The University of Sydney

MATH1902 Linear Algebra (Advanced)

Semester 1 **Tutorial Exercises for Week 1** 2023

Each tutorial sheet starts with a summary of the important definitions, results, and formulae from the Monday lecture. The Tutorial Questions are intended to be done in the tutorial hour, and will cover the lecture material. The Extra Questions are intended to be done either in the tutorial hour, or else later in your own time. The Challenge Questions are tougher, but should be attempted. Some of these harder questions can be treated as discussion-style questions, rather than something that requires a careful written answer. Short answers are provided to selected exercises. **Solutions to all exercises will be available from the course webpage at the end of each week.**

Summary of definitions and results from Week 1

(i) A *vector* $\mathbf{v}$ is a directed line segment that corresponds to a displacement from a point A to a point B. The vector from A to B is denoted $\overrightarrow{AB}$, with A the *initial point* or *tail* and B the *terminal point* or *head*.

(ii) The set of all points in n-dimensional space corresponds to the set of all vectors with tails at the origin O. To each point A, there corresponds the vector $\overrightarrow{OA}$, called the *position vector* of A.

(iii) If A is the point in the plane with coordinates (a_1, a_2), then $\overrightarrow{OA} = [a_1, a_2]$. The entries a_1 and a_2 are the *components* of the vector $\overrightarrow{OA}$. We often use the *column vector* $\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$ rather than the *row vector* $[a_1, a_2]$. Similarly if A is the point in n-dimensional space with coordinates $(a_1, a_2, \ldots, a_n)$, then the vector $\overrightarrow{OA}$ is $[a_1, a_2, \ldots, a_n]$ or $\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$, and this vector has ith component a_i.

(iv) Two vectors are equal if and only if their corresponding components are equal. Geometrically, two vectors are equal if and only if they have the same length and the same direction, equivalently, we could translate one on to the other. A vector with its tail at the origin is in *standard position*.

(v) The set of all vectors with n components is denoted $\mathbb{R}^n$. This is the same as the set of all ordered n-tuples, written as row or column vectors.

(vi) If $\mathbf{u} = [u_1, u_2]$ and $\mathbf{v} = [v_1, v_2]$ then their *sum* $\mathbf{u} + \mathbf{v}$ is the vector $\mathbf{u} + \mathbf{v} = [u_1 + v_1, u_2 + v_2]$. Similarly in $\mathbb{R}^n$, if $\mathbf{u} = [u_1, u_2, \ldots, u_n]$ and $\mathbf{v} = [v_1, v_2, \ldots, v_n]$, then

$$\mathbf{u} + \mathbf{v} = [u_1 + v_1, u_2 + v_2, \ldots, u_n + v_n].$$

(vii) Head-to-Tail Rule for Vector Addition: Translate $\mathbf{v}$ so its tail is at the head of $\mathbf{u}$. Then $\mathbf{u} + \mathbf{v}$ is the vector from the tail of $\mathbf{u}$ to the head of $\mathbf{v}$.

(viii) Parallelogram Rule: If $\mathbf{u}$ and $\mathbf{v}$ are in standard position, then $\mathbf{u} + \mathbf{v}$ is the vector in standard position along the diagonal of the parallelogram determined by $\mathbf{u}$ and $\mathbf{v}$.

(ix) The *zero vector* in $\mathbb{R}^n$ is $\mathbf{0} = [0, 0, \ldots, 0] = \overrightarrow{OO}$. This vector has length 0 and its direction is not defined. For all vectors $\mathbf{u}$ in $\mathbb{R}^n$, $\mathbf{u} + \mathbf{0} = \mathbf{u}$.

(x) A *scalar* is a real number. Given a scalar c and a vector $\mathbf{v}$, the *scalar multiple* $c\mathbf{v}$ is the vector obtained by multiplying each component of $\mathbf{v}$ by c. So if $\mathbf{v} = [v_1, v_2]$ then $c\mathbf{v} = [cv_1, cv_2]$. Similarly in $\mathbb{R}^n$, if $\mathbf{v} = [v_1, v_2, \ldots, v_n]$ then $c\mathbf{v} = [cv_1, cv_2, \ldots, cv_n]$. For all vectors $\mathbf{v}$, $0\mathbf{v} = \mathbf{0}$ and $1\mathbf{v} = \mathbf{v}$, and for all scalars c and d, $c(d\mathbf{v}) = (cd)\mathbf{v}$. Geometrically, $c\mathbf{v}$ points in the same direction as $\mathbf{v}$ if $c > 0$ and the opposite direction if $c < 0$, and $c\mathbf{v}$ has length $|c|$ times the length of $\mathbf{v}$. Two vectors are scalar multiples of each other if and only if they are parallel.

(xi) The *negative* of a vector $\mathbf{v}$ is the vector $(-1)\mathbf{v} = -\mathbf{v}$. This has the same length as $\mathbf{v}$ but points in the opposite direction. For all vectors $\mathbf{v}$, $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$. If A and B are points then $\overrightarrow{AB} = -\overrightarrow{BA}$.

(xii) We define *vector subtraction* by $\mathbf{u} - \mathbf{v} = \mathbf{u} + (-\mathbf{v})$.

(xiii) Commutative Law of Addition: For all vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$, $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.

(xiv) Associative Law of Addition: For all vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ in $\mathbb{R}^n$, $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$.

(xv) Distributive Laws: For all scalars c and d and all vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$, $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$ and $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$.

(xvi) A vector $\mathbf{v}$ is a *linear combination* of vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ if there are scalars $c_1, c_2, \ldots, c_k$ such that

$$\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k.$$

The scalars $c_1, c_2, \ldots, c_k$ are the *coefficients* of this linear combination.

(xvii) Vector Space: Any set of vectors and scalars (either real or complex numbers) that satisfy the 8 rules stated above (commutative addition, associative addition, there is a zero vector $\mathbf{0}$ such that $\mathbf{0} + \mathbf{v} = \mathbf{v}$ for all $\mathbf{v}$, for every $\mathbf{v}$ there is the negative $-\mathbf{v}$ such that $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$, $a(b\mathbf{v}) = (ab)\mathbf{v}$, $1\mathbf{v} = \mathbf{v}$, and the two laws of distributivity) is called a vector space.

Tutorial Questions

1. Let $\mathbf{a} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ and $\mathbf{c} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

(i) Draw these vectors in standard position in $\mathbb{R}^2$.

(ii) Compute the vectors $\mathbf{a} + \mathbf{b}$, $\mathbf{b} + \mathbf{c}$ and $\mathbf{a} - \mathbf{c}$. How can these results be obtained geometrically?

(iii) Draw the vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ with their tails at the point $(2, -1)$.

2. Let $\mathbf{a} = [0, 2, 1]$, $\mathbf{b} = [1, 2, \frac{1}{3}]$ and $\mathbf{c} = [-1, -\frac{1}{2}, 5]$.

(i) Draw these vectors in standard position in $\mathbb{R}^3$.

(ii) Compute the vectors $2\mathbf{a} + 3\mathbf{b}$ and $-\mathbf{a} + 4\mathbf{b} - \mathbf{c}$.

3. If the vector $\mathbf{v}$ has length 2, find the length of the vector $\mathbf{u}$ in each of the following cases.

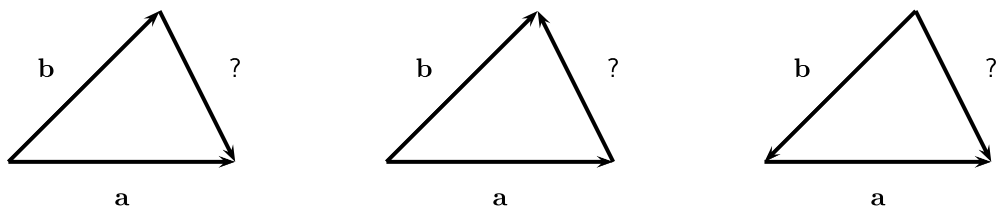
(i) $\mathbf{u} = 3\mathbf{v}$

(ii) $\mathbf{u} = \frac{1}{2}\mathbf{v}$

(iii) $\mathbf{u} = -3\mathbf{v}$

(iv) $\mathbf{v} = 3\mathbf{u}$

4. In each diagram below, find the unknown vector in terms of $\mathbf{a}$ and $\mathbf{b}$.



5. Solve for $\mathbf{x}$ in terms of $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ in each case.

(i) $\mathbf{v} + \mathbf{x} = \mathbf{u} - \mathbf{w}$

(ii) $\mathbf{v} - \mathbf{x} = \mathbf{w} - \mathbf{u}$

(iii) $2\mathbf{v} + \mathbf{x} = 2\mathbf{w} - 2\mathbf{u} - \mathbf{x}$

6. A balloon experiences two forces, a buoyancy force of 8 newtons vertically upwards and a wind force of 6 newtons acting horizontally to the right. Calculate the magnitude and direction of the resultant force.

7. * Let $\mathbf{u} = [3,1]$ and $\mathbf{v} = [-1,1]$. Show that the vector $\mathbf{w} = [-7,-1]$ can be expressed as a linear combination of $\mathbf{u}$ and $\mathbf{v}$, and draw a picture to illustrate this geometrically.

8. Prove the associative law for vector addition: for all vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ in $\mathbb{R}^n$,
$$(\mathbf{a}+\mathbf{b})+\mathbf{c} = \mathbf{a}+(\mathbf{b}+\mathbf{c}).$$

9. Prove (by contradiction) that in any vector space the zero vector is unique (only using the 8 axioms of a vector space).

Extra Questions

10. Express $2\mathbf{a} - 3\mathbf{b}$ in terms of $\mathbf{u}$ and $\mathbf{v}$, and simplify, when $\mathbf{a} = \mathbf{u}+\mathbf{v}$ and $\mathbf{b} = 3\mathbf{u} - 2\mathbf{v}$.

11. Let $ABCDEF$ be a regular hexagon and put $\mathbf{a} = \overrightarrow{AB}$ and $\mathbf{b} = \overrightarrow{BC}$. Find vector expressions in terms of $\mathbf{a}$ and $\mathbf{b}$ for the displacements $\overrightarrow{CD}$, $\overrightarrow{DE}$, $\overrightarrow{EF}$ and $\overrightarrow{FA}$.

12. A plane travels 20km in the direction $30°$ north of east and then 10 km southeast. Use trigonometry and your calculator to find the final distance and direction of the aircraft from the starting position.

13. Prove the following distributive laws:
    (i) For all scalars c and all vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$, $c(\mathbf{u}+\mathbf{v}) = c\mathbf{u} + c\mathbf{v}$.
    (ii) For all scalars c and d and all vectors $\mathbf{u}$ in $\mathbb{R}^n$, $(c+d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$.

Challenge Questions

14. Prove that $a\mathbf{0} = \mathbf{0}$ for any scalar a (only using the 8 axioms of a vector space).

15. Prove that $0\mathbf{v} = \mathbf{0}$ for any vector $\mathbf{v}$ (only using the 8 axioms of a vector space).

16. Prove that $(-1)\mathbf{v}$ is the additive inverse of $\mathbf{v}$ (only using the 8 axioms of a vector space). Note the distinction here between the additive inverse $-\mathbf{v}$ and the scalar multiplication by minus one $(-1)\mathbf{v}$.

17. Show that the complex numbers $x+iy$ form a vector space where the scalars are the real numbers.

18. Consider vectors of the form $[u_1,\ldots,u_n]$ and scalars λ with the usual arithmetic. Write $\mathbb{C}$ for the set of complex numbers. Which of the following three combinations gives a vector space (explain why or why not):
    (i) $u_i \in \mathbb{C}, \lambda \in \mathbb{C}$
    (ii) $u_i \in \mathbb{R}, \lambda \in \mathbb{C}$
    (iii) $u_i \in \mathbb{C}, \lambda \in \mathbb{R}$

Short answers to selected exercises

1. $\mathbf{a}+\mathbf{b} = \begin{bmatrix}5\\2\end{bmatrix}$, $\mathbf{b}+\mathbf{c} = \begin{bmatrix}2\\3\end{bmatrix}$ and $\mathbf{a}-\mathbf{c} = \begin{bmatrix}3\\-1\end{bmatrix}$.

2. $2\mathbf{a}+3\mathbf{b} = [3,10,3]$ and $-\mathbf{a}+4\mathbf{b}-\mathbf{c} = [5,\frac{13}{2},-\frac{14}{3}]$.

3. (i) 6 (ii) 1 (iii) 6 (iv) $2/3$

4. From left to right, $\mathbf{a}-\mathbf{b}$, $\mathbf{b}-\mathbf{a}$ and $\mathbf{a}+\mathbf{b}$.

5. (i) $\mathbf{x} = \mathbf{u}-\mathbf{v}-\mathbf{w}$ (ii) $\mathbf{x} = \mathbf{u}+\mathbf{v}-\mathbf{w}$ (iii) $\mathbf{x} = -\mathbf{u}-\mathbf{v}+\mathbf{w}$

6. 10 newtons, $53°$ to the horizontal towards the right.

7. $\mathbf{w} = -2\mathbf{u}+\mathbf{v}$

10. $2\mathbf{a}-3\mathbf{b} = -7\mathbf{u}+8\mathbf{v}$

11. $\overrightarrow{CD} = \mathbf{b}-\mathbf{a}$, $\overrightarrow{DE} = -\mathbf{a}$, $\overrightarrow{EF} = -\mathbf{b}$, $\overrightarrow{FA} = \mathbf{a}-\mathbf{b}$.

12. final distance 25 km, final direction $7°$ north of east.