1 Sommes-Produits-Factorielles

Exercice 1 : Calculer

$$\sum_{i=0}^{5} u_i = u_0 + u_1 + u_2 + u_3 + u_4 + u_5$$

,

$$\sum_{i=3}^{7} u_i = u_3 + u_4 + u_5 + u_6 + u_7$$

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$$\sum_{i=3}^{7} (u_i + i) = (u_3 + 3) + (u_4 + 4) + (u_5 + 5) + (u_6 + 6) + (u_7 + 7)$$

,

$$\prod_{i=0}^{3} u_i = u_0 \times u_1 \times u_2 \times u_3$$

,

$$\prod_{i=3}^{6} u_i = u_3 \times u_4 \times u_5 \times u_6$$

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$$\sum_{i=0}^{4} u_{3i} = u_0 + u_3 + u_6 + u_9 + u_{12}$$

et

$$\prod_{i=2}^{4} u_{i+1} = u_3 \times u_4 \times u_5$$

pour chacune des suites suivantes :

a) $u_n = 1$ pour tout $n \geq 0$.

- $\sum_{i=0}^{5} u_i = \sum_{i=0}^{5} 1 = 1 + 1 + 1 + 1 + 1 + 1 = 6$
- $\sum_{i=3}^{7} u_i = 1 + 1 + 1 + 1 + 1 = 5$
- $\sum_{i=3}^{7} (u_i + i) = \sum_{i=3}^{7} (1 + i) = 4 + 5 + 6 + 7 + 8 = 22$
- $\prod_{i=0}^{3} u_i = 1$
- $\prod_{i=3}^{6} u_i = 1$
- $\sum_{i=0}^{4} u_{3i} = 1$
- $\prod_{i=2}^{4} u_{i+1} = 1$

b) $u_n = n$ pour tout $n \geq 0$.

- $\sum_{i=0}^{5} u_i = 0 + 1 + 2 + 3 + 4 + 5 = 15$
- $\sum_{i=3}^{7} u_i = 3 + 4 + 5 + 6 + 7 = 25$
- $\sum_{i=3}^{7} (u_i + i) = (3 + 3) + (4 + 4) + (5 + 5) + (6 + 6) + (7 + 7) = 50$
- $\prod_{i=0}^{3} u_i = 0 \times \cdots = 0$
- $\prod_{i=3}^{6} u_i = 3 \times 4 \times 5 \times 6 = 360$
- $\sum_{i=0}^{4} u_{3i} = 0 + 3 + 6 + 9 + 12 = 30$

- $\prod_{i=2}^{4} u_{i+1} = 3 \times 4 \times 5 = 60$

c) $u_n = n - 2$ pour tout $n \geq 0$.

- $\sum_{i=0}^{5} u_i = -2 - 1 + 0 + 1 + 2 + 3 = 6$
- $\sum_{i=3}^{7} u_i = 1 + 2 + 3 + 4 + 5 = 15$
- $\sum_{i=3}^{7} (u_i + i) = (1 + 3) + (2 + 4) + (3 + 5) + (4 + 6) + (5 + 7) = 40$
- $\prod_{i=0}^{3} u_i = -2 \times -1 \times 0 \times 1 = 0$
- $\prod_{i=3}^{6} u_i = 1 \times 2 \times 3 \times 4 = 24$
- $\sum_{i=0}^{4} u_{3i} = -2 + 1 + 4 + 7 + 10 = 20$
- $\prod_{i=2}^{4} u_{i+1} = 1 \times 2 \times 3 = 6$

d) $u_n = 2n + 1$ pour tout $n \geq 0$.

- $\sum_{i=0}^{5} u_i = 1 + 3 + 5 + 7 + 9 + 11 = 36$
- $\sum_{i=3}^{7} u_i = 7 + 9 + 11 + 13 + 15 = 55$
- $\sum_{i=3}^{7} (u_i + i) = 10 + 13 + 16 + 19 + 22 = 80$
- $\prod_{i=0}^{3} u_i = 1 \times 3 \times 5 \times 7 = 105$
- $\prod_{i=3}^{6} u_i = 7 \times 9 \times 11 \times 13 = 9009$
- $\sum_{i=0}^{4} u_{3i} = 1 + 7 + 13 + 19 + 25 = 65$
- $\prod_{i=2}^{4} u_{i+1} = 7 \times 9 \times 11 = 693$

e) $u_n = n^2$ pour tout $n \geq 0$.

- $\sum_{i=0}^{5} u_i = 0^2 + 1^2 + 2^2 + 3^2 + 4^2 + 5^2 = 55$
- $\sum_{i=3}^{7} u_i = 3^2 + 4^2 + 5^2 + 6^2 + 7^2 = 135$
- $\sum_{i=3}^{7} (u_i + i) = (3^2 + 3) + (4^2 + 4) + (5^2 + 5) + (6^2 + 6) + (7^2 + 7) = 160$
- $\prod_{i=0}^{3} u_i = 0^2 \times = 0$
- $\prod_{i=3}^{6} u_i = 3^2 \times 4^2 \times 5^2 \times 6^2 = 129600$
- $\sum_{i=0}^{4} u_{3i} = 0^2 + 3^2 + 6^2 + 9^2 + 12^2 = 270$
- $\prod_{i=2}^{4} u_{i+1} = 3^2 \times 4^2 \times 5^2 = 3600$

Exercice 2 : Simplifier :
On a $\dfrac{7!}{5!} = \dfrac{7 \times 6 \times 5!}{5!} = 7 \times 6 = 42$; $\dfrac{10!}{7!} = \dfrac{10 \times 9 \times 8 \times 7!}{7!} = 10 \times 9 \times 8 = 720$;
$\dfrac{(n+1)!}{(n-1)!} = \dfrac{(n+1) \times n \times (n-1)!}{(n-1)!} = (n+1)n$;

Exercice 3 : En remarquant que $\dfrac{1}{k(k+1)} = \dfrac{1}{k} - \dfrac{1}{k+1}$, déterminer :

a) On a

$$\sum_{k=1}^{100} \frac{1}{(k(k+1))} = \sum_{k=1}^{100} \left(\frac{1}{k} - \frac{1}{k+1} \right) = \frac{1}{1} - \frac{1}{101} = \frac{100}{101}.$$

b) On a

$$\sum_{k=3}^{45} \frac{1}{(k(k+1))} = \sum_{k=3}^{45} \left(\frac{1}{k} - \frac{1}{k+1} \right) = \frac{1}{3} - \frac{1}{46} = \frac{43}{138}.$$

Exercice 4 : Déterminer :

a) On a

$$\sum_{k=1}^{100} \frac{1}{\sqrt{k+1}+\sqrt{k+2}} = \sum_{k=1}^{100} \frac{\sqrt{k+1}-\sqrt{k+2}}{(\sqrt{k+1})^2-(\sqrt{k+2})^2} = \sum_{k=1}^{100}(-\sqrt{k+1}+\sqrt{k+2}) = \sqrt{102}-\sqrt{2}.$$

b) On a

$$\sum_{k=3}^{45} \frac{1}{\sqrt{k+1}+\sqrt{k+2}} = \sum_{k=3}^{45} \frac{\sqrt{k+1}-\sqrt{k+2}}{(\sqrt{k+1})^2-(\sqrt{k+2})^2} = \sum_{k=3}^{45}(-\sqrt{k+1}+\sqrt{k+2}) = \sqrt{47}-\sqrt{4}.$$

Exercice 5 : Déterminer :

a) On a $\sum_{k=1}^{100} kk! = \sum_{k=1}^{100}((k+1)-1)k! = \sum_{k=1}^{100}(k+1)k!-k!) = \sum_{k=1}^{100}((k+1)!-k!) = 101!-1! = 101!-1$.

b) $\sum_{k=3}^{45} kk! = \sum_{k=3}^{45}((k+1)-1)k! = \sum_{k=3}^{45}(k+1)k!-k!) = \sum_{k=3}^{45}((k+1)!-k!) = 46!-3!$.

Exercice 6 : Calculer :

a) On a $\prod_{i=1}^{n-1}(i) = 1 \times 2 \times \cdots \times n-1 = (n-1)!$.

b) On a $\prod_{i=1}^{n-1}(n-i) = (n-1) \times (n-2) \times \cdots (n-(n-1)) = (n-1)!$.

c) On a $\prod_{i=1}^{2n}(i^2-1) = (1^2-1) \times \cdots = 0 \times \cdots = 0$.

d) On a $(\prod_{i=1}^{2n}(2i)) \prod_{i=0}^{2n}(2i+1) = (2 \times 4 \times \cdots \times 2(2n))(1 \times 3 \times \cdots \times (2(2n)+1)) = 1 \times 2 \times 3 \times (4n) \times (4n+1) = (4n+1)!$.