since their distributions are centered at θ. The estimator $\hat{\Theta}_1$ has a smaller variance than $\hat{\Theta}_2$ and is therefore more efficient. Hence our choice for an estimator of θ, among the three considered, would be $\hat{\Theta}_1$.

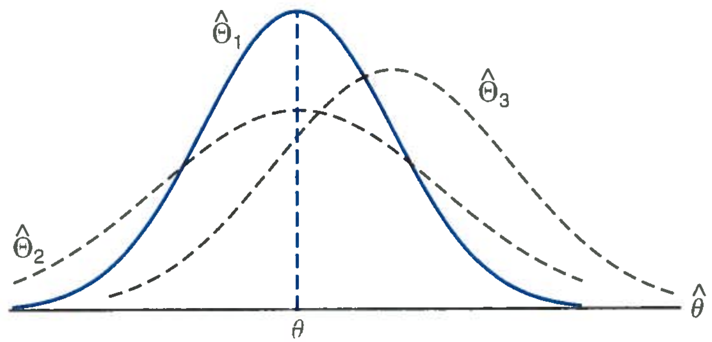

Figure 9.1: Sampling distributions of different estimators of θ.

For normal populations one can show that both $\bar{X}$ and $\tilde{X}$ are unbiased estimators of the population mean μ, but the variance of $\bar{X}$ is smaller than the variance of $\tilde{X}$. Thus both estimates $\bar{x}$ and $\tilde{x}$ will, on the average, equal the population mean μ, but $\bar{x}$ is likely to be closer to μ for a given sample, and thus $\bar{X}$ is more efficient than $\tilde{X}$.

The Notion of an Interval Estimate

Even the most efficient unbiased estimator is unlikely to estimate the population parameter exactly. It is true that our accuracy increases with large samples, but there is still no reason why we should expect a **point** estimate from a given sample to be exactly equal to the population parameter it is supposed to estimate. There are many situations in which it is preferable to determine an interval within which we would expect to find the value of the parameter. Such an interval is called an **interval estimate**.

Interval Estimation

An interval estimate of a population parameter θ is an interval of the form $\hat{\theta}_L < \theta < \hat{\theta}_U$, where $\hat{\theta}_L$ and $\hat{\theta}_U$ depend on the value of the statistic $\hat{\Theta}$ for a particular sample and also on the sampling distribution of $\hat{\Theta}$. Thus a random sample of SAT verbal scores for students of the entering freshman class might produce an interval from 530 to 550 within which we expect to find the true average of all SAT verbal scores for the freshman class. The values of the endpoints, 530 and 550, will depend on the computed sample mean $\bar{x}$ and the sampling distribution of $\bar{X}$. As the sample size increases, we know that $\sigma_{\bar{X}}^2 = \sigma^2/n$ decreases, and consequently our estimate is likely to be closer to the parameter μ, resulting in a shorter interval. Thus the interval estimate indicates, by its length, the accuracy of the point estimate. An engineer will gain some insight into the population proportion defective by taking a sample and computing the *sample proportion defectives*. But an interval estimate might be more informative.

Interpretation of Interval Estimates

Since different samples will generally yield different values of $\hat{\Theta}$ and, therefore, different values $\hat{\theta}_L$ and $\hat{\theta}_U$, these endpoints of the interval are values of corresponding random variables $\hat{\Theta}_L$ and $\hat{\Theta}_U$. From the sampling distribution of $\hat{\Theta}$ we shall be able to determine $\hat{\Theta}_L$ and $\hat{\Theta}_U$ such that the $P(\hat{\Theta}_L < \theta < \hat{\Theta}_U)$ is equal to any positive fractional value we care to specify. If, for instance, we find $\hat{\Theta}_L$ and $\hat{\Theta}_U$ such that

$$P(\hat{\Theta}_L < \theta < \hat{\Theta}_U) = 1 - \alpha$$

for $0 < \alpha < 1$, then we have a probability of $1-\alpha$ of selecting a random sample that will produce an interval containing θ. The interval $\hat{\theta}_L < \theta < \hat{\theta}_U$, computed from the selected sample, is then called a $100(1-\alpha)\%$ **confidence interval**, the fraction $1-\alpha$ is called the **confidence coefficient** or the **degree of confidence**, and the endpoints, $\hat{\theta}_L$ and $\hat{\theta}_U$, are called the lower and upper **confidence limits**. Thus, when $\alpha = 0.05$, we have a 95% confidence interval, and when $\alpha = 0.01$ we obtain a wider 99% confidence interval. The wider the confidence interval is, the more confident we can be that the given interval contains the unknown parameter. Of course, it is better to be 95% confident that the average life of a certain television transistor is between 6 and 7 years than to be 99% confident that it is between 3 and 10 years. Ideally, we prefer a short interval with a high degree of confidence. Sometimes, restrictions on the size of our sample prevent us from achieving short intervals without sacrificing some of our degree of confidence.

In the sections that follow we pursue the notions of point and interval estimation, with each section representing a different special case. The reader should notice that while point and interval estimation represent different approaches to gain information regarding a parameter, they are related in the sense that confidence interval estimators are based on point estimators. In the following section, for example, we should see that the estimator $\bar{X}$ is a very reasonable point estimator of μ. As a result, the important confidence interval estimator of μ depends on knowledge of the sampling distribution of $\bar{X}$.

We begin in the following section with the simplest case of a confidence interval. The scenario is simple and yet unrealistic. We are interested in estimating a population mean μ and yet σ is known. Clearly if μ is unknown, it is quite unlikely that σ is known. Any historical information that produced enough information to allow the assumption that σ is known would likely have produced similar information about μ. Despite this argument, we begin with this case because the concepts and indeed the resulting mechanics associated with confidence interval estimation remain the same when more realistic situations surface later in Section 9.4 and beyond.

9.4 Single Sample: Estimating the Mean

The sampling distribution of $\bar{X}$ is centered at μ and in most applications the variance is smaller than that of any other estimators of μ. Thus the sample mean $\bar{x}$ will be used as a point estimate for the population mean μ. Recall that $\sigma_{\bar{X}}^2 = \sigma^2/n$, so that a large sample will yield a value of $\bar{X}$ that comes from a sampling distribution with a small variance. Hence $\bar{x}$ is likely to be a very accurate estimate of μ when n is large.

Let us now consider the interval estimate of μ. If our sample is selected from a normal population or, failing this, if n is sufficiently large, we can establish a confidence interval for μ by considering the sampling distribution of $\bar{X}$.

According to the central limit theorem, we can expect the sampling distribution of $\bar{X}$ to be approximately normally distributed with mean $\mu_{\bar{X}} = \mu$ and standard deviation $\sigma_{\bar{X}} = \sigma/\sqrt{n}$. Writing $z_{\alpha/2}$ for the z-value above which we find an area of $\alpha/2$, we can see from Figure 9.2 that

$$P(-z_{\alpha/2} < Z < z_{\alpha/2}) = 1 - \alpha,$$

where

$$Z = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}}.$$

Hence

$$P\left(-z_{\alpha/2} < \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} < z_{\alpha/2}\right) = 1 - \alpha.$$

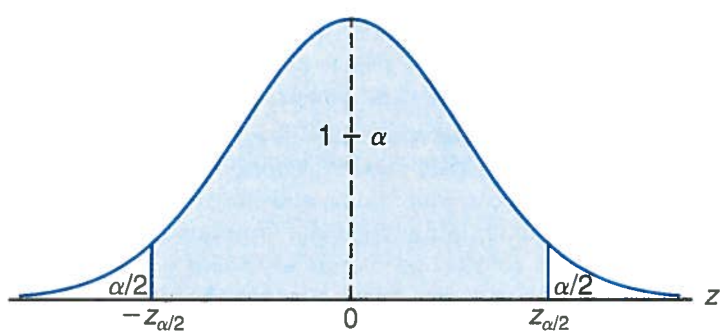

Figure 9.2: $P(-z_{\alpha/2} < Z < z_{\alpha/2}) = 1 - \alpha$.

Multiplying each term in the inequality by $\sigma/\sqrt{n}$, and then subtracting $\bar{X}$ from each term and multiplying by -1 (reversing the sense of the inequalities), we obtain

$$P\left(\bar{X} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}} < \mu < \bar{X} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right) = 1 - \alpha.$$

A random sample of size n is selected from a population whose variance σ^2 is known and the mean $\bar{x}$ is computed to give the $100(1-\alpha)\%$ confidence interval below. It is important to emphasize that we have invoked the central limit theorem above. As a result it is important to note the conditions for applications that follow.

Confidence Interval of μ; σ Known

If $\bar{x}$ is the mean of a random sample of size n from a population with known variance σ^2, a $100(1-\alpha)\%$ confidence interval for μ is given by

$$\bar{x} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}} < \mu < \bar{x} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}},$$

where $z_{\alpha/2}$ is the z-value leaving an area of $\alpha/2$ to the right.

For small samples selected from nonnormal populations, we cannot expect our degree of confidence to be accurate. However, for samples of size $n \geq 30$, with the shape of distributions not too skewed, sampling theory guarantees good results.

Clearly, the values of the random variables $\hat{\Theta}_L$ and $\hat{\Theta}_U$, defined in Section 9.3, are the confidence limits

$$\hat{\theta}_L = \bar{x} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}} \quad \text{and} \quad \hat{\theta}_U = \bar{x} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}.$$

Different samples will yield different values of $\bar{x}$ and therefore produce different interval estimates of the parameter μ as shown in Figure 9.3. The circular dots at the center of each interval indicate the position of the point estimate $\bar{x}$ for each random sample. Most of the intervals are seen to contain μ, but not in every case. Note that all of these intervals are of the same width, since their widths depend only on the choice of $z_{\alpha/2}$ once $\bar{x}$ is determined. The larger the value we choose for $z_{\alpha/2}$, the wider we make all the intervals and the more confident we can be that the particular sample selected will produce an interval that contains the unknown parameter μ.

Example 9.2: The average zinc concentration recovered from a sample of zinc measurements in 36 different locations is found to be 2.6 grams per milliliter. Find the 95% and 99% confidence intervals for the mean zinc concentration in the river. Assume that the population standard deviation is 0.3.

Solution: The point estimate of μ is $\bar{x} = 2.6$. The z-value, leaving an area of 0.025 to the right and therefore an area of 0.975 to the left, is $z_{0.025} = 1.96$ (Table A.3). Hence the 95% confidence interval is

$$2.6 - (1.96)\left(\frac{0.3}{\sqrt{36}}\right) < \mu < 2.6 + (1.96)\left(\frac{0.3}{\sqrt{36}}\right),$$

which reduces to $2.50 < \mu < 2.70$. To find a 99% confidence interval, we find the z-value leaving an area of 0.005 to the right and 0.995 to the left. Therefore, using Table A.3 again, $z_{0.005} = 2.575$, and the 99% confidence interval is

$$2.6 - (2.575)\frac{0.3}{\sqrt{36}} < \mu < 2.6 + (2.575)\frac{0.3}{\sqrt{36}},$$

or simply

$$2.47 < \mu < 2.73.$$

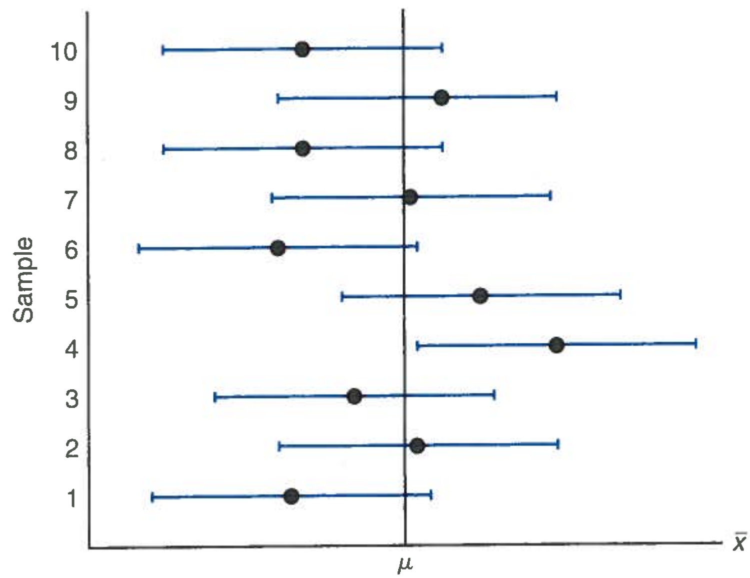

Figure 9.3: Interval estimates of μ for different samples.

We now see that a longer interval is required to estimate μ with a higher degree of confidence. ■

The $100(1-\alpha)\%$ confidence interval provides an estimate of the accuracy of our point estimate. If μ is actually the center value of the interval, then $\bar{x}$ estimates μ without error. Most of the time, however, $\bar{x}$ will not be exactly equal to μ and the point estimate is in error. The size of this error will be the absolute value of the difference between μ and $\bar{x}$, and we can be $100(1-\alpha)\%$ confident that this difference will not exceed $z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$. We can readily see this if we draw a diagram of a hypothetical confidence interval as in Figure 9.4.

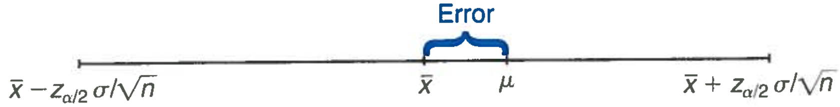

Figure 9.4: Error in estimating μ by $\bar{x}$.

Theorem 9.1: If $\bar{x}$ is used as an estimate of μ, we can then be $100(1-\alpha)\%$ confident that the error will not exceed $z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$.

In Example 9.2 we are 95% confident that the sample mean $\bar{x} = 2.6$ differs from the true mean μ by an amount less than 0.1 and 99% confident that the difference is less than 0.13.