon Words

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Lyndon Words

Definition (Concatenation)

Let $u = a_1, \ldots, a_i, \ldots, a_k$ and $v = b_1, \ldots, b_j, \ldots, b_\ell$.
The concatenation of u and v, denoted uv, is equal to the word $a_1, \ldots, a_i, \ldots, a_k, b_1, \ldots, b_j, \ldots, b_\ell$.

$u = UP$, $v = MC$, $uv = UPMC$

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Lyndon Words

Definition (Lexicographic Order)

Let A be an alphabet totally ordered by $\prec$, *i.e.*, $a_0 \prec a_1 \prec \ldots \prec a_n$.

A word $u = a_0 a_1 \ldots a_s$ is said to be *lexicographically smaller than or equal to a word* $v = b_0 b_1 \ldots b_t$*, denoted by* $u \preceq v$*, iff:*

- *either* u *is a prefix of* v*,*
- *or,* $\exists k : \forall i \in [1, \ldots, k-1], a_i = b_i$ *and* $a_k \prec b_k$*.*

$ab \preceq abc$ $\quad abc \preceq abc$ $\quad \epsilon \preceq abc$ $\quad abc \preceq def$

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Lyndon Words

Definition (Primitive Word)

A word u is said to be *primitive* iff $u = v^k \Rightarrow k = 1$. Otherwise, u is said to be *periodic*.

Primitive Words

ab *dabcbc* *dcba*

Periodic Words

d^8 *bcbc* ϵ

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Definition (Rotation)

A word u is said to be a *rotation* of a word v iff there exists two words x, y such that $u = xy$ and $v = yx$.

$$u = abcd \text{ and } v = cdab$$
$$u = abcd \text{ and } v = bcda$$

Definition (Minimality)

A word u is said to be a *minimal* iff u is lexicographically smaller than any of its rotations.

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$$u = ab\color{red}{cd} \text{ and } v = \color{red}{cd}ab$$
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Lyndon Words

Definition (Lyndon Word)

A word u is a *Lyndon word* iff u is primitive and minimal.

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abc ($abc \preceq cab$ and $abc \preceq bca$)

Not a Lyndon Word

bca ($bca \succ abc$)

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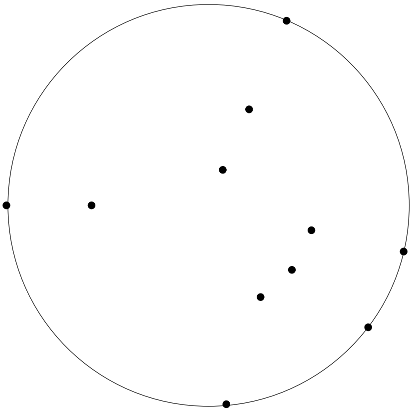

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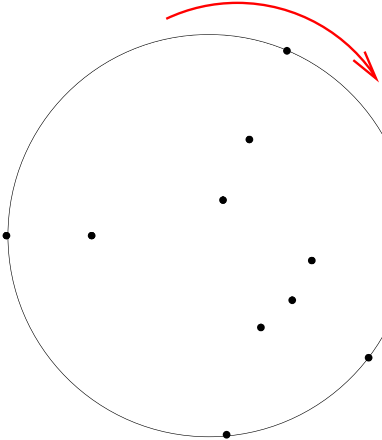

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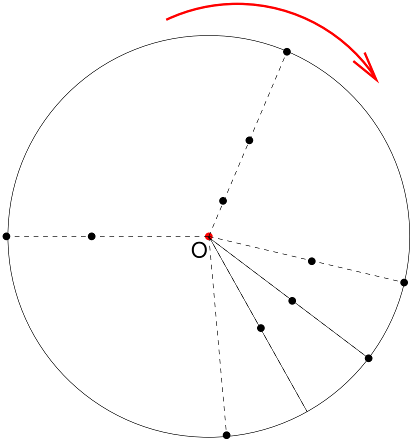

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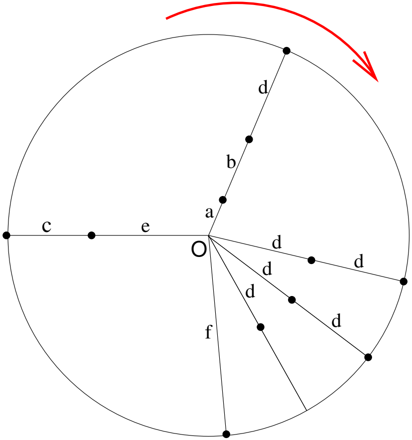

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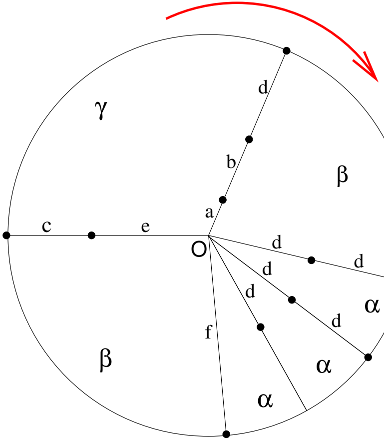

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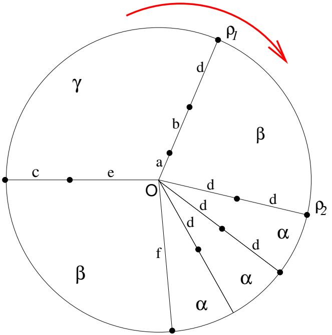

$$W(\rho_1) = (abc,\beta)(d^2,\alpha)^2(d,\alpha)(f,\beta)(ec,\gamma)$$
$$W(\rho_2) = (d^2,\alpha)^2(d,\alpha)(f,\beta)(ec,\gamma)(abc,\beta)$$

Lemma

If two distinct radii ρ_1 and ρ_2 exist such that both $W(\rho_1)$ and $W(\rho_2)$ are Lyndon words, then $\forall\rho$, $W(\rho) = (0,0)$.

Lemma ($\Rightarrow$)

If there exists a radius ρ such that $W(\rho)$ is a Lyndon word, then the robots are able to deterministically agree on the same leader.

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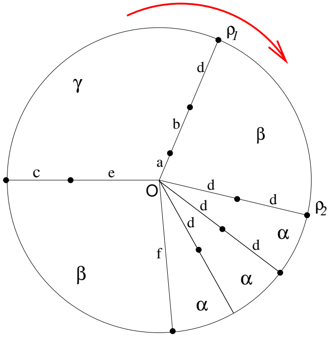

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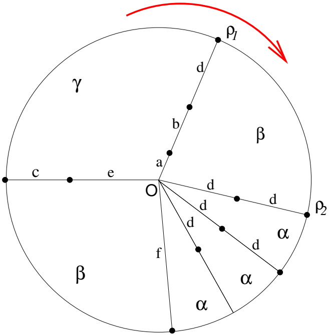

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If there exists a radius ρ such that $W(\rho)$ is a Lyndon word, then the robots are able to deterministically agree on the same leader.

Leader Election with Chirality

Lemma ($\Leftarrow$)

If there exists no radius ρ such that $W(\rho)$ is a Lyndon word, then the robots are not able to deterministically agree on the same leader.

Property

[Lothaire 1983]

If no rotation of a work u is a Lyndon word, then u is periodic.

Leader Election with Chirality

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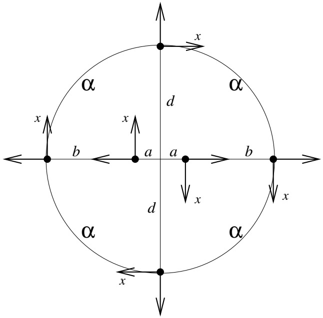

Leader Election with Chirality

Theorem

Assuming chirality, a swarm of robots is able to deterministically agree on the same leader if and only if there exists a radius ρ such that $W(\rho)$ is a Lyndon word.

Leader Election without Chirality

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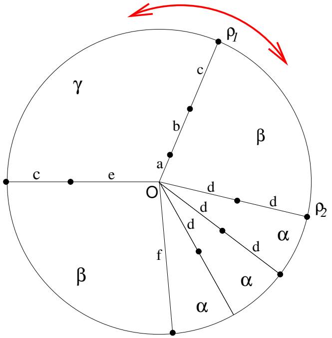

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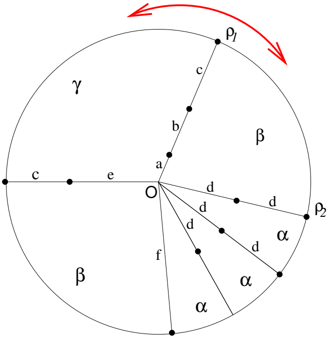


For each ρ, there are 2 ways to compute $W(\rho)$

$W(\rho_1) =$

either

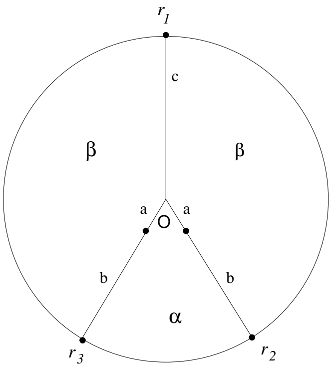
$(abc, \beta)(d^2, \alpha)^2(d, \alpha)(f, \beta)(ec, \gamma)$

or

$(abc, \gamma)(ec, \beta)(f, \alpha)(d, \alpha)(d^2, \alpha)(d^2, \beta)$

depending on either ↻ or ↺, respectively.

Leader Election without Chirality



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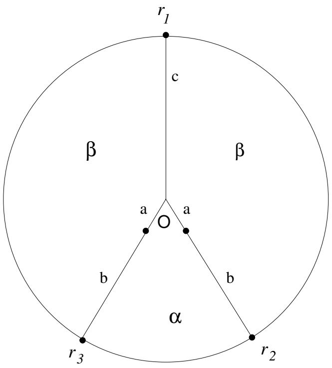
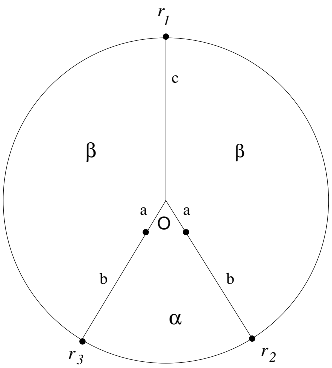


The word $W(\rho_2)^{\circlearrowright} = W(\rho_3)^{\circlearrowleft} = (ab, \alpha)(ab, \beta)(c, \beta)$ is a Lyndon word.

Leader Election without Chirality



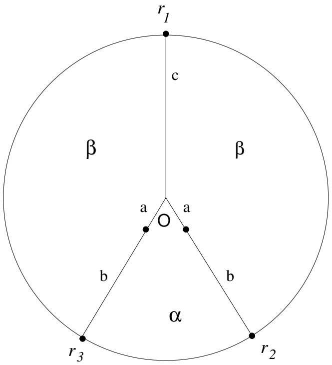
Definition (Type of Symmetry)

A radius ρ_i is of Type (of symmetry) 0 if there exists no radius ρ_j such that $W(\rho_i)^{\circlearrowleft} = W(\rho_j)^{\circlearrowright}$. Otherwise, ρ_i is said to be of Type 1.

A radius of Type *t* is said to be *t*-symmetric.

ρ_1 is 0-symmetric.
ρ_2 and ρ_3 are 1-symmetric.

Leader Election without Chirality



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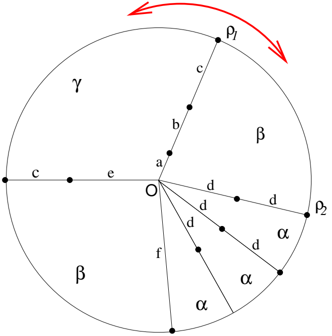
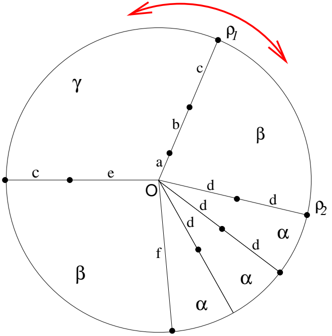


For each radius ρ_i, every robot computes $W(\rho_i)^{\circlearrowright}$ and $W(\rho_i)^{\circlearrowleft}$ of the form $(\mathit{type}, \mathit{radiusword}, \mathit{angle})$.

Leader Election without Chirality



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$$W(\rho_1)^{\circlearrowright} = (0, abc, \beta)(0, d^2, \alpha)^2(0, d, \alpha)(0, f, \beta)(0, ec, \gamma)$$
$$W(\rho_1)^{\circlearrowleft} = (0, abc, \gamma)(0, ec, \beta)(0, f, \alpha)(0, d, \alpha)(0, d^2, \alpha)(0, d^2, \beta)$$

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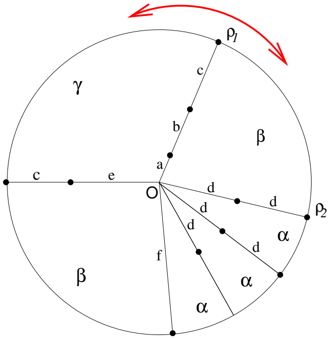


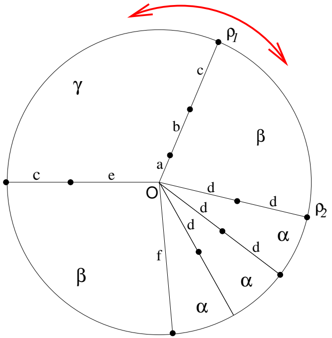
Lemma

If two distinct radii ρ_1 and ρ_2 exist such that both $W(\rho_1)$ and $W(\rho_2)$ are Lyndon words, then $\forall \rho,\ W(\rho) = (0,0,0)$.

Lemma

If there exists a pair of radii $\{\rho_1, \rho_2\}$ so that $W(\rho_i)^{\circlearrowright}$ or $W(\rho_i)^{\circlearrowleft}$ is a Lyndon word ($i \in \{1,2\}$), then the robots are able to deterministically agree on the same leader if and only if ρ_1 and ρ_2 are 0-symmetric.

Leader Election without Chirality



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Lemma

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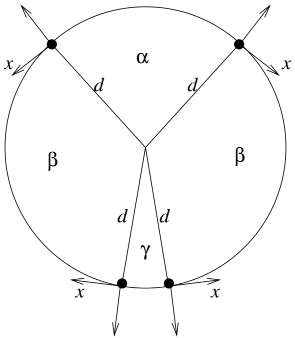
Leader Election without Chirality



No leader exists.

Lemma

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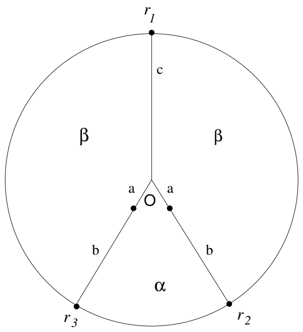

The robot on ρ_1 is the leader.

Lemma

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Leader Election without Chirality

Theorem

Assuming no chirality, a swarm of robots is able to deterministically agree on the same leader if and only if there exists a radius ρ *such that* $W(\rho)$ *is a 0-symmetric Lyndon word.*