Reminder: Start the Recording!

You can book the time for attending the presentation on your labs Project.
Best Regards
Valentin

Declarative Problem Solving with Mixed-Integer Linear Programming

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Outline

Declarative Problem Solving: A New Perspective

- Shift the focus to **what the problem is** and how we can **describe it clearly**.
- *Forget about the process*—the magic happens in defining the problem, not how the solution is obtained!
  - No need to dive into complex algorithms.
  - No need for heavy, intricate math.
  - Simply focus on capturing the essence of the problem with precision.
- Let the tools handle the "how"—your job is to **describe the "what"**.

Mixed-Integer Linear Programming: A Powerful Approach

- Define **what we aim to optimize**: a clear, linear objective function that captures our goals.
- Set the stage by framing the **constraints** as linear inequalities—this is the space where solutions live.
- Merging integer decisions and linear constraints offers flexibility to model real-world scenarios with precision and efficiency.

Demo: Gaming Bar Setup

Optimize the setup of Arcade Machines and VR Stations to maximize the entertainment value while staying within budget and space constraints.

Gaming Bar Setup: Facts

Attribute	Arcade Machine	VR Station
Cost [$]	200	500
Space [m^2]	5	3
Entertainment Value	80	90

Resource	Value
Total Budget [$]	4000
Total Space [m^2]	50

Table: Summary of Costs, Space Requirements, Entertainment Values, and Available Resources

The Origins of Linear Programming

- LP originated in the 1940s during World War II.
- It was developed as a tool to improve military logistics, optimize production, and allocate resources.
- The U.S. Air Force played a crucial role in pushing for mathematical solutions to complex logistical problems.

Early Applications of Linear Programming

- During WWII, LP was used to solve transportation and supply chain problems.
- Military planning: determining optimal ways to allocate resources and personnel.
- Post-War: LP found applications in various industries, including agriculture, energy, and manufacturing.

George Dantzig and the Simplex Method

- George Dantzig, a prominent American mathematician, is considered the "father" of Linear Programming.
- In 1947, Dantzig developed the Simplex Method, a groundbreaking algorithm to solve LP problems.
- The Simplex Method efficiently navigates the feasible region of solutions to find the optimal point.

Fun Fact (?)

Dantzig is said to have solved two unsolved problems in statistics by mistake, thinking they were homework!

The Role of Computers in LP

- Early LP problems were solved manually, which was time-consuming and error-prone.
- The advent of computers in the 1950s revolutionized LP, allowing large-scale problems to be solved quickly.
- LP became a cornerstone of operations research and optimization algorithms.

Real-World Applications of MILP, ILP, and LP

Applications in Operations Research, Business, and Economics:

- **Supply Chain Optimization:**
  - Optimize inventory levels, production schedules, and distribution routes.
- **Financial Portfolio Optimization:**
  - Maximize returns and minimize risk by selecting the optimal mix of investments.
- **Manufacturing Scheduling:**
  - Schedule production processes to optimize resource use and meet deadlines.
- **Transportation Planning:**
  - Design optimal routes for logistics and transportation, minimizing costs and time.
- **Workforce Scheduling:**
  - Create efficient work schedules for employees, adhering to regulations and minimizing costs.

Boolean Business Rules

- Business decisions often involve logical rules (e.g., if conditions are met, then actions are taken).
- These rules can often be represented using boolean (true/false) variables.
- Mixed Integer Linear Programming (MILP) allows us to model such rules using binary variables.
- Common examples:
  - Constraints involving "if-then" logic.
  - If chemical A is used, then chemical B must not be used.
  - Combinations of conditions using logical operators like AND, OR, and IMPLIES.

Big-M Method: Real to Boolean Translation

Problem: We want to define a binary variable y such that:

$$y = \begin{cases} 0 & \text{if } x = 0 \\ 1 & \text{if } x > 0 \end{cases}$$

Solution: Use the **Big-M method** with the following constraints:

$$\epsilon y \leq x \leq My$$

where M is a large constant, and ϵ is a small positive constant.

Modeling Logical Operators with MILP

Logical operators like AND, OR, IMPLIES, and NOT can be formulated using binary variables and constraints in MILP.

- NOT: $\neg x$
$$1 - x$$
- AND: $x_1 \wedge x_2$
$$x_1 + x_2 \geq 2$$
- OR: $x_1 \vee x_2$
$$x_1 + x_2 \geq 1$$
- IMPLIES: $x_1 \implies x_2$
$$x_1 \leq x_2$$

Demo: Sudoku

Sudoku is a popular logic-based puzzle where the objective is to fill a 9x9 grid with digits from 1 to 9. The grid is divided into 9 smaller 3x3 subgrids, and each row, column, and subgrid must contain every digit exactly once.

Initial Setup:

5	3			7				
6			1	9	5			
	9	8					6	
8				6				3
4			8		3			1
7				2				6
	6					2	8	
			4	1	9			5
				8			7	9

Tools and Solvers

Open Source Solvers

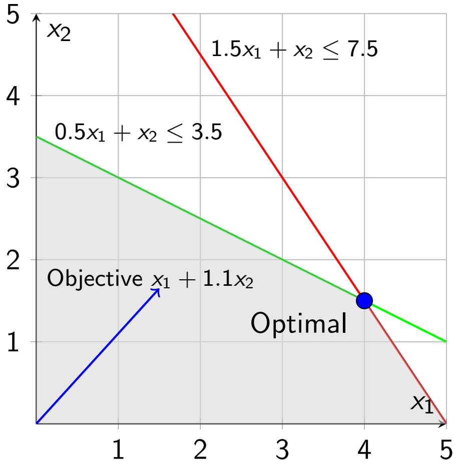
- **GLPK:** Free, basic solver.
- **CBC:** Free, suitable for large problems.
- **Google OR-Tools:** Free, comprehensive optimization tools.

Commercial Solvers

- **CPLEX:** Expensive, industry-standard solver.
- **Gurobi:** Expensive, known for high performance.
- **MOSEK:** Cost varies, solves a wide range of problem types.

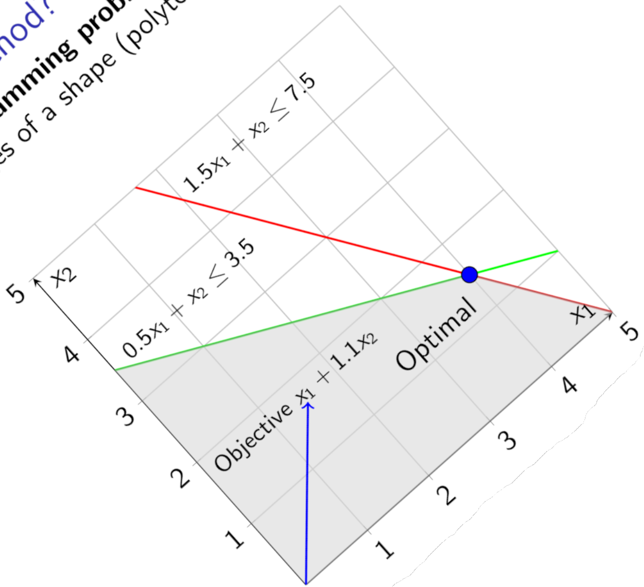
What is the Simplex Method?

- Solves **linear programming problems**.
- Moves along edges of a shape (polytope) to find the **best solution**.



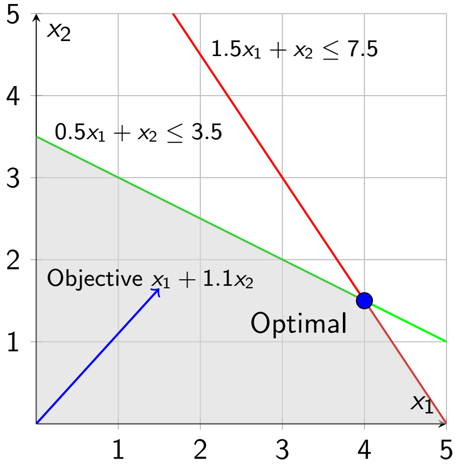
Simplex Method?

- ...es **linear programming problems**.
- ...oves along edges of a shape (polytope) to find t...
- **solution**.



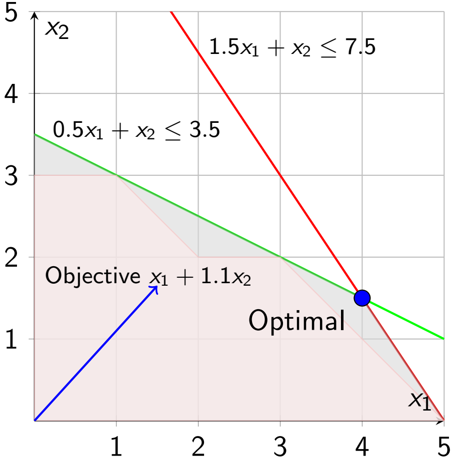
Why it Works

- The **best solution** is always at a vertex.
- Simplex moves only to better solutions until the optimal one is found.



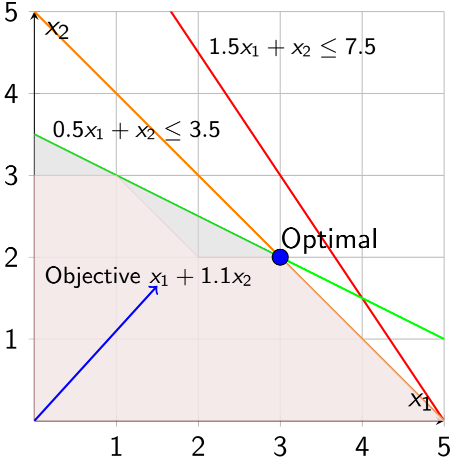
Cutting Planes Method

- Start with Relaxed Problem: Solve without integer constraints.
- Identify Fractional Solutions: Solutions where integer constraints aren't met.

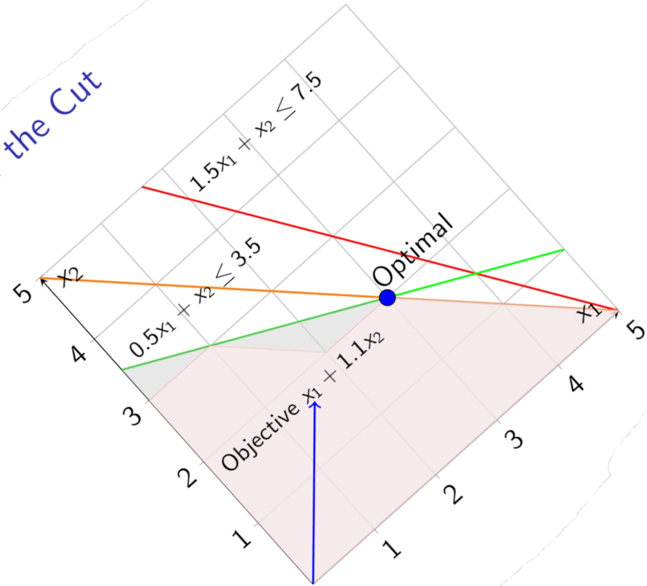


Cutting Planes Method

- Generate Cuts: Add constraints to exclude these fractional solutions.
- Refine Solution: Solve the modified problem.
- Iterate: Repeat until finding an integer solution or no further cuts help.

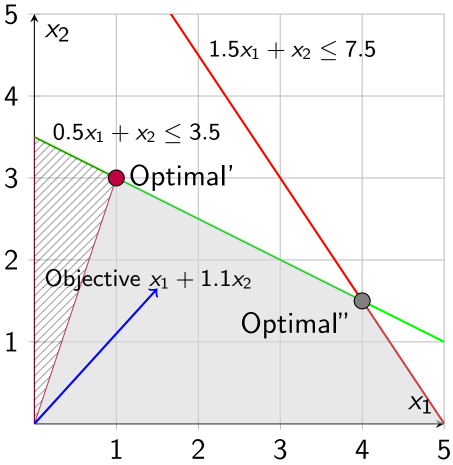


lem after the Cut



Branch and Bound Method

- Branch: Split the problem into smaller subproblems.
- Bound: Update the bounds for each subproblem.
- Prune: Eliminate subproblems that won't improve the best solution.



Branch and Cut Method

- Combine Branch and Bound with Cutting Planes:
  - Start with a relaxed MILP problem.
  - Apply Cutting Planes to refine each subproblem.
  - Use Branch and Bound for splitting.
- Process:
  - Branch: Divide into subproblems.
  - Cut: Add constraints to tighten the solution space.
  - Bound: Update and prune based on bounds.

Demo: Steel blending problem

- Objective: Blend different types of steel to produce 25 tons of steel with a specific chemical composition.
- The resulting steel should contain:
  - 5% carbon
  - 5% molybdenum
- The goal is to minimize the total cost of blending.

Reference

Reference: Carl-Henrik Westerberg, Bengt Bjorklund, and Eskil Hultman, "An Application of Mixed Integer Programming in a Swedish Steel Mill."
Source: *Interfaces*, February 1977 Vol. 7, No. 2 pp. 39–43.
DOI: https://doi.org/10.1287/inte.7.2.39

Available Steel Ingots for Purchase

- Four steel ingots are available for purchase.
- Only one of each ingot can be used.

Ingot	**1**	**2**	**3**	**4**
Weight (Tons)	5	3	4	6
% Carbon	5	4	5	3
% Molybdenum	3	3	4	4
Cost (per Ton)	\$350	\$330	\$310	\$280

Table: Available Steel Ingots

Alloy and Scrap Steels for Purchase

- Three grades of alloy steel and one grade of scrap steel are available for purchase.
- These steels can be purchased in fractional amounts.

Type	**Alloy 1**	**Alloy 2**	**Alloy 3**	**Scrap**
% Carbon	8	7	6	3
% Molybdenum	6	7	8	9
Cost (per Ton)	$500	$450	$400	$100

Table: Alloy and Scrap Steels

Available Steel Ingots, Alloys, and Scrap

Type	Weight [Tons]	Carbon [%]	Molybdenum [%]	Cost [$\frac{1}{\text{Ton}}$]
Ingot 1	5	5	3	$350
Ingot 2	3	4	3	$330
Ingot 3	4	5	4	$310
Ingot 4	6	3	4	$280
Alloy 1		8	6	$500
Alloy 2		7	7	$450
Alloy 3		6	8	$400
Scrap		3	9	$100

Table: Available Steel Ingots, Alloys, and Scrap Steels

Q&A and Wrap-Up

- Open Floor for Questions
- Your Takeaways
- Summary