Georg Cantor and Pope Leo XIII: Mathematics, Theology, and the Infinite

Author(s): Joseph W. Dauben

Source: *Journal of the History of Ideas*, Jan. - Mar., 1977, Vol. 38, No. 1 (Jan. - Mar., 1977), pp. 85-108

Published by: University of Pennsylvania Press

Stable URL: https://www.jstor.org/stable/2708842

GEORG CANTOR AND POPE LEO XIII: MATHEMATICS, THEOLOGY, AND THE INFINITE

BY JOSEPH W. DAUBEN*

Georg Cantor (1845–1918) is remembered chiefly for his creation of transfinite set theory, which revolutionized mathematics by making possible a new, powerful approach to understanding the nature of the infinite.[1] But Cantor's concerns extended well beyond the purely technical content of his research, for he responded seriously to criticism from philosophers and theologians as he sought to advance and to refine his transfinite set theory. He was also keenly aware of the ways in which his work might in turn aid and improve both philosophy and theology.[2] Prompted by a strong belief in the role set theory could play in helping the Roman Catholic Church to avoid misinterpreting the nature of infinity, he undertook an extensive correspondence with Catholic theologians, and even addressed one letter and a number of his pamphlets directly to Pope Leo XIII. Consequently, for anyone interested in the history of science and its cultural dimensions in the nineteenth century, Cantor's life and work illustrate the way in which such multiple

*I am grateful to the Faculty Research Foundation of the City University of New York for helping to support this research. I am particularly indebted to Oberstudienrat Wilhelm Stahl, executor of the surviving literary *Nachlass* of Georg Cantor, for access to the letter books in which Cantor drafted much of his correspondence, and to the American Academy in Rome and the Biblioteca Vaticana for their help in making their facilities available to me. Unless otherwise noted, *all citations* from the publications of Georg Cantor are taken from the edition of his collected works by Ernst Zermelo: *Gesammelte Abhandlungen mathematischen und philosophischen Inhalts* (Berlin, 1932; repr. Hildesheim, 1966), hereafter Cantor (1932). Cantor's papers are designated by their original years of publication. Thus "Cantor (1872), 93" refers to Cantor's article of 1872 *as given in Zermelo's edition* on page 93.

[1]Set theory, as created by Cantor, initiated systematic study of the number and order relations of general collections or sets of objects. He showed how it was possible to define the "number" of objects in infinite sets, and he developed a transfinite arithmetic capable of dealing with infinitely large numbers and describing the properties of infinitely large collections, like sets of all natural, rational, or real numbers. For recent and informative studies of Cantor's life and work cf. H. Meschkowski, *Probleme des Unendlichen. Werk und Leben Georg Cantors* (Braunschweig, 1967), hereafter Meschkowski (1967), and his article on Cantor in *Dictionary of Scientific Biography,* ed. C. C. Gillispie (New York, 1971), III, 52–58; I. Grattan-Guinness, "Towards a Biography of Georg Cantor," *Annals of Science,* **27** (1971), 345–96, hereafter Grattan-Guinness (1971).

[2]Specific aspects of Cantor's philosophical interests and their significance for the development of his mathematics are discussed in J. Dauben, "Georg Cantor's Philosophy of Mathematics: the Irrational and the Transfinite Numbers," *Proc. of the XIIIth International Congress of the History of Science* (Moscow, 1974), 86–93, hereafter Dauben (1974).

interests combine to reinforce and to influence one another. This paper attempts to bring such diverse threads together, and to show specifically how mathematics, philosophy, and religion did act and react upon each other in the development of Cantor's own mathematical research. Particular emphasis is given to the interaction between transfinite set theory and the interest of Catholic theologians working in the spirit of Pope Leo XIII's encyclical *Aeterni Patris*.[3]

1. *Cantor's Radical Innovation.*—Cantor's introduction of the actual infinite was a radical departure from traditional practice, even dogma. It was an idea which mathematicians, philosophers, and theologians in general had repudiated since the time of Aristotle.[4] Philosophers and mathematicians rejected completed infinities largely because of the apparent logical paradoxes they seemed to generate. Theologians represented another tradition of opposition to the actual infinite, regarding it as a direct challenge to the unique and absolute infinite nature of God. St. Thomas, in particular, had argued against the possibility of any absolute infinity,[5] and yet Cantor's new theory produced precisely what Thomas had denied. Similarly, the great German mathematician Carl Friedrich Gauss expressed in most authoritative terms his opposition to the use of such infinities in mathematics in a celebrated letter to Heinrich Schumacher:

> But concerning your proof, I protest above all against the use of an infinite quantity (*Grösse*) as a *completed* one, which in mathematics is never allowed. The infinite is only a *Façon de parler,* in which one properly speaks of limits.[6]

[3]Leo XII issued *Aeterni Patris* in 1879. For translations of the encyclical and bibliographic details, cf. note 41 below.

[4]For some indication of Cantor's estimation of the most significant opponents of the actual infinite in philosophy and theology, cf. his *Grundlagen einer allgemeinen Mannichfaltigkeitslehre, ein mathematisch-philosophischer Versuch in der Lehre des Unendlichen* (Leipzig, 1883), hereafter Cantor (1883), and reprinted in Cantor (1932), 165–208, esp. sections 5 through 8. Cantor later returned to these ideas in his "Über die verschiedenen Standpunkte in bezug auf das aktuelle Unendliche," *Zeitschrift für Philosophie und philosophische Kritik,* **91** (1887), 81–125; **92** (1888), 240–65; hereafter Cantor (1887/88) and reprinted in Cantor (1932), 378–439. These articles were later issued together in Cantor's *Gesammelte Abhandlungen zur Lehre vom Transfiniten* (Halle, 1890).

[5]Cantor (1887/88), 399.

[6]Gauss wrote to Schumacher from Göttingen, July 12, 1831. See letter #396 (Gauss' letter #177) in K. F. Gauss, *Briefwechsel zwischen C. F. Gauss und H. C. Schumacher,* ed. C. A. F. Peters (Altona, 1860), II, 269. Gauss went on to explain that by speaking of limits, mathematicians could avoid paradoxical results involving the infinite in an actual sense. Given any sequence with a limit, it was possible to determine elements of the sequence as arbitrarily close to the limit as one liked. Thus 3 is not as close to the true value of π as is 3.14, and 3.14159 is still closer. By adding additional places to the right of the decimal, it is possible to approximate the true value of π as closely as one likes. But Gauss insisted that one could not assume all the terms of the decimal expansion to be given to determine π exactly. To do so would involve an infinite number of terms and thus comprise an actually infinite set of numbers, which Gauss refused to allow in rigorous mathematics.

Cantor believed, on the contrary, that with his newly found transfinite distinctions between the potential and actual infinite, there was no reason to retain the old objections, and that it was possible to answer mathematicians like Gauss, philosophers like Aristotle, and theologians like Aquinas in terms which even they would find impossible to reject. In the process, Cantor was led to consider not only the epistemological problems his new transfinite numbers raised, but to formulate as well an accompanying metaphysics.[7] In fact, Cantor argued convincingly that the idea of the actual infinite was implicitly part of any view of the potential infinite, and that the only reason mathematicians had avoided using the actual infinite was because they were unable to see how the well-known paradoxes of the infinite, celebrated from Zeno to Bolzano, could be understood and avoided.[8] He argued that once the self-consistency of his transfinite numbers was recognized, they could not be refused a place alongside the other accepted but once disputed members of the mathematical family, for example, the irrational and the complex numbers. In fact, one of the best examples Cantor had in mind was the way in which the irrational numbers had come to be accepted as a legitimate part of mathematics. As it turned out, Cantor placed the fate of his entire theory of transfinite numbers upon the acceptability of his theory of irrationals.[9]

2. *Consistency: the Importance of Cantor's Formalism.*—Cantor was explicit in describing the very direct relationship he saw between his

[7] Dauben (1974).

[8] Perhaps the most direct example illustrating a paradox of the infinite is the fact that there are as many even integers as there are even and odd integers together. This can be seen from the correspondence which shows that every even integer can be uniquely matched with exactly one integer, as follows:

$$\begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & & n & \text{(all integers)} \\ | & | & | & | & | & \cdots & | & \cdots \\ 2 & 4 & 6 & 8 & 10 & & 2n & \text{(all even integers)} \end{array}$$

The two sets are said to be equivalent, or equipotent, and the apparent "paradox" depends upon the infinity of both sets. Of equal interest is Bolzano's analysis of "the striking example which was set forth by *Galilei* in his *Discorsi e dimostrazioni matematiche,* no doubt solely for the purpose of stimulating reflexion, to the effect that the circumference of a circle is equal in magnitude to its centre," 46 of B. Bolzano, *Paradoxien des Unendlichen,* ed. F. Přihonsky and photo-reproduced from the first edition of 1851 (Berlin, 1899); trans. D. A. Steele, *Paradoxes of the Infinite* (New Haven, 1950), 140–41; also Cantor (1883), 180.

[9] Irrational numbers are those like 2 and π which cannot be expressed simply as the ratio of two integers a/b. Cantor's theory takes infinite sequences of numbers to define irrational numbers, much as the infinite decimals 1.414. . . or 3.14159. . . may be taken to determine $\sqrt{2}$ or π, respectively. Similarly, Cantor likened his first transfinite number ω to the first number coming after all the *finite* natural numbers 1, 2, 3, . . . , n, . . . , ω, and thus it was defined by an infinite sequence, as were his irrational numbers. Moreover, ω could even be taken to be the *limit* (in a certain sense, as Cantor noted) of the infinite sequence 1, 2, 3, ... For his development of these ideas in the *Grundlagen,* see Cantor (1883), 195.

theory of transfinite numbers and his theory of irrationals:

The transfinite numbers themselves are in a certain sense *new irrationals*, and in fact I think the best way to define the *finite* irrational numbers is entirely similar; I might even say in principle it is the same as my method described above for introducing transfinite numbers. One can certainly say that the transfinite numbers *stand or fall* with the finite irrational numbers; they are alike in their most intrinsic nature (*inneresten Wesen*); for any numbers like these are definite, delineated (*abgegrenzte*) forms or modifications (ἀφωρισμένα) of the actual infinite.[10]

Cantor had first outlined his theory of the irrationals as part of a paper on trigonometric series in 1872. There he had used infinite sequences taken as equivalence classes in order to define the real numbers in general.[11] Later, in describing the operative features of this theory, Cantor recalled that the fundamental sequences were actually infinite sets used to define irrational numbers.[12] Having already characterized the transfinite numbers as direct generalizations or abstractions from actually infinite sets, all Cantor needed was the acceptance of his theory of irrational numbers. By implication, that would involve acceptance of the existence of actually infinite sets, and the acceptability of his transfinite numbers would follow immediately.

But what was the basis upon which the irrational numbers were to be deemed acceptable as a legitimate part of conventional mathematics? Cantor argued forcefully that the only grounds of consequence concerned the *consistency* of the new concepts themselves:

In particular, in introducing new numbers, mathematics is only obliged to give definitions of them, by which such a definiteness and, circumstances permitting, such a relation to the older numbers are conferred upon them that in given cases they can definitely be distinguished from one another. As soon as a number satisfies all these conditions, it can and must be regarded as existent and real in mathematics. Here I perceive the reason why one has to regard the rational, irrational, and complex numbers as just as thoroughly existent as the finite positive numbers.[13]

Logical consistency was the touchstone Cantor applied to any new theory before declaring its concepts "existent" mathematically. Though Cantor had argued the theoretical consistency of his transfinite num-

[10]Cantor (1887/88), 395–96.

[11]Cantor, "Über die Ausdehnung eines Satzes aus der Theorie der trigonometrischen Reihen," *Mathematische Annalen*, **5** (1872), 123–32, hereafter Cantor (1872), rpt. in Cantor (1932), 92–102; and section 9 of Cantor (1883), 183–90. Dauben, "The Trigonometric Background to Georg Cantor's Theory of Sets," *Archive for History of Exact Sciences*, **7** (1971), 181–216, hereafter Dauben (1971).

[12]Cf. Cantor's remarks in a letter to the Italian mathematician Giulio Vivanti, Cantor (1887/88), 410.

[13]Cantor (1883), 182.

bers, he was not content to leave that as their only justification. He had cleverly devised a series of arguments leading directly from the generally accepted irrational numbers to the corresponding mathematical reality of the transfinite numbers as well. Realizing that more mathematicians might find it initially easier to admit the consistency and reality of the irrational numbers, Cantor suggested that it was then only a short step, but a direct and necessary one, to accepting his new transfinite numbers without reservation.

One of Cantor's strongest arguments in support of the legitimacy of his new theory stemmed from his formalist position[14] concerning the nature of mathematical existence. In his treatment of transfinite numbers, as in his theory of irrational numbers, he could not avoid deep concern for matters of ontology and the philosophy of number generally. But from the outset, Cantor faced formidable opposition from a dedicated finitist, Leopold Kronecker,[15] with whom Cantor had studied during his student days in Berlin. Kronecker came to be one of the most ardent foes of transfinite set theory, and he firmly believed that mathematics should ultimately rely upon nothing but finite numbers and only a finite number of arithmetic operations. Like Gauss, Kronecker feared that were the actual infinite allowed, the way was then open to paradoxes which would challenge the certainty of mathematics. This left him irreconcilably opposed to Cantor's theory of the infinite.

3. *The Limitations of Kronecker's Finitism.*—Cantor's *Grundlagen einer allgemeinen Mannichfaltigkeitslehre* (1883), the first comprehensive presentation of his new ideas, was consciously designed, in part, as a reaction to the attitude that mathematicians should admit nothing but

[14]Cantor's brand of formalism was founded upon his belief that mathematical ideas have to be taken as real and fully acceptable parts of mathematics once their internal conceptual *consistency* had been established. As an example, he stressed in the *Grundlagen* that once mathematicians had established the rational, irrational, and complex numbers as being entirely consistent with the rest of established mathematics, they were then compelled to admit the mathematical reality of these new numbers. Similarly, Cantor hoped that the consistency of his transfinite numbers would be taken as confirmation of their reality and legitimacy in mathematics. For further details, Cantor (1883), 181–83, and Dauben (1974).

[15]For consideration of the relationship between Kronecker and Cantor: A. Fraenkel, "Georg Cantor," *Jahresbericht der Deutschen Mathematiker-Vereinigung,* **39** (1930), 198–99, 209–10, hereafter Fraenkel (1930). This biography also appeared separately as *Georg Cantor* (Leipzig, 1930), and in an abridged version in Cantor (1932), 452–83. For the references cited above in Fraenkel (1930) see Cantor (1932), 458–59, 464–66; also Grattan-Guinness (1971), 352–53. Schoenflies felt it was no exaggeration to say that Kronecker regarded Cantor, in his roles as scholar and teacher, as a "corrupter of youth," A. Schoenflies, "Die Krisis in Cantors mathematischem Schaffen," *Acta Mathematica,* **50** (1927), 2. For Kronecker's position concerning the way in which he felt mathematics should build upon finite numbers and operations only, cf. the edition of his collected works by K. Hensel, *Leopold Kronecker's Werke* (Leipzig, 1895–1931), III, 253; also Meschkowski (1967), 67–69, 134–39.

the integers and finite arithmetic combinations of them. This attitude, defended by Kronecker, regarded any concept, theorem, or proof requiring an infinite number of elements, operations, or arguments as inadmissible. In keeping with this rejection of the infinite, Kronecker had come to believe that the only numbers actually existing were the finite whole numbers.[16] There were exceptions. The rational numbers could be given a similar reality, but the irrational numbers should be taken in a purely formal sense as marks or symbols introduced merely for purposes of calculation; they represented nothing more than certain configurations of numbers from the set of natural numbers.[17] The most devastating aspect of Kronecker's program was its refusal to recognize any sort of limiting process in the traditional sense of the infinitesimal calculus. Not only the irrational numbers, but function theory itself would be regarded as legitimate and admissible only insofar as the various theorems were demonstrable in terms of finite numbers of arguments based on the finite whole numbers. Cantor's reservations were clear:

> To this interpretation [Kronecker's] of pure mathematics, though I cannot agree with it, certain advantages are undeniably bound, which I want to emphasize here. The fact that some of the most respected mathematicians of the present ascribe to it also attests to its significance.[18]

Cantor recognized that in terms of the certainty and absolute correctness of any proof, the finitist position was beyond reproach.[19] But it also put many of the most useful and suggestive developments beyond mathematics. Weierstrassian analysis, and certainly the bulk, if not all, of Cantor's own work, was seriously open to question once such a finitist position had been accepted. Therefore there was more to Cantor's combativeness than mere polemic. His reservations reflected not only his own belief in the validity of analysis and his new theory of transfinite numbers, they were as well representative of the struggle to keep the value of his life's work alive.

Cantor argued that the difficulties with Kronecker's position were numerous. Assuming that only the natural numbers be taken as the elements of mathematical investigation, no proof requiring more than a finite number of elements or operations could be deemed rigorous and

[16]Cantor's discussion of Kronecker's views: in section 4 of the *Grundlagen,* Cantor (1883), 172–75.

[17]Kronecker once wrote that if only he lived long enough, he would show that arithmetic could show the way, and in fact the rigorous way, for analysis. Moreover, he claimed that in so doing, arithmetic would reveal all of the errors in the logic of "so-called analysis." For the details of Kronecker's attitude towards both analysis and set theory, cf. the remarks he made in a letter to H. A. Schwarz, Dec. 28, 1884, now part of the *Nachlass Schwarz* of the Akademie der Wissenschaften der DDR, Berlin.

[18]Cantor (1883), 173.

[19]Cantor acknowledged this in the *Grundlagen* explicitly: Cantor (1883), 173.

mathematically acceptable. Any gaps that might occur in analysis during translation into strictly finite terms would either have to be filled with arithmetic arguments alone, or the theorem at hand would have to be discarded. Feasibility was the real criterion, for Cantor, in rejecting Kronecker's entire approach. Cantor was nevertheless generous in his evaluation of the position itself.

It is not to be denied that in this way the proof of many theorems may be perfected and other methodological improvements may also be brought about in various parts of analysis; one also sees a guarantee against any kind of absurdity or error in the observance of the basic principles following from that point of view [meaning Kronecker's].[20]

Cantor was willing to take such certainty of argument as a guiding principle. Finitism ought to serve as a regulator, to dispel the sham of mathematical speculation and conceptual fantasy. In the transcendental there always seemed to be the fear of an "anything possible,"[21] and this would be eliminated by the program of finitism. But Cantor felt that the project as Kronecker had envisioned it was both presumptuous and immoderate. Worst of all, there was no productive principle to follow, no guide that might be applied to the ends of innovative research. If the future of mathematics were to depend upon the finitist demands which Kronecker, for one, was determined to follow, then it could be nothing but barren. The narrowness of Kronecker's vision precluded any expansion of the mathematical horizon. Though present knowledge could be reworked and verified by this finitist revisionism, that was the sole extent of its usefulness. Fortunately, Cantor could say that such stringent and confining rules were rarely taken so literally.

4. *The Nature of Mathematics*.—Throughout his *Grundlagen*,[22] Cantor explained as carefully and as explicitly as he could the justification for his set theory in the face of criticism, especially from Kronecker. It is interesting to see the sort of response Kronecker's hostility had generated, and the uses to which Cantor put new formulations in the attempt to support his theory on strong philosophical grounds. The battle against a strictly finite mathematics demanded a careful presentation of questions ranging from the nature of the new transfinite numbers to their existence.

Cantor reinforced his study of the philosophical status of his new transfinite numbers with a simple analysis of the familiar and accepted whole numbers. Both finite and infinite integers could be considered in essentially two ways. Insofar as they were considered as well-defined in the mind, distinct from all other components of thought, they served in a relational sense to modify the substance of thought itself. This Cantor

[20] Cantor (1883), 173.

[21] *Ibid.*

[22] Section 4 of Cantor (1883), 171–75.

described as their *intrasubjective* or *immanent* reality.[23] In contrast to this immanent reality, numbers could also assume an external reality via objects of the physical world. Cantor explained further that this second sort of reality of the whole numbers arose from their reality as expressions or images of processes in the realm of physical phenomena. This aspect of the whole numbers, whether finite or infinite, Cantor termed *transsubjective* or *transient*.[24]

Cantor claimed reality for both the physical and ideal halves of his approach to the number concept. The dual realities, in fact, were always found in a joined sense, insofar as a concept possessing an immanent reality always possessed a transient reality as well.[25] It was one of the most difficult problems of metaphysics to find the determining features of the connection between the two kinds of reality. Cantor ascribed the coincidence of the dual aspects of the reality of numbers to the unity of the universe itself.[26] This left mathematicians with the important result that it was possible to study only the immanent realities, without having to confirm or conform to any subjective content. This placed mathematics apart from all other sciences for which natural phenomena provided objective, physical objects and events for scientific scrutiny. It gave mathematics an independence that was to imply great freedom for mathematicians in the creation of mathematical concepts. It was on these grounds that Cantor offered his now famous dictum that the essence of mathematics is its freedom. As Cantor put it in the *Grundlagen:*

> Because of this extraordinary position which distinguishes mathematics from all other sciences, and which produces an explanation for the relatively free and easy way of pursuing it, it especially deserves the name of *free mathematics,* a designation which I, if I had the choice, would prefer to the now customary "pure" mathematics.[27]

Mathematics was therefore absolutely free in its development, and bound only to the requirement that its concepts permit no internal contradiction, but that they follow in definite relation to previously given

[23]Cantor distinguished between the "*intrasubjektive* oder *immanente Realität*" of numbers as they assumed a definite place in one's understanding through definitions, and the "*transsubjektive* oder auch *transiente Realität* der ganzen Zahlen," as they represented in the mind processes and relations in the external world. Cantor further specified the meaning he attached to "intrasubjektive" reality as follows: "What I here call the 'intrasubjektive' or 'immanente' reality of concepts or ideas ought to agree with the meaning of 'adequate' in the sense that this word is used by Spinoza when he says: Eth. pars II def. IV, 'Per ideam adaequatam intelligo ideam, quae, quatenus in se sine relatione ad objectum consideratur, omnes verae ideae proprietates sive dominationes intrinsecas habet,' " Cantor (1883), 206. Cantor explains his distinction between "immanent" and "transient" realities in section 8 of the *Grundlagen,* Cantor (1883), 181–83; J. Dauben (1974).

[24]Cantor (1883), 181. [25]*Ibid.* [26]Cantor (1883), 182.

[27]*Ibid.*

definitions, axioms, and theorems. On these grounds, what were the criteria for introducing new numbers? The matter rested entirely upon the consistency of definitions. So long as new numbers were distinct and could be distinguished from other kinds of number, as well as from each other, then a new number was defined, and must be taken as existing. Earlier in the *Grundlagen* Cantor had already prepared the way for this more abstract, philosophical argument by offering an example. The acceptability of rational, irrational, and complex numbers as existing and proper in mathematics was due to nothing more than the way in which they conformed to certain specified criteria.[28] Any individual number, taken arbitrarily, was unique and distinct from all others. In conjunction with other elements it was part of a larger, mathematically consistent system. These were the fundamental requirements, Cantor claimed, which any new sort of number had to satisfy before it could be considered as real and existing mathematically. In similar terms, Cantor was to argue for the acceptability of his transfinite numbers.

The only possible objection to Cantor's doctrine of freedom in mathematics, as he saw it, concerned the possible indiscriminate creation of new ideas. But there were, he insisted, correctives nevertheless. If an idea was fruitless or unnecessary, this quickly became evident, and by reason of failure, was abandoned or forgotten.[29] The alternative, supported by Kronecker and his followers, was a very dangerous one from Cantor's viewpoint. Any restriction or narrowness in mathematics would have direct and obvious consequences. Controls and artificial philosophical presuppositions retarded, or worse, prevented any growth of mathematical knowledge.

Cantor appealed to the great, legendary figures of the history of mathematics to lend support to his contention. Without the freedom to construct new ideas and connections in mathematics, Gauss, Hermite, and Riemann would never have made the significant advances they did. Kummer would never have been able to formulate his ideal numbers, and consequently the world would be in no position, Cantor added with a note of cunning, to appreciate the work of Kronecker and Dedekind.[30] Thus, strictly in terms of his philosophy of mathematics, Cantor was convinced that time would eventually corroborate his belief that transfinite set theory was perfectly consistent, rigorous, and acceptable as part of mathematics. Just as the irrational and complex numbers had found their way into the corpus of approved mathematical entities, Cantor was certain that his transfinite numbers would likewise find their place.

But Cantor also believed in the acceptability of his work for reasons that he never emphasized in print. There was another, very different dimension to his work that has never received much notice, but one which

[28] *Ibid.* [29] *Ibid.* [30] Cantor (1883), 183.

was indispensable to his conviction that his research was legitimate, even absolutely true, despite the steadfast opposition of mathematicians like Kronecker. In order to appreciate Cantor's deeper reasons for faith in the ultimate correctness of transfinite set theory, it is first necessary to consider a more personal side of Cantor's mathematics. Though nowhere in any of his published work did he ever discuss his private views in any detail, he did refer on occasion to such matters in his correspondence. One of the best examples is a letter Cantor drafted to his friend, the French mathematician Charles Hermite.

5. *Cantor's Letter to Hermite: Mathematics and God.*—In November 1895 Cantor wrote to Hermite, in part to explain several basic aspects of his philosophy of mathematics.[31] Though the original is now lost, the letter survives in a draft version, and is of interest because Cantor devoted nearly two pages to epistemological questions concerning the nature of numbers and their mode of existence. Hermite had expressed his own opinion on such matters in an earlier letter:

> The [whole] numbers seem to me to be constituted as a world of realities which exists outside of us with the same character of absolute necessity as the realities of nature, of which understanding is given to us by our senses.[32]

Cantor went even further. He insisted that the reality and absolute legitimacy of the natural numbers was much greater than any based on their existence in the real world and perceived through one's senses. He explained to Hermite that this was so for one very simple reason. Both separately and collectively as an infinite totality, the natural numbers "exist at the highest level of reality as eternal ideas in the *Divine Intellect* (*Intellectu Divino*)."[33] Cantor added that this came to him as no new revelation. In fact, he had expressed views similar to Hermite's at the very outset of his career, in 1869, in one of three theses at the end of his *Habilitationsschrift*. The third of these (which he publicly defended) reads as follows: "Integer numbers constitute a *unity* composed of laws and relations *in a manner similar to those of celestial bodies*."[34]

Years later, Cantor said he realized that Saint Augustine had expressed similar thoughts in his *De civitate Dei;* he even went so far as to footnote an entire excerpt in one of his later publications, specifically part of Augustine's Chapter 19 of Book XII: "Contra eos, qui dicunt ea, quae infinita sunt, nec Dei posse scientia comprehendi."[35] In attempting

[31]Cantor to Hermite, Nov. 30, 1895, in Meschkowski (1967), 262–63.

[32]*Ibid.* Hermite's original is unknown. It does not appear among the collection of papers and manuscripts of Cantor's surviving *Nachlass*. Meschkowski (1967), 262.

[33]*Ibid.*

[34]"Numeros integros *simili modo atque corpore coelestia totum quoddam* legibus et relationibus compositum efficere," G. Cantor, *De transformatione formarum ternariarum quadraticarum* (Halle, 1869), in Cantor (1932), 62. The underlining is based upon Cantor's letter to Hermite, in Meschkowski (1967), 262.

[35]Cantor to Hermite, Nov. 30, 1895, in Meschkowski (1967), 262. Cantor reproduced the passage from St. Augustine in a lengthy note to section 5 of his "Mitteilungen," Cantor (1887/88), 401.

to counter a variety of views opposed to the existence of the infinite, Cantor concluded that the transfinite numbers were just as possible and existent as the finite numbers: "Therefore the transfinite species are just as much at the disposal of the intentions of the Creator and His absolute boundless will as are the finite numbers."[36]

Thus the efficacy of Cantor's theory was ultimately referred to the Divine Intellect where the Transfinitum, all the transfinite numbers, existed as eternal ideas. This was a strong form of Platonism, but one to which Cantor repeatedly turned for support. The religious connections were as significant to him as the mathematical ones, and in the final analysis, Cantor could only be sure of the propriety of his abstractions because they found their ideal representation in the mind of God. Hermite placed the reality of number in the concrete and absolute reality he found in the perceptual world. Cantor needed to go much further, and found his most important source of inspiration and reassurance in God's infinite capabilities.[37]

The letters Cantor wrote to Roman Catholic clerics and theologians profoundly reveal the importance he attributed to the inextricable bond he felt between his mathematical ideas and his theological beliefs. In 1888, for example, he wrote to the Neo-Thomist priest Ignatius Jeiler.[38] Cantor was anxious that his view of the infinite be carefully studied by the Roman Catholic Church in order that it might be taken constructively and not be interpreted as being at variance with Church doctrine. Specifically, he was worried that unless the Church took notice of the implications of his research concerning the nature of infinity, it might fall into dangerous errors of interpretation and doctrine:

> In any case it is necessary to undertake a serious examination of the latter question concerning the truth of the Transfinitum, for were I correct in asserting its truth in terms of the possibility of the Transfinitum, then there would be (without doubt) a certain danger of religious error for those of the opposite opinion, since: *error circa creaturas redundat in falsam de Deo scientiam* (*Summa contra Gent.* II,3).[39]

Cantor thus emerges as a modern Galileo. Both felt it was their duty to make the Church accept the reality of a world God had made, and to

[36]Cantor (1887/88), 404.

[37]*Ibid.*, 405–06.

[38]J. Bendiek, "Ein Brief Georg Cantors an P. Ignatius Jeiler, O.F.M.," *Franziskanische Studien*, **47** (1965), 65–73, hereafter Bendiek (1965). Something must be said of Cantor's own religious position. He was baptized an Evangelical Lutheran, and given very strict instruction by his father in matters of religion. But his mother was a Roman Catholic, and later in life Georg explained his interest in problems of Catholic theology by reference to his mother's Catholicism. But in 1896 Cantor could declare that he belonged to no denomination, nor counted himself a part of any organized or formal church. H. Meschkowski, "Aus den Briefbüchern Georg Cantors," *Archive for History of Exact Sciences*, **2** (1965), 515, hereafter Meschkowski (1965); also Meschkowski (1967), 122–28, and Grattan-Guinness (1971), 371.

[39]Cantor to Jeiler, Pfingsten, 1888, in Bendiek (1965), 65–73.

face the fact that man had been given the capacity to understand it. The Church could only come to ruin if it stubbornly ignored the necessary logic of the reasoning mind. Like Galileo, Cantor was certain that there could be no doubt of the correctness of his theory once it had been carefully studied and understood.

However, before it is possible to appreciate the response of Catholic intellectuals and theologians to Cantor's provocative and challenging theory of the mathematical infinite, it is necessary to understand the new climate of interest in scientific ideas generally aroused and encouraged by Pope Leo's attempt to promote a revival of Thomistic philosophy, which he did in his encyclical *Aeterni Patris*, issued in 1879, just five years before the first major publication of Cantor's transfinite set theory in 1883.

6. *Pope Leo XIII and the Encyclical* AETERNI PATRIS.—Pope Leo XIII's interest in reassessing the directions of Christian philosophy, in attempting to reconcile the new and often perplexing discoveries of science with scripture and authorities of the Church, encouraged a number of Catholic intellectuals to study the various branches of natural science in detail.[40] By virtue of one of his most influential encyclicals, *Aeterni Patris*,[41] Leo XIII was to arouse interest in scientific ideas generally, and from one quarter at least, interest in Cantor's mathematics in a very direct, if somewhat surprising way.

[40]The best short introduction to Leo XIII's place in modern European history is in C. J. H. Hayes, *A Generation of Materialism* (New York, 1940), esp. 141–48. Leo XIII's concern for science was indicated by his support of a well-trained staff of physicists and the purchase of the most modern equipment available for the astronomical observatory at the Vatican. As for the problems science posed to religion, Darwin's theory of evolution was a major concern. It was an obvious target of Leo XIII's appeals for Christian philosophers to reassess the foundations and principles of modern science, particularly when it seemed ready to support materialism and atheism. Cf. P. J. Muldoon, *The Great White Shepherd of Christendom: His Holiness Pope Leo XIII* (Chicago, 1903), Ch. 27, 404. Indicative is Thesis 10 from the "Index Thesium" of T. Pesch, *Institutiones Philosophiae Naturalis* (Freiburg, 1897), vol. 1: "Th. 10: Apparet in rebus naturalibus veri fines sive *causae finales;* unde *systema Darwini,* adeo hisce temporibus celebratum, falsum esse legitima conclusione infertur." Cf. W. Ong, *Darwin's Vision and Christian Perspectives* (New York, 1960); H. de Dorlodot, *Le Darwinisme au Point de Vue de l'Orthodoxie Catholique* (Brussels, 1921), trans E. Messenger, *Darwinism and Catholic Thought* (London, 1922). For additional studies of Leo XIII: J. Galland, *Papst Leo XIII. Festschrift zum goldenen Priester-Jubiläum des h. Vaters* (Paderborn, 1880), hereafter Galland (1880); B. O'Reilly, *Life of Leo XIII* (New York, 1887).

[41]The best commentary on *Aeterni Patris* is still by F. Ehrle, *Zur Enzyklika "Aeterni Patris"; Text und Kommentar,* edited in a later edition of F. Pelster (Rome, 1954); A. Alexander, "Thomas Aquinas and the Encyclical Letter," *The Princeton Review,* 7 (March, 1880), 245–61. For English translations: "The Study of Scholastic Philosophy. *Aeterni Patris*. August 4, 1879," *The Great Encyclical Letters of Pope Leo XIII,* J. J. Wynne, ed. (New York, 1903), hereafter Wynne (1903). A conveniently available translation of *Aeterni Patris,* annotated by E. Gilson, is in J. Maritain, *St. Thomas Aquinas* (New York, 1958), hereafter Maritain (1958).

Pope Leo XIII had been educated by Jesuits in Viterbo before elevation to the Archbishopric of Perugia,[42] an office he held from 1846 until 1878. It was under the influence of his brother Giuseppe, a Jesuit professor at the local seminary, that Leo XIII became interested in Thomistic philosophy and established in 1859 an Academy of St. Thomas. As Archbishop, Leo XIII issued an important series of pastoral letters in the years 1876-77 devoted to matters of the Church and civilization, in which he argued that the Church had to enter the current of modern times, or be left behind.[43] To Leo XIII, science was one of the most immediate representatives of modernity. But his pastoral letters were only a presentiment of the encyclicals to follow.

Leo XIII was meant to be a transitional Pope, but his Pontificate persisted for twenty-five years. Among the first of his pronouncements was the Encyclical, *Aeterni Patris*, delivered on August 4, 1879. Urged by his brother, by then Cardinal Giuseppe Pecci, Leo XIII sought renewal of philosophical thought along the lines of a new Thomism.[44] Seen as an opposition to political and social liberalism, Leo XIII's new program included attempts to revitalize and modernize the thought of the Church by reorganizing the Academy of St. Thomas, and by nominating Désiré Mercier (later to become one of the prominent figures of the Neo-Thomist movement), to a chair of Thomism at the University of Louvain. In fact, the *Institut Supérieur de Philosophie* at Louvain was organized by Mercier under the auspices of the Pope himself.[45]

The basic position of the Neo-Thomists may be characterized succinctly, if somewhat too simply, by their conviction that contemporary evils were the result of false philosophy. Improper or incorrect views of nature, they held, resulted in two consequences: atheism and materialism. Since the time of Hobbes, science had been repeatedly charged with these two undesirable offspring, but rarely was science then called upon to fill the gap and rebuke the dreaded Leviathans of

[42]For details concerning the revival of Thomism in the nineteenth century: B. M. Bonansea, "Pioneers of the Nineteenth Century Scholastic Revival in Italy," *The New Scholasticism*, **27** (1954), 1–37; J. Collins, "Leo XIII and the Philosophical Approach to Modernity," *Leo XIII and the Modern World*, E. T. Gargan, ed. (New York, 1961), 181–209; J. L. Perrier, *The Revival of Scholastic Philosophy in the Nineteenth Century* (New York, 1909), hereafter Perrier (1909); P. Wyser, "Der Thomismus," *Bibliographische Einführungen in das Studium der Philosophie*, **15/16** (1951), esp. section 40: "Die Thomistische Philosophie des 19. und 20. Jahrhunderts," 46–53; and H. J. John, *The Thomist Spectrum* (New York, 1966).

[43]Leo XIII, *La Chiesa e la Civilità; lettera pastorale per la quaresima 1877, diocesi di Perugia* (Perugia, 1877), trans. *The Church and Civilization. Pastoral Letters for Lent 1877–1878* (New York, 1878).

[44]Perrier (1909), esp. ch. 9.

[45]Hugon, É, "Les Services Rendus à la Cause Thomiste par son Éminence le Cardinal Mercier," *Revue Thomiste*, **7** (July–Aug., No. 28, 1924), 333–39; L. de Raeymaeker, *Le Cardinal Mercier et l'Institut Supérieur de Louvain* (Louvain, 1952).

atheism and materialism on its own grounds. *Aeterni Patris* was clear on one point in particular: that science could profit from scholastic philosophy, and in the process further the ideals and goals of the Church itself.

For, the investigation of facts and the contemplation of nature is not alone sufficient for their profitable exercise and advance; but, when facts have been established, it is necessary to rise and apply ourselves to the study of the nature of corporeal things, to inquire into the laws which govern them and the principles whence their order and varied unity and mutual attraction in diversity arise. To such investigations it is wonderful what force and light and aid the Scholastic philosophy, if judiciously taught, would bring.[46]

In dealing with the metaphysics of science, Neo-Thomism was expected to guide the human understanding of the natural and spiritual world in the proper and unobjectionable direction prescribed by the Church, specifically along the lines defined implicitly in the course of *Aeterni Patris*. But the program Leo XIII envisioned was less a subordination of philosophy and Thomistic teachings to science than an attempt to show how science ought to proceed and reconcile itself with the true principles of Christian philosophy. On April 21, 1878, in his first Papal Encyclical, *Inscrutabili*, Leo wrote:

Above all [education] must be wholly in harmony with the Catholic faith in its literature and system of training, and chiefly in philosophy, upon which the foundation of other sciences in great measure depends.[47]

But there was opposition to Leo XIII's efforts to commend the study of St. Thomas as the epitome of true philosophy. Worse yet, there was resistance among the clergy in Rome. As a result, it was suggested that a free course on Thomism be established; thus a priest with a high reputation as both a scientist and a philosopher, Father Cornoldi, was imported from Bologna to establish Thomism in the Papal city.[48] Cornoldi went so far as to promise that "in the *Summa Theologica* was to be found the key to all the difficulties of modern science."[49] In stressing the harmony between Thomism and science, Cornoldi paved the way for renewed interest among churchmen in the affairs of science. Shortly after Cornoldi had begun his lectures in Rome, the encyclical *Aeterni Patris* was issued.

The impetus Leo XIII gave to scholarly and scientific study among intellectuals of the Church cannot be overestimated. Of special significance was the interest his encyclical generated among Germans who were to become interested in reconciling Cantor's work on absolute infinities with the doctrines of Catholicism. Of particular importance in this respect was Constantin Gutberlet.

[46]Leo XIII, *Aeterni Patris,* Maritain (1958), 207.

[47]Leo XIII, *Inscrutabili,* trans. "On the Evils Affecting Modern Society. *Inscrutabili.* April 21, 1878," Wynne (1903), 9–21.

[48]Perrier (1909), 161–64.

[49]Cornoldi, quoted in Perrier (1909), 162.

7. *Gutberlet and the Appreciation of Cantor's Work in Germany.*—Constantin Gutberlet[50] studied philosophy and theology at the Roman College from 1856 to 1862, visiting simultaneously at the German University in Rome. From 1862 to 1865 he lectured on the natural sciences at the seminary in Fulda, of which he was later made director. In 1866 he became a professor of philosophy, apologetics, and dogma, and in 1888 he founded the review journal, *Philosophisches Jahrbuch der Görres-Gesellschaft,* a leader in the dissemination of Neo-Scholastic thought. But of special significance was an article he published in 1886 drawing upon Cantor's set theory in a defense of his own views on the theological and philosophical nature of the infinite.[51]

Gutberlet realized that the study of infinity had entered a new phase with the appearance of Cantor's mathematical and philosophical studies. The question uppermost in Gutberlet's mind concerned the challenge of mathematical infinity to the unique, absolute infinity of God's existence. As it turned out, correspondence over this question increasingly encouraged Cantor to consider the theological aspects of his theory of the transfinite numbers. It was Cantor's claim that instead of diminishing the extent of God's nature and dominion, the transfinite numbers actually made it all the greater.[52]

It is not important to follow the details of Gutberlet's article. It is useful to know only that it was, in its broadest outline, an attempt to reinstate and support the arguments he had advanced several years earlier on the existence of the actual infinite. Gutberlet's ideas had been attacked severely by another German, Caspar Isenkrahe.[53] Isenkrahe argued that the actual infinite was contradictory and consequently any attempts to support it were necessarily hopeless. The discussion was couched in terms of the infinite duration and eternity of the world, and raised objections to the actual infinite made by thinkers as diverse in time as Thomas and Herbert.[54] Gutberlet presented Cantor's work in order to support his own claims in the face of opposition from other theologians of his day, such as Isenkrahe. Gutberlet hoped that a quick

[50]For biographical data about Gutberlet: Perrier (1909), 199.

[51]C. Gutberlet, "Das Problem des Unendlichen," *Zeitschrift für Philosophie und philosophische Kritik,* **88** (1886), 179–223, hereafter Gutberlet (1886).

[52]Cantor's letter to Gutberlet, Jan. 24, 1886, in Cantor (1887/88), 396–98.

[53]C. Isenkrahe, "Das Unendliche in der Ausdehnung. Sein Begriff und seine Stützen," *Zeitschrift für Philosophie und philosophische Kritik,* **86** (1885), 73–111; 145–98.

[54]For some indication of the context and flavor of the arguments on infinite duration, cf. the first pages of section 2 of Gutberlet (1886), 195–98. The problem of eternity and the infinity of space is also discussed by N. Y. S. Désiré, "La Nature de l'Espace d'après les Théories Modernes depuis Descartes," *Mémoires, Académie Royale de Belgique: Classe des Lettres et des Sciences morales et politiques et Classe des Beaux-Arts III* (1908); R. P. Phillips, *Modern Thomistic Philosophy* (London, 1941), esp. 164–72; T. Greenwood, "La nature du transfini," *Revue de l'Université d'Ottawa,* **14** (1944), 109–34, **15** (1945), 147–85.

recapitulation of Cantor's mathematics would be sufficient to allow any reader to decide

> if they were correct, when they supposed they could dispose of my theory of actually infinite magnitude [Grösse] so easily. Above all we [Gutberlet] now want to explain the Cantorian theory and then to defend our conception against criticism, which this journal published, with Cantor's corresponding interpretation of infinite magnitude.[55]

The use Gutberlet made of Cantor's ideas is of some interest. Particularly when it came to defending the existence of the absolute infinite, Gutberlet used a ploy which Cantor had not publicly used in any of his writings, one which was reminiscent of Berkeley's use of God as a guarantor of the reality of the external world. In short, Gutberlet argued that God Himself insured the existence of Cantor's transfinite numbers.

> But in the absolute mind the entire sequence is always in actual consciousness, without any possibility of increase in the knowledge or contemplation of a new member of the sequence.[56]

God could similarly be called upon to insure the ideal existence of infinite decimals, the irrational numbers, the true and exact value of π, and so on. God was also able to solve the problem of the continuum hypothesis;[57] at the same time God insured the concreteness and objectivity of the cardinal number representing the collection of all real numbers. At one point, Gutberlet even argued that since the thought ascribed to God was believed to be unchanging, then the collection of God's thoughts must comprise an absolute, infinite, complete, closed set. Again Gutberlet offered this as direct evidence for the reality of concepts like Cantor's transfinite numbers. Either one assumed the existence and reality of the actual infinite, or one was obliged to give up the infinite intellect and eternity of the absolute mind of God.

Thus Gutberlet called upon Cantor's analysis of the infinite to support further his own theological use of actually infinite numbers. In the process he encouraged Cantor's interest in the philosophical and theological aspects of his work. That Gutberlet was prepared to argue the objective possibility of the transfinite numbers on the basis of the infinite intellect of God must have appealed to the mind of a religious man like Cantor.[58] It was also a complimentary approach to Cantor's own

[55] Gutberlet (1886), 180 [56] *Ibid.*, 206.

[57] The continuum hypothesis is Cantor's celebrated conjecture that the total number of all real numbers is given by the second of his transfinite cardinal numbers $\aleph_1$. Algebraically the continuum hypothesis claims that $2^{\aleph_0} = \aleph_1$. For details: K. Gödel, "What is Cantor's Continuum Problem?" *American Mathematical Monthly*, **54** (1947), 515–25, reprinted in a revised and expanded version in P. Benacerraf and H. Putnam, eds., *Philosophy of Mathematics, Selected Readings* (New Jersey, 1964), 258–73.

[58] See the analysis of Cantor's religious sentiments in Meschkowski (1967), 122–29, and Meschkowski (1965), 514–15.

Platonism, in which the legitimacy of the actual infinite was conceived as existing in at least the immanent world of the mind in terms of the consistent forms of reason alone.[59]

8. *Infinity, Pantheism, and the Response of the Neo-Thomists.*—Though Gutberlet was one of Germany's leading Neo-Thomists[60] he was by no means the only example of a philosopher of the Church who was interested in Cantor's mathematics. Cantor counted among his correspondents Tillman Pesch, Thomas Esser, Joseph Hontheim, and Ignatius Jeiler. All were closely engaged with the revival of scholastic philosophy in the spirit of *Aeterni Patris.* Pesch[61] and Hontheim[62] were associated with a group of Jesuits at Maria-Laach, where a series of important contributions to Neo-Thomism were published under the title *Philosophia Lacensis*. Pesch, in a work which attempted to describe the foundations of a Thomistic cosmology, the *Institutiones Philosophiae Naturalis,* was chiefly concerned with the progress of science in the nineteenth century.[63] Hontheim, writing for the same series, published a work on mathematics and logic in 1895: *Der logische Algorithmus*. Jeiler[64] was a prominent figure among another group of Neo-Thomists at Quaracchi, near Florence, and at Leo XIII's commission, he undertook a new edition of the works of St. Bonaventura as a result of *Aeterni Patris*.

Equally representative of the Church's interest in Cantor's mathematics was Thomas Esser, a Dominican in Rome. Apparently

[59]Gutberlet (1886), 206: "Aber im absoluten Geiste ist immerdar die ganze Reihe im actualen Bewusstsein," and 207: "Kennt der absolute Geist alle Ziffern der Zahl in seiner Erkenntnis und also eine solche objective gegeben."

[60]Perrier (1903), 199. For specific treatment of Neo-Thomism in Germany: T. Wehofer, "Die geistige Bewegung im Anschluss an die Thomas-Encyclica Leo XIII vom 4. August, 1879," *Vorträge und Abhandlungen herausgegeben von der Leo-Gesellschaft,* 7 (Vienna, 1897) 1–26; B. Jansen, "The Neo-Scholastic Movement in Germany," ch. VIII in S. J. Zybura, ed., *Present-Day Thinkers and the New Scholasticism* (London, 1926), 250–75.

[61]T. Pesch, *Institutiones philosophiae naturalis secundum principia S. Thomae Aquinatis* (Freiburg in Breisgau, 1880); Perrier (1909), 198.

[62]J. Hontheim, *Institutiones theodicaeae, sive Theologiae naturalis secundum principia S. Thomae Aquinatis ad usum scholasticum* (Freiburg in Breisgau, 1893), and *idem, Der logische algorithmus in seinem Wesen, in seiner Anwendung und in seiner philosophischen Bedeutung* (Berlin, 1895); Perrier (1909), 198; J. Ternus, "Ein Brief Georg Cantors an P. Joseph Hontheim, S.J.," *Scholastik,* **4** (1929), 561–71, hereafter Ternus (1929).

[63]However, it appears that Pesch tried to reconcile each of two groups at Rome's Gregorian University, one before and one after the promulgation of *Aeterni Patris:* "On ne sait toutefois comment il les concilie; son traité est d'ailleurs moins une oeuvre de synthèse qu'un catalogue de solutions dressé suivant les méthodes rigoureuses de la science allemande," G. van Riet, *L'Épistémologie Thomiste* (Louvain, 1946), 121.

[64]Jeiler, *S. Bonaventurae principia de concursu Dei generali ad actiones causarum secundarum collecta et S. Thomae doctrina confirmata* (Ad Claras Aquas, 1897); Galland (1880), 150.

Esser represented a group of Dominicans who were engaged in a careful study of the theological implications of Cantor's work. In 1896 Cantor described their project in a letter to Jeiler.

Now everything concerning this question (I tell you this in confidence) will of course be examined by the Dominicans in Rome, who are conducting a scholarly [*wissenschaftliche*] correspondence with me about it which will be directed by Father Thomas Esser, O.Pr.[65]

An important concern of the Catholic intellectuals who knew of Cantor's work involved the question of whether the transfinite numbers could be said to exist *in concreto*. Gutberlet was always clear that he differed fundamentally with Cantor on the matter, admitting the actual infinite as a "possible," and even real in the immanent, non-physical dimensions of God's mind.[66] But Gutberlet, like his teacher Cardinal Franzelin,[67] denied the possibility of a concrete, objective Transfinitum for reasons that make the Church's concern for Cantor's work understandable.

Cardinal Johannes Franzelin, a leading Jesuit philosopher and Papal theologian to the Vatican Council, responded to Cantor's belief that the Transfinitum existed in *natura naturata* by explaining that it was a dangerous position to hold. Franzelin held that any belief in a concrete Transfinitum "could not be defended and in a certain sense would involve the error of Pantheism."[68] Pantheism had long been anathema to the Roman Church, but was only condemned formally in 1861 by decree of Pius IX.[69] Spinoza, a philosopher Cantor had studied carefully, used the *natura naturans/natura naturata* distinction in a form similar to that of his heretical forerunner Giordano Bruno.[70] Both had been led to advocate a monistic philosophy of substance identifying God with the natural world. The question of the infinite was an easy touchstone identifying pantheistic doctrines. Any attempt to correlate God's infinity with a concrete, temporal infinity suggested Pantheism. Thus infinite space and infinite duration, in both of which the infinite

[65]T. Esser, *Die Lehre des hl. Thomas von Aquino über die Möglichkeit einer anfangslosen Schöpfung* (Münster, 1895); Cantor's letter to Esser is in Bendiek (1965), 65–73.

[66]Gutberlet, "Rezension von A. Fraenkel: Einleitung in die Mengenlehre," *Philosophisches Jahrbuch der Görres-Gesellschaft,* **32** (1919), 364–70; for the problem of the existence of the actual infinite *in concreto:* Cantor's letter to Gutberlet of Jan. 24, 1886, Cantor (1887/88), 396–98.

[67]For Franzelin's position: Cantor (1887/88), 385–86.

[68]Franzelin's letter to Cantor in Cantor (1887/88), 385.

[69]Pius IX, *Die Encyclica Papst Pius IX vom 8. Dez. 1864* (*Syllabus Errorum*) (Stimmen aus Maria-Laach, Freiburg in Breisgau, 1866–69).

[70]Cantor discussed such matters, for example, in Cantor (1886), 375; (1887/88), 385–87, 399–400; cf. his note to section 4 of the *Grundlagen,* Cantor (1883), 205, and H. Brunnhofer, *Giordano Bruno's Weltanschauung und Verhängnis* (Leipzig, 1892).

was predicated of objects in the natural world, were held to be inadmissible on theological grounds. Any actual infinity *in concreto,* in *natura naturata,* was presumably identifiable with God's infinity, in *natura naturans*. Cantor, by arguing his actually infinite transfinite numbers *in concreto,* seemed to be aiding the cause of Pantheism.

These were the grounds upon which Gutberlet and Franzelin objected to Cantor's new, concrete infinities. Cantor, however, was able to add a distinction that was to satisfy at least some theologians, and in particular Franzelin. On Jan. 22, 1886, he wrote to the Cardinal and explained that in addition to differentiating between the infinite in *natura naturans* and in *natura naturata,* he further distinguished between an "Infinitum aeternum increatum sive Absolutum," reserved for God and his attributes, and an "Infinitum creatum sive Transfinitum," evidenced throughout created nature and exemplified in the actually infinite number of objects in the universe.[71] Cantor added that the important difference between absolute *infinitum* and actual *transfinitum* should not be forgotten. Cantor's clarifications turned Franzelin's reluctance into an imprimatur of sorts, when Franzelin chose to endorse Cantor's distinctions as follows:

> Thus the two concepts of the Absolute-Infinite and the Actual-Infinite in the created world or in the *Transfinitum* are essentially different, so that in comparing the two one must only describe the former as *properly infinite*, the latter as improperly and equivocally infinite. When conceived in this way, so far as I can see at present, there is no danger to religious truths in your concept of the *Transfinitum*.[72]

Cantor was always proud of the acceptance his new theory had found in the estimation of Franzelin, and would frequently remind his friends in the Church, through correspondence, that he had been assured on the Cardinal's authority that the theory of transfinite numbers posed no theological threats to religion.[73] In fact, Cantor believed that the real existence of the Transfinitum further reflected the infinite nature of God's existence. Cantor even devised a pair of arguments from which the existence of transfinite numbers *in concreto* could be deduced on both *a priori* and *a posteriori* grounds. *A priori,* the concept of God led directly on the basis of the perfection of God's being to the possibility and necessity of the creation of a Transfinitum.[74] Approaching the same conclusion with *a posteriori* arguments, Cantor believed that the assumption of a Transfinitum in *natura naturata* followed because the complete explanation of natural phenomena was impossible

[71]Cantor's letter to Franzelin, Cantor (1887/88), 399–400.

[72]Franzelin's response was included as part of the introduction to Cantor (1887/88), 385–86.

[73]Cantor's letter to Hontheim, Dec. 21, 1893, in Ternus (1929), 570.

[74]Cantor to Franzelin, letter of Jan. 22, 1886, in Cantor (1887/88), 400.

on exclusively finite assumptions.[75] Either way, Cantor felt he had demonstrated the necessity of accepting the Transfinitum *in concreto,* and he was not reluctant to call upon the nature and attributes of God in order to do so.

9. *Christian Philosophy and its Importance for Cantorian Set Theory.*—Cognizance of Cantor's infatuation with Catholic theology and the application of set theory to a revision of Thomistic philosophy does not explain why he was so avid in pursuing the theological possibilities his work might suggest. Certainly his motives, both psychological and logical, were complex. But some explanation is necessary in order to understand why Cantor should have corresponded at length with ranking members of the Roman Catholic Church, including the Pope, and why he even sought at one point to abandon mathematics entirely in order to teach philosophy instead. To do so it is first necessary to appreciate in more detail the nature and extent of opposition to the early work Cantor published relating to set theory and to his transfinite numbers.

Despite the recognition Cantor's work has received in the twentieth century, his own contemporaries for the most part were either uninterested or openly hostile. Leading the opposition, as noted above, was the Berlin mathematician, Kronecker. Not only was he adamantly opposed to Cantor's new theory of the infinite, but at one point, in 1877, Kronecker even tried to prevent the publication of a paper Cantor had written which showed that spaces of different dimension could be uniquely matched in a one-to-one correspondence regardless of the dimensions involved.[76] Only after intervention on Cantor's behalf by Weierstrass was the result eventually published.[77] But the entire episode was a dramatic signal that Cantor's work could not expect to find either easy or sympathetic acceptance.

Much has been written concerning the years 1884–85 as constituting some sort of watershed in Cantor's life.[78] Within the space of three years he published his first major defense of the theory of transfinite

[75]Cantor (1887/88), 400.

[76]Cantor, "Ein Beitrag zur Mannichfaltigkeitslehre," *Journal für die reine und angewandte Mathematik,* **84** (1878) 242–58, in Cantor (1932), 119–33, was ready for publication and sent to the editors of Crelle's *Journal* on July 12, 1877, but it did not appear immediately. For details concerning Kronecker's part in trying to prevent the paper's publication: Fraenkel (1930), 10; Meschkowski (1967), 40. Cf. references by Cantor to the matter in letters to Dedekind, esp. that of Oct. 23, 1877, in the edition of their correspondence by J. Cavaillès and E. Noether, *Briefwechsel Cantor-Dedekind* (Paris, 1937), 40.

[77]In a lecture to a small group in Braunschweig, 1897, Cantor recalled that Weierstrass had interceded on Cantor's behalf to urge publication of the paper: Fraenkel's references to notes of the lecture taken by Staeckel, but now lost, in Fraenkel (1930), 265–66; in Cantor (1932), 458.

[78]Esp. Schoenflies (1927).

numbers,[79] finished a series of articles dealing with linear point sets,[80] established his continuum hypothesis, refuted it, and finally admitted that no progress had been made.[81] It was also during this period that Cantor experienced a serious nervous breakdown, the first of many which were to plague him with increasing severity for the rest of his life.[82]

Throughout these difficult years one of the few important influences Cantor felt was that of the Swedish mathematician Mittag-Leffler, who had been among the first to realize the importance of Cantor's work.[83] Mittag-Leffler was the founder of an important journal, *Acta Mathematica,* and as its editor he solicited French translations of Cantor's major early papers in an attempt to make them more widely known at a time when few mathematicians outside of Germany could read German.[84] But in 1885, as Cantor was preparing a series of articles on what he believed to be a promising new theory of order types for the *Acta Mathematica,* Mittag-Leffler advised him to withhold publication.[85] Thus, when Cantor already felt abandoned and frustrated at nearly every turn, Mittag-Leffler's rejection of his articles on type-theory came as a severe blow. As a result, Cantor chose never to publish in the *Acta Mathematica* again.[86] Feeling isolated and alone at his university in Halle, he sought to give up his teaching of mathematics and to abandon any publication in mathematical journals. Instead, he began to teach philosophy, and to correspond with theologians who provided a natural outlet for Cantor's need to communicate the importance and implications of his work.[87]

In turn, Cantor's contact with Catholic theologians may have made his own religious sympathies all the stronger. By the early part of 1884,

[79] *Grundlagen,* Cantor (1883).

[80] Cantor, "Ueber unendliche, lineare Punktmannichfaltigkeiten" (Part 6), *Mathematische Annalen,* **23** (1884), 453–88, in Cantor (1932), 210–44.

[81] See two of Cantor's letters to Mittag-Leffler: Aug. 26, 1884, in Meschkowski (1967), 242–43, and Nov. 14, 1884, in Schoenflies (1927), 17.

[82] For details: Schoenflies (1927), Grattan-Guinness (1971), 368–69.

[83] G. Mittag-Leffler, "Sur la représentation analytique des fonctions monogénès uniformes d'une variable indépendante," *Acta Mathematica,* **4** (1884), 1–79; "Démonstration nouvelle du théorème de Laurent," *ibid.* 80–88.

[84] Cantor's earliest papers were translated into French for the *Acta Mathematica,* **2** (1883), 305–414.

[85] The paper was to have appeared in two parts: Mittag-Leffler's letter to Cantor, March 9, 1885, in I. Grattan-Guinness, "An Unpublished Paper by Georg Cantor: 'Principien einer Theorie der Ordnungstypen. Erste Mittheilung,' " *Acta Mathematica,* **124** (1970), 65–107, esp. 102.

[86] Cantor recalled the events explicitly in a letter to the Italian mathematician Gerbaldi, Jan. 11, 1896, in Grattan-Guinness (1970), 104: "Von den *Acta Mathematica* will ich aber natürlich nichts mehr wissen!"

[87] Cantor to Gerbaldi in Grattan-Guinness (1970), 104, and Grattan-Guinness (1971), 366–67.

he could write to Mittag-Leffler that he was not the creator of his new work, but merely a reporter. *God* had provided the inspiration, leaving Cantor responsible only for the way in which his articles were written, for their style and organization, but not for their content.[88] Apparently hoping to disassociate himself as much as possible from having to assume responsibility for his controversial research, Cantor was trying to shield himself from the criticism his transfinite numbers were destined to generate. Psychologically, the letter to Mittag-Leffler is revealing because it clearly demonstrates that Cantor had reached the point, *before* his first nervous breakdown in the late spring of 1884, when he was no longer anxious to take credit for his work, but was willing to place the burden of responsibility for the provocative new ideas elsewhere.

Cantor also believed in the absolute truth of his set theory because it had been revealed to him, as he once told Mittag-Leffler, from God directly.[89] Thus he may have seen himself not only as God's messenger, accurately recording, reporting, and transmitting the newly revealed theory of the transfinite numbers, but as God's ambassador as well. If so, Cantor would not only have felt it appropriate, but more accurately, his duty, to use the knowledge which was his by the grace of God to prevent the Church from committing any grave errors with respect to doctrines concerning the nature of infinity. In writing to Jeiler during Whitsuntide, 1888, Cantor declared:

> I entertain no doubts as to the truth of the transfinites, which I have recognized with God's help and which, in their diversity, I have studied for more than 20 years; every year, and almost every day brings me further in this science.[90]

Cantor was even more direct in a letter written to Hermite during January 1894, in which he claimed that it was God's doing that led him away from serious mathematics to concerns of theology and philosophy:

> Now I only thank God, the all-wise and all-good, that He always denied me the fulfillment of this wish [for a position at University in either Göttingen or Berlin], for He thereby constrained me, through a deeper penetration into theology, to serve Him and His Holy Roman Catholic Church better than I would have been able to with my probably weak mathematical powers through an exclusive occupation with mathematics.[91]

At one stroke, Cantor signaled the many disappointments and doubts accumulated over more than two decades. His lack of confidence in himself and his mathematical powers reflects the frustration he must have felt at being unable to solve the continuum hypothesis, com-

[88]Cantor to Mittag-Leffler, Jan. 31, 1884, in Schoenflies (1927), 15–16.

[89]Cantor's letter to Mittag-Leffler of Dec. 23, 1883, and of Jan. 31, 1884, in Schoenflies (1927), 15–16.

[90]Cantor to Jeiler, in Bendiek (1965), 68.

[91]Cantor's letter to Hermite, Jan. 22, 1894, in Meschkowski (1965), 514–15.

pounded by the disastrous effects both the relentless attacks from Kronecker and Mittag-Leffler's response to his work on order types had occasioned. Realizing that no positions were ever going to be offered him in either Göttingen or Berlin,[92] Cantor turned to other interests less demanding than his mathematics, and more positively reinforcing. Later he interpreted his disaffection from mathematics and his deepening interest in philosophy and theology as the work of God. By the end of his life, in the spirit of *Aeterni Patris,* Cantor saw himself as the servant of God, a messenger or reporter who could use the mathematics he had been given to serve the Church. As he told Esser in early February 1896: "From me, Christian Philosophy will be offered for the first time the true theory of the infinite."[93]

Cantor had given up his mathematical colleagues, and had found both consolation and inspiration among theologians and philosophers of the Church. Religion renewed his confidence, and sustained his belief in the truth and significance of his research. Inspired and helped by God, Cantor was sure that his work *was* of consequence, despite the failure of mathematicians to understand the importance of his discoveries.

10. *Conclusion*—At the very beginning of his last major work, his *Beiträge zur Begründung der transfiniten Mengenlehre,* Cantor placed three aphorisms chosen to reflect his basic philosophy of mathematics in general, and his transfinite set theory in particular. The last of these aphorisms was a single line from the *Bible:*

> The time will come when these things which are now hidden from you will be brought into the light.[94]

Perhaps nothing summarized so well Cantor's attitude towards his work and the way in which his theological and philosophical understanding was related directly to his life's work in mathematics. The line from Corinthians directly embodied his conviction that set theory, despite original opposition and rejection by those like Kronecker, would one day be brought into the light and enjoy both success and acceptance. But the aphorism also suggested the way in which Cantor regarded his

[92]Cantor's letter to Mittag-Leffler, Jan. 1. 1884, in Schoenflies (1927), 3–4; and Grattan-Guinness (1971), 358–59.

[93]From part of a letter dated Feb. 15, 1896, from Cantor to Esser, in Meschkowski (1965), 513.

[94]"Veniet tempus, quo ista quae nunc latent, in lucem dies extrahat et longioris aevi diligentia," the *Bible,* I Corinthians 4:5, cited in G. Cantor, "Beiträge zur Begründung der transfiniten Mengenlehre" (Part I), *Mathematische Annalen,* **46** (1895), 481–512, in Cantor (1932), 282–311; trans. by P. E. B. Jourdain, *Contributions to the Founding of the Theory of Transfinite Numbers* (Chicago, 1915). The first aphorism Cantor chose was Newton's famous line from the "General Scholium" to Book III of his *Principia:* "Hypotheses non Fingo." For discussion of the significance of Cantor's choice: J. Dauben, " 'Hypotheses non Fingo': Georg Cantor's Set Theory and His Philosophy of the Infinite," *Proceedings of the XIV International Congress of the History of Science* (Tokyo, 1976).

knowledge of the infinite as being revealed knowledge, conferred upon him by the grace of God. It was God who ultimately guaranteed the absolute correctness, necessity, and uniqueness of Cantor's theory.

Thus Cantor believed that his work eventually would have to be acknowledged as providing the only possible arithmetic embracing both the finite and the infinite. God was the source of inspiration and the ultimate guarantor of the necessary truth of Cantor's research. It was as if Cantor regarded himself as specially chosen. If so, it was a belief he had perhaps recognized even as a young man. For in 1862 he had written to his father, who had just consented to his son's pursuing a career in mathematics, in order to explain that "my soul, my entire being lives in my calling; whatever one wants and is able to do—and to which an unknown, secret voice calls him—that he will surely carry through to success."[95]

Thus, from the very beginning, Cantor had recognized a special force, a secret voice, unknown, and yet drawing him relentlessly to the study of mathematics. Later generations might forget the philosophy, smile at his abundant references to St. Thomas and the Church fathers, overlook his metaphysical pronouncements, and miss entirely the deeply religious roots of Cantor's later faith in the veracity of his work. But these all contributed to Cantor's resolve not to abandon his transfinite numbers. Instead, his determination seems actually to have been strengthened in the face of opposition. His forbearance, as much as anything else he might have contributed, insured that set theory would survive the early years of doubt and denunciation to flourish as a vigorous, revolutionary force in scientific thought of the twentieth century.

At the same time, Cantor was able to offer guidance and support to those like Constantine Gutberlet who were anxious to enrich and to strengthen Catholic doctrine concerning the infinite. Cantor showed how his interpretation of the absolute status of his transfinite numbers was no danger to religion. He was always happy to remind others that Cardinal Franzelin had considered his theory and pronounced that it could not be taken to support pantheistic doctrines. Above all, as Pope Leo XIII urged in *Aeterni Patris,* the study of the philosophical status of scientific thought could be used to show that conclusions of science and religion were indeed compatible. Gutberlet believed the two could support each other, and Neo-Thomists of the late nineteenth century found that even Cantor's revolutionary and disconcerting transfinite set theory might be reconciled with Catholic understanding of the nature of infinity.

Herbert H. Lehman College, City University of New York.

[95]Cantor in a letter to his father, May 25, 1862, in Meschkowski (1967), 5.