The University of Sydney
School of Mathematics and Statistics

Solutions to Practice Class 1

MATH2069: Discrete Mathematics and Graph Theory — Semester 1, 2023

For some questions with a numerical answer, the answer is indicated in two forms, e.g. $\frac{12}{2} = 6$. *This is to give an indication of one way to obtain the answer; in a quiz, either form on its own would suffice, e.g.* $\frac{12}{2}$ *alone or* 6 *alone.*

1. Complete the following definitions:

(a) A function $f : A \to B$ is injective if

for any $x, y \in A$ the equality $f(x) = f(y)$ implies $x = y$

(b) A function $f : A \to B$ is surjective if

for any $y \in B$ there exists $x \in A$ such that $f(x) = y$

(c) A function $f : A \to B$ is bijective if

for any $y \in B$ there exists a unique $x \in A$ such that $f(x) = y$

2. Complete the following principles:

(a) (Difference Principle) If A is a subset of the finite set X, then

$$|X \setminus A| \;=\; |X| - |A|$$

(b) (Pigeonhole Principle) If $f : X \to Y$ is a function between nonempty finite sets, then there is some $y \in Y$ such that

$$|f^{-1}(y)| \;\geq\; \left\lceil \frac{|X|}{|Y|} \right\rceil$$

3. Find a function $f : \mathbb{N} \to \mathbb{N}$ which is

(a) injective but not surjective.

$$\boxed{f(n) = 2n}$$

(b) surjective but not injective.

$$\boxed{f(2n) = n \text{ and } f(2n+1) = 0}$$

(c) bijective and has the property that $f(n) \neq n$ for all n.

$$\boxed{f(2n) = 2n+1 \text{ and } f(2n+1) = 2n}$$

4. Suppose there are 135 students enrolled in a class. From that information alone, you can deduce that:

(a) there must be at least $\boxed{\lceil \frac{135}{7} \rceil = 20}$ of them who were born on the same day of the week;

(b) there must be at least $\boxed{\lceil \frac{135}{26} \rceil = 6}$ whose surnames begin with the same letter;

(c) there must be at least $\boxed{0}$ of them who are left-handed;

(d) there must be at least $\boxed{230}$ days this year which are not the birthday of any of them.

5. Count the subsets of the set {A,B,C,D,E,F,G}:

(a) which contain D

$2^6 = 64$

(b) which do not contain C

$2^6 = 64$

(c) which contain exactly 2 elements

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6. If a function $f : A \to B$ is bijective, we define its inverse as the function $f^{-1} : B \to A$ by the rule $f^{-1}(y) = x$ if $y = f(x)$. For each of the following functions determine whether it is bijective. If so, write a formula for its inverse.

(a) $f : \mathbb{R} \to \mathbb{R}, \quad f(x) = x^2 + 1;$

not bijective since $f(1) = f(-1) = 2$

(b) $f : \mathbb{R} \to \mathbb{R}, \quad f(x) = x^3 + 1;$

bijective, $f^{-1}(y) = (y-1)^{\frac{1}{3}}$

(c) $f : \mathbb{N} \to \mathbb{N}, \quad f(n) = n^3 + 1;$

not bijective since $f(n) \neq 0$ for any n

(d) $f : \mathbb{R} \to \mathbb{Z}, \quad f(x) = \lfloor x \rfloor;$

not bijective since $f(1/2) = f(0) = 0$

(e) $f : \mathbb{R} \to \mathbb{R}, \quad f(x) = \dfrac{x + \lfloor x - 7.5 \rfloor}{2};$

not bijective since $f(x) \neq 4.25$ for any x

(f) $f : \mathbb{Z} \to \mathbb{Z}, \quad f(n) = \dfrac{n + \lfloor n - 7.5 \rfloor}{2};$

bijective, $f^{-1}(n) = n + 4$

7. For each of the following statements, write T for true or F for false.

(a) The Tower of Hanoi number h_n, in binary, consists of n ones.	T
(b) Among any 36 people, there are 6 who were born on Monday.	F
(c) If $\lvert X\rvert > \lvert Y\rvert$, any function $f : X \to Y$ must be surjective.	F
(d) If $\lvert X\rvert < \lvert Y\rvert$, no function $f : X \to Y$ can be surjective.	T
(e) If $\lvert X\rvert > \lvert Y\rvert$, no function $f : X \to Y$ can be injective.	T
(f) If $\lvert X\rvert < \lvert Y\rvert$, any function $f : X \to Y$ must be injective.	F
(g) Any injective function $f : X \to X$ is bijective.	F