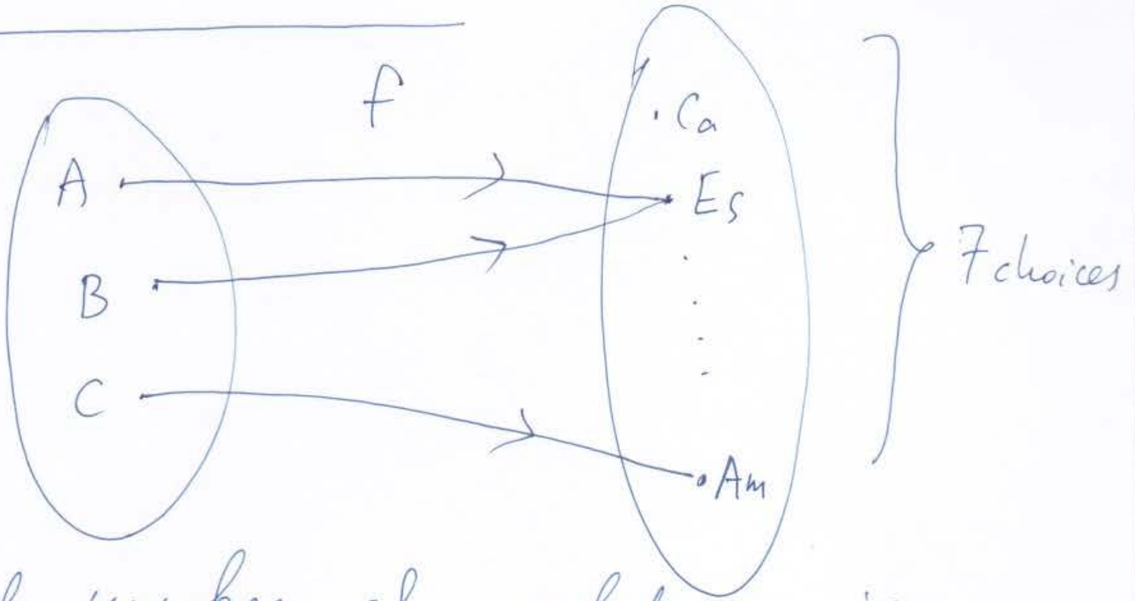
Ordered selections

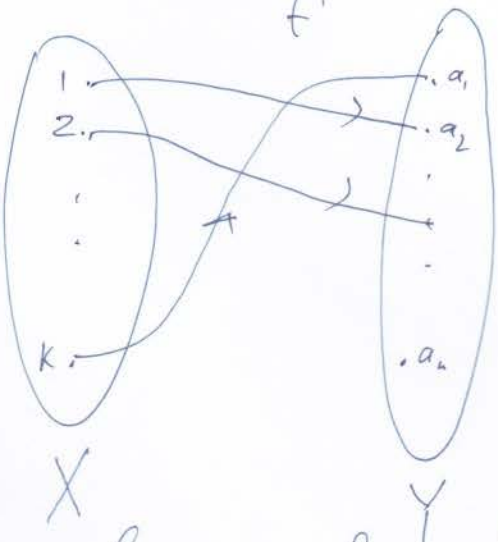
Ex



The total number of selections is $7 \times 7 \times 7 = 7^3$ by the Product Principle.

Thm If X and Y are sets with $|X| = k$ and $|Y| = n$, then the number of functions $f: X \to Y$ is n^k.

Proof



Indeed, the number of functions is found by $\underbrace{n \times n \times \cdots \times n}_{k} = n^k$. $\square$

Every such function corresponds to an <u>ordered selection</u> of k things from n possibilities:

$$f \longleftrightarrow (y_1, y_2, \dots, y_k),$$

each $y_i \in Y$.

<u>Thm</u> The number of subsets of the set $\{1, 2, \dots, n\}$ is 2^n.

<u>Proof</u>. For any subset $A \subset \{1, 2, \dots, n\}$ consider the function

$$f_A : \{1, 2, \dots, n\} \longrightarrow \{0, 1\},$$

with $$f_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases}$$

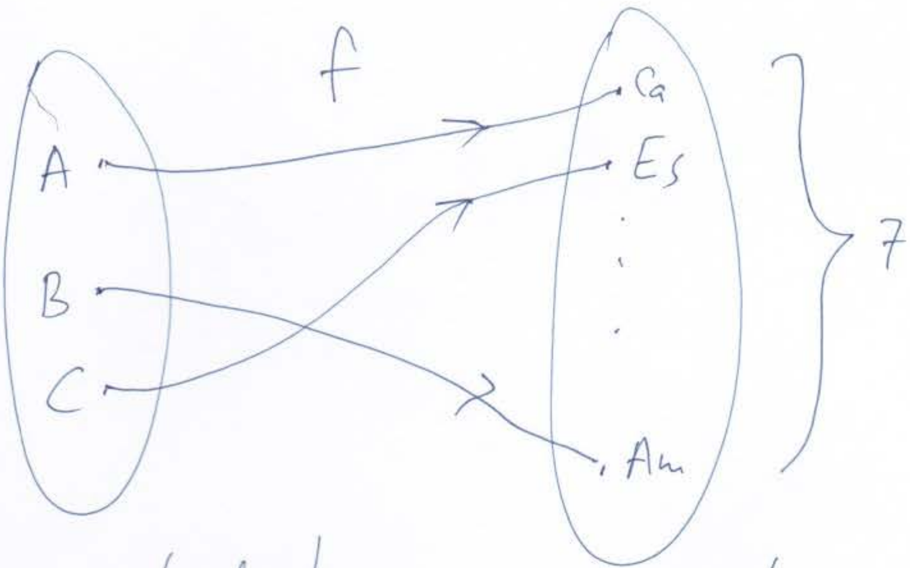
The number of subsets equals the number of functions

$$\{1, 2, \dots, n\} \longrightarrow \{0, 1\}$$

which is 2^n by the previous theorem.

$\square$

Ex

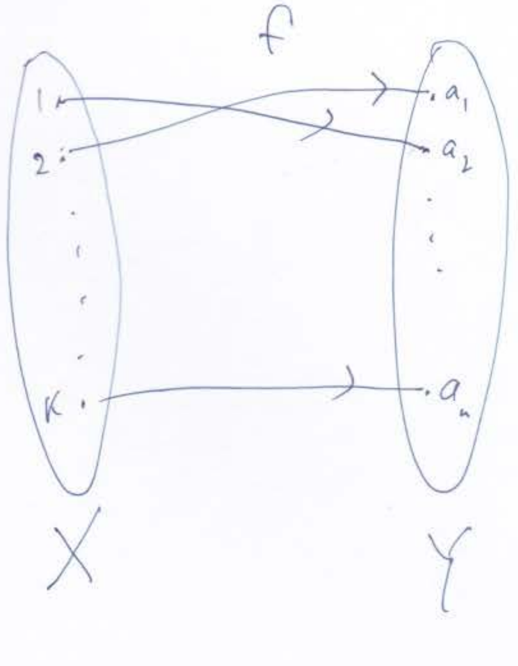


If the students must choose different types of coffee, then the number of selections is

$$7 \times 6 \times 5 = 210.$$

Thm The number of injective functions $f : X \to Y$, where $|X| = k$ and $|Y| = n$, equals

$$n(n-1) \times \cdots \times (n-k+1).$$



If $k > n$ then the number is 0.

This agrees with the formula:

$$n(n-1)\times\cdots\times\underline{\underline{0}}\times(-1)\cdots = 0.$$

We write

$$n_{(k)} = \underbrace{n(n-1)\cdots(n-k+1)}_{k \text{ factors}},$$

This is called the <u>falling factorial</u>.

<u>Def</u> Any bijection $f: X \to X$ is called a <u>permutation</u> of X.

If X is a finite set, then any injective function $f: X \to X$ is bijective.

<u>Corollary</u>. The number of permutations of the set $\{1, 2, \dots, k\}$ is

$$k_{(k)} = k(k-1)\cdots 1$$

which is $k! = 1 \cdot 2 \cdot \cdots \cdot k$.

We set $0! = 1$.

<u>Ex</u> The permutations of the set $\{1, 2, 3\}$ are

$$123, \quad 132, \quad 213, \quad 231, \quad 312, \quad 321,$$

we have $3! = 1 \cdot 2 \cdot 3 = 6$ permutations.

Note that

$$n_{(k)} = n(n-1) \cdots (n-k+1) = \frac{n!}{(n-k)!}.$$

We set $(-m)! = \infty$ for $m = 1, 2, 3, \ldots$.

<u>Ex</u> In how many ways can a group of $2n$ people be split into n pairs for games of tennis?

$$\underbrace{* \quad * \quad \cdots\cdots \quad *}_{2n}$$

There are $2n-1$ choices for the opponent of the first person.

$$\underbrace{* \quad * \quad \cdots \quad *}_{2n-2}$$

There are $2n-3$ choices for the opponent of the first person.

Hence, the answer is

$$(2n-1)(2n-3)\cdots 3\cdot 1 =: (2n-1)!!\,.$$