The University of Sydney
School of Mathematics and Statistics

# Tutorial 2 (Week 3)

MATH2069: Discrete Mathematics and Graph Theory — Semester 1, 2023

*More difficult questions are marked with either * or **. Those marked * are at the level which MATH2069 students will have to solve in order to be sure of getting a Credit, or to have a chance of a Distinction or High Distinction.*

**1.** Suppose you have 7 different ornaments to put on your mantelpiece.

(a) If you want to use all of them, how many possible arrangements are there?

(b) If you want to use 6 of them, how many possible arrangements are there?

(c) If you can use all, some, or none of them, how many possible arrangements are there?

*(d) Divide your answer to part (c) by your answer to part (a); this ratio measures how much extra freedom you get by not necessarily using all the ornaments. Notice that it agrees with $e$ up to four decimal places. Is this a coincidence?

**2.** An ordinary knock-out singles tennis tournament (with no seeds or byes) consists of a series of rounds. In each round, the remaining players play against each other in pairs, with the losers being eliminated and the winners going through to the next round; in the last round, the only two remaining players play the final match to determine the winner of the tournament. Suppose that there are 7 rounds.

(a) How many players are there at the start of the tournament?

(b) How many matches are played in total?

(c) Before the tournament starts, the organizers need to construct the draw, which specifies who plays who in the first round, and then which first-round winners play which other first-round winners in the second round, and so on. Of course, the organizers don't know who the first-round winners will be, so in the draw they are just thought of as "the winner of the match between player X and player Y", and so forth. How many possible draws are there? Use the fact, proved in lectures, that the number of ways to group $2k$ people into $k$ pairs is $\frac{(2k)!}{2^k k!}$. (The answer is too large to evaluate, so just leave it as a fraction of expressions involving a factorial.)

(d) Suppose that after the tournament is finished, the only things that are recorded are the draw and the winners of each match. How many different such records are possible?

**3.** Let $X = \{m, m+1, \cdots, n-1, n\}$ be a set of consecutive positive integers, and $A$ the subset of $X$ consisting of those elements which are multiples of 3. You would expect $|A|$ to be about $\frac{|X|}{3}$, but the latter is not always an integer, so you may have to round it up or down.

(a) Give an example where $|A|$ does not equal $\lfloor \frac{|X|}{3} \rfloor$.

(b) Show that if $X = \{1, 2, \cdots, n\}$, then $|A| = \lfloor \frac{n}{3} \rfloor$.

(c) Hence give a formula for $|A|$ when $X = \{m, m+1, \cdots, n\}$ and $m \geqslant 2$.

***4.** If $m$ is a positive integer, write $v_2(m)$ for the exponent of the highest power of 2 that divides $m$. For example:

$$\begin{aligned} v_2(m) = 0 &\iff m \text{ is odd}, \\ v_2(m) = 1 &\iff m \text{ is even but not a multiple of 4}, \\ v_2(m) = 2 &\iff m \text{ is a multiple of 4 but not of 8, etc.} \end{aligned}$$

(a) Show that $v_2(mm') = v_2(m) + v_2(m')$ for all integers $m, m'$.

(b) Prove that for all positive integers $n$,

$$v_2(n!) = \left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{n}{2^2} \right\rfloor + \left\lfloor \frac{n}{2^3} \right\rfloor + \left\lfloor \frac{n}{2^4} \right\rfloor + \cdots\cdots.$$

(Note that although this sum appears to go on forever, all the terms will be zero from a certain point on, because when $2^k > n$ we have $\lfloor \frac{n}{2^k} \rfloor = 0$.)

(c) Deduce that $v_2((2n)!) = n + v_2(n!)$. How could this be proved another way?

***5.** Let $\mathcal{F}_n$ be the set of functions $f : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ such that if $i < j$ then $f(i) \leqslant f(j)$, and for all $i$, $f(i) \leqslant i$. Given two rows of boxes with $n$ boxes in each row,

| | | | |
|---|---|---|---|
| | | $\cdots$ | |
| | | $\cdots$ | |

a *standard tableau* is obtained by placing the numbers $1, 2, \ldots, 2n$ bijectively in the boxes so that the numbers increase from left to right in each row and so that each number in the bottom row is larger than the number in the box above it. Let $\mathcal{T}_n$ be the set of standard tableaux with entries $1, 2, \ldots, 2n$. Construct a bijection $\mathcal{F}_n \to \mathcal{T}_n$. [The cardinality $|\mathcal{F}_n| = |\mathcal{T}_n|$ is known as the $n$-th *Catalan number*].

****6.** Use the Pigeonhole Principle to prove that for any odd integer $m \geqslant 3$, at least one of the numbers $2^2 - 1, 2^3 - 1, \cdots, 2^{m-1} - 1$ is divisible by $m$.

**Selected numerical answers:**

**1.** 5040, 5040, 13700.