Georg Cantor at the Dawn of Point-Set Topology

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Abstract

A first course in point-set topology can be challenging for the student because of the abstract level of the material. In an attempt to mitigate this problem, we use the history of point-set topology to obtain natural motivation for the study of some key concepts. In this paper, we study an 1872 paper by Georg Cantor. We will look at what Cantor was attempting to accomplish and see how the now familiar concepts of a point-set and derived set are natural answers to his problem. We emphasize ways to utilize Cantor's methods in order to introduce point-set topology to the student.

1 Introduction

In the introduction of his book *Introduction to Phenomenology*[23], Msgr. Robert Sokolowski writes

> Mathematicians ... tend to absorb the writings of their predecessors directly into their own work. They do not comment on the writings of earlier mathematicians, even if they have been very much influenced by them. They simply make use of the material that they find in the authors they read. When advances are made in mathematics, later thinkers condense the findings and move on. Few mathematicians study works from past centuries; compared with contemporary mathematics, such older writings seem to them almost like the work of children.

As a philosopher, Msgr. Sokolowski is accustomed to the traditional methods of teaching philosophy to undergraduates– start with Plato, Aristotle and the other ancients, continue with developments though the Scholastic and Enlightenment eras, and then show how modern philosophy builds upon all that has gone before. He must be puzzled, then, by the lack of attention to historical development of ideas that generally attends to the teaching of mathematics. He perceives that something important is missing, and he is correct.

In recent years, interest has grown considerably in developing an historical approach to the teaching of mathematics. Victor Katz has edited an anthology of articles giving different perspectives on the development of mathematics in general from an historical point of view [16]. Authors, such as Brian Hopkins [14], have written recent textbooks introducing discrete mathematics from an historical point of view while other authors, such as Klyve, Stemkoski, and Tou, focus on the work of Euler [17]. There is also interest in the historical development of certain areas of math. David Bressoud ([2],[3]) has written two textbooks introducing analysis from an historical perspective, while Adam Parker has complied an ODE original sources bibliography containing many of the original papers in differential equations. [1].

This is the first paper in a planned series of several that outline ways to introduce point-set topology concepts motivated by their place in history. To borrow a phrase from David Bressoud, it is an "attempt to let history inform pedagogy" [2, p. vii]. A growing collection of the historic papers that are important to the development of point-set topology may be found on the author's web site[2]. This paper focuses on the seminal work of Georg Cantor (1845-1918), a German mathematician most well-known for his contributions to the foundations of set theory, but whose contributions to point-set topology are not very well known. Cantor's works are collected in [8]. For complete biographical information, see Dauben's definitive work [11].

First, a few words about why point-set topology is an important topic to understand from an historical point of view. Other authors have advanced many good reasons to study mathematics historically in general. Beyond these reasons, an historical approach to point-set topology should help a beginning student grasp and become inter-

[1]Available online at http://userpages.wittenberg.edu/aparker/originalsources.html. Accessed on 3/5/2012

[2]http://webpages.ursinus.edu/nscoville/HistoryProject.html

ested in this area of mathematics, which is notoriously inaccessible to beginners. To the student, analysis is easily seen as a kind of "super-calculus" and abstract algebra can be motivated by discussing symmetries of objects. But when a course in point-set topology begins merely by defining a topology and giving several examples, it can be very difficult for students to grasp the general concept or how topology connects with all the other mathematics they have learned. As we will see below, introducing point-set topology through its historical development motivates the student to consider the idea of "nearness without distance" as well as immediately places the subject within the larger mathematical world.

Point-set topology, which was originally called analysis situs or analysis of position, grew out of analysis. In discussing Cauchy's contribution to the foundations of analysis, Manheim [19, p. 26]writes that

> The conceptual difficulties associated with the word *limit* derived from attempts to define it in terms of magnitude rather than aggregation. The unsatisfactory results of these endeavors led first to the formalism of Euler and later to that of Lagrange. . . [so] A new approach was required, an approach which recognized both the fundamental role of the limit concept and its basic arithmetical nature. (emphasis original)

Here we see an attempt to isolate the limit concept from geometric and physical intuition. The now famous Weierstrass function [1, Section 5.4](which is both nowhere differentiable and everywhere continuous) verified that such an isolation was needed [3]. The equally famous Bolzano-Weierstrass Theorem today cannot be discussed without a knowledge of limit points, but a limit point was never defined as a concept[4] by Weierstrass. Thus there are several places where one can see the arguable beginnings of point-set topology. Cantor's 1872 paper [6], the main focus of this paper, is chosen as our object of study

[3] Hykšová [15] argues that although Bolzano's paper proving the existence of a nowhere differentiable everywhere continuous function was not published until 1930, Bolzano had constructed such a function before 1834. In any event, Weierstrass had lectured on the function now bearing his name in the 1860s and had published a paper on this function in 1872 [24].

[4] It was not uncommon to work with a concept without having an explicit definition. See [12]

because it begins with a very clear and well-defined problem in analysis and solves this problem by introducing the derived set, a purely point-set notion. Thus it builds a "motivational bridge" between familiar concepts in analysis to a new concept in point-set topology, addressing the problem of point-set topology being disconnected from other branches of math.

This paper has a two-fold purpose. First, it introduces the reader to Cantor's 1872 paper and in particular, his need for a theory of "distance without nearness" in order to solve an analysis problem. Second, the paper provides information to the teacher who would like to introduce point-set topology in an historical context, motivated by some of the questions that were popular at the time. Our two objectives are combined in Section 6, where we present Cantor's main theorem in a way that should be accessible and provide motivation for the study of point-set topology. We begin with a brief discussion of the mathematical climate at the time of Cantor's 1872 paper. We use modern notation and parlance to convey Cantor's ideas, whenever doing so would make it easier for the modern reader of mathematics to understand and when there is no possibility of losing any of Cantor's original meaning or intent.

2 Background

The primary focus of Cantor's paper is not point-set concepts. Rather, he was concerned with a certain theorem in Fourier Series. Inspired by the work of Heine [13], Cantor was able to weaken conditions for which the Fourier series of a function is unique. He first did this in an 1870 paper [4] and using the same technique weakened the conditions further in an 1871 paper [5]. Both of these papers were precursors to his 1872 paper. For an in-depth discussion of the mathematics in these and Cantor's other papers around that same time, see Dauben [10]. Heine's 1870 work showed that if a function is almost everywhere continuous and its trigonometric series converges uniformly, then the Fourier series is unique. As Dauben points out, "Requiring almost-everywhere continuity and uniform convergence, Heine's theorem invited direct generalizations." These generalizations would be taken up by Cantor. Important for our purposes is that Cantor developed a proof technique in his 1870 paper and modified it only slightly while weakening his hypotheses in the 1872 paper. More specifically, Cantor showed that under certain hypotheses, the trigonometric representation of a function remains unique even when convergence or representation of the function was given up on certain infinite subsets of $(0, 2\pi)$. [5] The nature and construction of the particular kind of infinite set for which Cantor's 1872 theorem held interest us here. For now we see point-sets first defined, and in hindsight it is not surprising that point-sets went on to be studied in their own right.

3 Real Numbers

In order to modify the proof in his 1870 paper, Cantor needed to develop his own theory of the real numbers. Although our purpose in examining his theory of real numbers is to demonstrate the emergence of point-set topology, Cantor's construction is interesting in its own right.[6] Weierstrass apparently had a theory of the reals at this point[7], but Euclid is cited by Cantor as the definitive source for the theory of real numbers. He writes "For comparison ... we mention the book "Elements of Euclid" which remains the decisive treatment of the subject" [6, p. 127]. Still, Cantor was not fully satisfied with this theory, and he desired a more solid foundation for standard operations (addition etc.) performed on the real numbers. To this end, he defined a sequence of rational numbers $a_1, a_2, \ldots, a_n, \ldots$ to possess **numberness** (Zahlengroße) if for every $\epsilon > 0$, there exists an integer N such that whenever $n \geq N$, $|a_{n+m} - a_n| < \epsilon$ for m any positive integer [8]. We associate to any sequence possessing numberness a symbol b, which we refer to as a **number value.** Cantor writes "This property of sequence a_n I express in the words a_n has a certain limit b." In other words, by the symbol $\lim a_n$, Cantor means the number value b associated to that sequence a_n. Such a sequence Cantor called **funda-**

[5] Cantor denotes the interval $(0, 2\pi)$ by $(0, \ldots, 2\pi)$.

[6] Even Bertrand Russell begrudges Cantor a compliment when he writes "The theory of Cantor ... with all the requisite clearness, lends itself more easily to the interpretation which I advocate, and is specially designed to *prove* the existence of limits." (emphasis original) [22] p.283

[7] See [10] footnote 47 for details about Weierstrass' contribution along with references such as [21]. None of Weierstrass' writing on the theory of reals seems to have survived, and we only know of it through his students. Since his students all seem to have published their teacher's theory after 1872, it may not have been well known when Cantor wrote his paper.

[8] In modern terminology, we would use the term Cauchy sequence.

mental. He denotes the collection of all such fundamental sequences by B. We will see in Section 4 that every real number corresponds to a fundamental sequence of B. Cantor goes on to show how we may for now at least conceptualize B as the real numbers. However, this is only for intuitive purposes. Cantor is explicit about this last point when he writes "Now these words initially have no other meaning except as an expression for *those* properties of the sequence, and from the fact that we associate to the series $a_1, a_2, \ldots, a_n$ a special character b, it follows that with various series, various characters $b, b', b^{''}, \ldots$ are formed." Immediately after making this definition, Cantor is quick to note that the number value b is simply a formal symbol associated to the sequence $a_1, a_2, \ldots, a_n, \ldots$. This is important to note, as Cantor did not wish to fall into the error of assuming the existence of limits of sequences of real numbers. He simply associates a symbol to any fundamental sequence. Next Cantor defines a total ordering on the set B. Let b, b' be number values with corresponding fundamental sequences $a_1, a_2, \ldots, a_n, \ldots$ and $a'_1, a'_2, \ldots, a'_n, \ldots$ respectively. Then one of the following three relations must hold:

1. For every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that for every $n \geq N$ we have $a_n - a'_n < \epsilon$.

2. There is a rational $\epsilon > 0$ and $N \in \mathbb{N}$ such that for every $n \geq N$, we have $a_n - a'_n > \epsilon$.

3. There is a rational $\epsilon > 0$ and $N \in \mathbb{N}$ such that for every $n \geq N$, we have $a_n - a'_n < -\epsilon$.

In the first case, Cantor defines $b = b'$, in the second $b > b'$, and in the third $b < b'$. It is important to again stress that Cantor is defining formal symbol relations. The symbol $=$ is an equivalence relation on B (although Cantor does not use this terminology), and we continue to write B for the set of equivalence classes of number values under the relation $=$. Technically speaking, then, an element $b \in B$ is an equivalence class of fundamental sequences.

Now Cantor is ready to define the operations of addition, subtraction, multiplication and addition in B. For conceptual purposes, we may think of this as defining said operations on all real numbers, but technically speaking we are only formally symbol pushing. Let the number values $b, b', b^{''}$ correspond to the fundamental sequences

$$a_1, a_2, \dots$$
$$a_1', a_2', \dots$$
$$a_1'', a_2'', \dots$$

respectively. If

$$\begin{aligned}
\lim(a_n \pm a_n' - a_n'') &= 0 \\
\lim(a_n \cdot a_n' - a_n'') &= 0 \\
\lim\left(\frac{a_n}{a_n'} - a_n''\right) &= 0[\text{for } a_n' \neq 0],
\end{aligned}$$

then we write $b \pm b' = b''$, $bb' = b''$, $\frac{b}{b'} = b''$, respectively.

After defining operations on B, Cantor constructs the set C from B in an analogous way that he defined B from the rationals. That is, he considers all sequences $b_1, b_2, \dots$ of B that the limit of $b_{n+m} - b_n$ equals 0 for some fixed value of m. We associate a symbol c to such a sequence and define relations[9] among such c as we did in B. Continuing in this manner, Cantor is able to construct the sets $D, E, \dots$ consisting of equivalence classes of fundamental sequences made up of members of the previous set. Cantor remarks that "It is reserved for me to come back to all these conditions on another occasion in more detail." This statement is a precursor to Cantor's theory of transfinite numbers.

4 Number line vs. Numberness

Cantor creates a bijection between the number line and number values. He calls an element on the number line a "point" and an element of B (a number value) a value. We thus seek a bijection between values and points. Again, the idea here is that after we have established the bijection, we will then have a rigorous foundation to justify arithmetic operations on the real numbers.

Fix a point o on the number line (think of this as the origin). To show that the number line injects into the collection of number values,

[9]We use the term relations as a generic word to mean any of $=, <, >, \pm, \cdot$, and $\div$.

Cantor first considers the case when the distance from the given point to o has a rational relationship. If a on the number line is rational, then we may associate the constant sequence possessing numberness $a, a, \ldots,$ to the rational number a on the number line. This seemingly obvious fact is not without its criticisms. As Russell ([22] p. 285) points out

> There is absolutely nothing in the above definition of the real numbers [number values] to show that a is the real number defined by a fundamental series [sequence] whose terms are all equal to a. The only reason why this seems self-evident is, that the definition by limits is unconsciously present, making us think that, since a is plainly the limit of a series whose terms are all equal to a, therefore a must be the real number defined by such a series.

However, this may only be problematic when dealing with the arithmetic of number values, not defining a bijection with the reals. So if a is rational, we associate a with the sequence $a, a, \ldots$ (whatever symbol the limit symbol of the sequence may be). Now suppose that an element b on the number line is irrational. Cantor asserts that "It is always possible to give a sequence

$$a_1, a_2, \ldots \tag{1}$$

[so that the] distance to the determined point to the point o is equal to b where b is the corresponding numerical quantity of sequence 1." In other words, Cantor associates the equivalence class $[a_n] \in B$ to b.

Cantor took the converse, that "the geometry of the straight line is complete," as an axiom. He writes "... to make the geometry of the straight line complete is only to add an axiom, which simply consists in [declaring that] any numerical quantity belongs to a certain point of the straight line ... I call this theorem an *axiom* because it is in its nature to not generally be provable." (emphasis original)

Before diving into topology, Cantor points out that this bijection is helpful only for conceptual purposes.

> Given the previous,we now assign number values to the points on the line. For clarity (not that it is essential), we use this notion in the following and have, when we speak of points, values in mind by which they are given.

Notice that once again, Cantor stresses that thinking of points on the real line as number values is "for clarity" and not essential to what he is doing.

Though it may seem like the work we have done in Sections 3 and 4 is not at all related to topology, the building of such a rigorous foundation was necessary to make the following definitions appearing in Section 5 below. These abstract away the distance between real numbers but kept the nearness between real numbers precisely by making the distinction between "points" and "values."

5 Topology

Before Cantor proves his main theorem, he gives several definitions which today we would recognize as belonging to the area of point-set topology. Recall Cantor's distinction between values and points above. He first defines a **value set** to be a finite or infinite set of values (number values). He then defines a **point set** to be a finite or infinite set of points. Modern mathematics tends to view the term "point-set" as synonymous with "open set." But Cantor's original understanding of point-set is any subset of the real line thought of as being in a one to one correspondence with a set of symbols where you can "do" arithmetic. In fact, it is interesting to note, as G.H. Moore points out [20], that Cantor never used the idea of an open set. Today it is unheard of to even mention topology without thinking about open sets, yet the concept of open set as we know it took dozens of years to develop (again, see [20] for an excellent discussion of the history of open sets).

With a view towards generalizing his theorem, Cantor then defines a **cluster point** or **limit point**[10] of a point set P as "a point of the line situated in such a way that each neighborhood of it contains *infinitely* many points of P." (emphasis original) A **neighborhood** of a point is "any interval that has the point *as its interior*[11]." (emphasis original) Now that he has defined limit point, Cantor is able to partition points of a point set P as either a limit point of P or not a limit point of P. In this way, we he defines the **first derived set** of P, denoted P', to be the set of all limit points of P. We may then define the **second derived set** of P, denoted P'', as the first derived set of

[10]This is its earliest known published definition of limit point.

[11]Interior point is not defined in this paper, but is defined in Cantor's 1879 paper [7].

the first derived set. Continuing in this manner, Cantor defines $P^{(v)}$, the v^{th} **derived set** of P, noting that $P^{(k)}$ may be empty for some k. This allows Cantor to define P to be a **point set of the** v^{th} **kind** whenever $P^{(v)}$ is finite (and hence $P^{(v+1)} = \emptyset$).

6 The main Theorem and Teaching

We present the argument of Cantor's main theorem in a way that is amenable to introducing the subject of point-set topology to junior or senior mathematics majors. Cantor states his main theorem as follows:

Theorem 1 *If there is an equation of the form*

$$0 = C_0 + C_1 + \ldots + C_n + \ldots, \tag{2}$$

where $C_0 = \frac{1}{2}d_0; C_n = c_n \sin(nx) + d_n \cos(nx)$*, for all values of* x *except those which correspond to points in the interval* $[0, 2\pi]$ *which give a point set* P *of the* v^{th} *kind, where* v *signifies any large number, then*

$$d_0 = 0, c_n = d_n = 0.$$

The fact that point-set topology grew out of analysis and in particular trigonometric representation of a function immediately poses a difficulty in an historic introduction to point-set topology. A standard math curriculum usually places trigonometric representation of a function (Fourier series) in an upper level or second course in differential equations, a course which surprisingly few topology students take. Thus one would need to spend some time motivating interest in the question of trigonometric representation. Considering a trigonometric representation of a function as an analogous idea to a power series representation of a function from Calculus II can help the student see why it may be useful to represent a function in this way. For example, we may remind the student that under reasonable hypotheses, we may differentiate or integrate the power series of a function term by term. One can also mention some of the vast applications of Fourier series in physics [18]. Once the importance of such a representation has been established, the existence may be assumed and the question of when it converges is a natural question to consider. In order to show uniqueness, let

$$\frac{1}{2}b_0 + \sum_{i=1}^{\infty} a_n \sin(nx) + b_n \cos(nx)$$

and

$$\frac{1}{2}b'_0 + \sum_{i=1}^{\infty} a'_n \sin(nx) + b'_n \cos(nx)$$

be two representations for $f(x)$. We follow Cantor's argument and use his above notation. If we set $d_0 = b_0 - b'_0$, $c_n = a_n - a'_n$ and $d_n = b_n - b'_n$, then $0 = \frac{1}{2}d_0 + \sum_{i=1}^{\infty} c_n \sin(nx) + d_n \cos(nx)$. We further simplify by writing $C_0 = \frac{1}{2}d_0$ and $C_n = c_n \sin(nx) + d_n \cos(nx)$ so that $0 = C_0 + C_1 + C_2 + \ldots + C_n + \ldots$. Define $F(x) = C_0 \frac{x^2}{2} - C_1 - \frac{C_2}{2^2} - \ldots - \frac{C_n}{n^2} - \ldots$. Then $F(x)$ is continuous, and if $F(x)$ can be shown to be linear in the sense that $F(x) = cx + c'$ for real values c and c', then Cantor was able to prove that $c_n = d_n = 0$ so that the representation is unique. We are thus interested in conditions for which $F(x)$ is linear.

Given the above setup, Cantor showed in both his April 1870 paper [4] and the 1871 Notiz [5] that if convergence is given up on a finite number of points in (p, q), then $F(x)$ is linear on (p, q). Cantor notes this in his 1872 paper by stating the following:

(A) *"If there is an interval (p, q) in which only a finite number of points of the set P lie, then $F(x)$ is linear in this interval."*

Hence the challenge that lay ahead of him was to extend this theorem to certain kinds of infinite sets for which convergence or representation is given up. As we now show, a point-set of the v^{th} kind fits this description. At this point, it is worth outlining the important features to emphasize to the students.

- A trigonometric series representation for a function $f(x)$ is important.
- Thus, the uniqueness of such a series is important.
- Cantor was able to show that if a certain function $F(x)$ (based on $f(x)$) is linear, then uniqueness follows.
- Cantor was able to show that if we give up convergence or representation for finitely many points, then $F(x)$ is linear.

Our task is to create an infinite set P on which we give up convergence or representation on $(0, 2\pi)$ such that we can break up $(0, 2\pi)$

into subintervals, each of which has only a finite number of points that give up convergence or representation. Then on each of those subintervals, we apply (A) to obtain the result. It should be noted that the key to understanding this inductive proof is to understand the base case. Once the base case is understood, the inductive step is simply formalizing the rest of the argument apart from any new insights or techniques. Following the proof with a picture lends good geometric as well.

Let P be a point-set of the v^{th} kind so that $P^{(v)}$ is finite and $P^{(v+1)} = \emptyset$ and suppose that we give up convergence on P. Let (p, q) contain a finite number of points of P. By (A), $F(x)$ is linear on (p, q). We will show that $F(x)$ is linear on $(0, 2\pi)$ by induction on k, where k represents the k^{th} derived set of P, $P^{(k)}$. Consider any subinterval (p', q') of (p, q) which contains a finite number of points $x'_0, x'_1, \ldots, x'_v \in P'$, where $x'_0 < x'_1 < \ldots < x'_v$. In general (p', q') may contain infinitely many points of P, so Cantor cannot apply (A). In particular, any subinterval (x'_j, x'_{j+1}) may contain infinitely many points of P. However, any proper subinterval $(s, t) \subset (x'_j, x'_{j+1})$ does not contain a limit point of P since by assumption there are no limit point of P between x'_j and x'_{j+1}. Thus by definition of the first derived set of P, (s, t) contains only finitely many points of P. Hence, while we cannot apply (A) directly to the interval (p', q'), we can apply it to (s, t) to conclude that $F(x)$ is linear on (s, t). Since the endpoints s, t can be made arbitrarily close to x'_j and x'_{j+1}, $F(x)$ is linear over each subinterval $(x'_j, x'_{j+1}) \subset (p', q')$. It follows that $F(x)$ is linear over all of (p', q').

Assume that $F(x)$ is linear on any interval containing finitely many points of $P^{(k-1)}$. We wish to show that $F(x)$ is linear on any subinterval, say (p^k, q^k), which contains finitely many points of $P^{(k)}$. Let $x^k_0, x^k_1, \ldots, x^k_s$ be finitely many points of $P^{(k)}$ contained in (p^k, q^k) with $p^k < x^k_0$ and $x^k_s < q^k$. Let (c, d) be a proper subinterval of (x^k_j, x^k_{j+1}) with $x^k_j < c$ and $d < x^k_{j+1}$. Then (c, d) contains only finitely many points of $P^{(k-1)}$, for otherwise (c, d) would contain a point of $P^{(k)}$. By the inductive hypothesis, $F(x)$ is linear on (c, d). As above, this implies that $F(x)$ is linear on all subintervals (x^k_j, x^k_{j+1}) and $F(x)$ is linear on (p^k, q^k).

A short presentation of the above theorem, along with the relevant definitions and emphasis on the four bullet points, should provide strong motivation for the study of concepts in point-set topology.

Although we took great care in discussing Cantor's theory of the reals, these details can safely be swept under the rug when attempting to coalesce the necessary background for point-set motivation into as concise and pithy a presentation as possible.

With the usefulness of the limit point and derived set in mind, we end this section with a quote from Roger Cooke [9]

> Seen in this context (rather than in the usual unmotivated classroom setting of point-set topology) the concept of a limit point and derived set are completely natural, almost inevitable, results of the attempt to decide the question of uniqueness of trigonometric series.

7 Conclusion

Cantor's 1872 paper made an important contribution towards the development of point-set topology. His detailed and meticulous construction of the real numbers made it possible to build a rigorous foundation for what we might now refer to as nearness without distance. These ideas can be used to place point-set topology within the larger mathematical world and motivate the study of topology via analysis. It is our hope that this beautiful branch of mathematics will not be lost on students because it appears to them as being totally removed from what they believe to be "real" mathematics.

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