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Sub- and super-diffusion on Cantor sets: Beyond the paradox

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ABSTRACT

There is no way to build a nontrivial Markov process having continuous trajectories on a totally disconnected fractal embedded in the Euclidean space. Accordingly, in order to delineate the diffusion process on the totally disconnected fractal, one needs to relax the continuum requirement. Consequently, a diffusion process depends on how the continuum requirement is handled. This explains the emergence of different types of anomalous diffusion on the same totally disconnected set. In this regard, we argue that the number of effective spatial degrees of freedom of a random walker on the totally disconnected Cantor set is equal to $n_{sp} = [D] + 1$, where $[D]$ is the integer part of the Hausdorff dimension of the Cantor set. Conversely, the number of effective dynamical degrees of freedom (d_s) depends on the definition of a Markov process on the totally disconnected Cantor set embedded in the Euclidean space E^n ($n \geq n_{sp}$). This allows us to deduce the equation of diffusion by employing the local differential operators on the F^{α}-support. The exact solutions of this equation are obtained on the middle-ϵ Cantor sets for different kinds of the Markovian random processes. The relation of our findings to physical phenomena observed in complex systems is highlighted.

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1. Introduction

Diffusion phenomena play an important role in physics, chemistry, technological processes, and biology [1–4]. Classical diffusion is associated with a Brownian motion. The laws of Brownian motion crucially rely on the hypothesis that the steps of a random walker are small (with finite variance) and uncorrelated. Whenever these assumptions are violated, the standard diffusion picture breaks down, leading to emergence of an anomalous diffusion [5]. In contrast to the Gaussian diffusion, which is characterized by the linear dependence of the mean square displacement on the time, the anomalous diffusion is non-universal in that it involves a parameter called the fractal dimension of a random walk [2]. Accordingly, for long enough times the mean square displacement of an ensemble of walkers behaves asymptotically as

$$< r^2 >= K_{\gamma} t^{\gamma}, \tag{1}$$

where K_{γ} is the generalized diffusion coefficient and $\gamma = 2/D_w$ is the diffusion exponent. In the case of normal diffusion $\gamma = 1$ whereas the sub- and super-diffusion are characterized by $\gamma < 1$ and $\gamma > 1$ respectively [1–5]. The generalized Einstein law provides the relation between the fractal dimension of random walk D_w, electric resistance exponent ζ and spectral dimension of medium d_s, which reads as $D_w = 2\zeta/(2 - d_s)$ [6]. Furthermore, many types of fractals (but not all) obey the Alexander–Orbach relation

$$D_w = \frac{2D}{d_s}, \tag{2}$$

where D is the fractal (e.g. Hausdorff, box-counting, or self-similarity) dimension, such that $\zeta = D_w - D$ [5,6]. For path-connected fractals $d_s \leq D$ and so $\gamma \leq 1$ while $D_w \geq 2$ [7]. Accordingly, the random walk on path-connected fractals has become a paradigmatic model for the sub-diffusion observed in a great variety of physical systems (see Refs. [1–9] and references therein). On the other hand, the super-diffusion is commonly associated with Lévy flights [9,10]. At the same time, characteristic features of many physical systems can be modeled using totally disconnected fractals, e.g. Cantor sets embedded in E^n [11–14]. The ternary Cantor set of real numbers was introduced by Georg Cantor in order to illustrate the statement that a perfect set does not need to be everywhere dense. Geometrically, it is constructed by iterative deletion of open middle-third intervals from remaining intervals of the previous iteration, starting from the unit interval $[0, 1]$ ad infinitum. The ternary Cantor set is a self-similar, totally discon-

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nected, bounded, compact, perfect set which is nowhere dense on $[0, 1]$. Its Hausdorff dimension is equal to the similarity dimension $D = \ln 2/\ln 3$. The middle-ϵ Cantor sets with the Hausdorff dimension

$$D = \frac{\ln 2}{\ln 2 - \ln(1-\epsilon)} \tag{3}$$

can be constructed by the iterative deletion of open intervals of length equal to the ϵ fraction of the length of intervals remaining after the previous iteration [15]. The Cantor set in E^n (also called the Cantor dust) with the Hausdorff dimension $0 < D < n$ can be constructed either by similar iterative method, starting from n-dimensional unit cube $[0, 1]^n$, or as the Cartesian product of n orthogonal Cantor sets of the same or different Hausdorff dimensions. Every Cantor set (dust) is homeomorphic to the ternary Cantor set [16]. Furthermore, it has been proved (see Ref. [17]) that any compact metric space is a continuous image of the ternary Cantor set. Due to these remarkable properties the totally disconnected Cantor sets have found a celebrated place in mathematical analysis and its applications. In particular, stochastic processes on Cantor sets are an active topic of research [18–31]. In this regard, the ternary Cantor set is interesting because the diffusion on it exhibits some apparent paradoxes. Specifically, in some works, (see Refs. [18–24]) it was suggested that the Markov processes defined on the totally disconnected Cantor set lead to the super-diffusion allied with the set's self-similarity. Conversely, it was argued (see [28–31]) that the ultrametric properties of the totally disconnected Cantor sets stipulate the sub-diffusive nature of the Markov processes. It has been also recognized that local and average measurements can display different asymptotic behavior [12,33]. Accordingly, alternative models of the anomalous diffusion on the ternary Cantor set were suggested and disputed in the literature (see, for example, Refs. [32–36] and references therein). In this work, we argue that diffusion on the totally disconnected Cantor set can be either the super-, or the sub-diffusion, depending on how the random walk is constructed. In this regard, we stress that the number of effective dynamical degree of freedom of a random walker on the Cantor set depends on the Markov process definition, while the number of effective spatial degree of freedom is equal to the minimum dimension of the Euclidean space in which the Cantor set can be bi-Lipschitz embedded. This explains the emergence of different types of the anomalous diffusion on the ternary Cantor set. The diffusion equation describing different types of diffusion is deduced and solved for the middle-ϵ Cantor sets. The physical implications of these findings are discussed. The rest of the paper is organized as follows. Sec. 2 is devoted to the analysis of different Markov processes on the totally disconnected Cantor set. The processes associated with the sub- and super diffusion on the ternary Cantor set are outlined. The predicted values for the diffusion exponents are compared with available data from numerical simulations of Lévy flights and walks on the Cantor sets. In Sec. 3 the equation of diffusion is deduced using the algorithmic F^α-calculus based on the Riemannian-like method. The relations between the orders of F^α-derivatives and the numbers of the effective degrees of freedom are established. The probability density distributions functions are found for different types of diffusion on the middle-ϵ Cantor sets. Some physical phenomena associated with different types of anomalous diffusion on the Cantor sets are highlighted. The main findings and conclusions are outlined in Sec. 4.

2. Markov processes on Cantor sets

It is a straightforward matter to understand that there is no way to build a nontrivial Markov process having continuous trajectories on a totally disconnected Cantor set embedded in the Euclidean space E^n. Therefore, in order to delineate the diffusion process, one needs to relax the continuum requirement. In this way, different properties of the ternary Cantor set were explored to define the Markov random processes on it. Specifically, the Brownian motion on the totally disconnected fractal set was introduced by Fujita [18]. This model was used to determine the spectral dimension of the Cantor set from the growth order of eigenvalues of the Brownian motion generator. Further, the transition probability densities for generalized one-dimensional diffusion processes were studied in [19]. Later, Takahashi and Tamura [20] have defined diffusion processes on totally disconnected self-similar fractal sets as the limits of suitably scaled random walk. On the other hand, Evans [21] has introduced Markov processes with stationary independent increments taking values in a non-discrete, locally compact, metrizable, totally disconnected Abelian Cantor group. Further, Aldous and Evans [22] have used the Dirichlet form methods to construct and analyze a general class of reversible Markov processes with totally disconnected state space. At the same time, Freiberg [23] has introduced the measure theoretic Dirichlet forms on compact subsets of the real line. Using the technique of Dirichlet–Neumann-bracketing, he has obtained the estimates of the eigenvalue counting functions of the associated measure geometric Laplacians. Using the Dirichlet form technique, Karwowski [24] has developed a model of diffusion on the real line with jumps on the Cantor set which preserve the ultrametric feature of random process on unit ball of 2-adic numbers. In this regard, it is pertinent to note that the Markov processes studied in Refs. [18–24] lead to the super-diffusion on the ternary Cantor set.

Alternatively, Lobus [25] has constructed a strong Markov processes on Cantor sets from the Wiener processes by means of time change, killing, and space transformation. Bhamidi et al. [26] have proposed an analogue of Brownian motion that has as its state space an arbitrary closed subset of the line that is unbounded above and below. It was shown that there is a unique such process, which turns out to be automatically a reversible Feller–Dynkin Markov process. This process is martingale and has the identity function as its quadratic variation process. So, it is continuous in the sense that its sample paths don't skip over points. It was also recognized that the totally disconnected Cantor sets exhibit a natural ultrametric structure [27,28]. An analogue of the Riemannian geometry for an ultrametric Cantor set was suggested in Ref. [29]. Consequently, Pearson and Bellissard [29] have defined a probability measure, Dirichlet forms, and associated analogue of the Laplace–Beltrami operator. In this way, it was deduced that the diffusion on the ultrametric Cantor set exhibits nonstandard sub-diffusive behavior, as $t \to 0$. Later, Bakhtin [30] has defined an analog of Brownian motion on the triadic Cantor set by introducing requirements on the Markov semigroup, most important of which are isometry invariance, and scale invariance. This allows to build up the jump statistics, while the generators of the symmetric self-similar Markov process play the role of the Laplacian on the Cantor set. More recently, Kigami [31] has shown that a transient random walk on a tree induces a Dirichlet form on its Martin boundary, which is the Cantor set. This was used to define an intrinsic metric on the Cantor set associated with the random walk. Accordingly, the harmonic measure and induced Dirichlet form on the Cantor set were explicitly expressed in terms of the effective resistances. Consequently, the asymptotic behaviors of the heat kernel, the jump kernel, and moments of displacements were obtained. In this regard, it is pertinent to point out that, in contrast to Markov processes studied in Refs. [18–24], the Markov processes defined in Refs. [25–31] lead to the sub-diffusion on the ternary Cantor set.

In this background, we stress that, generally, the type of diffusion is determined by the numbers of effective dynamical and spatial degrees of freedom of a random walker (see Refs. [37,38]).

The number of effective dynamical degrees of freedom of the random walker is equal to the spectral dimension of the fractal d_s [39], while the numbers of effective spatial degrees of freedom n_{sp} is equal to the number of directions that the random walker experiences on the fractal [37,38,40]. In Ref. [37] it was deduced that

$$d_s + n_{sp} = 2d_l, \tag{4}$$

where d_l is the connectivity dimension of the fractal, also called as the chemical [41,42] or spreading dimension [43]. The connectivity dimension is related to the Hausdorff dimension D as $d_l = D/d_{min}$, where d_{min} is the fractal (Hausdorff) dimension of the minimum path between two randomly chosen points on the fractal [37–43]. For the random walk on an Euclidean pattern the numbers of spatial and dynamical degrees of freedom are equal. Namely, $d_s = n_{sp} = D = d_l = d$, where d is the topological dimension of the pattern. For the random walk on path-connected fractals in E^n, generally, $d \leq d_s \leq d \leq n_{sp} \leq n$, while n_{sp} can be less than, equal to, or larger than $D \leq n$ [37,40].

Conversely, for the totally disconnected fractals (e.g. the middle-ϵ Cantor sets) the dimension numbers d_l, d_{min}, and d_s are not uniquely defined, rather than they are dependent of how the discontinuity is handled. In this regard, it has been proved (see Ref. [44]) that the self-similar ultrametric Cantor set is bi-Lipschitz embeddable in $E^{[D]+1}$, where $[D]$ denotes the integer part of the Hausdorff dimension of the set.[1] Therefore, the number of effective spatial degrees of freedom of the random walker on the self-similar ultrametric Cantor set in E^n is generally,

$$n_{sp} \leq [D] + 1 \leq n, \tag{5a}$$

while $0 < D < n$ (see Refs. [37,38]). In this work, we will consider only the random walks for which

$$n_{sp} = [D] + 1 = n, \tag{5b}$$

while the equality (4) holds. Accordingly, the random process on the Cantor set can be characterized by the intrinsic time τ (see, for example, Refs. [9,10,37,38]) which scales with the physical time in the embedding Euclidean space t as

$$\tau \propto t^{\beta}, \tag{5c}$$

where

$$\beta = \frac{d_s}{n_{sp}} = \frac{2d_l}{n} - 1 \leq 1 \tag{5d}$$

is the intrinsic time exponent [37]. Bellow we define three different kinds of the Markovian random processes which obey Eq. (5b), but lead to different types of diffusion on the ternary Cantor set.

In this regard, we noted that Freiberg and Zahle [45] have considered the jumps between the endpoints of the cutouts and have constructed the Laplace operator which generates a Hunt process on the support of the linear Cantor measure. It is a straightforward matter to understand that the instantaneous jumps (Lévy flights) on the totally disconnected Cantor set in E^n imply that $d_{min} = D/n$ [34]. Consequently, in this case, the connectivity dimension of the Cantor set is equal to

$$d_l = n_{sp} = n \tag{6a}$$

and so, from Eq. (4) it follows that the spectral dimension is equal to

$$d_s = 2d_l - n = n, \tag{6b}$$

such that, the intrinsic time of the Lévy flights coincide with the physical time in E^n. Notice that Eq. (6a) and (6b) are consistent with the model of super-diffusion suggested in Ref. [34]. Namely, in Ref. [34] it was shown that the Gaussian random walk in the chemical space of the Cantor set leads to the Lévy flights obeying Eq. (6). Consequently, the super-diffusion on the Cantor set embedded in the Euclidean space is characterized by the diffusion exponent

$$\gamma = \frac{1}{d_{min}} > 1 \tag{6c}$$

and the stretched exponential probability density distribution function. In this regard, it is noteworthy that the value of $D_w = 2d_{min} = \ln 4/\ln 3 \approx 1.26$ for the ternary Cantor set is consistent with the value $D_w \approx 1.3$ obtained from numerical simulations in Ref. [32].

Conversely, the symmetric self-similar Markov processes defined in [30] are associated with the Lévy walks on the Cantor set with a constant velocity. It is a straightforward matter to understand that, in this case, $d_{min} = 1$. Accordingly, the connectivity dimension of the Cantor set is equal to its similarity dimension, that is

$$d_l = D, \tag{7a}$$

and so, from Eq. (4) it follows that the spectral dimension is

$$0 < d_s = 2D - n < D. \tag{7b}$$

Consequently, in the case of Lévy walk with the constant velocity, the intrinsic time exponent (5d) is equal to $\beta = 2D/n - 1 < 1$. Notice that the first inequality in Eq. (7b) implies that the symmetric self-similar Lévy walk on the totally disconnected Cantor set can be constructed if and only if $D > n/2 = ([D] + 1)/2$. Hence, the Lévy walks on the Cantor sets lead to the sub-diffusion characterized by the diffusion exponent

$$\gamma = 2 - \frac{n}{D} < 1. \tag{7c}$$

In this regard, it is worthy to note that note that for the sub-diffusion on the Cantor dust with $D = \ln 4/\ln 3$ in E^2 equation (7c) predicts the value $\gamma = 2 - \ln 3/\ln 2 \approx 0.42$, which is not so different from the numerical estimation $\gamma \approx 0.5$ reported in Ref. [36] for the Lévy walks on the pre-fractal Cantor dust.

On the other hand, in the case of random processes defined in Ref. [20] as limits of suitably scaled random walks on the totally disconnected self-similar set, the number of effective dynamical degrees of freedom of the random walker is equal to

$$d_s = \frac{2D}{D+1} < 2, \tag{8a}$$

while the relation (2) holds (see Ref. [20]) and so, from Eqs. (4) and (5b) it follows that

$$D < d_l = \frac{n + (n+2)D}{2(D+1)} < n, \tag{8b}$$

while $d_{min} = D/d_l < 1$. In this case, the diffusion exponent is equal to

$$\gamma = \frac{2}{1+D} \tag{8c}$$

and so, the scaled random walk leads to the super-diffusion on the Cantor set in E^1, whereas in E^n $(D > 1)$ Eq. (8c) predicts the sub-diffusion. Notice that the intrinsic time of the scaled random

[1] Notice that the Hausdorff dimension of the totally disconnected Cantor set in E^n can be either less, or larger than one.

Table 1
Markovian random processes defined on a totally disconnected Cantor set embedded in E^n, the corresponding number of effective dynamical degrees of freedom of the random walker d_s, fractal dimension of the minimum path d_{min}, and anomalous diffusion exponent γ.

Markovian random process on $C \subset E^n$		d_s	d_{min}	γ	Type of diffusion observed in numerical simulations
Lévy flights [45]		n	D/n	Eq. (6c)	Super-diffusion [32]
Lévy walk [30]		Eq. (7b)	1	Eq. (7c)	Sub-diffusion [36]
Scaled random walk [20]	$n = 1$	Eq. (8a)	Eq. (8b)	Eq. (8c)	Super-diffusion
	$n > 1$				Sub-diffusion

walk on the Cantor set scales with the physical time in E^n with the intinsic time exponent $\beta = 2D/[n(D+1)]$.

So, one can define three different Markovian random processes on the same Cantor set. These processes are characterized by different numbers of effective dynamical degrees of freedom of the random walker (see Table 1), while the number of effective spatial degrees of freedom is fixed by the requirement (5b). Accordingly, the Lévy walk and scaled walk on the Cantor set in both lead to the sub-diffusion, whereas the Lévy flights and scaled walks in produce the super-diffusion. So, the paradox regarding the diffusion on the totally disconnected Cantor set is resolved. We recall that different types of diffusion pre-fractal Cantor sets were observed in numerical simulations with different probability rules mimicking the corresponding random process (see Table 1). Notice that a range of power-law behavior of walker displacement on a pre-fractal Cantor set depends on the number of iterations, while the diffusion coefficient depends on the system size [36].

3. Diffusion equation on Cantor sets

Functions defined on fractals are essentially non-analytic and non-differentiable in the conventional sense [16]. Accordingly, in order to deal with problems on fractals within a continuum framework, the non-analytic functions are commonly approximated by analytical envelopes (see Ref. [46]). The analytical envelopes provide overall approximations for continuous but nowhere differentiable functions defined on the path-connected fractals [38,46–55]. However, in contrast to the continuous functions on the path-connected fractals, the functions defined on a totally disconnected fractal are pointwise functions which are almost everywhere discontinuous [16]. So, the use of analytical envelopes of the pointwise functions leads to loss of many essential features of physical phenomena on the totally disconnected fractals. Fortunately, the remarkable properties of Cantor sets allow development of suitable calculus on fractals which are homeomorphic to the ternary Cantor set. In this way, Freiberg [56] has introduced a class of measure geometric Laplacians defined as the second order differential operators with respect two atomless finite Borel measures with compact supports. The corresponding theoretic Dirichlet forms on compact subsets of the real line were defined in Ref. [57].

On the other hand, using the Riemannian-like method, Parvate and Gangal [58,59] have developed the algorithmic F^α-calculus which can be used for analysis on the totally disconnected Cantor sets (as well as on the self-avoiding fractal curves [60]). The corresponding F^α-measure of the Cantor set was defined in Ref. [61]. This allows to formulate and solve the boundary values problems on Cantor sets in E^n [58–62]. In order to deduce the diffusion equation on the Cantor sets, let us first briefly recall the basic definitions of the F^α-calculus. If $C \subset I$ is the totally disconnected Cantor set on closed interval $I = [a, b]$, then we can define the flag function

$$\theta(C, I) = \begin{cases} 1 & \text{if } C \cap I \neq \emptyset \\ 0 & \text{otherwise.} \end{cases} \tag{9a}$$

Let us also define a subdivision of I as a finite set of points $P = \{a = x_0, x_1, \ldots, x_n = b\}$, with components $P_i = [x_i, x_{i+1}]$, while $x_i < x_{i+1}$. Then for a given C and P we can define the function

$$\sigma^\alpha(C, P) = \sum_{i=0}^{n-1} \frac{(x_{i+1} - x_i)^\alpha}{\Gamma(\alpha+1)} \theta(C, P_i), \tag{9b}$$

where $\Gamma(\ldots)$ is the Gamma function and $a < b$, whereas $\sigma^\alpha(C, P) = 0$, if $a = b$. Accordingly, the coarse-grained mass of $C \cap I$ is equal to $\gamma_\delta^\alpha(C, a, b) = \inf_{\{P : |P_i| \le \delta\}} \sigma^\alpha(C, P)$, where $\delta > 0$ and $|P_i| = \max_i P_i$. Consequently, the mass function is defined as

$$\gamma^\alpha(C, a, b) = \lim_{\delta \to 0} \gamma_\delta^\alpha(C, a, b) \ge 0. \tag{9c}$$

Notice that $\gamma^\alpha(C, a, b)$ always exists and $\gamma^\alpha(C, a, b) = \gamma^\alpha(C, a, c) + \gamma^\alpha(C, c, b)$, where $a < c < b$, while $\gamma^\alpha(C, a, b) = 0$, if $C \cap [a, b] = 0$ [58]. Other properties of $\gamma^\alpha(C, a, b)$ are described in Refs. [58–63]. The γ-dimension of set $C \subseteq [a, b]$ is defined as

$$\begin{aligned} D_\gamma &= \inf\{\alpha : \gamma^\alpha(C, a, b) = 0\} \\ &= \sup\{\alpha : \gamma^\alpha(C, a, b) = \infty\}. \end{aligned} \tag{10}$$

It was proved that for the compact sets (e.g. Cantor sets in E^n) the γ-dimension D_γ is equal to the Hausdorff dimension D, that is

$$D_\gamma = D \tag{11}$$

while, generally, $D_\gamma \ge D$ [58]. The integral staircase function of order α for the Cantor set C is defined in Refs. [58–63] as

$$S^\alpha(x) = \begin{cases} \gamma^\alpha(C, a_0, x), & \text{if } x \ge a_0 \\ -\gamma^\alpha(C, x, a_0), & \text{otherwise,} \end{cases} \tag{12}$$

where a_0 is an arbitrary but fixed real number chosen according to convenience. It was proved that if $\gamma^\alpha(C, a, b)$ is finite and $0 < \alpha \le 1$, then for any pair of $x_1 < x_2 \in (a, b)$ the following statements hold: a) $S^\alpha(x)$ is continuous increasing function of x in (a, b), b) $S^\alpha(x) = const$ in the interval $[x_1, x_2]$, if $C \cap (x_1, x_2) = 0$, and c) $S^\alpha(x_2) - S^\alpha(x_1) = \gamma^\alpha(C, a, b)$. The correspondence between sets and their staircase functions is many to one. Notice that, as the mass function is atomless, the removing a single point from C does not change its value [58]. The F^α-derivative of function $f(x \in C)$ at point x_1 is defined as

$$D^\alpha f(x) = \begin{cases} C\text{-}\lim_{y \to x} \frac{f(x_2) - f(x_1)}{S^\alpha(x_2) - S^\alpha(x_1)}, & \text{if } x_1 \in C, \\ 0, & \text{otherwise} \end{cases} \tag{13}$$

if the limit exists. In a view of Eqs. (10), (11), and the assumption (5b), the order of F^α-derivative existing for functions defined on the Cantor set in E^n is equal to

$$\alpha = \frac{D}{n_{sp}} = \frac{D}{n}, \tag{14}$$

where $n_{sp} = [D] + 1 = n$ (see also Ref. [64]). Bellow, we also need to use the Dirac delta function which is defined on the Cantor set as follows:

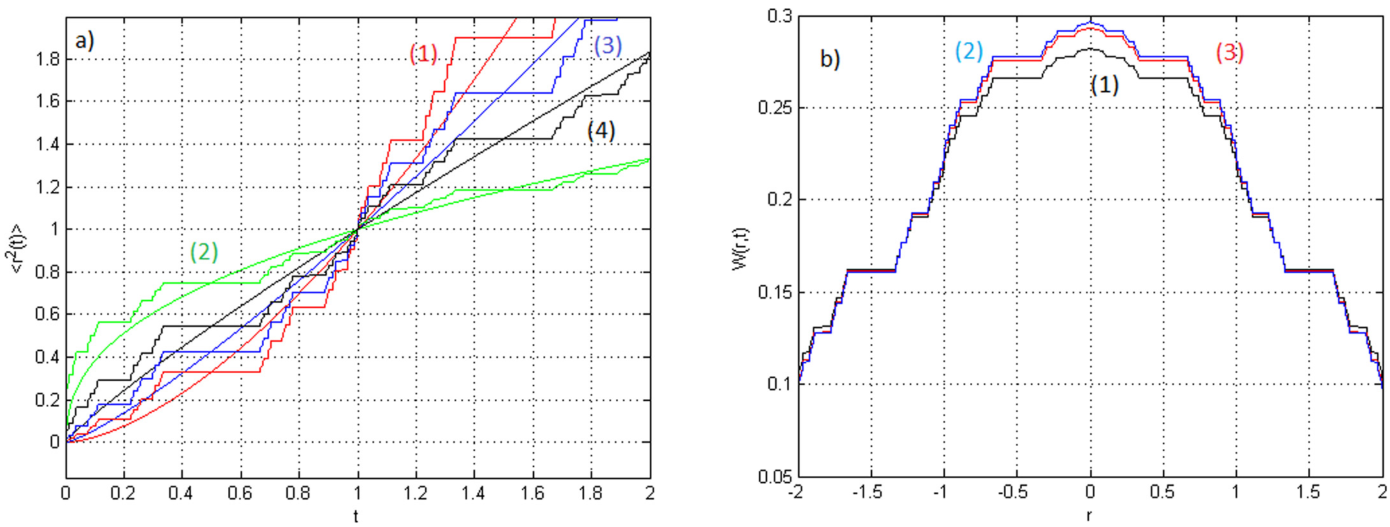

Fig. 1. a) The mean square displacement versus time for diffusion on the ternary Cantor set for different types of diffusion associated with: Lévy flights (1), Lévy walks (2), and scaled random walk E^1 (3) and E^2 (4). Continuous curves—the power law fitting with Eq. (1); b) The density distribution function $W(r,t)$ for different types of diffusion on the ternary Cantor set at time $t = 1$ for: Lévy flights (1), Lévy walks (2), and scaled random walk (3).

$$\delta^{\alpha}(x) = \begin{cases} \infty, & x = 0; \\ 0, & x \neq 0, \end{cases} \qquad \text{while} \quad \int_{-\infty}^{+\infty} \delta^{\alpha}(x) d^{\alpha}x = 1, \tag{15}$$

where F^{α}-integral is defined on the Cantor set (see Ref. [58]).

Now let us consider a random process defined on the Cantor set embedded in E^n. This process is characterized by the fractal-like intrinsic time $\tau = S^{\beta}(t)$ providing that the relation (5c) holds. The conservation of probability implies that the conditional probability $W(r,t)$ to find the walker at distance $S^{\alpha}(r)$ from the origin after the elapsed intrinsic time $S^{\beta}(t)$ should satisfy the continuity equation

$$D_t^{\beta} W(r,t) = -D_r^{\alpha} J(r,t), \tag{16a}$$

where

$$J = -K_{\beta}^{\alpha} D_r^{\alpha} W(r,t) \tag{16b}$$

is the net current, while K_{β}^{α} is the coefficient of diffusion and $r = (\sum_i^n x_i^2)^{1/2}$. Although, in some approaches to diffusion on fractals it was assumed that the diffusion coefficient is a space and/or time dependent, we think that the requirement that $K_{\beta}^{\alpha} = const$ is more appropriate, because so we need not to introduce an additional fitting parameter. Accordingly, the diffusion equation on the Cantor set gets the form

$$D_t^{\beta} W(r,t) = D_r^{\alpha} \left[K_{\beta}^{\alpha} D_r^{\alpha} W(r,t) \right], \tag{17}$$

while β and α are defined by Eqs. (5d) and (14), respectively. In this regard, it is pertinent to note that using the F^{α}-calculus [58–64] the diffusion equation on Cantor sets in $\gamma > 1$ can be written in the Cartesian coordinates, while the diffusion coefficient can be different in different directions. The corresponding solution of this equation can be easily obtained by using the conjugacy with ordinary calculus (see Ref. [59]). Here, we only illustrate that the probability distribution function and the mean squared displacement defined on Cantor sets behave as devil's-staircase-like functions of spatial coordinates. Three of four examples correspond to the Cantor set on line and one to the isotropic Cantor dust in plane. Accordingly, in the last case we use the polar coordinates with $r \in [0, \infty]$. The solution of Eq. (17) with the initial condition

$$W(r,0) = \delta_F^{\alpha}(r) \tag{18}$$

has the Gaussian form

$$W(r,t) = \frac{S^{\beta}(t)^{-1/2}}{\sqrt{4\pi K_{\beta}^{\alpha}}} \exp\left[\frac{-S^{\alpha}(r)^2}{4K_{\beta}^{\alpha} S^{\beta}(t)} \right], \tag{19}$$

where $S^{\alpha}(r)$ and $S^{\beta}(t)$ play the role of the spatial and time variables, respectively. Accordingly, the mean square displacement of the walkers on the Cantor set scales with the intrinsic time as

$$< S^{\alpha}(r)^2 >= K_{\beta}^{\alpha} S^{\beta}(t) \tag{20}$$

and so, the diffusion on the Cantor set is always Gaussian with respect to the F^{α}-measures of the set and the intrinsic time of the walk. Consequently, with respect to the physical time and the Cartesian coordinates in the embedding Euclidean space E^n, the type of diffusion depends on the ratio β/α. Taking into account that

$$< S^{\alpha}(r)^2 > \propto < r^{2\alpha} > \quad \text{and} \quad S^{\beta}(t) \propto t^{\beta}, \tag{21}$$

from Eqs. (1), (5c), (14), and (22) it follows that

$$\frac{\beta}{\alpha} = \gamma = \frac{2}{D_w}, \tag{22}$$

while the diffusion exponents for the Lévy flights and walks, and the scaled random walk on the Cantor set are defined by Eqs. (6c), (7c), and (8c), respectively. Specifically, the intrinsic time of the Markovian Lévy flights on the Cantor set is governed by $\beta = 1$ and so, it coincides with the physical time in the embedding space E^n. Accordingly, the Markovian Lévy flights on the Cantor set lead to the super-diffusion in the embedding space E^n (see Fig. 1a) with the diffusion exponent defined by Eq. (6c). The graphs of $W(r,t)$ for the Markovian Lévy flights on the Cantor sets with different Hausdorff dimensions in are shown in Figs. 1b and 2. Notice that the density distribution function in the embedding Euclidean space E^n can be fitted by the stretched exponential function

$$W(r,t) = \frac{t^{-n/2}}{\sqrt{4\pi K}} \exp\left[\frac{-r^{2D/n}}{4Kt} \right], \tag{23}$$

which resembles the propagator envelope on path-connected fractals, rather than the propagator of the annealed Lévy motion (see

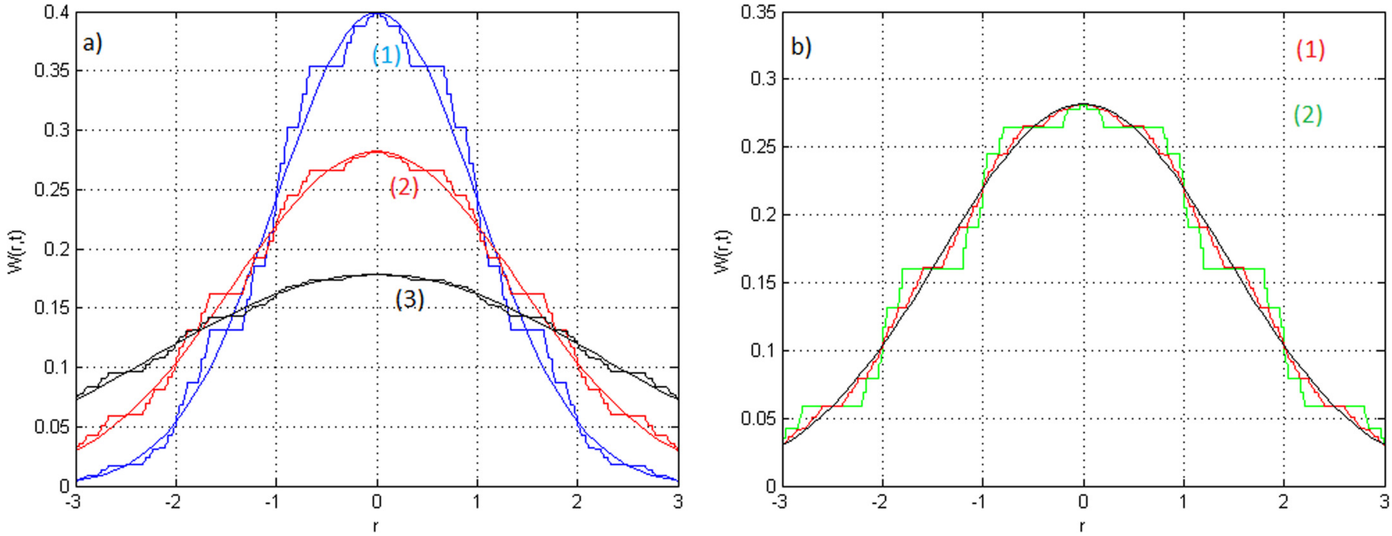

Fig. 2. The graphs of $W(r,t)$ for the Markovian Lévy flights on the middle-ϵ Cantor sets: a) for $D = \ln 2/\ln 3$ at times 0.5 (1), 1 (2), and 2.5 (3); b) at time $t = 1$ for $D = \ln 2/\ln 5$ (1) and $D = \ln 2/\ln 2.5$ (2).

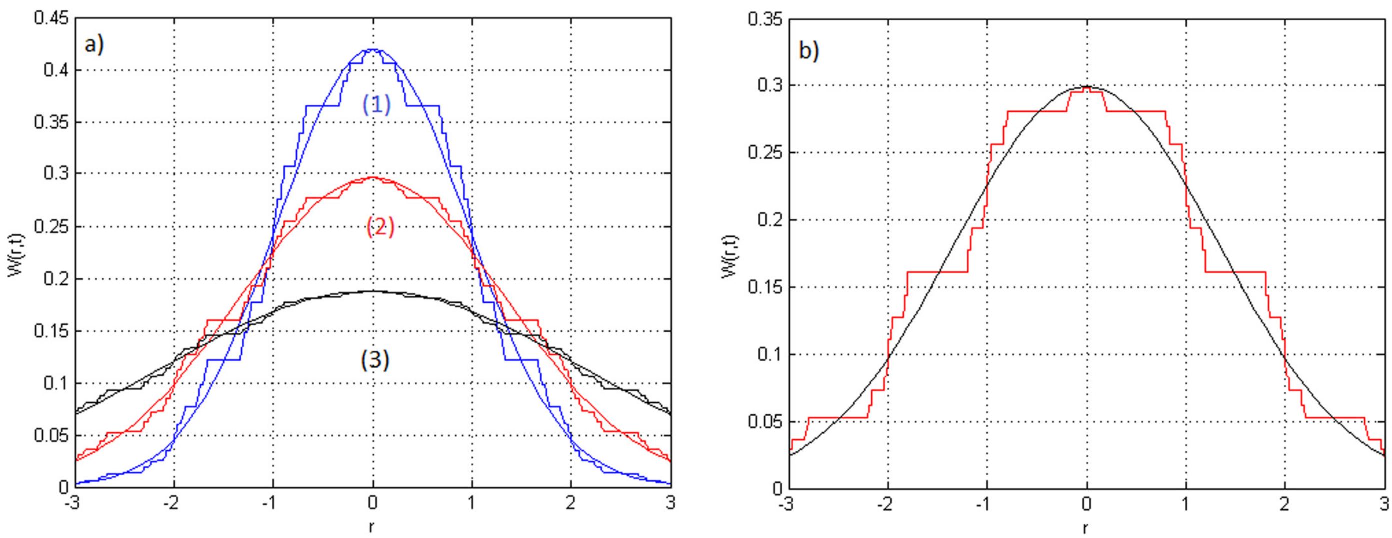

Fig. 3. The graphs of $W(r,t)$ for the Markovian Lévy walks on the middle-ϵ Cantor sets: a) for $D = \ln 2/\ln 3$ at times 0.5 (1), 1 (2), and 2.5 (3); b) at time $t = 1$ for $D = \ln 2/\ln 2.5$. Continuous curve represent analytical envelopes of $W(r,t)$.

Ref. [34]). Furthermore, in contrast to models with the propagator having a power-law asymptotic behavior, the super-diffusion ruled by the Lévy flights obeying Eq. (6a) is characterized by the diffusion exponent (6c) determined by the fractal dimension of the minimum path on the Cantor set. In this regard, it is pertinent to point out that diffusion equation (17) together with Eqs. (14) and (22) can be used to describe the acceleration of chemical wave fronts in the Belousov–Zhabotinsky reactions in the quasi-two-dimensional chaotic flow[2] (for details see Ref. [34]).

The Markovian Lévy walk obeying Eqs. (7) on the Cantor set in E^n has the intrinsic time $S^\beta(t)$ governed by the intrinsic time exponent

$$\beta = \frac{2D}{n} - 1 = \alpha - 1, \tag{24a}$$

while $n/2 < D < n$,

$$D_w = \frac{2D}{2D - n} > 2, \tag{24b}$$

and α is defined by Eq. (14). Accordingly, in this case, the diffusion equation (17) describes the sub-diffusion on the Cantor sets embedded in E^n (see Fig. 1a). The graphs of $W(r,t)$ for the Lévy walk on the Cantor sets with different Hausdorff dimensions are shown in Figs. 1b and 3. The sub-diffusion associated with the Lévy walk on the pre-fractal Cantor sets was studied in Ref. [36].

In the case of the scaled random walk obeying Eq. (8) on the Cantor set in E^n, by the intrinsic time exponent

$$\beta = \frac{2D}{n(1+D)} = \frac{2\alpha}{1+n\alpha}, \tag{25}$$

while α is given by Eq. (14) and the fractal dimension of the random walk is equal to $D_w = 1 + D$. So, the scaled random walk on the Cantor set in E^1 leads to the super-diffusion with the diffusion exponent defined by Eq. (8c). Conversely, in the space with $n \geq 2$, the scaled random walk on the Cantor set leads to a sub-diffusion (see Fig. 1a). The graphs of $W(r,t)$ for the scaled random walk on the Cantor sets with different Hausdorff dimensions in E^1 and E^2 are shown in Figs. 1a and 4. In this regard, we noted that, when

[2] In Ref. [65] it was found that $\gamma = 1.3 \pm 0.1$ is independent of the Lévy index $0.8 \leq \mu \leq 1.6$ and the density distribution function has the form of Eq. (23).

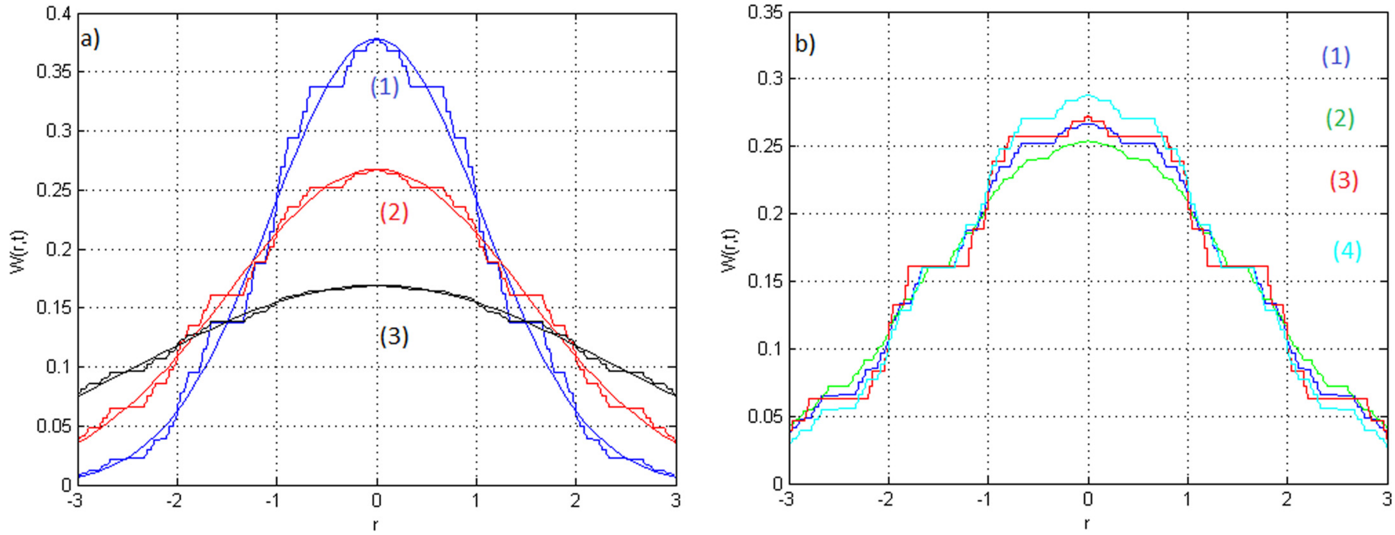

Fig. 4. The graphs of $W(r,t)$ for the scaled Markovian random walk on the middle-ϵ Cantor sets: a) for $D = \ln 2/\ln 3$ at times 0.5 (1), 1 (2), and 2.5 (3); b) at time $t = 1$ for $n = 1$ and $D = \ln 2/\ln 3$ (1), $D = \ln 2/\ln 5$ (2), $D = \ln 2/\ln 2.5$ (3), and for $n = 2$ and $D = \ln 4/\ln 3$ (4).

the electric current through the fractal-like cellular solids studied in Ref. [11] is governed by the scaled random walk obeying Eq. (8), the Einstein relation implies that the electric resistivity exponent is equal to $\zeta = 1$, while a non-stationary diffusion current is expected to scale as $J \propto t^{-\nu}$, where $\nu = d_s/2$, while the spectral dimension is defined by Eq. (8a). Accordingly, for the cellular solids with the ternary Cantor set structure we expect $\nu = 0.38$, that is consistent with $\nu = 0.36$ obtained in Ref. [11] by direct numerical evaluation.

4. Conclusions

In this work, we stress that the type of diffusion is determined by the ratio of numbers of effective spatial and dynamical degrees of freedom of a random walker on the fractal. We also argue that this ratio depends on the definition of the Markov process on the Cantor set. In this way, we find the numbers of effective degrees of freedom for three kinds of Markovian random processes early defined on the Cantor set. This allows us to explain the emergence of different types of anomalous diffusion on the ternary Cantor set. Specifically, we find that the Markovian Lévy flights and walks on the Cantor set lead to the super- and sub-diffusion, respectively. On the other hand, the scaled random walk leads to the super-diffusion on the Cantor set on line, but to the sub-diffusion on the Cantor set (dust) on the plane. The theoretical predictions are compared with the available results of numerical simulations reported in the literature.

Further we deduce the equation of diffusion on the totally disconnected fractals by employing the local differential operators on the F^α-support. The relation between the orders of F^α-derivatives and the numbers of effective degrees of freedom are established. The solutions for different types of diffusion on the middle-ϵ Cantor sets are obtained. In this regard, we point out that different types of diffusion on the Cantor sets can be related to different physical phenomena observed in complex systems. So, our findings provide a novel insight to this subject.

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