Addendum: A guide for Methies
2024-25

MATHEMATICS CLUB

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v1.0

§1 Complex Integrals!

Complex integrals are studied quite often in the world of complex analysis. There exist a certain class of functions which behave very nicely called 'entire' functions. An interesting property of these functions is that they satisfy the relation:

$$n(\gamma; a)f(a) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-a}dz$$

Here $\int_\gamma$ denotes a line integral along the closed curve γ. If you haven't encountered line integrals in your first semester yet here is a quick guide to line integrals. Line integrals find an enormous amount of use in engineering. You may have encountered them in Ampere's law in High school.

$n(\gamma; a)$ denotes the number of times the closed curve γ loops around the point a in the counterclockwise direction. If the point a lies outside the curve γ then $n(\gamma; a) = 0$. This quantity is what is known as the winding number of a point with respect to a curve.

If I tell you that the function $f : \mathbb{C} \to \mathbb{C}, f(z) = \sin z$ is an 'entire' function, can you try to evaluate the integral now?

§2 Feynman for ya!

The Feynman method is a powerful technique often helpful in evaluating tricky integrals. The technique involves the introduction of a new variable and then differentiating under the integral using the Leibniz rule to simplify the problem.

Example:
Let's say we wish to evaluate the integral:

$$\int_0^\infty \frac{\sin(ax)}{x}dx \quad a > 0.$$

The main idea behind Feynman's technique is to introduce something that makes evaluation of the integral after differentiation simpler one step in the future. We introduce an auxiliary term which on differentiation would make the x in the denominator vanish and make the integral easier to evaluate.

We define:

$$I(t) = \int_0^\infty \frac{e^{-tx}\sin(ax)}{x}dx$$

$$\implies I'(t) = \int_0^\infty -e^{-tx}\sin(ax)dx = -\left[\frac{e^{-tx}}{a^2+t^2}\left(t\sin(ax) - a\cos(ax)\right)\right]_0^\infty$$

$$\implies I'(t) = -\frac{a}{a^2+t^2}$$

$$\implies I(t) = -\tan^{-1}\left(\frac{x}{a}\right) + c$$

Note that $\lim\limits_{t\to\infty} I(t) = 0$, so $c = \frac{\pi}{2}$ $\quad (\because \ a > 0)$

Thus, $\boxed{\int_0^\infty \frac{\sin(ax)}{x}dx = I(0) = \frac{\pi}{2}}$

§3 Death by Intuition

You learnt about vectors in physics. Now let's play with them in math. Let us define an operator $\Lambda : \mathbb{V} \times \mathbb{V} \rightarrow \mathbb{V}$ ($\mathbb{V}$ is what is called a vector space and can be thought of as a domain comprising of vectors) which operates on 2 vectors and satisfies the following properties:

(a) Λ is distributive i.e. $\mathbf{u}\Lambda(\mathbf{x}+\mathbf{y}) = \mathbf{u}\Lambda\mathbf{x} + \mathbf{u}\Lambda\mathbf{y} \quad \text{for } \mathbf{x}, \mathbf{y}, \mathbf{u} \in \mathbb{V}$

(b) Λ is associative i.e. $\mathbf{x}\Lambda(\mathbf{y}\Lambda\mathbf{z}) = (\mathbf{x}\Lambda\mathbf{y})\Lambda\mathbf{z} \quad \text{for } \mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{V}$.

(c) $|\mathbf{e_1}\Lambda\mathbf{e_2}\Lambda.....\Lambda\mathbf{e_n}| = 1$ ($\mathbf{e_i}$ are orthogonal unit vectors in $\mathbb{V}$ i.e. $\mathbf{e_i} \cdot \mathbf{e_j} = 0 \quad \forall\ i \neq j \leq$. You would have encountered in Cartesian 3D, $(\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3})$=$(\mathbf{i}, \mathbf{j}, \mathbf{k})$)

(d) Λ is anti-symmetric i.e.
$(\mathbf{v_1}\Lambda\mathbf{v_2}\Lambda.....\mathbf{v_{k-1}}\Lambda\mathbf{v_k}\Lambda\mathbf{v_{k+1}}\Lambda\mathbf{v_{k+2}}.....\Lambda\mathbf{v_n}) = -(\mathbf{v_1}\Lambda\mathbf{v_2}\Lambda.....\mathbf{v_{k-1}}\Lambda\mathbf{v_{k+1}}\Lambda\mathbf{v_k}\Lambda\mathbf{v_{k+2}}.....\Lambda\mathbf{v_n}) \quad \forall\ \mathbf{v_k} \in \mathbb{V}$

§4 Proof Theory!

Mathematics thrives on proofs. Much like in life, where you don't quite believe your friend until they show you solid evidence to back up their wild claims. Let's explore some of the elegant (and occasionally quirky) techniques that mathematicians love to use:

- ~~**Proof by Intuition**~~ - All of us have been there, when something just "feels" correct but we aren't able to prove it rigorously:) More formally this is known as a **Direct Proof**.
- **Proof by Induction** - You've most likely heard of this. The basic idea works by proving that if something holds for a proposition (technical term for a statement) P(n), prove ~~somehow~~ that it holds for P(n+1) as well.
- **Reductio ad absurdum** - A fancy way of saying proof by contradiction! Just tell your friend that he's wrong and then let him prove how he cannot be wrong.
- **Proof by Exhaustion** - Yes you got that right, a tiring method asking you to prove that something is true by showing that it is true for each and every case that could possibly be considered.
- **Proof by Contraposition** - This is true because our dear DCs could prove it's true. Help prove us right!
- **Providing a Counterexample** - The most sophisticated way of saying, 'you're wrong!'
- **Proof by Intimidation** - It is true because I said so. You get it, don't you? I hope you do. I wouldn't like to be in the shoes of someone who doesn't, if I were you.

§5 Torn Realities

This section has a lot for you to read. So bear with us! An interesting ability of math is to describe the deformation of shapes using the framework of algebra. A nice class of deformations are what are called continuous deformations where we can shrink or twist the shape but without tearing.

Let us study the deformations of a disk onto the space occupied by itself. A nice way to visualize this would be to imagine a piece of elastic cloth cut into a circular shape which you deform along its area without affecting its boundaries.

Compactly one would write such a deformation as a map $f : D \rightarrow D$ and say that f is continuous. Here D refers to a closed disk, that is a disk along with its boundary ∂D. Some intuitive observations that you can make about such continuous maps are the following:

(a) If $f : D \to D$ is a continuous map, then $f(x) = x$ for atleast one point $x \in D$. Think of this as there being at least one immovable point in any continuous map from a disk onto itself. For example the rotation of the disk about its centre has an immovable point - the centre itself.

(b) There cannot exist an onto continuous map $g : D \to \partial D$. That is there does not exist any continuous map from the disk to the entirety of its boundary. You cannot stretch the interior of the elastic cloth to the exterior without creating holes in it and ripping it apart!

(c) If $h : D \to D$ is a continuous map, then all points in a neighbourhood of x get mapped to a neighbourhood of $h(x)$. That is all points close to each other in the input space get mapped close to each other in the output space. Otherwise remember the cloth analogy, you'll be creating many tears in it.

You have to show that just as a consequence of (b) being true, (a) must also now be true. Try to think along the lines of working in the world of contra-positives. Imagine you have been transported into a world by an evil villain where statement (a) is no longer true. Then we want you to show that if continuous maps of the disk onto itself do not have any fixed points, then you can construct a continuous map from the disk to its entire boundary. You may ignore the fact that the monstrous villain is still alive and you may or may not reach back home.