The University of Sydney
School of Mathematics and Statistics

Practice Class 5

MATH2069: Discrete Mathematics and Graph Theory — Semester 1, 2023

1. Consider the recurrence relation $a_n = 4a_{n-1} - 4a_{n-2} + 3n + 2$, for $n \geq 2$.

(a) If p_n is a particular solution of this recurrence, the general solution is

$$\boxed{a_n = \qquad\qquad\qquad}$$

where b_n is a general solution of the homogeneous recurrence relation, i.e. $b_n = 4b_{n-1} - 4b_{n-2}$.

(b) The characteristic polynomial of the homogeneous recurrence is $x^2 - 4x + 4$. Hence

$$\boxed{b_n = \qquad\qquad\qquad}$$

(c) A particular solution of the form $p_n = An + B$ is

$$\boxed{p_n = \qquad\qquad\qquad}$$

(d) Find the solution of the original recurrence when $a_0 = 15$, $a_1 = 21$:

$$\boxed{a_n = \qquad\qquad\qquad}$$

2. Write down the general solution of each of the following recurrence relations, by finding a particular solution of the stated form.

(a) $a_n = 3a_{n-1} + 2$ for $n \geq 1$; $p_n = A$.

$$\boxed{a_n = \qquad\qquad\qquad}$$

(b) $a_n = 2a_{n-1} + n + 1$ for $n \geq 1$; $p_n = An + B$.

$$\boxed{a_n = \qquad\qquad\qquad}$$

(c) $a_n = 3a_{n-1} - 2^n$ for $n \geq 1$; $p_n = A2^n$.

$$\boxed{a_n = \qquad\qquad\qquad}$$

3. If $A(z)=\sum_{n=0}^{\infty} a_n\, z^n$ and $B(z)=\sum_{n=0}^{\infty} b_n\, z^n$ are two formal power series, complete the definitions of the following formal power series:

$A(z)+B(z)\ =$

$A(z)B(z)\ =$

$A'(z)\ =$

4. For each of the sequences (a)–(g), write the number (i)–(vii) of its generating function.

(a) $1, 2, 2^2, 2^3, \cdots$ ☐

(b) $1, 1, 1, 1, \cdots$ ☐

(c) $3, 2, 1, 0, 0, 0, \cdots$ ☐

(d) $1, 2, 3, 4, 5, \cdots$ ☐

(e) $0, 1, 2, 3, 4, \cdots$ ☐

(f) $0, 0, 1, 2, 3, 4, \cdots$ ☐

(g) $2, 3, 4, 5, 6, \cdots$ ☐

(i) $\dfrac{1}{1-z}$

(ii) $\dfrac{1}{(1-z)^2}$

(iii) $\dfrac{1}{1-2z}$

(iv) $\dfrac{z^2}{(1-z)^2}$

(v) $3+2z+z^2$

(vi) $\dfrac{z}{(1-z)^2}$

(vii) $\dfrac{2-z}{(1-z)^2}$

5. For each of the following power series, give a formula for the coefficient of z^n.

(a) $\dfrac{3}{1-z}$

(b) $\dfrac{1}{1-3z}$

(c) $\dfrac{3}{1+z}$

(d) $\dfrac{3}{(1+z)^2}$

(e) $\dfrac{1}{(1-3z)^2}$

6. Give formulas for the following power series in terms of $A(z) = \sum_{n=0}^{\infty} a_n\, z^n$.

(a) $\displaystyle\sum_{n=0}^{\infty}(a_0 a_n + a_1 a_{n-1} + \cdots + a_{n-1}a_1 + a_n a_0)z^n$

(b) $\displaystyle\sum_{n=0}^{\infty}(a_0 + a_1 + \cdots + a_n)z^n$

(c) $\displaystyle\sum_{n=0}^{\infty}(n+1)a_{n+1}\, z^n$

(d) $\displaystyle\sum_{n=0}^{\infty}\left(\frac{a_0}{n!} + \frac{a_1}{(n-1)!} + \cdots + \frac{a_{n-1}}{1!} + \frac{a_n}{0!}\right) z^n$

7. For each of the following statements, write T for true or F for false.

(a) The sequence $3, -3, 3, -3, \cdots$ has generating function $\frac{3}{1+z}$.

(b) $3 + z$ is an example of a formal power series.

(c) $\sum_{n=1}^{\infty} a_{n-1} z^n = \sum_{n=0}^{\infty} a_n z^{n+1}$.

(d) $\frac{1}{1+3z} = \sum_{n=0}^{\infty} 3^n z^n$.

(e) $\frac{1}{(1-2z)^2} = \sum_{n=0}^{\infty} (n+1) 2^n z^n$.

(f) $\frac{1}{(1-z)^3} = \sum_{n=0}^{\infty} \binom{n}{2} z^n$.

(g) If $\sum_{n=0}^{\infty} a_n z^n = \frac{z}{1-3z}$, then $a_{900} = 3^{901}$.

(h) If $A'(z) = 0$, then $A(z) = a_0$ for some constant a_0.

(i) For $n \geq 1$, the coefficient of z^n in $\frac{4-6z}{1-2z}$ is 2^n.

(j) For $n \geq 2$, the coefficient of z^n in $\frac{1-z^2}{1-z}$ is 1.