COMP2823

Lecture 3: Trees
[GT 2.3 & 20.2]

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Some content is taken from material provided by the textbook publisher Wiley.

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Agenda: Trees

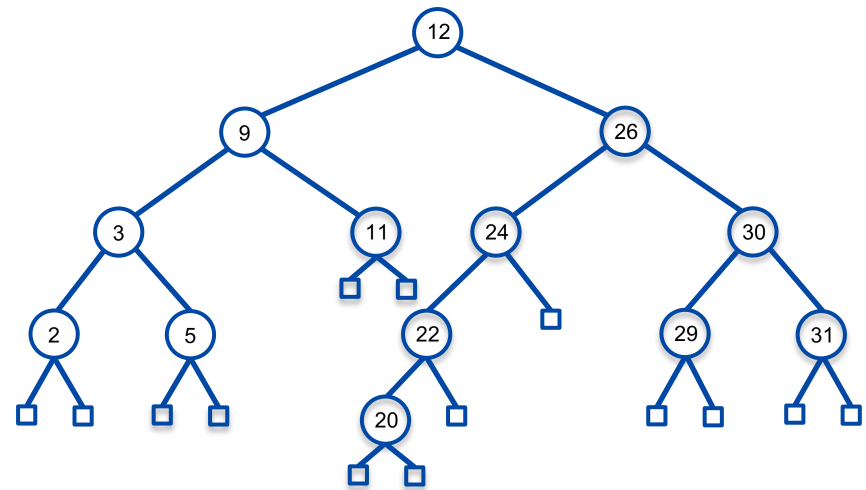
- Definition and terminology
- Applications
- Tree ADT
- Tree traversal algorithms
- Binary trees
- Implementing trees
- Recursive code on trees

Trees

leaves

root

root



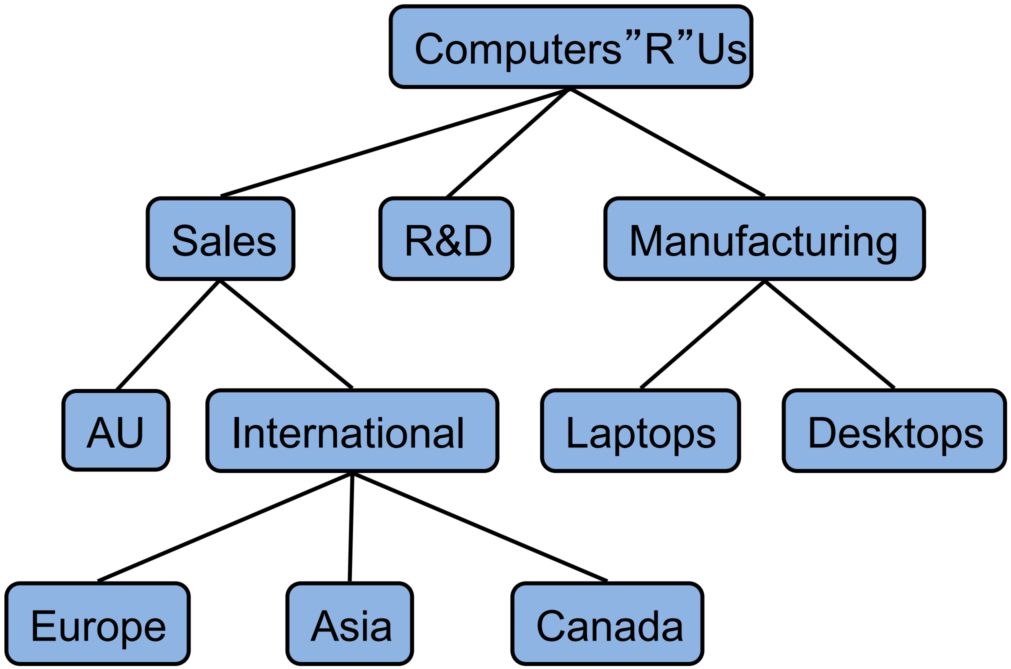
leaves

What is a Tree

In computer science, a tree is an abstract model of a hierarchical structure

A tree consists of nodes with a parent-child relation

- if u is parent of v, then v is a child of u
- a node has at most **one** parent in a tree
- a node can have zero, one or more children



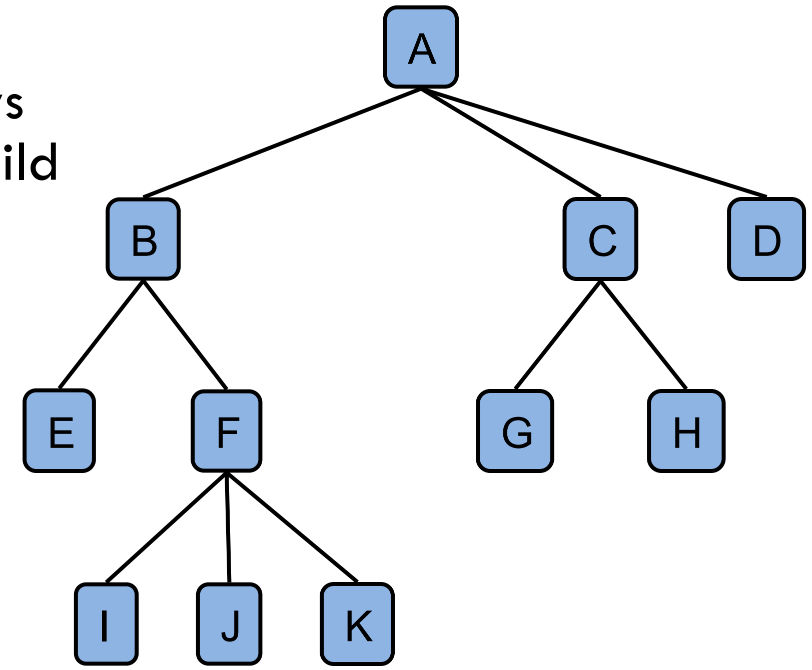
Applications:

- Organization charts
- File systems
- Phrase structure

Formal definition

A tree T is made up of a set of nodes endowed with parent-child relationship with following properties:

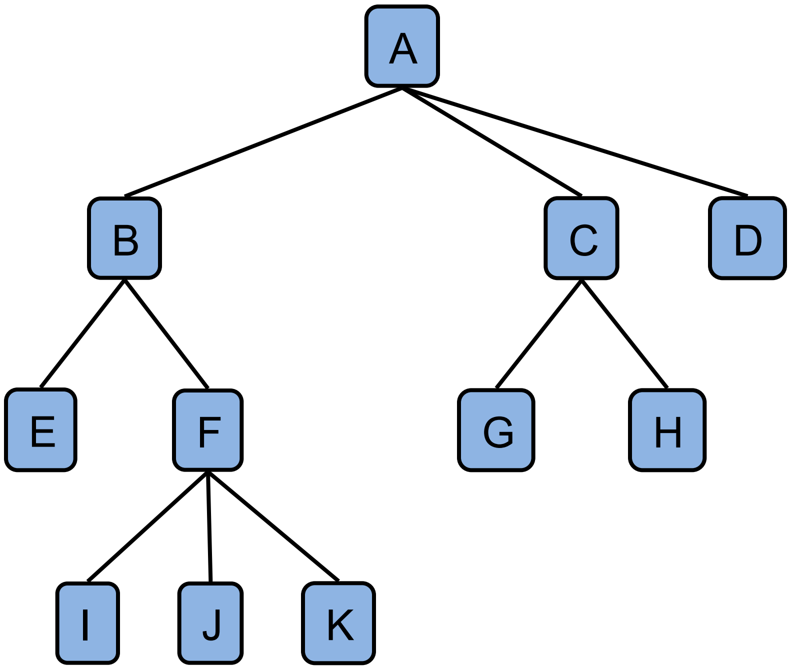
- If T is non-empty, it has a special node called the root that has no parent
- Every node v of T other than the root has a unique parent
- Following the parent relation always leads to the root (i.e., the parent-child relation does not have "cycles")



Tree Terminology

Depending on where they are in the tree, we classify nodes into:

- Root: node without parent (e.g., A)
- Internal node: node with at least one child (e.g., A, B, C, F)
- External/leaf node: node without children (e.g., E, I, J, K, G, H, D)



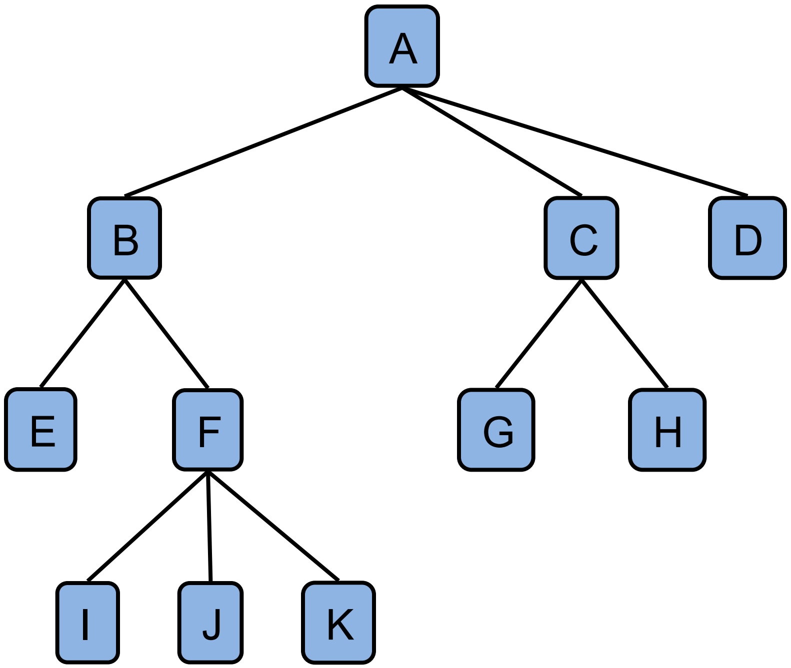
Tree Terminology

We can extend the parent-child relation to capture indirect relations:

- Ancestors: parent, grandparent, great-grandparent, etc. (e.g., ancestors of F are A, B)

- Descendants: child, grandchild, great-grandchild, etc. (e.g., descendants of B are E, F, I, J, K)

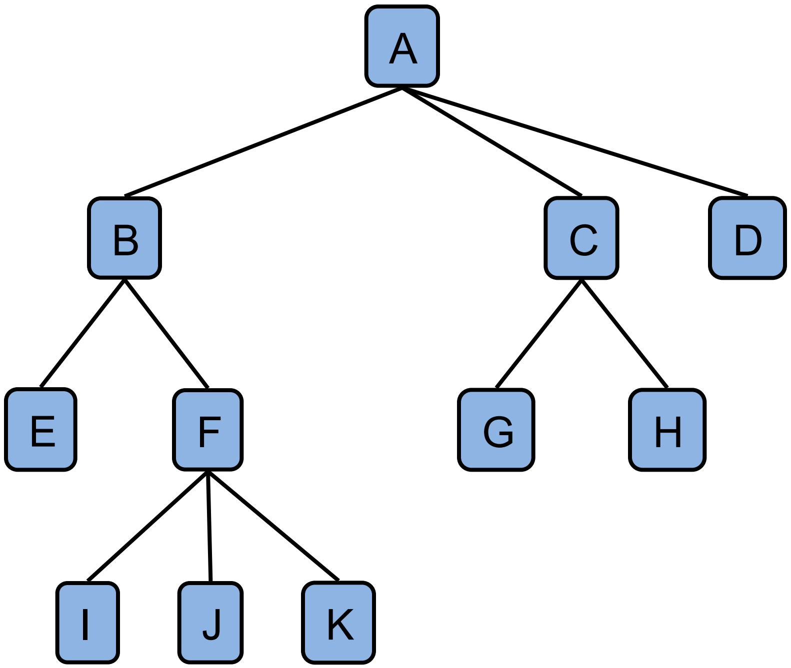
- Two nodes with the same parent are siblings (e.g., B and D)



Tree Terminology

More fine-grained location concepts:

- Depth of a node: number of ancestors not including itself (e.g., depth(F) = 2)
- Level: set of nodes with given depth (e.g., {E, F, G, H} are level 2)
- Height of a tree: maximum depth (e.g., 3)



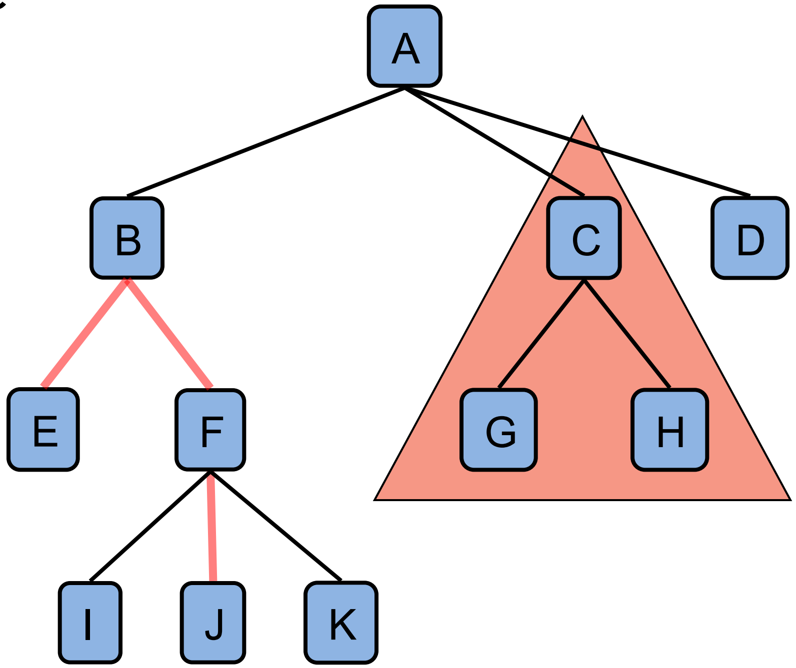
Tree Terminology

Substructures of a tree:

– Subtree: tree made up of some node and its descendants. (e.g., subtree rooted at C is {C, G, H})

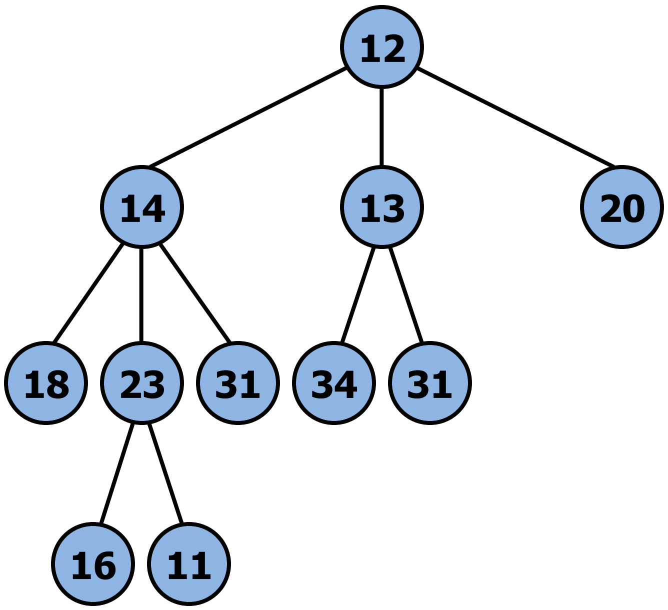
– Edge: pair of nodes (u, v) such that one is the parent of the other

– Path: sequence of nodes such that 2 consecutive nodes in the sequence have an edge (e.g., <E, B, F, J>).



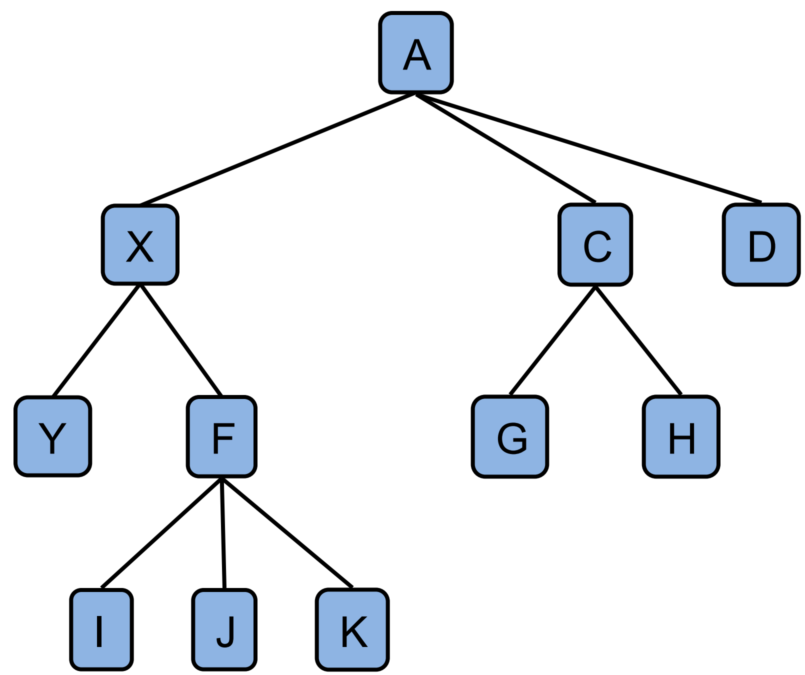
Examples

- Node 14 has depth … 1
- The tree has height … 3
- Subtree rooted at node 14 has height … 2
- Any subtree of a leaf has height … 0
- The root has depth … 0



Tree facts

- If node X is an ancestor of node Y, then Y is a descendant of X.
- Ancestor/descendant relations are transitive
- Every node is a descendant of the root
- There may be nodes where neither is an ancestor of the other
- Every pair of nodes has at least one common ancestor.
- The lowest common ancestor (LCA) of x and y is a node z such that z is the ancestor of x and y and no descendant of z has that property



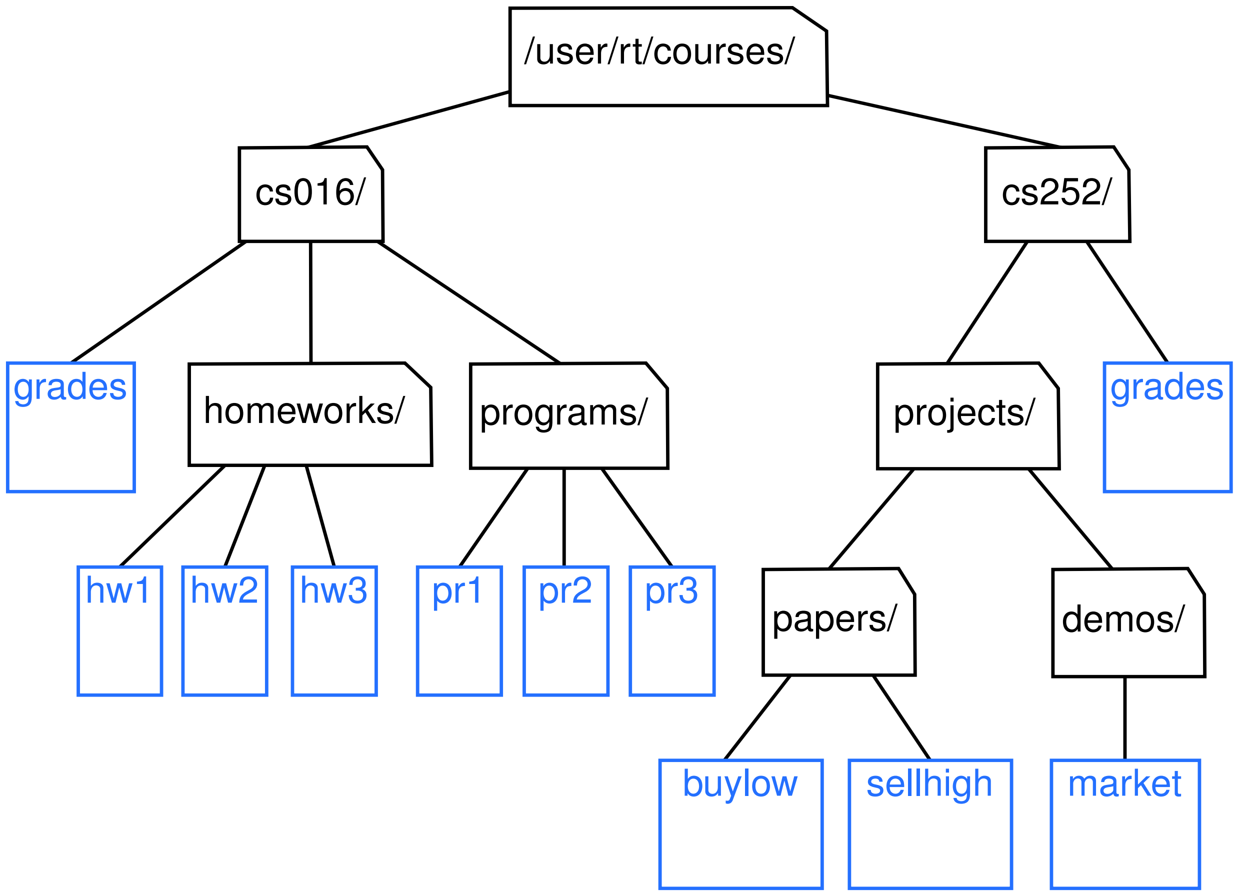
Ordered Trees

Sometimes order of siblings matter

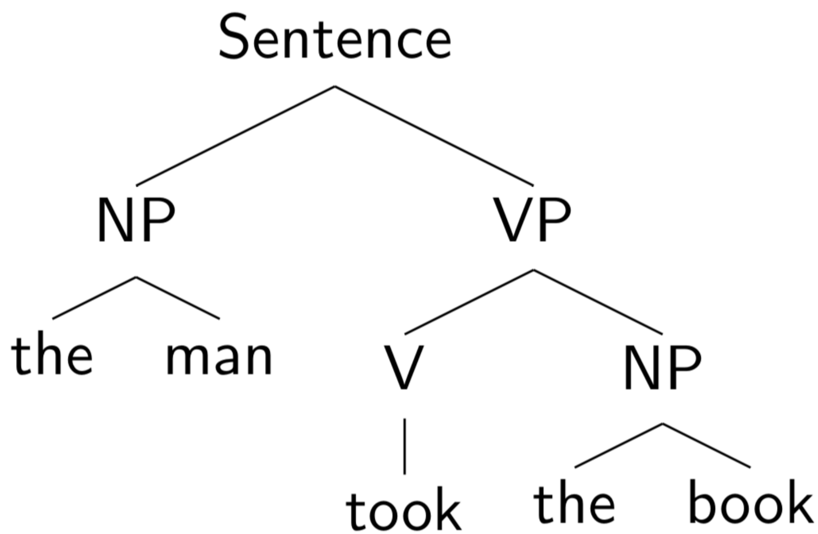
In an ordered tree there is a prescribed order for each node's children

In a diagram this ordering is usually represented by the left to right arrangement of the nodes

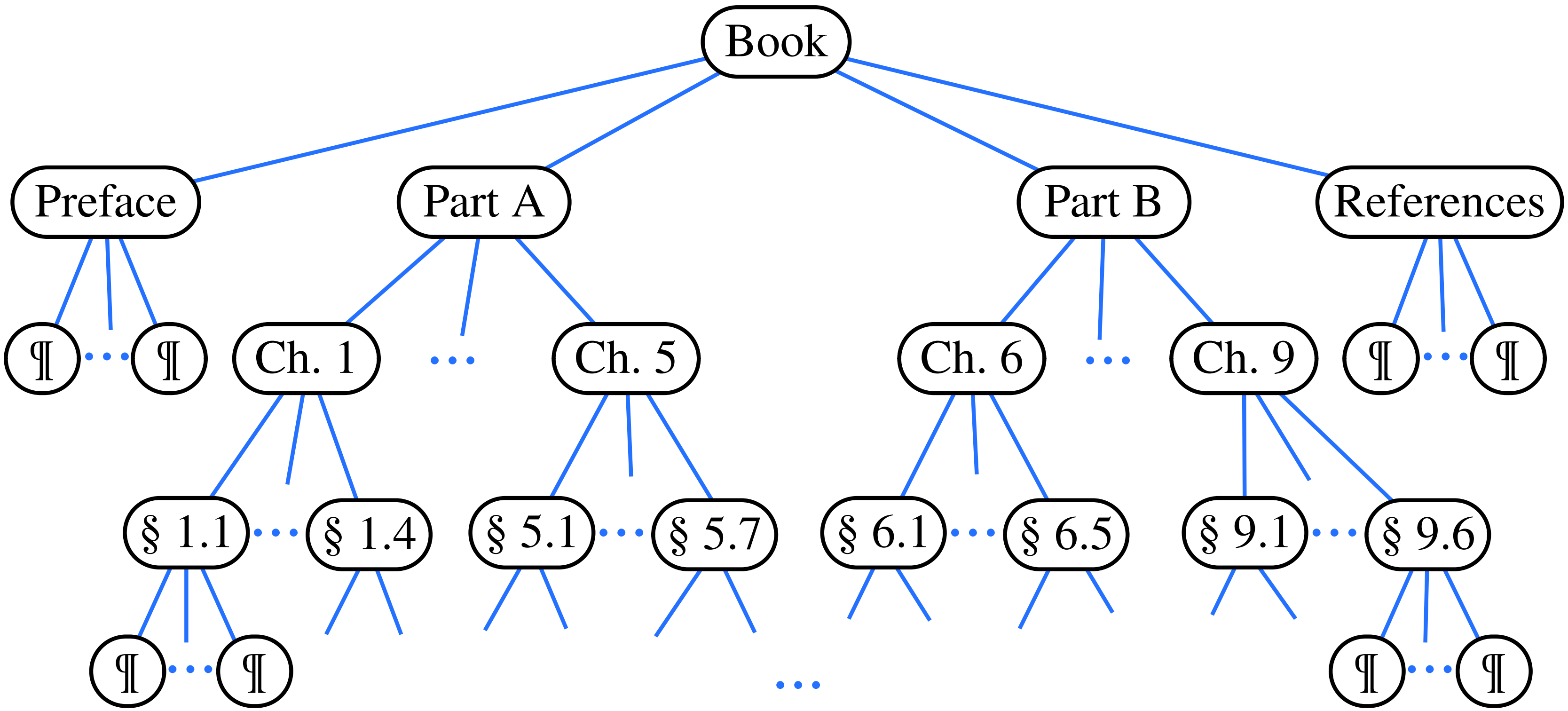
Application: OS file structure



Application: Phrase structure tree



Application: Document structure



Tree ADT

- Position as Node abstraction
- Generic methods:
  - integer size()
  - boolean isEmpty()
  - Iterator iterator()
  - Iterable positions()
- Access methods:
  - Position root()
  - Position parent(p)
  - Iterable children(p)
  - Integer numChildren(p)

- Query methods:
  - boolean isInternal(p)
  - boolean isExternal(p)
  - boolean isRoot(p)

- Additional update methods may be defined by data structures implementing the Tree ADT

Node object

Node object implementation typically has the following attributes:

- value: the value associated with this Node
- children: set or list of children of this Node
- parent: (optional) the parent of this Node

```
def is_external(v)
   # test if v is a leaf
   return v.children.is_empty()

def is_root(v)
   # test if v is the root
   return v.parent = null
```

Traversing trees

A traversal visits the nodes of a tree in a systematic manner

When traversing a simpler structure like a list there is one natural traversal strategy (forward or backwards)

Trees are more complex and admit more than one natural way:

- pre-order
- post-order
- in-order (for binary trees)

Preorder Traversal

To do a preorder traversal starting at a given node, we visit the node <u>before</u> visiting its descendants

```
def pre_order(v)
   visit(v)
   for each child w of v
      pre_order(w)
```

If tree is ordered visit the child subtrees in the prescribed order

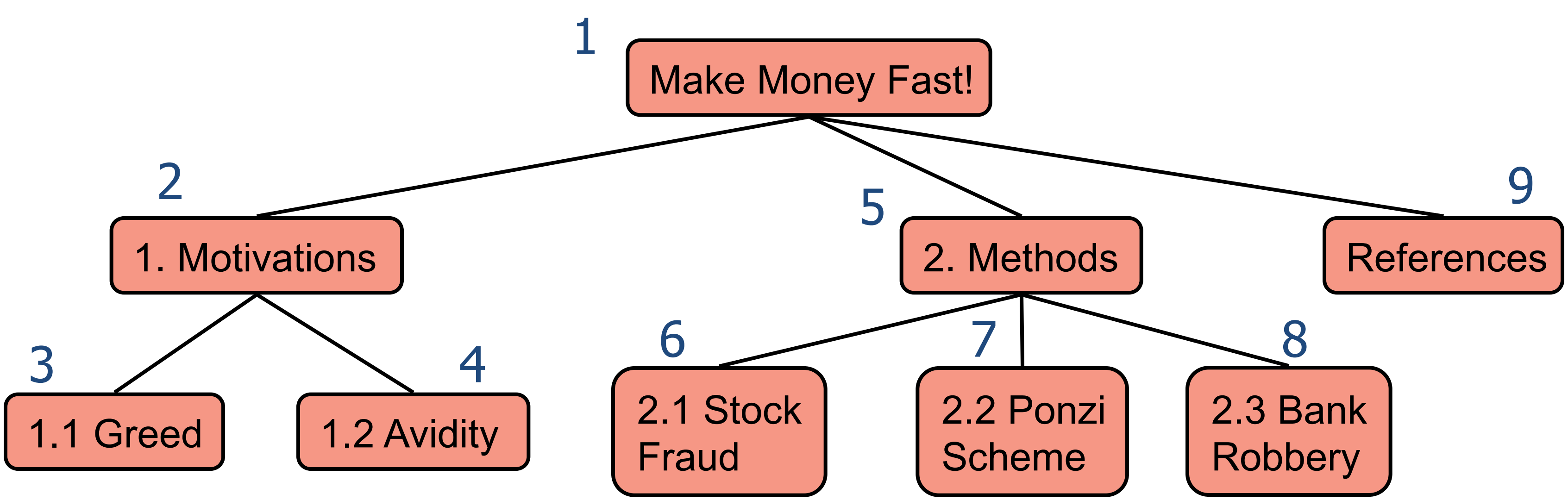
Visit does some work on the node:

- print node data
- aggregate node data
- modify node data

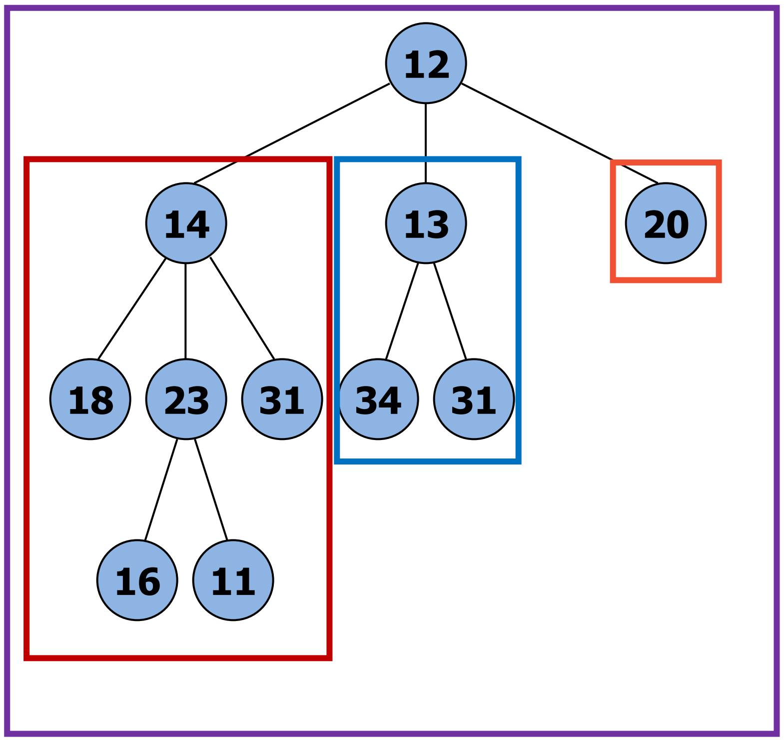
Preorder Traversal Example

Nodes are numbered in the order they are visited when we call `pre_order()` at the root

```
def pre_order(v)
    visit(v)
    for each child w of v
        pre_order(w)
```



Preorder Traversal Example



visit order

12
14
18
23
16
11
31
13
34
31
20

Preorder traversal of subtree

Postorder Traversal

To do a postorder traversal starting at a given node, we visit the node <u>after</u> its descendants

```
def post_order(v)
   for each child w of v
      post_order(w)
   visit(v)
```

If tree is ordered visit the child subtrees in the prescribed order

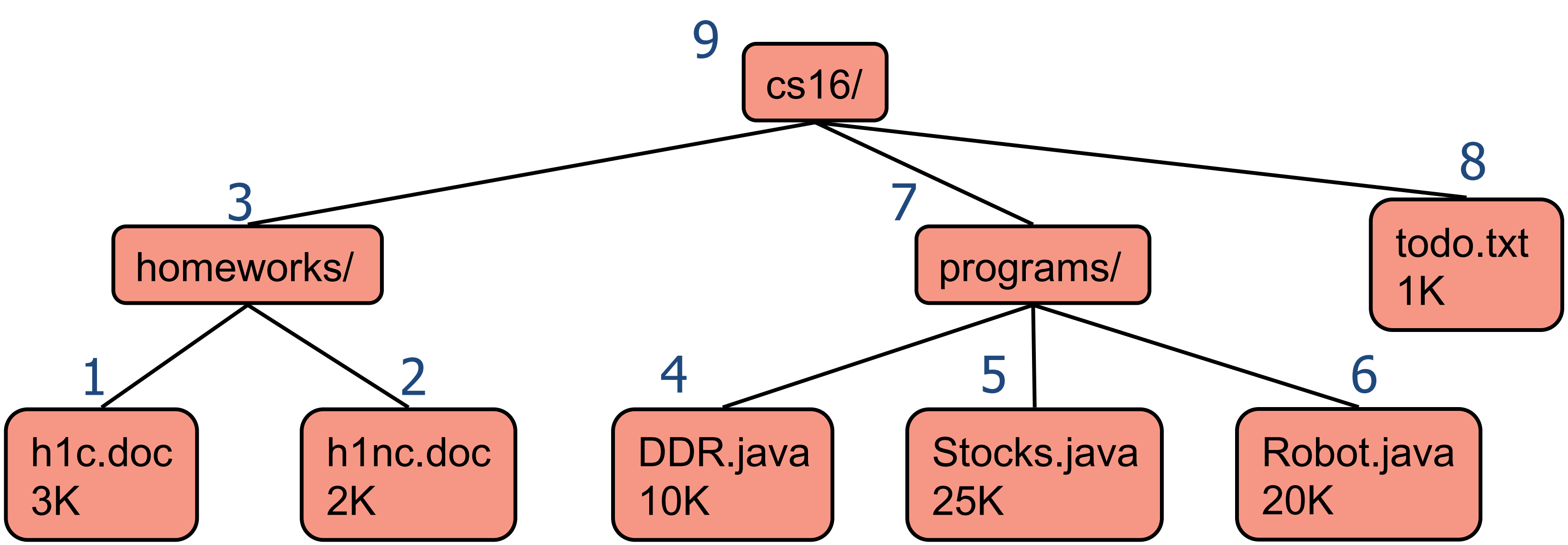
Visit does some work on the node:

- print node data
- aggregate node data
- modify node data

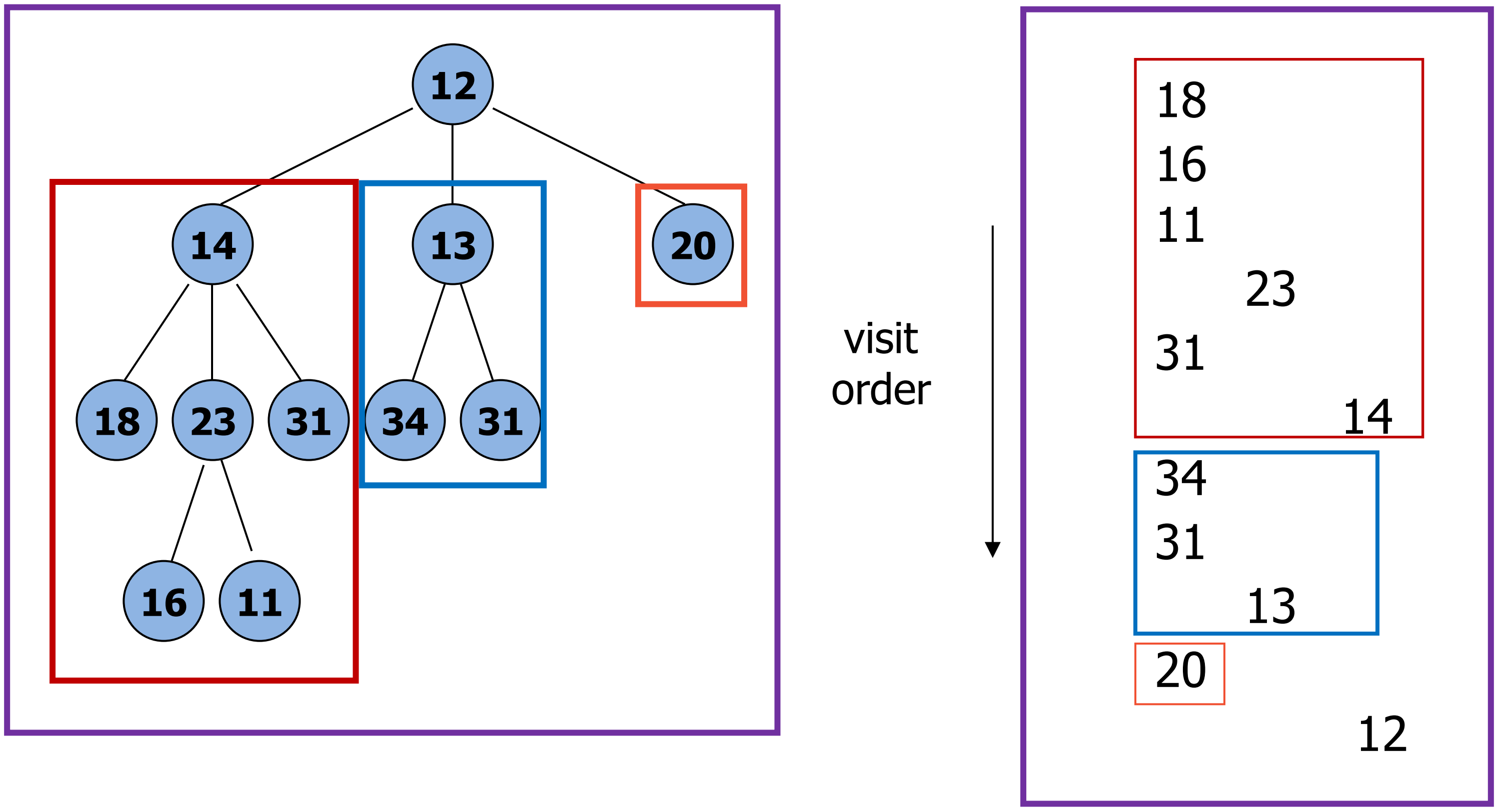
Postorder Traversal

Nodes are numbered in the order they are visited when we call `post_order()` at the root

```
def post_order(v)
   for each child w of v
      post_order(w)
   visit(v)
```



Traversing in postorder

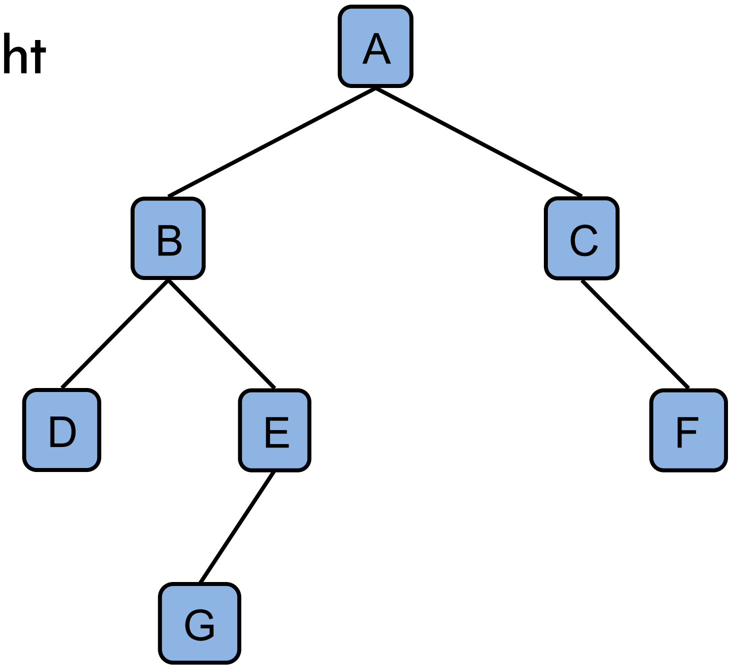


Postorder traversal of subtree

Binary Trees

A binary tree is an ordered tree with the following properties:

- Each internal node has at most two children
- Each child node is labeled as a left child or a right child
- Child ordering is left followed by right



The right/left subtree is the subtree root at the right/left child.

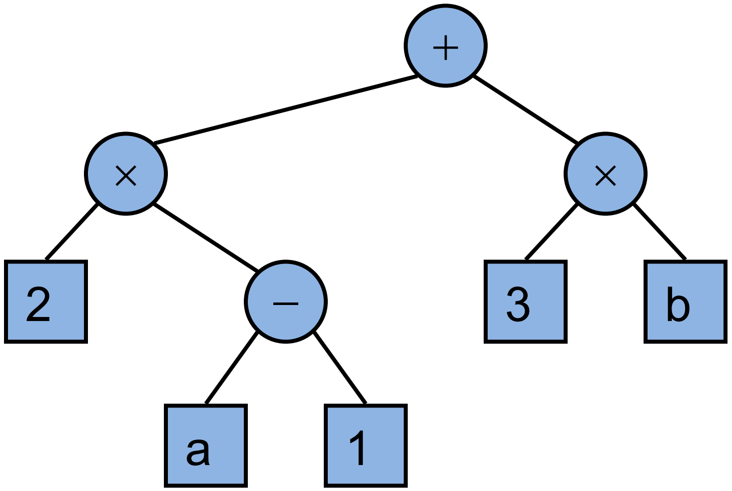
We say the tree is proper if every internal node has two children

Binary tree application: Arithmetic expression tree

Binary tree associated with an arithmetic expression

- internal nodes: operators
- external nodes: operands

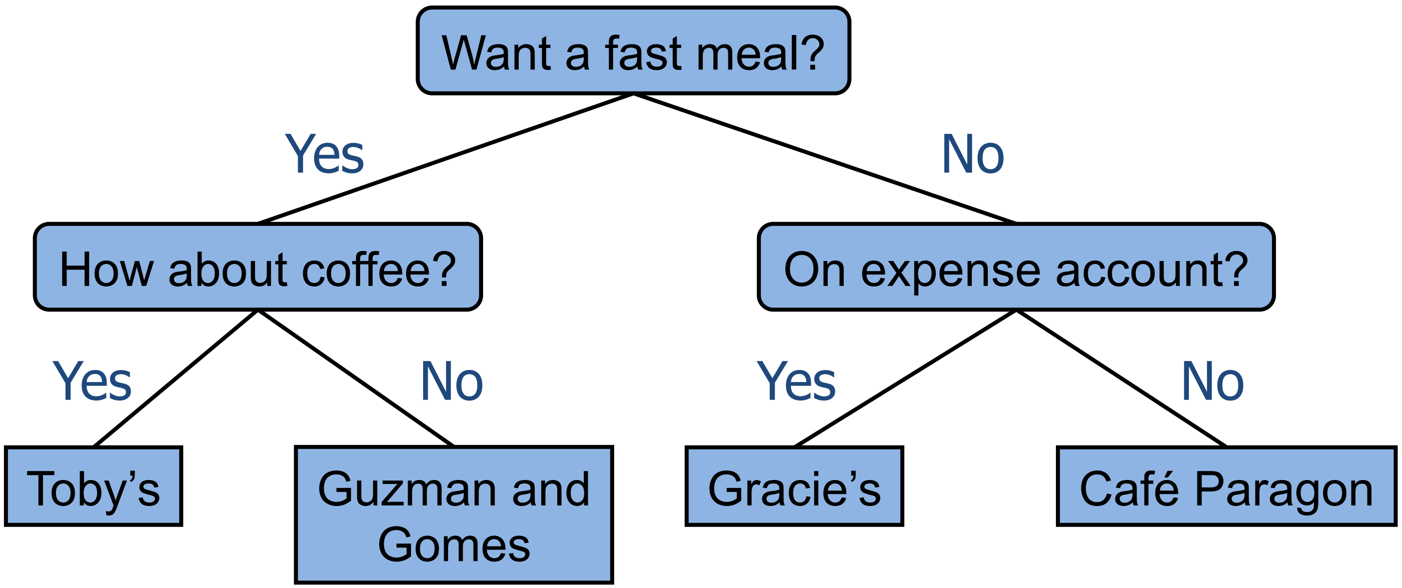
Example: Arithmetic expression tree for $(2 \times (a - 1) + (3 \times b))$



Binary tree application: Decision trees

Tree associated with a decision process
- internal nodes: questions with yes/no answer
- external nodes: decisions

Example: dining decision



Binary Tree Operations

- A binary tree extends the Tree operations, i.e., it inherits all the methods of a tree.
- Update methods may be defined by data structures implementing the binary tree
- Additional methods:
  - position leftChild(p)
  - position rightChild(p)
  - position sibling(p)

return null when there is no left, right, or sibling of p, respectively

Node object

Node object implementation typically has the following attributes:

- value: the value associated with this Node
- left: left child of this Node
- right: right child of this Node
- parent: (optional) the parent of this Node

```
def is_external(v)
    # test if v is a leaf
    return v.left = null and v.right = null
```

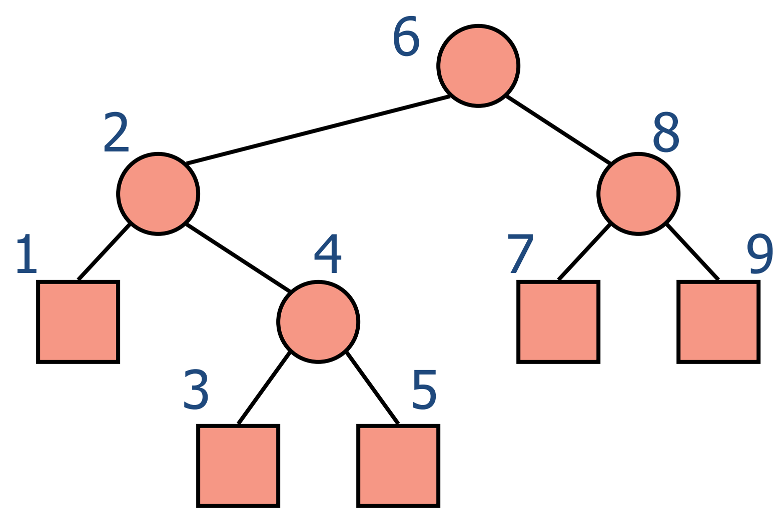
Inorder Traversal

To do an inorder traversal starting at a given node, the node is visited <u>after</u> its left subtree but <u>before</u> its right subtree

Visit does some work on the node:

- print node data
- aggregate node data
- modify node data

```
def in_order(v)
   if v.left ≠ null then
      in_order(v.left)
   visit(v)
   if v.right ≠ null then
      in_order(v.right)
```

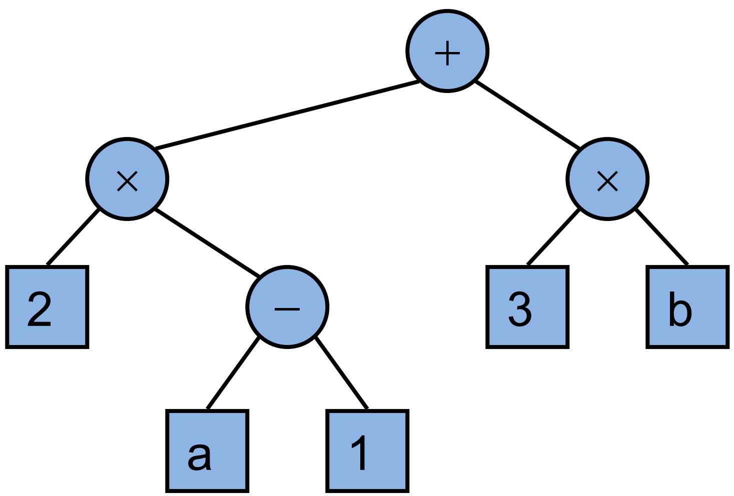


Print Arithmetic Expressions

Extended inorder traversal:

- print operand or operator when visiting node
- print "(" before left subtree
- print ")" after right subtree

```
def print_expr(v)
   if v.left ≠ null then
      print("(")
      print_ expr(v.left)
   print(v.element)
   if v.right ≠ null then
      print_expr(v.right)
      print (")")
```



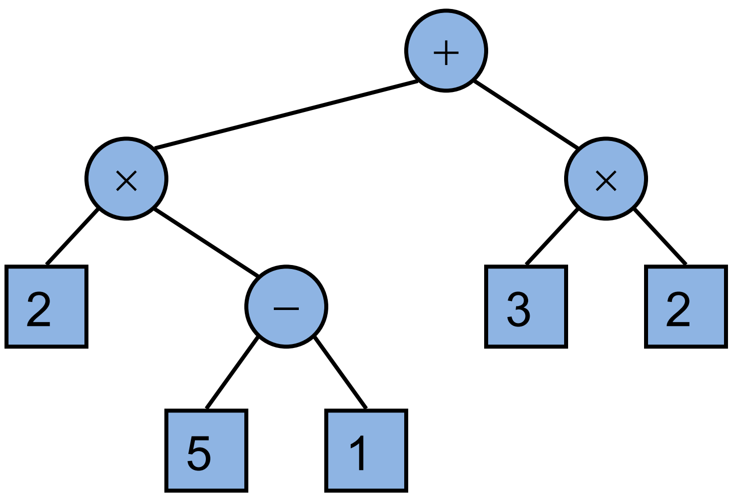
$$((2 \times (a - 1)) + (3 \times b))$$

Evaluate Arithmetic Expressions

Extended postorder traversal

- recursive method returning the value of a subtree
- when visiting an internal node, combine the values of the subtrees

```
def eval_expr(v)
  if v.is_external() then
    return v.element
  else
    x ← eval_expr(v.left)
    y ← eval_expr(v.right)
    ⊕ ← v.element
    return x ⊕ y
```

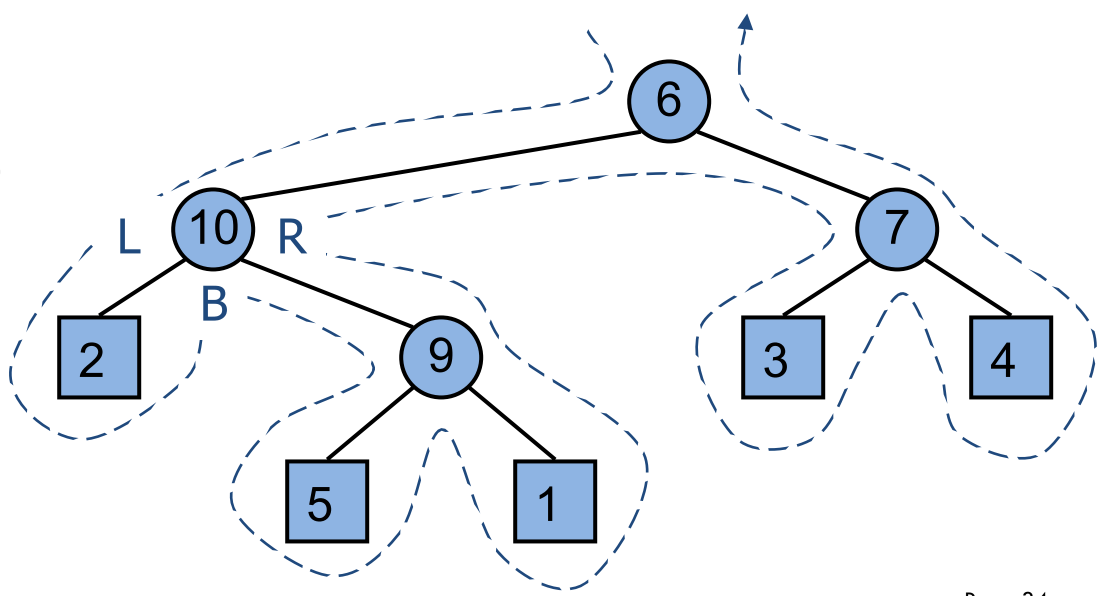


Euler Tour Traversal

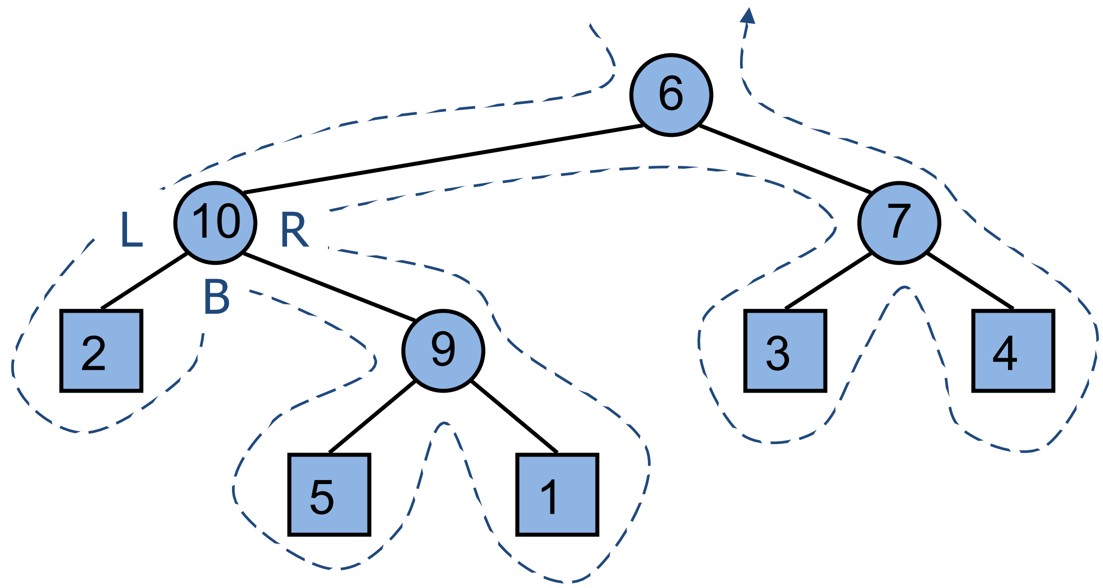
Generic traversal of a binary tree. Includes as special cases the preorder, postorder and inorder traversals

Walk around the tree, keeping the tree on your left, and visit each node three times:

- on the left (preorder)
- from below (inorder)
- on the right (postorder)



Euler Tour Traversal



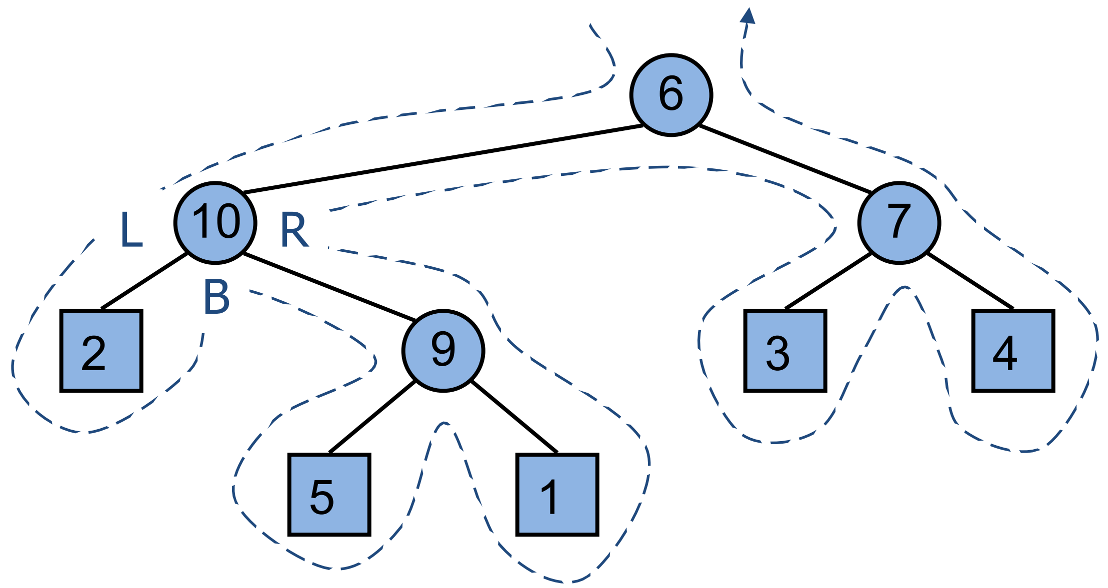
6,10,2,2,2,10,9,5,5,5,9,1,1,1,9,10,6,7,3,3,3,7,4,4,4,7,6

Preorder (first visit):

Inorder (second visit):

Postorder (third visit):

Euler Tour Traversal



6,10,2,2,2,10,9,5,5,5,9,1,1,1,9,10,6,7,3,3,3,7,4,4,4,7,6

Preorder (first visit): 6, 10, 2, 9, 5, 1, 7, 3, 4

Inorder (second visit): 2, 10, 5, 9, 1, 6, 3, 7, 4

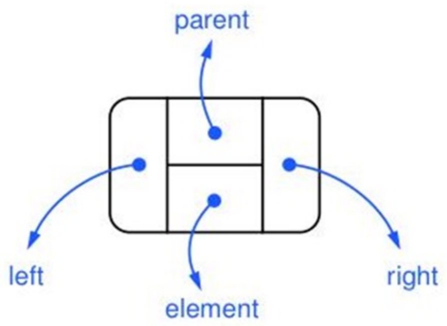
Postorder (third visit): 2, 5, 1, 9, 10, 3, 4, 7, 6

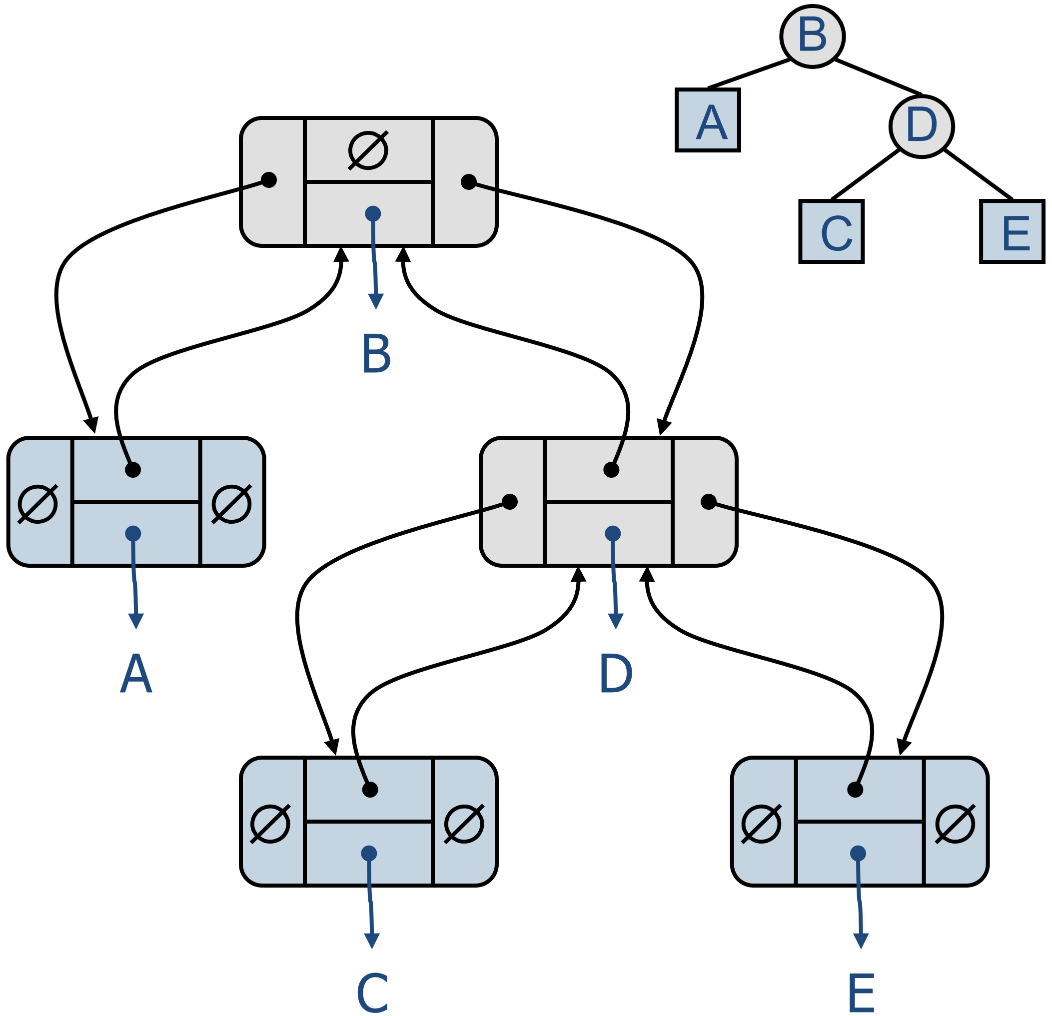
Linked Structure for Binary Trees

A node is represented by an object storing

- Element
- Parent node
- Left child node
- Right child node

Node objects implement the Position ADT

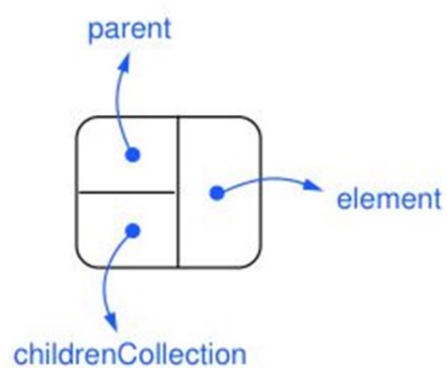


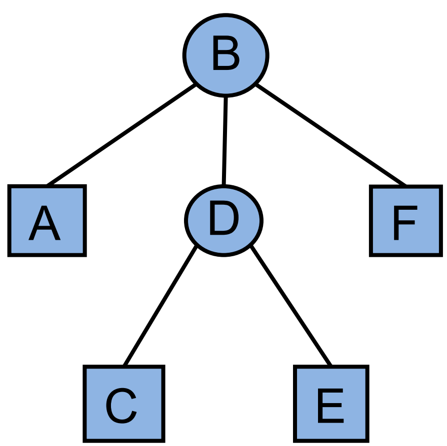


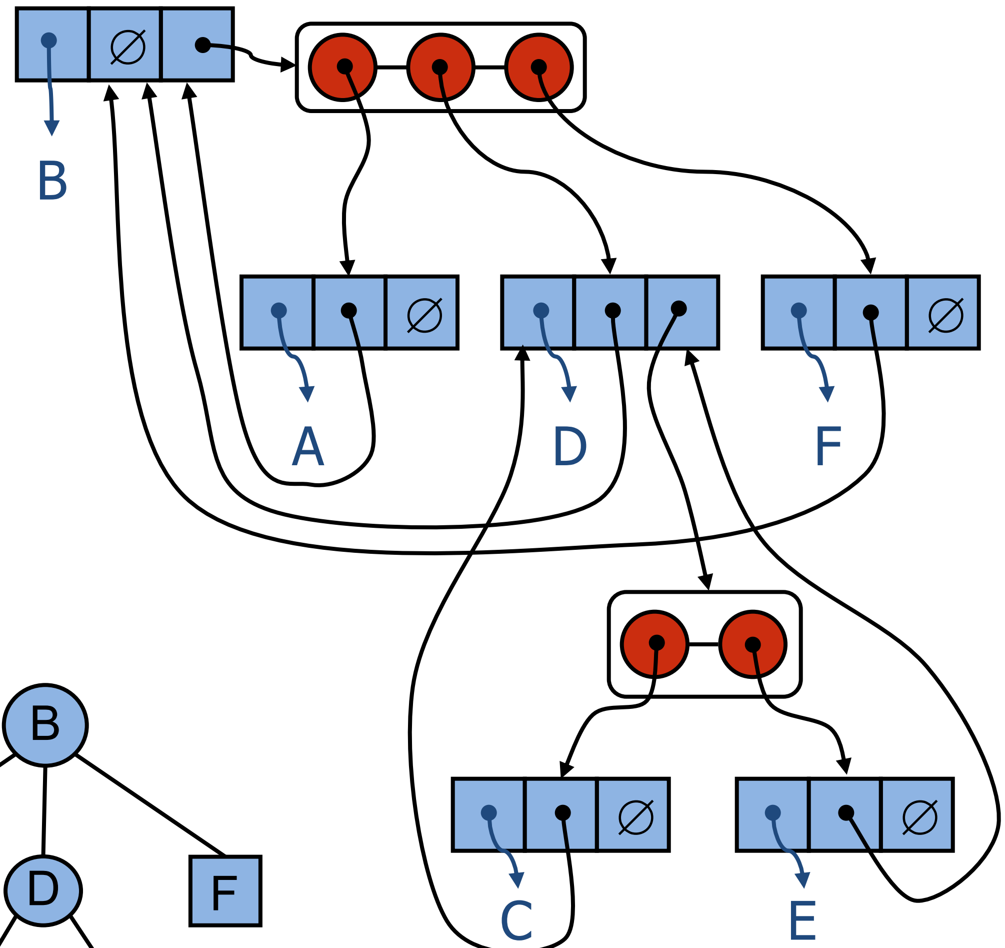
Linked Structure for General Trees

A node is represented by an object storing
- Element
- Parent node
- Sequence of children

Node objects implement the Position ADT







Examples of recursive code on trees

Calculating depth

```
def depth(v)
   if v.parent = null then
      return 0
   else
      return depth(v.parent) + 1
```

Calculating height

```
def height(v)
   if v.isExternal() then
      return 0
   else
      h ← 0
      for each child w of v
         h ← max(h, height(w))
      return h + 1
```

Complexity analysis of recursive algorithms on trees

Sometimes, the method may call itself on all children

- In worst case, do a call on every node
- If the work done, *excluding the recursion*, is constant per call, then the total cost is linear in the number of nodes

Sometimes, the method calls itself on at most one child

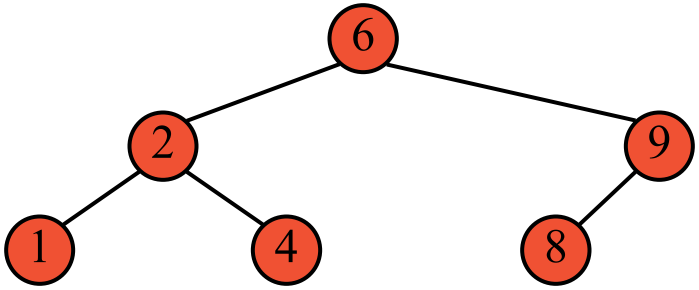
- In worst case, do one call at each level of the tree
- If the work done, *excluding the recursion*, is constant per call, then the total cost is linear in the height of the tree

Binary Search Tree

So far we've been focused on the structure of the tree. The real usefulness of trees hinges on the values we store at each element and how these values are laid out.

BST is a data structure for storing values that can be sorted. These values are laid out so that an in-order traversal of the BST visits the values in sorted order.

Can search for elements and insert/delete operations run in O(log n) time provided the tree is "balanced". More on that next week!

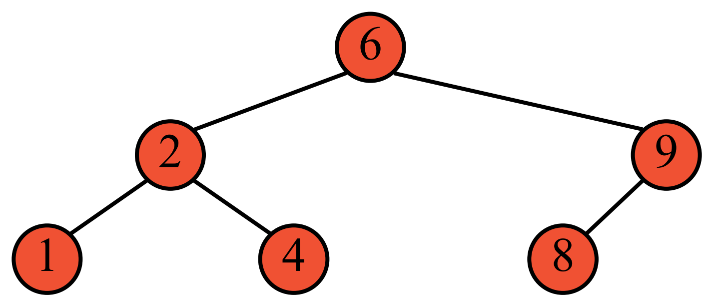


Binary Search Trees (BST)

A binary search tree is a binary tree storing keys (or key-value pairs) satisfying the following BST property

> For any node v in the tree and
> any node u in the left subtree of v and
> any node w in the right subtree of v,
>
> key(u) < key(v) < key(w)

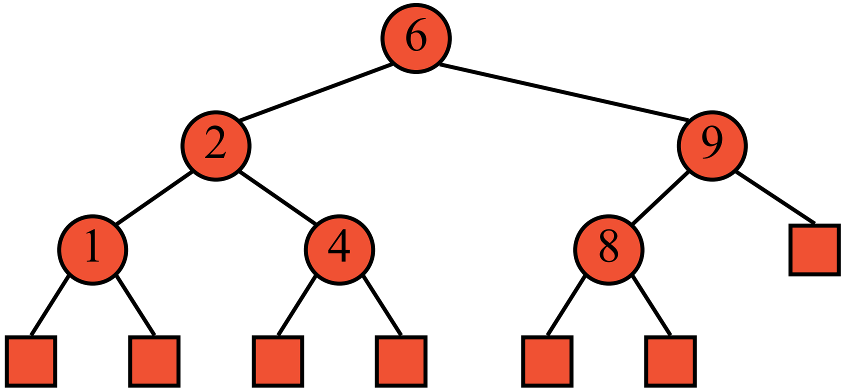
Note that an inorder traversal of a binary search tree visits the keys in increasing order.



BST Implementation

To simplify the presentation of our algorithms, we only store keys (or key-value pairs) at internal nodes

External nodes do not store items (and with careful coding, can be omitted, using null to refer to such)

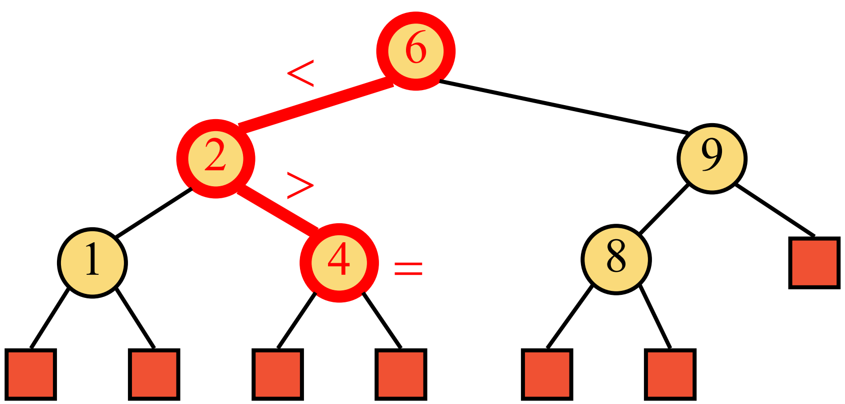


Searching with a Binary Search Tree

To search for a key k, we trace a downward path starting at the root

To decide whether to go left or right, we compare the key of the current node v with k

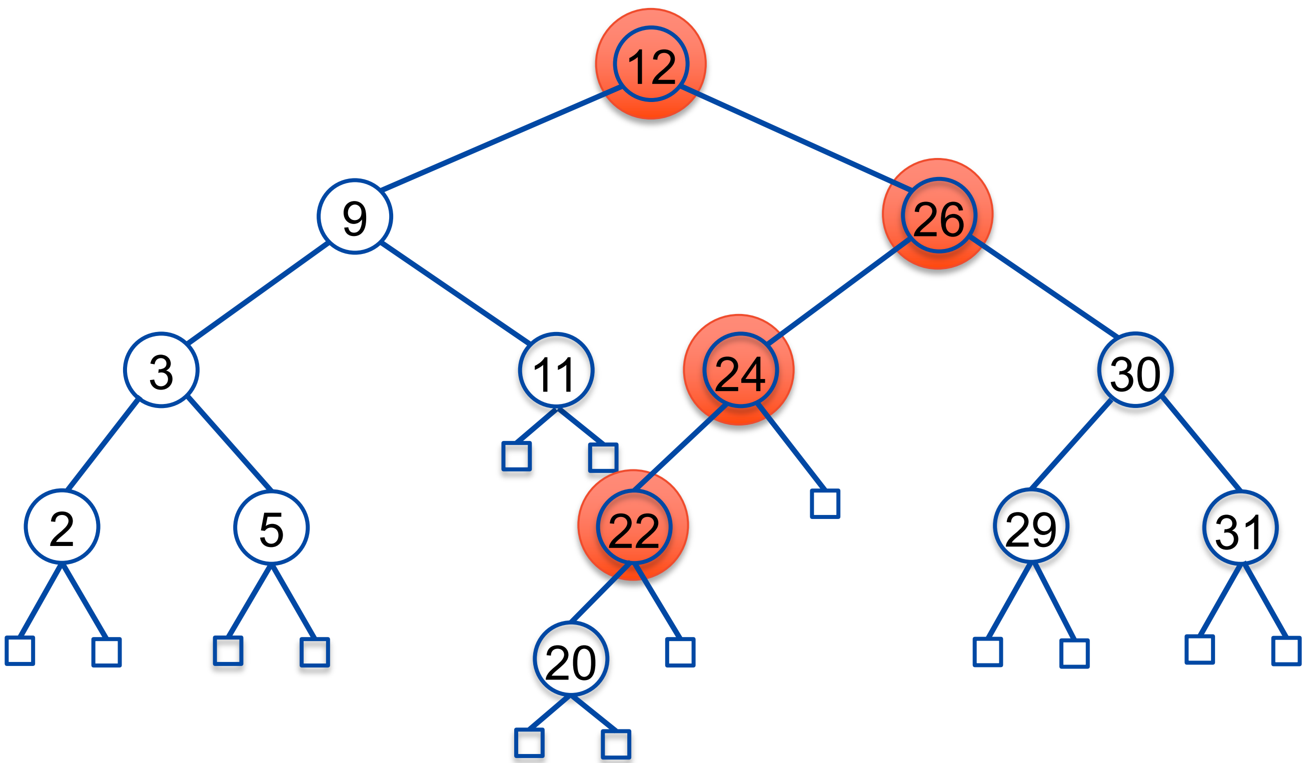
If we reach an external node, this means that the key is not in the data structure




```
def search(k, v)
   if v.isExternal() then
      # unsuccessful search
      return v
   if k = key(v) then
      # successful search
      return v
   else if k < key(v) then
      # recurse on left subtree
      return search(k, v.left)
   else
      # that is  k > key(v)
      # recurse on right subtree
      return search(k, v.right)
```

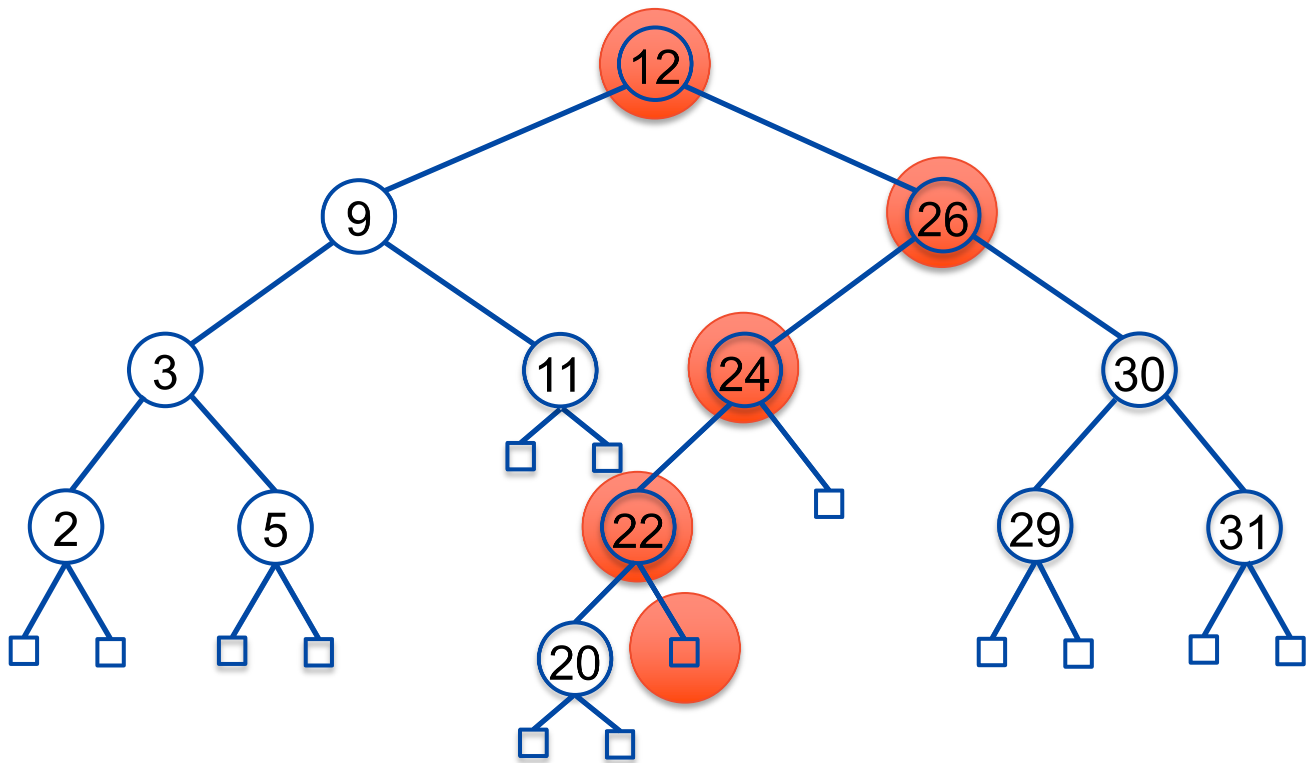
Example: Find 22

S={2,3,5,9,11,12,20,22,24,26,29,30,31}



Example: Find 23

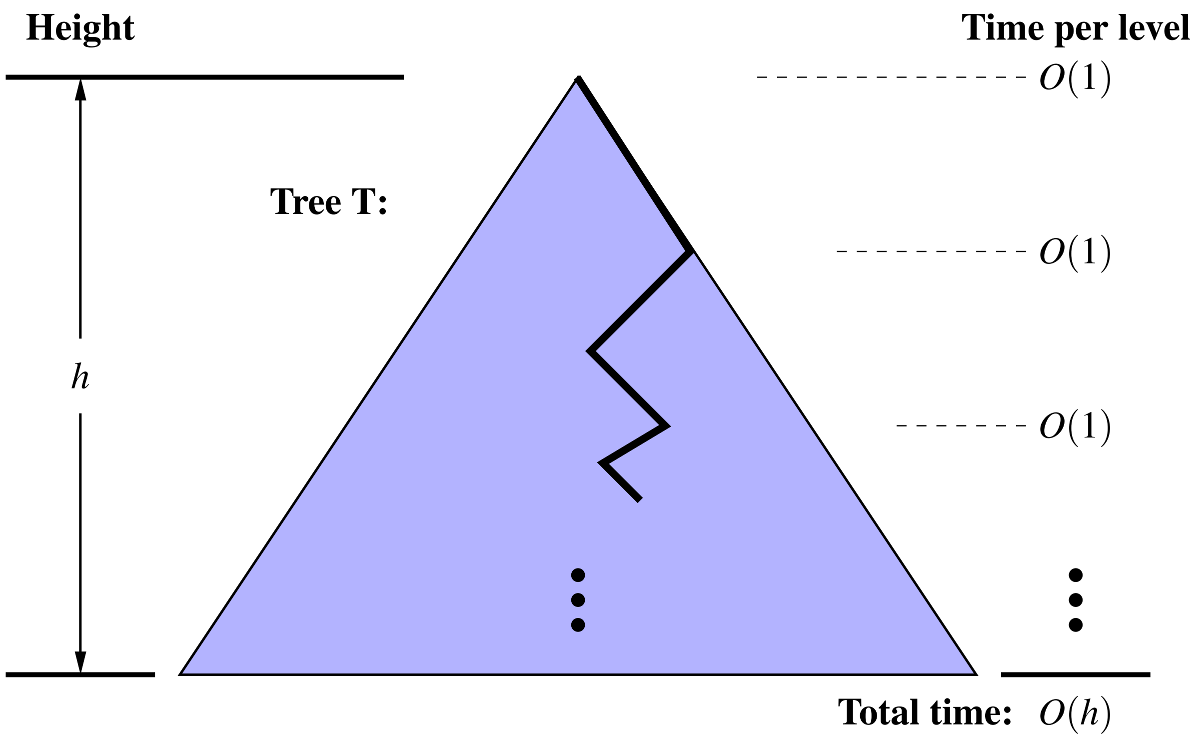
S={2,3,5,9,11,12,20,22,24,26,29,30,31}



Analysis of Binary Tree Searching

Runs in O(h) time, where h is the height of the tree

Next week we will see how to balance a BST so that h = O(log n) even if we insert and delete nodes



B-trees

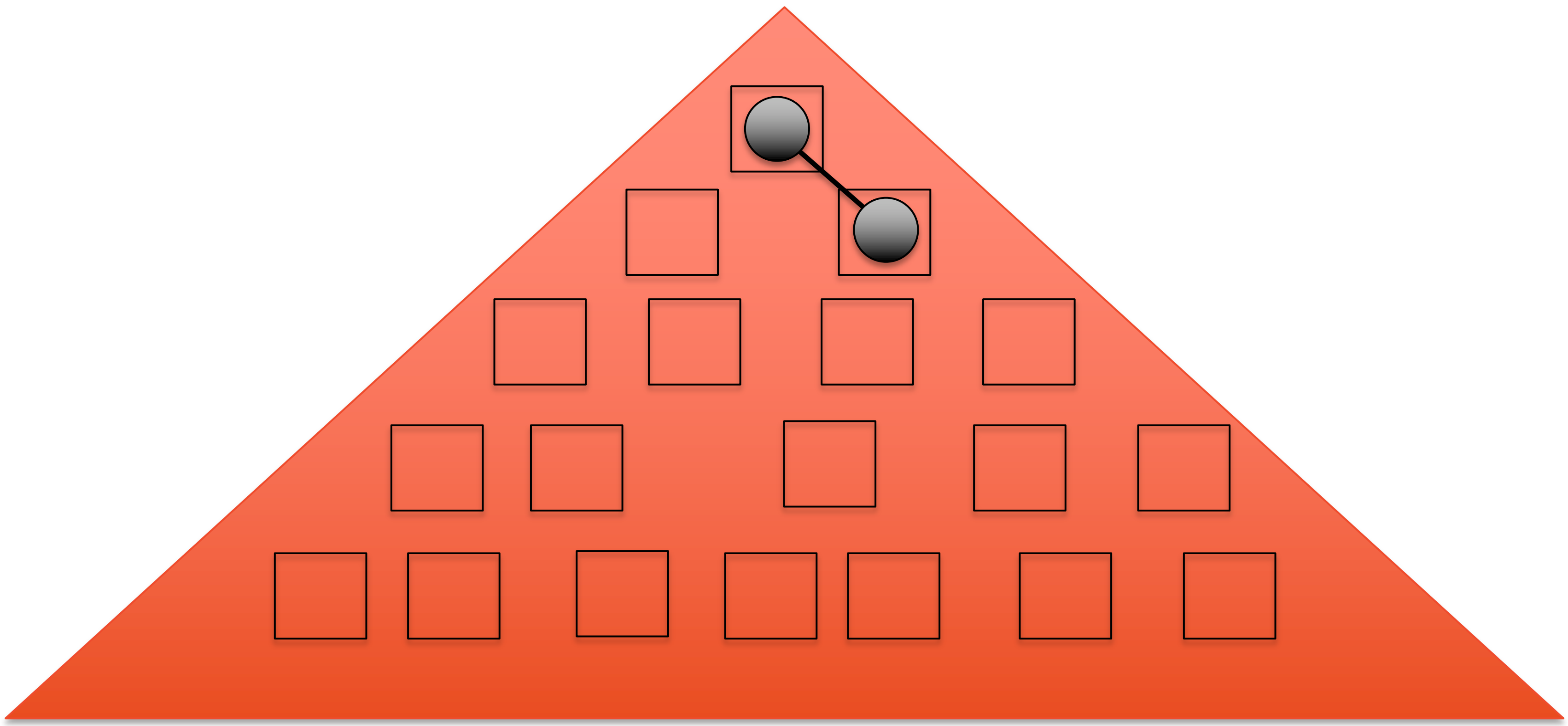
Database systems use a generalization of BST to index data and allow fast lookups.

Two key features of external memory complicate the task:

- External memory is orders of magnitude slower than internal memory
- External memory is transferred in blocks of size B (order of 10^3 bytes)

Our goal is to minimize the number of transfers (a.k.a. I/O operation)

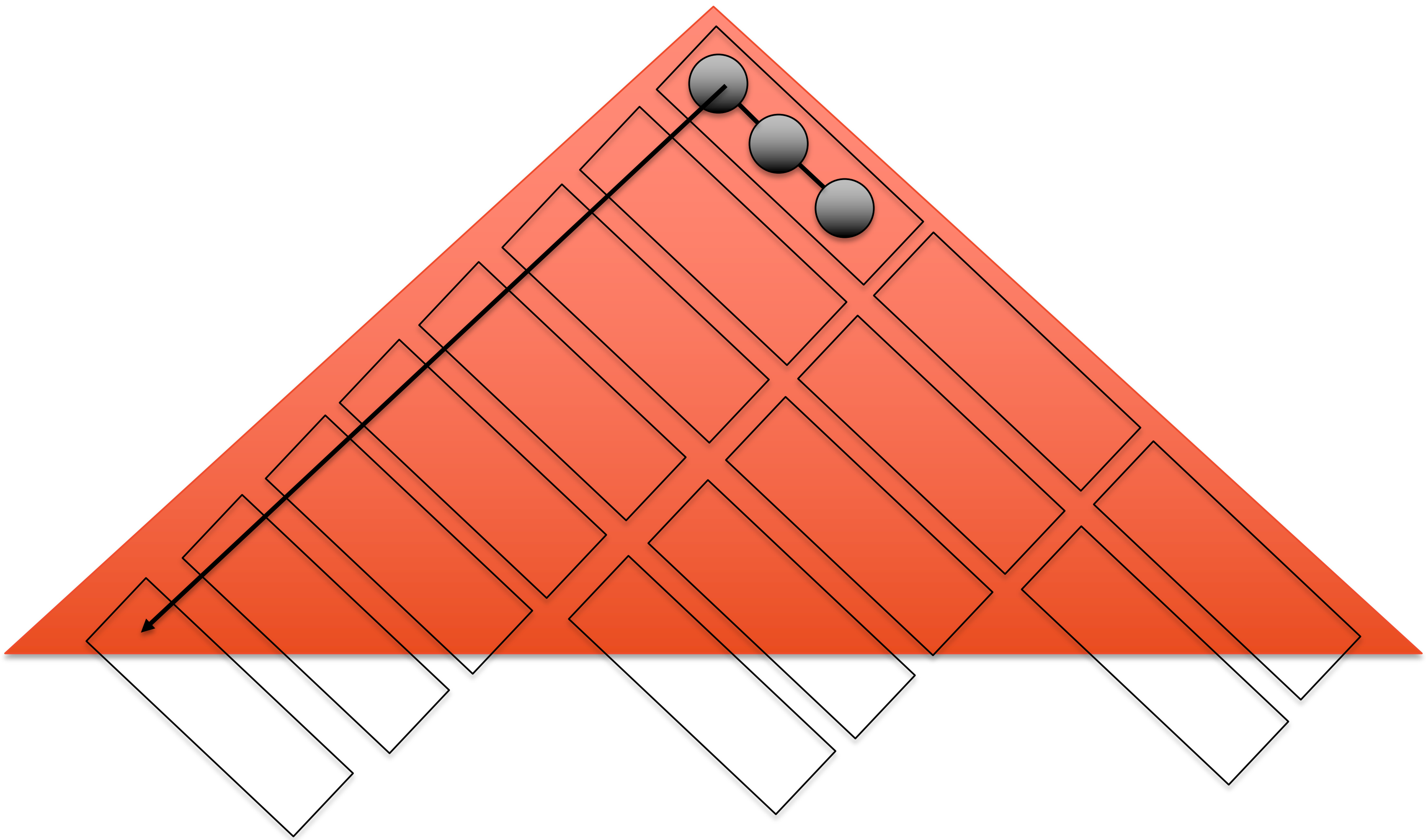
How do you store a BST for fast searching?



Store as BST would

If we implemented BST as is in external memory we need $\log_2 n$ I/Os even if the tree is perfectly balanced. When n is very large and the throughput of queries is high, this becomes too slow, so databases systems resort to a different data structure.

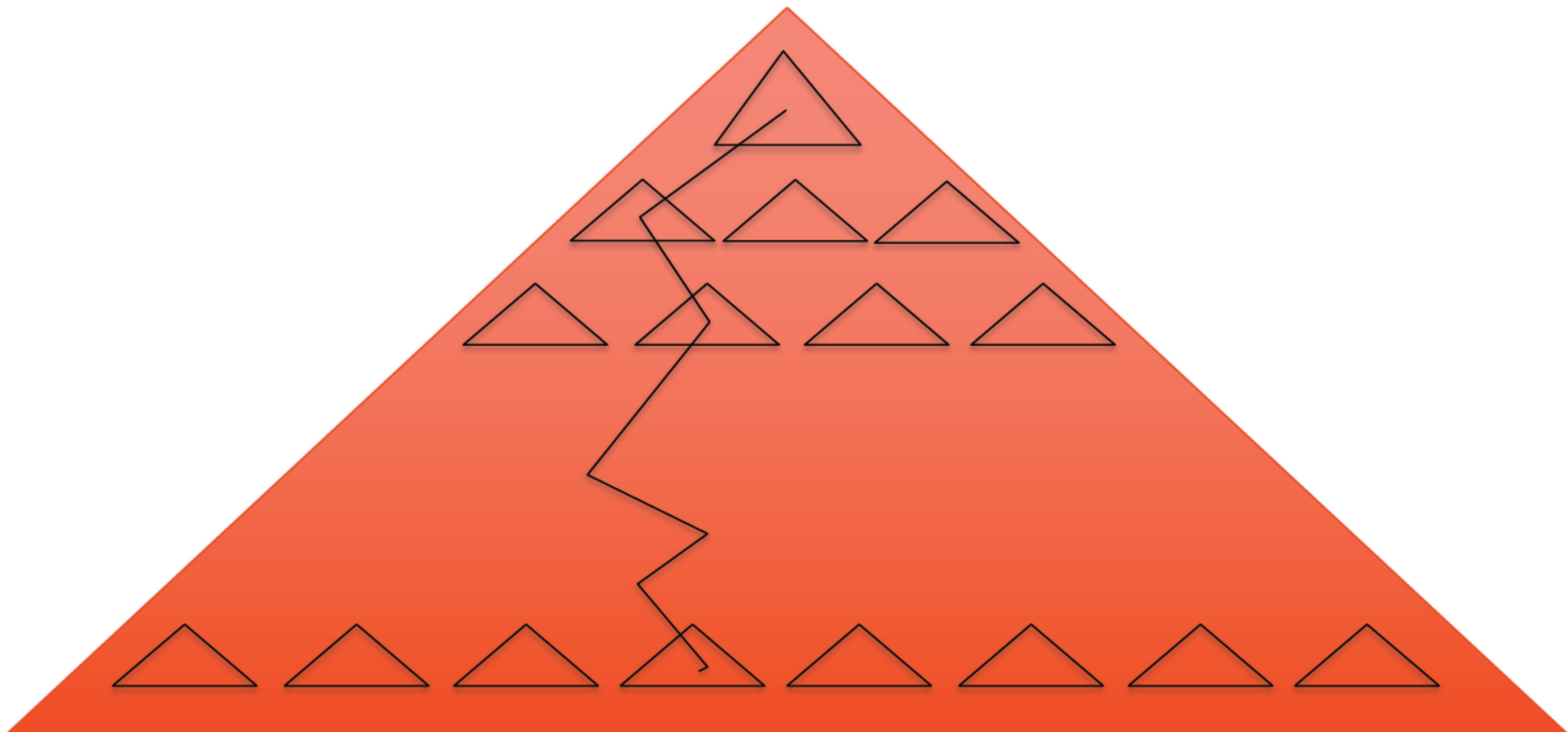
How do you store BSTs for fast searching?



High-level idea behind B-trees

Group tree into chunks of size B and layout each chunk in its own external memory block.

Number of I/Os is equal to the number of chunks we need to fetch.

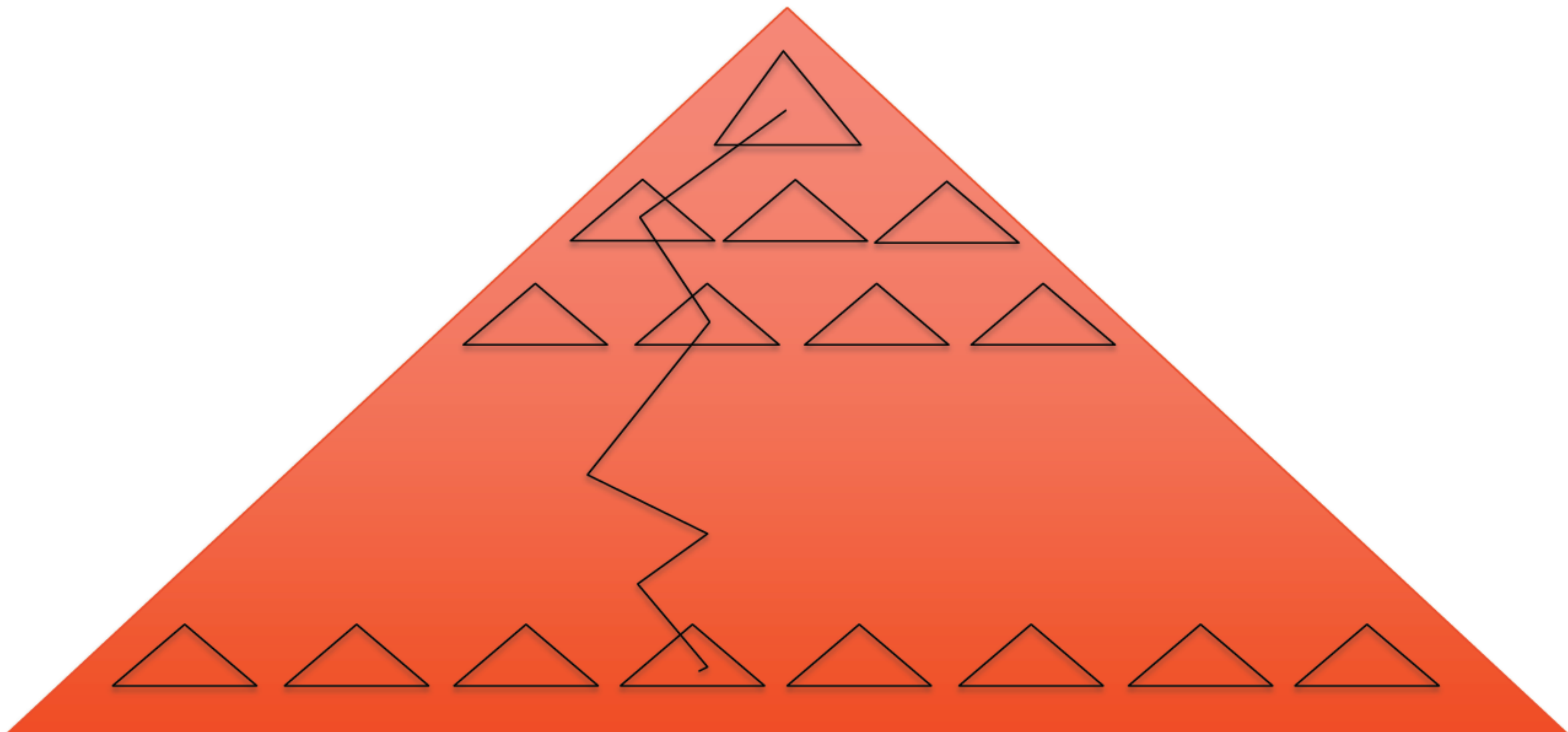


Performance of B-trees

Each block has close to B nodes, so each chunk has height close to $\log_2 B$

Thus, the number of I/Os is close to $\log_2 n / \log_2 B$ or equivalently $\log_B n$

Recall that $B \approx 10^3$, so $\log_2 B \approx 10$



B-trees