THE UNIVERSITY OF SYDNEY
SCHOOL OF MATHEMATICS AND STATISTICS

Practice Class 4

MATH2069: Discrete Mathematics and Graph Theory — Semester 1, 2023

1. Give recursive definitions of the following sequences.

(a) The sequence of powers of 2: $2^0 = 1$, and for $n \geq 1$,

$2^n =$

(b) The Catalan numbers: $c_0 = 1$, and for $n \geq 1$,

$c_n =$

2. The following is meant to be a proof by induction that

$$\frac{1}{1\times 2} + \frac{1}{2\times 3} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}$$

for all $n \geq 1$, but the lines are jumbled.

Suppose $n \geq 2$ and the claim holds for $n-1$. **(1)**

$= \dfrac{n-1}{n} + \dfrac{1}{n(n+1)} = \dfrac{(n-1)(n+1)+1}{n(n+1)}$ **(2)**

The base case is true because **(3)**

This establishes the inductive step and completes the proof. **(4)**

Then $\dfrac{1}{1\times 2} + \dfrac{1}{2\times 3} + \cdots + \dfrac{1}{(n-1)n} + \dfrac{1}{n(n+1)}$ **(5)**

$\dfrac{1}{1\times 2} = \dfrac{1}{2} = \dfrac{1}{1+1}$ **(6)**

$= \dfrac{n^2}{n(n+1)} = \dfrac{n}{n+1}$ **(7)**

What is the correct order of the lines?

3. Consider the sequence defined by $a_0 = 0$, $a_n = a_{n-1} + 2n$ for $n \geq 1$.

(a) Unravelling gives the non-closed formula

$a_n =$ for all $n \geq 0$.

(b) Summing the arithmetic progression gives the closed formula

$a_n =$ for all $n \geq 0$.

(c) Check that the formula in the previous part satisfies the recurrence relation:

is indeed true.

4. Consider the recurrence relation $a_n = a_{n-1} + 6a_{n-2}$, for $n \geq 2$.

(a) What is the characteristic polynomial?

(b) What are the roots of this polynomial? $x =$

(c) Write down the general solution:

$a_n =$

(d) Find the solution when $a_0 = 1$, $a_1 = -1$:

$a_n =$

5. Consider the recurrence relation $a_n = -2a_{n-1} - a_{n-2}$, for $n \geq 2$.

(a) What is the characteristic polynomial?

(b) What are the roots of this polynomial? $x =$

(c) Write down the general solution:

$a_n =$

(d) Find the solution when $a_0 = 1$, $a_1 = -3$:

$a_n =$

6. Consider the recurrence relation $a_n = -a_{n-2}$, for $n \geq 2$.

(a) What is the characteristic polynomial?

(b) What are the roots of this polynomial? $x =$

(c) Write down the general solution:

$a_n =$

(d) If $a_0 = 0$ and $a_1 = 1$, what is a_7?

$a_7 =$

7. For each of the following statements, write T for true or F for false.

(a) Apart from 0 and 1, every Fibonacci number is prime.

(b) For all $n \geq 7$, we have $3^n < n!$.

(c) If $a_0 = 3$ and $a_n = 2a_{n-1}$ for $n \geq 1$, then $a_n = 3 \times 2^n$.

(d) $1 + 3 + 5 + 7 + \cdots + 99 = 99^2$.

(e) $1^3 + 2^3 + 3^3 + \cdots + 99^3$ is a perfect square.

(f) $a_n = 3a_{n-2}$ is a second-order homogeneous linear recurrence.

(g) The roots of the quadratic $x^2 - x - 1$ are $\pm(\frac{1+\sqrt{5}}{2})$.

(h) If $a_0 = 4$ and $a_n = 2a_{n-1} + 1$ for $n \geq 1$, then $a_5 = 159$.

(i) If $a_0 = 2$, $a_1 = 0$, and $a_n = a_{n-2}$ for $n \geq 2$, then $a_n = 1 + (-1)^n$.

(j) $a_n = n^2 2^n$ is a solution of $a_n = 4a_{n-1} - 4a_{n-2}$ $(n \geq 2)$.