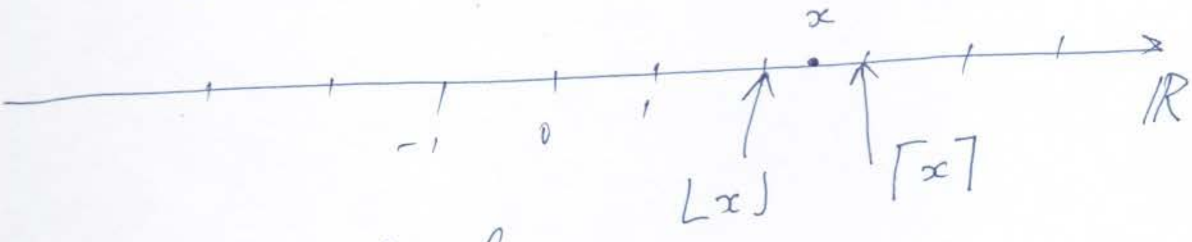
Pigeonhole Principle (cont'd)

<u>Notation</u>: For any $x \in \mathbb{R}$

set $\lfloor x \rfloor$ = x rounded down to the nearest integer
"floor of x"

$\lceil x \rceil$ = x rounded up to the nearest integer.
"ceiling of x"



In particular,
$$\lfloor x \rfloor = \lceil x \rceil = x \text{ for } x \in \mathbb{Z}.$$

<u>Thm</u> (Pigeonhole Principle).
Let $f: X \to Y$ be a function between two finite nonempty sets.
Then there exists $y \in Y$ s.t.
$$\left| f^{-1}(y) \right| \geq \left\lceil \frac{|X|}{|Y|} \right\rceil .$$
In particular, if $|X| > |Y|$, then there exist $x_1 \neq x_2$ in X s.t. $f(x_1) = f(x_2)$.

Ex. Suppose there are 60 pigeons and 25 holes.

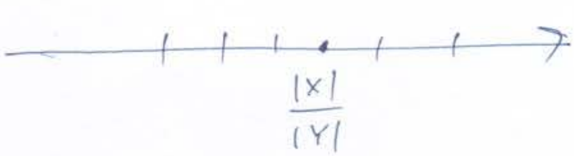
Note $|X| = 60$, $|Y| = 25$.

$$\frac{|X|}{|Y|} = \frac{60}{25} = 2.4 \Rightarrow \left\lceil \frac{|X|}{|Y|} \right\rceil = 3.$$

Proof. Argue by contradiction.

Suppose $|f^{-1}(y)| < \left\lceil \frac{|X|}{|Y|} \right\rceil$ for all $y \in Y$.

Then $|f^{-1}(y)| < \frac{|X|}{|Y|}$.



We have

$$\sum_{y \in Y} |f^{-1}(y)| < \sum_{y \in Y} \frac{|X|}{|Y|} =$$

$$= \frac{|X|}{|Y|} \cdot |Y| = |X|.$$ This contradicts the relation $|X| = \sum_{y \in Y} |f^{-1}(y)|$. $\square$

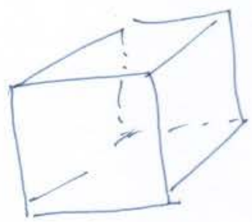
<u>Thm</u> (Overcounting Principle).

Let X and Y be finite sets and m a positive integer. Suppose there is a function $f: X \to Y$ s.t. for all $y \in Y$ we have $|f^{-1}(y)| = m$. Then $|Y| = \frac{|X|}{m}$.

<u>Proof</u>. We have

$$|X| = \sum_{y \in Y} |f^{-1}(y)| = \sum_{y \in Y} m = m|Y|. \quad \square$$

<u>Ex</u>. How many edges does a cube have?

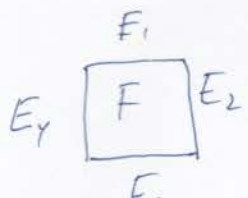


If we take $6 \times 4 = 24$, this would overcount the number of edges by the factor of 2.

Hence, the number is $\frac{24}{2} = 12$.

Here $X = \{(\text{Face}, \text{Edge})\}$,

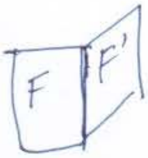
$Y = \{\text{Edges}\}$



$$f : X \to Y$$

$$f : (\text{Face}, \text{Edge}) \longmapsto \text{Edge}$$

$$|f^{-1}(\text{Edge})| = 2.$$



$$|X| = 6 \times 4 = 24, \quad |Y| = \frac{24}{2} = 12.$$

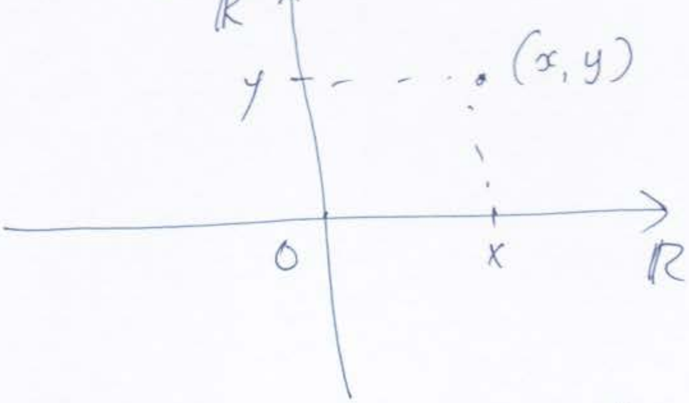
Product Principle

Def. For two sets X and Y

set $X \times Y = \{(x, y) \mid x \in X \text{ and } y \in Y\}$.

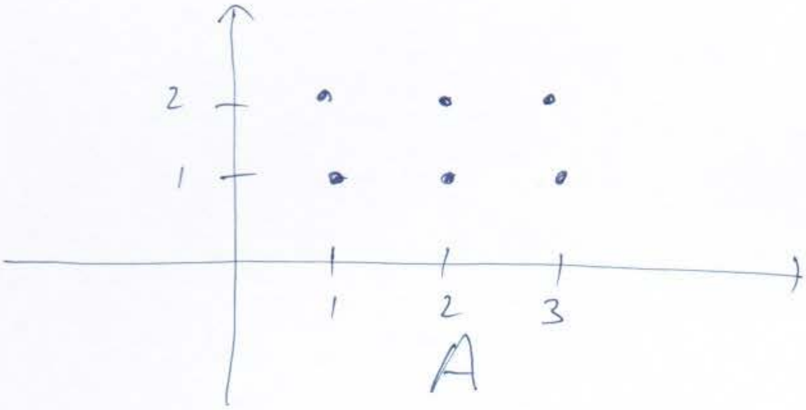
This is the Cartesian product of X and Y.

Ex 1) $\mathbb{R} \times \mathbb{R} = \mathbb{R}^2$, the set of all pairs (x, y), $x, y \in \mathbb{R}$.



2) $A = \{1, 2, 3\}$

$B = \{1, 2\}$



More generally, if $X_1, \dots, X_n$ are sets, then

$$X_1 \times X_2 \times \dots \times X_n = \{(x_1, x_2, \dots, x_n) \mid x_i \in X_i\}.$$

<u>Thm</u> (Product Principle).
If $X_1, \dots, X_n$ are finite sets, then

$$|X_1 \times \dots \times X_n| = |X_1| \cdot |X_2| \cdots |X_n|.$$

Take $n=2$ for the proof.

$$X \times Y = \{(x, y) \mid x \in X, y \in Y\}.$$

The number of such pairs is

$$\underbrace{|Y| + |Y| + \dots + |Y|}_{|X|} = |X| \cdot |Y|.$$

<u>Ex.</u> How many five-digit numbers are there?

$$* \quad * \quad * \quad * \quad *$$

$X_1 = \{1, 2, \dots, 9\}$, $X_2 = X_3 = X_4 = X_5 = \{0, 1, \dots, 9\}$.

$9 \times 10 \times 10 \times 10 \times 10$ is the number.

The Product Principle applies in the situations where the choices at any step depend on the selections made at the previous steps.

The key assumption is that the number of choices ~~at~~ is the same.

<u>Ex.</u> Count the number of five-digit numbers which have distinct digits.

$$* \quad * \quad * \quad * \quad *$$

$\{1, 2, \dots, 9\}$ $\quad 9 \quad 8 \quad 7 \quad 6$

The number is $9 \times 9 \times 8 \times 7 \times 6$
$= 27216$.