The University of Sydney

MATH1902 Linear Algebra (Advanced)

Semester 1 **Tutorial Solutions for Week 1** 2023

Tutorial Questions

1. Let $\mathbf{a} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ and $\mathbf{c} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

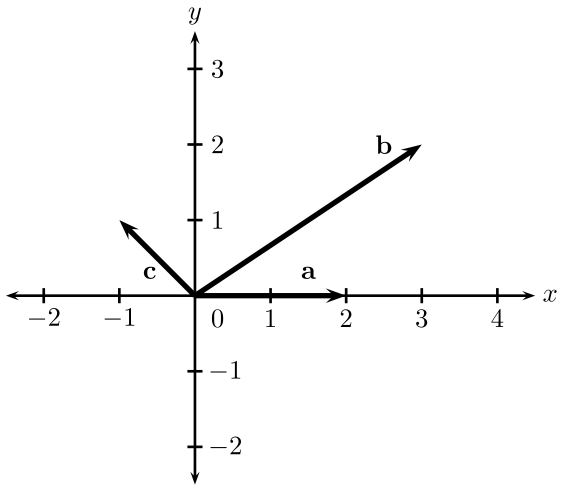
(i) Draw these vectors in standard position in $\mathbb{R}^2$.

(ii) Compute the vectors $\mathbf{a}+\mathbf{b}$, $\mathbf{b}+\mathbf{c}$ and $\mathbf{a}-\mathbf{c}$. How can these results be obtained geometrically?

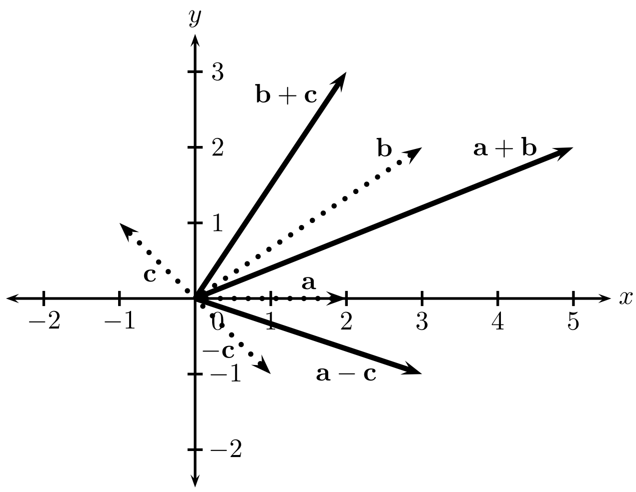
(iii) Draw the vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ with their tails at the point $(2, -1)$.

Solution:

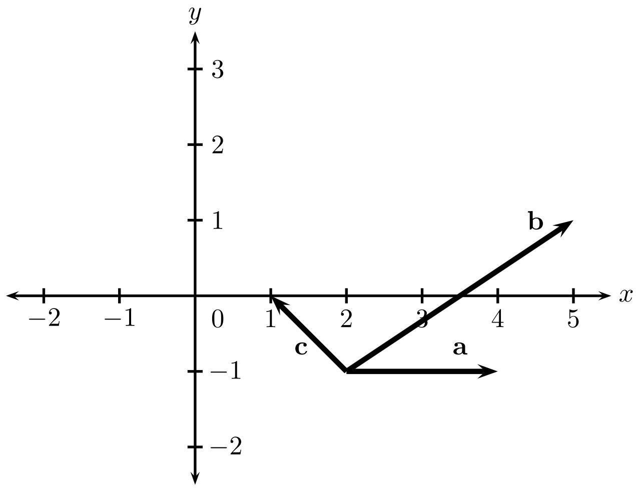
(i)



(ii) $\mathbf{a}+\mathbf{b} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$, $\mathbf{b}+\mathbf{c} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and $\mathbf{a}-\mathbf{c} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$. You can obtain these results geometrically by drawing $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $-\mathbf{c}$ in standard position, and then using the head-to-tail law or parallelogram law for vector addition. (Note that $\mathbf{a}-\mathbf{c} = \mathbf{a}+(-\mathbf{c})$.)



(iii)



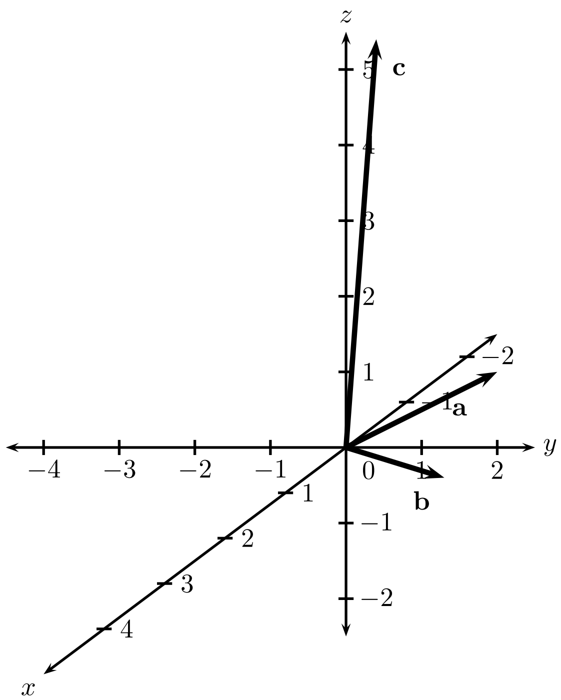
2. Let $\mathbf{a} = [0, 2, 1]$, $\mathbf{b} = [1, 2, \frac{1}{3}]$ and $\mathbf{c} = [-1, -\frac{1}{2}, 5]$.

(i) Draw these vectors in standard position in $\mathbb{R}^3$.

(ii) Compute the vectors $2\mathbf{a} + 3\mathbf{b}$ and $-\mathbf{a} + 4\mathbf{b} - \mathbf{c}$.

Solution:

(i) The difficulty of drawing vectors accurately even in $\mathbb{R}^3$ is an important reason for working algebraically when studying vectors in $\mathbb{R}^n$.



(ii) $2\mathbf{a} + 3\mathbf{b} = [3, 10, 3]$ and $-\mathbf{a} + 4\mathbf{b} - \mathbf{c} = [5, \frac{13}{2}, -\frac{14}{3}]$.

3. If the vector $\mathbf{v}$ has length 2, find the length of the vector $\mathbf{u}$ in each of the following cases.

(i) $\mathbf{u} = 3\mathbf{v}$

(ii) $\mathbf{u} = \frac{1}{2}\mathbf{v}$

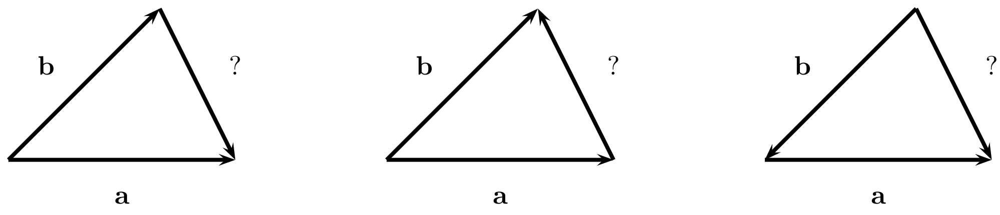
(iii) $\mathbf{u} = -3\mathbf{v}$

(iv) $\mathbf{v} = 3\mathbf{u}$

Solution:

(i) 6
(ii) 1
(iii) 6
(iv) 2/3

4. In each diagram below, find the unknown vector in terms of $\mathbf{a}$ and $\mathbf{b}$.



Solution: From left to right, $\mathbf{a} - \mathbf{b}$, $\mathbf{b} - \mathbf{a}$ and $\mathbf{a} + \mathbf{b}$.

5. Solve for $\mathbf{x}$ in terms of $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ in each case.

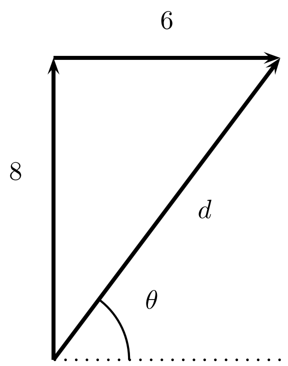
(i) $\mathbf{v} + \mathbf{x} = \mathbf{u} - \mathbf{w}$
(ii) $\mathbf{v} - \mathbf{x} = \mathbf{w} - \mathbf{u}$
(iii) $2\mathbf{v} + \mathbf{x} = 2\mathbf{w} - 2\mathbf{u} - \mathbf{x}$

Solution:

(i) $\mathbf{x} = \mathbf{u} - \mathbf{v} - \mathbf{w}$
(ii) $\mathbf{x} = \mathbf{u} + \mathbf{v} - \mathbf{w}$
(iii) $\mathbf{x} = -\mathbf{u} - \mathbf{v} + \mathbf{w}$

6. A balloon experiences two forces, a buoyancy force of 8 newtons vertically upwards and a wind force of 6 newtons acting horizontally to the right. Calculate the magnitude and direction of the resultant force.

Solution:



By Pythagoras $d = \sqrt{8^2 + 6^2} = 10$. If θ is the angle to the horizontal then $\cos\theta = 6/10$, yielding an angle $\theta \approx 53°$. Thus the resultant force is 10 newtons in a direction 53° to the horizontal, towards the right.

7. * Let $\mathbf{u} = [3, 1]$ and $\mathbf{v} = [-1, 1]$. Show that the vector $\mathbf{w} = [-7, -1]$ can be expressed as a linear combination of $\mathbf{u}$ and $\mathbf{v}$, and draw a picture to illustrate this geometrically.

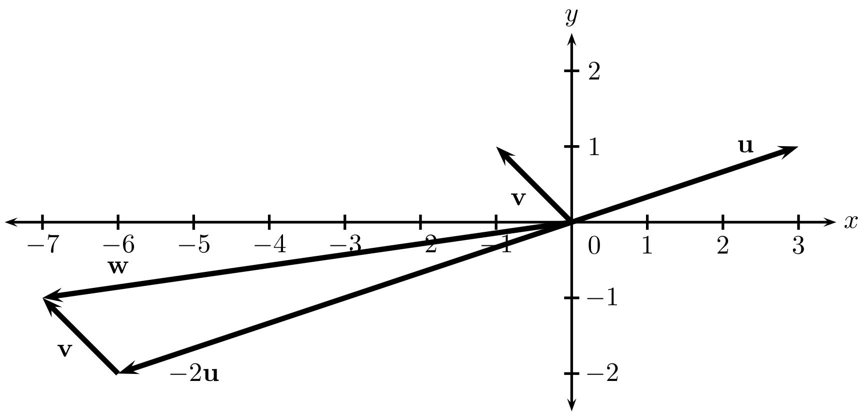
Solution: We want to find scalars c_1 and c_2 so that $\mathbf{w} = c_1\mathbf{u} + c_2\mathbf{v}$. Now

$$c_1\mathbf{u} + c_2\mathbf{v} = [3c_1, c_1] + [-c_2, c_2] = [3c_1 - c_2, c_1 + c_2]$$

and $\mathbf{w} = [-7, -1]$. So by comparing components, we need to solve the simultaneous equations

$$3c_1 - c_2 = -7 \quad \text{and} \quad c_1 + c_2 = -1]$$

These have (unique) solution $c_1 = -2$ and $c_2 = 1$, so $\mathbf{w} = -2\mathbf{u} + \mathbf{v}$. (You should check this answer.)



8. Prove the associative law for vector addition: for all vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ in $\mathbb{R}^n$,

$$(\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c}).$$

Solution: Let $\mathbf{a} = [a_1, a_2, \dots, a_n]$, $\mathbf{b} = [b_1, b_2, \dots, b_n]$ and $\mathbf{c} = [c_1, c_2, \dots, c_n]$. Then

$$\begin{aligned}
(\mathbf{a} + \mathbf{b}) + \mathbf{c} &= ([a_1, a_2, \dots, a_n] + [b_1, b_2, \dots, b_n]) + [c_1, c_2, \dots, c_n] \\
&= [a_1 + b_1, a_2 + b_2, \dots, a_n + b_n] + [c_1, c_2, \dots, c_n] \\
&= [(a_1 + b_1) + c_1, (a_2 + b_2) + c_2, \dots, (a_n + b_n) + c_n] \\
&= [a_1 + (b_1 + c_1), a_2 + (b_2 + c_2), \dots, a_n + (b_n + c_n)] \text{ since addition of scalars is associative} \\
&= [a_1, a_2, \dots, a_n] + [b_1 + c_1, b_2 + c_2, \dots, b_n + c_n] \\
&= [a_1, a_2, \dots, a_n] + ([b_1, b_2, \dots, b_n] + [c_1, c_2, \dots, c_n]) \\
&= \mathbf{a} + (\mathbf{b} + \mathbf{c})
\end{aligned}$$

as required.

9. Prove (by contradiction) that in any vector space the zero vector is unique (only using the 8 axioms of a vector space).

Solution: Assume that there are two distinct zero-vectors $\mathbf{0}$ and another one $\tilde{\mathbf{0}} \neq \mathbf{0}$. By definition they must satisfy

$$\mathbf{v} + \mathbf{0} = \mathbf{v} \tag{1}$$

and

$$\mathbf{v} + \tilde{\mathbf{0}} = \mathbf{v}, \tag{2}$$

respectively, for all $\mathbf{v}$ in the vector space. Setting $\mathbf{v} = \tilde{\mathbf{0}}$ in (1) gives $\tilde{\mathbf{0}} + \mathbf{0} = \tilde{\mathbf{0}}$. Setting $\mathbf{v} = \mathbf{0}$ in (2) gives $\mathbf{0} + \tilde{\mathbf{0}} = \mathbf{0}$. Using commutativity of addition these two equations imply $\tilde{\mathbf{0}} = \mathbf{0}$, which is a contradiction, and hence the zero vector is unique.

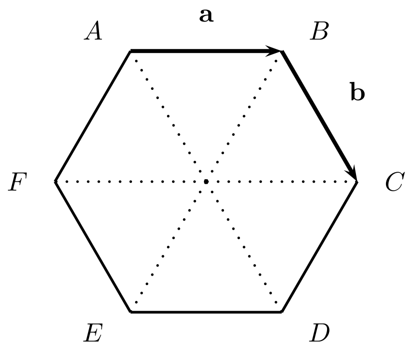
Extra Questions

10. Express $2\mathbf{a} - 3\mathbf{b}$ in terms of $\mathbf{u}$ and $\mathbf{v}$, and simplify, when $\mathbf{a} = \mathbf{u} + \mathbf{v}$ and $\mathbf{b} = 3\mathbf{u} - 2\mathbf{v}$.

Solution: $2\mathbf{a} - 3\mathbf{b} = 2(\mathbf{u} + \mathbf{v}) - 3(3\mathbf{u} - 2\mathbf{v}) = 2\mathbf{u} + 2\mathbf{v} - 9\mathbf{u} + 6\mathbf{v} = -7\mathbf{u} + 8\mathbf{v}$.

11. Let $ABCDEF$ be a regular hexagon and put $\mathbf{a} = \overrightarrow{AB}$ and $\mathbf{b} = \overrightarrow{BC}$. Find vector expressions in terms of $\mathbf{a}$ and $\mathbf{b}$ for the displacements $\overrightarrow{CD}$, $\overrightarrow{DE}$, $\overrightarrow{EF}$ and $\overrightarrow{FA}$.

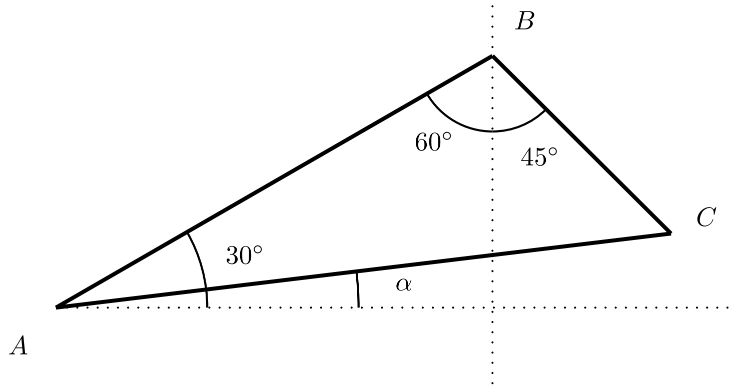
Solution:



We get $\overrightarrow{CD} = \mathbf{b} - \mathbf{a}$, $\overrightarrow{DE} = -\mathbf{a}$, $\overrightarrow{EF} = -\mathbf{b}$ and $\overrightarrow{FA} = \mathbf{a} - \mathbf{b}$.

12. A plane travels 20km in the direction 30° north of east and then 10 km southeast. Use trigonometry and your calculator to find the final distance and direction of the aircraft from the starting position.

Solution:



We have $|\overrightarrow{AB}| = 20$ and $|\overrightarrow{BC}| = 10$. By the Cosine Rule,

$$|\overrightarrow{AC}| = \sqrt{20^2 + 10^2 - 2(10)(20)\cos 105^\circ} \approx 25.$$

By the Sine Rule,

$$\sin(30^\circ - \alpha) = \frac{10 \sin 105^\circ}{|\overrightarrow{AC}|},$$

from which it follows that

$$30^\circ - \alpha \approx 23^\circ,$$

so that $\alpha \approx 7^\circ$. Hence the final distance and direction of the aircraft from the starting point are approximately 25 km and 7° north of east respectively.

13. Prove the following distributive laws:

(i) For all scalars c and all vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$, $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$.

(ii) For all scalars c and d and all vectors $\mathbf{u}$ in $\mathbb{R}^n$, $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$.

Solution:

(i) Let $\mathbf{u} = [u_1, u_2, \ldots, u_n]$ and $\mathbf{v} = [v_1, v_2, \ldots, v_n]$. Then

$$\begin{aligned} c(\mathbf{u} + \mathbf{v}) &= c([u_1, u_2, \ldots, u_n] + [v_1, v_2, \ldots, v_n]) \\ &= c[u_1 + v_1, u_2 + v_2, \ldots, u_n + v_n] \\ &= [c(u_1 + v_1), c(u_2 + v_2), \ldots, c(u_n + v_n)]. \end{aligned}$$

On the other hand

$$\begin{aligned}
c\mathbf{u}+c\mathbf{v} &= c[u_1,u_2,\ldots,u_n]+c[v_1,v_2,\ldots,v_n] \\
&= [cu_1,cu_2,\ldots,cu_n]+[cv_1,cv_2,\ldots,cv_n] \\
&= [cu_1+cv_1,cu_2+cv_2,\ldots,cu_n+cv_n] \\
&= [c(u_1+v_1),c(u_2+v_2),\ldots,c(u_n+v_n)] \quad \text{using the distributive law for scalars.}
\end{aligned}$$

Since both $c(\mathbf{u}+\mathbf{v})$ and $c\mathbf{u}+c\mathbf{v}$ are equal to $[c(u_1+v_1),c(u_2+v_2),\ldots,c(u_n+v_n)]$, we have that $c(\mathbf{u}+\mathbf{v})=c\mathbf{u}+c\mathbf{v}$ as required. General comment: sometimes it is easier to prove an equality by getting two expressions to equal the same thing, rather than trying to work on just one of them until you obtain the other.

(ii) Let $\mathbf{u}=[u_1,u_2,\ldots,u_n]$. Then

$$\begin{aligned}
(c+d)\mathbf{u} &= (c+d)[u_1,u_2,\ldots,u_n] \\
&= [(c+d)u_1,(c+d)u_2,\ldots,(c+d)u_n] \\
&= [cu_1+du_1,cu_2+du_2,\ldots,cu_n+du_n] \quad \text{using the distributive law for scalars} \\
&= [cu_1,cu_2,\ldots,cu_n]+[du_1,du_2,\ldots,du_n] \\
&= c[u_1,u_2,\ldots,u_n]+d[u_1,u_2,\ldots,u_n] \\
&= c\mathbf{u}+d\mathbf{u}
\end{aligned}$$

as required.

Challenge Questions

14. Prove that $a\mathbf{0}=\mathbf{0}$ for any scalar a (only using the 8 axioms of a vector space).

Solution: Since $\mathbf{u}+\mathbf{0}=\mathbf{u}$ for any $\mathbf{u}$ we certainly have $\mathbf{0}+\mathbf{0}=\mathbf{0}$. Thus $a\mathbf{0}=a(\mathbf{0}+\mathbf{0})=a\mathbf{0}+a\mathbf{0}$ using distributivity. Now every vector $\mathbf{u}$ has an additive inverse $-\mathbf{u}$ and so we add $-a\mathbf{0}$ to both sides of the equation, which combines with one term on each side to give $\mathbf{0}$ and hence $\mathbf{0}=a\mathbf{0}+\mathbf{0}=a\mathbf{0}$ because $\mathbf{0}$ is the neutral element for vector addition.

15. Prove that $0\mathbf{v}=\mathbf{0}$ for any vector $\mathbf{v}$ (only using the 8 axioms of a vector space).

Solution: Using the scalar identity $0=0+0$ write $0\mathbf{v}=(0+0)\mathbf{v}=0\mathbf{v}+0\mathbf{v}$ using the distributive law. Now add the additive inverse $-0\mathbf{v}$ of $0\mathbf{v}$ to both sides. This will combine with one term on each side to give $\mathbf{0}$, and hence $\mathbf{0}=0\mathbf{v}+\mathbf{0}$ and by the property that $\mathbf{0}$ is the neutral element for vector addition we find $\mathbf{0}=0\mathbf{v}$. Quite similar in style to the previous problem.

16. Prove that $(-1)\mathbf{v}$ is the additive inverse of $\mathbf{v}$ (only using the 8 axioms of a vector space). Note the distinction here between the additive inverse $-\mathbf{v}$ and the scalar multiplication by minus one $(-1)\mathbf{v}$.

Solution: We want to prove that $\mathbf{v}+(-1)\mathbf{v}=\mathbf{0}$. Thus $\mathbf{v}+(-1)\mathbf{v}=(1+(-1))\mathbf{v}=0\mathbf{v}=\mathbf{0}$ using the result from the previous problem.

17. Show that the complex numbers $x+iy$ form a vector space where the scalars are the real numbers.

Solution: A quick proof is to observe that we can identify a complex number $x+iy$ with a vector in $\mathbb{R}^2$ given by $[x,y]$. Since we know that vectors in $\mathbb{R}^2$ satisfy the vector space axioms the same follows for complex numbers. Or we could write out very explicitly, e.g. for commutativity $x+iy+a+ib=x+a+i(y+b)=a+x+i(b+y)=a+ib+x+iy$ reducing the problem to commutativity of real addition, and similarly for the other axioms. Note that the fact that the scalars are real as opposed to complex does not pose a problem, e.g., $c(d\mathbf{u})=c(d(x+iy))=c(dx+idy)=cdx+icdy=(cd)(x+iy)=(cd)\mathbf{u}$ as required.

18. Consider vectors of the form $[u_1,\ldots,u_n]$ and scalars λ with the usual arithmetic. Write $\mathbb{C}$ for the set of complex numbers. Which of the following three combinations gives a vector space (explain why or why not):

(i) $u_i\in\mathbb{C},\lambda\in\mathbb{C}$

(ii) $u_i \in \mathbb{R}, \lambda \in \mathbb{C}$

(iii) $u_i \in \mathbb{C}, \lambda \in \mathbb{R}$

Solution:

1. Since multiplication of complex numbers gives a complex number the axioms will hold in the same way as they hold for $\mathbb{R}^n$ with scalars from $\mathbb{R}$.
2. Here we encounter a problem because, e.g., scalar multiplication by $\lambda = i$ of a vector in $\mathbb{R}^n$ gives a vector with components that are purely imaginary, and hence the product is not in $\mathbb{R}^n$. Before we state the axioms of a vector space we need to say that the operations of vector addition and of scalar multiplication produce vectors in the same space we started with, and this is violated in this example.
3. At first sight this seems similar to the previous case, but it is different. Here we start with vectors with complex entries. Their addition gives vectors with complex entries. Multiplication by a real scalar again gives a vector with complex entries. So the operations do remain in the same space, and the axioms are satisfied. So this is a vector space. Writing $u_j = a_j + ib_j$ with real a_j, b_j one can think of the elements of this vector space as the real vectors $[a_1, b_1, a_2, b_2, \ldots, a_n, b_n]$, which are elements of $\mathbb{R}^{2n}$.

In general scalars must form a field, which is roughly a set with addition, multiplication and division as we know it for $\mathbb{R}$ and $\mathbb{C}$, but there are other fields. The 8 axioms may hold even when the scalars are not a field, e.g say the integers, which cannot generally be divided. In this particular case the resulting structure is not a vector space but a module, which is studied in general algebra as opposed to linear algebra.