The University of Sydney
School of Mathematics and Statistics

Solutions to Tutorial 2 (Week 3)

MATH2069: Discrete Mathematics and Graph Theory — Semester 1, 2023

1. Suppose you have 7 different ornaments to put on your mantelpiece.

(a) If you want to use all of them, how many possible arrangements are there?

Solution: There are 7 choices for which ornament goes first, then 6 choices for which goes next, then 5, etc. So the answer is $7! = 5040$.

(b) If you want to use 6 of them, how many possible arrangements are there?

Solution: By the same reasoning as in the previous part, the answer is $7 \times 6 \times 5 \times 4 \times 3 \times 2$, which is also 5040. It makes sense that the answer should be the same as the previous part, because you can imagine storing the unused ornament at the right-hand edge of the mantelpiece and thus 'using' it after all.

(c) If you can use all, some, or none of them, how many possible arrangements are there?

Solution: If you use k of the ornaments, then by the same reasoning as in the previous parts, the number of possible arrangements is $7_{(k)} = \frac{7!}{(7-k)!}$. So the answer is

$$\frac{7!}{0!} + \frac{7!}{1!} + \frac{7!}{2!} + \frac{7!}{3!} + \frac{7!}{4!} + \frac{7!}{5!} + \frac{7!}{6!} + \frac{7!}{7!} = 13700.$$

*(d) Divide your answer to part (c) by your answer to part (a); this ratio measures how much extra freedom you get by not necessarily using all the ornaments. Notice that it agrees with e up to four decimal places. Is this a coincidence?

Solution: The ratio is $\frac{13700}{5040} = 2.718253\cdots$, whereas $e = 2.718281\cdots$. This is no coincidence, because the ratio can also be written as

$$\frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!} + \frac{1}{7!},$$

i.e. the sum of the first eight terms of the series $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n!}$, which is well known to converge to e.

2. An ordinary knock-out singles tennis tournament (with no seeds or byes) consists of a series of rounds. In each round, the remaining players play against each other in pairs, with the losers being eliminated and the winners going through to the next round; in the last round, the only two remaining players play the final match to determine the winner of the tournament. Suppose that there are 7 rounds.

(a) How many players are there at the start of the tournament?

Solution: Every round halves the number of remaining players. So if you imagine starting at the end with the sole winner and going back in time, the number of players is multiplied by 2 for every round, and there must have been $2^7 = 128$ players at the start.

(b) How many matches are played in total?

Solution: The smart answer is that every match eliminates one player, and all but one of the original players gets eliminated, so there are 127 matches. An alternative method is to note that there are $2^6 = 64$ matches in the first round, $2^5 = 32$ matches in the second, and so on up to $2^0 = 1$ match in the last round (the final), so the answer is $2^6 + 2^5 + \cdots + 2^0$, which equals $2^7 - 1$ by the formula for the sum of a geometric progression.

(c) Before the tournament starts, the organizers need to construct the draw, which specifies who plays who in the first round, and then which first-round winners play which other first-round winners in the second round, and so on. Of course, the organizers don't know who the first-round winners will be, so in the draw they are just thought of as "the winner of the match between player X and player Y", and so forth. How many possible draws are there? Use the fact, proved in lectures, that the number of ways to group $2k$ people into k pairs is $\frac{(2k)!}{2^k k!}$. (The answer is too large to evaluate, so just leave it as a fraction of expressions involving a factorial.)

Solution: Since there are 128 players at the start, the number of possibilities for who plays who in the first round is $\frac{128!}{2^{64}\,64!}$. Then there are 64 players to be paired off in the second round (their actual identities may not be known, but they can be identified by what first-round match they won); the number of ways to do this is $\frac{64!}{2^{32}\,32!}$, and so forth. In the final round there are only two players, and only $\frac{2!}{2^1 1!} = 1$ way to pair them off, obviously. By the Product Principle, the total number of draws is the product of all these numbers, which is

$$\frac{128!\,64!\,32!\,16!\,8!\,4!\,2!}{2^{64+32+16+8+4+2+1}\,64!\,32!\,16!\,8!\,4!\,2!\,1!} = \frac{128!}{2^{127}}.$$

(d) Suppose that after the tournament is finished, the only things that are recorded are the draw and the winners of each match. How many different such records are possible?

Solution: Given the draw, the number of possible outcomes of the tournament is 2^{127}, because each of the 127 matches can have 2 possible winners. So by the answer to the previous part, the number of possible records is $\frac{128!}{2^{127}} 2^{127} = 128!$. This answer is of course the same as the number of ways of ordering the 128 players. The explanation for this is that you can use the record of the tournament to number the players, as follows. The winner of the final is player 1, and the loser of the final is player 2. The other semi-finalists are numbered 3 and 4, with player 3 being the one who lost to player 1 and player 4 being the one who lost to player 2. Then the other quarter-finalists are numbered 5 up to 8, with player $n+4$ being the one who lost to player n for $1 \leqslant n \leqslant 4$; and so on until all the players are numbered. This gives a bijection between the possible records and the numberings of the players. So the fact that there are 128! possible records also follows from the Bijection Principle.

3. Let $X = \{m, m+1, \cdots, n-1, n\}$ be a set of consecutive positive integers, and A the subset of X consisting of those elements which are multiples of 3. You would expect $|A|$ to be about $\frac{|X|}{3}$, but the latter is not always an integer, so you may have to round it up or down.

(a) Give an example where $|A|$ does not equal $\lfloor \frac{|X|}{3} \rfloor$.

Solution: One example is $X = \{3, 4, 5, 6\}$, where $A = \{3, 6\}$. Here $\lfloor \frac{|X|}{3} \rfloor = \lfloor \frac{4}{3} \rfloor = 1$, whereas $|A| = 2$.

(b) Show that if $X = \{1, 2, \cdots, n\}$, then $|A| = \lfloor \frac{n}{3} \rfloor$.

Solution: If a is a positive integer, then $3a \leqslant n$ if and only if $a \leqslant \lfloor \frac{n}{3} \rfloor$. So $A = \{3 \times 1, 3 \times 2, \cdots, 3 \times \lfloor \frac{n}{3} \rfloor\}$, and the result follows.

(c) Hence give a formula for $|A|$ when $X = \{m, m+1, \cdots, n\}$ and $m \geqslant 2$.

Solution: Let $Y = \{1, 2, \cdots, m-1\}$ and $Z = \{1, 2, \cdots, n\}$, and let B and C be the subsets of Y and Z respectively consisting of multiples of 3. Then X is the complement of Y in Z, and A is the complement of B in C. Hence by the Difference Principle,

$$|A| = |C| - |B| = \left\lfloor \frac{n}{3} \right\rfloor - \left\lfloor \frac{m-1}{3} \right\rfloor.$$

***4.** If m is a positive integer, write $v_2(m)$ for the exponent of the highest power of 2 that divides m. For example:

$$\begin{aligned} v_2(m) = 0 &\iff m \text{ is odd}, \\ v_2(m) = 1 &\iff m \text{ is even but not a multiple of 4}, \\ v_2(m) = 2 &\iff m \text{ is a multiple of 4 but not of 8, etc.} \end{aligned}$$

(a) Show that $v_2(mm') = v_2(m) + v_2(m')$ for all integers m, m'.

Solution: We can write $m = 2^{v_2(m)} m_0$ and $m' = 2^{v_2(m')} m_0'$ where m_0 and m_0' are odd. So $mm' = 2^{v_2(m)+v_2(m')} m_0 m_0'$ where $m_0 m_0'$ is odd, which shows that $v_2(mm') = v_2(m) + v_2(m')$.

(b) Prove that for all positive integers n,

$$v_2(n!) = \left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{n}{2^2} \right\rfloor + \left\lfloor \frac{n}{2^3} \right\rfloor + \left\lfloor \frac{n}{2^4} \right\rfloor + \cdots\cdots.$$

(Note that although this sum appears to go on forever, all the terms will be zero from a certain point on, because when $2^k > n$ we have $\lfloor \frac{n}{2^k} \rfloor = 0$.)

Solution: By definition, $n!$ is the product of the numbers $1, 2, 3, \cdots, n$. So by the previous part, $v_2(n!) = v_2(1) + v_2(2) + \cdots + v_2(n)$. Now imagine the numbers $1, 2, \cdots, n$ as positions in a line, with a stack of $v_2(i)$ blocks put at position i. We want to count the total number of blocks; we can do this by adding up the number of blocks in the bottom level, the number of blocks in the second level, and so on (once the level is sufficiently high, there will be no more blocks). The number of blocks in the kth level from the bottom is the number of elements $i \in \{1, 2, \cdots, n\}$ such that $v_2(i) \geqslant k$, which by definition is the number of multiples of 2^k in $\{1, 2, \cdots, n\}$. So there are $\lfloor \frac{n}{2^k} \rfloor$ blocks in the kth level, and adding up the levels gives the claimed formula.

(c) Deduce that $v_2((2n)!) = n + v_2(n!)$. How could this be proved another way?

Solution: By the previous part with n replaced by $2n$,

$$\begin{aligned} v_2((2n)!) &= \left\lfloor \frac{2n}{2} \right\rfloor + \left\lfloor \frac{2n}{2^2} \right\rfloor + \left\lfloor \frac{2n}{2^3} \right\rfloor + \left\lfloor \frac{2n}{2^4} \right\rfloor + \cdots \\ &= n + \left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{n}{2^2} \right\rfloor + \left\lfloor \frac{n}{2^3} \right\rfloor + \cdots \\ &= n + v_2(n!). \end{aligned}$$

Another proof of this fact comes from the observation made in lectures that $(2n)! = 2^n n!(2n-1)!!$, where $(2n-1)!!$ is the product of the odd integers from 1 up to $2n-1$. Since $(2n-1)!!$ is odd,

$$v_2(2^n n!(2n-1)!!) = v_2(2^n) + v_2(n!) + v_2((2n-1)!!) = n + v_2(n!) + 0,$$

as required.

***5.** Let $\mathcal{F}_n$ be the set of functions $f : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ such that if $i < j$ then $f(i) \leqslant f(j)$, and for all i, $f(i) \leqslant i$. Given two rows of boxes with n boxes in each row,

		$\cdots$	
		$\cdots$	

a *standard tableau* is obtained by placing the numbers $1, 2, \ldots, 2n$ bijectively in the boxes so that the numbers increase from left to right in each row and so that each number in the bottom row is larger than the number in the box above it. Let $\mathcal{T}_n$ be the set of standard tableaux with entries $1, 2, \ldots, 2n$. Construct a bijection $\mathcal{F}_n \to \mathcal{T}_n$. [The cardinality $|\mathcal{F}_n| = |\mathcal{T}_n|$ is known as the n-th *Catalan number*].

Solution: We may represent a function by the list of its values: $f(1)f(2)\ldots f(n)$. Begin with an increasing function f as described and then create a two-rowed tableau with first row

$$f(1), f(2)+1, f(3)+2, \ldots, f(n)+n-1$$

and whose second row consists of the remaining integers from $\{1, 2, \ldots, 2n\}$ in ascending order.

For $i < j$ we have $f(i) \leqslant f(j)$ and therefore $f(i)+i-1 < f(j)+j-1$. Thus the first row begins with 1 and is strictly increasing. Moreover, $f(n)+n-1 < 2n$ and therefore the last entry in the bottom row is $2n$. We need to see that the values increase down the columns. We can think of filling the tableau beginning at the leftmost column and proceeding to the right. When we come to fill the k-th column, the top row will be the one obtained by restricting f to $\{1, 2, \ldots, k\}$ and so the k-th entry in the bottom row must be at least $2k$.

Conversely, if we begin with a standard tableau with two rows of length n, we construct the increasing function from the first row of the tableau by subtracting $k-1$ from the value in the k-th box for $k = 1, \ldots, n$. The value in the top box of the k-th row is always less than $2k$; that is, $f(k)+k-1 < 2k$ and so $f(k) \leqslant k$, as required.

**6. Use the Pigeonhole Principle to prove that for any odd integer $m \geqslant 3$, at least one of the numbers $2^2 - 1, 2^3 - 1, \cdots, 2^{m-1} - 1$ is divisible by m.

Solution: Let $X = \{1, 2, \cdots, m-1\}$, and let $f : X \to X$ be the function which takes n to the remainder after dividing 2^n by m (it is impossible for this remainder to be 0, since no odd number greater than 1 can exactly divide a power of 2). Suppose that $f(n) = 1$ for some $n \in X$: then $2^n - 1$ is divisible by m, and we clearly cannot have $n = 1$, so the desired result follows. Henceforth we assume that $f(n) \neq 1$ for all $n \in X$, and try to find a contradiction. Our assumption means that the range of f is a subset of $Y = \{2, \cdots, m-1\}$, whose size is strictly smaller than $|X|$. By the Pigeonhole Principle, $f : X \to Y$ cannot be injective, so there must be two different elements $n_1 < n_2$ in X such that $f(n_1) = f(n_2)$. This means that 2^{n_1} and 2^{n_2} have the same remainder after division by m, so their difference $2^{n_2} - 2^{n_1} = 2^{n_1}(2^{n_2-n_1} - 1)$ has remainder 0, i.e. is divisible by m. Since m is odd, it must divide the $2^{n_2-n_1} - 1$ part, which means that $2^{n_2-n_1}$ has remainder 1. But $n_2 - n_1$ is clearly an element of X, and we have shown that $f(n_2 - n_1) = 1$, contradicting our assumption. This finishes the proof.