School of Mathematics and Statistics

MATH1023

Multivariable Calculus and Modelling

THE UNIVERSITY OF SYDNEY

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Acknowledgements

The material in these notes has been developed over many years by the following members of the School of Mathematics and Statistics:

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Introduction

Calculus is not only one of the fundamental branches of Mathematics, it is a success story of scientific reasoning. Its development in the XVII-th century was driven both by mathematical curiosity and practical problems. Many questions were related to the interpretation of infinity, the trajectory of cannon balls, and the movement of comets. The mathematical theory that emerged enabled the growth of modern science and lies at the foundation of most of the scientific and technological developments that happened ever since. The goal of this course is to further advance your mathematical knowledge about Calculus and to show how it is applied both as the fundamental language of science and as the computational tool to solve problems. The success of Calculus in combining mathematical theory and applications is a paradigmatic example of the deeper understanding of the world that we can achieve through Mathematics.

The course is divided in 12 chapters, each chapter corresponds roughly to the material you will learn in one week. It is evenly divided in two related parts:

- Chapters 1-6 focus on how to formulate and solve *differential equations*, one of the main tools in mathematical modelling.
- Chapters 7-10 focus on *multivariable calculus*, the extension of differential calculus to two (or more) variables.

0.1 Differential Equations

After learning so much about Calculus you may be asking yourself what more is there to learn? To answer this question, and to have an overview of what we will learn in this course, it is is useful to make an analogy to other experiences you had in Mathematics. After learning the simple arithmetic operations ("$+, -, \times, \div$") it was natural to use them in the formulation and solution of problems: *"Alice has three times as many apples as Bob; Bob has* 6 *apples; how many apples Alice has?"* was translated into $3 \times x = 6$ and solved for x. Other problems led to similar polynomial equations and solved using specific methods (an example is the quadratic equation $ax^2 + bx + c = 0$, for constant a, b, c). Similarly, differentiation in Calculus naturally leads to equations in which one or more derivatives appear. These are called *differential equations*. We often use a prime to denote the derivative $\frac{d}{dx}y(x) = y'(x)$, the rate of change of y in respect to x, so that simple forms of differential equations can be written as

(0.1a) $$\boxed{\frac{dy}{dx} = f(x, y) \quad \text{or} \quad y' = f(x, y)\ .}$$

While the solutions of the equations derived from arithmetic operations are numbers ($x = 2$ apples, in the example above), the solution of differential equations are functions ($y(x)$ in the equation above). Here and throughout these notes $y(x)$ express the fact that a quantity y *depends* on x, which is equivalent to say that y is a *function* of x. We also write $y = f(x)$ and draw graphs with x in the horizontal axis and y in the vertical axis (it is also common to think of y itself as the function, and thus write just $y(x)$ or $y = y(x)$). Typically the independent variable is taken to be x or t (time).

Models of falling bodies

Differential equations naturally appear in mathematical models in many different areas. For instance, according to Newton's second law of mechanics,

$$\text{Mass} \times \text{Acceleration} = \text{Force}.$$

If we apply this law to study how bodies with mass m fall close to the surface of the earth, we obtain that $ma = mg$ (where g is a constant). The acceleration a is the derivative of the velocity v and therefore we obtain a differential equation for the velocity

$$\boxed{\frac{dv}{dt} = g.} \tag{0.1b}$$

From what we know about differentiation, it is easy to see that $v(t) = gt$ satisfies the equation above, matching our knowledge that the velocity grows linearly in time. Consider now a better model of falling bodies, obtained including air resistance as a force $F_{air} = -bv^2$ opposing the increase in velocity (b is a constant). In this case the differential equation we obtain is

$$\boxed{\frac{dv}{dt} = g - kv^2,} \tag{0.1c}$$

where g and $k \equiv b/m$ are constants. The expected velocity of the body at time t is given by the solution $v(t)$ of this equation. In the first example, Eq. (0.1b), the solution was obtained as the integral (or anti-derivative) of the constant function g. In the second example, Eq. (0.1c), this process cannot be applied directly because the right hand side depends on the dependent variable v, which is what we are trying to find. In **Chapters 1-4** we will classify and learn how to solve different types of differential equations that can be written as in Eq. (0.1a), including Eq. (0.1c).

Higher-order differential equations

There are still many other important classes of differential equations that go beyond Eq. (0.1a). From Newton's law, if x measures position then acceleration is the second derivative of x

with respect to time. If the force depends on both position and time, $F = F(x,t)$, then this relation can be expressed as the differential equation

$$\boxed{m \times \frac{d^2x}{dt^2} = F(x,t).}$$

This is an example of a **second-order** differential equation, since it involves the second-order derivative of x in respect to t. In **Chapters 4-6** we will develop methods for equations of this type which allow us to study the case of oscillators and to incorporate the effect of friction, damping, and external forces varying with time.

The examples we have looked at so far involve a single independent variable (which is often, but not always, the time t) and a single unknown dependent variable (x in the previous examples) which we are trying to find as a function $x(t)$ of t. Other types of differential equations are mostly beyond the range of this course. However, it is worth looking at some examples because they give an idea of the scope of this branch of mathematics and motivate the study of multivariable calculus in the second part of this course.

Systems of Differential Equations

Sometimes we have several variables, all depending on a single independent variable. Just as in linear algebra, this situation may involve several equations which we have to solve *simultaneously*. This will be the subject of **Chapter 6**. One of the most famous examples of such a system of equations is the **Volterra Predator-Prey equations**. Here x and y are respectively measures of the populations of two animal species, where one of the species, y say, is the predator and other species x is its prey. We suppose that both x and y vary with time t. The pair of equations

$$\boxed{\begin{aligned} \frac{dx}{dt} &= Ax - Bxy\,, \\ \frac{dy}{dt} &= -Cy + Dxy\,, \end{aligned}}$$

where A, B, C, D are positive constants, has been suggested as a simple mathematical model of this situation. Where does such a model come from? If we forget the xy terms (or equivalently take $B + D = 0$) we are left with the two uncoupled equations

$$\frac{dx}{dt} = Ax\,, \qquad \frac{dy}{dt} = -Cy\,.$$

These equations reflect what might happen to the numbers of prey and predator if they never meet each other: the prey increases, and the predator dies out. In this case the two equations can be solved independently of the other. The xy terms represent the *interaction* of the two species. The equations (like the two species themselves) are no longer independent.

The extra term has the desired effect of benefiting the predator (increasing its growth rate) and disadvantaging the prey (decreasing its growth rate). Analysis of these equations and their various refinements leads to many fascinating insights into the behaviour of ecological systems.

0.2 Multivariable Calculus

In other situations there may be differential equations involving several independent variables. The simplest example are functions of two variables $z = f(x, y)$, which necessarily need to be represented in a 3-dimensional space x, y, z, as described in **Chapter 7**. For example, the temperature T along a uniform rod depends on both position x and time t. Then $T = T(x, t)$ and the **heat equation**

$$\frac{\partial^2 T}{\partial x^2} = k\frac{\partial T}{\partial t}$$

is a mathematical expression of physical laws which apply to the flow of heat in a solid body. Here k is a constant depending on the choice of units and the material of the rod. Solving the equation for a specified initial temperature distribution will allow us to predict the future temperature at different parts x of the rod and at different times t. Therefore, the solution is now represented in a 3-dimensional space T, x, t.

Here it is important to realise that the differential calculus we learnt so far for functions of one variable $y = f(x)$ cannot be directly applied to functions of two variables $z = f(x, y)$. How do we compute the derivative of such functions? What is the geometric interpretation of such derivatives? How do we compute maxima and minima of two-variable functions? The remaining part of the course will answer these questions by generalising the ideas you already know from single-variable calculus to multi-variable functions. In **Chapter 8** we will introduce the concept of *partial* derivative $\frac{\partial}{\partial x}$ and in **Chapter 11** this will be generalised to higher-order partial derivatives (such as $\frac{\partial^2}{\partial x^2}$). In fact, partial derivatives appear in the heat equation above, an example of *partial differential equation*, showing how multivariable calculus is essential to understand mathematical models of multivariable functions. The course will continue with further fundamental properties of multivariable calculus: more precise generalisations of the derivative $\frac{dy}{dx}$ – in **Chapters 10 and 9** – and optimisation of functions of two variables – discussed in **Chapter 12**.

Partial differential equations involving more than one independent variables appear in many other areas of Science and Engineering. Prominent examples include the **wave equation**:

$$\frac{\partial^2 y}{\partial x^2} = c^2\frac{\partial^2 y}{\partial t^2},$$

where c is a constant depending (e.g., the mass per unit length of a vibrating string and its tension); the equations which describe the phenomena of electromagnetism (Maxwell's Equations), quantum mechanics (Schrödinger's Equation), and fluids (Navier–Stokes Equation). The equations of fluid flow in particular are very difficult to analyse mathematically, but in many cases can be solved very accurately by powerful computers. Engineering design problems (such as those involving structural analysis or aircraft aerodynamics) which once required extensive model building and prototyping can be handled more efficiently by complex mathematical models based on differential equations and solved on supercomputers.

Summary of Introduction

- Differential equations are equations in which one or more derivatives appear. They appear naturally as 'mathematical models' in many areas of science and technology.
- The solutions of differential equations are functions. In simple cases, these function can be obtained by working backwards, using what we know about differentiation, or by simple integration.
- More complicated differential equations involve multi-variable functions. This leads to the analysis of *systems of differential equations* and may involve 'ordinary' or partial derivatives. Sometimes several differential equations need to be satisfied simultaneously.
- Multi-variable calculus is the extension of calculus to functions of more than one independent variable.

CHAPTER 1

Models and Differential Equations

From the earliest historical times to the present day, people have used mathematics in their efforts to measure, analyse, and predict more and more aspects of life around them. Indeed, mathematics developed and continues to evolve in response to this need. Although some of the problems confronting mathematicians today are very different from those tackled by the ancients, the approach has nevertheless changed very little.

The basic idea is to construct a mathematical model; that is, an equation or system of equations which describe as accurately as possible the relationships between the various quantities with which the problem deals. The equations may (preferably) be derived from some scientific theory covering the situation, or may simply be the result of analysis of collected data. Having obtained equations, the mathematician will attempt to solve them, and use the model to make predictions.

1.1 The development of models

Let us look first at how models are developed. A good illustration is provided by an example with which we began this course: the motion of a falling object. The story begins with Galileo Galilei at the beginning of the 17th Century. He knew of observations which suggested the distance travelled by an object falling from rest is proportional to the square of the time elapsed, and independent of the weight of the object. He tried over many years to develop a model of this motion by relating it to the notion of *uniform acceleration*. Galileo defined motion with uniform acceleration as that in which equal increments of velocity are acquired in equal increments of time. Introducing the symbols Δv and Δt for these increments, Galileo's definition can be expressed mathematically as

$$\Delta v = a\Delta t \quad \text{or as a ratio} \quad \frac{\Delta v}{\Delta t} = a, \tag{1.1a}$$

where the constant of proportionality a measures the acceleration. Equation (1.1a) is an example of a **difference equation** (so-called because Δv is the difference between the velocity at one point in time and the velocity Δt seconds later).

Galileo did not convert the ratio $\Delta v/\Delta t = a$ to a derivative by taking the limit as $\Delta t \to 0$; this step had to wait for the development of calculus by Newton and Leibniz half a century later.

However, the idea is now familiar and we can take the limit and convert Galileo's formula to the *differential equation*

(1.1b) $$\frac{dv}{dt} = a.$$

As we saw in the Introduction, it is easy to solve this equation and arrive at a differential equation governing the distance s fallen from rest:

(1.1c) $$v = \frac{ds}{dt} = at\,.$$

Solving this gives

$$s = \frac{1}{2}at^2,$$

for the distance s fallen from rest in agreement with Galileo's observations. These also suggested that the downward acceleration of all falling objects was the same, irrespective of its weight. Experiments show this acceleration $a = g \simeq 10\,\mathrm{m/s^2}$.

However, this model is only an approximation. In Galileo's era, time was generally measured by counting pulse beats. Galileo was one of the first to recognize that a swinging pendulum was a much better 'clock' and experiments became much more accurate thereafter. It then became clear that the model equation for falling objects should include a deceleration term which increases with the speed of the object.

The simplest model of the deceleration term is to consider it to be linear proportional to the velocity[1]. Considering that the velocity of the falling object is positive ($v > 0$), the new equation of motion for a falling object reads

(1.1d) $$\frac{dv}{dt} = g - kv,$$

wher the constant of proportionality $k > 0$ varies from object to object.

The differential equation (1.1d) is more complicated than (1.1b), since it involves v as well as dv/dt. In the next Chapter we will learn how to solve equations of this kind to find the solution

$$v(t) = \frac{g}{k}(1 - e^{-kt}).$$

Even if we don't know how to obtain the solution at this stage, we can already explore it. First, by introducing the given $v(t)$ in the left- and right-hand side of Eq. (1.1d) we can verify that it is indeed a solution. Second, we can see that instead of increasing indefinitely, the speed $v(t)$ tends to a finite limit g/k: $\lim_{t\to\infty} v(t) = \frac{g}{k}$. This value is the **terminal speed** of the falling object, the maximum speed it achieves. As consistency checks, note that introducing $v = \frac{g}{k}$ in Eq. (1.1d) leads to $\frac{dv}{dt} = 0$ (no acceleration) and that the ratio g/k has the dimension

[1]This corresponds to Stokes' law and is a good approximation for objects falling in viscous fluids. At speeds achieved by common objects falling through air or water, a deceleration proportional to the second power of the speed provides a more accurate description.

of velocity $[m/s]$ (the dimension of g is $[m/s^2$ and the dimension of k is $[1/s]$). Since the value of k is not a universal constant, the terminal speed will vary between objects. It is thus clear that this model describes a behaviour very different from that of Galileo's model when t become large.

Over small times the two models are indistinguishable to all intents and purposes (assuming $v(0) = 0$). Over larger times, the second represents the observed behaviour much better. However, the distances must not be too large. Medieval astronomers knew that a uniform acceleration model would not reproduce the motion of celestial bodies. The Moon is not falling freely towards the Earth with a constant acceleration due to gravity.

Newton provided the model for the motion of bodies on this scale. His theory of gravity from 1678 states that the weight of a object is not constant but varies inversely as the square of the distance between the object and the centre of the attracting body. If r is the radial distance from that centre to the object, the equation of motion is

$$m\frac{dv}{dt} = -\frac{GMm}{r^2}, \tag{1.1e}$$

where M is the mass of the attracting body (for terrestrial experiments this is the Earth) and G is a constant ($6.67 \times 10^{-11}\,\mathrm{N\,m^2/kg}$) known as the gravitational constant. Note that the sign of the right hand side is negative because the radial distance r is measured *outwards* and gravity acts *inwards*, that is, downwards.

Using chain rule and $v = \frac{dr}{dt}$ we obtain

$$\frac{dv}{dt} = \frac{dv}{dr}\frac{dr}{dt} = v\frac{dv}{dr}$$

and so (1.1e) can be written as

$$\frac{dv}{dr} = -\frac{GM}{vr^2}. \tag{1.1f}$$

This type of equation is also solved in Chapter 2; the solution yields the following formula for the velocity as a function of distance s fallen from rest at $r = R$,

$$v(s) = \sqrt{\frac{2GMs}{R(R-s)}}.$$

This is again a very different formula to that produced by Galileo's model, but then Galileo's model was not designed to apply over such large distances. He was interested in motion over only small heights above the ground, that is, to motion in which s is always much smaller than R. When $s \ll R$, Newton's model reproduces the form of Galileo's model.

For centuries Newton's law was thought to be the ultimate model of celestial dynamics. But it too becomes inadequate over velocities close to the speed of light $c \approx 3\ 10^8 m/s$ or in the vicinity of extremely small massive objects like black holes. At present, the most accurate model for the motion of a falling object is provided by Einstein's theory of general relativity (1916).

1.2 Limitations of models

No model reproduces *exactly* what is observed and no model provides more than an approximate description of the relationships between the quantities involved. This is patently true of economic laws and weather forecasts. It is not so generally recognized that it is also true of a subject like dynamics. Yet dynamics consists simply of a set of models of how objects are seen to move. Each model is useful only under circumstances that must be clearly stated.

There are hierarchies of models. For example, Newton's model is more general than Galileo's and embodies the earlier model as a limiting case. However, the later models do not entirely supersede the earlier because they are often more complicated in structure and more difficult to work with. So long as we are aware of the limits of their validity and are content to work within their limits of accuracy, the simple models are to be preferred. This is summed up in a precept known as 'Occam's Razor'—*Entia non sunt multiplicanda praeter necessitatem* (Do not introduce complications that are not really necessary)—which is attributed to William of Occam, who died in 1349.

The dangers of unnecessary complication can be illustrated by another dynamical example. We have seen that Newton's law of resistance and Newton's law of gravitation were developed to describe motion in two different regimes. Could not greater accuracy be obtained if they were combined? The model equation would then read

$$v\frac{dv}{dr} = -\frac{GM}{r^2} + kv^2 .$$

The air above the Earth's surface gets rapidly thinner with height; so it would be more realistic to assume that the resistive coefficient k is a function of height, $k = k(r)$, rather than a constant. If $k(r)$ is a simple enough function, this can be solved explicitly.

Now suppose that we wish to allow for the motion when speeds are not sufficiently high for Newton's law of resistance to apply. Then we might want to assume

$$v\frac{dv}{dr} = -\frac{GM}{r^2} + k(r)v^n ,$$

where the exponent n takes some value in the range 1–2. This is very little different to the previous equations. For $n = 0$ or $n = 2$ it can can easily integrated by the methods we are about to develop. Otherwise it becomes intractable to direct analysis.

This is as far as we can go in finding explicit solutions to the problem of an object falling through the atmosphere. To go further we must return to Galileo's starting point. The differential equation must be converted into a *difference* equation which is then solved numerically.

Adding a little extra complication made the problem intractable to any elementary analysis. This is not a great problem these days when powerful computers are available to produce numerical results. However, numerical solutions are always particular solutions appropriate to the chosen starting point. The computer does not allow us to find general solutions and examine the behaviour of a whole class of solutions. If one is interested in predicting *general* behaviour it is still better to try to understand a simpler model for which explicit solutions can be found.

1.3 What is a mathematical model?

To summarize the features of mathematical modelling let us return to Galileo's model of motion with constant acceleration.

- The process started with an observation, the motion of objects falling freely from rest. The observation provided the *stimulus* for the model.
- A mathematical relationship between the quantities involved in the observations was proposed. The symbolic equation was a guess at describing an underlying *organization* of the mass of observational data.
- Mathematical analysis was used (indeed, developed) to produce further relationships between the quantities concerned. These were the mathematical *consequences* of the proposed model relationship.
- The model was accepted because one of the derived relationships reproduced the original observational law. To this extent, the model showed a *correspondence* with reality. If that had not been the case, another model would have been sought.
- The original observation was not the only relationship produced by the model. The model also provided a different law of motion for the case of an object falling freely not from rest but with some initial velocity u. It led to *predictions* which could be tested by further observations.
- The model could be applied to other situations in which objects move with uniform acceleration. It had greater *generality* than the initial observation.
- Finally, the model could be applied to the special case in which the acceleration vanishes. Setting $a = 0$ recovers Galileo's law of uniform motion, proportional distances are covered in proportional times. The model provided a *link* between two previously unrelated forms of motion, uniform motion and motion with uniform acceleration.

The first four items in the list above demonstrate the purpose of a mathematical model. It allows a body of observational data to be systematized by means of mathematical formulae. The next two items give a clue to the importance of mathematical models. The most significant models go *beyond* what was originally envisaged. They provide not only a correspondence with the specific observation that was the stimulus, but make predictions which can be the subject of further experiment, and demonstrate links with other models which have been developed independently.

In the natural sciences, a theory is a series of linked models which between them cover a very wide range of observations. Sciences like physics, chemistry and engineering have progressed a long way towards being based on a few, all-embracing models (sometimes elevated to the status of "laws") from which all other models can be derived as special cases. Other sciences, particularly those dealing with much more complicated organic systems, still rely on a set of more or less unrelated models.

1.4 General Properties of Differential Equations

The differential equations introduced in the previous section have taken different forms. Put in terms of common variables x and y, (1.1c) is of the type

$$\frac{dy}{dx} = f(x),$$

(1.1d) is of the type

$$\frac{dy}{dx} = f(y),$$

whilst (1.1f) is of the type

$$\boxed{\frac{dy}{dx} = f(x, y).} \tag{1.4a}$$

Clearly the first two types are just special cases of the third type with $f(x, y)$ taken to be independent of y and x in each case.

We will refer to (1.4a) as the standard form for a first-order differential equation. The general classification of differential equations will be taken up in the next chapter. It is very important to be able to tell which type of differential equation you are dealing with because the method of solution will depend upon that identification. However, before looking at the various methods for solving such differential equations we will look at the general properties of the solutions, properties which are independent of the method of solution.

Solution Curves

As was noted in the Introduction a differential equation does not have just a single solution. The process of integration produces a constant of integration. This constant can be varied and may be chosen to meet other conditions imposed by the problem, e.g. the specification that the body fell from *rest* or that distance is measured from the point of release in the earlier examples.

General and particular solutions

- **The general solution** is the form of solution which incorporates the constants of integration.
- **The particular solution** is a form in which the constants are given particular values in order to meet further conditions.

Since the solutions of the differential equations encountered in the previous section are functions of a single variable $y(x)$, these ideas can be illustrated in a simple graphical manner.

The general solution is represented by a set of curves. In the case of (1.1c) or $\dfrac{dy}{dx} = kx$ the general solution is

$$y = \frac{kx^2}{2} + C\,,$$

which is a set of parabolas displaced vertically one above the other. Each parabola represents a particular solution. There is clearly just one such curve through each point (x_0, y_0) in the x, y-plane, that is, there is a *unique* particular solution passing through every point. We call this curve the **solution curve** passing through (x_0, y_0).

Direction Fields

In general it is not possible to find expressions for the solution of a differential equation in terms of familiar elementary functions such as log, exp, sin, cos, tan, $\ldots \sin^{-1}$, $\tan^{-1}$, $\ldots$. However, we can gain a very good idea of what the solution curves look like *without solving the equation*.

To do this, we make use of the fact that if we know that a solution curve passes through some point, (x_1, y_1) say, there is a simple geometric construction for drawing the tangent to the solution curve at this point. Since the slope of the tangent to the solution curve at this point is the derivative dy/dx, it is given by $f(x_1, y_1)$. Hence the tangent is the straight line

(1.4b) $$y = y_1 + f(x_1, y_1)(x - x_1)\,.$$

If we indicate this tangent on a graph by a short straight line segment through (x_1, y_1), we will have an approximation to the solution curve there. This can be repeated for any number of different points (x_1, y_1). The totality of all such line segments forms the **direction field** for the differential equation.

This suggests the following graphical construction of solution curves:

Direction field construction

- At a large number of points in the chosen region of the (x, y) plane, use (1.4b) to draw short segments of the tangent to the solution curves at those points.
- Then, starting from an initial point (x_0, y_0), construct the solution curve by drawing it parallel to the segment of direction field at each point through which it passes.

Example 1.4c As a simple example to illustrate these ideas, take the differential equation

(1.4d) $$\frac{dy}{dx} = 1 - x\,.$$

The direction field for this differential equation is shown in Figure 1.1.

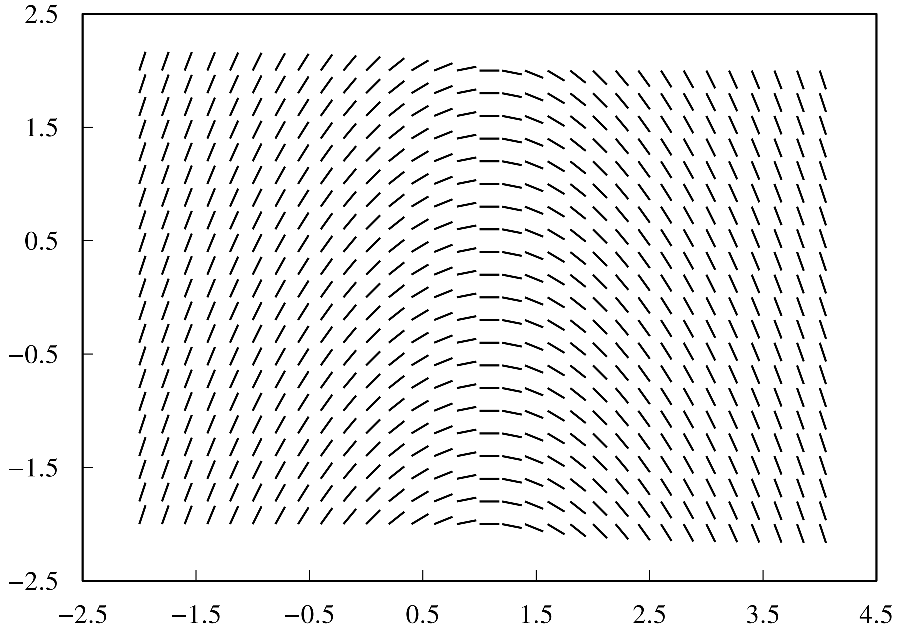

Figure 1.1:

It is easy to see that in the part of the (x, y) plane to the left of the vertical line $x = 1$, the slope of the direction field is always positive and that to the right of this line is always negative. Consequently the solution curves are increasing to the left of $x = 1$ and decreasing to the right of $x = 1$. Furthermore, in this special case where $f(x, y)$ is independent of y, the slope at any given point (x, y) is independent of the y value, so it is clear that the families of solution curves may be generated by shifting a given solution curve vertically upwards or downwards by a constant amount. In symbols, if $y = F(x)$ is a solution, so is $y = F(x) + C$, where C is a constant. ◇

Example 1.4e As a second example, let us consider a differential equation where $f(x, y)$ is a function of y alone. A simple example is provided by the logistic equation[2]

$$\frac{dy}{dx} = ky - ay^2 .$$

For the sake of a concrete example, let us take $k = 2$ and $a = 1$. In the positive quadrant $(x > 0, y > 0)$, the direction field for this differential equation then appears as in Figure 1.2.

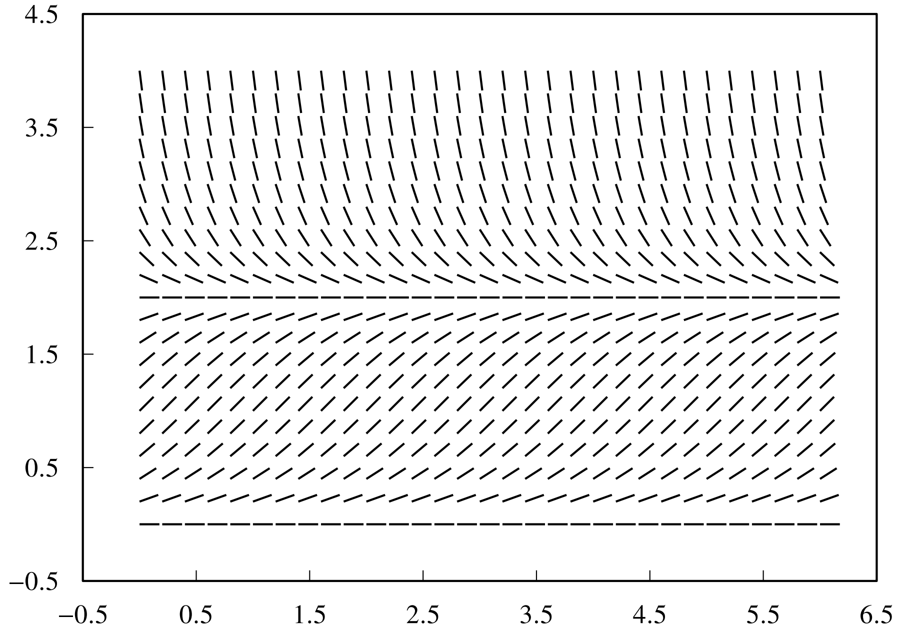

Figure 1.2:

It is clear that the line $y = 2$ divides the direction field into two parts. Above $y = 2$ the solution is always decreasing, and in between $y = 0$ and $y = 2$ the solution is always increasing. Furthermore, $y = 0$ and $y = 2$ are both solution curves. This leads us to guess (correctly) that a solution curve which starts off in the region $0 < y < 2$ always stays in that region, and that a solution which starts off in the region $y > 2$ always stays in that region.

The direction fields for all differential equations of the form $y' = g(y)$ have a geometrical symmetry which makes it easy to generate a whole family of solution curves once one solution curve is known: we simply shift the whole solution curve to the right or left by a constant amount. In symbols, if $y = F(x)$ is a solution, so is $y = F(x + C)$, where C is a constant. Note carefully the difference between this symmetry and the symmetry previously described for (1.4d). ◇

Example 1.4f As a more complicated example we take the differential equation

$$\frac{dy}{dx} = \frac{(x - 2y)(y - 1 + x)}{1 + x^2 + y^2} ,$$

[2]This equation receives detailed treatment in Chapter 2.

whose direction field is given in Figure 1.3. Can you sketch some approximate solution curves?

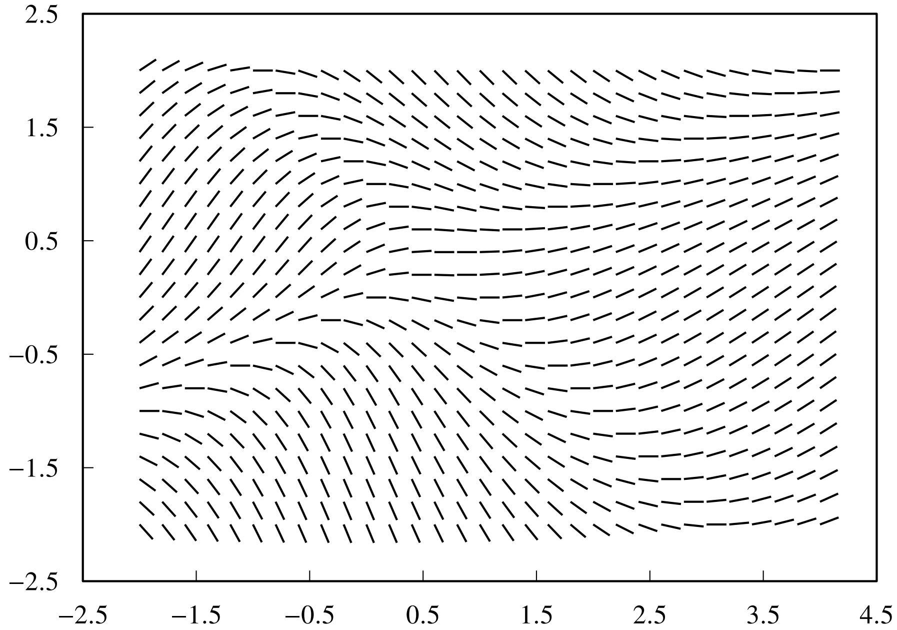

Figure 1.3:

◇

Numerical solutions: The procedure of estimating a particular solution of a differential equation from a direction field is similar to the procedure used to obtain numerical estimations of the solution using computers. Computers do not handle infinitesimally small quantities so that results of numerical computations always depend on some small parameter h (good numerical solutions can be made arbitrary similar to the true solutions by reducing h). Let us discuss one simple procedure to obtain an approximate solution of a differential equations such as Eq. (1.4a) passing through point (x_0, y_0). We consider $\frac{dy}{dx} \approx \frac{\Delta y}{\Delta x}$ and obtain the next point (x_1, y_1) of the solution as

$$\Delta y = f(x_0, y_0)\Delta x \Rightarrow y_1 - y_0 = f(x_0, y_0)(x_1 - x_0)$$

In the direction field, the value $f(x_0, y_0)$ corresponds to the slope of the line at the point (x_0, y_0). Identifying Δx with our small parameter h we obtain that

$$x_1 = x_0 + h,$$

$$y_1 = y_0 + f(x_0, y_0)h.$$

Repeating this procedure starting from (x_1, y_1) we obtain (x_2, y_2), and so on. Consequently, we can obtain a series of points $(x_0, y_0), (x_1, y_1), (x_2, y_2), \ldots$ which approximate the particular solution of the differential equation. This method to obtain numerical solutions of differential equations is known as Euler method. Many alternative methods exists and are widely used in numerous applications for which no analytical solution can be obtained.

Existence and Uniqueness

If we wanted to generate the solution curves for the above examples numerically there would be no difficulty in stepping along them from any given starting point. The derivative $f(x)$ gives the slope of the tangent to the curve at each point. If the x step is made very small the rise along the tangent approximates the rise of the curve $f(x)$—put more formally, the differential approximates the increment. Unfortunately, this 'nice' property of being able to generate the solution curves by tracing out the curve in space is not shared by all first-order differential equations of the form (1.4a).

These matters are the topic of the *Existence and Uniqueness Theorem*. This theorem has two parts, one relating to when solutions exist, the other to when they are unique. The proof lies outside the scope of a first-year course[3]. We shall simply quote the results.

- A solution *exists* over a certain domain of x and y if $f(x, y)$ is continuous in that domain.

- The solution is *unique* if the values of f do not fluctuate too wildly (the range of $f(x, y)$ must always be a finite multiple of the variation in y along the curve). More formally, the uniqueness of the solution is ensured if $\partial f/\partial y$ is continuous, where $\partial f/\partial y$ is the partial derivative derivative of f in respect to y (a concept that will be introduced in Chap. 8). If the derivative becomes infinite at any point there is potential trouble.

You need only remember that, for all practical purposes, the theorem guarantees that we are not wasting our time looking for a solution. Moreover, having found one, we need look no further for any others.

[3]For those interested in the (somewhat lengthy) proof it may be found in *Differential Equations and Their Application* by M. Braun (Springer-Verlag, 1984)

Summary of Chapter 1

- **The order** of a differential equation is the order of the highest derivative occurring.
- **The standard form** of the first-order differential equation is

$$\frac{dy}{dx} = f(x, y)\,.$$

All first-order differential equations should first be put in this form before proceeding.

- **The general solution** of a first-order equation is a set of curves generated by different choices of the single constant of integration.
- **The particular solution** is the curve passing through a specified point (x_0, y_0). There is *one* and *only one* particular solution through every point in general.
- **The direction field** provides a visual representation of a differential equation that allow us to obtain a good idea of particular solutions.

Exercises

1.1 A mothball initially has radius 0.5 cm and slowly evaporates.

a) If V denotes the volume of the mothball and r the radius, use the chain rule for differentiation to show that

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}.$$

b) Suppose that the rate of change of the volume is proportional to the surface area of the mothball. Express this condition as a differential equation for r as a function of t.

c) Find a formula for the radius as a function of time, assuming that after 30 days the radius is 0.25 cm. How long before the mothball disappears altogether?

1.2 A car is travelling at 100 km/h on a level road when it runs out of fuel. Its speed v starts to decrease according to the formula

$$\frac{dv}{dt} = -kv,$$

where k is a constant. One kilometre after running out of fuel its speed has fallen to 50 km/h. Use the chain rule substitution

$$\frac{dv}{dt} = \frac{dv}{ds}\frac{ds}{dt} = \frac{dv}{ds}v$$

to solve the differential equation.

a) How far will the car travel from the point where it runs out of fuel?
b) How long after running out of fuel will the car come to a stop?
c) Is the model reasonable?

CHAPTER 2

Separable Equations

2.1 Introduction

In this chapter we begin our systematic investigation of differential equations. Recall that a differential equation is an equation involving the derivatives (or the rates of change) of a function. If the dependent variable y is a function of a single independent variable x, that is $y = y(x)$, then the function will possess ordinary derivatives (not partial derivatives) and a differential equation for y will involve the derivatives dy/dx, d^2y/dx^2, and so on. For example,

$$\frac{d^4y}{dx^4} + 9x\frac{d^2y}{dx^2} + 5\cos y\left(\frac{dy}{dx}\right)^2 = 19$$

is a differential equation. The **order** of the equation is the order of the highest derivative occurring in it. In this example the order is four, since d^4y/dx^4 is a fourth-order derivative. We shall be concerned mainly with first-order equations and some second-order equations.

An example of a first-order equation is

$$\left(\frac{dy}{dx}\right)^7 + \frac{x^2}{y}\left(\frac{dy}{dx}\right)^2 = \cos^3 x\,.$$

This is a particularly nasty example because there is no way in which this can be re-arranged so that the derivative dy/dx is an explicit function of x and y, that is, it cannot be put in the standard form

$$\frac{dy}{dx} = f(x, y).$$

When this is the case, the **degree** of the equation is that of the highest power of the (highest) derivative. This is of seventh degree and will interest us no further. Here we shall deal only with first-order equations of the first degree:

$$\boxed{\frac{dy}{dx} = f(x, y).}$$

We shall concentrate exclusively on *how* you find a solution. Unfortunately, there is no universal method of solution. We can give recipes for solving certain standard types of equation. In each of these, the derivative is given by an expression with a particular form, that is, $f(x, y)$ has a special functional form. We shall go through these below.

However, many differential equations assume standard form only after some manipulation such as a change of variable, or multiplication by a suitable factor. Recognizing how to produce a standard type of equation is quite an art, and, like all art, it comes only with practice.

2.2 Simple First-Order Differential Equations

We have already looked at examples of the simplest type of differential equation, namely, those in which the derivative is expressed as a function of x alone:

$$\boxed{\frac{dy}{dx} = f(x).} \tag{2.2a}$$

An example is

$$\frac{dy}{dx} = x^2.$$

This simple type can be given a fairly complete mathematical treatment. By the Fundamental Theorem of Calculus we know that (2.2a) always has a solution $y = F(x)$, provided that f is continuous. The function F is an antiderivative of f and can be written formally as an indefinite integral of f,

$$F(x) = \int f(x)\,dx + C,$$

C being the constant of integration. In practice, given an equation of the form (2.2a) the procedure is to integrate with respect to x

$$\int \frac{dy}{dx}\,dx = \int f(x)\,dx,$$

to produce

$$\boxed{y = \int f(x)\,dx = F(x) + C.}$$

The solution of the differential equation reduces to a problem in integration. In the example we have

$$y = \int x^2\,dx = \frac{x^3}{3} + C.$$

A table of standard integrals is provided in the Appendix A. Techniques to integrate more complicated functions, and the theory of integral calculus, are the subject of MATH1021 *Calculus of One Variable*. We recommend you to revise this material in the MATH1021 course notes.

General solution

The *set* of solutions $y = F(x) + C$ produced as the constant of integration is varied forms the **general solution** of the differential equation (2.2a). Moreover, this set of solutions includes *all* the possible solutions of the differential equation. There are no other. This property is known as **completeness**. Since the proof is so simple, let us look at it here:

Proof. Suppose $F(x)$ is an antiderivative of f and let $y(x)$ be any solution of the differential equation (2.2a). Consider the function $G(x) = y(x) - F(x)$. Its derivative is just

$$\frac{dG}{dx} = \frac{dy}{dx} - \frac{dF}{dx} = f(x) - f(x) = 0;$$

that is, it vanishes everywhere. The only continuous function whose slope is everywhere zero is a constant function $G(x) = C$. (Recall the formal proof of this uses the Mean Value Theorem.) In other words, another solution of the differential equation (2.2a) can differ from a given antiderivative by only a constant.

Particular solution

The value of C will be fixed if it is required that y take a given value y_0 at some given value x_0 of x; that is, if the solution curve is required to pass through (x_0, y_0). Then substituting the given values

$$y_0 = F(x_0) + C,$$

we find that C is given by

$$C = y_0 - F(x_0).$$

Thus the solution satisfying this constraint is

$$y = F(x) + [y_0 - F(x_0)].$$

This is then the **particular solution** of the differential equation passing through (x_0, y_0).

The existance of an arbitrary constant C is the distinguishing feature between general and particular solutions. However, as we will see in later examples, the arbitrary constant C is not necessarily a simple additive constant.

Example 2.2b Find the general solution of

$$\frac{dy}{dx} = kx,$$

and the particular solution which satisfies $y = 1$ when $x = 0$.

This differential equation is of simple type so that its solution is obtained by integrating with respect to x,

$$y = \int kx\,dx = \frac{kx^2}{2} + C.$$

This is the general solution.

The value of C which gives the required particular solution is found by substituting $y = 1$ and $x = 0$,

$$1 = 0 + C.$$

Thus $C = 1$ gives the required particular solution

$$y = \frac{kx^2}{2} + 1.$$

◇

A form of differential equation very closely related to that just solved is that in which the derivative is given as a function of y alone,

$$\boxed{\frac{dy}{dx} = f(y).}$$

We can transform it into the type just described if we use the fact that

$$\frac{dy}{dx} = \frac{1}{dx/dy},$$

to rewrite it as

$$\frac{1}{dx/dy} = f(y),$$

or

$$\frac{dx}{dy} = \frac{1}{f(y)}.$$

This is now of the previous form with the dependent and independent variables interchanged. It may be solved by simply integrating with respect to y,

$$\boxed{x = \int \frac{dy}{f(y)}.}$$

Example 2.2c Find the general solution of

$$\frac{dy}{dx} = 3y^2.$$

Since the right-hand side is a function of y alone, we take the reciprocal of both sides and solve

$$\frac{dx}{dy} = \frac{1}{3y^2}.$$

Integrating with respect to y gives x as a function of y,

$$x = \int \frac{1}{3y^2}\,dy = -\frac{1}{3y} + C.$$

◇

2.3 Separable Differential Equations

More generally, it is very common to meet differential equations such as

$$\frac{dy}{dx} = f(x, y), \tag{2.3a}$$

where the given function $f(x, y)$ depends on both x and the sought-after function $y(x)$. There are many examples of this type later in the notes. A common beginner's mistake here is to integrate $f(x, y)$ with respect to x, treating y as a constant. The problem with this idea is that y is *not* a constant—it depends on x in a way we are trying to determine. The dependencies can be made clearer by writing (2.3a) in the form

$$\frac{d}{dx}y(x) = f\big(x, y(x)\big)\,.$$

We can also see the difficulty by the following calculation. Suppose that the result of integrating $f(x, y)$ with y constant is a function $F(x, y)$. Then $f(x, y)$ is the derivative of F when y is held constant. This is just the definition of the *partial derivative* with respect to x, $\frac{\partial}{\partial x}$ to be introduced in Chapter 8, and therefore

$$\frac{\partial F}{\partial x}(x, y) = f(x, y)\,.$$

This is alright so far, but we cannot now claim that $y = F(x, y)$ is a solution to the differential equation. The chain rules gives

$$\frac{dy}{dx} = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{dy}{dx} = f + \frac{\partial F}{\partial y}\frac{dy}{dx}\,.$$

and the appearance of the extra term on the right shows that the original equation is not satisfied.

There is no single method for the solution of (2.3a); the choice of approach depends on the particular form of the function $f(x, y)$. One situation, common in applications, occurs when the x and y dependence of f can be 'separated out' by writing $f(x, y)$ as a *product* $g(x)h(y)$. In this case the equation (2.3a) is said to be **separable**. We can rewrite it as

$$\frac{1}{h(y)}\frac{dy}{dx} = g(x)\,.$$

Since we know $g(x)$, we can integrate the right side of this equation with respect to x. What happens when we perform the same operation on the other side of the equation? Making explicit the dependence of y on x, the integral with respect to x of the left side is

$$\int \frac{1}{h(y(x))} y'(x)\, dx\, .$$

But this is now exactly the right form for integration by substitution, with new variable $y = y(x)$. The fact that y depends on x is now irrelevant, since in the transformed integral the variable of integration is y rather than x. Therefore

$$\int g(x)\, dx = \int \frac{1}{h(y(x))} y'(x)\, dx = \int \frac{1}{h(y)}\, dy\, .$$

Evaluation of the two integrals yields a relation between x and $y = y(x)$ involving one arbitrary constant. Generally this is an *implicit* formula for y in terms of x. It may be possible to solve it to get y *explicitly* as a function of x.

Although this argument is based on the substitution formula, it is very easy to remember how to apply it. Starting with the equation

$$\frac{dy}{dx} = g(x)h(y)$$

we move all the x terms to one side and all the y terms to the other, leaving

$$\frac{1}{h(y)}\, dy = g(x)\, dx\, .$$

Finally we write the integral sign on both sides

$$\boxed{\int \frac{1}{h(y)}\, dy = \int g(x)\, dx + C\, .}$$

The arbitrary constant is included as a reminder that the integrals are not determined uniquely. Although this way of arriving at the formula may be questionable mathematically, it is justified by its agreement with the substitution formula. This technique, applied to separable equations, is called **separation of variables**.

Example 2.3b Find the general solution of the differential equation

$$\cos x \frac{dy}{dx} - y^2 \tan x = 0.$$

First put the equation into standard form as $dy/dx = f(x, y)$. Making dy/dx the subject, we get

$$\frac{dy}{dx} = \frac{y^2 \tan x}{\cos x}.$$

Simplify the right hand side if at all possible:

$$\frac{dy}{dx} = y^2 \tan x \sec x.$$

Now inspect the right hand side to see whether it can be written as a product of factors involving x only and y only. In this case, it is clearly possible as

$$\frac{dy}{dx} = (y^2)(\tan x \sec x).$$

Divide through by y^2 to separate the variables,

$$\frac{1}{y^2}\frac{dy}{dx} = \tan x \sec x$$

and now integrate both sides,

$$\int \frac{1}{y^2}\frac{dy}{dx}\, dx = \int \tan x \sec x\, dx,$$

or

$$\int \frac{1}{y^2}\, dy = \int \tan x \sec x\, dx.$$

Both integrations may now be performed (using tables if necessary):

$$-\frac{1}{y} + C_1 = \sec x + C_2.$$

Although we have introduced constants of integration on both sides, the two are not really necessary because the solution may be re-arranged as

$$-\frac{1}{y} = \sec x + (C_2 - C_1),$$

and the difference between two arbitrary constants is just some other constant, C say. Thus a constant of integration need only be introduced on *one side* and the general solution can be written

$$-\frac{1}{y} = \sec x + C.$$

$\diamond$

We could tidy up the solution in the last example by making y the subject,

$$y = -\frac{1}{\sec x + C}.$$

This gives us an explicit expression for $y = F(x)$. However, *this is not usually possible in the case of separable equations*. Most often, the answer will be an implicit expression of the form $G(y) = F(x)$. So do not worry if it is not obvious how to find $y(x)$ explicitly.

When tidying up, be careful *not* to (incorrectly) rewrite the solution as

$$y = -\frac{1}{\sec x} - C = -\cos x - C.$$

This is not a solution of the original differential equation: differentiate it and see.

Note. Simple differential equations in which the derivative is a function of x or y alone are just special cases of separable equations. We do not really need to distinguish the types.

Example 2.3c Use separation of variables to find a solution to the equation of exponential growth:

$$\frac{dx}{dt} = kx\,,$$

where k is a positive constant. In particular, find a solution $x = x(t)$ satisfying the condition $x(0) = a$, where a is a constant. Assume that $x > 0$.

Separate the x and t variables to give

$$\frac{1}{x}\,dx = k\,dt\,.$$

Now integrate, so

$$\int \frac{1}{x}\,dx = k \int dt\,,$$

The two indefinite integrals are easily evaluated, giving

$$\ln|x| = kt + C\,,$$

where C is an arbitrary constant. This equation defines x implicitly in terms of t, but it is easy to solve for x in order to get an explicit relation. Just apply the exponential function to both sides and use the fact that it is inverse to the logarithm. Thus

$$x = e^{kt+C} = e^{kt}e^{C} = (e^{C})e^{kt} = Ae^{kt}\,,$$

where we define the new arbitrary constant $A \equiv e^{C}$. Note that by allowing $A > 0$ and $A < 0$ we automatically consider both $x > 0$ and $x < 0$ cases, effectively getting rid of $|x|$.

Finally, using the initial condition $x = a$ when $t = 0$, we obtain $A = a$ and thus

$$y = ae^{kt}\,.$$

Substitution into the original equation confirms that this is a solution. ◇

2.4 Newtonian Dynamics

Newtonian dynamics is one of the most complete and successful mathematical models ever proposed and forms the basis of almost all branches of science. It provides also some of the best examples of the importance of differential equations for our understanding of the natural world. Several examples of its many areas of application are worked through here.

Newtonian dynamics is based on Newton's law of motion[1]:

$$\boxed{\frac{d}{dt}(mv) = F.} \tag{2.4a}$$

This states that the rate of change of momentum mv is equal to the applied force F. The momentum is the product of mass m and velocity v.

When the mass is constant, equation (2.4a) reduces to

$$\boxed{m\frac{dv}{dt} = F.}$$

The rate of change of velocity is the acceleration a,

$$\boxed{a = \frac{dv}{dt}.}$$

[1]Isaac Newton (1642—1727) was born in the year of Galileo's death. Newton took the ideas about acceleration and developed them to form an all-embracing dynamical theory of matter that survived until the beginning of last century. In the course of this work he invented the differential calculus (discovered independently by Leibniz at the same time) to deal with rates of change.

The basis of Newtonian dynamics is the concept of force. Acceleration is deemed to be the result of the action of force. Force is a result of the mutual interaction of two objects—the force exerted by one object on another is equal and opposite to that exerted by the second on the first. It is observed though that the accelerations produced by these equal and opposite forces are not always the same. The ratio of the accelerations gives the inverse ratio of a property of the objects which we call mass. With mass thus defined, Newton's law of motion can be written

$$F = ma,$$

where F is the force, a the acceleration and m the mass.

We now recognize many kinds of force. When an object is acted upon by several forces, it is observed that it moves as though under the action of a single force equal to the vector sum of the individual forces. If the resultant force vanishes, the object experiences no acceleration. It may then remain at rest or in uniform (constant speed) motion in a straight line.

Once the forces F and the mass m are specified, the motion may be found by integrating to obtain $v(t)$. Moreover, we can often go further. Recall the definition of velocity as rate of change of position x,

$$\boxed{v = \frac{dx}{dt}.}$$

Having found $v(t)$, we can then integrate *again* to find $x(t)$.

Force as function of t or v

An object falling through air close to the surface of the Earth is accelerated by its weight force and decelerated by air resistance. The weight force is $W = mg$ where m is the mass of the object and g is the 'acceleration due to gravity'. If the object is large and light so that it falls slowly, the resistance is $R = mk|v|$ where v is the velocity and k is a constant depending on the properties of the object and the medium through which it falls[2].

Example 2.4b An object falls from rest through air that provides a resistance $mk|v|$. Find the subsequent velocity and distance fallen as functions of time.

If the object is falling downwards let us measure its position x downwards from the initial point. Then the equation of motion is

$$m\frac{dv}{dt} = mg - mkv.$$

(Note that the second term is negative since m, k, v are all positive by assumption and we require a deceleration—a negative acceleration.)

Before starting to solve such an equation, get rid of the factor m which is common to all terms:

$$\frac{dv}{dt} = g - kv.$$

This is a separable equation and may be integrated immediately,

$$\begin{aligned} t &= \int \frac{dv}{g - kv} \\ &= -\frac{1}{k} \ln |g - kv| + C, \end{aligned}$$

[2]The value of k is given by Stokes' law, $k = 18\nu\rho_s/d^2\rho_o$, where ρ_o and d are the density and diameter of the object and ρ_s and ν are the density and (kinematic) viscosity of the surrounding medium. Typical numbers for density in units of $\mathrm{kg/m^3}$ are 1.25 for air, 10^3 for water, 10^4 for metals. The viscosity of air is $10^{-5}\ \mathrm{m^2/s}$, that of water is $10^{-6}\ \mathrm{m^2/s}$. This form of the resistance force is valid if a quantity known as the Reynolds' number $R = vd/\nu$ is less than about 10. It requires slow speeds v, large diameters d and high viscosity ν (as in heavy oils).

or, re-arranging as before,

$$g - kv = Ae^{-kt},$$

where $A = \pm e^C$ is an arbitrary constant[3]. If $v = 0$ when $t = 0$ we must have $g = A$ so that

$$v = \frac{g}{k}(1 - e^{-kt}).$$

Having found $v(t)$ we can now proceed to the next step, writing

$$v = \frac{dx}{dt} = \frac{g}{k}(1 - e^{-kt}).$$

This is of simple type. Integrating with respect to t gives

$$\begin{aligned} x &= \int \left(\frac{g}{k} - \frac{g}{k}e^{-kt}\right) dt \\ &= \frac{g}{k}t + \frac{g}{k^2}e^{-kt} + C. \end{aligned}$$

If $x = 0$ when $t = 0$ we must have $0 = g/k^2 + C$, hence the required solution is

$$x = \frac{gt}{k} - \frac{g}{k^2}(1 - e^{-kt}).$$

◇

Here we may note that the velocity does not increase without limit as it does in the case of free fall without air resistance. Instead, as $t \to \infty$ the exponential term will vanish so that $v \to g/k$. This is known as the **terminal velocity** of the object.

These results look quite different to the usual constant acceleration formulae with which you are familiar. However, we may use the series expansion of the exponential function to write

$$\begin{aligned} e^{-kt} &= 1 + (-kt) + \frac{(-kt)^2}{2!} + \ldots \\ &= 1 - kt + \frac{k^2t^2}{2} - \ldots \end{aligned}$$

Substituting, we find

$$\begin{aligned} v &= \frac{g}{k}(1 - e^{-kt}) \\ &= \frac{g}{k}\left(1 - 1 + kt - \frac{k^2t^2}{2} + \ldots\right) \\ &= gt - \frac{gkt^2}{2} + \ldots. \end{aligned}$$

[3]The absolute value in the logarithmic function allow for $A > 0$ and $A < 0$, even if e^C is strictly positive.

For very low values of the resistance we need retain only the first few terms of the expansion. As $k \to 0$ we recover

$$v = gt,$$

which is Galileo's result for motion with constant acceleration.

The demonstration that in the limit $k \to 0$ we recover

$$x = \frac{gt^2}{2},$$

is left as an exercise.

Force as a function of x

It often happens that the forces acting on an object depend on the position of the object rather than time or velocity. In such cases the equation of motion

$$m\frac{dv}{dt} = F(x),$$

cannot be integrated directly because we now have *three* variables v, t and x. We must first reduce the number to two. This is achieved by using the chain rule to rewrite the acceleration as

$$\boxed{\frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = \frac{dv}{dx}v.}$$

In this form the derivative no longer involves t explicitly. Hence we can rewrite the differential equation in integrable form as

$$mv\frac{dv}{dx} = F(x).$$

Example 2.4c Given Newton's law of gravitation

$$m\frac{dv}{dt} = -\frac{GMm}{r^2},$$

find $v(r)$.

First cancel the factor of m from both sides, leaving

$$\frac{dv}{dt} = -\frac{GM}{r^2}.$$

Now we want to find $v(r)$ so we must rewrite the differential equation in terms of r as the independent variable using

$$\frac{dv}{dt} = v\frac{dv}{dr}.$$

This gives

$$v\frac{dv}{dr} = -\frac{GM}{r^2}.$$

This is a separable equation which is already separated, so we can integrate immediately to find

$$\int v\,dv = -\int \frac{GM}{r^2}\,dr$$
$$\frac{v^2}{2} = \frac{GM}{r} + C\,,$$

or, writing $B = 2C$,

$$v^2 = \frac{2GM}{r} + B\,.$$

◇

We can use this formula to look at motion close to the surface of the Earth. If we let R be the radius of the Earth, close to the surface we may write $r = R + x$ where x/R is a small quantity and x is measured *upwards*, that is, in the direction opposite to that in the previous example. Then

$$\begin{aligned} v^2 &= \frac{2GM}{R+x} + B \\ &= \frac{2GM}{R}\left(1 + \frac{x}{R}\right)^{-1} + B\,. \end{aligned}$$

In the second line we have rewritten the expression in the form suitable for application of the binomial theorem. Applying the theorem to $(1 + x/R)^{-1}$ gives

$$v^2 = \frac{2GM}{R}\left(1 - \frac{x}{R} + \dots\right) + B\,.$$

Sufficiently close to the surface, that is, making x/R sufficiently small, we can ignore all higher-order terms and write

$$v^2 \simeq \frac{2GM}{R} + B - \frac{2GM}{R^2}x.$$

Suppose now that $v = 0$ when $x = 0$, then $0 = (2GM/R) + B$ and the result is

$$v^2 \simeq -\frac{2GM}{R^2}x.$$

This is just Galileo's result for free fall under *constant* gravity with $g = GM/R^2$. Thus Newton's law not only contains Galileo's law as a special case but also gives us a means of evaluating the 'acceleration due to gravity'. Putting in the appropriate values for the Earth, $M = 5.98 \times 10^{24}$ kg, $R = 6.37 \times 10^6$ m, we obtain $g = 9.8\,\mathrm{m/s}$ at the surface of the Earth.

As a second example of the use of the formula for $v(r)$, consider an object projected upwards from the surface of the Earth $r = R$ with velocity V.

Then $V^2 = (2GM/R) + B$, so that the particular solution is

$$v^2 = V^2 - \frac{2GM}{R} + \frac{2GM}{r}.$$

If we now look at the maximum distance attained by the object, that is, the point where $v = 0$ so that it stops moving outwards, this is given by

$$0 = V^2 - \frac{2GM}{R} + \frac{2GM}{r_{max}}.$$

This can be re-arranged to give

$$r_{max} = \frac{2GM}{(2GM/R) - V^2}.$$

As V^2 increases the maximum distance r_{max} also increases (naturally!). Note, however, that the distance becomes indefinitely large as $V^2 \to (2GM/R)$. An object projected at this speed will continue to travel outwards for ever, thus escaping from the gravitational field of the Earth. The critical value $V = \sqrt{2GM/R}$ is known as the **escape velocity** of the Earth.

Summary of Chapter 2

- *Simple* differential equations of the form

$$\frac{dy}{dx} = f(x),$$

may be integrated immediately with respect to x,

$$y = \int f(x)\,dx\,.$$

- The variant

$$\frac{dy}{dx} = f(y),$$

must first be inverted and then integrated with respect to y,

$$x = \int \frac{1}{f(y)}\,dy\,.$$

- If $f(x,y)$ can be factorized into functions of x and y alone,

$$\frac{dy}{dx} = g(x)h(y),$$

the equation is *separable*. The function of y is removed to the left side, and then both sides are integrated,

$$\int \frac{1}{h(y)}\,dy = \int g(x)\,dx\,.$$

Exercises

2.1 Find the general solutions and sketch the solution curves of the differential equations:

a) $\dfrac{dy}{dx} = e^x$

b) $\dfrac{dy}{dx} = \sin x$

c) $\dfrac{dy}{dx} = \sinh x$

2.2 Find the particular solutions satisfying the given conditions:

a) $\dfrac{dy}{dx} = 1 + \cos x + \cos^2 x\,, \qquad y = 2$ when $x = 0$.

b) $\dfrac{dy}{dx} = 1 - 2x - 3x^{-2}$, $\quad y = -1$ when $x = 1$.

c) $e^{2x}\dfrac{dy}{dx} + 1 = 0$, $\quad y \to 2$ as $x \to \infty$.

d) $\dfrac{dy}{dx} = \left(x + \dfrac{1}{x}\right)^2$, $\quad y = 1$ when $x = 1$.

e) $\dfrac{dy}{dx} = x\sqrt{1 + x^2}$, $\quad y = -3$ when $x = 0$.

f) $x^2 e^{x^2}\dfrac{dy}{dx} = x^3$, $\quad y(0) = 1$.

2.3 Find the general solution of

a) $x^3\dfrac{dy}{dx} = 2x^2 + 5$

b) $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{1 + 3x}}$

c) $\cos^2 x\dfrac{dy}{dx} = \sin x$

2.4 If $y = 3$ when $x = 3$ and $dy/dx = (x^2 + 1)/x^2$, find the value of y when $x = 1$.

2.5 The graph of $y = f(x)$ passes through the point $(9, 4)$. Also, the line tangent to the graph at any point (x, y) has the slope $3\sqrt{x}$. Find $f(x)$.

2.6 Solve the differential equations

a) $\dfrac{dy}{dx} = y^2$

b) $\dfrac{dy}{dx} = \dfrac{4(1 + y^2)^{3/2}}{y}$

2.7 Find the general solution of the differential equation $y^\nu \dfrac{dy}{dx} = 1$, where ν is a real number. Write out the solutions explicitly in the cases when $\nu = -1$, $\nu = -3$, $\nu = -\dfrac{1}{2}$.

2.8 Find the particular solutions of

a) $y - x\dfrac{dy}{dx} = 1 + x^2\dfrac{dy}{dx}$, $\quad y = 3/2$ when $x = 1$

b) $(1 + x^2)^2\dfrac{dy}{dx} + xy^2 = 0$, $\quad y \to \infty$ as $x \to 0$

c) $y\cos^2 x\dfrac{dy}{dx} = \tan x + 2$, $\quad y = 2$ when $x = \pi/4$.

d) $x(x + 1)\dfrac{dy}{dx} = y(y + 1)$, $\quad y(1) = 2$

2.9 Solve the following differential equations

a) $\dfrac{dy}{dx} = \cot x \cot y$

b) $(x^2+1)\dfrac{dy}{dx} + xy = xy^2$

c) $(1+x)^2\dfrac{dy}{dx} + y^2 = 1$

2.10 According to Galileo's law the motion of a freely falling object is governed by the equation[4]

$$\frac{dv}{dt} = g\,,$$

where g is the constant 'acceleration due to gravity' and position x is measured downwards. Show that for an object falling from rest at the origin,

$$v = gt\,, \quad x = \frac{gt^2}{2}\,, \quad v^2 = 2gx\,.$$

2.11 What is Galileo's law of motion for a freely falling object when position x is measured *upwards*? If the object is initially at the point x_0 with velocity u, find $v(t)$, $x(t)$ and $v(x)$.

2.12 The equation of motion of an object thrown upwards through air producing a resistance of $mk|v|$ is

$$m\frac{dv}{dt} = -mg - mkv\,,$$

where position x is measured upwards. If the object is initially at $x = 0$ with velocity u, find:

a) $v(t)$, $x(t)$ and $x(v)$.

b) The maximum height reached and the time taken to reach it.

---

[4]Galileo Galilei lived from 1564 to 1642 in an era when many of the principles of mechanics now taught in high school were beginning to emerge from a mass of observations of objects in motion—falling bodies, projectiles and planets, for example.

The story that Galileo himself studied the motion of falling objects by dropping them from the top of the Leaning Tower of Pisa is certainly apocryphal because the law governing the motion of an object dropped from rest was well known by 1600. Observation had shown that objects fall the same distance in the same time and that the distance fallen is proportional to the square of the time elapsed.

Galileo set out to develop a model for this motion based upon the assumption of uniform acceleration. He worked on the problem for over 30 years and finally published his results in 1638.

CHAPTER 3

Applications of Separable Equations

It is often the case that the form of the function $f(x, y)$ is found from measurement, that is, from experimental data. It is then our task to find $y(x)$. This type of problem is often found in biological applications. Sometimes, the form of the derivative is suggested by some guiding principle and it is our task to explore the consequences. Economic applications fall into this category. Finally, there is sometimes a fully developed theory underlying the form of the derivative. We can then often construct the derivative ourselves and extend the theory still further.

3.1 Constructing the Differential Equation

Experiment or observation rarely provides the functional form of the derivative directly. More often, they provide a description of the trends in the data. If there are two variables x and y, there will be trends in the rate of change of y with x associated with varying x and with varying y. Sometimes, the dy/dx will be found to depend on only one variable. Then the construction is particularly simple.

Let us look at such a simple example first.

Radioactive Decay

Suppose that we are told that the rate of decay of radioactive atoms in a sample is proportional to the number of radioactive atoms present in the sample at any time[1]. We can construct the differential equation governing the number of radioactive atoms present in the sample as follows.

First, we need some symbols to represent the various quantities involved. A good choice here will help you to remember easily what they refer to. For example, let N denote the number of

[1]Radioactive atoms are atoms which spontaneously change the structure of their nucleus. The commonest modes of decay involve the ejection of an alpha particle (a nucleus of helium consisting of two protons and two neutrons) or the conversion of a neutron into a proton or *vice versa*. When the number of protons in the nucleus changes the chemical properties of the atom change; hence the atom becomes a new chemical element.

The processes by which radioactive decay occur involve very high energies, much higher than the energies commonly involved in natural processes on Earth. Thus the rate of radioactive decay is largely independent of environmental conditions and is found to be a constant for any particular atomic species.

atoms at time t. Then we are told that the rate of change of N with time is proportional to N itself, that is,

$$\frac{dN}{dt} \propto N.$$

This equation considers N to be a real variable and it is thus valid only for the density of matter (large N or continuum limit), when the discreteness of matter can be ignored (in reality, N is an integer and the equation is thus an approximation).

We can convert this into a differential equation by introducing a *constant of proportionality*. Let this be k, so that

$$\frac{dN}{dt} = kN\,.$$

This is correct as far as it goes, but it does not use all the information available. We also know that the radioactive atoms *decay*. Their numbers therefore decrease, and dN/dt must be negative. Since N must always be positive, this requires k to be negative. It is easy to forget this, but we can emphasize that the rate of change is negative by writing

$$\boxed{\frac{dN}{dt} = -kN\,,}$$

where k is now a *positive* constant. This constant k is the **decay constant**. Once we have the governing equation, it is a simple matter to solve it to find $N(t)$. This will be pursued in a more general context later in this chapter.

Physiological Response

An example of a more complicated dependence is provided by Stevens (1906–1973) who proposed a law which relates the change in physiological response to a change in sensory stimulus. It states that the rate of change of response with stimulus is proportional to the response and inversely proportional to the stimulus.[2]

Again we first need to introduce sensible symbols such as R for response and S for stimulus. Then the law expresses *two* dependencies. The rate of change dR/dS is directly proportional to response R, i.e.,

$$\frac{dR}{dS} \propto R$$

and is inversely proportional to the stimulus S, i.e.,

$$\frac{dR}{dS} \propto \frac{1}{S}\,.$$

[2]Note that this does not imply that the response decreases with increasing stimulus, which is obviously contrary to experience. It is the rate of change dR/dS that decreases with increasing stimulus. There are clear advantages to such physiological behaviour. We are sensitive to weak stimuli whilst being protected from dangerously large stimuli.

We must now combine these two dependencies. A little thought should convince you that there is only one way to combine them—by multiplying them—i.e.,

$$\frac{dR}{dS} \propto \frac{R}{S} .$$

This form then produces the correct dependence on S when R is held constant and the correct dependence on R when S is held constant. This would not have been the case if the two dependencies had been added.

Now we can convert the proportionality into an equation by introducing a constant of proportionality, n say. Then Steven's law provides the differential equation

$$\frac{dR}{dS} = n\frac{R}{S},$$

where n is a constant.

To solve this equation, note that the equation is in standard form and that the expression for the derivative can be factorized. Separating the variables and integrating gives

$$\int \frac{dR}{R} = \int \frac{n}{S}\, dS.$$

Performing the integrations produces the result

$$\ln|R| = n\ln|S| + C = \ln|S|^n + C.$$

This can be written more compactly by exponentiating both sides and introducing a new constant $D = e^C$. (Note that D is a positive constant.) Then

$$|R| = e^{(\ln|S|^n + C)} = e^{\ln|S|^n} e^C = D|S|^n.$$

This form is still cumbersome because of the presence of absolute values. They can be removed by introducing yet another constant A such that $A = D > 0$ when R and S have the same sign and $A = -D < 0$ when R and S have the opposite sign. Then we may write, finally,

$$R = AS^n .$$

(Check each possible sign combination of R and S if you are not convinced.)

The values of the constants n and A in the above problem are obtained by fitting data.

In Table 3.1 the first two columns give measurements of the response to brightness changes. The third column gives the values predicted by the Stevens' formula taking $n = 0.36$ and $A = 10$.

The disadvantage of empirical formulae is that several different ones can give almost the same results. The fourth column of the table gives the values predicted by the formula

$$R = 10\ln S - 1\,,$$

which is a version of Fechner's law which states that the rate of change of response with stimulus is just inversely proportional to the stimulus.

S	R	STEVENS	FECHNER
10	21	21.0	22.0
20	27	26.9	29.0
30	33	31.2	33.0
40	36	34.6	35.9
50	38	37.5	38.1
60	40	40.0	39.9

Table 3.1:

3.2 Models of Growth

A common application of mathematics is the modelling of various type of 'growth processes'. Models of this type specify how fast some quantity is changing as a function of the current value of that quantity. If the quantity is x and the time is t, this can be expressed mathematically as a differential equation of the form

$$\boxed{\frac{dx}{dt} = f(x)\,.} \tag{3.2a}$$

In this situation the most common type of problem involves finding an explicit formula for x in terms of t, given that it has a certain value at $t = 0$. Examples of such models occur in many branches of science. These include

- Radioactive decay—already mentioned above;
- Population growth—how fast a population is growing depends on its current size;
- Chemical reactions—the speed with which a reaction proceeds depends on the concentration of the reactants.

We will concentrate on the first two models. Chemical models can be found in the set of exercises.

Constant Growth

The simplest kind of growth process occurs when x increases at a constant rate. The differential equation takes the form

$$\frac{dx}{dt} = k\,,$$

where k is a constant. The most general solution,

$$x(t) = kt + C\,,$$

is found immediately by integration. Note that the constant C is equal to the value of x at $t = 0$. This type of growth process is also called *straight-line* or *linear* growth, since the graph of $x(t)$ against t is simply a straight line of slope k intercepting the line $t = 0$ at $x = C$. See Figure 3.1.

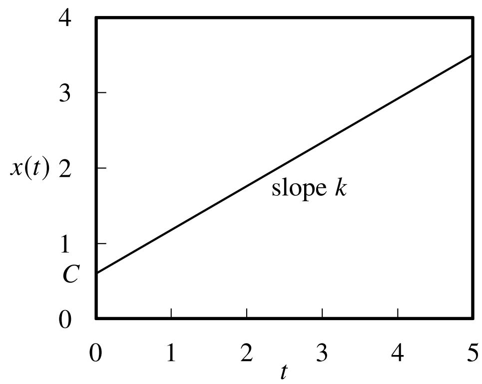

Figure 3.1:

Exponential Growth

This growth model is often cited in connection with population growth. It is based on the idea that the growth rate is *proportional* to the size of the population. Birth and death rates, for example, are often expressed as births or deaths per annum per 1000 population. Suppose the birth rate is B and the death rate is D. If $x(t)$ is the population at time t then the change in population over a short time interval from t to $t + \Delta t$ is approximately

$$\Delta x = (B - D) \times \frac{x(t)}{1000} \times \Delta t\,.$$

Then

$$\frac{\Delta x}{\Delta t} = \frac{B - D}{1000} \times x(t) = kx(t)\,,$$

say, where $k = (B - D)/1000$. In the limit as $\Delta t \to 0$ this becomes the differential equation

(3.2b) $$\boxed{\frac{dx}{dt} = kx\,.}$$

Solutions of (3.2b) have already been investigated. Generally any solution to (3.2b) is of the form

$$x(t) = Ae^{kt}$$

where A is a constant, equal to the value of x at $t = 0$.

The appearance of the exponential function gives this type of growth pattern its name of *exponential* growth. The most prominent feature of exponential growth is that $x(t)$ *doubles* at regular intervals. It is easy to determine the length of these intervals: we look for a constant a for which

$$x(t + a) = 2 \times x(t)\,.$$

From the formula for $x(t)$, we then get

$$Ae^{k(t+a)} = Ae^{kt}e^{ka} = 2Ae^{kt}\,.$$

The factor Ae^{kt} cancels out, leaving $e^{ka} = 2$, or

$$\boxed{a = \frac{1}{k}\ln 2\,.}$$

This model of population growth was suggested by Thomas Malthus[3] in 1798.

If $k < 0$ then $x(t)$ *decreases* with t. The process is then called **exponential decay**. In the population example this happens when the death rate exceeds the birth rate. The previous calculation is still valid, but gives a negative value of a. This implies that the population *halves* in size at regular intervals of $-\ln 2/k$. For a radioactive isotope undergoing exponential decay this period of time is called the **half-life** of the isotope.

Figures 3.2 and 3.3 respectively show exponential growth and exponential decay. In Figure 3.2 the intervals in which $x(t)$ doubles from 1 to 2 and from 2 to 4 are indicated. These are of equal length, as predicted. In Figure 3.3 the magnitude of k is twice as large, but negative. This gives a half-life which is half the length of the doubling period in the first figure.

---

[3]Thomas Robert Malthus (1766–1834) was an English political scientist, the first known professor of history and political economy. He was employed by the East India Company and authored the book *Principles of Political Economy* in 1820.

He lived during the time of rapid population growth fuelled by the industrial and agricultural revolutions. He was greatly concerned about the consequences of this growth and introduced the concept of the 'struggle for existence'. His writings provided Charles Darwin, who was at that time developing his theory of evolution, with a key to the mechanism of evolution, namely, natural selection.

In 1798 he wrote an *Essay on the Principle of Population*. Here he expounded the ideas current at the time that population increased exponentially whilst food supply increased only linearly. These were intuitive models and were given no mathematical basis. Malthus was extremely pessimistic because observations at that time clearly indicated exponential growth, and he could foresee the dire consequences of such growth. His suggestions for alleviating the situation were extremely unsympathetic!

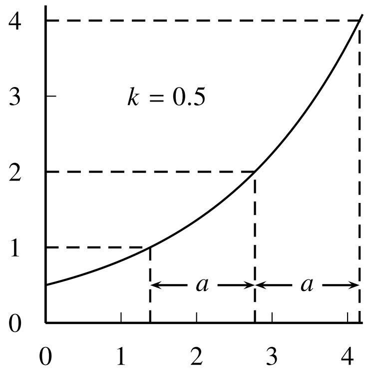

Figure 3.2:

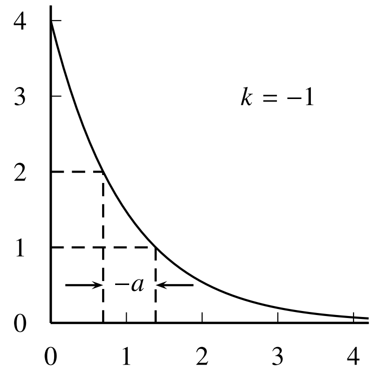

Figure 3.3:

Logistic Growth

Nothing can go on doubling forever (at least here on Earth). For this reason true exponential growth cannot be a realistic model of a growth process for all values of t. Eventually this kind of growth will be checked by physical limits. A small animal population with unlimited resources of food and space may well show exponential growth initially. As the population becomes larger, development of food shortages, overcrowding and other negative effects will act to reduce the growth rate. Any attempt to construct a long-term mathematical model for such a population should take this into account. We therefore look for a way to modify the equation of exponential growth to produce a more realistic model.

The main new feature we require is that the 'birth-rate' k should decrease as the population x increases. Thus k is no longer constant. We write it as a function $g(x)$ of x instead. Then our differential equation becomes

$$\frac{dx}{dt} = g(x) \times x\,.$$

What form should the function $g(x)$ take? Near $x = 0$ we should have $g(x) \approx k$, the unchecked growth rate. The reflects the fact that a small population can grow exponentially. But as x increases $g(x)$ should decrease. The simplest formula which exhibits this type of behaviour is

$$g(x) = k - ax\,,$$

where a is a positive constant. This leads to the differential equation

(3.2c)
$$\boxed{\frac{dx}{dt} = kx - ax^2\,.}$$

This is known as the **logistic equation**. Here is the Macquarie Dictionary definition:

> **logistics** *n.*, the branch of military science concerned with the mathematics of transportation and supply. . .

It is also sometimes called the *Verhulst*[4] equation. Is it a reasonable model? Even without solving the differential equation we can say quite a bit about the solutions. Note that the value $x = k/a$ is significant, since then $g(x) = 0$. Therefore $dx/dt = 0$ for this value of x. In particular this means that taking $x(t)$ to be *constant* and equal to k/a gives a solution to the differential equation—both sides will always be zero. This suggests a population 'in equilibrium' with the available resources. If $x > k/a$ the formula for $g(x)$ shows that the growth rate will be negative. This implies a population too large for available resources, leading to declining numbers.

We solve (3.2c) by separation of variables. We can simplify the formulas slightly by introducing a new constant $b = k/a$, so $kx - ax^2 = ax(b - x)$. Separation of x and t terms followed by integration then gives

$$\int \frac{dx}{x(b-x)} = \int a\,dt\,. \tag{3.2d}$$

We now use partial fractions[5] to evaluate the integral on the left.

We start with the partial fraction formula

$$\frac{1}{x(b-x)} = \frac{1/b}{x} + \frac{1/b}{b-x}\,.$$

This allows us to rewrite the integral in (3.2d) in the form

$$\int \frac{dx}{x(b-x)} = \frac{1}{b}\int \frac{dx}{x} + \frac{1}{b}\int \frac{dx}{b-x}\,.$$

This in turn integrates to

$$\frac{1}{b}\big(\ln|x| - \ln|b-x|\big) = \frac{1}{b}\ln\left|\frac{x}{b-x}\right|\,.$$

Equation (3.2d) then becomes

$$\frac{1}{b}\ln\left|\frac{x}{b-x}\right| = at + C\,,$$

where C is the arbitrary constant of integration. Multiplying through by b and using the fact that $ab = k$ gives

$$\ln\left|\frac{x}{b-x}\right| = kt + C\,,$$

[4]Around 1840 Verhulst and Quetelet developed the idea that as the population increased in size obstacles to unfettered growth would arise. Pierre-François Verhulst (1804–1849) was a Belgian mathematician. In 1846 he put forward the specific argument that new farmland would cease to become available to accommodate the growth in farming families. Others argued more generally that as living conditions become sufficiently unfavourable (with the exhaustion of food and other resources, for example) the death rate would outweigh the birth rate and the net rate would become negative.

[5]See the MATH1021 Course Notes for a more general theory of this method.

where b has been absorbed into the arbitrary constant C. Our aim is to find a formula for $x(t)$, so we have to 'solve' this equation for x in terms of t. First apply the exponential function to both sides, to give

$$\left|\frac{x}{b-x}\right| = e^C e^{kt}$$

or, equivalently,

$$\frac{x}{b-x} = \pm e^C e^{kt}.$$

Since C is an arbitrary constant, we can think of $\pm e^C$ as another arbitrary constant K, subject only to the condition $K \neq 0$. In fact K is then related to the initial value $x(0)$ of x by

$$K = \frac{x(0)}{b - x(0)}. \tag{3.2e}$$

The constant K is therefore positive or negative depending on whether the initial population is less than or greater than the 'equilibrium value' b. It is now easy to rearrange the equation into the form

$$x(t) = \frac{bKe^{kt}}{1 + Ke^{kt}}.$$

We can see the implications of this more easily by dividing through by Ke^{kt}, giving

$$\boxed{x(t) = \frac{b}{1 + K^{-1}e^{-kt}}.}$$

As $t \longrightarrow \infty$, so $e^{-kt} \longrightarrow 0$ (assuming $k > 0$). Therefore $x(t) \longrightarrow b$. The model implies that the population will approach the equilibrium value b as t increases. We can also use (3.2e) to replace K^{-1} by an expression in b and $x(0)$. This gives

$$x(t) = \frac{bx(0)}{x(0) + \big(b - x(0)\big)e^{-kt}},$$

showing directly how the solution depends on the starting point $x(0)$.

Figure 3.4 illustrates the behaviour of $x(t)$ for $b = 2$ and various initial values $x(0)$.

3.3 Flow problems

Both problems considered in this section involve separable equations describing the flow of fluids, where the following notation is used:

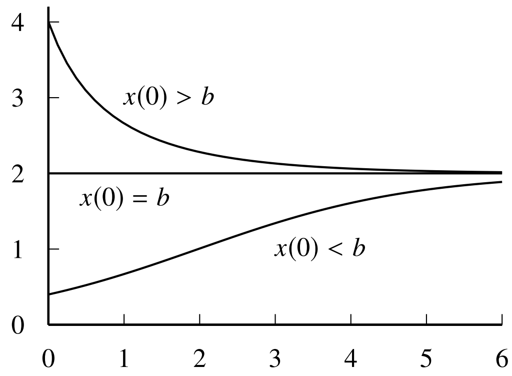

Figure 3.4:

Some terms used in applications.

density (mass/volume), $\rho = \dfrac{m}{V}$

concentration (mass/volume), $c = \dfrac{\text{mass of additive}}{\text{volume of bulk substance}}$

concentration (volume/volume), $c = \dfrac{\text{volume of additive}}{\text{volume of bulk substance}}\,.$

Flow from a Container

If a container full of liquid has a hole in it, the liquid will of course flow out. You may have noticed that the rate of outflow is greater when the hole is well below the surface than when it is close to the surface. In fact, when the hole in the container is small, Torricelli's law[6] asserts

[6]Evangelista Torricelli (1608–1647) was an Italian scientist who became first Galileo's assistant and then his successor in Florence. His major work, *Opera Geometrica*, was published in 1644. Despite its title, it reflected the preoccupations of applied mathematicians of the time. The second part concerned the motion of objects under gravity and was strongly influenced by Galileo's work. It included the motion of a projectile and tables of trigonometric functions for gunners. It also included observations of the manner in which water flowed, under the influence of gravity, from containers. This work led Mach to call Torricelli the father of hydrodynamics.

If the hole in the container is very small, the bulk of the liquid in the container can be considered to remain stationary, so that the liquid lost from the hole appears to come from the top of the container where the level is dropping. The speed of the outflow is then given by imagining the surface layer to fall freely from rest down to the hole. Galileo's formula then yields

$$v^2 = 2gh\,,$$

where g is the acceleration due to gravity.

a simple relationship between the speed of the outflow v and the height h of the surface of the liquid above the hole:

$$v^2 \propto h\,.$$

Equivalently, the rate of outflow is proportional to the *square root* of the surface height. The rate of loss of liquid from the container is the volume flowing out in unit time. This is va, where a is the cross-sectional area of the hole.

Example 3.3a A dam is built with a flat bottom and vertical sides. Water flows out of a sluice at the bottom of the dam wall. The speed of the outflow obeys Torricelli's law which states that the speed is proportional to the square root of the height of the water level above the sluice. If the water level falls to half its original value after 10 days, how long will it then take to empty?

The outflow changes the volume of water in the dam, and this in turn affects the level of the water. Let $V(t)$ be the volume at time t.

The volume decreases at the rate at which water flows out. We are given the fact that the speed of outflow is proportional to the square root of the height of the water level. Let v be the speed and h the height of the water level above the sluice. Then

$$v \propto \sqrt{h}\,.$$

The volume of water lost through the sluice in unit time is proportional to the speed of outflow. Thus

$$\frac{dV}{dt} \propto \sqrt{h}\,,$$

or

$$\frac{dV}{dt} = -k\sqrt{h}\,.$$

Here we have introduced a constant of proportionality and a negative sign because the outflow is decreasing the volume.

One further step is required before we have a usable differential equation. The left side of the preceding equation involves the volume of water, while the right side uses the height of the water. Since we are interested ultimately in the height, we need to write V in terms of h.

There is no information given about the shape of the dam apart from the facts that its bottom is level and that the walls are vertical (our diagram has rectangular cross section only for convenience). Otherwise the shape of the dam is irrelevant. Figure 3.5 illustrates the situation.

The important thing is that the area of the dam in horizontal cross section is the same at all heights. Let this area be A. Then the volume of water V present when the level is at height h is

$$V = Ah\,.$$

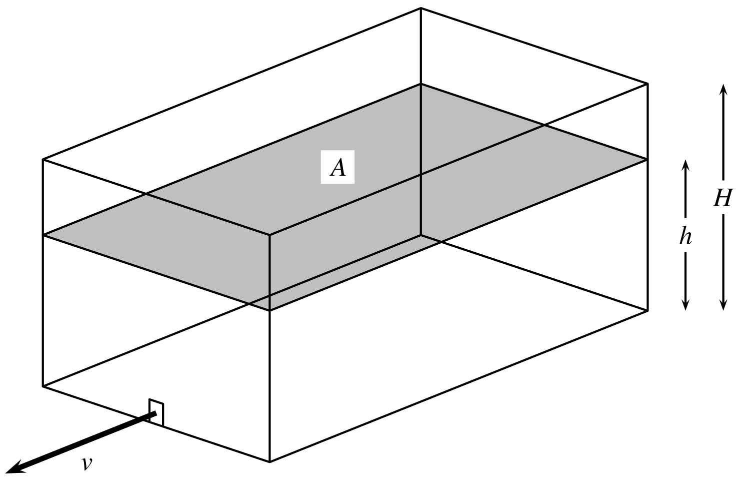

Figure 3.5:

Using this expression to substitute for V in the model equation produces

$$\frac{d}{dt}(Ah) = -k\sqrt{h}\,.$$

Since A is a constant in this example, the model equation becomes a differential equation governing h,

$$A\frac{dh}{dt} = -k\sqrt{h}\,,$$

or

$$\frac{dh}{dt} = -k'\sqrt{h}\,,$$

where the constant factor has been written $k/A = k'$ for convenience.

This equation can be separated and integrated:

$$\int \frac{1}{\sqrt{h}}\,dh = -k' \int dt\,,$$

leading to

$$2\sqrt{h} = -k't + C\,,$$

or

$$h(t) = \left(\frac{-k'}{2} + \frac{C}{2}\right)^2\,.$$

This is the general solution. The particular solution that applies in this case is found from the remaining information.

The question states that the level has fallen by half over 10 days. Since we are not told the initial level, we must assign it some value. Let H represent the initial height of the water (at

$t = 0$). Then the height must be $H/2$ when $t = 10$, the units being days. The statements in these two sentences provide *two* conditions on the solution found above. At $t = 0$ we must have

$$2\sqrt{H} = C\,,$$

and at $t = 10$ we must have

$$2\sqrt{\frac{H}{2}} = -10k' + C\,.$$

Hence

$$C = 2\sqrt{H}\,,$$

and

$$k' = \frac{1}{10}(2 - \sqrt{2})\sqrt{H}\,.$$

If these values are substituted, the solution to this problem becomes (after some simplification)

$$\sqrt{h} = \sqrt{H}\left[1 - (2 - \sqrt{2})\frac{t}{20}\right]\,.$$

Note that the *only* unknown quantity to appear here is the initial height H. The area of the dam and the original constant of proportionality are not required.

The final step provides the answer to the question. The dam becomes empty when $h = 0$. This occurs after a time T given by

$$0 = \sqrt{H}\left[1 - (2 - \sqrt{2})\frac{T}{20}\right]\,,$$

from which we find

$$T = \frac{20}{2 - \sqrt{2}}\,.$$

This value is independent of the initial height of the level assumed. Evaluation of this expression shows that the dam empties after a total of 34.14 days, or 24.14 days after it was half full. ◇

Pollution Problems

A whole class of applications deal with the amount of additive or impurity present in a substance. The **concentration** of the additive is defined as the ratio of the mass of the additive to the mass (or volume) of the substance. When defined as the ratio of masses, the concentration is a dimensionless quantity. If expressed as the ratio of the mass of pollutant to the total volume, it has units of mass per unit volume. Concentration models occur in many areas of biophysics, engineering and chemistry.

Example 3.3b A lake has a volume of $V\,\mathrm{km}^3$ and is fed and drained by a river with a flow rate of $r\,\mathrm{km}^3/\mathrm{yr}$. A chemical company builds a plant upstream which discharges pollutant into the river where it produces a concentration of $c_1\,\mathrm{kg}/\mathrm{km}^3$.

If the lake was initially polluted with a concentration c_0 kg/km^3, find the concentration in the lake t years later.

We must first decide whether to work in terms of the mass of additive present or its volume. Since the concentrations are specified in units of mass per unit volume, the natural variable is mass. So let the mass of pollutant in the lake at time t be M. Then the mass will change as a result of pollutant being brought in by the inflow and carried out by the outflow, that is,

$$\frac{dM}{dt} = \text{MASS RATE IN} - \text{MASS RATE OUT}\,.$$

Figure 3.6 will help to fix ideas. Now the mass rates in and out are the product of the

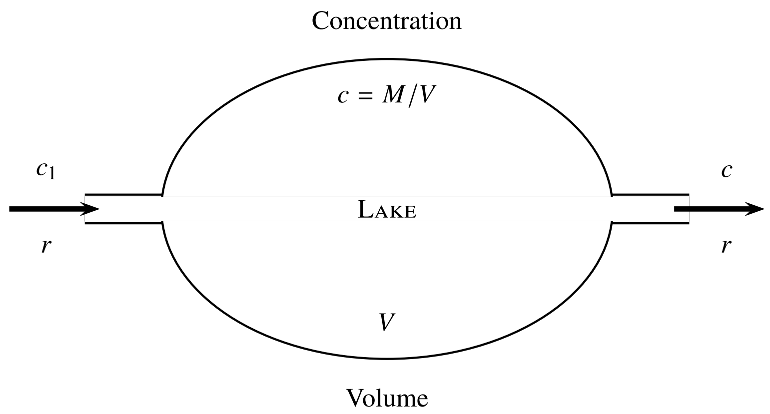

Figure 3.6:

concentration (mass per unit volume) and the flow rate (volume per unit time). Hence the mass rate in is $c_1 r$ kg/yr.

The mass rate out cannot be calculated immediately because the concentration of pollutant in the outflowing river is not stated. However, it is a reasonable assumption that the inflow stirs the lake water and mixes the pollutant fairly uniformly throughout the lake. The concentration in the outflow would then be simply the average concentration in the lake, $c = M/V$. The mass flow out is then $(M/V)r$ and the governing equation becomes

$$\boxed{\frac{dM}{dt} = c_1 r - \frac{r}{V}M\,.}$$

We could solve this separable equation to find the mass of pollutant in the lake and then calculate the corresponding concentration. However, it is more direct to proceed one step further and rewrite this equation in terms of the required quantity, the concentration.

The mass is related to the concentration by $M = cV$. In this case the volume V is constant:

$$\begin{aligned}\frac{dV}{dt} &= \text{VOLUME FLOW RATE IN} - \text{VOLUME FLOW RATE OUT}\\ &= r - r\\ &= 0\,.\end{aligned}$$

Thus $dM/dt = V(dc/dt)$, and the mass balance equation can be rewritten as

$$\boxed{\frac{dc}{dt} = \frac{r}{V}(c_1 - c)\,.}$$

This is another separable equation, and may be integrated to give

$$c = c_1 + Ce^{-(r/V)t}\,.$$

If the initial concentration of pollutant is c_0, setting $c = c_0$ at $t = 0$ shows that $C = c_0 - c_1$. Hence the concentration of pollutant as a function of time is

$$c(t) = c_1 + (c_0 - c_1)e^{-(r/V)t}\,.$$

◇

Summary of Chapter 3

Suggestions on how to solve model problems:

- Work out what variables are needed for the problem.
- Try to visualize the problem and draw a diagram based on what the problem tells you.
- If the problem involves the position of objects, introduce a coordinate system and indicate the axes on your diagram.
- If the equation governing the problem is not given, write down in words the processes which govern the rate of change. For example:

  Rate of change of number with time = number present at time t

- Using the information given, convert each term into mathematical form (such as $dN/dt = \ldots$, for example) introducing any new quantities by symbols at this stage.
- If the information is given in terms of proportionality, convert it into an equation by using a constant of proportionality.
- If the resulting equation contains only two variables plus constants, go ahead and solve it. If it still contains other variables, look for some subsidiary equations that will allow you to eliminate variables.
- The general solution should contain one unknown constant of integration but may contain other (proportionality) constants.
- Substitute information about specific values to evaluate the constants, find the particular solution, and answer the question.

Exercises

3.1 Use partial fractions to find the particular solutions of

a) $\dfrac{dy}{dx} = y^2 - 1,\ y(0) = 2;$

b) $\dfrac{dy}{dx} = 4 - y^2,\ y(0) = 6;$

c) $\dfrac{dy}{dx} = y^2 + 3y + 2, \; y(0) = 0.$

3.2 Fechner's law states that
$$\frac{dR}{dS} = \frac{n}{S}.$$
Determine $R(S)$.

(An example of its application to describe the response of the eye to brightness is given in Table 1 above.)

3.3 A generalization of Stevens' law (which includes Fechner's law as a special case) is
$$\frac{dR}{dS} = n\frac{R^p}{S^q},$$
where $n > 0$, $0 < p \leq 1$ and $0 < q \leq 1$ are constants. Find $R(S)$ given that $R = 0$ when $S = 0$.

3.4 The Loewenstein model of physiological reaction is
$$\frac{dR}{dS} = \frac{R}{(1 + \alpha S)S}$$
where α is a positive constant. Find $R(S)$.

3.5 a) If there are initially N_0 atoms of radioactive material with decay constant k in a sample, find $N(t)$.

b) A sample of lunar rock was found to contain equal numbers of potassium and argon atoms. Assuming that all the argon is the result of the radioactive decay of potassium (which has a half-life of 1.28×10^9 years) and that one in every nine potassium atom disintegrations produces one argon atom, determine the age of the lunar rock.

3.6 Charcoal from Stonehenge was found to contain C^{14}/C^{12} at a level of 63% that of the present day value[7]. The half-life of C^{14} is 5730 years. Show that the sample is 3820 years old.

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[7]The most commonly used means of absolute dating in archaeology measures the radioactive decay of C^{14}. The basis of this method is the following.

Radioactive carbon (C^{14}) is produced in the Earth's atmosphere by cosmic rays (high-energy protons) colliding with atoms of the atmosphere. The atomic fragments produced on impact include some neutrons. These neutrons can be captured by nitrogen (N^{14}) atoms in the atmosphere. In the process a proton is ejected. The overall mass remains the same (14) but the number of protons decreases by one so that the atom slips one place in the chemical table from nitrogen to carbon. The radioactive carbon in the atmosphere then decays slowly, with a half-life of 5730 years, as the neutrons emit a positron and antineutrino in order to revert to a proton. The atoms then become N^{14} again.

The amount of C^{14} in the atmosphere is established by balancing the cosmic ray production rate against the spontaneous decay rate. The ratio of radioactive carbon to stable carbon (C^{12}) is therefore a function of the amount of N^{14} and C^{12} in the atmosphere, as well as the cosmic ray flux. These quantities are assumed to have remained unchanged over historic time, maintaining the C^{14}/C^{12} ratio in the atmosphere at its present day value

3.7 The number N of bacteria in a culture grew according to the Malthusian law. The value of N was initially 100 and increased to 332 in one hour. What was the value of N after $1\frac{1}{2}$ hours?

3.8 A model for bacterial growth was proposed by Smith in 1963. His observations suggested that bacterial populations demonstrate that growth is inhibited by an extra term proportional to the rate of population change. Thus

$$\frac{dN}{dt} = \gamma N\left(1 - \frac{N}{M} - \frac{\mu}{M}\frac{dN}{dt}\right),$$

where μ is another positive constant.

Solve the differential equation, given that the initial population size is N_0. What is the limiting population size?

3.9 A model for growth which is influenced by seasonal factors is provided by writing

$$\frac{dN}{dt} = (\gamma + \mu\cos\omega t)N\,,$$

where γ, μ and ω are constants. Find $N(t)$ given that the initial population is N_0.

Describe the behaviour of the solutions when $|\gamma| < |\mu|$ and $|\gamma| > |\mu|$.

3.10 An equation governing the growth of fish was proposed in 1951 by von Bertalanffy. It is based on metabolic balance. Chemical energy is gained by the body through the walls of the gut. The rate is proportional to the surface area, that is, roughly to $V^{2/3}$ where V is the volume of the fish. Chemical energy is used by cells throughout the body, so that the rate of loss is proportional to V. Any excess chemical energy goes into cell growth, dV/dt. Since the mass m is itself proportional to V, the rate of mass change can be written as

$$\frac{dm}{dt} = \alpha m^{2/3} - \beta m\,,$$

where α and β are positive constants. Find $m(t)$ given that the initial mass is zero.

[Hint: use the substitution $x = m^{1/3}$ to perform the integration.]

3.11 If interest is compounded continuously at a rate of 10% per year, what will an initial investment P_0 be worth after one year? What is the effective yearly rate of interest?

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of about 3.6×10^{-13}.

Radioactive carbon is chemically identical to normal carbon and combines with oxygen to form carbon dioxide in the same proportion. Living beings then respire and ingest radioactive carbon and accumulate the same fraction in their tissues. After death, however, carbon is no longer assimilated and the radioactive carbon which decays is not renewed. The C^{14}/C^{12} ratio in the material then drops exponentially from the atmospheric value. The amount of time which has elapsed since the death of the animal or plant can then be found quite simply by comparing the two values.

3.12 An investor has \$1000 with which to open an account and plans to add \$1000 per year. All funds in the account will earn 10% interest per year, continuously compounded.

a) Assuming that the added deposits are also made continuously, find the differential equation governing the sum in the account x at time t.

b) How many years would it take for the account to reach \$100000?

3.13 The spread of innovation (in agriculture and industry) has been successfully modelled by assuming that the rate of spread is proportional to both the number of people already having adopted the new system and to the number of people who have not yet adopted it.

If M is the potential market for the innovation and $N(t)$ the number who have already adopted it, then

$$\frac{dN}{dt} = aN(M - N),$$

where a is a constant. This has the same form as the Verhulst population equation.

Now suppose that the effect of advertising is added. Suppose further that it contributes to the rate of adoption in proportion to the number of people at whom the advertising is aimed, that is, the number $M - N$ who have still to change to the new system. The new model equation then reads

$$\frac{dN}{dt} = aN(M - N) + b(M - N),$$

where b is another constant.

Solve this equation to find $N(t)$, given that $N(0) = N_0$.

3.14 Repeat the dam problem supposing that the water level was a fraction f of the initial level after T_0 days. Show that the dam the empties after $T_0/(1 - \sqrt{f})$ days.

3.15 Modify the previous problem to find the height of water in the dam as a function of time if there is a constant flow of water into the dam. Suppose that the inflow occurs at a rate r and that the initial height of the water level is h_0.

3.16 Liquid runs out of the hole at the bottom of an upright funnel whose angle at the apex is 2α at a rate governed by Torricelli's law. If the hole has area a and the level of the liquid is initially a height H above the apex, find the time taken for the liquid to all run out.

3.17 Drugs such as penicillin are gradually removed from the bloodstream at a rate proportional to the mass present in the blood at any instant.

a) If m is the mass of the drug and V the volume of blood, find the equation governing the concentration $c = m/V$ of the drug in the blood.

b) Solve the equation, given that the initial concentration is c_0.

3.18 In a dilute sugar solution the rate of decrease of concentration is proportional to the concentration c. If $c = 0.01\,\mathrm{g/cm^3}$ at time $t = 0$ and $c = 0.005\,\mathrm{g/cm^3}$ at time $t = 4$ hours, show that the concentration after 10 hours will be $0.00177\,\mathrm{g/cm^3}$.

3.19 The steady infusion of a substance such as glucose into the bloodstream produces a constant term in the equation governing the amount of that substance in the blood,

$$\frac{dm}{dt} = s - km\,.$$

Show that the amount of substance present a time t after infusion commences, assuming that there is an initial mass m_0 already in the blood, is

$$m = \frac{s}{k} - \left(\frac{s}{k} - m_0\right) e^{-kt}\,.$$

What happens when t becomes large?

3.20
a) What is the limiting concentration achieved after a long time in the polluted lake of the second example?
b) If the chemical plant were to shut down and the source of pollution eliminated, show that it would take a time $(V/r)\ln 10$ for the pollution level to fall by a factor of 10.
c) Calculate this time for the case of Lake Ontario, given that $V = 1600\,\mathrm{km^3}$ and $r = 209\,\mathrm{km^3/year}$. The cleaning-up of the Great Lakes was a long-term project!

3.21 Solve the equation for the mass of pollutant in the lake as function of time and verify that it leads to the same result for the concentration.

3.22 A swimming pool holds 40000 litres of water with a chlorine concentration of 0.01‰. An overflow maintains the volume of water constant. Water with a chlorine concentration of 0.001‰ is now pumped in at a rate of 20 L/min.
a) What is the chlorine concentration in the pool after one hour?
b) When will the concentration have dropped to 0.002‰?

3.23 A tank initially contains 500 L of fresh water. A pipe is opened which admits polluted water at 4 L/min. At the same time, a drain is opened to allow 3 L/min of the mixture to leave the tank. If the inflowing polluted water contains 0.01 kg/L of impurity, what is the mass of impurity in the tank after 100 minutes?

CHAPTER 4

Linear Differential Equations

In this chapter we go beyond the case of first-order separable equations treated so far. *Linear* differential equations build an important solvable class. We start with first-order equations and then discuss the properties of solutions of higher-order equations.

4.1 First order Linear Differential Equations

Beyond separable equations

On the one hand, even simple differential equations of the form $dy/dx = f(x, y)$ are too general to admit of a single method of solution. On the other hand, the cases treated so far all reduce to the separable equation $dy/dx = g(x)f(x)$. Between these two types we can still identify certain types of equations with their own methods of solution.

One of the most common and important special types is the **linear differential equation**. Here we discuss only *first-order* linear equations. These are equations of the form

$$\boxed{\frac{dy}{dx} + p(x)y = q(x)\,.} \tag{4.1a}$$

The equation is *linear* because when written out as a polynomial in y and dy/dx the differential equation has these terms occurring *only* in a linear fashion. There are however no requirements that either $p(x)$ or $q(x)$ should be linear functions of x. We only require that $p(x)$ and $q(x)$ are sufficiently smooth to be integrable.

Method of Solution – Integrating Factor

The great breakthrough in solving these equations occurred about 1692 when Leibniz discovered that the *general* solution of this *general* linear differential equation of the first-order could always be found. The most straightforward way of obtaining the general solution is as follows.

Suppose we can find a function $r(x)$ such that, when multiplying both sides of equation (4.1a) by $r(x)$ the left side becomes the derivative of the product $r(x)\,y$, That is, such that

$$r(x)\frac{dy}{dx} + r(x)p(x)y = \frac{d}{dx}\,(r(x)y)\,. \tag{4.1b}$$

If we can find $r(x)$ that satisfies that condition, the differential equation becomes

$$\frac{d}{dx}\left(r(x)y\right) = r(x)q(x)\,. \tag{4.1c}$$

Integrating both sides of (4.1c) with respect to x gives

$$r(x)\,y = \int r(x)q(x)\,dx + C, \quad C = \text{ arbitrary constant}$$

and dividing both sides by $r(x)$, gives the solution

$$y(x) = \frac{1}{r(x)}\int r(x)q(x)\,dx + \frac{C}{r(x)}. \tag{4.1d}$$

The function $r(x)$ is called an **integrating factor**. To calculate the integrating factor we expand the right side of (4.1b) using the product rule to get

$$r(x)\frac{dy}{dx} + r(x)p(x)y = r(x)\frac{dy}{dx} + \frac{dr}{dx}y, \tag{4.1e}$$

and cancel terms to get the following equation,

$$\frac{dr}{dx} = r(x)p(x)\,. \tag{4.1f}$$

This is a separable differential equation for $r(x)$ which can be solved as follows,

$$\begin{aligned}
\frac{dr}{dx} = r(x)p(x) \implies & \frac{dr}{r(x)} = p(x)\,dx \\
\implies & \int \frac{dr}{r(x)} = \int p(x)\,dx + C \\
\implies & \ln|r(x)| = \int p(x)\,dx + C \\
\implies & r(x) = Ae^{\int p(x)\,dx},
\end{aligned} \tag{4.1g}$$

where $A = e^C$. Since any solution $r(x)$ of equation (4.1f) will work, we may take the constant of integration $C = 0$, which is equivalent to taking $A = 1$. The integrating factor is thus

$$\boxed{r(x) = e^{\int p(x)\,dx}.}$$

From equations (4.1d) and (4.1g), we see that the general solution of the linear differential equation (4.1a) may be written as

$$\boxed{y = e^{-\int p(x)\,dx}\int e^{\int p(x)\,dx}q(x)\,dx + Ce^{-\int p(x)\,dx}.} \tag{4.1h}$$

Example 4.1i Find the general solution of

$$(t^2-1)\frac{dx}{dt} = t + 1 - 2x\,.$$

It is clear that this equation is linear, so first put the equation in the appropriate standard form (4.1a), i.e.,

$$\frac{dx}{dt} + \frac{2}{t^2-1}x = \frac{t+1}{t^2-1}.$$

Hence

$$p(t) = \frac{2}{t^2-1} \quad \text{and} \quad q(t) = \frac{t+1}{t^2-1}\,.$$

Note that for this problem the independent variable is t and the dependent variable is x.

First evaluate the integrating factor,

$$r(t) = \exp\left(\int p(t)\,dt\right) = \exp\left(\int \frac{2}{t^2-1}\,dt\right)\,.$$

(**Note carefully the notation used here:** $\exp x \equiv e^x$. The exp notation is both useful and widely used when the exponent is complicated.)

Performing the integration, we find

$$r(t) = \exp\left(\ln\left|\frac{t-1}{t+1}\right|\right) = \frac{t-1}{t+1}.$$

(As we noted above, it is sufficient to take the constant of integration to be 0 in the result for the integrating factor $r(t)$, since *any* nonzero integrating factor will do.)

This can now be substituted into

$$\frac{d}{dt}[r(t)x] = r(t)q(t)\,,$$

giving

$$\frac{d}{dt}\left[\left(\frac{t-1}{t+1}\right)x\right] = \left(\frac{t-1}{t+1}\right)\left(\frac{t+1}{t^2-1}\right).$$

Cancelling factors on the right and making use of the fact that $t^2-1 = (t-1)(t+1)$, we find that

$$\frac{d}{dt}\left[\left(\frac{t-1}{t+1}\right)x\right] = \frac{1}{t+1}.$$

We can now integrate with respect to t to obtain

$$\left(\frac{t-1}{t+1}\right)x = \int \frac{dt}{t+1} = \ln(t+1) + C.$$

The general solution is therefore

$$x = \left(\frac{t+1}{t-1}\right)(\ln(1+t) + C)\,.$$

◇

Example 4.1j The flow examples treated in the last part of the previous chapter lead to differential equations that are both separable and linear. Verify that applying the method derived above for linear equations leads to the same solutions obtained previously using the method for separable equations.

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4.2 Higher-order differential equations

So far we have discussed only methods to solve first-order differential equations. We now will overcome this limitation and consider also higher-order differential equations, i.e., equations involving higher-order derivatives such as $\frac{d^2y}{dx^2}$. An important property of the solution of first-order equations is that they involve one integration constant which is fixed by specifying the solution value at one point. In order to understand what happens in higher-order differential equations, we first consider a simple example.

Consider the second-order equation

$$\frac{d^2y}{dx^2} - \frac{1}{x}\frac{dy}{dx} = 0\,.$$

By writing

$$w = \frac{dy}{dx}\,,$$

this becomes a first-order equation for w,

$$\frac{dw}{dx} = \frac{w}{x}\,.$$

This is easily solved to give $w = Cx$, where C is an arbitrary constant. Thus

$$\frac{dy}{dx} = Cx\,,$$

which may now be integrated, giving

$$y = \frac{1}{2}Cx^2 + D\,,$$

where D is a second arbitrary constant. This form of the solution is called the **general solution**. Any solution of this form, where C and D are completely arbitrary, will satisfy the equation.

Second-order equations thus involve *two* constants of integration and both need to be determined. This generally requires two conditions on the solution $y(x)$ (and possibly its derivative) to be specified. Suppose for instance we know $y(0) = 0$ and $y(1) = 1$. Substituting these conditions into the solution gives

$$0 = D, \qquad 1 = C/2 + D\,.$$

Thus, for this **particular solution**, $C = 2$ and $D = 0$, and so

$$y = x^2.$$

Boundary Values and Initial Values

When information about y is given at two specific locations, as above, the subsidiary conditions are referred to as **boundary conditions**. An alternative possibility is that y and dy/dx are specified at one point; in this case the subsidiary conditions are known as **initial conditions**.

This terminology arises from dynamics; if at time $t = 0$ the position x and velocity $v = dx/dt$ of a particle are known, Newton's law allows us to integrate the *second-order* equation

$$\frac{dv}{dt} = \frac{d^2x}{dt^2} = \frac{F}{m},$$

using the values at $t = 0$ as initial conditions to pick out that particular solution which the particle subsequently follows. In the example above, we might have had $y = dy/dx = 1$ when $x = 1$. The first condition would require, on substitution,

$$\frac{1}{2}C + D = 1\,.$$

To satisfy the second, we must first differentiate the general solution to find dy/dx, namely,

$$\frac{dy}{dx} = Cx\,.$$

(In fact, this had been obtained already on the way to the general solution. This is not usually the case, though, in solving second-order equations.) The second condition is now seen to require

$$1 = C.$$

Together, the two conditions give $C = 1$ and $D = 1/2$ so that the particular solution would then be

$$y = \frac{1}{2}x^2 + \frac{1}{2}\,.$$

Note that the two parts of the general solution associated with each arbitrary constant, namely $y = Cx^2/2$ and $y = D$, each separately satisfy the original differential equation. The general solution of a second-order equation always involves two arbitrary constants like this; in any particular case they are determined by boundary or initial conditions.

This result can be generalized:

> The general solution to an nth-order ordinary differential equation always involves n arbitrary constants.

The proof of this result lies outside the scope of this course, but loosely speaking it arises because each successive integration of a function gives rise to one more arbitrary constant; thus integrating n times (which is effectively what we do when we solve an nth-order differential equation) gives rise to n constants.

Notation. In dealing with differential equations of second and higher orders, the standard derivative notation $d^n y/dx^n$ becomes very cumbersome. It is usual to abbreviate the derivatives using the $'$ (prime) notation in which

$$\boxed{\frac{dy}{dx} = y', \quad \frac{d^2y}{dx^2} = y'', \quad \frac{d^3y}{dx^3} = y''', \quad \text{etc.}}$$

The dy/dx notation was coined by Leibniz. Newton, who developed the calculus independently, used the dot form that is still used in dynamics to indicate time derivatives,

$$\boxed{\frac{dx}{dt} = \dot{x}, \qquad \frac{d^2x}{dt^2} = \ddot{x}\,.}$$

Example 4.2a Solve the third-order differential equation

$$\frac{d^3y}{dx^3} = x\,,$$

given that $y(0) = 0, \quad y(1) = y'(1) = 1.$

This equation can be integrated directly,

once:

$$\frac{d^2y}{dx^2} = y'' = \frac{1}{2}x^2 + C_1$$

twice:

$$\frac{dy}{dx} = y' = \frac{1}{6}x^3 + C_1 x + C_2$$

and a third time:

$$y = \frac{1}{24}x^4 + \frac{1}{2}C_1 x^2 + C_2 x + C_3\,.$$

This is the general solution. As promised, there are three arbitrary constants. To evaluate these with the given boundary conditions, we substitute into the solution:

$$\begin{aligned} y(0) = 0 &\implies 0 = 0 + 0 + 0 + C_3 \\ y(1) = 1 &\implies 1 = \frac{1}{24} + \frac{1}{2}C_1 + C_2 \\ y'(1) = 1 &\implies 1 = \frac{1}{6} + C_1 + C_2 \,. \end{aligned}$$

These give $C_3 = 0$, $C_1 = -1/4$, $C_2 = 13/12$. Thus the particular solution satisfying these boundary conditions is

$$y = \frac{1}{24}x^4 - \frac{1}{8}x^2 + \frac{13}{12}x \,.$$

◇

This equation was easy to solve. However, as with first-order problems, an equation picked at random is unlikely to be soluble in terms of the elementary functions. Indeed, many equations which arise commonly in applications have no such solution. In this case we fall back on methods of approximating solutions; by series expansions, by asymptotic analysis, or numerically, using a computer. For certain solutions of key equations this process has been carried out and the properties of these 'special functions' have been extensively studied: see any text of Mathematical Physics or Advanced Engineering.

Examples

a) $y'' - 2y' - 3y = 0$.
The coefficients of y and its derivatives are all constants in this equation. It is said to be *linear with constant coefficients*. We describe a method to solve such equations in the next section.

b) $y'' + \dfrac{1}{x}y' + \left(1 - \dfrac{n^2}{x^2}\right)y = 0$.
This is Bessel's differential equation which occurs frequently in problems involving circular symmetry. This again a *linear* equation in y and its derivatives in which the coefficients are functions of the independent variable x. Its solutions are given in terms of Bessel Functions, whose Taylor series can be explicitly determined.

c) $y'' + \dfrac{2}{x}y' + y^n = 0$.
This equation is no longer *linear* if $n \neq 0, 1$; it arises in trying to model the internal structure of a star, and is called the Lane-Emden equation. Elementary solutions exist only for $n = 0$, 1 or 5. For other values, solutions are studied by approximation methods.

4.3 General form of linear equations

The general form of an nth-order linear differential equation is:

$$P_n(x)\frac{d^n y}{dx^n} + P_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \ldots + P_1(x)\frac{dy}{dx} + P_0(x)y = F(x),$$

where $P_i(x)$ for $i = 0, 1, \ldots, n$ are arbitrary (continuous) functions of x. For simplicity we may write this equation as

(4.3a) $$\boxed{L^{(n)}(y(x)) = F(x),}$$

where $L^{(n)}$ is the linear operator of order n defined as

$$L^{(n)} \equiv P_n(x)\frac{d^n}{dx^n} + P_{n-1}(x)\frac{d^{n-1}}{dx^{n-1}} + \ldots + P_1(x)\frac{d}{dx} + P_0(x).$$

When $F(x) = 0$ the right hand side of Eq. (4.3a) vanishes and we say the equation is *homogeneous*. Otherwise, when $F(x) \neq 0$, we say that the equation is *inhomogeneous*. In Chapter 4 we studied first-order ($n = 1$) inhomogeneous equations.

There is no universally valid procedure to obtain the *general* solution of linear differential equations of arbitrary order. Our approach here is to consider properties of the solution that help us in this task. In particular, consider that $y_1(x)$ and $y_2(x)$ are solutions of a *homogeneous* linear differential equations

$$L^{(n)}(y(x)) = 0$$

The following two general results hold:

$$\boxed{Ay_1(x) \text{ is also a solution, for arbitrary } A \in \mathbb{R}}$$

Proof: By hypothesis $L^{(n)}(y_1) = 0$. Using the linearity of the equation we can write $L^{(n)}(Ay_1) = AL^{(n)}(y_1) = A0 = 0$, and thus Ay_1 is also a solution.■

$$\boxed{y_1(x) + y_2(x) \text{ is also a solution.}}$$

Proof: By hypothesis $L^{(n)}(y_1) = 0$ and $L^{(n)}(y_2) = 0$. Using the linearity of the equation we can write $L^{(n)}(y_1 + y_2) = L^{(n)}(y_1) + L^{(n)}(y_2) = 0 + 0 = 0$, and thus $y_1 + y_2$ is also a solution.■

If we know n (different) solutions $y_1(x), y_2(x), \ldots, y_n(x)$, the combination of the two results above show that

$$y(x) = A_1 y_1(x) + A_2 y_2(x) + \ldots A_n y_n(x)$$

is also a solution. This solution has n arbitrary constants and thus, using the main result of the previous section, we know that it is the *general* solution of the nth order homogeneous linear equation. Let us see how this can be helpful in a concrete example:

Example 4.3b Find the general solution of

$$\frac{d^2y}{dx^2} + y = 0. \tag{4.3c}$$

This second order equation is linear and homogeneous. The solution $y(x)$ should be such that after differentiating it twice we obtain the same function but with a negative sign. This is precisely the property of the functions $y(x) = \sin x$ and $y(x) = \cos x$, which are both particular solutions of Eq. (4.3c) (check it!). Using the general results above we obtain that

$$y(x) = A_1 \sin x + A_2 \cos x,$$

is also a solution of Eq. (4.3c) (check it!). Since this solution has two arbitrary constants, A_1 and A_2, and Eq. (4.3c) is a second-order differential equation, we know this is *the* general solution we were seeking. ◇

This example shows how the general results of this section help us: the difficult problem of finding a general solution is reduced to the simpler problem of finding particular solutions. The same solution strategy will be used in the next chapter to find general solutions of second-order ($n = 2$) homogeneous equations with constant coefficients (i.e., constant $P_0(x), P_1(x)$, and $P_2(x)$).

Summary of Chapter 4

First order linear differential equations

The following steps summarize the procedure to find the solution of first order linear equations:

- Write the equation in the standard form of the *linear* differential equation

$$\frac{dy}{dx} + p(x)y = q(x)\,.$$

- Compute the *integrating factor* (no integration constant needed here)

$$r(x) = e^{\int p(x)\,dx}\,.$$

- Multiply all terms in the differential equation by $r(x)$ to obtain

$$\frac{d}{dx}[r(x)y] = r(x)q(x)\,,$$

- Integrate with respect to x (a constant of integration is required):

$$r(x)y = \int r(x)q(x)\,dx\,.$$

Higher order differential equations

- The general solution of a *second-order* differential equation,

$$\frac{d^2y}{dx^2} = f(y, dy/dx, x)\,,$$

  involves *two* constants of integration.

- A particular solution requires two pieces of information. If y is specified at two values of x, we have *boundary* conditions. If y and dy/dx are specified at a single value of x, we have *initial* conditions.

- Higher-order derivatives are commonly written as

$$\frac{dy}{dx} = y'\,, \qquad \frac{d^2y}{dx^2} = y''\,, \quad \text{etc.}$$

- If $y_1(x)$ and $y_2(x)$ are solutions of a linear homogenous equation, $Ay_1(x) + By_2(x)$ is also a solution (for arbitrary constants A, B).

Exercises

4.1 Find the general solutions of

a) $\dfrac{dy}{dx} - 2y = 3$

b) $\dfrac{dx}{dt} - tx = t$

c) $\dfrac{dy}{dx} = \dfrac{4x^3 - y}{x}$

d) $\dfrac{dy}{dx} + 2y = e^{-x}$

e) $\dfrac{dx}{dt} + 2tx = 2t^3$

f) $x^2\dfrac{dy}{dx} + (1 - 2x)y = x^2$

4.2 Solve $\dfrac{dy}{dx} = \dfrac{x - y - x^3}{2x}$.

4.3 Find the particular solutions of

$$\begin{aligned}
\frac{dy}{dx} + y\tan x &= \sec x\,, & y &= 2 \quad \text{when} \quad x = 0 \\
\frac{dy}{dx} &= \frac{2y}{x} + x^4\,, & y &= 1 \quad \text{when} \quad x = 1 \\
\frac{dx}{dt} + 4x &= e^{-4t}\sin 2t\,, & x &= \frac{1}{2} \quad \text{when} \quad t = 0 \\
\cos x\,\frac{dy}{dx} + y &= \cos^3 x\,, & y &= \frac{1}{2} \quad \text{when} \quad x = 0 \\
(1 + x)\,\frac{dy}{dx} + y &= 3x^2\,, & y &= 2 \quad \text{when} \quad x = 0 \\
(1 + x^2)\,\frac{dy}{dx} + 2xy &= 4 + 2x\,, & y &= 4 \quad \text{when} \quad x = 0
\end{aligned}$$

4.4 Solve the differential equation

$$x\frac{dy}{dx} + y = e^x - xy\,.$$

Show that $y = (e^x + e^{-x})/2x$ is a particular solution provided $x \neq 0$.

4.5 Find the general solution of $\dfrac{dy}{dx} + 2y\tan x = \sin x$.

4.6 The loss of heat from an object to its surroundings is commonly governed by what is known as Newton's law of cooling. This states that the rate of loss of heat is proportional to the temperature difference between the object and the surroundings. Thus

$$\frac{dT}{dt} = -K[T - T_0(t)]\,,$$

where T is the temperature of the object, T_0 is the temperature of the surroundings (and may vary with time) and K is a constant specific to the object in question.

An object in room at a constant temperature of 15°C is initially at a temperature of 75°C. It is then cooling at a rate of 2°C/min. What will be its temperature after 15 minutes? What time will elapse before the temperature falls to 35°C?

4.7 When a condenser of capacity C is charged, the potential difference across the condenser at time t is given by

$$CR\frac{dV}{dt} = E - V\,,$$

where E is the emf of the battery and R is a non-inductive resistance. If $V = 0$ at $t = 0$, find $V(t)$ and $V(\infty)$.

4.8 Find the general solution of

$$\frac{d^4y}{dx^4} = 0\,.$$

What is the particular solution satisfying $y(0) = y'(0) = y''(0) = y'''(0) = 1$?

CHAPTER 5

Second-Order Linear Equations and Oscillations

In this chapter we discuss the general solution of second-order linear equations. These equations appear in numerous problems in Physic and Engineering and allow us to understand the different types of (damped) oscillatory behaviour.

5.1 Homogeneous, 2^{nd}-Order, Constant Coefficient Equations

We are interested in the solutions $y(x)$ of

$$\boxed{\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = 0\,,}$$

which is the general form of a homogeneous, 2^{nd}-order, linear differential equation with constant coefficients.

The solution rests on the fundamental property of the exponential function that it differentiates into a multiple of itself:

$$\frac{d}{dx}e^{mx} = me^{mx}\,; \qquad \frac{d^2}{dx^2}e^{mx} = m^2e^{mx}\,, \quad \text{etc.}$$

Thus, we shall assume at the outset that the solution takes the form

$$y = Ce^{mx}\,,$$

where m is a constant which may be real or complex. (An alternative method which does not make this assumption but *proves* that it is so, is given below.) Substituting this form in the differential equation gives

$$C(m^2e^{mx}) + aC(me^{mx}) + bCe^{mx} = 0,$$

or

$$(m^2 + am + b)Ce^{mx} = 0\,.$$

Since a nontrivial solution requires $C \neq 0$ and $e^{mx} > 0$, for this form to work we must choose m so that

$$\boxed{m^2 + am + b = 0\,.}$$

This is known as the **characteristic equation** or **auxiliary equation**. Since it is a quadratic equation we can simply write down its solutions,

$$m_{1,2} = \frac{-a \pm \sqrt{a^2 - 4b}}{2}.$$

If $a^2 \neq 4b$, these values are distinct, and we have found two solutions to the original equation, $Ce^{m_1 x}$ and $De^{m_2 x}$, where C and D are arbitrary constants. In fact these two solutions may be combined. If we put

$$y = Ce^{m_1 x} + De^{m_2 x},$$

and substitute we find that the differential equation is satisfied (because each term separately satisfies the equation: note that this would not be true is the differential equation were not *linear* in y and its derivatives). Thus the *general solution* of the homogeneous second-order equation with constant coefficients is

$$\boxed{y = Ce^{m_1 x} + De^{m_2 x}\,.}$$

5.2 Nature of Solutions

The nature of this general solution depends on the values of a and b. The following three different cases are possible:

Case 1 $a^2 > 4b$, **Case2** $a^2 < 4b$, **Case 3** $a^2 = 4b$.

Case 1. $a^2 > 4b$.

In this case m_1 and m_2 are real numbers and the interpretation of the general solution

$$y = Ce^{m_1 x} + De^{m_2 x}$$

is straightforward.

If both m_1 and m_2 are positive, each term shows exponential growth (curves A and B in Figure 5.1). The combination of terms will be dominated at large x by the faster growing exponential term (curve A). If $m_2 > m_1$, the solution will tend to $+\infty$ if the coefficient of the dominant m_2 term is positive, i.e. if $D > 0$ (dashed curve $A + B$), and to $-\infty$ if $D < 0$ (dashed curve $B - A$).

If both m_1 and m_2 are negative, each term will decay exponentially (curves A and B in Figure 5.2). The combination will always decay to zero for large x, where the solution is dominated by the slowest decaying term (curve A). If $m_2 < m_1$, the curve will tend to zero from above if the coefficient of the dominant m_2 term is positive, i.e. $D > 0$ (dashed curve $A + B$), and from below if $D < 0$ (dashed curve $B - A$).

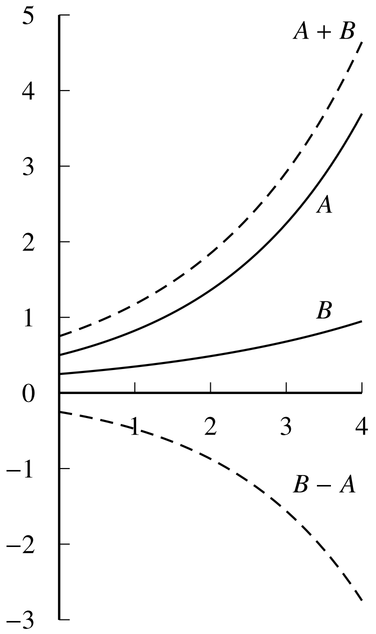


Figure 5.1:

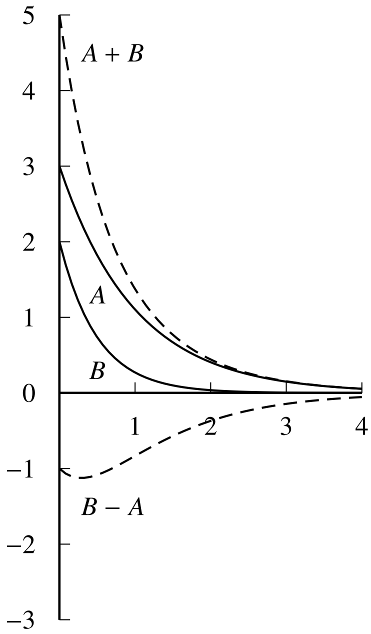


Figure 5.2:

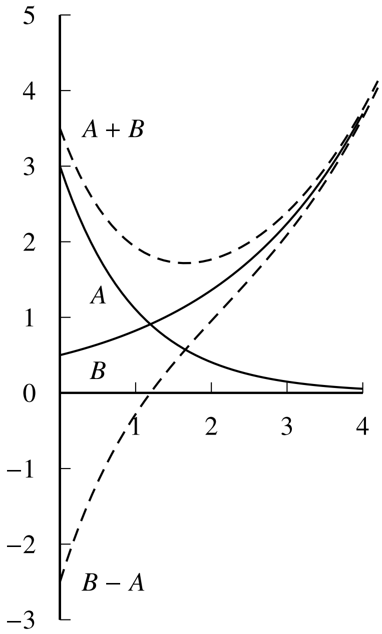


Figure 5.3:

Note that the combination of terms may produce neither a strictly increasing function nor a strictly decreasing function if the coefficients of the two terms have opposite sign.

There is even greater variety in the behaviour of the solutions if the roots m_1 and m_2 are of opposite sign. The one term will show exponential growth (curve B in Figure 5.3) and the other exponential decay (curve A). If $m_1 < 0$ and $m_2 > 0$ the combination of terms is always dominated by the growing m_2 term for large x unless its coefficient is identically zero, i.e. $D = 0$. The behaviour close to the origin depends on whichever term has the larger coefficient. In Figure 5.3 the decaying m_1 term has the larger coefficient, i.e. initial value, and the combined curves follow curve A, with the appropriate sign.

Case 2. $a^2 < 4b$.

Now m_1 and m_2 are complex conjugates (for real a, b). The solutions may now be written as

$$m_1 = \frac{-a + 2ik}{2}, \quad m_2 = \frac{-a - 2ik}{2},$$

where $2k = \sqrt{4b - a^2}$. In this case the general solution can be rewritten as

$$y = e^{-ax/2}\left(Ce^{ikx} + De^{-ikx}\right).$$

This can be put in a more suggestive form using Euler's theorem,

$$\begin{aligned} e^{ikx} &= \cos kx + i \sin kx \\ e^{-ikx} &= \cos kx - i \sin kx, \end{aligned}$$

as

$$\boxed{y = e^{-ax/2}(E\cos kx + F\sin kx)\,,}$$

where $E = C + D$ and $F = i(C - D)$. Since C and D were arbitrary, we might just as well use E and F.

The general solution then shows that as well as growing or decaying exponentially (depending on the sign of a), the solutions will oscillate because of the sine and cosine terms.

These solutions are sketched in Figure 5.4 and Figure 5.5.

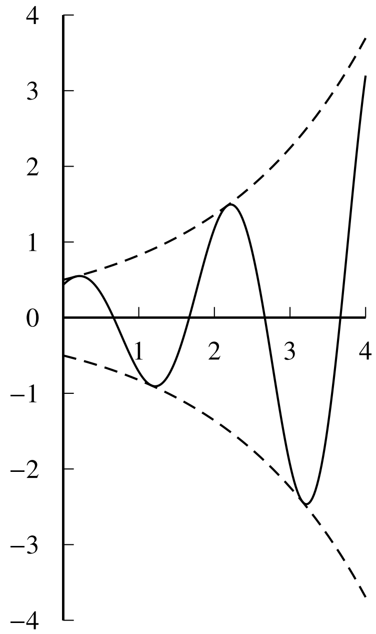


Figure 5.4:

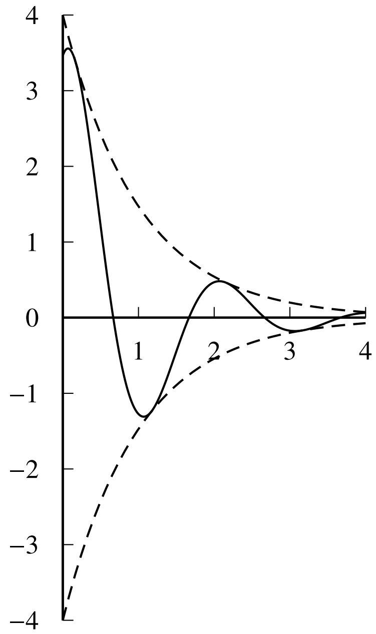


Figure 5.5:

The case when $a < 0$ is shown in Figure 5.4. To understand this solution, first note that we may rewrite the oscillatory part of the solution as

$$\begin{aligned}
E\cos kx + F\sin kx &= \sqrt{E^2+F^2}\left(\frac{E}{\sqrt{E^2+F^2}}\cos kx + \frac{F}{\sqrt{E^2+F^2}}\sin kx\right)\\
&= A\left(\cos\phi\cos kx - \sin\phi\sin kx\right)\\
&= A\cos(kx+\phi)\,,
\end{aligned}$$

where we have introduced $A = \sqrt{E^2+F^2}$ and the angle ϕ such that

$$\cos\phi = \frac{E}{\sqrt{E^2+F^2}}\,, \quad \text{and} \quad \sin\phi = -\frac{F}{\sqrt{E^2+F^2}}\,.$$

Since E and F uniquely determine A and ϕ and A and ϕ uniquely determine E and F, the general solution may be written in the alternative form

$$\boxed{y = Ae^{-ax/2}\cos(kx + \phi)\,.}$$

Now the cosine term will vary between $+1$ and -1 as x varies, hence

$$-Ae^{-ax/2} \le Ae^{-ax/2}\cos(kx + \phi) \le Ae^{-ax/2}\,.$$

Hence the solution curve $y(x)$ will lie between the two exponential curves $y = \pm Ae^{-ax/2}$ which are the dashed curves in Figure 5.4. Moreover, when the cosine term equals ± 1 the solution curve will have the same value as the bounding exponential curves and will touch them. In other words, the solution curve oscillates between the two growing exponential curves.

When $a > 0$, similar reasoning shows that the solution curve oscillates between two decaying exponentials, as shown in Figure 5.5.

A special case arises if $a = 0$ in which case the general solution is purely sinusoidal,

$$\boxed{y = E\cos kx + F\sin kx = A\cos(kx + \phi)\,,}$$

where $k = \sqrt{b}$, as shown in Figure 5.6.

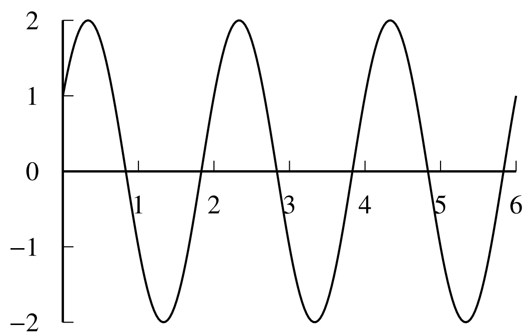


Figure 5.6:

Case 3. $a^2 = 4b$.

In this case, the auxiliary equation has only one root $-a/2$ so that $y = Ce^{-ax/2}$ is a solution of the differential equation. But there should be a second arbitrary constant; where is the solution associated with that?

There is a procedure which, given that we know one solution, enables us to find a second one. This is to put y equal to the known solution times an unknown function which we intend to find. Thus here we put

$$y = e^{-ax/2}f(x)\,,$$

and find a new equation for $f(x)$. Substituting into the original equation using

$$y' = -\frac{a}{2}e^{-ax/2}f(x) + e^{-ax/2}f'(x)$$

$$y'' = \frac{a^2}{4}e^{-ax/2}f(x) - ae^{-ax/2}f'(x) + e^{-ax/2}f''(x),$$

gives, on cancelling the exponential factor,

$$f''(x) + f'(x)(-a+a) + f(x)\left(\frac{a^2}{4} - \frac{a^2}{2} + b\right) = 0\,.$$

Note now that in precisely the case $a^2 = 4b$ that we are interested in, the term in $f(x)$ vanishes, and we are left with

$$f''(x) = \frac{d^2 f}{dx^2} = 0\,.$$

This has the general solution, found by integrating successively,

$$f(x) = Gx + H\,,$$

where G and H are arbitrary constants. Thus our new solution is

$$y = (Gx + H)e^{-ax/2}\,.$$

If we note that this solution contains the previous solution as the special case in which $G = 0$, we see that this form, now containing precisely *two* arbitrary constants, is in fact the general solution to the original problem,

$$\boxed{y = Gxe^{-ax/2} + He^{-ax/2}\,.}$$

This solution is derived rigorously below. Some textbooks produce it by guesswork; the method given here has the advantage that it works for other equations with non-constant coefficients.

Examples 5.2a Solve the following equations with the given initial or boundary conditions,

i) $y'' + 3y' + 2y = 0$ with initial conditions $y(0) = y'(0) = 1$,

ii) $y'' - 2y' + y = 0$ with boundary conditions $y(0) = 0$, $y(1) = 1$,

iii) $y'' + 9y = 0$ with initial conditions $y(0) = 1$, $y'(0) = 0$.

(i) Solutions are proportional to e^{mx}, with auxiliary equation

$$m^2 + 3m + 2 = 0\,.$$

This has roots $m = -2$ and $m = -1$, so that the general solution (this is **Case 1** above) is

$$y = Ce^{-2x} + De^{-x}\,.$$

To find the particular solution, the first condition $y(0) = 1$ requires

$$1 = C + D\,.$$

To use the second condition, $y'(0) = 1$, we need to calculate the derivative,

$$y' = \frac{dy}{dx} = -2Ce^{-2x} - De^{-x}\,.$$

Substitution of the condition then gives

$$1 = -2C - D\,.$$

Solving these simultaneous equations produces $C = -2$, $D = 3$, so that the required particular solution is

$$y = 3e^{-x} - 2e^{2x}\,.$$

(ii) Solutions are proportional to e^{mx}, with auxiliary equation

$$m^2 - 2m + 1 = 0\,.$$

The roots are coincident, the only solution being $m = 1$. This is therefore **Case 3** and the general solution is

$$y = Gxe^x + He^x\,.$$

Using the boundary conditions, $y(0) = 0$ requires

$$0 = H$$

and $y(1) = 1$ requires $1 = Ge$, or

$$G = e^{-1}\,.$$

Thus the required particular solution is

$$y = xe^{x-1}\,.$$

(iii) Solutions are proportional to e^{mx} where the auxiliary equation is

$$m^2 + 9 = 0\,.$$

This has roots $m = \pm 3i$. Here we have **Case 2**, so that the general solution is

$$y = Ce^{3ix} + De^{-3ix}$$

or, better,

$$y = E\cos 3x + F\sin 3x\,.$$

Applying the initial conditions, $y(0) = 1$ requires

$$1 = E + F\,.$$

The derivative is

$$y' = -3E\sin 3x + 3F\cos 3x\,,$$

so that $y'(0) = 0$ requires

$$0 = 3F\,.$$

Hence we must have $E = 1$ and $F = 0$. The required particular solution is

$$y = \cos 3x\,.$$

◇

You should note that the arithmetic used to evaluate the coefficients E and F in the last example used only real numbers. This is the great advantage of using the sin and cos form of the general solution. If we had used the complex exponential form, the arithmetic would have involved complex numbers. Try it!

5.3 The Harmonic Oscillator

Many systems *oscillate*, or vary periodically. Examples are musical instruments, ocean waves, hearts, clocks, mains (AC) electricity, structures such as bridges, 'boom-bust' economies, the number of sunspots on the Sun, the Earth's surface during an earthquake, and so on. The insights gained from studying any of these systems as a prototype are readily applicable to all the others.

Many applications lead to the equation in which the change in a quantity y, say, as a function of time t is given by

$$\ddot{y} + 2\gamma\dot{y} + \omega_0^2 y = 0.$$

To fix our ideas, suppose that y represents a physical displacement. This is the equation for a **harmonic oscillator**.

- When $\gamma = 0$, this is familiar simple harmonic motion.
- When $\gamma > 0$, this is a **damped harmonic oscillator**, with no forcing term.

Simple Harmonic Motion

Suppose damping is absent, that is, $\gamma = 0$. Then the governing equation becomes

$$\ddot{y} + \omega_0^2 y = 0\,.$$

This is known as the equation of *simple harmonic motion* (SHM). The acceleration is produced by a force which is proportional to the displacement and in the opposite direction. The general solution of this equation is

$$y = C\cos\omega_0 t + D\sin\omega_0 t \qquad \text{or alternatively} \qquad y = A\cos(\omega_0 t + \phi)\,.$$

In the latter form,

A is called the **amplitude**,,

ω_0 is called the **frequency** (measured in radians/second),

ϕ (which is between 0 and 2π) is called the **phase** of the oscillation,

The time $T = 2\pi/\omega_0$ is called the **period** of the oscillator.

Because the cosine function is periodic with period 2π in its argument, the value of y recurs every T seconds, where $\omega_0 T = 2\pi$. The frequency measured in completed cycles per second (a unit called the hertz, abbreviated Hz) is $\omega/2\pi$. The quantities are illustrated in Figure 5.7.

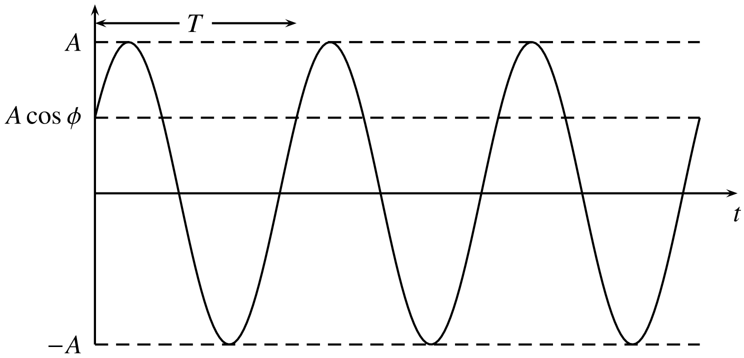


Figure 5.7:

The damped harmonic oscillator

When $\gamma \neq 0$, the governing equation can also be solved by the methods of the previous section. Three different kinds of physical behaviour are possible, associated with the general solutions

- $y = e^{-\gamma t}\left(Ce^{\kappa t} + De^{-\kappa t}\right) \quad (\gamma^2 > \omega_0^2) \quad \Longrightarrow$ **Supercritical damping**
- $y = e^{-\gamma t}\left(C\cos\kappa t + D\sin\kappa t\right) \quad (\gamma^2 < \omega_0^2) \quad \Longrightarrow$ **Subcritical damping**
- $y = e^{-\gamma t}\left(C + Dt\right) \quad (\gamma^2 = \omega_0^2) \quad \Longrightarrow$ **Critical damping**,

where $\kappa = \sqrt{|\gamma^2 - \omega_0^2|}$. In the first case,

$$y = Ce^{-(\gamma-\kappa)t} + De^{-(\gamma+\kappa)t}\,,$$

the exponents are both real. Moreover, both exponents are negative for $t > 0$ since $\kappa = \sqrt{\gamma^2 - \omega_0^2} < \gamma$. Thus both terms decay exponentially as $t \to \infty$ (as in Figure 5.2). This is called **supercritical damping**.

In the second case,

$$y = ae^{-\gamma t}\cos(\kappa t + \phi)\,.$$

The cosine term produces an oscillation but the amplitude is modulated by the $e^{-\gamma t}$ factor which decays exponentially as $t \longrightarrow \infty$. The oscillation thus dies away (as in Figure 5.5). This is **subcritical damping**.

In the third case, we need to note that $x^n e^{-x} \longrightarrow 0$ as $x \longrightarrow \infty$ for all powers x^n. The $te^{-\gamma t}$ term may increase initially but will eventually decay; the $e^{-\gamma t}$ term will always simply decay. This is **critical damping**.

Thus in all three cases, a positive damping $\gamma > 0$ causes the displacement to die away as $t \longrightarrow \infty$.

Summary of Chapter 5

- The only generally soluble second-order equations are *linear differential equations with constant coefficients*,

$$\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = f(x)\,.$$

  If $f(x) = 0$, the equation is *homogeneous*. If $f(x) \neq 0$ it is *inhomogeneous*.

- The solutions of *homogeneous* equations all take the form e^{mx} where m may be real, imaginary or complex. The values of m are the roots of the auxiliary equation

$$m^2 + am + b = 0\,.$$

- If $a^2 > 4b$, the roots m_1 and m_2 are real and distinct so that we can write

$$y = Ce^{m_1 x} + De^{m_2 x}\,.$$

  If $a^2 < 4b$, the roots are complex conjugates $m = -a/2 \pm ik$ so that we can write

$$\begin{aligned} y &= e^{-ax/2}\left(Ce^{ikx} + De^{-ikx}\right) \\ &= e^{-ax/2}\left(E\cos kx + F\sin kx\right)\,. \end{aligned}$$

  If $a^2 = 4b$, the repeated root is real, $m = -a/2$. The solution is then

$$y = Gxe^{mx} + He^{mx}\,.$$

- When $a = 0$ and $b > 0$, we have *simple harmonic motion*. When $a > 0$ and $b > 0$, we have a *damped harmonic oscillator*.

Exercises

5.1 Solve the following equations, giving the general solution and then the particular solution satisfying the given boundary or initial conditions.

a) $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} - 15y = 0\,, \quad y(0) = y'(0) = 1$

b) $\dfrac{d^2y}{dx^2} + 6\dfrac{dy}{dx} + 9y = 0\,, \quad y(0) = 0, y(1) = 3$

c) $\dfrac{d^2y}{dx^2} + 25y = 0\,, \quad y(0) = y(\pi/2) = 1$

d) $3\dfrac{d^2y}{dx^2} + 5\dfrac{dy}{dx} - 2y = 0\,, \quad y(0) = 1, y(2) = 1$

e) $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + 26y = 0\,, \quad y(0) = 0, y'(0) = 2$

(Give the general and particular solutions in both real and complex exponential form.)

f) $\dfrac{d^2y}{dx^2} - 99\dfrac{dy}{dx} - 100y = 0\,, \quad y(0) = 1, y'(0) = -1$

g) $\dfrac{d^2y}{dx^2} - 99\dfrac{dy}{dx} - 100y = 0\,, \quad y(0) = 1, y'(0) = -1.01\,.$

(Compare these last two cases; if you attempt to plot the particular solutions for x small and positive in each instance you will find that the first decays steadily to zero but the second will 'explode' on you. The difference in behaviour stems from the slight variation in the initial conditions. Equations like this are called *stiff*; they are hard to solve numerically, because slight errors get grotesquely magnified!)

5.2 **Alternative derivation of the general solution of homogeneous second-order equations with constant coefficients.** Show that the homogeneous second-order equation with constant coefficients

$$\frac{d^2y}{dx^2} + A\frac{dy}{dx} + By = 0$$

can be rewritten as

$$\frac{d}{dx}\Big(\frac{dy}{dx} - m_1 y\Big) - m_2\Big(\frac{dy}{dx} - m_1 y\Big) = 0\,,$$

where the constants m_1, m_2 are the roots of

$$m^2 + Am + B = 0\,.$$

Now, defining $p = \dfrac{dy}{dx} - m_1 y$, the second-order equation becomes a pair of first-order equations

$$\begin{aligned}\frac{dp}{dx} - m_2 p &= 0\\ \frac{dy}{dx} - m_1 y &= p\,.\end{aligned}$$

Solve the first equation for p, then substitute in the right-hand side of the second equation and solve for y.

Hence show that the general solution of the second-order equation is

$$y = \begin{cases} Ce^{m_1 x} + De^{m_2 x} & \text{if } m_1 \neq m_2 \\ (C + Dx)\, e^{mx} & \text{if } m_1 = m_2 = m. \end{cases}$$

5.3 Show the solution for simple harmonic motion may also be written as

$$y = a \cos\left(\omega_0 t + \phi\right),$$

where $a = \sqrt{C^2 + D^2}$ and $\tan\phi = -D/C$. Write down the corresponding form for $\dot{y}$, and plot a graph of both for the case $a = 1, \omega_0 = \pi, \phi = \pi/4$. What is the time before this solution repeats itself?

CHAPTER 6

More General Differential Equations

In this chapter we finish our study of differential equations by considering simple examples of two very large classes of differential equations. First, we discuss the case of *inhomogeneous* linear equations and show how they allow us to understand how resonances appear. Second, we consider examples in which more than one differential equation are coupled to each other, building a *system* of differential equations.

6.1 Inhomogeneous Linear Equations

The general form of an nth-order linear differential equation was introduced in Sec. 4.3 above as

$$L^{(n)}(y(x)) = F(x), \tag{6.1a}$$

where $L^{(n)}$ is the linear operator of order n defined as

$$L^{(n)} \equiv P_n(x)\frac{d^n}{dx^n} + P_{n-1}(x)\frac{d^{n-1}}{dx^{n-1}} + \ldots + P_1(x)\frac{d}{dx} + P_0(x).$$

In the last Chapter 5 (Sec. (5.1) we studied the second-order ($n = 2$) homogeneous equation ($F(x) = 0$) with constant coefficients (i.e., constant $P_0(x), P_1(x)$, and $P_2(x)$). Here we are interested in the *inhomogeneous* case $F(x) \neq 0$.

General solution

There is no universally valid procedure to find the *general* solution of *inhomogeneous* linear differential equations and we again search for general properties of the solution that can help us in this task. The following general theorem helps by showing how the solution can be decomposed into two simpler problems.

Theorem 6.1b *Let $y_p(x)$ be a particular solution of a linear inhomogeneous differential equation – Eq. (6.1a) – and $y_h(x)$ be the general solution of the corresponding homogeneous equation – obtained setting $F(x) = 0$ in Eq. (6.1a). The general solution $y(x)$ of Eq. (6.1a) can be written as a linear combination of these two solutions*

$$y(x) = y_p(x) + y_h(x). \tag{6.1c}$$

Proof: By hypothesis $L^{(n)}(y_h) = 0$ and $L^{(n)}(y_p) = F(x)$. Therefore, from the linearity of the operator ($L^{(n)}(y_1 + y_2) = L^{(n)}(y_1) + L^{(n)}(y_2)$), it follows that

$$L^{(n)}(y(x)) = L^{(n)}(y_p(x) + y_h(x)) = L^{(n)}(y_p(x)) + L^{(n)}(y_h) = F(x) + 0,$$

which shows that solution (6.1c) indeed satisfies Eq. (6.1a). As discussed in the beginning of the last chapter, the *general* solution of an nth-order differential equation has n arbitrary constants. By hypothesis, $y_h(x)$ is the *general* solution of the homogeneous equation and thus it already has n arbitrary constants. Therefore $y(x) = y_p(x) + y_h(x)$ is the general solution of the inhomogeneous equation (6.1a).

Remark: The procedure to obtain the particular solution $y_p(x)$ typically involves trying different functions and checking whether they satisfy the given differential equation. The help provided by the theorem above is that it tells us that we need only a *particular* solution (no arbitrary constants needed) because the general solution is obtained from the *general* solution of the homogeneous equation (which is usually easier to obtain).

Example 6.1d Solve $y'' + 9y = e^{5x}$.

The solution y_h of the homogeneous equation $y'' + 9y = 0$ is obtained using the method of the previous chapter as

$$y_h(x) = A\cos(3x) + B\sin(3x).$$

For a particular solution, because of the properties of the derivative of the exponential function, we try $y_p(x) = Ce^{5x}$. Substituting into the differential equation, we have

$$25Ce^{5x} + 9(Ce^{5x}) = e^{5x},$$

$$(34C - 1)e^{5x} = 0,$$

leading to $C = 1/34$. Thus a particular solution is $y_p(x) = e^{5x}/34$. Using Theorem 6.1b we find the general solution as

$$y(x) = y_h(x) + y_p(x) = A\cos(3x) + B\sin(3x) + e^{5x}/34.$$

◇

Second-order inhomogeneous linear equation

As an application of the general results of the previous section, consider a periodic forcing of the harmonic oscillator discussed at the end of the last chapter

(6.1e) $$\ddot{y} + \omega_0^2 y = f\cos(\omega t).$$

From a physical point of view, the oscillator has a natural frequency ω_0 and is forced externally with frequency ω and constant maximum intensity f, i.e., by a force $F(t) = f\cos(\omega t)$. From a mathematical point of view, the equation above is an example of a non-homogeneous, linear, second-order differential equation with constant coefficients.

We compute the general solution of Eq. (6.1e) following Theorem 6.1b. As discussed in the last chapter, the solution of the homogeneous equation is given by

$$y_h(t) = A\cos(\omega_0 t + \Phi),$$

where A, Φ are arbitrary constants. To compute the particular solution we try $y_p(t) = C\cos(\omega t)$ in Eq. (6.1e), which leads to

$$(-\omega^2 + \omega_0^2)C\cos(\omega t) = f\cos(\omega t) \Rightarrow C = \frac{f}{\omega_0^2 - \omega^2}$$

Therefore, we find that $y_p(t) = \frac{f}{\omega_0^2-\omega^2}\cos(\omega t)$ is a particular solution of Eq. (6.1e). Using Theorem 6.1b we conclude that the *general* solution of Eq. (6.1e) is

(6.1f)
$$\boxed{y(t) = A\cos(\omega_0 t + \Phi) + \frac{f}{\omega_0^2 - \omega^2}\cos(\omega t).}$$

The solution is thus a superposition of two oscillations, one with the natural frequency of the oscillator ω_0 and one with the frequency of the external forcing ω.

Resonance

What is the amplitude of the oscillations described in Eq. (6.1f)? For concreteness, consider the generic case in which ω_0/ω is an irrational number and that the oscillator at $t = 0$ is at rest ($v(0) = \dot{y}(0) = 0$) at $y(0) = 0$. This implies that $\Phi = 0$ and $A = -f/(\omega_0^2 - \omega^2)$ and thus corresponds to the particular solution of Eq. (6.1f) equal to

$$y(t) = \frac{f}{\omega_0^2 - \omega^2}(\cos(\omega t) - \cos(\omega_0 t)).$$

Since $-1 \leq \cos(\theta) \leq 1$, the maximum of $y(t)$ occurs when the effect of the two oscillations add to each other and therefore the maximum amplitude is given by

(6.1g)
$$\boxed{Amp(\omega) = \left|\frac{2f}{\omega_0^2 - \omega^2}\right|.}$$

The amplitude of the oscillation is thus proportional to the external forcing but it is inversely proportional to the difference of the square of the frequencies. While ω_0 is a property of the oscillator and is thus usually a constant, the frequency ω depends only on the external forcing and is thus a parameter that can be controlled without changing the oscillator itself.

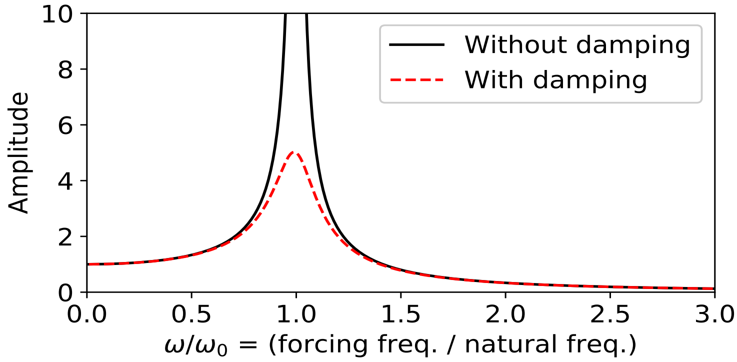


Figure 6.1: Amplitude of oscillation showing a resonance when $\omega = \omega_0$. The solid line is Eq. (6.1g), where the y−axis is $Amp(\omega)\omega_0^2/(2f)$. The dashed line is the amplitude of the damped ($\gamma = 0.1$) harmonic oscillator with periodic forcing $Amp(\omega) = f/\sqrt{(2\omega\gamma)^2 + (\omega^2 - \omega_0^2)^2}$, plotted for $f = 1$ and $\omega_0 = 1$.

It is thus natural to consider Eq. (6.1g) as a function of ω, as plotted in Fig. 6.1. When the forcing happens with a frequency similar to the natural frequency of the oscillators (when ω approaches ω_0) the amplitude of the oscillations grows dramatically. In fact, $Amp(\omega) \to \infty$ when $\omega \to \omega_0$. This phenomenon is known as **resonance**.

Resonance is a widespread effect in the Sciences and Engineering. In particular, mechanical resonances lead to growing vibrations and breakdown of structures so that avoiding mechanical resonances is a major concern in every construction project in civil or mechanical Engineering. Resonances appear also in all wave phenomena and are the key idea to generate sounds of a specific frequency (in musical instruments) and to tune electrical circuits to selected frequencies (in radios and TVs).

6.2 Systems of Equations

The examples considered so far involve a single dependent variable. More complex situations require more variables. Mathematical models of complicated systems in engineering or economics may involve hundred or even thousands of variables. A corresponding number of equations are needed to specify the relations between these variables. Just as we consider systems of linear equations in linear algebra, we can try to analyse systems of differential equations. As in linear algebra, it is generally impossible to solve the equations individually. They must be solved *simultaneously*. In many cases it is not possible to find explicit solutions, and we are forced to resort to numerical solution by computer. However, there are a few very simple types of system which can be solved directly.

With two dependent variables we generally require two equations to specify their behaviour.

We consider pairs of equations of the form

$$\boxed{\begin{aligned}\frac{dx}{dt} &= f(x,y,t)\\ \frac{dy}{dt} &= g(x,y,t),\end{aligned}}$$

In general each equation involves both x and y. Then they are said to be **coupled**. In such cases it is not possible to solve for one variable without taking account of the other. We single out three (I-III) special types of system, along with corresponding methods of solution.

I-Equations that may be solved independently

The simplest possible set of equations involves differential equations in which each contains only one dependent variable, thus

$$\boxed{\begin{aligned}\frac{dx}{dt} &= f(x,t)\\ \frac{dy}{dt} &= g(y,t).\end{aligned}}$$

These equations are **uncoupled**, and if they take one of the forms discussed in earlier chapters, they can be integrated independently of each other.

Example 6.2a Consider the motion of an object (projectile) falling freely but not necessarily vertically. This models the flight of a ball thrown into the air, for instance. If the x and y axes are taken to be in the horizontal plane and the z axis to point vertically upwards, the equations governing the motion are

$$\begin{aligned}\frac{dv_x}{dt} &= 0\\ \frac{dv_y}{dt} &= 0\\ \frac{dv_z}{dt} &= -g,\end{aligned}$$

where v_x, v_y and v_z are the components of velocity in the x, y and z directions respectively. These equations may be written more concisely in vector form

$$\frac{d\mathbf{v}}{dt} = \mathbf{g},$$

where the velocity vector is $\mathbf{v} = (v_x, v_y, v_z)$ and $\mathbf{g} = (0, 0, -g)$.

The vector equation may be integrated with respect to time to produce

$$\mathbf{v} = \int \mathbf{g}\,dt = \mathbf{g}t + \mathbf{C}.$$

The constant of integration in each equation has been written as a component of the vector constant $\mathbf{C}$. If the initial velocity is given as $\mathbf{u}$, then substitution of $\mathbf{v} = \mathbf{u}$ when $t = 0$ shows that $\mathbf{C} = \mathbf{u}$ and hence

$$\mathbf{v} = \mathbf{u} + \mathbf{g}t\,.$$

Now the velocity $\mathbf{v}$ is just the rate of change of the position vector $\mathbf{r}$, hence

$$\frac{d\mathbf{r}}{dt} = \mathbf{u} + \mathbf{g}t\,.$$

This may now be integrated with respect to t to yield

$$\mathbf{r} = \int (\mathbf{u} + \mathbf{g}t)dt = \mathbf{u}t + \frac{1}{2}\mathbf{g}t^2 + \mathbf{D}\,.$$

If we take the origin at the initial position the vector constant of integration may be evaluated as $\mathbf{D} = 0$. Hence the position vector of the object as a function of time is

$$\mathbf{r} = \mathbf{u}t + \frac{1}{2}\mathbf{g}t^2\,.$$

Note that if the initial velocity has no component in the y-direction, so that

$$\mathbf{u} = (u\cos\alpha, 0, u\sin\alpha)$$

where α is the usual angle of projection, then the y-component of the solution reduces to $y = 0$. This demonstrates that projectile motion under gravity takes place in two dimensions only—the third spatial coordinate may be simply ignored from the outset. ◇

II-Equations that may be solved successively

Equations of this type take the form

$$\boxed{\begin{aligned} \frac{dx}{dt} &= f(x,t) \\ \frac{dy}{dt} &= g(x,y,t)\,. \end{aligned}}$$

The first may be integrated directly to give $x = F(t)$ and then this form may be substituted into the second

$$\frac{dy}{dt} = g(F(t), y, t)\,.$$

This now is a first-order differential equation in y and t alone and may be solved, hopefully, by standard means. This type of system is quite common. It occurs in variable mass problems for example.

III-Equations for which one variable can be eliminated

In a general system of first-order differential equations each variable will appear in each derivative. No equation can be solved independently of the others. This is the case of **coupled** equations.

One form of coupled equations which is quite tractable is the example of *linear equations with constant coefficients*. These can be written as

$$\boxed{\begin{aligned}\frac{dx}{dt} &= ax + by\\ \frac{dy}{dt} &= cx + dy\,.\end{aligned}}$$

In this case, we can eliminate one of the variables to produce a differential equation of standard form whose solution is known.

The elimination is a three-step process. Firstly, either equation is differentiated with respect to t. Suppose we work with the first. Its derivative is

$$\frac{d^2x}{dt^2} = a\frac{dx}{dt} + b\frac{dy}{dt}\,.$$

Substitute for dy/dt using the second equation, giving

$$\frac{d^2x}{dt^2} = a\frac{dx}{dt} + b(cx + dy)\,.$$

Finally use the original form of the first equation to eliminate y, leaving

$$\frac{d^2x}{dt^2} = a\frac{dx}{dt} + bcx + d\left(\frac{dx}{dt} - ax\right),$$

or

$$\frac{d^2x}{dt^2} - (a+d)\frac{dx}{dt} + (ad - bc)x = 0\,.$$

This is a second-order differential equation with constant coefficients. It can therefore be solved by the means described in Chapter 5.

Unfortunately, most second-order equations that result from the elimination of one variable *cannot* be solved in this way. In general, solutions cannot be obtained analytically if the equations are coupled in such a way that they cannot be solved simultaneously or successively. This does not mean that we cannot discover a great deal about the behaviour of the solutions of such equations. There are analytical techniques which will reveal the nature of the solutions without being able to write down the solution explicitly. These, however, take us beyond the scope of this course.

Summary of Chapter 6

- The general solution of inhomogeneous linear differential equations can be written as
$$y(x) = y_p(x) + y_h(x),$$
where $y_h(x)$ is the solution of the corresponding homogeneous equation and $y_p(x)$ is a particular solution of the inhomogeneous equation.

- An example of second-order inhomogeneous linear differential equation is the harmonic oscillator with periodic forcing
$$\ddot{y} + \omega_0^2 y = f\cos(\omega t).$$
For $\omega \approx \omega_0$ the resulting oscillations show very large amplitude, a phenomenon known as **resonance**.

- A system of *two coupled* equations is written as
$$\frac{dx}{dt} = f(x, y, t), \qquad \frac{dy}{dt} = g(x, y, t),$$
and is generally intractable.

- In simple cases, one of the equations is independent of the other. The solution of this equation can then be introduced into the second. Sometimes a *substitution* for x and y may produce independent equations.

- In cases of *linear equations with constant coefficients*,
$$\frac{dx}{dt} = ax + by, \qquad \frac{dy}{dt} = cx + dy,$$
one variable may be eliminated by first *differentiating* one equation,
$$\frac{d^2x}{dt^2} = a\frac{dx}{dt} + b\frac{dy}{dt}$$
then substituting for dy/dt using the other equation,
$$\frac{d^2x}{dt^2} = a\frac{dx}{dt} + b(cx + dy),$$
and finally eliminating y by using the first equation again,
$$\frac{d^2x}{dt^2} = a\frac{dx}{dt} + bcx + bd\left(\frac{dx/dt - ax}{b}\right).$$
This produces a *second-order differential equation with constant coefficients*.

Exercises

6.1 Find the general solution $y(x)$ of the inhomogeneous equation

$$y'' + y' - 2y = x^2.$$

Hint: As a particular solution, try $y_p(x) = Ax^2 + Bx + C$.

6.2 Find the general solution of

$$\frac{d^2y}{dx^2} = e^{2x}.$$

What is the particular solution satisfying the boundary conditions $y(0) = 0$, $y(1) = 2$?

6.3 Find the solution of

$$\frac{d^3y}{dx^3} = \cos 2x + x^2 - 1$$

satisfying $y(0) = 0$, $y'(0) = 0$, $y''(0) = 1$.

6.4 Two species with populations x and y compete for their food supply. The equations describing the evolution of x and y are

$$\dot{x} = ax - by$$
$$\dot{y} = -cx + dy,$$

where a, b, c, d are positive constants. Explain why this is a reasonable model, and eliminate y to deduce that x satisfies the second-order equation

$$\ddot{x} - (a + d)\dot{x} + (ad - bc)x = 0.$$

Show that the solution is $x(t) = Ce^{m_1 t} + De^{m_2 t}$, where you are to give expressions for m_1 and m_2; show that of necessity they are real. Find a similar solution for y.

The values of the parameters for the two species are estimated as $a = d = 2$ and $b = c = 1$. Find the particular solutions valid for the initial conditions $x = 100, y = 200$ at $t = 0$. Determine the time until one species is eliminated. (Note that in this special case, the only difference between the species is that one starts out with more members than the other.)

6.5 Two species cooperate to maintain each other against their natural death rates according to the equations

$$\dot{x} = -ax + by$$
$$\dot{y} = cx - dy.$$

Proceeding as in the last example, show that unless $bc > ad$, both populations tend to extinction. If $a = 2, b = 4, c = 4$ and $d = 2$, determine the particular solutions for x and y if initially $x = 100, y = 200$. Deduce that as $t \to \infty$, the populations become equal.

6.6 The path of a long jumper is very flat and thus the speed which gives the deceleration due to air resistance may be approximated by the horizontal component of the velocity, leading to

$$\frac{d\mathbf{v}}{dt} = \mathbf{g} - k v_x \mathbf{v},$$

where k is a constant (air resistance), $\mathbf{g} = g\mathbf{k}$ is the acceleration due to gravity, $\mathbf{v} = v_x\mathbf{i} + v_z\mathbf{k}$, v_x is the velocity in the horizontal direction, and v_z is the velocity in the vertical direction. Solve these equations for the motion in the horizontal and vertical directions. Hence find the equation of the path.

CHAPTER 7

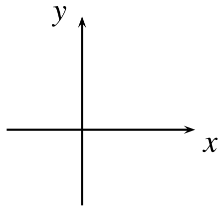
Curves and Surfaces in 3-Dimensional Space

We started this course with simple first-order differential equations and proceeded towards more general cases, such as higher-order and systems of coupled differential equations. Another important class of differential equations appearing in science involves functions of more than one independent variables. As motivated in the last part of the Introduction, these cases require a deeper understanding of multi-variable calculus. This is the goal of the second part of this course that starts here.

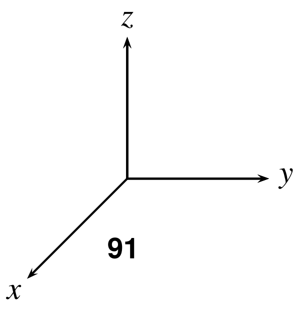
This chapter begins by introducing the 3-dimensional Cartesian coordinate system. We see how to plot points in such a system, and look at some examples of curves and surfaces in 3-dimensional space. We then move on to real-valued functions of two real variables, and their graphs. In general, the graph of such a function is a surface in 3-dimensional space.

7.1 Cartesian coordinates in 3 dimensions

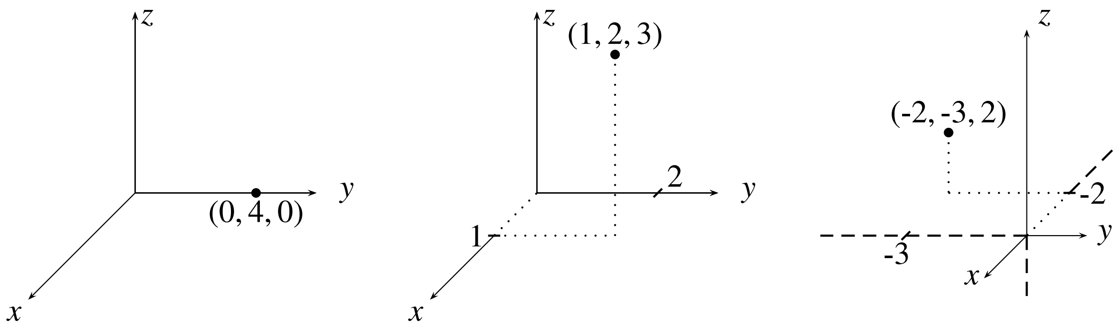
Think of the 2-dimensional Cartesian coordinate system, drawn in the usual fashion:



Now imagine a third axis, through the origin, perpendicular to the page you are looking at. The third axis is known as the z-axis. It is perpendicular to the plane determined by the x-axis and the y-axis, and the positive end of the z-axis is the one pointing up from the page. The usual way to draw these three axes is as follows:



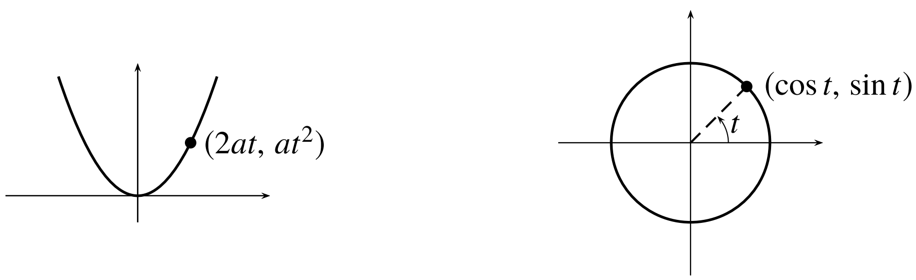
The set of all points $\{(x, y) \mid x \in \mathbb{R}, y \in \mathbb{R}\}$ in the 2-dimensional Cartesian plane is denoted by $\mathbb{R}^2$, and the set of all points $\{(x, y, z) \mid x \in \mathbb{R}, y \in \mathbb{R}, z \in \mathbb{R}\}$ is denoted by $\mathbb{R}^3$. Plotting points in $\mathbb{R}^3$ is entirely analogous to plotting points in $\mathbb{R}^2$. In two dimensions, the point $(2, 3)$, for example, is plotted by moving 2 units in the positive x-direction, and then 3 units in the positive y-direction. To plot the point $(1, 2, 3)$ in three dimensions, move 1 unit in the positive x-direction, 2 in the positive y-direction and 3 in the positive z-direction. This is shown in the following diagram, along with two other examples.



7.2 Curves in 3-dimensional space

In two dimensions, many curves can be represented by a single equation linking x and y. For example, $y = 3x + 2$ is the equation of a straight line; an equation of the form $y = f(x)$, where f is a function from $\mathbb{R}$ to $\mathbb{R}$, is some sort of a curve; $x^2 + y^2 = 1$ is the equation of a circle. (Note that the equation $x^2 + y^2 = 1$ does not represent a function of the form $y = f(x)$, since for each value of x in the open interval $(-1, 1)$ the equation gives two values for y.) In three dimensions, a single equation relating x, y and z will, in general, represent a surface. More than one equation is needed to represent a curve in three dimensions. The usual way to represent such a curve is by parametric equations.

You should already be familiar with the use of parametric equations to represent a curve in two dimensions. Examples are the representation of a parabola $x^2 = 4ay$ as $(2at, at^2)$, and the unit circle as $(\cos t, \sin t)$.



$\triangleright$ **Aside** Students should recognise that a parametric representation of a curve and a vector equation describing the curve are much the same thing. For example, the unit circle, given in parametric form as $x = \cos t$, $y = \sin t$, can be described by the vector equation $x\,\mathbf{i} + y\,\mathbf{j} = \cos t\,\mathbf{i} + \sin t\,\mathbf{j}$. In other words, $\cos t\,\mathbf{i} + \sin t\,\mathbf{j}$ is simply the position vector of the point $(\cos t, \sin t)$ on the unit circle. $\triangleleft$

In general, if I is an interval of $\mathbb{R}$, and f and g are functions from I to $\mathbb{R}$, the parametric equations $x = f(t)$, $y = g(t)$ represent a curve in the plane. The curve is the locus of the points with coordinates $\big(f(t), g(t)\big)$, as t varies over I.

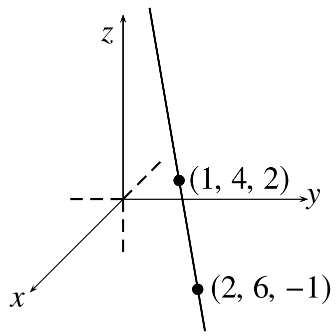
Now, if h is another function from I to $\mathbb{R}$ then the parametric equations $x = f(t)$, $y = g(t)$, $z = h(t)$ for $t \in I$, represent a curve in 3-dimensional space. This curve is the locus of the points P with coordinates $\big(f(t), g(t), h(t)\big)$, as t varies over I. (The position vector of P is $f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$.)

Examples 7.2a

i) The parametric equations

$$x = 1 + t, \quad y = 4 + 2t, \quad z = 2 - 3t$$

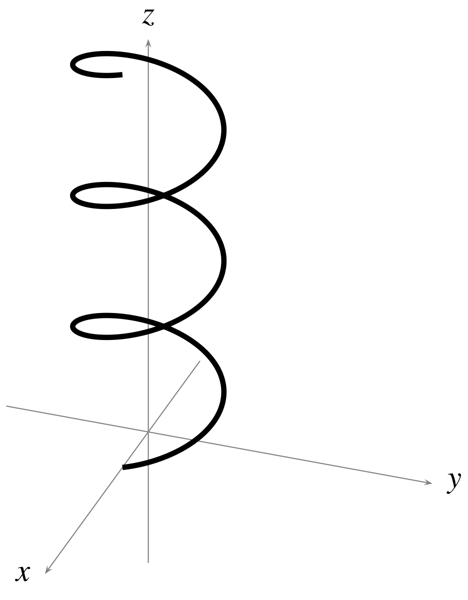
represent a straight line in $\mathbb{R}^3$. (Note that each of the three equations is linear in t.) This line passes through the points $(1, 4, 2)$ (when $t = 0$), and $(2, 6, -1)$ (when $t = 1$).



ii) The parametric equations

$$x = \cos t, \quad y = \sin t, \quad z = t$$

represent a helix in 3-dimensional space. This curve winds around the surface of a cylinder of radius 1, with the z-axis as its central axis.



It may be helpful to think of this as follows. In two dimensions (where $z = 0$), the point $(\cos t, \sin t)$ is at $(1, 0)$ when $t = 0$, and as t increases the point moves around the unit circle, returning to $(1, 0)$ when $t = 2\pi$, and then moving round the circle again as t continues to increase. In three dimensions, the point $(\cos t, \sin t, t)$ is at $(1, 0, 0)$ when $t = 0$, and as t increases it moves in a circular fashion but is "lifted" by the z-coordinate to a height equal to the current value of t. Hence, when $t = 2\pi$ for example, the point is at $(1, 0, 2\pi)$, directly above where it was at $t = 0$; when $t = 4\pi$ the point is at $(1, 0, 4\pi)$, and so on.

◇

7.3 Planes

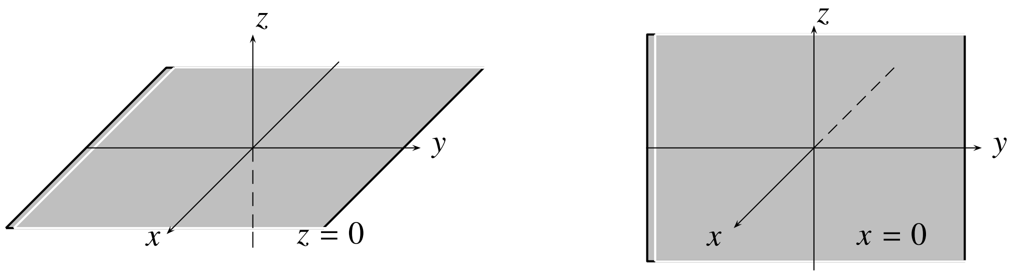
The (Cartesian) equation of a plane in 3-dimensional space is of the form

$$ax + by + cz = d,$$

where a, b, c and d are constants such that not all of a, b and c are equal to zero. What we mean by this is that if we were to plot all the points in the 3-dimensional coordinate system that satisfied the equation $ax + by + cz = d$ (for some values of a, b, c and d), then the points would all lie in a plane. (Compare the equation $ax+by+cz = d$ with the equation $ax+by = c$ of a straight line in two dimensions.)

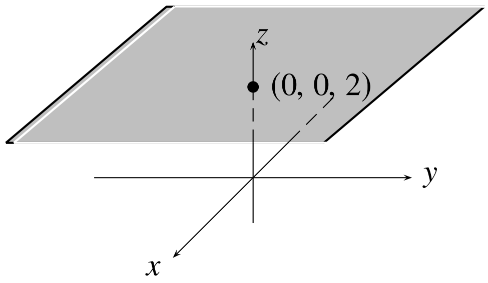
Examples 7.3a

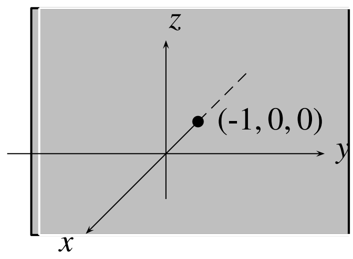
i) Consider all the points in the 3-dimensional coordinate system such that $z = 0$. That is, all the points $(x, y, 0)$, where x and y can take any value. It should be clear that these points form the xy-plane, since we can move anywhere in the x-direction and anywhere in the y-direction, but we cannot move up or down. So the graph of the equation $z = 0$ is the xy-plane. Similarly, $x = 0$ represents the yz-plane, and $y = 0$ the xz-plane.



ii) The equation $z = 2$ represents a horizontal plane (parallel to the xy-plane), through the point $(0, 0, 2)$.

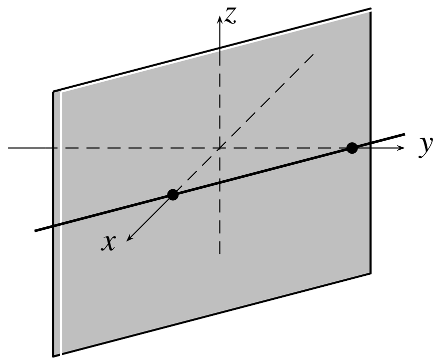
The equation $x = -1$ represents a vertical plane (parallel to the yz-plane), through the point $(-1, 0, 0)$.



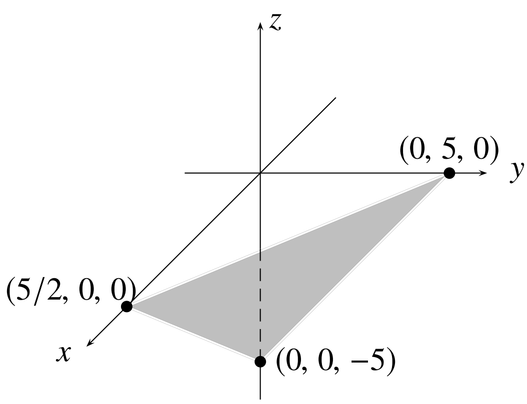


iii) The equation $2x + y = 5$ represents a plane. Since z is not mentioned in the equation, any point (x, y, z) which satisfies $2x + y = 5$ will lie in the plane, regardless of the value of z. In particular, the points such that $2x + y = 5$ and $z = 0$ lie in the plane. Such points are precisely those which form the straight line $2x + y = 5$ in the xy-plane.

Thinking about this line, and the fact that z can take any value, we see that the plane is the vertical plane which passes through the line $2x + y = 5$ in the xy-plane.



iv) Since a plane is determined by any three non-collinear points, the simplest way to sketch a plane is often to find three non-collinear points which lie in it. Consider, for example, the plane with equation $2x + y - z = 5$. Three points which lie in this plane are $(5/2, 0, 0)$, $(0, 5, 0)$ and $(0, 0, -5)$; these are the points where this plane meets the x, y and z–axes, respectively. The graph of a triangular region of this plane is therefore as shown:



◇

7.4 Other surfaces

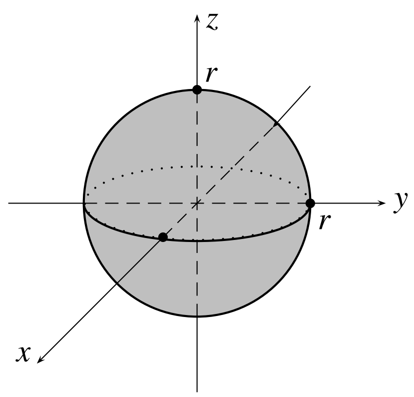
As mentioned above, an equation involving the variables x, y, and z will, in general, represent a surface. (That is, if we have an equation relating x, y, and z, and we plot all the points (x, y, z) satisfying that equation, then the result will generally be a surface.) Unless the equation is relatively simple, it is not easy to sketch the corresponding surface, and it is not expected that you should be able to do so. There are some simple surfaces, however, with which familiarity is useful. It is also useful to be able to analyse a surface in terms of its cross-sections. (This will be particularly so when we look at partial derivatives in Chapter 8.)

A horizontal cross-section of a surface is obtained by intersecting the surface with a horizontal plane. In the 3-dimensional coordinate system horizontal planes have equation $z = a$, for some constant a. So we can find the equation of a horizontal cross-section by substituting $z = a$ into the equation of the surface. In general, such a substitution will result in an equation relating x and y, representing a curve in the plane $z = a$.

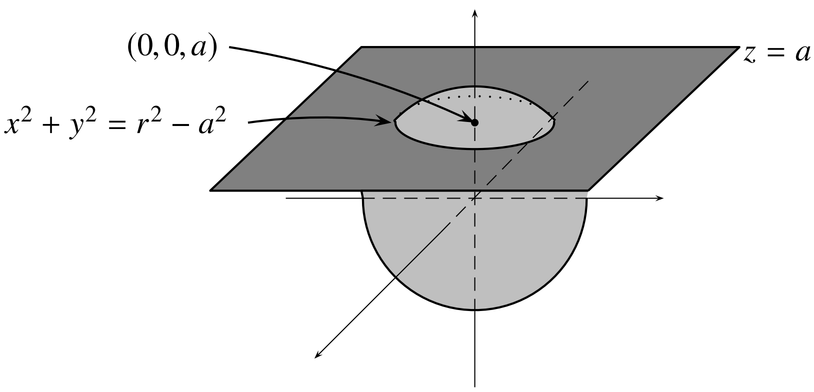
Vertical cross-sections are obtained by intersecting the surface with vertical planes. The general equation of a vertical plane in our coordinate system is $ax + by = c$ (where a and b are not both zero). However, it is usually sufficient to look at those vertical cross-sections obtained by intersecting a surface with a vertical plane of the type $x = c$ (parallel to the yz-plane), or $y = d$ (parallel to the xz-plane).

Examples 7.4a

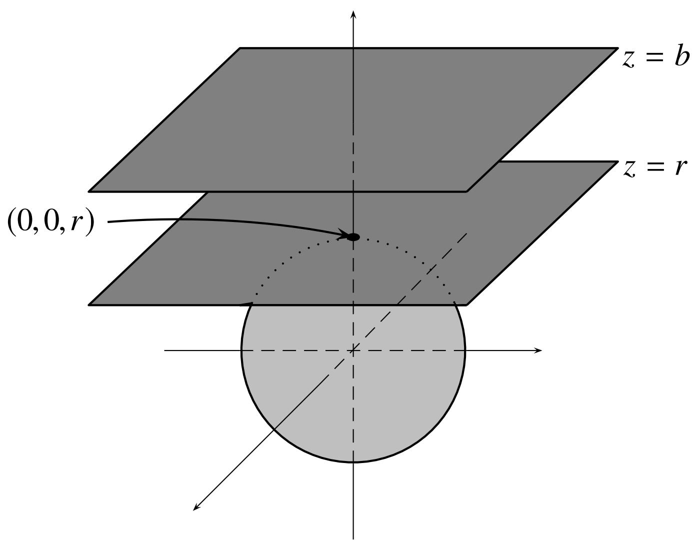
i) A sphere with centre $(0, 0, 0)$ and radius r has equation $x^2 + y^2 + z^2 = r^2$.



Consider the points on this sphere such that $z = a$, where a is some constant such that $-r < a < r$. These points lie in the plane $z = a$, and satisfy $x^2 + y^2 = r^2 - a^2$. That is, they form a circle in the plane $z = a$, with centre $(0, 0, a)$ and radius $\sqrt{r^2 - a^2}$. This circle is a horizontal cross-section of the sphere.

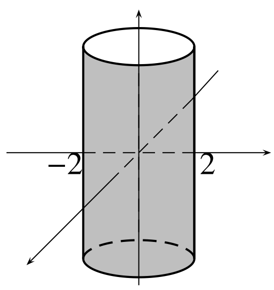


Now consider the intersection of the sphere with the plane $z = r$. This plane sits on top of the sphere, and its intersection with the sphere is the single point $(0, 0, r)$. Substituting $z = r$ into the equation $x^2 + y^2 + z^2 = r^2$ gives $x^2 + y^2 = 0$, and of course the only values satisfying this equation are $x = 0$ and $y = 0$. Consider also a plane $z = b$ where $b > r$. Such a plane does not intersect the sphere at all, and substituting $z = b$ into $x^2 + y^2 + z^2 = r^2$ gives the equation $x^2 + y^2 = r^2 - b^2 < 0$, which is satisfied by no values of x and y.



Note that the vertical cross-sections of the sphere obtained by letting x or y equal some constant between $-r$ and r are also circles.

ii) An infinite cylinder with radius 2 and central axis the z-axis has equation $x^2 + y^2 = 4$. To see this, consider the fact that the equation $x^2 + y^2 = 4$ represents a circle with radius 2 and centre $(0, 0)$ in two dimensions. Now, moving into three dimensions, we retain the relationship between x and y, and at the same time allow z to take any value. We therefore simply allow the circle to exist in any plane $z = a$, for any real number a, and hence obtain the cylinder as shown.



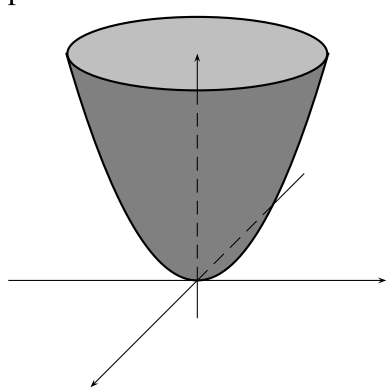
Note that all horizontal cross-sections are simply a copy of the circle $x^2 + y^2 = 4$.

What are the vertical cross-sections of this cylinder?

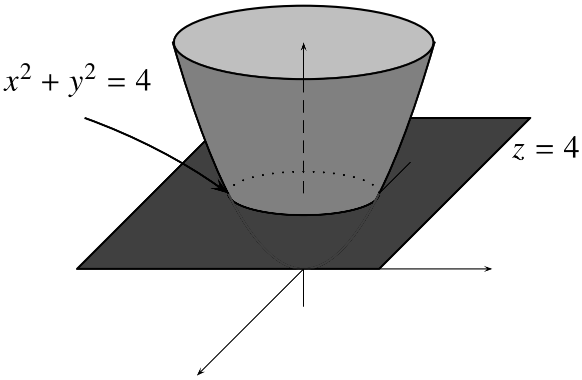
iii) Consider the equation $z = x^2 + y^2$. What is its graph? Firstly, note that $z \geq 0$ for all values of x and y, so the surface with equation $z = x^2 + y^2$ lies entirely above the xy-plane, except for the point $(0, 0, 0)$. (The point $(0, 0, 0)$ lies on the surface, since $z = 0$ when $x = 0$ and $y = 0$.) The horizontal cross-sections of the surface are circles, since if we are in the plane $z = a$, where $a > 0$, then $x^2 + y^2 = a$. As z increases, we have circles of increasing radius as horizontal cross-sections.

If we now keep x fixed at some value b, say, we obtain the vertical cross-section resulting from the intersection of the surface $z = x^2 + y^2$ and the plane $x = b$. This vertical cross-section is a curve in the plane $x = b$ with equation $z = b^2 + y^2$. It is a parabola. Note that if $b = 0$, we are talking about the plane $x = 0$, (that is, the yz-plane), and the cross-section is the parabola $z = y^2$.

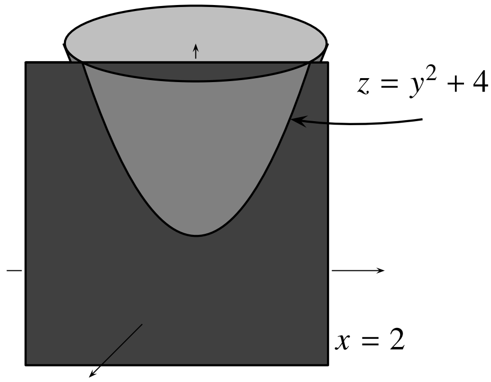
Similarly, keeping y fixed will give us vertical cross-sections in planes parallel to the xz-plane, and these are also parabolas. The surface is known as a "paraboloid".



The paraboloid $z = x^2 + y^2$

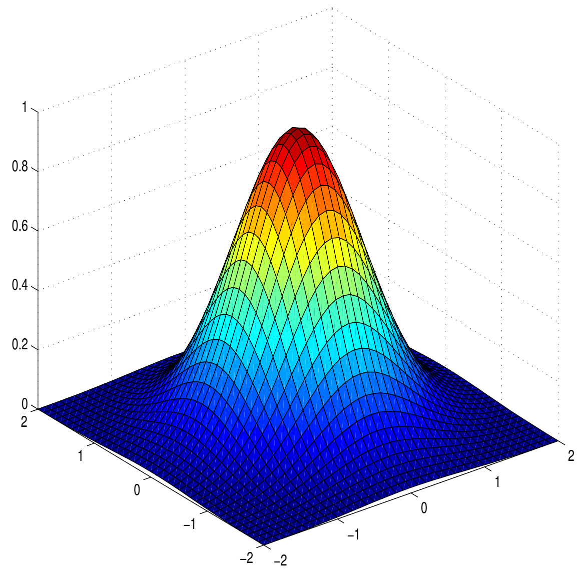


Horizontal cross-sections are circles.



Vertical cross-sections are parabolas.

iv) The following graph is of the surface given by $z = e^{-(x^2+y^2)}$.



Note that the exponential function is always positive, so the surface lies entirely above the xy-plane. Also, if $x = y = 0$ then $z = 1$, and for all other values of x and y, $e^{-(x^2+y^2)}$ is a positive number less than 1.

Horizontal cross-sections are circles. For example, show that the intersection of the plane $z = \frac{1}{2}$ with the surface is the circle $x^2 + y^2 = \ln 2$.

The vertical cross-section in the plane $x = 0$ is the curve $z = e^{-y^2}$, while that in the plane $y = 0$ is $z = e^{-x^2}$.

$\diamond$

7.5 Functions of two real variables

There are many quantities which depend on more than one variable. A simple example is the volume V of a cylinder, which depends on the height h and the radius r of the cylinder.

The formula $V = \pi r^2 h$ is a rule for calculating V, given r and h. The rule uses a pair of real numbers (one for r and one for h) as input, and outputs exactly one real number for any input.

Recall that the definition of a function does not stipulate that the domain ought to be any particular type of set. There is nothing to prevent us, therefore, defining a function on a set of pairs of numbers. All we require is a rule which assigns *exactly one* output for *each* input. So we may consider the rule for the volume of a cylinder as a function, with domain the set of pairs (r, h), and outputs in $\mathbb{R}$, given by the rule $f(r, h) = \pi r^2 h$.

If we are using this rule to calculate the volume of a cylinder, then we would only want to use positive real numbers for r and h, and so the domain, or set of inputs, would be the set $D = \{(r, h) \mid r, h \in \mathbb{R}, r > 0, h > 0\}$. The set D is a subset of the set of all pairs of real numbers, $\mathbb{R}^2 = \{(x, y) \mid x \in \mathbb{R}, y \in \mathbb{R}\}$. Geometrically, D is the set of all points in the first quadrant of the Cartesian plane.

The function $f : D \to \mathbb{R}$, given by $f(r, h) = \pi r^2 h$ is an example of a "real-valued function of two real variables". It requires a pair of two real numbers as input, and it outputs a real number. In this particular example we chose to call the variables r and h, since we were thinking of them as representing radius and height. The names we choose to give variables in describing a function are quite arbitrary, however. We could equally well have written the rule as $f(x, y) = \pi x^2 y$.

Functions of two variables
A "real-valued function of two real variables" is a function $f : D \to \mathbb{R}$ where D is a subset of $\mathbb{R}^2$ and f is a rule which assigns exactly one real number, denoted by $f(x, y)$, to each element $(x, y) \in D$.

Note that the domain D is a region in $\mathbb{R}^2$. The range of a function $f : D \to \mathbb{R}$, where $D \subseteq \mathbb{R}^2$, is the set of real numbers $\{f(x, y) \mid (x, y) \in D\}$. In other words, the range is the set of all possible values taken by $f(x, y)$ as (x, y) varies over D.

Examples 7.5a In each of these examples, the domain of f is all of $\mathbb{R}^2$. So in each case f is a function from $\mathbb{R}^2$ to $\mathbb{R}$. See if you can work out the range of f in each example.

i) $f(x, y) = 2x + y - 5$

ii) $f(x, y) = x^2 + y^2$

iii) $f(x, y) = 3$

iv) $f(x, y) = 2x$

v) $f(x, y) = e^{-(x^2+y^2)}$ ◇

If we describe a function simply by giving a formula for $f(x, y)$, without specifying a domain, it is to be assumed that the domain is the "natural domain"; that is, the set of all pairs (x, y) in $\mathbb{R}^2$ for which the expression yields a unique real number for the value of $f(x, y)$.

Examples 7.5b

i) The natural domain of the function $f(x, y) = \sqrt{x} + \sqrt{y}$ is the region

$$\{ (x, y) \in \mathbb{R}^2 \mid x \geq 0 \text{ and } y \geq 0 \},$$

(that is, the first quadrant including the boundary), and the range of f is the interval $[0, \infty)$.

ii) The natural domain of the function $f(x, y) = \ln(x + y)$ is the region

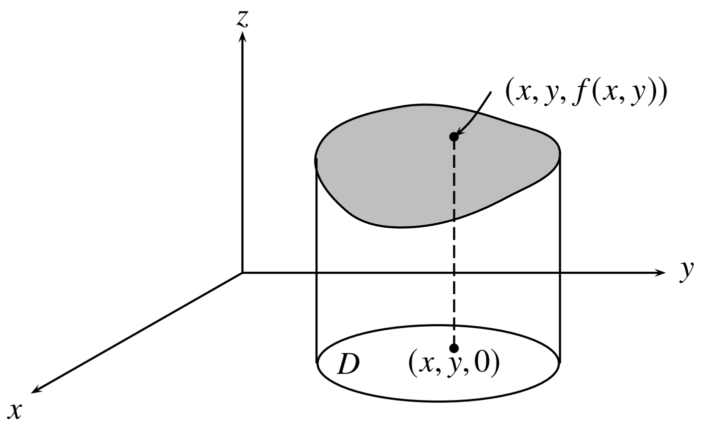
$$\{ (x, y) \in \mathbb{R}^2 \mid x + y > 0\}$$

and the range of f is the whole real line $\mathbb{R}$. Can you sketch the domain? ◇

7.6 The graph of a function of two variables

Recall that the graph of a function of one real variable is the set of points (x, y) in the Cartesian plane such that $y = f(x)$.

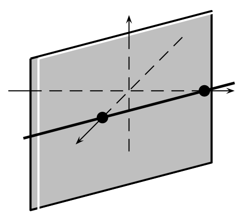
The graph of a function of two real variables is, similarly, the set of points (x, y, z) in 3-dimensional space such that $z = f(x, y)$, for each (x, y) in the domain of f. That is, it is "the surface with equation $z = f(x, y)$". Typically, this is a surface sitting "over" the domain D of f.



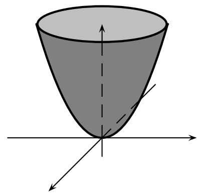
Note that for each point (x, y) in the domain of some function f, there will be exactly one point $(x, y, z) = (x, y, f(x, y))$. (Why is that?)

Examples 7.6a

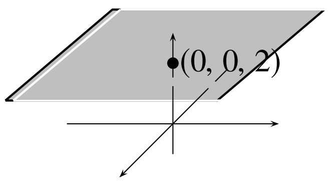
i) The graph of $f(x, y) = 2x + y - 5$ is the plane with equation $z = 2x + y - 5$, or $2x + y - z = 5$. We saw a sketch of this plane in Example 4.3a iv).



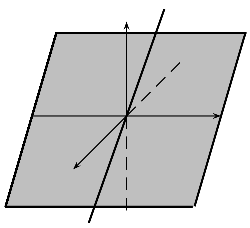
ii) The graph of $f(x, y) = x^2 + y^2$ is the paraboloid with equation $z = x^2 + y^2$, illustrated in Example 4.4a iii).



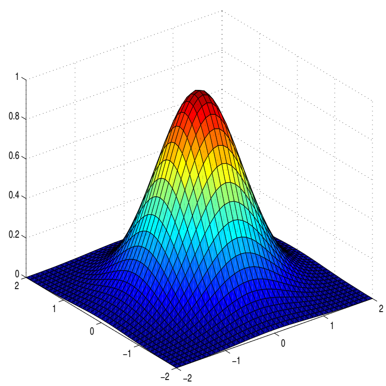
iii) The graph of the function $f(x, y) = 2$ is the horizontal plane, through the point $(0, 0, 2)$, with equation $z = 2$.



iv) The graph of $f(x, y) = -2x$ is also a plane, with equation $z = -2x$, or $2x + z = 0$. This plane passes through the line $z = -2x$ in the xz-plane, as shown.



v) The graph of $f(x, y) = e^{-(x^2+y^2)}$ is the surface with equation $z = e^{-(x^2+y^2)}$, illustrated in Example 4.4a iv).



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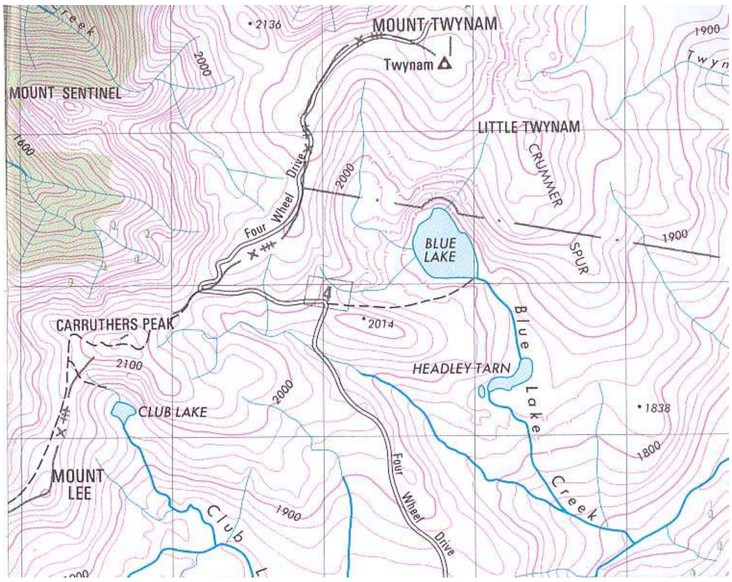
Some of the surfaces we discussed in Section 4.4 are not graphs of functions. For example, the unit sphere with centre $(0, 0, 0)$ (and radius 1) has equation $x^2 + y^2 + z^2 = 1$, but this equation does not express z as a function of x and y. Solving the equation for z gives $z = \pm\sqrt{1 - x^2 - y^2}$; that is, for each (x, y) such that $x^2 + y^2 < 1$ there are two values of z. The upper hemisphere is the graph of a function $g(x, y) = \sqrt{1 - x^2 - y^2}$, and the lower hemisphere is also the graph of a function, $h(x, y) = -\sqrt{1 - x^2 - y^2}$, but the sphere itself is not the graph of a function. The same phenomenon occurs with the circle $x^2 + y^2 = r^2$ in $\mathbb{R}^2$. Notice that a vertical line through a sphere hits it twice, and that the vertical line test applies in $\mathbb{R}^3$ as well as in $\mathbb{R}^2$. That is, if any vertical line intersects a surface more than once, then that surface is not the graph of a function.

Similarly, the cylinder with equation $x^2 + y^2 = 4$ is not the graph of a function. In this case, z takes infinitely many values for each (x, y) such that $x^2 + y^2 = 4$.

7.7 Level curves

As we saw in Section 7.4, consideration of a surface's cross-sections often helps in visualising the surface. In order to visualise the graph of a function of two variables, a similar approach is to draw the so-called "level curves", or "contour lines". This is the same method as that used to draw the contour maps carried by hikers (where the surface being represented is a relatively small, undulating portion of the earth's surface, often a mountain range). The diagram below is a contour map of part of the Kosciuszko alpine area.



In such a map, every point on the same contour line has the same elevation above sea level. (Hence the alternate name "level curve".) In other words, each level curve is the horizontal cross-section of the surface at a particular height, but drawn at sea level rather than on the surface. If you visualise each level curve being lifted to the height indicated, then you should be able to get a good idea of the shape of the terrain. The level curves are taken at equally spaced heights, and so the surface is steep when level curves are close together, and flatter where the curves are further apart.

We can use precisely this idea to draw the level curves of a function $f(x, y)$. It is simply a matter of finding horizontal cross-sections of the surface $z = f(x, y)$, and then drawing them all in the xy-plane.

Level curves
A "level curve" of a function $f(x, y)$ is a curve with equation $f(x, y) = c$ (drawn in the xy-plane), where c is a constant in the range of f.

Note: The reason for choosing c in the range of f is that if it does not belong to the range then the plane $z = c$ does not intersect the surface at all, and so there is no level curve. Even when c does belong to the range of f it sometimes happens that the corresponding horizontal cross-section is not a curve; it may consist of just one isolated point, or a set of isolated points. For example the intersection of the plane $z = 0$ and the surface $z = x^2 + y^2$ is just the single point $(0, 0, 0)$.

The point of drawing a diagram showing the level curves (or contour diagram) of a function $f(x, y)$ is to help us visualise the surface $z = f(x, y)$. It is therefore important to choose the level curves $f(x, y) = c$ at equally spaced values of c, and to label each curve we draw in the xy-plane with the corresponding value of c, indicating the height above the xy-plane of the corresponding cross-section.

Examples 7.7a

i) The graph of the function $f(x, y) = x^2 + y^2$ is the paraboloid $z = x^2 + y^2$, as we saw in Example 7.4a iii).

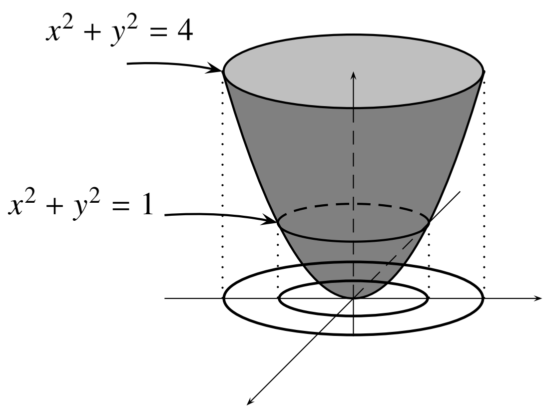
   The level curve obtained by taking the cross-section at height c has equation

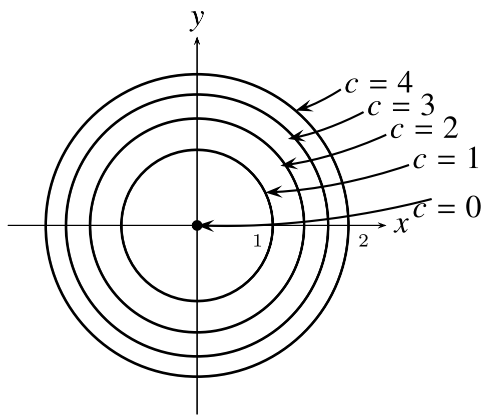
   $$f(x, y) = x^2 + y^2 = c.$$

   Note that the range of f is the interval $[0, \infty)$, so we want $c \geq 0$. Choosing the equally spaced values $c = 0, 1, 2, 3, 4, \ldots$, we see that the level curves have equations

   $$x^2 + y^2 = 0, \quad x^2 + y^2 = 1, \quad x^2 + y^2 = 2, \quad x^2 + y^2 = 3, \quad x^2 + y^2 = 4, \ldots.$$

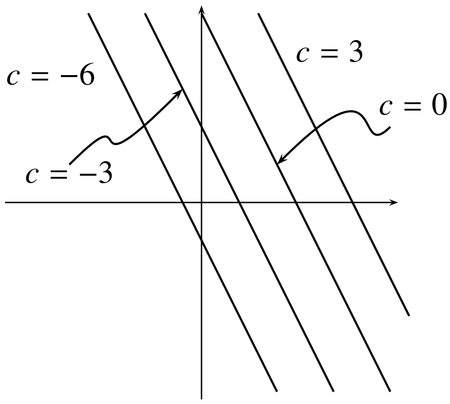
   The equation $x^2 + y^2 = 0$ represents the single point $(0, 0, 0)$. The other equations represent concentric circles, centre $(0, 0)$, with radii: $1, \sqrt{2}, \sqrt{3}, \sqrt{4}, \ldots$. Notice that the circles get closer together as the radius increases, showing that the paraboloid gets steeper as z increases.



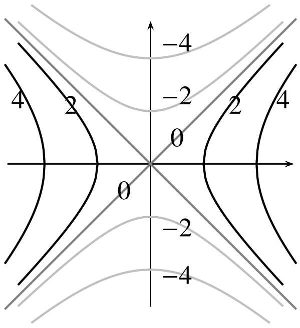


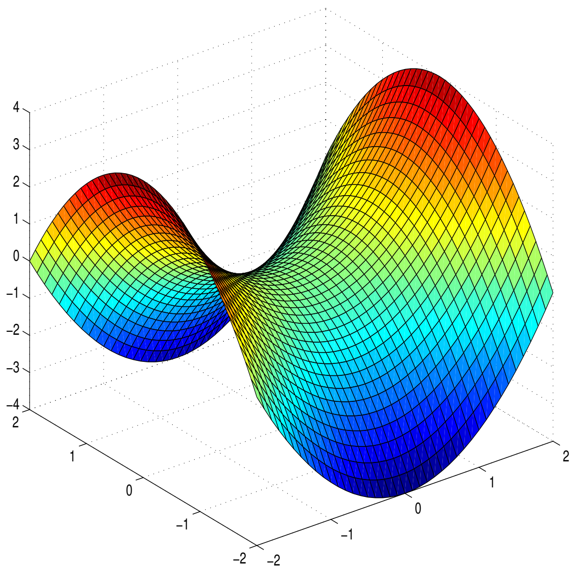
Level curves for $f(x, y) = x^2 + y^2$

ii) The graph of the function $f(x, y) = 2x + y - 5$ is the plane $z = 2x + y - 5$. The level curves of $f(x, y) = 2x + y - 5$ are the straight lines $2x + y - 5 = c$. The diagram shows the lines for $c = -6, -3, 0, 3, \ldots$. Note that the range of f is $\mathbb{R}$, so we can choose any values for c.

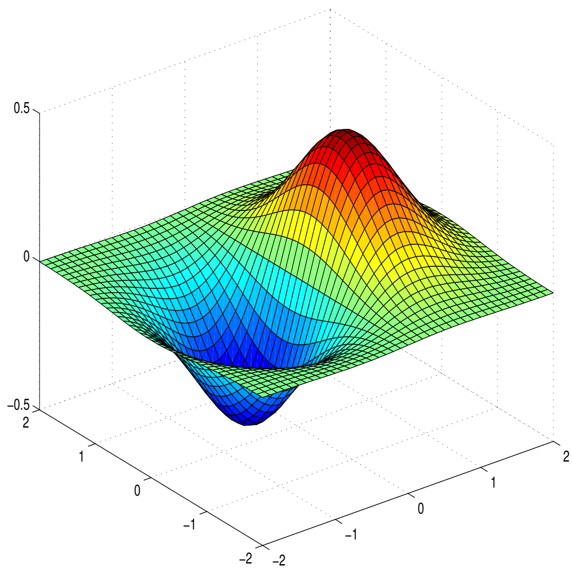


iii) Here is a contour map, and a graph, of $f(x, y) = x^2 - y^2$. The level curve of height c has equation $c = x^2 - y^2$, or $y = \pm\sqrt{x^2 - c}$. When $c = 0$ this gives the straight lines $y = \pm x$. For $c \neq 0$ the level curves are hyperbolas $y = \pm\sqrt{x^2 - c}$. The point $(0, 0, 0)$ on the surface $z = x^2 - y^2$ is known as a saddle point.





iv) This example is the function $f(x,y) = xe^{-(x^2+y^2)}$.



(You are not expected to be able to draw diagrams like this!) $\diamond$

Summary of Chapter 7

a) The 3-dimensional Cartesian coordinate system was introduced.
b) Curves in space were studied in terms of their parametric. representations $x = f(t), \quad y = g(t), \quad z = h(t)$ where t is a parameter.
c) The following surfaces were considered:

- Planes $ax + by + cz = d$
- Sphere $(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2$, centre (a, b, c), radius r
- Circular cylinder $x^2 + y^2 = r^2$, radius r
- Paraboloid $z = x^2 + y^2$

d) The process of taking horizontal and vertical cross-sections was used to visualize surfaces.
e) Functions of two real variables were defined as $f : D \to \mathbb{R}$, where $D \in \mathbb{R}^2$ is the domain (the set of $(x, y) \in \mathbb{R}^2$ points in which $f(x, y)$ is defined) and the range of f is a subset of $\mathbb{R}$ (the set of all possible values taken by $f(x, y)$)
f) Several examples of functions of two variables and their graphs $z = f(x, y)$ were presented.
g) Level curves $f(x, y) = c, \quad c =$ constant, were introduced to help in the visualisation of graphs of functions of two variables.

Exercises

7.1 For each of the following functions find the domain and range, sketch some level curves and finally try to sketch the surface $z = f(x, y)$.

a) $f(x, y) = x + y$
b) $f(x, y) = y^3 - x^3$
c) $f(x, y) = \sqrt{x + y}$
d) $f(x, y) = \ln\left(9 - x^2 - y^2\right)$
e) $f(x, y) = \cos\left(x^2 + y^2\right)$
f) $f(x, y) = \dfrac{1}{3x - y}$
g) $f(x, y) = xy$
h) $f(x, y) = y - 2x^2$

7.2 The following equations in the three variables x, y, z correspond to surfaces in space but do not determine z as a function of x and y. Nevertheless they do have level curves corresponding to $z = c$, for c a constant. Sketch the level curves for a selection of z-values. Can you tell from the level curves that a surface is not the graph of a function?

a) $z^2 + x^2 + y^2 = 25$
b) $x^2 + y^2 = 9$
c) $(x - z)^2 + y^2 = 4$
d) $x^2 + y^2 = z^2 - 1$

7.3 a) Sketch the domain of the function $z = \ln\left(3 - x^2 - y^2\right)$ and write down its range. How does z behave as (x, y) approaches the circle $x^2 + y^2 = 3$ from the inside?

b) Sketch the domain of the function $z = \sqrt{x - 2y + 2}$ and, on a separate diagram, draw the level curves corresponding to equally spaced heights, $z = 0, 1, 2, 3$.

7.4 a) Sketch the domain and write down the range of the function

$$z(x, y) = \ln\left(36 - 4x^2 - 9y^2\right).$$

b) Sketch the domain and write down the range of the function

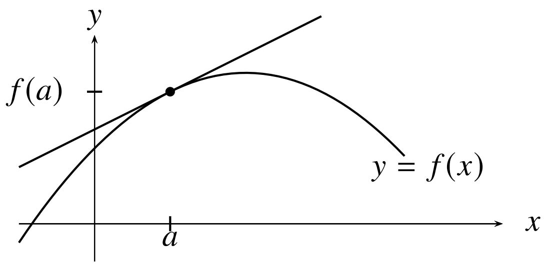
$$z(x, y) = \frac{1}{\sqrt{2x - y + 1}}$$

On a separate diagram, draw the level curves corresponding to the equally spaced heights, $z = 1, 2, 3$.

CHAPTER 8

Partial Derivatives and Tangent Planes

The derivative of a function f of one variable at a point $x = a$ in its domain can be interpreted as the slope of the tangent line to the graph $y = f(x)$ at the point $(a, f(a))$.



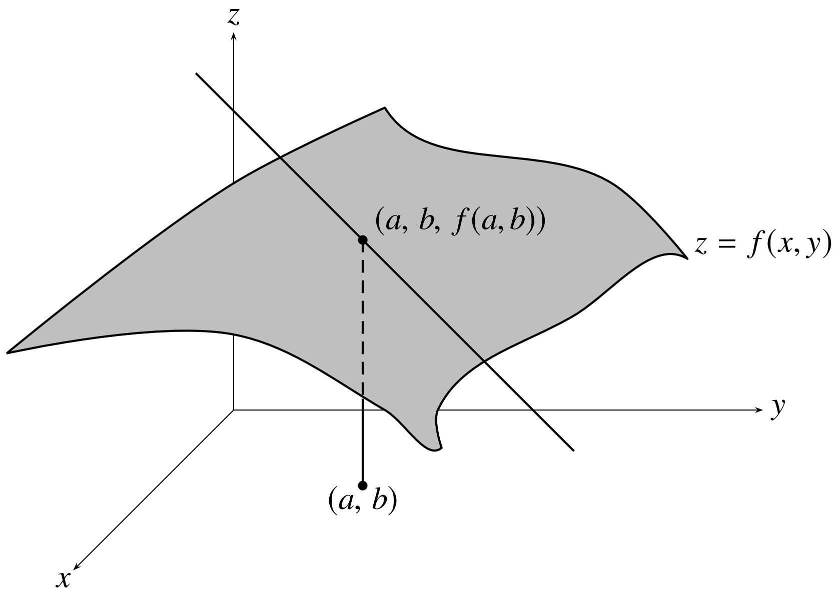
For example, the function $f(x) = x^2 - 2x - 1$ has derivative $f'(x) = 2x - 2$ and so the slope of the tangent to the graph of $f(x)$ at the point $(3, 2)$ is 4.

In this Chapter we consider how these ideas generalize to functions of two variables (as defined in Chapter 7). Do graphs of functions of two variables have tangents? If so, how are the slopes of these tangents defined? Can these slopes be found by differentiation? How do we find a derivative of a function of two variables?

8.1 Tangents to graphs of functions of two variables

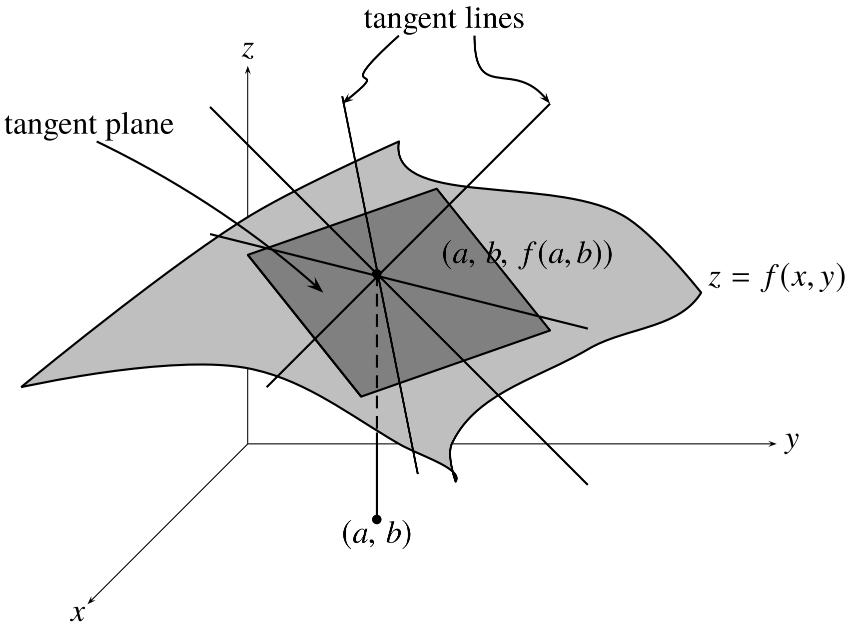
It is harder to draw tangents to a curved surface than tangents to a curve in the plane. Fortunately we live in a three-dimensional world with an abundance of curved surfaces. You will find it helpful in understanding the discussion in this section if you have to hand a curved surface, such as an upturned bowl, balloon or basketball, a straight rod, such as a pencil or a ruler and a flat planar surface, such as a piece of cardboard or a hardcover book.

Consider the graph of a function of two variables $z = f(x, y)$ and a point on the surface given by $x = a,\ y = b$. If the surface is continuous and smooth (that is with no sharp edges) at (a, b), then we can draw a tangent to the surface at the point $(a, b, f(a, b))$ in space.



The slope of that tangent is given by "rise over run" where now "run" is horizontal distance in the xy plane and "rise" is the change in z value on the tangent.

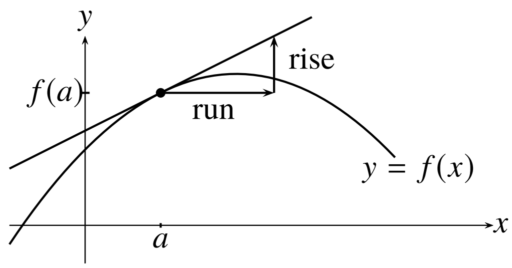
If you explore this geometrical idea using your pencil as a tangent line on a curved surface you will very quickly see that there is more than one tangent line to a given point on the surface and that different tangent lines may have different slopes. In fact, there is an infinite number of tangent lines through a point on a smooth, continuous surface. All of these tangent lines, however, are contained in a unique plane which is tangent to the surface at that point. You can see this by using a sheet of cardboard or a flat book as the tangent plane.



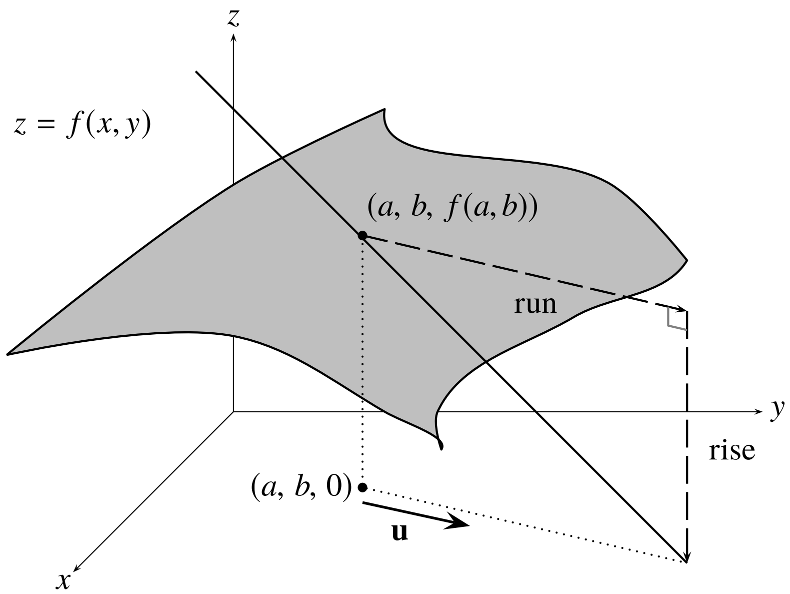
So, for a surface which is the graph of a function of two variables, the tangent plane completely specifies all tangent lines. All these lines lie in the plane and run through the point of tangency. So how can we find a tangent plane?

Recall that to specify a plane you must specify the orientation of the plane, usually given by a vector normal to the plane, and a point on the plane. For a tangent plane, the point of tangency is the easiest point on the plane to find. There are several ways to find the orientation of the plane, but it is sufficient to know the slopes of two of the tangent lines that lie in the plane. How do we find these slopes?

To answer this question we need to know exactly what we mean by the slope of a line in Cartesian space. The gradient or slope of a line in the Cartesian plane is often said to be given by "rise over run", but what exactly is meant by rise and run in space? For functions of one real variable the run is the distance measured in the positive x direction and the rise in the distance measured in the positive y direction. This is illustrated in the diagram below.



In Cartesian space the rise is the distance measured in the positive z direction. The run, however, is measured along a vector that lies in the xy plane. The diagram below shows the tangent vector in the direction **u**.

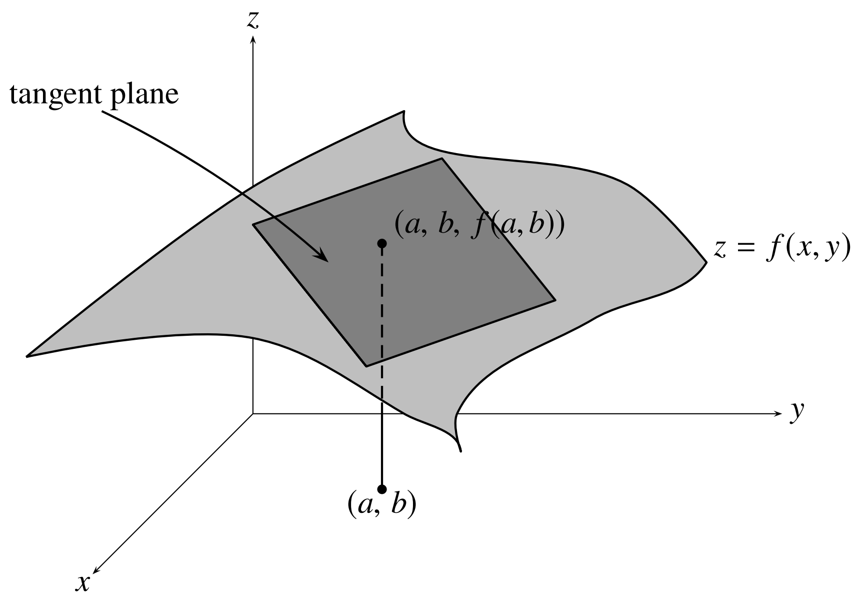


The line in the diagram has a negative slope; as we move in the positive **u** direction along the line, the height of the line above the xy plane decreases. If we specify the direction of the run to be $-\mathbf{u}$ then the line would have positive slope.

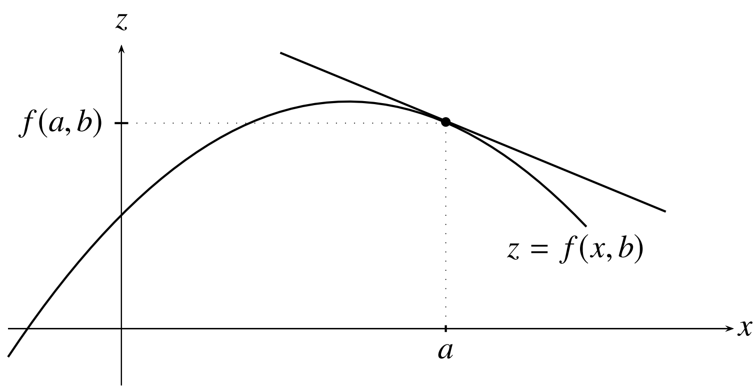
Two special directions in Cartesian space are the direction of the positive x axis and the direction of the positive y axis. The vector $\mathbf{i}$ specifies the positive x direction and the vector $\mathbf{j}$ gives the positive y direction. It is easy to find the slope of a tangent line in these directions by extending the idea of a derivative of a function of one variable. Let's briefly revise that definition now.

8.2 Definition of first-order partial derivatives

In this section we calculate the slope of the tangent line to a surface in space. Consider a surface given by $z = f(x, y)$, with a tangent plane at $(a, b, f(a, b))$.



Let us take a cross-section of the surface and its tangent plane through $(a, b, f(a, b))$, in the plane $y = b$ which is parallel to the xz plane. We can plot this cross-section on the x and z axes:



The curve in this diagram is the graph of the function $z = f(x, b)$. Because y is fixed at b, $f(x, b)$ is a function of one variable only so we can write $f(x, b) = g(x)$. The tangent line on the diagram is the cross-section through the tangent plane at $(a, b, f(a, b))$ in the vertical plane $y = b$; its slope is the same as the slope of the tangent to the curve $z = g(x)$ at $x = a$. Using the definition of the derivative of a function of one variable, the slope of this line is

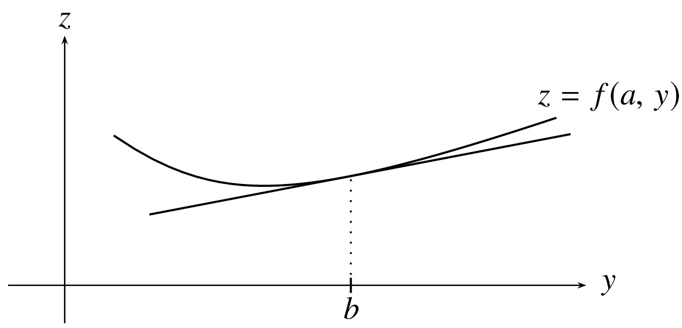
$$\frac{d}{dx}g(x)\Big|_{x=a} = \lim_{h\to 0}\left(\frac{g(a+h) - g(a)}{h}\right),$$

or

$$\frac{d}{dx}(f(x,b))\Big|_{x=a} = \lim_{h\to 0}\left(\frac{f(a+h,b) - f(a,b)}{h}\right).$$

We define this limit to be the **partial derivative of f with respect to x evaluated at** $x = a$. It is written $f_x(a, b)$ or $\frac{\partial f}{\partial x}|_{(x,y)=(a,b)}$. The partial derivative with respect to x gives the slope of the tangent line to the surface $z = f(x, y)$ which lies in the direction of the positive x axis in a vertical plane, with y fixed.

Let us now take a cross-section through the surface $z = f(x, y)$ and its tangent plane, in the vertical plane $x = a$, which is parallel to the yz plane. In this cross-section, x is fixed with value a and so we can plot the cross-section on the y and z axes.



The curve on this plot is the graph of the function $z = f(a, y) = G(y)$, which is the graph of a function of one variable. Hence the slope of the tangent line is

$$\frac{d}{dy}G(y)\Big|_{y=b} = \frac{d}{dy}(f(a,y))\Big|_{y=b} = \lim_{h\to 0}\left(\frac{f(a,b+h) - f(a,b)}{h}\right).$$

This is the **partial derivative of f with respect to y evaluated at** $y = b$. We write $f_y(a, b)$ or $\frac{\partial f}{\partial y}|_{y=b}$. The partial derivative with respect to y gives the slope of the line tangent to the surface $z = f(x, y)$ which lies in the direction of the positive y axis in a vertical plane, with x fixed.

To sum up:

Partial derivatives
Let f be a function of two variables.

a) The "partial derivative of f with respect to x" is the function of two variables given by

$$\frac{\partial f}{\partial x} = f_x(x, y) = \lim_{h \to 0} \left(\frac{f(x + h, y) - f(x, y)}{h} \right).$$

b) The "partial derivative of f with respect to y" is the function of two variables given by

$$\frac{\partial f}{\partial y} = f_y(x, y) = \lim_{h \to 0} \left(\frac{f(x, y + h) - f(x, y)}{h} \right).$$

8.3 Calculating first-order partial derivatives

We have now defined the partial derivatives of a function of two variables, $f(x, y)$. These partial derivatives give the gradient of the tangent line to the graph of $f(x, y)$ in the direction of the positive x axis in a vertical plane (when y is fixed) and in the direction of the positive y axis in a vertical plane (when x is fixed). Given a formula for $f(x, y)$, how can we find these derivatives in practice?

When we defined the partial derivative, we fixed one variable so that the function of two variables looked like a function of one variable. Then the derivative was defined in the same way as a derivative of a function of one variable. To calculate the partial derivative of $f(x, y)$ with respect to x, we take y to be constant and differentiate with respect to x only. For example, consider the function

$$f(x, y) = x^2 + x^3 y - y^2 + 3.$$

If we think of y as constant and differentiate with respect to x we have

$$\frac{\partial f}{\partial x} = f_x(x, y) = 2x + 3x^2 y.$$

Because y is regarded as constant, the term y^2 has zero derivative with respect to x. To calculate the partial derivative with respect to y, we take x to be constant and differentiate with respect to y only:

$$\frac{\partial f}{\partial y} = f_y(x, y) = x^3 - 2y.$$

Note that it is important to use the correct notation for partial derivatives. The special symbol ∂ and the subscript notation f_x and f_y, indicate that the function which is being differentiated is a function of more than one variable and so has more than one derivative. We write $\dfrac{df}{dx}$ or $f'(x)$ only when f is a function of one variable. The derivative of a function of one variable is known as an **ordinary derivative**.

The example of calculating first-order partial derivatives that we gave above was a simple one, as it didn't involve composite functions. Before tackling some more complicated examples, it will be useful to revise the derivatives of elementary functions of one variable and the rules for differentiation. These are given below, and should be familiar to you from high school calculus.

Some ordinary derivatives – revision

$f(x)$	$f'(x)$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$
e^x	e^x
$\ln x$	$1/x \quad (x > 0)$
x^a	$ax^{a-1} \quad (a \text{ constant})$

Rules for finding ordinary derivatives – revision

If $f : I \longrightarrow \mathbb{R}$ is a real-valued function of one variable and $f'(x)$ is defined for all x in the open interval I, we say f is *differentiable on I*. Suppose that f and g are differentiable functions of one variable, on the interval I. The following well–known formulas give the derivatives of kf, $f + g$, fg, f/g and $f \circ g$.

The constant multiple rule

$$\frac{d}{dx}(kf) = kf', \quad k \text{ constant}$$

The sum rule

$$\frac{d}{dx}(f+g) = f' + g'$$

The product rule

$$\frac{d}{dx}(fg) = f'g + fg'$$

The quotient rule

$$\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{f'g - fg'}{g^2}$$

The chain rule

$$\frac{d}{dx}f(g(x)) = f'(g(x))g'(x)$$

Examples 8.3a Check the following ordinary derivatives.

i) $\frac{d}{dx}\left(4x^3 - 2x + 4\cos x - \frac{1}{e^x + x}\right) = 12x^2 - 2 - 4\sin x + \frac{e^x + 1}{(e^x + x)^2}$.

ii) $\frac{d}{dx}\left(4xe^{x^2}\right) = 4e^{x^2} + 8x^2e^{x^2}$.

iii) $\frac{d}{dx}\left(3\sin(2x+1)\right) = 6\cos(2x+1)$.

iv) $\frac{d}{dx}\left(\frac{7x+1}{3x^2-1}\right) = -\frac{21x^2 + 6x + 7}{(3x^2-1)^2}$. ◇

Now let's return to the computation of some more first-order partial derivatives. Since we regard y as a constant when calculating f_x, and x as a constant when calculating f_y, the problem reduces essentially to the problem of calculating a ordinary derivative, and so we can use the same rules for differentiation as in the case of ordinary derivatives. The following examples will illustrate the idea.

Examples 8.3b

i) Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ for $z = e^{2y}\cos(x+y)$.

$$\frac{\partial z}{\partial x} = e^{2y}(-\sin(x+y)) = -e^{2y}\sin(x+y) \text{ and } \frac{\partial z}{\partial y} = 2e^{2y}\cos(x+y) - e^{2y}\sin(x+y)$$

As you can see in this example, the product rule and the chain rule for differentiation work in exactly the same way for partial differentiation as for ordinary differentiation. All you have to remember is that one variable is held constant while you differentiate with respect to the other. The quotient rule also works as usual, as the next example shows.

ii) If $g(x,y) = \dfrac{x+y}{x^2+y^2}$ find $g_x(2,0)$ and $g_y(1,1)$.

$$g_x(x,y) = \frac{(x^2+y^2) - 2x(x+y)}{(x^2+y^2)^2} = \frac{y^2 - 2xy - x^2}{(x^2+y^2)^2} \quad \text{and so} \quad g_x(2,0) = \frac{-4}{16} = \frac{-1}{4}$$

$$g_y(x,y) = \frac{(x^2+y^2) - 2y(x+y)}{(x^2+y^2)^2} = \frac{x^2 - 2xy - y^2}{(x^2+y^2)^2} \quad \text{and so} \quad g_y(1,1) = \frac{-2}{4} = \frac{-1}{2}.$$

iii) Find $f_x(x,y)$ and $f_y(x,y)$ if $f(x,y) = x^4 + x^2y^3 + y^5$. Hence find $f_x(1,-1)$ and $f_y(1,-1)$.

$$f_x(x,y) = 4x^3 + 2xy^3 \text{ and } f_y(x,y) = 3x^2y^2 + 5y^4$$

So $f_x(1,-1) = 4 + (-2) = 2$ and $f_y(1,-1) = 3 + 5 = 8$.

iv) Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ for $z = \cos(xy + y^2)$.

$$\frac{\partial z}{\partial x} = -y\sin(xy + y^2)$$

and

$$\frac{\partial z}{\partial y} = -(x+2y)\sin(xy + y^2).$$

v) Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ for $z = \cos(\sin(3x + 5y))$.

$$\frac{\partial z}{\partial x} = -3\sin(\sin(3x+5y))\cos(3x+5y)$$

and

$$\frac{\partial z}{\partial y} = -5\sin(\sin(3x+5y)\cos(3x+5y)).$$

vi) The volume of a circular cylinder is given by $V = \pi r^2 h$ where r is the radius of the circular cross-section and h is the height of the cylinder. Find $\dfrac{\partial V}{\partial r}$ and $\dfrac{\partial V}{\partial h}$.

$$\frac{\partial V}{\partial r} = 2\pi rh \quad \text{and} \quad \frac{\partial V}{\partial h} = \pi r^2.$$

◇

8.4 Tangent planes

This chapter began with a discussion about tangents to surfaces that are graphs of functions of two variables. We saw that there are an infinite number of tangent lines to a point on a smooth surface but only one tangent plane. The tangent plane to a point on a surface is not only an interesting mathematical object, but also very useful in making numerical or algebraic approximations to functions of two variables. This will be further developed in Chapter 9.

The general form of the equation of a plane is $\alpha x + \beta y + \gamma z = \delta$ where α, β, γ and δ are constants and not all of α, β and γ are equal to zero. An alternative expression for the equation of a plane is

$$\alpha(x - p) + \beta(y - q) + \gamma(z - r) = 0$$

where (p, q, r) is a point on the plane. (You can see that this reduces to the form above by setting $\delta = \alpha p + \beta q + \gamma r$.)

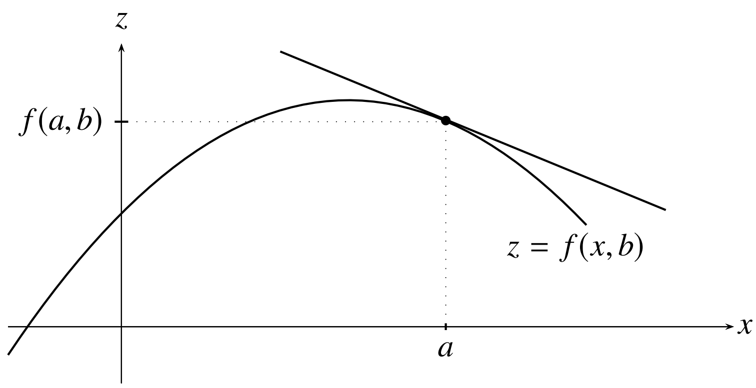
In particular, we are interested in the equation of the tangent plane to the surface $z = f(x, y)$ at the point with coordinates $(a, b, f(a, b))$, so the equation we seek is of the form

$$\alpha(x - a) + \beta(y - b) + \gamma(z - f(a, b)) = 0 \qquad (*)$$

for some α, β, γ.

[We may assume that $\gamma \neq 0$, since if $\gamma = 0$ the equation of the tangent plane contains no z term and is therefore a vertical tangent plane (for example, the tangent plane to a sphere at a point on the equator). Such a surface is not given by a function of two variables, so we exclude this possibility.]

In order to find the values of α, β and γ, we consider the vertical cross-sections through the surface and its tangent plane at $(a, b, f(a, b))$ that were drawn in Section 8.2.



In the first cross-section y is fixed; that is $y = b$, a constant. The intersection of the vertical plane $y = b$ and the tangent plane $(*)$ is a straight line whose equation is found by substituting $y = b$ into $(*)$. We obtain

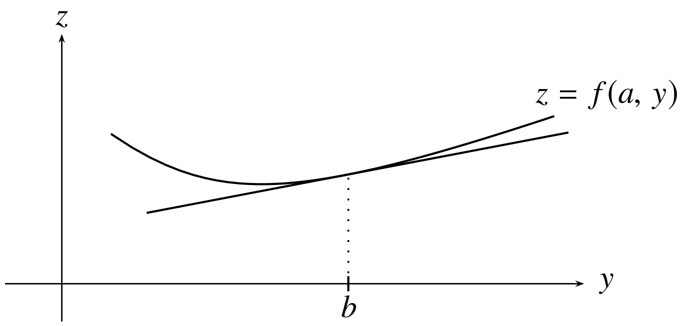
$$\alpha(x - a) + \gamma(z - f(a, b)) = 0.$$

This last equation rearranges to give

$$z = -\frac{\alpha}{\gamma}(x - a) + f(a, b),$$

from which we see that the slope of the line of intersection is $-\alpha/\gamma$. But we already know the slope of this line of intersection: it is the first partial derivative $f_x(a, b)$.
Therefore $f_x(a, b) = -\alpha/\gamma$, that is, $\alpha = -\gamma f_x(a, b)$.

In the second cross-section $x = a$ is fixed, as illustrated in the following diagram,



The intersection of the vertical plane $x = a$ and the tangent plane $(*)$ is a straight line whose equation is found by substituting $x = a$ into $(*)$ to give

$$\beta(y - b) + \gamma(z - f(a, b)) = 0.$$

This rearranges as follows:

$$z = -\frac{\beta}{\gamma}(y - b) + f(a, b),$$

from which we see that the slope of the line of intersection is $-\beta/\gamma$. As in the previous case, we already know the slope of this line of intersection: it is the first partial derivative $f_y(a, b)$.
Therefore $f_y(a, b) = -\beta/\gamma$, that is, $\beta = -\gamma f_y(a, b)$.

Substituting these values of α and β into $(*)$ and cancelling γ, we obtain the equation of the tangent plane,

$$-f_x(a, b)(x - a) - f_y(a, b)(y - b) + (z - f(a, b)) = 0.$$

This can be rearranged to give the more useful form shown in the following box. You should memorise this formula.

Tangent plane

The equation of the "tangent plane" to the surface $z = f(x,y)$ at the point $(a,b,f(a,b))$ is

$$z - f(a,b) = f_x(a,b)(x-a) + f_y(a,b)(y-b).$$

Examples 8.4a

i) Find the equation of the tangent plane to the surface $z = f(x,y) = x^2y^3$, at the point $x = 3$, $y = -1$.

$$f_x(x,y) = 2xy^3 \quad \text{so} \quad f_x(3,-1) = -6.$$

$$f_y(x,y) = 3x^2y^2 \quad \text{so} \quad f_y(3,-1) = 27.$$

At $(3,-1)$, $z = f(3,-1) = -9$ and so the equation of the tangent plane is
$z + 9 = -6(x-3) + 27(y+1)$ or, rearranging, $6x - 27y + z = 36$.

ii) What is the equation of the tangent plane to the surface $z = x^2 + 2y^2 - 6x - 4y + 15$ at $x = 3$ and $y = 1$?

Let $f(x,y) = x^2 + 2y^2 - 6x - 4y + 15$, then $f_x(x,y) = 2x - 6$ so $f_x(3,1) = 0$ and $f_y(x,y) = 4y - 4$ so $f_y(3,1) = 0$. The z coordinate value is $f(3,1) = 4$ and so, since both partial derivatives are zero at $(3,1)$ the equation of the tangent plane is $z - 4 = 0$ or $z = 4$. That is, the tangent plane is the horizontal plane which is four units above the xy coordinate plane.

What does the horizontal tangent plane imply about the shape of this surface near $(3,1,4)$? You may like to consider the cross-sections of the surface with $x = 3$ and $y = 1$. Can you think of other cases where a tangent plane is horizontal?

iii) Find the equation of the tangent plane to $z = \cos(x^2 + y^2)$ at $(\sqrt{\frac{\pi}{2}}, 0, 0)$.

The z coordinate of the point of tangency has been given here and so only the partial derivatives need to be found. Let $f(x,y) = \cos(x^2 + y^2)$. Then

$$f_x(x,y) = \frac{\partial z}{\partial x} = -2x\sin(x^2+y^2) = -2\sqrt{\frac{\pi}{2}}\sin(\frac{\pi}{2}) = -\sqrt{2\pi},$$

when $x = \sqrt{\pi/2}$ and $y = 0$.

$$f_y(x,y) = \frac{\partial z}{\partial y} = -2y\sin(x^2+y^2) = 0,$$

when $x = \sqrt{\pi/2}$ and $y = 0$.

Therefore the equation of the tangent plane is $z = -\sqrt{2\pi}(x - \sqrt{\pi/2})$.

What does it mean in this example that there is no y term in the equation of the tangent plane? Try to visualise the surface, perhaps using level curves or cross-sections. Where does the tangent plane lie in relation to the surface and the coordinate axes? ◇

Summary of Chapter 8

- Given a function $f(x, y)$ we defined the first-order partial derivatives

$$f_x(x, y) = \frac{\partial f}{\partial x} \quad \text{and} \quad f_y(x, y) = \frac{\partial f}{\partial y}$$

- The equation of the tangent plane to the surface $z = f(x, y)$ at the point $(a, b, f(a, b))$ is

$$z - f(a, b) = f_x(a, b)(x - a) + f_y(a, b)(y - b).$$

Exercises

8.1 Find f_x and f_y for each of the following functions of two variables:

a) $f(x, y) = 2x^2 + 7y - xy^3$

b) $f(x, y) = \sin y \, \cos x$

c) $f(x, y) = e^{x^2+y} + y - 1$

d) $f(x, y) = xe^x + (x + y)e^y$

e) $f(x, y) = \dfrac{1}{x^2 y}$

f) $f(x, y) = \tan(x + y) + 3\ln(x^4 + y^4)$

g) $f(x, y) = \sqrt{x^2 + 8y^4}$

h) $f(x, y) = \dfrac{\cos y}{\sqrt{x^2 + y^2}}$

8.2 Find the slopes of the curves of intersection of the surface $z = f(x, y)$ with the planes perpendicular to the x axis and y axis respectively at the given point.

a) $z = 2x^2 y$ at $(1, 1)$

b) $z = \sin(4x + y)$ at $(0, \frac{\pi}{2})$

c) $z = \dfrac{x^2 + \cos y}{1 - x}$ at $(2, 0)$

8.3 For each of the surfaces $z = f(x, y)$ given, evaluate the partial derivatives f_x and f_y at the point P. Hence find the equation of the tangent plane to the surface at point P.

a) $z = \sqrt{4 - x^2 - y^2}$; $P = (1, 1, \sqrt{2})$

b) $z = ye^x - 1$; $P = (1, 0, -1)$

c) $z = \sin x \cos y$; $P = \left(\frac{\pi}{2}, \frac{\pi}{2}, 0\right)$

8.4 Is there a function f such that $\frac{\partial f}{\partial x} = e^x \sin y$ and $\frac{\partial f}{\partial y} = e^x \cos y$?

8.5 Differentiate the following.

a) $f(x) = e^{x+5}$

b) $f(x) = (\ln 4)e^x$

c) $f(x) = xe^x$

d) $f(x) = \dfrac{x^2 + 5x + 2}{x + 3}$

e) $f(x) = (x + 1)^{99}$

f) $f(x) = xe^{-x^2}$

g) $f(t) = \tan t$

h) $f(t) = e^{\cos t}$

i) $f(t) = e^{t \cos 3t}$

j) $f(t) = \ln\big(\cos(1 - t^2)\big)$

k) $f(x) = (x + \sin^5 x)^6$

l) $f(x) = \sin(\sin(\sin x))$

m) $f(x) = \sin(6 \cos(6 \sin x))$

8.6 For each of the following functions f, find $f(f'(x))$ and $f'(f(x))$.

a) $f(x) = \dfrac{1}{x}$,

b) $f(x) = x^2$,

c) $f(x) = 2$,

d) $f(x) = 2x$.

CHAPTER 9

Further Applications of the Partial Derivative

We've already discussed the process of calculating partial derivatives of functions of two variables. If a function f depends on x and y then we can calculate partial derivatives f_x and f_y, but what if the input variables x and y are themselves functions of other variables, and we want derivatives with respect to these? Can we use partial derivatives to approximate a function of two variables?

In this chapter we introduce techniques for making linear approximations to a function for two variables and for calculating partial derivatives using variations of the chain rule.

9.1 Linear approximation: differentials

In this section we present a method of approximating a function of two variables close to a given point. Let us start by defining increments and differentials for a function of one variable, before looking at functions of two variables.

A differentiable[1] function $f(x)$ can be approximated, close to the point $x = a$, by the tangent to the curve at that point. The equation of this tangent line is $y = f(a) + f'(a)(x - a)$ which can be written as the linear function

$$L(x) = f(a) + f'(a)(x - a),$$

called the **linearisation** of the function f at a. The idea is to approximate the function f near $x = a$ by its linearisation, that is,

$$f(x) \approx L(x) \quad \text{near} \quad x = a.$$

Example 9.1a Find the linearisation of the function $f(x) = \sqrt{x+3}$ at $x = 1$ and use it to calculate approximations of the numbers $\sqrt{3.98}$ and $\sqrt{4.05}$.

To calculate approximations of the given numbers, we first need to choose a point near $x = 1$ to evaluate the function. Writing $3.98 = 0.98 + 3$ gives us the clue that we need to choose $x = 0.98$. Similarly, writing $4.05 = 1.05 + 3$ means that $x = 1.05$.

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[1]As discussed in Sec. 11.3 below, $f(x, y)$ is *differentiable at the point* (a, b) if f_x and f_y exist near (a, b) and are continuous at (a, b).

Now, the linearisation of f at $x = 1$ is

$$\begin{aligned} L(x) &= f(1) + f'(1)(x-1) \\ &= 2 + \frac{1}{4}(x-1) \\ &= \frac{7}{4} + \frac{x}{4}. \end{aligned}$$

So we can approximate $f(x)$ as follows:

$$f(x) = \sqrt{x+3} \;\approx\; \frac{7}{4} + \frac{x}{4} \quad \text{when} \quad x \quad \text{is near } 1.$$

Therefore,

$$\sqrt{3.98} = \sqrt{0.98+3} \;\approx\; \frac{7}{4} + \frac{0.98}{4} = 1.995$$

and

$$\sqrt{4.05} = \sqrt{1.05+3} \;\approx\; \frac{7}{4} + \frac{1.05}{4} = 2.0125.$$

Because $x = 0.98$ and $x = 1.05$ are near $x = 1$, the approximations are reasonably good. If fact, using a calculator we get the exact values $\sqrt{3.98} = 1.99499$ and $\sqrt{4.05} = 2.01246$ which, after rounding to four decimal places, coincide with the approximate values. ◇

Differentials of functions of one variable

Differentials provide an alternative formulation of linear approximations that sometimes simplifies calculations such as the one above. Recall the alternative notation of the definition of derivative as a function in terms of increments Δx and Δy, given by

$$f'(x) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} \quad \text{where} \quad \Delta y = f(x + \Delta x) - f(x).$$

From the meaning of limit we see that for Δx "small", the ratio $\dfrac{\Delta y}{\Delta x}$ is close to the derivative $f'(x)$, that is,

$$\frac{\Delta y}{\Delta x} \approx f'(x) \quad \text{and therefore} \quad \Delta y \approx f'(x)\, \Delta x.$$

The last equation tells us that if x changes by a *small* amount Δx, then y will change by approximately the amount $\Delta y \approx f'(x)\, \Delta x$. This motivates the introduction of the concept of differential.

Differential – One variable functions
Let $y = f(x)$ where f is a differentiable function and let Δx be any nonzero real number. Then

a) The differential dx is a variable given by $dx = \Delta x$.
b) The differential dy of the function f is another function given by

$$dy = f'(x)\, dx \quad \text{with alternative notation} \quad df = f'(x)\, dx.$$

The differential of a function of one variable is itself a function of *two variables*; both x and dx are needed to evaluate the differential.

Note that if the definitions seem somewhat artificial it is because they have been introduced so that we can manipulate the symbols dx and dy and treat the notation for the derivative $\frac{dy}{dx}$ as a *"ratio"*, since in terms of differentials we have

$$\frac{dy}{dx} = \frac{f'(x)\,dx}{dx} = f'(x).$$

Example 9.1b Calculate the differential df of the function $f(x) = x^3 + 5x^2$.

Since $f'(x) = (3x^2 + 10x)$ then $df = f'(x)dx = (3x^2 + 10x)dx$. ◇

Example 9.1c Write down the differential df of $f(x) = 2x\cos x$ and find an expression for the differential in terms of dx when $x = 0$.

Since $f'(x) = 2\cos x - 2x\sin x$ then $df = f'(x)dx = \big(2\cos x - 2x\sin x\big)dx$ and when $x = 0$ the differential becomes $df = 2dx$. ◇

Relationship between the increment Δy and the differential dy

Referring to the figure below, consider the function $y = f(x)$ and let x_0 be a fixed number where $f'(x_0)$ exists. For any value of Δx, we have $dx = \Delta x$. The equation

$$dy = f'(x_0)\; dx$$

is the equation of the tangent to the curve with slope $f'(x_0)$ in the coordinates (dx, dy).

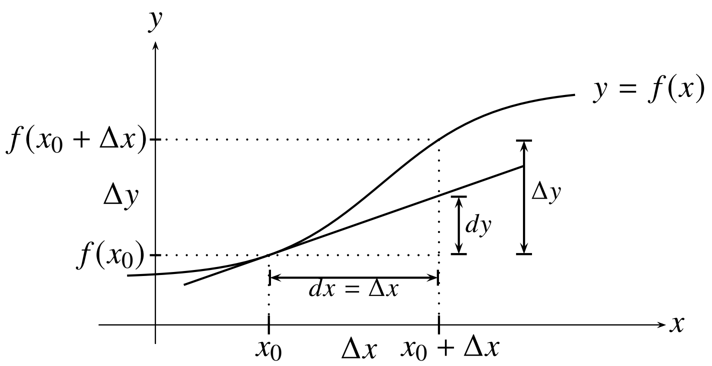
Observe carefully in the diagram that $\Delta y = f(x_0 + \Delta x) - f(x_0)$ changes along the curve $y = f(x)$ while dy changes along the tangent line. Therefore, assuming small values of $dx = \Delta x$, the values of dy and Δy become closer together, and we can say that

$$\Delta y \approx dy.$$

From the diagram we also note that $f(x_0 + \Delta x) = f(x_0) + \Delta y$ and using the approximation $\Delta y \approx dy$ we obtain $f(x_0 + \Delta x) = f(x_0) + \Delta y \approx f(x_0) + dy$. That is,

(9.1d) $$f(x_0 + \Delta x) \approx f(x_0) + dy.$$

What this relation is telling us is that, if we have the value of a function $f(x_0)$ at a point x_0, to find an approximate value at a nearby point $x_0 + \Delta x$ we only have to find $dy = f'(x_0)\Delta x$ using the simpler equation of the tangent line.



Example 9.1e

Given the function $f(x) = \sqrt{3x+4}$, use differentials to find an approximate value for $f(7.1)$.

We first need to choose a point to evaluate the differential. We can see that $f(7)$ is easy to evaluate since $f(7) = 5$, therefore a sensible choice is $x_0 = 7$. Then the differential in x is $dx = \Delta x = 7.1 - 7 = 0.1$ and the derivative

$$f'(x) = \frac{3}{2\sqrt{3x+4}} \qquad \text{and so} \qquad f'(7) = \frac{3}{10} = 0.3.$$

Hence

$$dy = f'(x_0)dx = 0.3 \times 0.1 = 0.03.$$

This differential is the approximate *change* in $f(x)$ between $x = 7$ and $x = 7.1$ and so an approximation for $f(7.1)$, as given by equation (9.1d), is

$$f(7.1) = f(x_0 + \Delta x) = f(7 + 0.1) \approx f(7) + dy = 5 + 0.03 = 5.03.$$

Using a calculator gives $f(7.1) = 5.02991$ to six significant figures and so the approximation using differentials is accurate to three significant figures. ◇

Example 9.1f Find the approximate error in the area of a circle of radius $30cm$, if there is an error of $\pm 0.2cm$ in the measurement of its radius.

Let $A = \pi r^2$ be the area of the circle, where r is its radius. The error is approximated by the differential of A: $dA = \frac{dA}{dr}dr = 2\pi r dr$. Here $r = 3$ and $|dr| = 0.2$. The magnitude of the error in A is

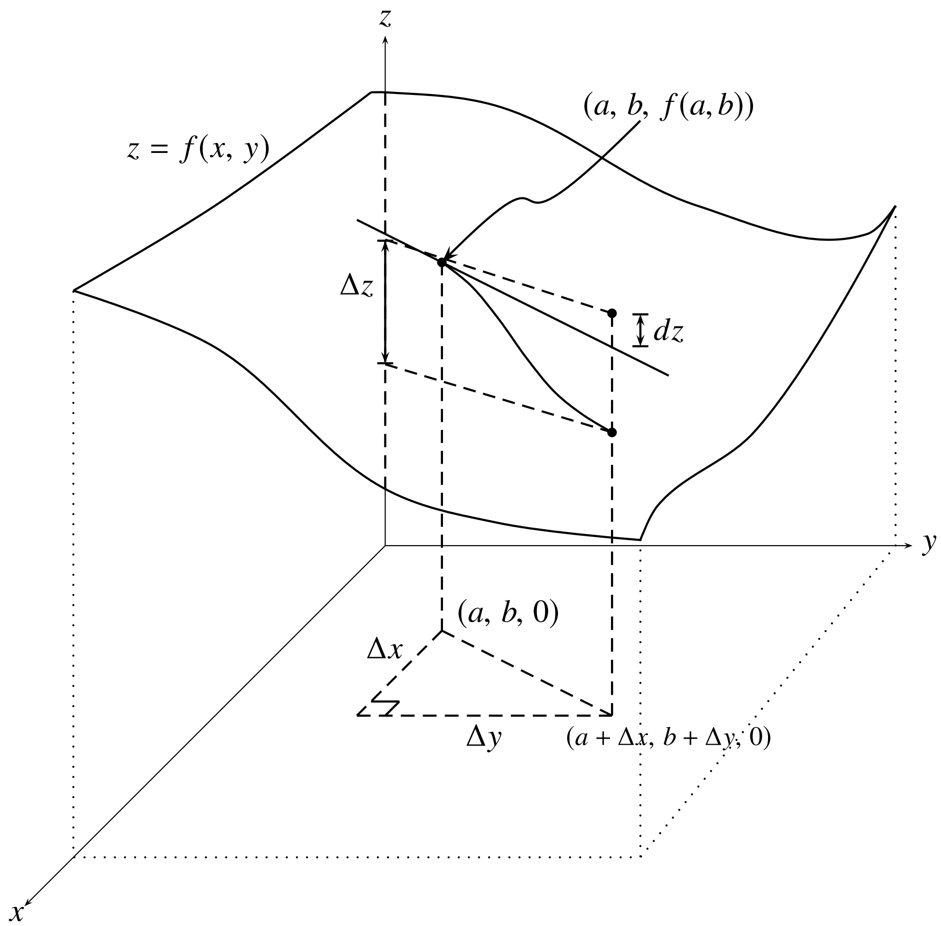
$$|dA| = \left|\frac{dA}{dr}dr\right| = \left|\frac{dA}{dr}\right||dr| = 60\pi \times 0.2 = 12\pi \approx 37.7.$$

Therefore the estimated error in the area of the circle is approximately $37.7cm^2$. This amounts to a relative error of $\frac{|dA|}{|A|} = \frac{12\pi}{900} \approx 4.2\%$, larger than the relative error of the radius $\frac{|dr|}{r} = \frac{0.2}{30} \approx 0.6\%$. ◇

Differentials of functions of two variables

The linear approximation of a differentiable function of two variables $f(x, y)$ near the point (a, b) is not a tangent line, but the tangent plane to the surface $z = f(x, y)$ at (a, b). In a similar way to a function of one variable, we measure the change in z as we move from one input value (a, b) to another input value (s, t).

The *actual* change in z, $f(s, t) - f(a, b) = \Delta z$ (or Δf) is called the "increment", as before. The *approximate* change in z, given by the change measured along the tangent plane, is the "differential" dz or df. The value of z on the tangent plane at (s, t) is given by $f(a, b) + dz$. This approximates $f(s, t)$.



For a function of two variables, both x and y change as we move from (a, b) to (s, t). Therefore, there are two differentials:

$$dx = \Delta x = s - a, \text{ and } dy = \Delta y = t - b.$$

These can be rewritten as $s = a + dx$ and $t = b + dy$.

A formula for the differential of a function of two variables can be derived using the equation of the tangent plane which we derived earlier. The equation of the plane tangent to $z = f(x, y)$

at $(a, b, f(a, b))$ is

$$z - f(a, b) = f_x(a, b)(x - a) + f_y(a, b)(y - b).$$

Substituting $z = f(a, b) + dz$, $x = s = a + dx$ and $y = t = b + dy$ gives

$$dz = f_x(a, b)dx + f_y(a, b)dy = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$$

where it is understood that the partial derivatives are evaluated at $x = a$ and $y = b$.

As with a function of one variable, the differential of a function of two variables need not be evaluated for particular values of x and y or dx and dy. When a differential of a function of two variables is left in terms of these variables, it is a function of four variables.

Differential – Two variable functions
The differential dz of a function of two variables $z = f(x, y)$ is given by

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy.$$

Examples 9.1g

i) Write down the differential of the function $f(x, y) = x^2e^{3y}$. Use this expression to find an approximate value for $f(3.8, 0.1)$. We have

$$df = f_x(x, y)dx + f_y(x, y)dy = 2xe^{3y}dx + 3x^2e^{3y}dy.$$

To approximate $f(3.8, 0.1)$ we first find a point near $(3.8, 0.1)$ where it is easy to evaluate the function $f(x, y)$ and its derivatives. One such point is $(4, 0)$.

Then $dx = 3.8 - 4 = -0.2$ and $dy = 0.1 - 0 = 0.1$. So

$$\begin{aligned} df &= f_x(4, 0)dx + f_y(4, 0)dy \\ &= 8 \times -0.2 + 48 \times 0.1 \\ &= 3.2. \end{aligned}$$

Now $f(4, 0) = 16$ and so $f(3.8, 0.1) \approx f(4, 0) + dz = 16 + 3.2 = 19.2$.

ii) Before you begin reading this example, think about what happens to the tangent plane approximation as the distance gets larger between the point of tangency and the point where the approximation is being taken. How can you explain your conclusion?

Write down an expression for the differential of $f(x, y) = 2x \cos y - y$ at $(0, 0)$. Find the value of the differential at the following points: $(0.2, 0.2)$, $(0.5, 0.5)$, $(1, 1)$, $(-0.2, 0)$,

$(-1,0)$, $(1,-1)$ and $(-1,1)$. Plot these points on the plane, marking the value of the differential next to the point.

$$df = 2\cos y dx + (-2x\sin y - 1)dy = 2dx - dy \text{ when } x = 0 \text{ and } y = 0.$$

Does this confirm or contradict your ideas at the start of the example?

iii) Suppose that the function values and partial derivatives are known for the function $z = g(x,y)$ at the point $(-7,3)$. They are $g(-7,3) = 24$, $g_x(-7,3) = -5$ and $g_y(-7,3) = 11$. Using differentials, estimate the value of $g(x,y)$ at $(-6.5,2.7)$.

Finding the differentials dx and dy gives

$$dx = -6.5 - (-7) = 0.5 \text{ and } dy = 2.7 - 3 = -0.3.$$

At $(7,3)$

$$dg = g_x(-7,3)dx + g_y(-7,3)dy = -5 \times 0.5 + 11 \times -0.3 = -5.8.$$

The differential dg gives the approximate *change* in the value of $g(x,y)$ between $(7,3)$ and $(-6.5,2.7)$. We use this to find the approximate *value* of $g(-6.5,2.7)$:

$$g(-6.5,2.7) \approx g(-7,3) + dg = 24 - 5.8 = 18.2.$$

iv) The volume of a circular cylinder is given by $V = \pi r^2 h$ where r is the radius of the circular cross-section and h is the length of the cylinder. A metal machine component has the shape of a cylinder, with external radius 3cm and height 8cm. The cylinder has a base but no top. The metal forming the component is 4mm thick. Use differentials to estimate the volume of metal in the component.

The volume of metal is given by dV. If dr and dh are the width of metal in the cylinder in the radial and vertical directions respectively then

$$dV = \frac{\partial V}{\partial r}dr + \frac{\partial V}{\partial h}dh = 2\pi r h dr + \pi r^2 dh.$$

Working in units of centimetres, we have $r = 3$, $h = 8$, $dr = dh = -0.4$. (We take the differentials as negative because the width of metal is inside the external surface. If we wanted to work out the internal volume of the component using differentials then the sign of dV is important.) Evaluating dV we have $dV = -22.8\pi \approx -71.6$. Therefore the amount of metal in the component is approximately 72 cm^3.

Try to estimate the internal volume of the component using differentials.

v) A rectangle is measured to have length 4m and height 2.5m. The relative error of each measurement is at most 0.3%. Estimate the maximum percentage error in the area using differentials.

Let the area of the rectangle be A, its length ℓ and its height h. (Note: if you are not given pronumerals as part of the statement of a problem then it is essential that you define any symbols which you introduce.) So $A = \ell h$ and

$$dA = \frac{\partial A}{\partial \ell} d\ell + \frac{\partial A}{\partial h} dh = h d\ell + \ell dh.$$

If the relative error in both the length and height is 0.3%, or 0.003 as a decimal, we have $|d\ell|/\ell = |dh|/h = 0.003$. We seek to find $|dA|/A$:

$$\begin{aligned} \frac{|dA|}{A} &= \left|\frac{hd\ell}{\ell h} + \frac{\ell dh}{\ell h}\right| = \left|\frac{d\ell}{\ell} + \frac{dh}{h}\right| \\ &\leq \left|\frac{d\ell}{\ell}\right| + \left|\frac{dh}{h}\right| \\ &= |0.003| + |0.003| = 0.006. \end{aligned}$$

(Here we use the "triangle inequality", $|a+b| \leq |a| + |b|$ to simplify the absolute values on the right-hand side.)

Therefore, the maximum relative error in the area of the rectangle is 0.006 as a decimal, or 0.6%, as a percentage. ◇

9.2 The chain rule

The chain rule for differentiation is needed whenever a function depends on a variable, which in turn depends on another variable. For example, using a function of one variable, we could have $y = f(x)$ and $x = g(t)$. Ultimately, y is a function of t and we can then calculate the total derivative $\dfrac{dy}{dt}$. This can be expressed in a diagram:

$$\begin{array}{rcl} & y & \\ f & \Big| & \text{depends on} \\ & x & \\ g & \Big| & \text{depends on} \\ & t & \end{array}$$

The derivative of y with respect to t can be calculated using the chain rule for a function of one variable:

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}.$$

We often use this idea when differentiating functions such as $y = \sin(t^2)$. Here we could write $x = t^2$ so $y = \sin x$ and then

$$\frac{dy}{dt} = (\cos x)(2t) = 2t \cos t^2$$

as expected.

When applying the chain rule to functions of two variables $f(x, y)$ it is instructive to consider two cases:

The total derivative rule.

The first case in which the chain rule can be used in functions of two variables corresponds to $z = f(x, y)$ and $x = g(t)$, $y = h(t)$, or in diagram form:

$$\begin{tikzcd} & z \arrow[dl, dash, "f"'] \arrow[dr, dash, "f"] & \\ x \arrow[d, dash, "g"'] & & y \arrow[d, dash, "h"] \\ t & & t \end{tikzcd}$$

The diagram shows that at an intermediate level z depends on both x and y, but ultimately z depends only on t; once we know t then we can calculate x and y and hence z. So z is fundamentally a function of one variable, t. Therefore $\frac{dz}{dt}$ is an ordinary derivative, also called a "total derivative". How is this total derivative calculated?

Suppose that we focus on a particular value of t, say $t = t_0$. We want to calculate $\frac{dz}{dt}$ at the point $t = t_0$. Let $x_0 = g(t_0)$ and $y_0 = h(t_0)$. An increment in t produces increments in x and y, which in turn produce an increment in z, which is well approximated by dz.

$$\Delta z \approx dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy,$$

where the partial derivatives are understood to be evaluated at (x_0, y_0). Since $x = g(t)$ and $y = h(t)$ we see that

$$\Delta x \approx dx = \frac{dx}{dt}dt = \frac{dx}{dt}\Delta t \text{ and } \Delta y \approx dy = \frac{dy}{dt}dt = \frac{dy}{dt}\Delta t.$$

(Here the derivatives are ordinary derivatives since x and y are both functions of one variable, and these derivatives are understood to be evaluated at t_0.) Substituting for Δx and Δy gives

$$\Delta z \approx \frac{\partial z}{\partial x}\frac{dx}{dt}\Delta t + \frac{\partial z}{\partial y}\frac{dy}{dt}\Delta t.$$

If we divide both sides by Δt we obtain

$$\frac{\Delta z}{\Delta t} \approx \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}.$$

As $\Delta t \to 0$, the left hand side becomes $\left.\dfrac{dz}{dt}\right|_{t=t_0}$ and the approximation becomes exact, that is, both sides become equal. This gives us the first variation of the familiar chain rule, called the Total Derivative Rule, which holds for all values of t.

The Total Derivative Rule
Suppose that $z = f(x, y)$ where $x = g(t)$ and $y = h(t)$ and f, g and h are all differentiable functions. Then the "total derivative" of z is given by

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}.$$

Examples 9.2a

i) Find the total derivative of z with respect to t when $z = \cos x \sin y$ and $x = t^2 + t$ and $y = t^3 + 3t$.

   Calculating the derivatives:

$$\frac{\partial z}{\partial x} = -\sin x \sin y, \qquad \frac{\partial z}{\partial y} = \cos x \cos y$$

$$\frac{dx}{dt} = 2t + 1, \qquad \frac{dy}{dt} = 3t^2 + 3.$$

   Applying the total derivative rule:

$$\begin{aligned} \frac{dz}{dt} &= \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} \\ &= (-\sin x \sin y)(2t + 1) + (\cos x \cos y)(3t^3 + 3) \\ &= -(2t + 1)\sin(t^2 + t)\sin(t^3 + 3t) + (3t^2 + 3)\cos(t^2 + t)\cos(t^3 + 3t). \end{aligned}$$

ii) Two trains are on the same line. Train A is a km from the terminus, and train B is b km from the terminus. The directed distance between them is $D = a - b$. The position of each train changes with time so $a = f(t)$ and $b = g(t)$. Write down an expression for the rate of change of the distance between the trains.

   The rate of change of the distance between the trains is $\frac{dD}{dt}$. Using the total derivative rule:

$$\frac{dD}{dt} = \frac{\partial D}{\partial a}\frac{da}{dt} + \frac{\partial D}{\partial b}\frac{db}{dt} = \frac{da}{dt} - \frac{db}{dt}.$$

   Here we do not have an expression for $\dfrac{da}{dt}$ and $\dfrac{db}{dt}$ and so these derivatives remain in the expression for $\dfrac{dz}{dt}$.

iii) A particle is moving on a closed curve on the surface $z = x^2 + y^2$. The projection of the particle's path on the xy plane is given by the parametric equations $x = 2\cos t + 1$, $y = 2\sin t$. (This curve is simply the circle of radius 2 and centre $(1, 0)$.) Find the highest point that is reached by the particle.

Here, if t is known, we can calculate both x and y and hence z, so there must exist a total derivative $\frac{dz}{dt}$. The height of the particle above the xy plane is given by z and so the maximum height will occur when $\frac{dz}{dt} = 0$. To find $\frac{dz}{dt}$ we first find the partial derivatives of $z = x^2 + y^2$:

$$\frac{\partial z}{\partial x} = 2x, \quad \frac{\partial z}{\partial y} = 2y$$

$$\frac{dx}{dt} = -2\sin t, \quad \frac{dy}{dt} = 2\cos t.$$

Using the total derivative rule:

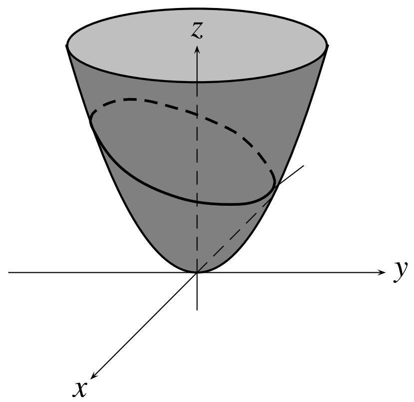
$$\frac{dz}{dt} = -4x\sin t + 4y\cos t = 4(-(2\cos t + 1)\sin t + 2\sin t\cos t) = -4\sin t.$$

So $\dfrac{dz}{dt} = 0$ whenever $\sin t = 0$, that is when $t = 0$ or π. At one of these values of t the path on the surface reaches maximum height; at the other value the path reaches its minimum height above the xy plane. Which value of t gives the maximum height?

Because $\dfrac{dz}{dt}$ is itself a function of one variable we can easily find the second derivative:

$$\frac{d^2z}{dt^2} = -4\cos t = \begin{cases} -4 \text{ when } t = 0, \\ 4 \text{ when } t = \pi. \end{cases}$$

Therefore there is a maximum at $t = 0$ where the second derivative is negative. When $t = 0$, $x = 3$ and $y = 0$ and hence $z = 9$. So the greatest height about the xy plane attained by the particle is 9 units.



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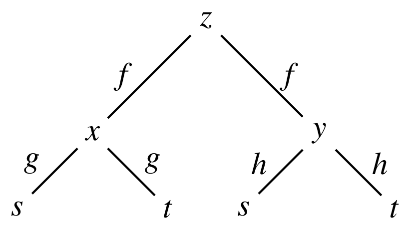
The chain rule for partial derivatives.

The second case in which the chain rule can be used in functions of two variable happens when $z = f(x, y)$ but now $x = g(s, t)$ and $y = h(s, t)$ are also functions of two variables. Diagramatically, the relationship between the variables looks like this:



Here z is a function of both s and t because to calculate z we need x and y, and these both depend on s and t. Hence z has *partial* derivatives with respect to s and t.

The chain rule

Suppose that $z = f(x, y)$ and $x = g(s, t)$ and $y = h(s, t)$ where f, g and h are differentiable functions of two variables. Then

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

Examples 9.2b

i) Find the partial derivatives of z with respect to s and t if $z = \ln(3x + 2y)$ and $x = s + t$ and $y = s - t$.

$$\frac{\partial z}{\partial x} = \frac{3}{3x + 2y}, \quad \frac{\partial z}{\partial y} = \frac{2}{3x + 2y},$$
$$\frac{\partial x}{\partial s} = 1, \quad \frac{\partial x}{\partial t} = 1,$$
$$\frac{\partial y}{\partial s} = 1, \quad \frac{\partial y}{\partial t} = -1.$$

Using the chain rule for partial derivatives:

$$\frac{\partial z}{\partial s} = \frac{3}{3x + 2y} + \frac{2}{3x + 2y} = \frac{5}{3(s + t) + 2(s - t)} = \frac{5}{5s + t},$$
$$\frac{\partial z}{\partial t} = \frac{3}{3x + 2y} - \frac{2}{3x + 2y} = \frac{1}{5s + t}.$$

ii) If $z = x^2 + y^2$ and x and y are given by $x = r\cos\theta$, $y = r\sin\theta$, find $\dfrac{\partial z}{\partial r}$ and $\dfrac{\partial z}{\partial \theta}$.

$$\frac{\partial z}{\partial x} = 2x, \quad \frac{\partial z}{\partial y} = 2y$$

$$\frac{\partial x}{\partial r} = \cos\theta, \quad \frac{\partial x}{\partial \theta} = -r\sin\theta$$

$$\frac{\partial y}{\partial r} = \sin\theta, \quad \frac{\partial y}{\partial \theta} = r\cos\theta.$$

Therefore the partial derivatives of z with respect to r and θ are:

$$\frac{\partial z}{\partial r} = 2x\cos\theta + 2y\sin\theta = 2r\cos^2\theta + 2r\sin^2\theta = 2r$$

$$\frac{\partial z}{\partial \theta} = -2xr\sin\theta + 2yr\cos\theta = -2r^2\cos\theta\sin\theta + 2r^2\sin\theta\cos\theta = 0.$$

The surface $z = x^2 + y^2$ is the cup-shape or paraboloid of Example 9.2a(iii). If r is the distance from the origin to a point in the xy plane and θ is the angle around the origin in the xy plane, then we can interpret these derivatives geometrically. The derivative $\dfrac{\partial z}{\partial r}$ gives the rate of change of the height of the surface as we move outward from the origin. As $\dfrac{\partial z}{\partial r} = 2r$ is independent of θ, the height of the surface above the xy plane increases at the same rate regardless of the direction in which we move away from the origin. That is, the surface is rotationally symmetric. Since $\dfrac{\partial z}{\partial \theta} = 0$, if we move in a circle around the origin, the height of the surface is constant. This is equivalent to saying that the level curves of the surface are circles with centre $(0, 0)$.

iii) One important application of the chain rule for partial derivatives is changing the variables in expressions involving derivatives. This is often done when changing an expression written in terms of Cartesian coordinates to an expression written in terms of polar coordinates. Here is an example of this.

If $z = f(x, y)$ and $x = r\cos\theta$ and $y = r\sin\theta$, show that

$$\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = \left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2$$

Since we do not know what $z = f(x, y)$ is explicitly, it is easiest to start by finding expressions for $\dfrac{\partial z}{\partial r}$ and $\dfrac{\partial z}{\partial \theta}$.

$$\frac{\partial x}{\partial r} = \cos\theta, \quad \frac{\partial x}{\partial \theta} = -r\sin\theta, \quad \frac{\partial y}{\partial r} = \sin\theta, \quad \frac{\partial y}{\partial \theta} = r\cos\theta.$$

By the chain rule

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\cos\theta + \frac{\partial z}{\partial y}\sin\theta$$

$$\frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x}(-r\sin\theta) + \frac{\partial z}{\partial y} r\cos\theta.$$

Substituting into the right hand side of the equation above:

$$\begin{aligned}
\left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2 &= \left(\frac{\partial z}{\partial x}\cos\theta + \frac{\partial z}{\partial y}\sin\theta\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial x}(-r\sin\theta) + \frac{\partial z}{\partial y} r\cos\theta\right)^2 \\
&= \left(\frac{\partial z}{\partial x}\right)^2\cos^2\theta + 2\left(\frac{\partial z}{\partial x}\right)\left(\frac{\partial z}{\partial y}\right)\cos\theta\sin\theta + \left(\frac{\partial z}{\partial y}\right)^2\sin^2\theta \\
&\quad + \left(\frac{\partial z}{\partial x}\right)^2\sin^2\theta - 2\left(\frac{\partial z}{\partial x}\right)\left(\frac{\partial z}{\partial y}\right)\cos\theta\sin\theta \\
&\quad + \left(\frac{\partial z}{\partial y}\right)^2\cos^2\theta \\
&= \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2,
\end{aligned}$$

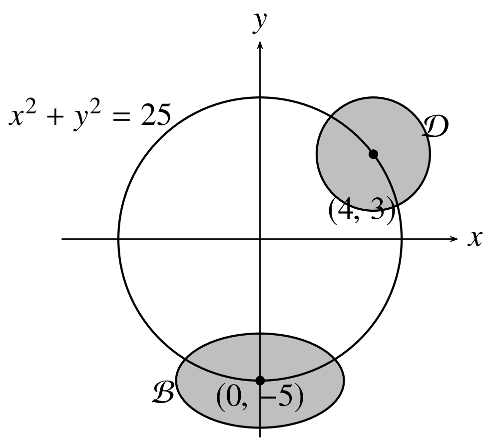
using $\cos^2\theta + \sin^2\theta = 1$. This is what we were required to show. ◇

9.3 Implicit differentiation using the chain rule

Theorem 9.3a (The Implicit Function Theorem)
Let C be a curve in two dimensions given by an equation of the form $f(x,y) = k$, where k is a constant and $f(x,y)$ is a differentiable function. Suppose that (a,b) is a point on the curve (that is, $f(a,b) = k$), with $f_y(a,b) \neq 0$. Then there exists a region $\mathcal{D}$ around (a,b) such that the part of the curve C in region $\mathcal{D}$ is the graph of a differentiable function $y = g(x)$.

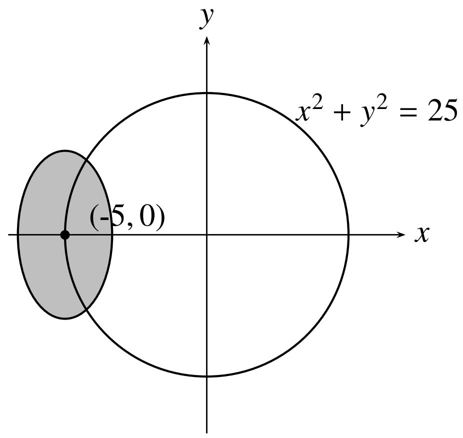
We shall not give a proof of this theorem, but instead will illustrate it with an example.

Consider the circle $x^2 + y^2 = 25$. This equation has the form $f(x,y) = k$, where $k = 25$ and $f(x,y) = x^2 + y^2$. So $f_y(x,y) = 2y$. The point $(4,3)$ lies on this circle and $f_y(4,3) = 6 \neq 0$. Therefore there is a region $\mathcal{D}$ around this point where the curve is the graph of a differentiable function. You can easily show that the rule for this function is $y = \sqrt{25 - x^2}$. One such possible region is shown in the diagram below.

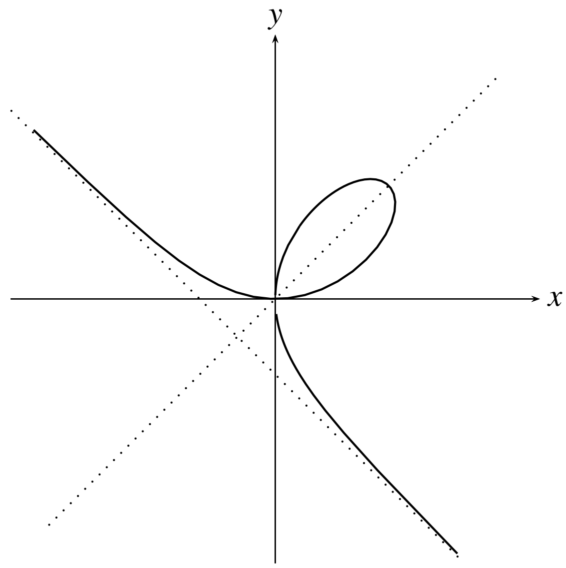


If we had chosen a different point, such as $(0, -5)$ then we would choose a different region, region $\mathcal{B}$, where $g(x) = -\sqrt{25 - x^2}$.

Suppose on the other hand, we choose to look near the point $(-5, 0)$. We see that $f_y(-5, 0) = 0$. No matter how small we draw a region containing this point, there will always be some values of x for which there are two y values for a single x value. Hence no part of the curve containing this point can be the graph of a function.



When $f(x, y) = k$ implicitly defines a differentiable function of one variable, $y = g(x)$, how can we find the derivative of $g(x)$? In the case of the circle above, this can be easily done by making y the subject and differentiating. Consider however the curve $x^3 + y^3 - 3xy = 0$, known as the folium of Descartes and illustrated below. It is not obvious how we can make y the subject in this case.



Let us return to the total derivative rule. We will seek to differentiate the function of two variables $f(x, y)$ where $y = g(x)$. In fact, x can also be thought of as a function of itself, say $h(x) = x$, and so the variables depend on one another in the following way:

$$
\begin{tikzcd}[column sep=small]
& z \arrow[dl, dash, "f"'] \arrow[dr, dash, "f"] & \\
x \arrow[d, dash, "h"'] & & y \arrow[d, dash, "g"] \\
x & & x
\end{tikzcd}
$$

We now differentiate both sides of the equation $f(x, y) = k$ with respect to x, remembering that k is a constant and using the total derivative rule for $f(x, y)$. This gives

$$\frac{\partial f}{\partial x}\frac{dh}{dx} + \frac{\partial f}{\partial y}\frac{dg}{dx} = 0.$$

Remembering that $\dfrac{dh}{dx} = 1$ since $h(x) = x$, rearranging this equation gives

$$\frac{dg}{dx} = \frac{-\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}.$$

Recalling that $y = g(x)$, and reverting to the other notation for partial derivatives, we get a simple expression for the derivative of a function defined implicitly.

> **Implicit differentiation**
> Suppose that a function $y = g(x)$ can be defined implicitly by $f(x, y) = k$, where f is differentiable and k is constant, and that $f_y(x, y) \neq 0$. Then the derivative of $g(x)$ is given by
>
> $$\frac{dy}{dx} = \frac{-f_x(x, y)}{f_y(x, y)}.$$

Examples 9.3b

i) Using this formula for the implicit derivative, it is easy to find the derivative of the function defined by $f(x, y) = x^3 + y^3 - 3xy = 0$ which we considered above. Here $f_x(x, y) = 3x^2 - 3y$ and $f_y(x, y) = 3y^2 - 3x$ so

$$\frac{dy}{dx} = \frac{-(3x^2 - 3y)}{3y^2 - 3x}.$$

ii) Use implicit differentiation to find the equation of the tangent line to the curve $e^x + y^3 = xy^2 + 9$ at $(0, 2)$.

It is important that we write the equation in the form $f(x, y) = k$. In this example, rewriting the equation of the curve in this way gives $e^x + y^3 - xy^2 = 9$ and so we set $f(x, y) = e^x + y^3 - xy^2$. Hence

$$f_x(x, y) = e^x - y^2, \quad f_y(x, y) = 3y^2 - 2xy,$$

and so $f_x(0, 2) = -3$ and $f_y(0, 2) = 12$. The slope of the tangent at $(0, 2)$ is

$$\frac{dy}{dx} = \frac{-f_x(0, 2)}{f_y(0, 2)} = \frac{-(-3)}{12} = \frac{1}{4}.$$

The tangent passes through the point $(0, 2)$ and so, using the point-gradient form of the line, its equation is $y - 2 = \frac{1}{4}(x - 0)$ or $4y - x - 8 = 0$.

iii) Proof that the gradient is perpendicular to level curves (stated in Sec. 10.4). Each level curve can be thought of as the graph of a function of one variable $y = g(x)$ defined implicitly by the equation $f(x, y) = c$. Using implicit differentiation, the slope of the tangent to the level curve passing through (a, b) is

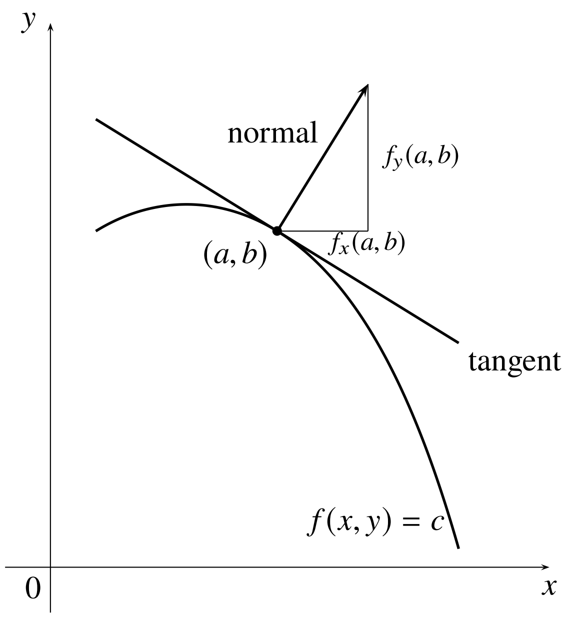
$$g'(a) = \frac{-f_x(a, b)}{f_y(a, b)}.$$

Hence the slope of the normal at the point (a, b) is

$$\frac{-1}{g'(a)} = \frac{f_y(a, b)}{f_x(a, b)}.$$

Therefore, a vector which is normal to $f(x, y) = c$ at (a, b) must have **i** and **j** components in the ratio $f_x(a, b)/f_y(a, b)$. One such vector is $f_x(a, b)\mathbf{i} + f_y(a, b)\mathbf{j}$. This is simply $\nabla f(a, b)$. Therefore the direction of the gradient vector at (a, b) is normal to the level curve $f(x, y) = c$ at (a, b).

◇



You may have previously learnt other techniques for finding implicit derivatives. For example, it is possible to find the equation of the tangent line to the curve in the previous example in the following way. Since $e^x + y^3 = xy^2 + 9$, differentiating each side with respect to x (using the chain rule and regarding y as an unknown function of x) gives

$$e^x + 3y^2 \frac{dy}{dx} = y^2 + x(2y) \frac{dy}{dx}$$

which rearranges to

$$\frac{dy}{dx} = \frac{y^2 - e^x}{3y^2 - 2xy}.$$

Thus at $(0, 2)$, we see that $\dfrac{dy}{dx} = \dfrac{1}{4}$, leading to the same tangent line equation as before.

The new approach outlined in this section, using partial derivatives, is recommended because it is very straightforward to use and leaves you less prone to careless errors than other methods.

Summary of Chapter 9

- The differential of a function $z = f(x, y)$ is defined as

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy.$$

- If $z = f(x, y)$, $x = g(t)$ and $y = h(t)$, the total derivative of z is

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}.$$

- The chain rule says that if $z = f(x, y)$ and $x = g(s, t)$ and $y = h(s, t)$, then

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$
$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

- If $y = g(x)$ is defined implicitly by $f(x, y) = k$, where k is constant, and if $f_y(x, y) \neq 0$, then

$$\frac{dy}{dx} = \frac{-f_x(x, y)}{f_y(x, y)}.$$

Exercises

9.1 Using the differential of an appropriate function of one variable, find an approximation to $(15)^{\frac{1}{4}}$.

9.2 Write down an expression for the differential of $f(x, y) = xy^{\frac{1}{2}}$ at $(1, 1)$. Use this to find an approximation to $f(1.5, 0.9)$.

9.3 A rectangle, sides x and y, is enlarged by a small amount so that its dimensions are now $x + \Delta x$ and $y + \Delta y$. Find the increment of its area ΔA and the differential dA. When would dA cease to be a good estimate for ΔA?

9.4 The dimensions of a rectangular block of wood are measured as 30cm, 40cm and 12cm. If there are possible errors of 2cm, 1cm and 0.5cm respectively, estimate the possible error in surface area. (Hint: Note that surface area is a function of three variables, but the same principles apply as for a function of two variables.)

9.5 Find $\dfrac{dz}{dt}$ for the following functions

a) $z = x^3y$, $x = t$ and $y = 2t^2$.

b) $z = e^y \cos 3x$, $x = 1 - t^2$ and $y = \ln t$.

c) $z = \ln\big(x(1 + y)\big)$, $x = e^{t^2}$ and $y = e^{4t} - 1$.

d) $z = \tan^{-1} x + \tan^{-1} y$, $x = \dfrac{t-1}{t+1}$ and $y = \dfrac{t+1}{t-1}$.

9.6 Using the chain rule find $\dfrac{\partial z}{\partial t}$ and $\dfrac{\partial z}{\partial u}$ for the following.

a) $z = 2y + x^3$, $x = t^{\frac{1}{2}} + u$ and $y = 3t^2 - u^2$.

b) $z = \sin(x + y)$, $x = t^2u$ and $y = \cos 2t + \sin 2u$.

9.7 Let $z = f(x, y)$ where $x = \alpha u + \beta v$, $y = \gamma u + \delta v$ where α, β, γ and δ are constants. Show that

$$\frac{\partial^2 z}{\partial u^2} = \alpha^2\frac{\partial^2 f}{\partial x^2} + 2\alpha\gamma\frac{\partial^2 f}{\partial x \partial y} + \gamma^2\frac{\partial^2 f}{\partial y^2}$$
$$\frac{\partial^2 z}{\partial v^2} = \beta^2\frac{\partial^2 f}{\partial x^2} + 2\beta\delta\frac{\partial^2 f}{\partial x \partial y} + \delta^2\frac{\partial^2 f}{\partial y^2}$$
$$\frac{\partial^2 z}{\partial v \partial u} = \alpha\beta\frac{\partial^2 f}{\partial x^2} + (\alpha\delta + \beta\gamma)\frac{\partial^2 f}{\partial x \partial y} + \gamma\delta\frac{\partial^2 f}{\partial y^2}$$

9.8 A function $f(x, y)$ is homogeneous of degree n if $f(tx, ty) = t^n f(x, y)$. Show that $f(x, y) = x^3 + 3x^2y + \dfrac{y^9}{x^6}$ is homogeneous of degree 3.

Now let f be a homogeneous function of degree n. By writing $a = tx$ and $b = ty$, show that

$$\frac{d}{dt} f(a, b) = x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nt^{n-1} f.$$

By setting $t = 1$ show that

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = n f.$$

This is Euler's Theorem.

9.9 Show that Laplace's equation

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0,$$

when written in terms of plane polar coordinates becomes

$$\frac{\partial^2 f}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{1}{r}\frac{\partial f}{\partial r} = 0.$$

9.10 Find $\frac{dy}{dx}$ when y is defined implicitly by $f(x, y) = 0$ where

$$f(x, y) = \ln\left(\frac{x}{x^2 + y^2}\right).$$

9.11 Use implicit differentiation to find the slope of the tangent to the curve

$$3x^4 + y^4 = (3x + y)^2$$

at the point $(2, 1)$.

9.12 In order to determine the angle θ which a sloping plane ceiling makes with the horizontal floor, an equilateral triangle of side-length l is drawn on the floor and the height of the ceiling above the three vertices is measured to be a, b and c. Show that

$$\tan^2\theta = 4\big(a^2 + b^2 + c^2 - ab - bc - ac\big)/3l^2.$$

(Hints: Use the vector and scalar products; θ is the angle between the normal to the plane and the vertical; $\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$; $\tan^2\theta = \sec^2\theta - 1$.)

9.13 Use the implicit function rule to find the derivative $\frac{dy}{dx}$ when y is defined implicitly by the relation

$$4x^3y^2 + \sin^{-1}(xy) = 2.$$

9.14 C is a path on the surface $z = \frac{x^3}{a^3} + \frac{y^3}{b^3}$ (a and b are positive constants) joining the points $A = (a, 0, 1)$ and $B = (0, b, 1)$. The projection of C onto the xy-plane is that part of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ lying in the first quadrant. Find the coordinates of the point P of C which has smallest height above the xy-plane.

9.15 Use the total derivative rule to calculate the derivative $\frac{dw}{dt}$ in terms of t when it is given that

$$w = x^4 - y^4, \qquad x = \cos t, \qquad y = \sin t.$$

Simplify your answer as far as possible.

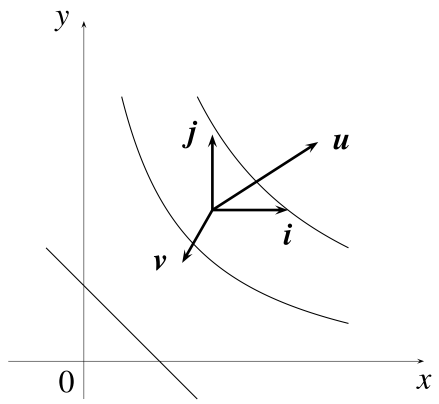
CHAPTER 10

Directional Derivatives and the Gradient Vector

In Chapter 8 we explored what was meant by the slope of a line that was tangent to a surface $z = f(x, y)$. We saw that each point on the surface had infinitely many tangent lines. We showed how the slopes of two of the lines, in the vertical planes parallel to the xz and yz coordinate planes, can be found using partial differentiation. But how do we find the slope of the other tangent lines? How do we find the rate of change of $f(x, y)$ if both x and y are changing? (Remember that in a partial derivative either x or y is held constant.) In this chapter we will explore these questions and see how the derivatives of a function of two variables can give a great deal of information about the function at a given point.

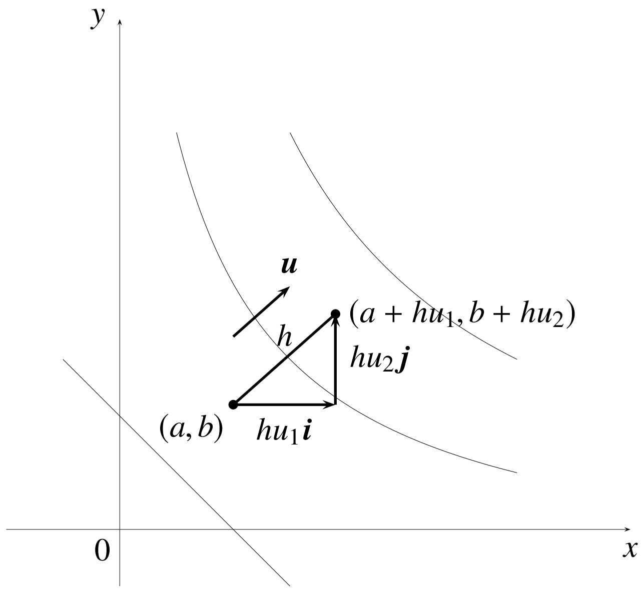
10.1 Defining the directional derivative

The diagram below shows some level curves of a surface $z = f(x, y)$. Using partial derivatives, we can find the rate of change of a function in the positive x direction, given by $\mathbf{i}$, and in the positive y direction, given by $\mathbf{j}$. But how can we find the rate of change in directions $\mathbf{u}$ or $\mathbf{v}$?



In this section we will find an expression for the derivative of a function $f(x, y)$ at a point (a, b) in the direction of an arbitrary vector $\mathbf{u}$. This derivative is called the "directional derivative" in the direction $\mathbf{u}$. We use the notation $D_{\mathbf{u}} f(a, b)$ for the directional derivative in the direction of $\mathbf{u}$, but the notations $f_{\mathbf{u}}(a, b)$ and $\dfrac{\partial f}{\partial \mathbf{u}}$ are also used. We will derive the directional derivative in two different ways. Each of these approaches gives a different insight into what is going on.

Let us consider first the surface $z = f(x, y)$ represented by its level curves. We want to find the directional derivative at (a, b) in the direction of $\mathbf{u}$. Let $\hat{\mathbf{u}} = u_1\mathbf{i} + u_2\mathbf{j}$ be the unit vector (that is, a vector of length 1) in the direction of $\mathbf{u}$. A point that is h units away from (a, b) in the direction of $\mathbf{u}$ will have coordinates $(a + hu_1, b + hu_2)$. This is illustrated below.



The value of the function at a point distance h from (a, b) in the direction of $\mathbf{u}$ will be $f(a + hu_1, b + hu_2)$, hence the average change in the function going from $f(a, b)$ to $f(a + hu_1, b + hu_2)$ is

$$\frac{f(a + hu_1, b + hu_2) - f(a, b)}{h}.$$

The numerator of this expression is simply $\Delta f \approx f_x(a, b)\Delta x + f_y(a, b)\Delta y$, the increment of $f(x, y)$. If h is small then the differential is a good approximation to the increment. The smaller h becomes, the better this approximation becomes. So we have

$$f(a + hu_1, b + hu_2) - f(a, b) = \Delta f \approx df = f_x(a, b)dx + f_y(a, b)dy.$$

Here $dx = \Delta x = hu_1$ and $dy = \Delta y = hu_2$. Taking limits of both sides as $h \longrightarrow 0$ gives

$$D_{\mathbf{u}}f = \lim_{h \to 0} \frac{f(a + hu_1, b + hu_2) - f(a, b)}{h} = \lim_{h \to 0} \frac{f_x(a, b)hu_1 + f_y(a, b)hu_2}{h}.$$

Taking this limit we get an expression for the directional derivative at (a, b) in terms of the partial derivatives of f at (a, b) and the unit vector $\hat{\mathbf{u}} = u_1\mathbf{i} + u_2\mathbf{j}$.

Directional derivative
The "directional derivative" of a differentiable function $f(x, y)$ in the direction of $\mathbf{u}$ at the point (a, b) is

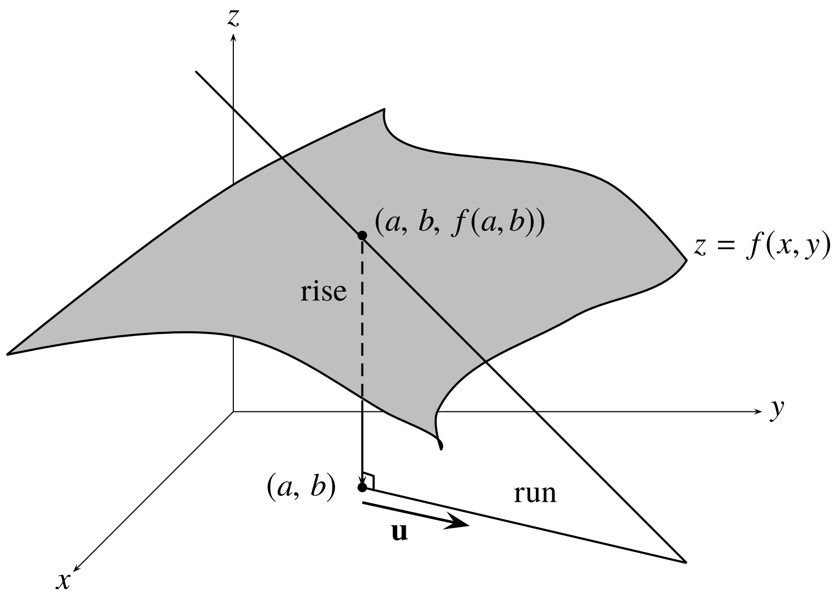
$$D_{\mathbf{u}} f = f_x(a, b)u_1 + f_y(a, b)u_2,$$

where $\hat{\mathbf{u}} = u_1\mathbf{i} + u_2\mathbf{j}$ is the unit vector in the direction of $\mathbf{u}$.

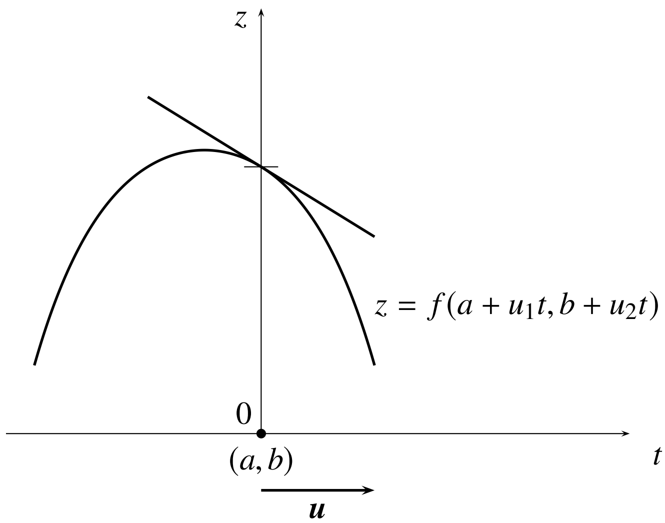
(The directional derivative of f in the direction of $\mathbf{u}$ is of course the same as the directional derivative of f in the direction of $k\mathbf{u}$ for any positive constant k. The important thing to note is that the values u_1 and u_2 used in the definition are the components of the *unit length* vector in the direction of $\mathbf{u}$.)

Let us derive this expression in a different way, using the cross-sections of a surface. This is similar to the derivation of partial derivatives in Chapter 8.

The direction of any tangent line to the surface $z = f(x, y)$ at a point $(a, b, f(a, b))$ can be specified by the vector $\mathbf{u}$ in the xy plane. The vertical plane in which the tangent lies also contains the vector $\mathbf{u}$. The vector $\mathbf{u}$ gives the direction from (a, b) in which the horizontal distance, the "run" is measured.



Now consider the cross-section of the surface and its tangent plane in the vertical plane which contains both the vector $\mathbf{u}$ and the tangent in the direction of $\mathbf{u}$. This is shown in the next diagram, where the origin is taken at the point (a, b) and the horizontal axis is in the direction of **u**. This horizontal axis is parallel to neither the x axis nor the y axis, so we have labelled it t. The vertical axis is parallel to the z axis. So, in tz coordinates, the cross-section looks like this:

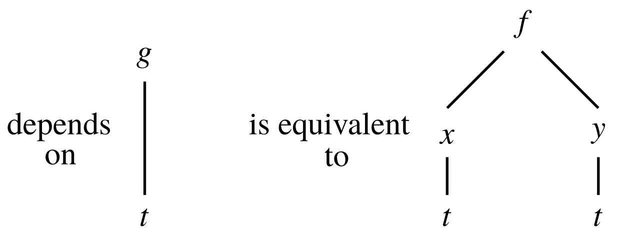


The t axis is the set of all points in the xy plane where $x = a + tu_1$ and $y = b + tu_2$. This lets us work out x and y for any t if we know $\hat{\mathbf{u}}$ and (a, b).

We want to find the slope of the tangent in the cross-section above. To do this we need to know the equation of the curve in terms of t. We can write $z = g(t)$ but how do we find $g(t)$?

We know that $z = g(t)$ is a cross-section of the surface $z = f(x, y)$ and that $x = a + u_1 t$ and $y = b + u_2 t$. So $g(t) = f(a + u_1 t, b + u_2 t)$ or $g(t) = f(x, y)$ where $x = a + u_1 t$ and $y = b + u_2 t$.

The slope of the tangent is given by $g'(0)$ where $g'(t) = \dfrac{dg}{dt}$. Now the value of g depends on the value of f, which in turn depends on the values of x and y; x and y both depend on t.



So by the total derivative rule – which will be discussed in Sec. 9.2 below – we have

$$\begin{aligned}\frac{dg}{dt} = \frac{df}{dt} &= \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} \\ &= \frac{\partial f}{\partial x}u_1 + \frac{\partial f}{\partial y}u_2\end{aligned}$$

using the fact that $\dfrac{dx}{dt} = u_1$ and $\dfrac{dy}{dt} = u_2$. If we evaluate this at $t = 0$, when $x = a$ and $y = b$ we find that the slope of the tangent is

$$g'(0) = f_x(a,b)u_1 + f_y(a,b)u_2.$$

This is the directional derivative of $f(x,y)$ at (a,b) in the direction of $\mathbf{u}$. As we expect, it agrees with the previous derivation.

For any given $\mathbf{u}$ the directional derivative can be written as a function of x and y instead of being evaluated at a point (a,b). That is

$$D_{\mathbf{u}}f(x,y) = f_x(x,y)u_1 + f_y(x,y)u_2$$

where u_1 and u_2 are the $\mathbf{i}$ and $\mathbf{j}$ components of the unit vector $\hat{\mathbf{u}}$ in the direction of $\mathbf{u}$.

Example 10.1a Find the directional derivative of $f(x,y) = x^3 - 3xy + 4y^2$ in the direction of $\mathbf{u} = 3\mathbf{i} + 4\mathbf{j}$. What is $D_{\mathbf{u}}f(2,1)$?

The partial derivatives of f are $f_x(x,y) = 3x^2 - 3y$ and $f_y(x,y) = -3x + 8y$. For the directional derivative we need $\hat{\mathbf{u}}$, the unit vector in the direction of $\mathbf{u}$.

$$\hat{\mathbf{u}} = \frac{\mathbf{u}}{|\mathbf{u}|} = \frac{3\mathbf{i} + 4\mathbf{j}}{\sqrt{4^2 + 3^2}} = \frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{j}.$$

The directional derivative of $f(x,y)$ in the direction of $3\mathbf{i} + 4\mathbf{j}$ is

$$\begin{aligned} D_{\mathbf{u}}f(x,y) &= f_x(x,y)u_1 + f_y(x,y)u_2 \\ &= \frac{3}{5}(3x^2 - 3y) + \frac{4}{5}(-3x + 8y) \\ &= \frac{1}{5}(9x^2 + 23y - 12x). \end{aligned}$$

To find $D_{\mathbf{u}}f(2,1)$ we evaluate $D_{\mathbf{u}}f(x,y)$ at $(2,1)$ to get $D_{\mathbf{u}}f(2,1) = 7$. ◇

Note: If $|\mathbf{u}| \neq 1$ it is essential to convert $\mathbf{u}$ into a unit vector in the same direction before calculating the directional derivative.

10.2 Finding directional derivatives using the gradient vector

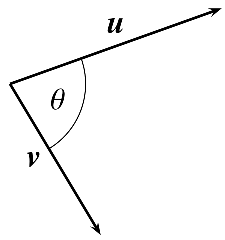
The "scalar product" or "dot product" of two vectors $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$ and $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j}$ is given by

$$\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2.$$

A scalar product takes two vectors and produces a scalar (that is, a real number) as output. There is an alternative definition of $\mathbf{u} \cdot \mathbf{v}$ which uses geometrical ideas:

$$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}|\,|\mathbf{v}| \cos\theta$$

where θ is the angle between the two vectors when placed tail to tail, and $|\mathbf{u}| = \sqrt{u_1^2 + u_2^2}$ and $|\mathbf{v}| = \sqrt{v_1^2 + v_2^2}$ are the magnitudes of $\mathbf{u}$ and $\mathbf{v}$.



If we look carefully at the expression for the directional derivative,

$$D_{\mathbf{u}}f(x,y) = f_x(x,y)u_1 + f_y(x,y)u_2$$

we see that the directional derivative looks like a scalar product of two vectors, $\hat{\mathbf{u}} = u_1\mathbf{i} + u_2\mathbf{j}$ and $f_x(x,y)\mathbf{i} + f_y(x,y)\mathbf{j}$. The latter vector is known as the "gradient vector" or the "gradient of f".

Gradient vector
If $f(x,y)$ is a differentiable function of two real variables, then the "gradient of f" is

$$\text{grad}f(x,y) = \nabla f(x,y) = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} = f_x(x,y)\mathbf{i} + f_y(x,y)\mathbf{j}.$$

Both grad$f$ and ∇f are used in mathematics textbooks. In this course, however, we will generally use the notation ∇f.

▷ **Aside** The symbol ∇ in known as "del" and represents a vector operator. A vector operator is essentially a mathematical rule. The rule for the del operator is sometimes written as

$$\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y}.$$

We can now easily write the directional derivative in terms of the gradient vector.

Directional derivative
Using the gradient vector, the directional derivative is expressed in the form

$$D_{\mathbf{u}}f(x,y) = \nabla f(x,y) \cdot \hat{\mathbf{u}}$$

where $\nabla f(x,y) = \text{grad}f(x,y)$ is the gradient of $f(x,y)$ and $\hat{\mathbf{u}}$ is a unit vector in the direction of $\mathbf{u}$.

▷ **Aside** In fact the directional derivative is most commonly written as a scalar product $\nabla f(x, y) \cdot \hat{\mathbf{u}}$ when it is used in applied mathematics. The notation $D_{\mathbf{u}} f$ is seldom used.

Examples 10.2a

i) Find $\nabla f(x, y)$ when $f(x, y) = e^x \sin y$. Hence find the gradient of f at $(0, 0)$.

The partial derivatives are $f_x(x, y) = e^x \sin y$ and $f_y(x, y) = e^x \cos y$. Hence

$$\nabla f = e^x \sin y\mathbf{i} + e^x \cos y\mathbf{j}.$$

Evaluating this expression at $(0, 0)$ gives $\nabla f(0, 0) = \mathbf{j}$.

ii) Find the directional derivative of $f(x, y) = x^3 - 3xy + 4y^2$ in the direction of $\mathbf{u} = 3\mathbf{i} + 4\mathbf{j}$.

This is the same as Example 8.1a, but we will use the gradient of $f(x, y)$ this time. We know already that $\hat{\mathbf{u}} = \frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{j}$. The partial derivatives are $f_x(x, y) = 3x^2 - 3y$ and $f_y(x, y) = -3x + 8y$ and so

$$\nabla f(x, y) = (3x^2 - 3y)\mathbf{i} + (-3x + 8y)\mathbf{j}.$$

Hence

$$\begin{aligned} D_{\mathbf{u}} f(x, y) &= \nabla f(x, y) \cdot \hat{\mathbf{u}} \\ &= \Big((3x^2 - 3y)\mathbf{i} + (-3x + 8y)\mathbf{j}\Big) \cdot \left(\frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{j}\right) \\ &= \frac{3}{5}(3x^2 - 3y) + \frac{4}{5}(-3x + 8y) \\ &= \frac{1}{5}(9x^2 + 23y - 12x). \end{aligned}$$

iii) Consider the function $g(x, y) = y/x$. Find $\nabla g(-1, 2)$ and hence find the directional derivative of g at $(-1, 2)$ in the direction of $\mathbf{v} = \mathbf{i} - \mathbf{j}$.

$$g_x(x, y) = \frac{-y}{x^2} \quad g_y(x, y) = \frac{1}{x},$$

so

$$g_x(-1, 2) = -2 \quad g_y(-1, 2) = -1.$$

Therefore $\nabla g(-1, 2) = -2\mathbf{i} - \mathbf{j}$. The directional derivative is given by

$$D_{\mathbf{v}} g(-1, 2) = \nabla g(-1, 2) \cdot \hat{\mathbf{v}}.$$

Since $\mathbf{v}$ is not a unit vector we must find $\hat{\mathbf{v}}$, the unit vector in the direction of $\mathbf{v}$. Now

$|\mathbf{v}| = \sqrt{1^2 + (-1)^2} = \sqrt{2}$ and so $\hat{\mathbf{v}} = \dfrac{1}{\sqrt{2}}\mathbf{i} - \dfrac{1}{\sqrt{2}}\mathbf{j}$. Hence

$$\begin{aligned} D_{\mathbf{v}}g(-1,2) &= \nabla g(-1,2) \cdot \hat{\mathbf{v}} \\ &= (-2\mathbf{i} - \mathbf{j}) \cdot \left(\frac{1}{\sqrt{2}}\mathbf{i} - \frac{1}{\sqrt{2}}\mathbf{j}\right) \\ &= \frac{-2}{\sqrt{2}} + \frac{1}{\sqrt{2}} \\ &= \frac{-1}{\sqrt{2}}. \end{aligned}$$

This means that the slope of the tangent to the surface $z = g(x, y) = \dfrac{y}{x}$ at $(-1, 2)$ in the direction of $\mathbf{i} - \mathbf{j}$ is $-1/\sqrt{2}$. ◇

> **Important**
> You should remember that the gradient of a function of two variables is always a *vector* and that the directional derivative is always a *scalar*.

10.3 Interpreting the gradient vector

The gradient vector does not represent the slope of a tangent or a straightforward rate of change, but it is a package of useful information. This information is easily extracted from the gradient vector and can be used in many different contexts. In fact, the gradient vector and the del operator are very widely used in applied mathematics, particularly in physics and engineering.

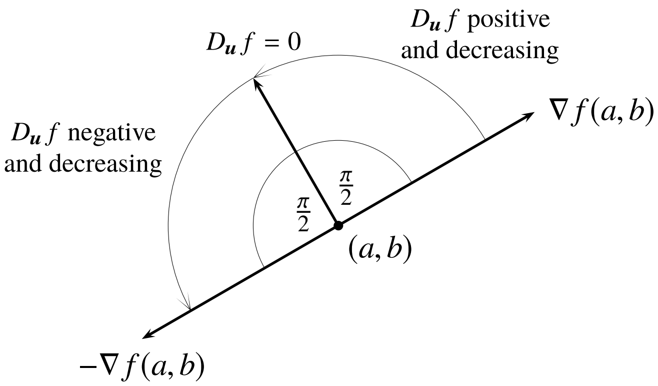
We have already seen that the gradient vector is useful in calculating the directional derivative. Using the geometric definition of the scalar product, the directional derivative at a point (a, b) can be written as

$$D_{\mathbf{u}}f(a,b) = \nabla f(a,b) \cdot \hat{\mathbf{u}} = |\nabla f|\,|\hat{\mathbf{u}}| \cos\theta,$$

where θ is the angle between ∇f and $\hat{\mathbf{u}}$. (Note that in this context θ does not have a positive or negative direction, unlike the argument of a complex number.) Now, $|\hat{\mathbf{u}}| = 1$ and so

$$D_{\mathbf{u}}f(a,b) = |\nabla f|\ \cos\theta.$$

What happens to $D_{\mathbf{u}}f(a,b)$ as we take the directional derivative in different directions, as θ varies? As θ increases from 0 to π, $\cos\theta$ decreases from 1 at $\theta = 0$ through 0 at $\theta = \pi/2$ to -1 at $\theta = \pi$. Therefore $D_{\mathbf{u}}f(a,b)$ will have its greatest positive value when $\mathbf{u}$ is in the same direction as $\nabla f(a,b)$. In this case, the directional derivative $D_{\mathbf{u}}f(a,b)$ is simply the magnitude of the gradient $|\nabla f(a,b)|$. If $\mathbf{u}$ is perpendicular to the gradient vector then $D_{\mathbf{u}}f(a,b) = 0$. The directional derivative is negative when $\pi/2 < \theta \leq \pi$. In particular, it will take its largest negative value when $\mathbf{u}$ is in the opposite direction to $\nabla f(a,b)$. These facts are illustrated in the following diagram:



Geometrically this means that, at a particular point on a surface, the tangent with the greatest positive slope lies in the vertical plane which contains the gradient vector. This greatest slope is equal to the magnitude of the gradient vector. The tangent with zero slope lies in the direction at right angles to the gradient vector and the tangent with the most negative slope lies in the opposite direction to the gradient direction and has slope $-|\nabla f(a,b)|$.

Examples 10.3a

i) Consider $f(x,y) = x^2y$ at the point $(3,-3)$. What is the direction of the maximum rate of change? What is the maximum rate of change?

   The direction of the maximum rate of change is given by the direction of the gradient vector:

   $$\nabla f(x,y) = 2xy\mathbf{i} + x^2\mathbf{j}. \quad \text{Hence } \nabla f(3,-3) = -18\mathbf{i} + 9\mathbf{j} = 9(-2\mathbf{i} + \mathbf{j}).$$

   The magnitude of the maximum change is given by the magnitude of the gradient vector:

   $$|\nabla f(3,-3)| = \sqrt{(-18)^2 + 9^2} = 9\sqrt{(-2)^2 + 1^2} = 9\sqrt{5}.$$

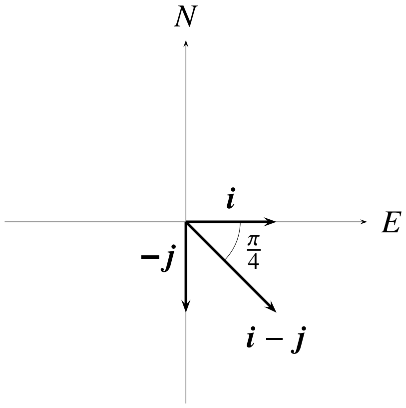
   Therefore the direction of greatest change at $(3,-3)$ is $-2\mathbf{i} + \mathbf{j}$ and the magnitude of greatest change is $9\sqrt{5}$.

ii) Find the gradient of $f(x,y) = y^3\cos x$ at $(0,2)$. Give the direction of greatest increase, of greatest decrease and of zero change.

   The gradient of $f(x,y)$ is $\nabla f(x,y) = -y^3\sin x\mathbf{i} + 3y^2\cos x\mathbf{j}$. Hence $\nabla f(0,2) = 12\mathbf{j}$. Therefore the direction of steepest increase is $12\mathbf{j}$ or just $\mathbf{j}$. The direction of greatest decrease is opposite to the direction of $\nabla f(0,2)$, that is $-\mathbf{j}$. The directional derivative is zero in the direction perpendicular to the direction of greatest increase, and so zero change occurs in the directions $\mathbf{i}$ or $-\mathbf{i}$.

iii) At a particular point on a mountain, the steepest gradient is 1 in 5 towards the south east. That is, for every five metres of horizontal travel your height above sea level increases by one metre. If you walk directly east how steeply will you be going up?

We first need to describe this problem in mathematical terms. Let the surface of the mountain be given by $z = f(x, y)$ where the positive x direction is due east and the positive y direction is due north. The direction of steepest climb is south east, which is in the direction of $\mathbf{i} - \mathbf{j}$. This means that $\nabla f(x, y)$ is in the direction of $\mathbf{i} - \mathbf{j}$.



The steepest climb is 1 in 5 or 0.2. Therefore $|\nabla f(x, y)| = 0.2$ at the particular point where we are standing. The direction due east, given by the vector $\mathbf{i}$, is at angle $\pi/4$ to the direction of the gradient vector. We want to find the directional derivative in this direction. This will be

$$|\nabla f(x, y)| \cos(\pi/4) = 0.2/\sqrt{2} = 1/(5\sqrt{2}) \approx 1/7.$$

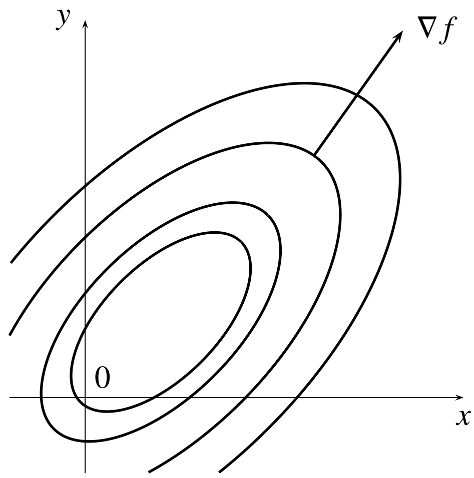
Therefore, the steepness of the climb due east is approximately 1 in 7.

◇

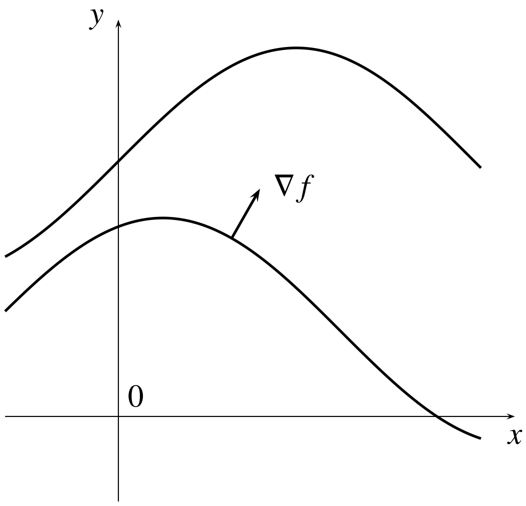
10.4 The gradient vector and level curves.

In Chapter 4 we introduced sets of level curves as a convenient way to visualise graphs of functions of two variables. If we have a surface $z = f(x, y)$ then the surface can be represented as a family of level curves of the form $f(x, y) = c$ where c is a constant in the range of f. Usually we choose a set of equally spaced z values and draw a "contour map" of the function.

The magnitude of the gradient vector gives the greatest rate of increase or the slope of the steepest tangent. Hence if $|\nabla f(x, y)|$ is large then the level curves will be close together. If $|\nabla f(x, y)|$ is small, the surface will be comparatively flat and the level curves will be far apart.



If $|\nabla f(x,y)|$ is small, the surface will be comparatively flat and the level curves will be far apart.



While the magnitude of the gradient is related to how close the level curves are from each other, what is the relation between the direction of the gradient vector and the level curves? Imagine you are walking in the mountains keeping the height constant and thus following a level curve. By opening your arms you will be pointing to the steepest slopes of the mountain, the direction of the gradient of the function $z = f(x,y)$. We thus conclude that:

> The direction of the gradient vector is normal to the level curves.

A formal derivation of this important result requires the concept of implicit differentiation introduced in Sec. 9.3). Each level curve can be thought of as the graph of a function of one variable $y = g(x)$ defined implicitly by the equation $f(x,y) = c$. Using implicit differentiation, the slope of the tangent to the level curve passing through (a,b) is

$$g'(a) = \frac{-f_x(a,b)}{f_y(a,b)}.$$

Hence the slope of the normal at the point (a, b) is

$$\frac{-1}{g'(a)} = \frac{f_y(a,b)}{f_x(a,b)}.$$

Therefore, a vector which is normal to $f(x, y) = c$ at (a, b) must have **i** and **j** components in the ratio $f_x(a, b)/f_y(a, b)$. One such vector is $f_x(a, b)\mathbf{i} + f_y(a, b)\mathbf{j}$. This is simply $\nabla f(a, b)$. Therefore the direction of the gradient vector at (a, b) is normal to the level curve $f(x, y) = c$ at (a, b).

Generally speaking, we can find a normal to any curve in the plane by writing the equation of the curve in the form $f(x, y) = c$ and then finding ∇f at that point. Furthermore, the directional derivative in a direction tangential to the level curve will be zero. (Remember level curves connect points at the same height above or below the xy plane.) As expected, this agrees with the result that the directional derivative is zero at angle $\pi/2$ to the gradient vector.

Examples 10.4a

i) Find a vector normal to the level curve of $z = x^2 + y^3$ when $x = -1$ and $y = 2$.

Let $f(x, y) = x^2 + y^3$. Then such a vector is given by $\nabla f(-1, 2)$.

$$\nabla f(x,y) = 2x\mathbf{i} + 3y^2\mathbf{j}, \quad \text{so} \quad \nabla f(-1,2) = -2\mathbf{i} + 12\mathbf{j} = 2(-\mathbf{i} + 6\mathbf{j}).$$

The vector $-\mathbf{i} + 6\mathbf{j}$ is normal to the level curve of $z = x^2 + y^3$ at $(-1, 2)$.

ii) Show that the curve $y^2 + y + 1 = e^{x^2}$ meets the y axis perpendicularly at $(0, 0)$.

Clearly the curve $y^2 + y + 1 = e^{x^2}$ intersects the y axis at $(0, 0)$.

Let $f(x, y) = y^2 + y + 1 - e^{x^2}$. Then $\nabla f(x, y) = -2xe^{x^2}\mathbf{i} + (2y + 1)\mathbf{j}$. Hence $\nabla f(0, 0) = \mathbf{j}$. This vector is normal to the curve and hence must be parallel to any line that is normal to the curve. Therefore the y axis is normal to the curve $y^2 + y + 1 = e^{x^2}$ at $(0, 0)$.

iii) Find a vector (in terms of x and y) normal to the level curves of $f(x, y) = y - x(x^2 - 1)$. Use this vector to find points where a level curve is tangential to the x axis in the xy plane.

If a curve is tangential to the x axis then, at the point of tangency, $y = 0$ and the normal to the curve must be parallel to the y axis. That is, the **i** component of the normal must be zero.

The gradient vector $\nabla f(x, y) = (-3x^2 + 1)\mathbf{i} + \mathbf{j}$. For the **i** component of this vector to be zero, $-3x^2 + 1 = 0$, or $x = \pm 1/\sqrt{3}$.

Hence the level curves of $f(x, y) = y - x(x^2 - 1)$ meet the x axis tangentially at $(-1/\sqrt{3}, 0)$ and $(1/\sqrt{3}, 0)$.

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Summary of Chapter 10

The gradient vector of $f(x, y)$ at the point (a, b) is

$$\nabla f(a, b) = f_x(a, b)\mathbf{i} + f_y(a, b)\mathbf{j}$$

which can be used to find the following:

- The *directional derivative* at (a, b) in the direction of $\mathbf{u}$ is given by $D_{\mathbf{u}}(a, b) = \nabla f(a, b) \cdot \hat{\mathbf{u}}$ where $\hat{\mathbf{u}}$ is the unit vector in the direction of $\mathbf{u}$.
- The *magnitude of steepest increase* of $f(x, y)$ at (a, b) is $|\nabla f(a, b)|$.
- The *direction of steepest increase* of $f(x, y)$ at (a, b) is $\nabla f(a, b)$. This can be used to find the direction of steepest decrease $-\nabla f(a, b)$ and the direction where there is no change (perpendicular to $\nabla f(a, b)$).
- The *normal* at (a, b) to the curve $f(x, y) = c$, where c is a constant, is given by $\nabla f(a, b)$.

Exercises

10.1 Calculate the directional derivatives of the following functions in the direction $\mathbf{u}$ and at the given point P.

a) $z = e^x \cos y$, $\mathbf{u} = 2\mathbf{i} + \mathbf{j}$ and $P = (0, 0)$.

b) $z = \ln \sqrt{x^2 + y^2}$, $\mathbf{u} = 3\mathbf{i} + 6\mathbf{j}$ and $P = (3, 4)$.

c) $z = x^2 + 2y^2$, $\mathbf{u} = \mathbf{i} + \mathbf{j}$ and $P = (1, 1)$.

d) $z = xy$, $\mathbf{u} = 10\mathbf{i} + 11\mathbf{j}$ and $P = (1, -1)$.

10.2 Consider a function $z = f(x, y)$, $f_x(a, b) = 1$ and $f_y(a, b) = -2$.

a) In what direction $\mathbf{u}$ does $D_{\mathbf{u}} f = 0$?

b) In which direction is $D_{\mathbf{u}} f$ largest?

10.3 Find the gradient of the following functions.

a) $f(x, y) = 2x^4 y^3$.

b) $f(x, y) = e^{y^2} \cos(x + 1)$.

c) $f(x, y) = 5x^2 y - \ln(x + y) + \sin x$.

10.4 For each of the following functions, use the gradient to find

(i) the direction of greatest slope at P,

(ii) the magnitude of the greatest slope at P,

(iii) the directional derivative at the point P in the given direction $\mathbf{u}$.

a) $f(x,y) = \sqrt{x^2 + y^2}$, $P = (1,2)$ and $\mathbf{u} = 3\mathbf{i} - \mathbf{j}$.

b) $f(x,y) = \sin^{-1}\left(\dfrac{x}{y}\right)$, $P = (1,2)$ and $\mathbf{u} = \mathbf{i} + 2\mathbf{j}$.

c) $f(x,y) = xy + 1$, $P = (-1,2)$ and $\mathbf{u} = -\mathbf{i} - \mathbf{j}$.

d) $f(x,y) = e^{x^2+y^2}$, $P = (1,0)$ and $\mathbf{u} = -2\mathbf{i} + 3\mathbf{j}$.

10.5 Using the gradient vector, find a normal vector to the following curves at the given point.

a) $x^2 + y^2 = 5$ at $P = (1,2)$.

b) $\sin x + \sin y = \frac{1+\sqrt{3}}{2}$ at $P = (\frac{\pi}{6}, \frac{\pi}{3})$.

c) $x^3 + y^3 = -9$ at $P = (-2,-1)$.

10.6 Let ϕ be the function $\phi(x,y) = ye^x + \cos x$.

a) Find $\nabla\phi$.

b) Find the directional derivative at $(0,1)$ in the direction of $2\mathbf{i} + \mathbf{j}$.

c) What is the direction of greatest change at $(0,1)$?

10.7 A lizard is crawling on a flat floor. When it crawls west it senses that the temperature increases at a rate of 0.04°C per centimetre. If it crawls north the temperature decreases at a rate of 0.01°C percentimetre.

a) What will be the rate of change if the lizard walks east?

b) If the lizard crawls north west what will be the rate of change?

c) If the lizard wants to remain at the same temperature in what direction should it move?

10.8 Show that the gradient operator has the following properties.

a) $\nabla(f+g) = \nabla f + \nabla g$.

b) $\nabla(\alpha f) = \alpha\nabla f$, where α is a constant.

c) $\nabla(fg) = f\nabla g + g\nabla f$.

CHAPTER 11

Second-Order Partial Derivatives and Continuity

11.1 Second-order partial derivatives

All the partial derivatives we calculated in the previous chapter are "first-order partial derivatives"; that is, each derivative was obtained by differentiating the function once only. Just as with ordinary derivatives, there are higher-order partial derivatives as well. To calculate second-order partial derivatives we just take partial derivatives of the first-order partial derivatives. These can be written using subscript notation, for example $f_{xx}(x, y)$, or using the special ∂ symbol, for example $\dfrac{\partial^2 f}{\partial x^2}$. The two notations are interchangeable and you will need to be familiar with both of them.

The four "second-order partial derivatives" of a function f of two variables x and y are

$$(f_x(x,y))_x = f_{xx}(x,y) = f_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2}$$

$$(f_x(x,y))_y = f_{xy}(x,y) = f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x}$$

$$(f_y(x,y))_x = f_{yx}(x,y) = f_{yx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y}$$

$$(f_y(x,y))_y = f_{yy}(x,y) = f_{yy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2}$$

The derivatives f_{xy} and f_{yx} are called "mixed derivatives".

Note that the order in which the differentiation is done is written differently in the two different notations. In the f_x type of notation, the first differentiation is the leftmost in the subscript. For example f_{xy} means differentiate f first with respect to x and then differentiate the result with respect to y. In the other notation the rightmost differentiation is performed first, so that $\dfrac{\partial^2 f}{\partial x \partial y}$ means to differentiate first with respect to y and then with respect to x.

Of course, having calculated second-order partial derivatives, we may then take partial derivatives of these to find third-order partial derivatives, and so on. We can calculate

$$f_{xxx},\ f_{yyy},\ f_{xyx},\ f_{yyx},\ f_{yxx},\ f_{xxy},\ f_{xyy},\ f_{yxy}.$$

Example 11.1a Consider $f(x, y) = \sin xy$. Check that

$$f_{xyx}(x, y) = -2y \sin xy - xy^2 \cos xy = f_{yxx}(x, y)$$

and also

$$f_{xxx}(x, y) = -y^3 \cos xy, \qquad f_{yyy}(x, y) = -x^3 \cos xy.$$

◇

Examples 11.1b

i) Find all second-order partial derivatives of $f(x, y) = 3x^7y^5 - 2x^4y$.

Now,

$$f_x(x, y) = 21x^6y^5 - 8x^3y$$

and so

$$\begin{aligned} f_{xx}(x, y) &= \frac{\partial}{\partial x}\left(21x^6y^5 - 8x^3y\right) \\ &= 126x^5y^5 - 24x^2y \end{aligned}$$

and

$$\begin{aligned} f_{xy}(x, y) &= \frac{\partial}{\partial y}\left(21x^6y^5 - 8x^3y\right) \\ &= 105x^6y^4 - 8x^3. \end{aligned}$$

Similarly,

$$f_y(x, y) = 15x^7y^4 - 2x^4$$

and so

$$f_{yx}(x, y) = 105x^6y^4 - 8x^3 \quad \text{and} \quad f_{yy}(x, y) = 60x^7y^3.$$

ii) Find all second-order partial derivatives of $f(x, y) = ye^{\sin x}$.

We have

$$f_x(x, y) = ye^{\sin x} \cos x$$

and so

$$\begin{aligned} f_{xx}(x,y) &= \frac{\partial}{\partial x}\left(ye^{\sin x}\cos x\right) \\ &= y\left(e^{\sin x}\cos^2 x - e^{\sin x}\sin x\right) \end{aligned}$$

and

$$\begin{aligned} f_{xy}(x,y) &= \frac{\partial}{\partial y}\left(ye^{\sin x}\cos x\right) \\ &= e^{\sin x}\cos x. \end{aligned}$$

Similarly,

$$f_y(x,y) = e^{\sin x}$$

and so

$$f_{yx}(x,y) = e^{\sin x}\cos x \qquad \text{and} \qquad f_{yy}(x,y) = 0.$$

◇

You will have noticed that in both these examples, $f_{xy} = f_{yx}$. This is no coincidence! It is always true that $f_{xy} = f_{yx}$ whenever the mixed partial derivatives of f satisfy certain *continuity conditions*. Continuity of a function of one variable is a simple concept to grasp intuitively if the domain is an interval; such a function is continuous if its graph can be drawn without lifting the pen from the paper.

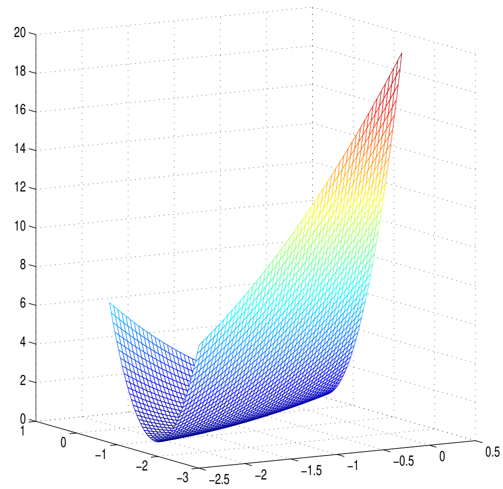
However the mathematical definition of continuity requires the notion of "limit", which will be extensively used below. Before we give the famous theorem in which the equality of the mixed partial derivatives is stated, let us spend some time understanding what is meant by the limit of a function of two variables and the continuity of functions of two variables.

11.2 Limits of functions of two variables

We'll start with an example. Consider the function $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ given by

$$f(x,y) = x^2 - 4xy + 2y + 5y^2 + 1.$$

The diagram below illustrates part of the surface.



What is the limit of $f(x, y)$ as (x, y) approaches $(0, 0)$? That is, what is

$$\lim_{(x,y)\to(0,0)} x^2 - 4xy + 2y + 5y^2 + 1\,?$$

The first thing to think about is what this limit means. Again, we can fall back on the intuitive definition of a limit discussed in the previous chapter.

Suppose $f(x, y)$ is defined at each point of an open disc containing the point (c, d), except possibly at (c, d) itself.
By our informal, intuitive definition, $\lim_{(x,y)\to(c,d)} x^2 - 4xy + 2y + 5y^2 + 1 = \ell$ if $f(x, y)$ is as close as we like to ℓ for all (x, y) sufficiently close to, but not equal to, (c, d).

Now, for fixed values of x and y, $f(x, y)$ is just a number; so $f(x, y)$ approaches ℓ if $|f(x, y) - \ell| < \epsilon$ whenever (x, y) is "close enough" to (c, d). The point (x, y) is close to (c, d) if the distance from (x, y) to (c, d) in the xy plane is "small"; that is if $\sqrt{(x - c)^2 + (y - d)^2}$ is small.

Putting these observations together we arrive at more precise definition of a limit for a function of two variables.

> **Limit of $f(x, y)$**
> Suppose that ℓ is a real number. Then the limit of $f(x, y)$, as (x, y) approaches (c, d), is equal to ℓ if for each $\varepsilon > 0$ there exists a number $\delta > 0$ such that
>
> $$|f(x, y) - \ell| < \varepsilon \quad \text{whenever} \quad 0 < \sqrt{(x - c)^2 + (y - d)^2} < \delta.$$

Notice that if we set $y = d$ then $\sqrt{(x-c)^2 + (y-d)^2} = \sqrt{(x-c)^2} = |x-c|$, so the definition of a limit for functions of two variables reduces to the definition for functions of one variable — as it should since we were guided by the same intuitive definition in both cases.

There are quite are few subtleties hidden in this definition. To see this let's look at the example $f(x, y) = x^2 - 4xy + 2y + 5y^2 + 1$, as (x, y) approaches $(0, 0)$, in more detail.

If $\lim\limits_{(x,y)\to(0,0)} f(x, y)$ exists then it should not matter how (x, y) approaches $(0, 0)$. For example, we could just set $x = 0$ and consider $\lim_{y\to0} f(0, y)$. We could also set $y = 0$ and consider $\lim_{x\to0} f(x, 0)$; or we could set $x = y$ and consider $\lim_{x\to0} f(x, x)$.

If $\lim\limits_{(x,y)\to(0,0)} f(x, y)$ exists then *all of these limits must be the same.* In fact, all of these limits are equal in this case — as the following calculations show.

$$(x = 0) \qquad \lim_{(x,y)\to(0,0)} f(x, y) = \lim_{y\to0} f(0, y) = \lim_{y\to0} 2y + 5y^2 + 1 = 1$$

$$(y = 0) \qquad \lim_{(x,y)\to(0,0)} f(x, y) = \lim_{x\to0} f(x, 0) = \lim_{x\to0} x^2 + 1 = 1$$

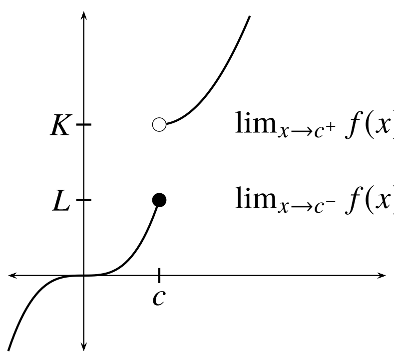
$$(x = y) \qquad \lim_{(x,y)\to(0,0)} f(x, y) = \lim_{x\to0} f(x, x) = \lim_{x\to0} x^2 - 4x^2 + 2x + 5x^2 + 1 = 1$$

More generally, you can check that for any $m \in \mathbb{R}$ if (x, y) approaches $(0, 0)$ along the line $y = mx$ then $\lim\limits_{(x,y)\to(0,0)} f(x, y) = 1$.

Indeed, even more than this is true: if a limit exists then we must be able to compute the value of the limit by letting (x, y) travel along *any* path that approaches the point $(0, 0)$. For this particular function, $\lim_{(x,y)\to(0,0)} f(x, y)$ does actually exist and is equal to 1 — the same value as all of the limits above.

Of course, in general we cannot prove that $f(x, y)$ has a limit at point (c, d) simply by testing many different paths to the point, as there are infinitely many ways of letting (x, y) approach (c, d).

Example 11.2a With functions of one variable we also have to worry about what happens to the limit $\lim_{x\to c} f(x)$ as we approach $x = c$ from different directions; however, such limits are much easier to understand because we can only approach c from the left (that is, $x < c$) and from the right (that is, $x > c$). For example, consider the function $f(x)$ which has the following graph.



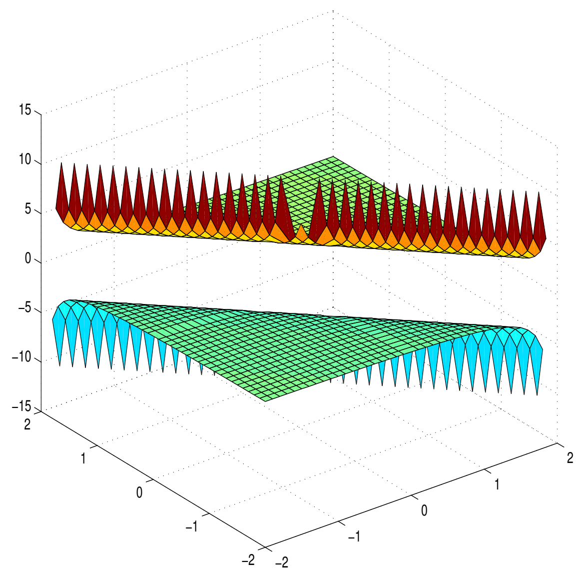
Suppose that we wanted to compute $\lim_{x\to c} f(x)$. If x approaches c along the x–axis from the *left* then $\lim_{x\to c^-} f(x) = L$; whereas, if x approaches c along the x–axis from the *right* then $\lim_{x\to c^+} f(x) = K$. Here we have used c^- and c^+ to indicate that x is approaches c from the left and right, respectively. As $L \neq K$ it follows that $\lim_{x\to c} f(x)$ does not exist. $\diamond$

Examples 11.2b

i) Convince yourself that $\lim_{(x,y)\to(c,d)} x = c$ and $\lim_{(x,y)\to(c,d)} y = d$.

ii) Now consider the limit
$$\lim_{(x,y)\to(0,0)} \frac{2x^2 + 3y^2}{x^2 + y^2}.$$
We claim that this limit does not exist. If this limit did exist then we would have to get the same value for the limit no matter how we let (x, y) approach $(0, 0)$; so let's compare what happens when $x = 0$ and when $y = 0$. When $x = 0$ the limit becomes $\lim_{y\to 0} \dfrac{3y^2}{y^2} = 3$; whereas when $y = 0$ we have $\lim_{x\to 0} \dfrac{2x^2}{x^2} = 2$. As we get two different answers, the original limit does not exist!

iii) Next consider the function
$$f(x, y) = \frac{1}{x + y}.$$
This function has domain $\{ (x, y) \in \mathbb{R}^2 \mid x + y \neq 0 \}$; that is, the plane minus the line $y = -x$. A diagram of part of the surface is given below.



Suppose that $(c, d) = (1, -1)$. Does $\lim_{(x,y)\to(1,-1)} f(x, y)$ exist? First we note that the point $(1, -1)$ is outside the domain of the function because it lies in the line $y = -x$ at which the function is not defined because of a zero in the denominator. This alone does not tell us anything about the existance of the limit, but it hints that the function diverges for points close to (c, d). Indeed, if (x, y) approaches $(1, -1)$ along any path in the domain which is "above" $(1,-1)$, in the sense that we always have $x + y > 0$, then values of $f(x, y)$ become extremely large and positive since $\dfrac{1}{x+y} > 0$ and the denominator is becoming increasingly small. On the other hand, if (x, y) approaches $(1,-1)$ along any path in the domain from "below", in the sense that $x + y < 0$, then values of $f(x, y)$ become extremely large and negative (since $\dfrac{1}{x+y}$ is always less than zero in this case). Therefore, we conclude that $\lim_{(x,y)\to(1,-1)} f(x, y)$ does not exist.

iv) We now give an example of a two-variable function which has a limit at a point not in its domain.

Consider the function given by $f(x, y) = \dfrac{x^4 + y^4}{x^2 + y^2}$. The point $(0, 0)$ is not in the domain but the function has limit 0 as $(x, y) \longrightarrow (0, 0)$. To see why, we use the device of changing to *polar coordinates*.

Recall that the position of any point $P(x, y)$ not situated at the origin can be described in terms of the distance r from the origin O to P and the angle θ which is measured from the positive x axis to the ray OP. Basic trigonometry shows that $x = r\cos\theta$ and $y = r\sin\theta$, and these relationships hold in any quadrant.

So, for any $(x, y) \neq (0, 0)$, let $x = r\cos\theta$ and $y = r\sin\theta$. Then

$$f(x, y) = \frac{x^4 + y^4}{x^2 + y^2} = \frac{r^4(\cos^4\theta + \sin^4\theta)}{r^2(\cos^2\theta + \sin^2\theta)} = r^2(\cos^4\theta + \sin^4\theta).$$

On any path, as $(x, y) \longrightarrow (0, 0)$, we must have $r \longrightarrow 0$, irrespective of the behaviour of θ. The value of $\cos^4\theta + \sin^4\theta$ lies in the interval $[0, 2]$ (for any θ), and is being multiplied by the quantity r^2 which is decreasing to zero. Hence as $r \longrightarrow 0$, $r^2(\cos^4\theta + \sin^4\theta) \longrightarrow 0$ and therefore $f(x, y) \longrightarrow 0$. (The argument we have employed here relies on a limit law known as the "squeeze law", which is discussed in the MATH1021 NOTES.)

◇

11.3 Continuity and differentiability of functions of two variables

We have seen that for functions of one and two variables, the intuitive notion of the limit of a function at a point is the unique number ℓ such that the function values can be made as close as we please to ℓ by selecting domain elements sufficiently close to the point, in any direction.

It is important to stress that when taking limits of a function, the point in question *need not be in the domain of the function* and that we are interested only in the tendency of the function *around the point*.

However, when the point in question *is* in the domain of the function, then the obvious questions arise: if $\lim_{x\to c} f(x)$ exists, does $\lim_{x\to c} f(x) = f(c)$ and if $\lim_{(x,y)\to(c,d)} f(x, y)$ exists, does $\lim_{(x,y)\to(c,d)} f(x, y) = f(c, d)$?

If the answer to these questions is yes, we say the functions are "continuous at the point".

Continuity of $f(x, y)$
A real valued function f of one real variable is said to be "continuous at the point" c in its domain if $\lim_{x \to c} f(x)$ exists and equals $f(c)$.

A real valued function f of two real variables is said to be "continuous at the point" (c, d) in its domain if $\lim_{(x,y)\to(c,d)} f(x, y)$ exists and equals $f(c, d)$.

Functions which are continuous at every point of their domains are said to be "continuous functions".

Most of the functions you will come across in this course are continuous functions. This is because the elementary functions e^x, $\ln x$, $\sin x$, $\cos x$, $\tan x$, x^a and so on are all continuous on their natural domains, as are the functions we create by adding, multiplying, dividing and composing such functions. We end this section with some more examples using functions of two variables.

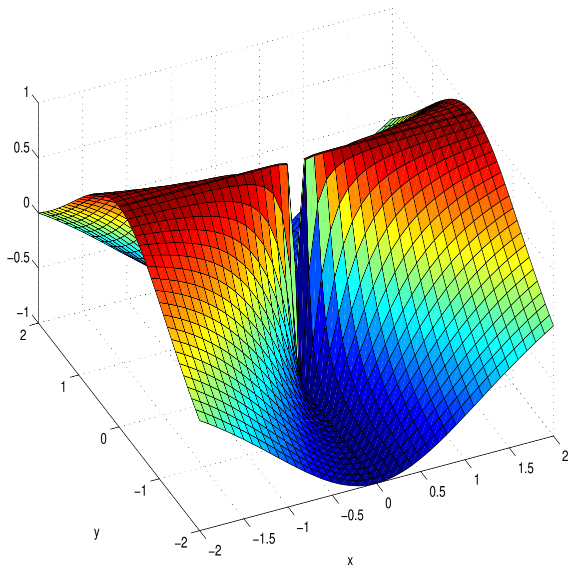
Example 11.3a Consider the function defined on $\mathbb{R}^2$ by

$$f(x, y) = \begin{cases} \dfrac{x^2 - 4y^2}{x - 2y}, & \text{if } x \neq 2y, \\ 4y, & \text{if } x = 2y. \end{cases}$$

Although this function is defined by different formulas in two different cases, it is indeed a continuous function on all of $\mathbb{R}^2$. To understand why, observe that $\dfrac{x^2 - 4y^2}{x - 2y}$ can be simplified to $\dfrac{(x - 2y)(x + 2y)}{x - 2y} = x + 2y$. That is, whenever (x, y) does *not* lie on the line $x = 2y$, the formula for the function is $f(x, y) = x + 2y$. On the other hand, when (x, y) *does* lie on the line $x = 2y$, then the formula is $f(x, y) = 4y = 2y + 2y = x + 2y$ (since $x = 2y$).

Therefore for all $(x, y) \in \mathbb{R}^2$, the function is given by the formula $f(x, y) = x + 2y$. Since this is a polynomial in x and y, the function f is also continuous on its domain. ◇

Example 11.3b Consider the function $f(x, y) = \dfrac{x^2 - y^2}{x^2 + y^2}$. This is a very interesting function. Note that $(0, 0)$ is the only point at which $f(x, y)$ is not defined. Is it possible to extend $f(x, y)$ to a continuous function on $\mathbb{R}^2$? That is, can we find some constant k such that $f(0, 0) = k$, so that f is continuous on $\mathbb{R}^2$?



Consider the behaviour of $f(x, y)$ as (x, y) approaches $(0, 0)$ along the line $y = mx$. If $y = mx$ then

$$f(x, y) = f(x, mx) = \frac{x^2 - m^2x^2}{x^2 + m^2x^2} = \frac{1 - m^2}{1 + m^2},$$

except when $(x, y) = (0, 0)$. Therefore, if we let (x, y) approach the point $(0, 0)$ by moving along the line $y = mx$ then

$$\lim_{\substack{(x,y)\to(0,0)\\ y=mx}} f(x, y) = \lim_{x\to 0} \frac{1 - m^2}{1 + m^2} = \frac{1 - m^2}{1 + m^2}.$$

So the limit depends on the slope of the line $y = mx$. In particular, different lines give different values for the limit, so $\lim_{(x,y)\to(0,0)} f(x, y)$ does not exist. (If this is not clear go back and review the section on limits of functions of two variables.) Consequently, there is no way that we can extend $f(x, y)$ to a continuous function on $\mathbb{R}^2$ because no number k has the property that $\lim_{(x,y)\to(0,0)} f(x, y) = k$. $\diamond$

Differentiable functions

Besides continuity, another important property of functions is *differentiability*. In functions of a single variable we thought of *differentiable* functions as functions with smooth graphs, which meant that the tangent line at a point stayed close to the curve of the function if we zoomed in close to that point. More precisely, we say that a function f of one variable is *differentiable at the point a* if $f'(a)$ exists. These ideas can be generalized to functions of two (or more) variables.

Differentiability of $f(x, y)$
A continuous function $f(x, y)$ is differentiable if it is *locally linear*, i.e. its graph looks more and more like its tangent plane if we zoom in on a point.
More formally, let $L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$ be the linearisation of $f(x, y)$ at the point (a, b). Then we say $f(x, y)$ is **differentiable** at (a, b) if

$$f(x, y) = L(x, y) + \epsilon_1(x - a) + \epsilon_2(y - b)$$

where ϵ_1 and ϵ_2 go to 0 as $(x, y) \to (a, b)$.

Notice that $z = L(x, y)$ in the above definition is just the equation of the tangent plane at (a, b). This definition implies that a differentiable function is one for which tangent plane approximates the graph well at the point of tangency. However, this can be a difficult definition to use in practice. Fortunately, there is a nice theorem that provides us with a sufficient condition to check for differentiability:

If the partial derivatives f_x and f_y exist near (a, b) and are continuous at (a, b), then f is differentiable at (a, b).

The problem of verifying the differentiability of f (at a given point) is thus reduced to the problem of verifying the continuity of its partial derivatives f_x and f_y (at the given point). Many of the results discussed in the previous chapters, such as the existance of linear approximations and the generalizations of the chain rule, are valid for differentiable functions only.

11.4 Equality of mixed derivatives

Recall that after completing some worked examples earlier in the chapter, using specific functions of two variables, we noticed that the mixed derivatives were equal: $f_{xy} = f_{yx}$. This result extends to all well-behaved functions of two variables! The crux of the matter is that the mixed derivatives are *continuous* at each point in some open region of the xy plane.

The Mixed Derivatives Theorem
Suppose that the domain of f is a region containing the point (a, b). If the functions f_{xy} and f_{yx} are both continuous in the region, then $f_{xy}(a, b) = f_{yx}(a, b)$.

This theorem guarantees that (provided the continuity conditions hold) the order of differentiation when calculating mixed derivatives is unimportant. You may choose to differentiate first with respect to x or with respect to y, whichever is easiest. The conditions of the Mixed Derivatives Theorem are true for almost all functions that you will see in this course.

Example 11.4a Confirm the conclusions of the Mixed Derivatives Theorem for the function given by $f(x, y) = \ln(x^2 + y)$.

We have $f_x = \dfrac{2x}{x^2 + y}$ and $f_y = \dfrac{1}{x^2 + y}$. Hence

$$f_{xy} = \frac{\partial}{\partial y}\left(\frac{2x}{x^2 + y}\right) = -\frac{2x}{(x^2 + y)^2}$$

and

$$f_{yx} = \frac{\partial}{\partial x}\left(\frac{1}{x^2 + y}\right) = -\frac{2x}{(x^2 + y)^2} = f_{xy}.$$

◇

Find the mixed derivatives of the function $f(x, y) = e^{x^2 - y^3}$. Do the conditions of the Mixed Derivatives Theorem hold for this function? Are the second-order mixed derivatives equal?

Summary of Chapter 11

- We defined the four second-order partial derivatives of $f(x, y)$

$$f_{xx}(x,y) = \frac{\partial^2 f}{\partial x^2}, \qquad f_{yy}(x,y) = \frac{\partial^2 f}{\partial y^2}$$

$$f_{xy}(x,y) = \frac{\partial^2 f}{\partial y \partial x} \qquad f_{yx}(x,y) = \frac{\partial^2 f}{\partial x \partial y}$$

- Limits, continuity, and differentibability of functions of two variables were introduced as a generalization of the one variable case.

- Strategies to evaluate the continuity of a function $f(x, y)$:

  - To show that $f(x, y)$ is **not** continuous at the point (a, b) it is enough to find two directions $(x, y) \to (a, b)$ such that

$$\lim_{\substack{(x,y)\to(a,b)\\ \text{direction 1}}} f(x,y) \neq \lim_{\substack{(x,y)\to(a,b)\\ \text{direction 2}}} f(x,y)$$

  - To show that $f(x, y)$ is continuous you can use the fact that elementary functions are continuous. You may need to simplify the original expression of $f(x, y)$ (e.g., use polar coordinates).

- The mixed derivatives theorem says that

$$f_{xy}(x,y) = f_{yx}(x,y)$$

provided the functions $f_{xy}(x, y)$ and $f_{yx}(x, y)$ are continuous in a region containing the point (x, y).

Exercises

11.1 Find all the first and second-order partial derivatives of the function,

$$R(x, y) = y^2 \ln x + \sin(x + 3y).$$

[There are six derivatives to calculate, namely R_x, R_y, R_{xx}, R_{xy}, R_{yx} and R_{yy}. You should get $R_{xy} = R_{yx}$.]

11.2 Find all the second-order partial derivatives of the function

$$f(x, y) = x \sin y - e^{xy},$$

and verify that $\dfrac{\partial^2 f}{\partial xy} = \dfrac{\partial^2 f}{\partial yx}$.

11.3 The equation

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$$

is called Laplace's equation in two dimensions. It arises in many areas of natural science. Solutions are called *harmonic functions*. Show that the following functions are harmonic.

a) $f(x, y) = e^{ax} \sin(ay)$

b) $f(x, y) = \ln \sqrt{x^2 + y^2}$.

11.4 The equation

$$\frac{\partial f}{\partial t} = K \frac{\partial^2 f}{\partial x^2}$$

is the heat equation or the diffusion equation describing the conduction of heat through a material or the diffusion of a substance through a fluid.

Show that the following functions satisfy the heat equation

a) $f(x, t) = t^{-\frac{1}{2}} e^{-ax^2/t}$ if $a = \dfrac{1}{4K}$

b) $f(x, t) = e^{-at} \sin(nx)$ if $a = n^2 K$.

11.5 Let $f(x, y)$ be any function with continuous n^{th}-order partial derivatives. How many of the n^{th}-order partial derivatives are distinct in general?
(Hint: First look at the number of distinct third partial derivatives.)

11.6 For each of the functions f whose formulas appear below, find all points in the plane where f is not defined. Is it possible to extend the domain of f to the entire plane (by defining values of $f(x, y)$ at points where it is currently undefined) in such a way that the 'new' function is continuous on the entire plane?

a) $f(x, y) = \dfrac{1}{xy}$

b) $f(x, y) = \begin{cases} 1, & \text{for } x > 0 \\ 0, & \text{for } x < 0 \end{cases}$

c) $f(x, y) = \ln|x + 2y|$

d) $f(x, y) = \dfrac{x^2 - y^2}{x - y}$

e) $f(x, y) = \dfrac{1}{y - 2x^2}$

f) $f(x, y) = \dfrac{x}{x^2 + y^2}$

g) $f(x, y) = \dfrac{x^3 + xy^2}{x^2 + y^2}$

h) $f(x, y) = \dfrac{3x^2 + 2y^2}{x^2 + y^2}$

11.7 Let $f(x, y) = \dfrac{xy}{x^4 + 2y^2}$.

a) Find the domain of $f(x, y)$.

b) What is the limit as $(x, y) \to (0, 0)$ along the line $y = x$?

c) What is the limit as $(x, y) \to (0, 0)$ along the line $y = 2x$?

d) Does $\lim_{(x,y)\to(0,0)} f(x, y)$ exist? Why or why not?

11.8 Show, by finding $\lim_{(x,y)\to(0,0)} f(x, y)$ along 2 paths of the form $y = mx$, that the function

$$f(x, y) = \begin{cases} 0 & \text{when } (x, y) = (0, 0) \\ \dfrac{3x^3 y}{x^6 + y^4} & \text{otherwise,} \end{cases}$$

is discontinuous at $(0, 0)$.

11.9 Sketch the graphs of the functions specified by the formulas. For each function, state whether $\lim_{x\to 1} f(x)$ exists, and if it does, find its value.

a) $f(x) = |x - 1|$

b) $f(x) = \begin{cases} 0, & \text{when } x = 1, \\ x, & \text{when } x \neq 1. \end{cases}$

CHAPTER 12

Optimizing Functions of Two Variables

Optimizing functions of one variable is an extremely useful application of calculus. The second derivative test identifies local maxima and minima, that can be used to determine global maxima and minima. This Chapter shows how these ideas can be generalized to functions of two variables. It is essential that you revise the standard tests for optimizing functions of one variable so we can tackle the more complicated case of testing for maxima and minima of functions of two variables.

12.1 Optimizing functions of two variables

Suppose that f is a function of two variables with domain some region A of the xy-plane. (You can think of the region A as a circular disc, a rectangle, or any simple, single area.)

Global extrema
A function f has a **global maximum** at the point $(a, b) \in A$ if $f(x, y) \leq f(a, b)$ for all (x, y) in its domain.
Similarly, f has a **global minimum** at the point $(a, b) \in A$ if $f(x, y) \geq f(a, b)$ for all (x, y) in its domain.

Local extrema
A function f has a **local maximum** (or **relative maximum**) at the point $(a, b) \in A$ if $f(x, y) \leq f(a, b)$ for all (x, y) in some open disc around (a, b), that is, when (x, y) is **near** the point (a, b).
Similarly, f has a **local minimum** (or **relative minimum**) at the point $(a, b) \in A$ if $f(x, y) \geq f(a, b)$ for all (x, y) in some open disc around (a, b).

Compare this with the same definition for functions of one variable (given in the MATH1021 NOTES). You'll notice that the concepts of "maximum" and "minimum" values are still the same; the difference comes in the requirement that the inequality holds in an open disc about (a, b) rather than an open interval on the real number line about a.

This is the natural progression from a 1-dimensional setting (the real number line) to a 2-dimensional setting (the xy plane), as we change from functions of one variable to functions of two variables.

Occasionally we can find information about maxima and minima of functions of two variables without any fancy techniques or formulas.

Example 12.1a Consider the function with domain $\mathbb{R}^2$ given by the formula

$$f(x, y) = x^2 - 4xy + 2y + 5y^2 + 1.$$

This formula can be simplified by completing the squares, to give

$$f(x, y) = (x - 2y)^2 + y^2 + 2y + 1 = (x - 2y)^2 + (y + 1)^2.$$

Therefore $f(x, y)$ is the sum of two squares and can never be negative. It achieves a global and local minimum value of 0 when $x - 2y = 0$ and $y + 1 = 0$ simultaneously, that is, at the point $(-2, -1)$. By keeping x fixed and taking y arbitrarily large it is clear that $(x-2y)^2$ and $(y+1)^2$, and hence $f(x, y)$, will be arbitrarily large, so f has no global maximum value. ◇

However for more complicated functions of two variables we need techniques involving the first and second partial derivatives.

Imagine a surface with a local maximum (which is shaped like a mountain top) or local minimum (shaped like the bottom of a bowl). Let us say that the surface is the graph of a function $f(x, y)$ and the maximum, or minimum is at the point (a, b). The plane tangent to the surface $z = f(x, y)$ at the point (a, b) is horizontal. Recall the equation of the tangent plane to the surface $z = f(x, y)$ at the point (a, b). It is

$$z - f(a, b) = f_x(a, b)\,(x - a) + f_y(a, b)\,(y - b).$$

The tangent plane at the point (a, b) where $f(x, y)$ has a local minimum or a local maximum is horizontal when its equation reduces to the simple form z = constant. This occurs when there are no terms in x and y in the equation, that is, when $f_x(a, b) = f_y(a, b) = 0$; in this case the tangent plane has equation $z = f(a, b)$.

Conditions for local extrema
If f is a function of two variables and has a local maximum or a local minimum at (a, b) then $f_x(a, b) = 0$ and $f_y(a, b) = 0$.

12.2 Critical points for functions of two variables

Here we extend the definition of critical points for functions of one variable, to the two-variable case for a function f whose domain is some region A of the xy plane.

> **Critical point**
> A point (a, b) in A where both $f_x(a, b) = 0$ and $f_y(a, b) = 0$ is called a "critical point" of the function f.
> A point where at least one of these first partial derivatives fails to exist is also called a critical point.

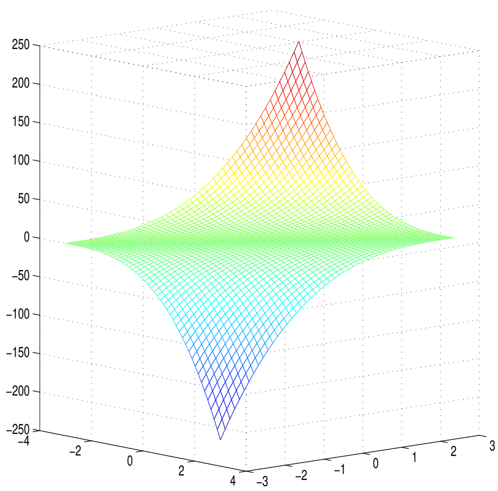
All points which give local maxima and minima of a continuous function of two variables are critical points. However, not all critical points are local maxima or minima. A saddle point, such as that illustrated in Example 7.7a(iii) is also a critical point, for instance.

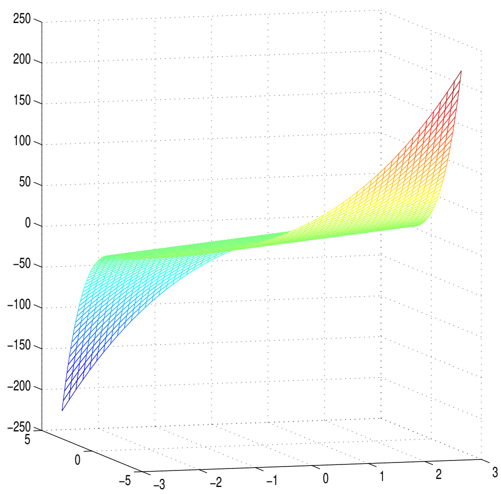
In that example, the formula is $f(x, y) = x^2 - y^2$, from which we obtain

$$f_x(x, y) = 2x \quad \text{and} \quad f_y(x, y) = -2y.$$

Thus $(0, 0)$ is the only critical point.

As with functions of one variable, there are other types of points where the first partial derivatives are zero. For example, the function giving the surface $z = (x - y)^3$ has a line of critical points. To see this, we calculate $f_x(x, y) = 3(x - y)^2$ and $f_y(x, y) = -3(x - y)^2$. The critical points are all points of the form (x, x), and these lie in the xy plane along the line $y = x$, with a z-value of 0. These points are analogous to horizontal points of inflection for a function of one variable. Two views of this surface from different positions are given below.





12.3 Test for maxima and minima

A test exists to determine the nature of a critical point of a function of two variables, although as you might expect, it is not as simple as the second derivative test in the one variable case.

Before we state the test, let's try to visualise a surface with a local minimum at (a, b) – think of the bottom of a bowl in space. If we keep y constant ($y = b$) and take a vertical cross-section through the surface $z = f(x, y)$ parallel to the x axis, the curve of intersection has a local minimum at $x = a$. To the left of a the slopes of the tangents $f_x(x, b)$ are negative, and to the right of a they are positive.

Therefore the slopes of the tangents are increasing as we move from left to right through a, that is $f_{xx}(a, b) > 0$. If we keep x fixed ($x = a$) and take a vertical cross-section parallel to the y axis, again the curve of intersection has a local minimum at $y = b$ and a similar conclusion is drawn, namely $f_{yy}(a, b) > 0$. In other words, at a point (a, b) where f has a minimum, both f_{xx} and f_{yy} are positive.

We can repeat this line of reasoning for a maximum point, and deduce that at such a point both f_{xx} and f_{yy} are negative.

Might this type of condition form our new second derivative test for a minimum/maximum? Almost, but not quite! It turns out that these conditions on f_{xx} and f_{yy} are not enough to guarantee a minimum/maximum and we need to involve the mixed partial derivative f_{xy} at (a, b) as well. The test requires the calculation of an expression known as the discriminant of f.

Discriminant of $f(x, y)$
The "discriminant" D of f at the point (a, b) is defined to be the expression

$$D(a, b) = f_{xx}(a, b) f_{yy}(a, b) - \left(f_{xy}(a, b)\right)^2 .$$

The second derivative test can now be stated:

> **Second derivative test**
> Let $f(x, y)$ be a function of two variables which has continuous second-order partial derivatives. Suppose (a, b) is a point such that $f_x(a, b) = f_y(a, b) = 0$.
> a) If the discriminant $D(a, b)$ is positive and $f_{xx}(a, b) > 0$ then there is a local minimum at (a, b).
> b) If the discriminant $D(a, b)$ is positive and $f_{xx}(a, b) < 0$ then there is a local maximum at (a, b).
> c) If the discriminant $D(a, b)$ is negative then there is a saddle point at (a, b).
> d) If the discriminant $D(a, b)$ is zero then we can't draw any conclusion without further work.

If you read the wording of this test carefully, you'll notice that it appears to give special prominence to the sign of $f_{xx}(a, b)$ while ignoring the sign of $f_{yy}(a, b)$ completely. However we have seen that at a minimum or maximum point, these particular second partial derivatives have the same sign and so only one is needed in the statement of the test.

Note that it is not possible for $D(a, b)$ to be positive and $f_{xx}(a, b) = 0$, for if $f_{xx}(a, b) = 0$ then $D(a, b) = -\left(f_{xy}(a, b)\right)^2 \le 0$.

Example 12.3a Find the discriminant D of f at each of its critical points when $f(x, y) = x^4 + y^4 - 4xy + 1$, and hence identify the critical points as local maxima, local minima or saddle points, if possible.

First we calculate the first and second partial derivatives.

$$\begin{aligned} f_x(x, y) &= 4x^3 - 4y, \\ f_y(x, y) &= 4y^3 - 4x, \\ f_{xx}(x, y) &= 12x^2, \\ f_{yy}(x, y) &= 12y^2, \\ f_{xy}(x, y) &= f_{yx}(x, y) = -4. \end{aligned}$$

To find critical points we solve $x^3 - y = 0$ and $y^3 - x = 0$ simultaneously. Substituting $y = x^3$ into the second equation gives

$$x^9 - x = x(x^8 - 1) = x(x^4 - 1)(x^4 + 1) = x(x^2 - 1)(x^2 + 1)(x^4 + 1) = 0,$$

which implies that $x = 0$ or $x = \pm 1$ (since x is real). Then since $y = x^3$, we must have $y = 0$ or $y = \pm 1$, respectively. There are three critical points, $(0, 0)$, $(1, 1)$ and $(-1, -1)$. At the point $(0, 0)$, all the above partial derivatives are 0 except for the last, $f_{xy}(0, 0) = -4$. Therefore

$$D(0, 0) = f_{xx}(0, 0) f_{yy}(0, 0) - \left(f_{xy}(0, 0)\right)^2 = -(-4)^2 = -16 < 0.$$

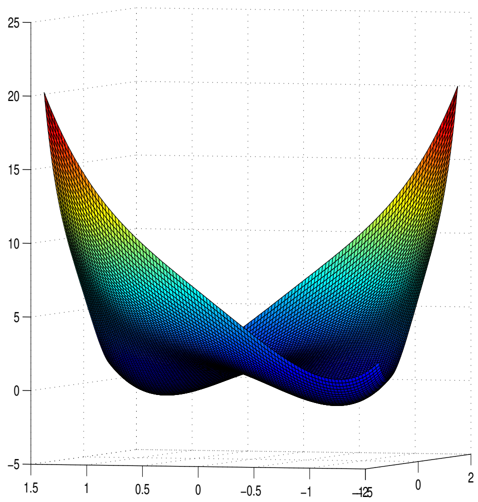
The test shows that there is a saddle point at $(0, 0)$.

At the point $(1, 1)$, we have $f_{xx}(1, 1) = f_{yy}(1, 1) = 12$ and $f_{xy}(1, 1) = -4$.

$$D(1, 1) = f_{xx}(1, 1)f_{yy}(1, 1) - \left(f_{xy}(1, 1)\right)^2 = 12 \times 12 - (-4)^2 = 128,$$

and since $f_{xx}(1, 1) = 12 > 0$ there must be a local minimum at $(1, 1)$. Similarly, you can confirm that there is also a local minimum at $(-1, -1)$.

The diagram below gives an illustration of the surface, showing the two local minima and the saddle point.



◇

The second derivative test may also be used to help calculate the shortest distance from a fixed point to a surface in space, as the next example shows.

Example 12.3b Find the shortest distance from the point with coordinates $(1, 4, 2)$ to the surface given by $z = x + y - 1$.

First note that the point does not lie on the surface (which in this case is a plane) so the shortest distance is not zero. If (x, y, z) is a point on the surface, then the distance d from $(1, 4, 2)$ to (x, y, z) is

$$d = \sqrt{(x-1)^2 + (y-4)^2 + (z-2)^2}.$$

Minimizing d is equivalent to minimizing d^2, which enables the algebra to be simplified.

Hence we may state the problem as follows: find the minimum value of $f(x, y) = (x-1)^2 + (y-4)^2 + (z-2)^2$ given that $z = x + y - 1$.

We substitute $z = x + y - 1$ into the expression for $f(x, y)$ and simplify:

$$\begin{aligned} f(x,y) &= (x-1)^2 + (y-4)^2 + (z-2)^2 \\ &= (x-1)^2 + (y-4)^2 + (x+y-3)^2 \\ &= x^2 - 2x + 1 + y^2 - 8y + 16 + x^2 + y^2 - 6y + 9 + 2xy - 6x \\ &= 2x^2 + 2y^2 + 2xy - 8x - 14y + 26 \end{aligned}$$

So our task is to find the minimum value of $f(x, y) = 2x^2 + 2y^2 + 2xy - 8x - 14y + 26$. First we calculate the partial derivatives.

$$f_x(x,y) = 4x + 2y - 8, \qquad f_y(x,y) = 4y + 2x - 14$$

$$f_{xx}(x,y) = 4, \qquad f_{yy}(x,y) = 4, \qquad f_{xy}(x,y) = 2.$$

The only critical point is $(1/3, 10/3)$ and at this point (indeed at every point) the discriminant equals

$$D(1/3, 10/3) = 4 \times 4 - 2^2 = 12 > 0.$$

Since the discriminant is positive and $f_{xx}(1/3, 10/3) = 4 > 0$, the second derivative test shows that there is a local minimum of $f(x, y)$ at $(1/3, 10/3)$. The minimum value of $f(x, y)$ is therefore $f(1/3, 10/3) = 12/9$ and hence the required shortest distance is $\sqrt{12/9} = 2/\sqrt{3}$.

This particular problem may also be solved using vector methods, since the surface is a plane whose normal vector is easily identifiable – you may like to try this and compare the two methods. ◇

Example 12.3c A rectangular box without a lid has volume 0.5 cubic metres. Find the dimensions which will give minimum surface area.

Let the length of the box be x metres, the depth y metres and the height z metres. Then the surface area A is $A = xy + 2xz + 2yz$. The volume is $xyz = 0.5$ and so (since none of the variables can be zero or negative in this problem) $z = \dfrac{1}{2xy}$. Substituting this expression for z into A, we obtain

$$A = xy + 2x \times \frac{1}{2xy} + 2y \times \frac{1}{2xy} = xy + \frac{1}{y} + \frac{1}{x}.$$

Our problem is to minimize $A = xy + \dfrac{1}{y} + \dfrac{1}{x}$, given that both x and y must be positive.

We begin with the calculation of the first partial derivatives.

$$A_x(x,y) = y - \frac{1}{x^2}, \qquad A_y(x,y) = x - \frac{1}{y^2}.$$

The critical points (x, y) satisfy $A_x(x, y) = 0$ and $A_y(x, y) = 0$; that is,

$$y = \frac{1}{x^2}, \qquad x = \frac{1}{y^2}.$$

Substituting $y = \dfrac{1}{x^2}$ into the second equation gives $x = \dfrac{1}{(1/x^2)^2} = x^4$, or equivalently, $x(x^3 - 1) = 0$. Since $x > 0$, the only solution of interest is $x = 1$ and then we see that $y = 1$ as well.

Therefore the only critical point of A is at $(1, 1)$, and for these values of x and y we have $z = \dfrac{1}{2xy} = \dfrac{1}{2}$. This gives a total surface area $A = 1 + 1 + 1 = 3$ square metres. To confirm that this is indeed a minimum, observe that

$$A_{xx}(x, y) = \frac{2}{x^3}, \qquad A_{yy}(x, y) = \frac{2}{y^3}, \qquad f_{xy}(x, y) = f_{yx}(x, y) = 1,$$

and so the discriminant at $(1, 1)$ is

$$D(1, 1) = 2 \times 2 - (1)^2 = 3 > 0.$$

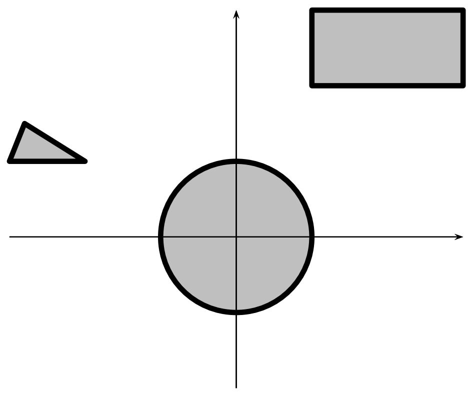
Since the discriminant is positive and $f_{xx}(1, 1) > 0$, we have a minimum value of A at $(1, 1)$.

◇

12.4 Global extrema on closed regions

Recall that to find the maximum and minimum values of a continuous function of one variable whose domain is a *closed interval*, we search for critical points *inside the interval*, evaluate the function values there, and then check function values *at the endpoints a* and b as well.

The situation is completely analogous in the two-variable case, when the function $f(x, y)$ has domain which is a *closed region* in $\mathbb{R}^2$. Three examples of closed regions in $\mathbb{R}^2$ are the disc, the rectangle and the triangle shown below. The essential feature of a simple closed region is that it includes both interior points and points on its boundary.



Global extrema in closed regions
To find the global maximum and minimum values of a continuous function of two variables whose domain is a closed region one needs to:

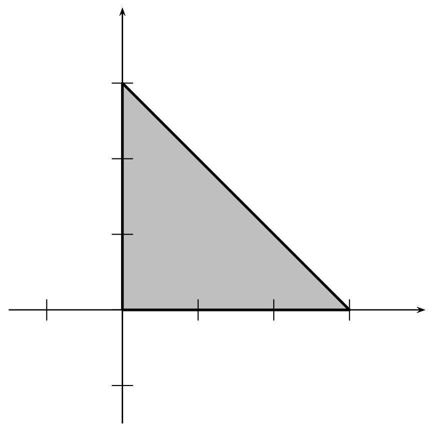
a) Search for critical points inside the region and evaluate the function values there;

b) Compute the maximum and minimum values of the function on the boundary of the region. This is done by describing the boundary as a parameterized curve $(x(t), y(t))$ and evaluating the function at the critical points of the function $f(x(t), y(t))$, which depends on t only.

The global maximum of f is the largest of all such values and the global minimum is the smallest.

In the second step considered above the boundary may be parameterized by a curve with multiple segments (e.g., the sides of a rectangle). In this case one needs to (i) consider each segment separately; and (ii) evaluate the function also at the boundary of each segment (e.g., at the vertices of the rectangle).

Example 12.4a Find the maximum and minimum values of $f(x, y) = x^2 - 2xy + 2y$ on the closed triangular region given by the vertices $(0, 0)$, $(3, 0)$, $(0, 3)$.



We'll begin by calculating the first partial derivatives of f.

$$f_x(x, y) = 2x - 2y, \qquad f_y(x, y) = -2x + 2.$$

Thus the only critical point is the point $(1, 1)$, which is an interior point of the region. At this point, the function has value $f(1, 1) = 1$. What are the function values on the boundary? There are three straight line segments which make up the boundary of the closed region.

On the horizontal line, all points have the form $(x, 0)$ where $0 \leq x \leq 3$. For such points,

$$f(x, 0) = x^2 - 2x \times 0 + 2 \times 0 = x^2,$$

which has maximum value 9 at the point $(3,0)$ and minimum value 0 at the origin.

On the vertical line, all points have the form $(0, y)$ where $0 \le y \le 3$. For such points,

$$f(0, y) = 0^2 - 2 \times 0 \times y + 2y = 2y,$$

which has maximum value 6 at the point $(0,3)$ and minimum value 0 at the origin.

On the oblique line, the line with equation $y = -x + 3$, all points have the form $(x, -x + 3)$ where $0 \le x \le 3$. For such points,

$$f(x, -x + 3) = x^2 - 2x(-x + 3) + 2(-x + 3) = 3x^2 - 8x + 6,$$

which (when $0 \le x \le 3$) has maximum value 9 at the point $(3,0)$ and minimum value $\dfrac{2}{3}$ at $(\dfrac{4}{3}, \dfrac{5}{3})$.

We conclude that on the given closed region, f has maximum value 9 at $(3,0)$ and minimum value 0 at the origin $(0,0)$.

Although we are not required to identify the nature of the interior critical point, it is very easy to do so, since

$$f_{xx}(x, y) = 2, \qquad f_{yy}(x, y) = 0, \qquad f_{xy}(x, y) = -2,$$

from which we see that the discriminant is

$$D(1,1) = 2 \times 0 - (-2)^2 = -4 < 0,$$

giving a saddle point at $(1,1)$.

◇

Notice that in the last example, the issue of finding the maximum and minimum values of $f(x, y)$ on the boundary lines of the region reduced to a one-variable max/min problem, as we were able to use the equations of the boundary lines to eliminate one of x or y. Naturally, when the boundary of the region is more complicated the algebra is correspondingly harder.

Summary of Chapter 12

- A point (a,b) is called a "critical point" of the function f if both $f_x(a,b) = 0$ and $f_y(a,b) = 0$ or at least one of these first partial derivatives fails to exist.
- If the function f has a *local* maximum or minimum at (a,b) then (a,b) is a critical point of f.
- The discriminant of f at the point (a,b) is defined as $D(a,b) = f_{xx}(a,b)f_{yy}(a,b) - \left(f_{xy}(a,b)\right)^2$.
- Suppose (a,b) is a point such that $f_x(a,b) = f_y(a,b) = 0$. The second derivative test says that
  a) If $D(a,b) > 0$ and $f_{xx}(a,b) > 0 \implies (a,b)$ is local minimum.
  b) If $D(a,b) > 0$ and $f_{xx}(a,b) < 0 \implies (a,b)$ is local maximum.
  c) If $D(a,b) < 0 \implies (a,b)$ is a saddle point.
  d) If $D(a,b) = 0$ no conclusion without further work.
- To find *global* maxima and minima in a closed region R
  a) Evaluate the function values at the critical points in R
  b) Evaluate the function values on the boundary of the region R
  c) Compare the values: The global maximum of f is the largest of all such values and the global minimum is the smallest.

Exercises

12.1 Find the derivatives of the functions given by the following formulas.

a) $7x^3 - 2x + \frac{1}{x}$.

b) $(x + \cos x)^3$.

c) $\sin\left(\cos(x^2)\right)$

d) $x^3 \sin x + e^{x \cos x}$

e) $7x^8 + \dfrac{1}{1 + \cos x^2}$.

12.2 Find the equation of the tangent line to $f(x) = 4xe^x + \cos x$ when $x = 0$.

12.3 Find the minimum and maximum values of the following functions on the closed interval $[-3, 3]$.

a) $f(x) = x^2 e^x$

b) $f(x) = 3x^2 - 6$

c) $f(x) = |x - 1| + 2$

d) $f(x) = \cos\left(\frac{x}{4}\right)$

12.4 Show that the polynomial

$$f(x) = 1 + x + x^3 + x^5 + x^7$$

is strictly increasing over the whole real line.

12.5 Find the value a such that the function

$$f(x) = x \ln x$$

is strictly decreasing for $x \in (0, a)$ and strictly increasing when $x > a$.

12.6 Find and classify the critical points of F, in each of the following cases. Note that in part d), the domain of f is restricted to points (x, y) inside a square of side length 4 centred at $(0, 0)$.

a) $f(x, y) = 4x^2 + y^2 + 8x - 8y$

b) $f(x, y) = x^3 + y^3 - 3xy$

c) $f(x, y) = y \sin x$

d) $f(x, y) = \sin x + \sin y$.

12.7 Find the global maximum and minimum values of $f(x, y) = \sin x + \sin y$ when f has domain the closed square of side length 4 centred at $(0, 0)$. (Hint: Refer to to part d) of the previous question.)

12.8 Find the critical points of f, given that $f(x, y) = \dfrac{y}{1 + x^2 + y^2}$. Identify them using the second derivative test.

12.9 Suppose that a rectangular box has length x metres, depth y metres and height z metres. If the diagonal joining opposite corners of the box (through the centre of the box) is of length 1 metre, what values of x, y, z give the maximum volume, and what is the maximum volume?

12.10 Find the point (x, y, z) on the surface given by $z = 1/(x^2 y^2)$ which is closest to the origin.

12.11 Find all critical points of the function given by $f(x, y) = x^3 y^2 (1 - x - y)$ when the domain of f is the first quadrant without the axes. Hence find the maximum value of the function on this domain.

12.12 Find the maximum and minimum values of the function given by

$$g(x, y) = (x^2 + 2y^2) e^{-(x^2 + y^2)},$$

where the domain is the entire xy plane.

12.13 Consider the curve in space given by the parametric equations

$$x = \cos t, \qquad y = \sin t, \qquad z = \sin \frac{t}{2},$$

where $t \in \mathbb{R}$. Which points on the curve are furthest from the origin and which are closest to the origin?

Note: The previous three problems are slightly modified versions of exercises appearing in *Calculus of Several Variables*, (second edition) by Serge Lang.

Appendix A

Table of Standard Integrals

1. $\displaystyle\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \qquad (n \neq -1)$

2. $\displaystyle\int \frac{dx}{x} = \ln|x| + C$

3. $\displaystyle\int e^x\,dx = e^x + C$

4. $\displaystyle\int \sin x\,dx = -\cos x + C$

5. $\displaystyle\int \cos x, dx = \sin x + C$

6. $\displaystyle\int \sec^2 x\,dx = \tan x + C$

7. $\displaystyle\int \operatorname{cosec}^2 x\,dx = -\cot x + C$

8. $\displaystyle\int \sinh x\,dx = \cosh x + C$

9. $\displaystyle\int \cosh x\,dx = \sinh x + C$

10. $\displaystyle\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\frac{x}{a} + C$

11. $\displaystyle\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$

12. $\displaystyle\int \frac{dx}{\sqrt{x^2 + a^2}} = \sinh^{-1}\frac{x}{a} + C$

13. $\displaystyle\int \frac{dx}{\sqrt{x^2 - a^2}} = \cosh^{-1}\frac{x}{a} + C \qquad (x > a > 0)$

Answers to selected exercises

Chapter 1

1. (a) Since $V = 4\pi r^3/3$ the formula for dV/dt follows immediately. (b) This says that

$$\frac{dV}{dt} = -4k\pi r^2$$

for some constant $k > 0$. We can now eliminate dV/dt from the two equations to leave

$$4\pi r^2 \frac{dr}{dt} = -4k\pi r^2$$

or $dr/dt = -k$. This shows that r is decreasing at a constant rate $-k$. (c) It is easy to see the solution to this differential equation. In fact all solutions are of the form $r(t) = -kt + \text{constant}$. The initial condition $r(0) = 0.5$ shows that the constant is 0.5. At $t = 30$ we have $r(30) = -30k + 0.5 = 0.25$, so $k = 0.25/30$ and the dependence of r on t is given by

$$r(t) = 0.5 - 0.25t/30.$$

Setting $r = 0$ gives $t = 30 \times 0.5/0.25 = 60$ days for the time at which the mothball disappears.
2. $v(s) = 100 - 50s, v(t) = 100e^{-50t}, s(t) = 2(1 - e^{-50t}$ (a) $2km$ (b) $v \to 0$ for $t \to \infty$ (c) Only for $s \leq 2$km because the velocity is always positive.

Chapter 2

1. (a) $y = e^x + C$ (b) $y = -\cos x + C$ (c) $y = \cosh x + C$
2. (a) $y = \frac{3}{2}x + \sin x + \frac{1}{4}\sin 2x + 2$ (b) $y = x - x^2 + 3x^{-1} - 4$ (c) $y = \frac{1}{2}e^{-2x} + 2$ (d) $y = \frac{1}{3}x^3 + 2x - x^{-1} - \frac{1}{3}$ (e) $y = \frac{1}{3}(1 + x^2)^{3/2} - \frac{10}{3}$ (f) $y = \frac{3}{2} - \frac{1}{2}e^{-x^2}$
3. (a) $y = 2\ln|x| - \frac{5}{2}x^{-2} + C$ (b) $y = \frac{2}{3}\sqrt{1 + 3x} + C$ (c) $y = \sec x + C$
4. $y(1) = \frac{1}{3}$.
5. The curve is $y = 2x^{3/2} - 50$.
6. (a) $y(x) = -1/(x + C)$ (b) $y(x) = \pm\sqrt{-1 + 1/(16(C - x)^2)}$
7. All solutions have the same form, $y^{\nu+1}/(\nu + 1) = x + C$, except when $\nu = -1$. The explicit solutions are: $y = Ae^x \quad y^2 = 1/(A - 2x), \quad y^{1/2} = \frac{1}{2}x + A$.
8. (a) $y - 1 = x/(x + 1)$ (b) $y = 2(1 + 1/x^2)$ (c) $y^2 = \sec^2 x + 4\tan x - 2$ (d) $y = -4x/(x - 3)$
9. The general solutions are (a) $\sin x \cos y = C$ (b) $(y - 1)/y = A\sqrt{x^2 + 1}$ (c) $(1 - y)/(1 + y) = Ae^{2/(1+x)}$ 11. $dv/dt = -g\,;\, v = -gt, \quad x = x_0 - gt^2/2, \quad v^2 = 2g(x_0 - x)$.
12. (a) $v(t) = \left(\frac{g}{k} + u\right)e^{-kt} - \frac{g}{k}, \quad x(t) = \frac{1}{k}\left(\frac{g}{k} + u\right)\left(1 - e^{-kt}\right) - \frac{gt}{k}$,
$x(v) = u - vk + \frac{g}{k^2}\ln g + kvg + ku$. (b) $x_{max} = uk - gk^2 \ln(1 + kug)\,, \quad t_{max} = 1k\ln(1 + kug)$.

Chapter 3

1. (a) $y = (3 + e^{2x})/(3 - e^{2x})$ (b) $y = (4 + 2e^{4x})/(2e^{4x} - 1)$ (c) $y = (2e^x - 2)/(2 - e^x)$

2. $R = n \ln S + C$. The visual magnitude scale for stars developed in antiquity was a logarithmic function of brightness.

3. $R = \left[\dfrac{n(1-p)S^{1-q}}{(1-q)}\right]^{1/(1-p)}$ if $p, q < 1$. 4. $R = AS1 + \alpha S$.

5. (b) 4.25×10^9 years, a good estimate of the age of the solar system.

7. $N = 605$.

8. $N(1 - N/M)^{\gamma\mu+1} = \dfrac{N_0}{(1 - N_0/M)^{\gamma\nu+1}} e^{\gamma t}$.

The limiting population size (as $t \to \infty$) is M.

9. $N = N_0 e^{\gamma t} e^{(\mu \sin \omega t)/\omega}$. If $\gamma > 0$ and $\gamma < |\mu|$ the curve oscillates up and down between limits which increase exponentially with time. If $\gamma > |\mu|$ the curve shows just ripples as it increases monotonically. If $\gamma < 0$ and $|\gamma| < |\mu|$ the curve oscillates up and down between limits with decay to zero exponentially with time (N is always positive though). If $|\gamma| > |\mu|$, the curve shows ripples as it decays monotonically.

10.

$$m = \left(\frac{\alpha}{\beta}\right)^3 \left(1 - e^{-\beta t/3}\right)^3 .$$

11. $1.105P_0$, 10.5%.

12. (a) $\dfrac{dx}{dt} = 0.10x + 1000$, (b) 23 years.

13. $N + (b/a)M - N = N_0 + (b/a)M - N_0 e^{(b+aM)t}$.

15. $t = 2Ak(\sqrt{h_0} - \sqrt{h}) - \dfrac{2Ar}{k^2} \ln(\dfrac{(R/k - \sqrt{h})}{(r/k - \sqrt{h_0})})$.

16. $(2\pi \tan^2 \alpha H^2/5a)\sqrt{H/2g}$.

17. $\dfrac{dc}{dt} = -kc$; $c(t) = c_0 e^{-kt}$.

19. $m \to s/k$.

20. (a) $c(\infty) = c_1$, (b) $t_{1/10} = (V/r) \ln 10$ (c) 17.6 years.

22. (a) 0.097% (b) 73.24 hours.

23. Hint: work in terms of concentration to obtain a separable equation. 3.11 kg.

Chapter 4

1. (a) $y = -\frac{3}{2} + Ce^{2x}$ (b) $x = -1 + Ce^{t^2/2}$ (c) $y = x^3 + C/x$ (d) $y = e^{-x} + Ce^{-2x}$ (e) $x = t^2 - 1 + Ce^{-t^2}$ (f) $y = x^2 + Cx^2 e^{1/x}$.

2. $y = \frac{1}{3}x - \frac{1}{7}x^3 + Cx^{-1/2}$.

3. $y = \sin x + 2\cos x$; $y = x^5/3 + \frac{2}{3}x^2$; $y = -\frac{1}{2}e^{-4t}\cos 2t + e^{-4t}$; $y = \cos x(\sin x - \frac{1}{4}\cos 2x + \frac{3}{4})/(1 + \sin x)$; $y = (x^3 + 2)/(1 + x)$; $y = (2 + x)^2/(1 + x^2)$.

4. The general solution is $y = (e^x + Ce^{-x})/2x$.

5. The general solution is $y = \cos x + C\cos^2 x$.

6. 51.4°C, 33 minutes.

7. $V(t) = E(1 - e^{-t/RC})$, $V(\infty) = E$.

8. General solution: $y = \dfrac{1}{6}Ax^3 + \dfrac{1}{2}Bx^2 + Cx + D$. Particular solution: $y = \dfrac{1}{6}x^3 + \dfrac{1}{2}x^2 + x + 1$.

Chapter 5

1. (a) G.S. $y = Ce^{-5x} + De^{3x}$; P.S. $y = (e^{-5x} + 3e^{3x})/4$. (b) G.S. $y = (C + Dx)e^{-3x}$; P.S. $y = 3xe^{3(1-x)}$. (c) G.S. $y = C\cos 5x + D\sin 5x$; P.S. $y = \cos 5x + \sin 5x$. (d) G.S. $y = Ce^{x/3}+De^{-2x}$; P.S. $(1 - e^{-4})e^{x/3} + (e^{2/3} - 1)e^{-2x}e^{2/3} - e^{-4}$. (e) G.S. $y = e^{-x}(Ce^{5ix}+De^{-5ix})$ (complex form); $y = e^{-x}(E\cos 5x + F\sin 5x)$ (real form).P.S. $y = e^{-x}(e^{5ix} - e^{-5ix})/5i$ (complex); $y = (2e^{-x}\sin 5x)/5$ (real). (f) G.S. $y = Ce^{-x} + De^{100x}$; P.S. $y = e^{-x}$. (g) G.S same as last question; P.S. $y = 1.00009901e^{-x} - 0.00009901e^{100x}$.

Chapter 6

1. $y(x) = y_h(x) + y_p(x) = C_1e^x + C_2e^{-2x} - 0.5x^2 - 0.5x - 0.75$

2. General solution: $y = \dfrac{1}{4}e^{2x} + Ax + B$. Particular solution: $y = \dfrac{1}{4}\big(e^{2x} + (9 - e^2)x - 1\big)$.

3. General solution: $y = -\dfrac{1}{8}\sin 2x + \dfrac{1}{60}x^5 - \dfrac{1}{6}x^3 + \dfrac{1}{2}Ax^2 + Bx + C$.

Particular solution: $y = -\dfrac{1}{8}\sin 2x + \dfrac{1}{60}x^5 - \dfrac{1}{6}x^3 + \dfrac{1}{2}x^2 + \dfrac{1}{4}x$.

6. $x = 1k\ln(1 + ku\cos\alpha t)$

$$z = \frac{g}{2(ku\cos\alpha)^2}\left[\left(1 + ku^2\sin 2\alpha g\right)\ln(1 + ku\cos\alpha t)\right.$$
$$\left. -\quad ku\cos\alpha t2(2 + ku\cos\alpha t)\right]$$
$$z = \frac{g}{2(ku\cos\alpha)^2}\left[\left(1 + ku^2\sin 2\alpha g\right)kx - \frac{1}{2}\left(e^{2kx} - 1\right)\right]$$

Chapter 7

1. (a) The domain is $\mathbb{R}^2$, the range is $\mathbb{R}$. (b) The domain is $\mathbb{R}^2$, the range is $\mathbb{R}$. (c) The domain is the set of all $(x, y) \in \mathbb{R}^2$ such that $x + y > 0$, the range is all non-negative real numbers. (d) The domain is the set of all $(x, y) \in \mathbb{R}^2$ such that $x^2 + y^2 < 9$, the range is the set of all real numbers less than or equal to $\ln 9$. (e) The domain is $\mathbb{R}^2$, the range is the set of all numbers of absolute value less than or equal to 1. (f) The domain is the set of all $(x, y) \in \mathbb{R}^2$ such that $x \neq \frac{y}{3}$, the range is $\mathbb{R}$. (g) The domain is $\mathbb{R}^2$, the range is $\mathbb{R}$. (h) The domain is $\mathbb{R}^2$, the range is $\mathbb{R}$.

3. (a) The domain is the set of points inside the circle $x^2 + y^2 = 3$. The range is the set of all numbers less than or equal to $\ln 3$. As (x, y) approaches the circle, $z \to -\infty$. (b) The domain is $\{(x, y) \in \mathbb{R}^2 \mid y \leq \frac{x+2}{2}$; that is, the points below and on the straight line $2y = x + 2$. The level curves are straight lines.

4. (a) The domain is the set of all $(x, y) \in \mathbb{R}^2$ such that $4x^2 + 9y^2 < 36$; the range is set of all numbers less than or equal to $\ln 36$. (b) The domain is the set of all $(x, y) \in \mathbb{R}^2$ such that $\mid y < 2x + 1$; the range is $(0, \infty)$.

Chapter 8

1. (a) $f_x = 4x - y^3$, $f_y = 7 - 3xy^2$ (b) $f_x = -\sin y\sin x$, $f_y = \cos y\cos x$ (c) $f_x = 2xe^{x^2+y}$, $f_y = e^{x^2+y} + 1$ (d) $f_x = e^x + xe^x + e^y$, $f_y = (1 + x + y)e^y$ (e) $f_x = -\dfrac{2}{x^3y}$, $f_y = -\dfrac{1}{x^2y^2}$ (f) $f_x = \sec^2(x + y) + \dfrac{12x^3}{x^4 + y^4}$, $f_y = \sec^2(x + y) + \dfrac{12x^3}{x^4 + y^4}$ (g) $f_x = \dfrac{x}{\sqrt{x^2 + 8y^4}}$, $f_y = \dfrac{16y^3}{\sqrt{x^2 + 8y^4}}$

(h) $f_x = -\dfrac{x\cos y}{(x^2+y^2)^{\frac{3}{2}}}$, $f_y = -\dfrac{\sin y}{\sqrt{x^2+y^2}} - \dfrac{y\cos y}{(x^2+y^2)^{\frac{3}{2}}}$

2. (a) $f_x(1,1) = 4$, $f_y(1,1) = 2$ (b) $f_x(0,\frac{\pi}{2}) = 0$,$f_y(0,\frac{\pi}{2}) = 0$ (c) $f_x(2,0) = 1$, $f_y(2,0) = 0$

3. (a) $f_x(1,1) = f_y(1,1) = -\dfrac{1}{\sqrt{2}}$; $x + y + \sqrt{2}z = 4$ (b) $f_x(1,0) = 0$, $f_y(1,0) = e$; $ey - z = 1$ (c) $f_x = 0$, $f_y = -1$; $y + z = \dfrac{\pi}{2}$

4. $f(x,y) = e^x \sin y$

5. (a) $f'(x) = e^{x+5}$ (b) $f'(x) = (\ln 4)e^x$ (c) $f'(x) = e^x + xe^x = (1+x)e^x$ (d) $f'(x) = \dfrac{(x+3)(2x+5)-(x^2+5x+2)}{(x+3)^2} = \dfrac{x^2+6x+13}{(x+3)^2}$ (e) $f'(x) = 99(x+1)^{98}$ (f) $f'(x) = e^{-x^2} - 2x^2e^{-x^2} = (1-2x^2)e^{-x^2}$ (g) $f'(t) = \dfrac{d}{dt}\left(\dfrac{\sin t}{\cos t}\right) = \dfrac{-\sin t\cdot(-\sin t)+\cos t\cdot\cos t}{\cos t\cdot\cos t} = \dfrac{1}{\cos^2 t} = \sec^2 t$ (h) $f'(t) = (-\sin t)e^{\cos t}$ (i) $f'(t) = (\cos 3t - 3t\sin 3t)e^{t\cos 3t}$ (j) $f'(t) = \dfrac{2t\sin(1-t^2)}{\cos(1-t^2)}$ (k) $f'(x) = 6(x+\sin^5 x)^5(1+5\sin^4 x\cos x)$ (l) $f'(x) = \cos(\sin(\sin x))\cos(\sin x)\cos x$ (m) $f'(x) = -36\cos(6\cos(6\sin x))\sin(6\sin x)\cos x$

6. (a) $f'(x) = -\dfrac{1}{x^2}$, so $f(f'(x)) = -x^2$ and also $f'(f(x)) = -x^2$. (b) $f'(x) = 2x$, so $f(f'(x)) = (2x)^2 = 4x^2$ and $f'(f(x)) = 2x^2$. (c) $f'(x) = 0$, so $f(f'(x)) = 2$ and $f'(f(x)) = 0$. (d) $f'(x) = 2$, so $f(f'(x)) = 4$ and $f'(f(x)) = 2$.

Chapter 9

1. Take $f(x) = x^{\frac{1}{4}}$. Then $df = \frac{1}{4}x^{-\frac{3}{4}}dx$ so $(15)^{\frac{1}{4}} = f(15) \approx f(16) + \frac{1}{4}(16)^{-\frac{3}{4}}(15-16) = 1\frac{31}{32}$.

2. $df = y^{\frac{1}{2}}dx + \frac{1}{2}xy^{-\frac{1}{2}}dy = dx + \frac{1}{2}dy$, so $f(1.5, 0.9) \approx f(1.5,1) + dy = 1.5 + (1)(1.5-1.5) + \frac{1}{2}(1)(0.9-1) = 1.45$.

3. $A = xy$, so $dA = ydx + xdx$ and $\Delta A = (x+\Delta x)(y+\Delta y) - xy = y\Delta x + x\Delta y$; dA stops being a good approximation to ΔA whenever Δx or Δy become too large.

4. Let the dimensions of the block be x, y and z. The surface area is $S = 2(xy + xz + yz)$ and $dS = 2(y+z)dx + 2(x+z)dy + 2(x+y)dz$, so the error in the surface is $dS \approx 2\left((52)2 + (42)1 + (70)(0.5)\right) = 362cm^2$.

5. (a) $10t^4$ (b) $6t^2\sin(3(1-t^2)) + \cos(3(1-t^2))$ (c) $2t+4$ (d) 0

10. $\frac{1-2x}{2y}$

11. $\frac{27}{5}$

13. $-\frac{12x^2y^2\sqrt{1-x^2y^2}+y}{8x^3y\sqrt{1-x^2y^2}+x}$

14. $(\frac{a}{\sqrt{2}}, \frac{b}{\sqrt{2}}, \frac{\sqrt{2}}{2})$

15. $-4\cos t\sin t$

Chapter 10

1. (a) $\frac{2}{\sqrt{5}}$ (b) $\frac{11}{25\sqrt{5}}$ (c) $3\sqrt{2}$ (d) $\frac{1}{\sqrt{221}}$

2. (a) $2\mathbf{i}+\mathbf{j}$ (b) $\mathbf{i}-2\mathbf{j}$

3. (a) $8x^3y^3\mathbf{i}+6x^4y^2\mathbf{j}$ (b) $-e^{y^2}\sin(x+1)\mathbf{i}+2ye^{y^2}\cos(x+1)\mathbf{j}$ (c) $(10xy-\frac{1}{x+y}+\cos x)+(5x^2-\frac{1}{x+y})\mathbf{j}$

4. (a) *(i)* $\mathbf{i}+2\mathbf{j}$ *(ii)* 1 *(iii)* $1/5\sqrt{2}$ (b) *(i)* $2\mathbf{i}-\mathbf{j}$ *(ii)* $\sqrt{5}/2\sqrt{3}$ *(iii)* 0 (c) *(i)* $2\mathbf{i}-\mathbf{j}$ *(ii)* $\sqrt{5}$ *(iii)* $-1/\sqrt{2}$ (d) *(i)* $\mathbf{i}$ *(ii)* $2e$ *(iii)* $-4e/5$

5. (a) $2\mathbf{i}+4\mathbf{j}$ (b) $\sqrt{3}/2\mathbf{i}+1/2\mathbf{j}$ (c) $12\mathbf{i}+3\mathbf{j}$
6. (a) $(ye^x-\sin x)\mathbf{i}+e^x\mathbf{j}$ (b) $3/\sqrt{5}$ (c) $\mathbf{i}+\mathbf{j}$ (the direction of $\nabla\phi$)
7. (a) Temperature decreases at a rate of 0.04°C per centimetre. (b) Temperature increases at a rate of 0.021°C per centimetre. (c) 0.245 radians west of north.

Chapter 11

1. $R_x=\frac{y^2}{x}+\cos(x+3y)$, $R_y=2y^2\ln x+3\cos(x+3y)$, $R_{xx}=1\frac{y^2}{x^2}-\sin(x+3y)$, $R_{xy}=R_{yx}=\frac{2y}{x}-3\sin(x+3y)$, $R_{yy}=2\ln x-9\sin(x+3y)$
2. $f_{xx}=-y^2e^{xy}$, $f_{xy}=f_{yx}=\cos y-e^{xy}-xye^{xy}$, $f_{yy}=-x\sin y-x^2e^{xy}$
3. (a) $f_{xx}=a^2e^{ax}\sin(ay)$ and $f_{yy}=-a^2e^{ax}\sin(ay)$, so $f_{xx}+f_{yy}=0$ (b) $f_{xx}=\dfrac{2y^2-2x^2}{(x^2+y^2)^2}$ and $f_{yy}=\dfrac{2x^2-2y^2}{(x^2+y^2)^2}$, so $f_{xx}+f_{yy}=0$
5. In general, there will be $n+1$ different n^{th}-order partial derivatives; namely, $\dfrac{\partial^n f}{\partial x^k\partial y^{n-k}}$, for $k=0,1,\ldots,n$. For example, if $n=3$ then the 4 derivatives $\dfrac{\partial^3 f}{\partial y^3}$, $\dfrac{\partial^3 f}{\partial x\partial y^2}$, $\dfrac{\partial^3 f}{\partial x^2\partial y}$ and $\dfrac{\partial^3 f}{\partial x^3}$ will all be different.
6. (a) $\{(x,y)\in\mathbb{R}^2 \mid x=0 \text{ or } y=0\}$; cannot be extended because $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist. (b) $\{(x,y)\in\mathbb{R}^2 \mid x=0\}$; cannot be extended (c) $\{(x,y)\in\mathbb{R}^2 \mid x=-2y\}$; cannot be extended (d) $\{(x,y)\in\mathbb{R}^2 \mid x=y\}$; these points can be added to the domain of $f(x,y)$ by defining $f(x,y)=2y$ for all points (x,y) on the line $y=x$. (e) $\{(x,y)\in\mathbb{R}^2 \mid y=2x^2\}$; cannot be extended (f) $\{(0,0)\}$; cannot be extended (g) $\{(0,0)\}$; this point can be added to the domain of $f(x,y)$ by defining $f(0,0)=0=\lim_{(x,y)\to(0,0)} f(x,y)$ (h) $\{(0,0)\}$; cannot be extended
7. (a) $\{(x,y)\in\mathbb{R}^2 \mid (x,y)\neq(0,0)\}$ (b) $\dfrac{1}{2}$ (c) $\dfrac{1}{4}$ (d) No. Taking the limit along different paths gives different answers and hence this limit does not exist.
9. (a) The limit exists and is equal to 0 (b) The limit exists and is equal to 1

Chapter 12

1. (a) $21x^2-2-\frac{1}{x^2}$ (b) $3(1-\sin x)(x+\cos x)^2$ (c) $\cos\left(\cos(x^2)\right)\times-\sin(x^2)\times 2x$ (d) $3x^2\sin x+x^3\cos x+e^{x\cos x}(-x\sin x+\cos x$ (e) $56x^7+\dfrac{2x\sin x}{(1+\cos x^2)^2}$ 2. $y=4x+1$
3. (a) Minimum: 0 at $x=0$; maximum $9e^3$ at $x=3$ (b) Minimum: -6 at $x=0$; maximum 21 at $x=\pm3$ (c) Minimum: 2 at $x=1$; maximum 6 at $x=-3$ (d) Minimum: 0.731 at $x=\pm3$; maximum 1 at $x=0$
4. $f'(x)=1+3x^2+5x^4+7x^6>0$ for all x
5. $a=e^{-1}$
6. (a) One critical point at $(-1,4)$ corresponding to a local minimum. (b) Two critical points: local minimum at $(1,1)$, saddle point at $(0,0)$. (c) Infinitely many critical points, all of the form $(n\pi,0)$, where n can be any integer, all saddle points. (d) Four critical points in the specified domain: local maximum of 2 at $(\pi/2,\pi/2)$, local minimum of -2 at $(-\pi/2,-\pi/2)$, and saddle points at $(-\pi/2,\pi/2)$ and $(\pi/2,-\pi/2)$. 7. The global max and min values are the same as the local max and min values found in the previous question. On the four sides of the

square, the largest function value is $1 + \sin 2 < 2$ and the smallest is $-1 - \sin 2 > -2$.
8. Maximum value is $1/2$, at the critical point $(0, 1)$. Minimum value is $-1/2$, at the critical point $(0, -1)$.
9. $x = y = z = \dfrac{1}{\sqrt{3}}$, max volume is $1/(3\sqrt{3})$.
10. $(2^{1/10}, 2^{1/10}, 2^{-2/5})$
11. One critical point at $(1/2, 1/3)$, with $D(1/2, 1/3) > 0$ and $f_{xx}(1/2, 1/3) < 0$. Hence f has a maximum at $(1/2, 1/3)$ and the maximum value is $1/432$.
12. The maximum value of g is $2/e$ at points $(0, \pm 1)$ and the minimum value of g is 0 at the origin.
13. Furthest points are $(-1, 0, \pm 1)$, and the closest point is $(1, 0, 0)$.