

Tutorial 1 (Week 2)

MATH2069: Discrete Mathematics and Graph Theory

Semester 1, 2023

*More difficult questions are marked with either * or **. Those marked * are at the level which MATH2069 students will have to solve in order to be sure of getting a Credit, or to have a chance of a Distinction or High Distinction.*

1. Given a set X and a subset $A \subseteq X$, define a function $f_A : X \rightarrow \{0, 1\}$ by setting $f_A(x) = 1$ if $x \in A$ and $f_A(x) = 0$ if $x \notin A$. If B is also a subset of X , define $(f_A f_B)(x) = f_A(x) f_B(x)$.
 - (a) Work out $f_A f_B$ when $X = \{1, 2, 3, 4, 5\}$, $A = \{2, 4, 5\}$ and $B = \{1, 2, 5\}$.
 - (b) What subset, if any, does $f_A f_B$ correspond to?
 - (c) Define f'_A by $f'_A(x) = 1 - f_A(x)$. What subset, if any, does f'_A correspond to?
 - (d) Form combinations of f_A and f_B which represent
 - (i) the *union* of A and B ;
 - (ii) the *symmetric difference* $A + B = (A \setminus B) \cup (B \setminus A)$ of A and B .
2. For the following sets X , Y and function $X \rightarrow Y$, determine whether the function is injective or surjective
 - (a) $X = \mathbb{N}$, $Y = \mathbb{N}$, $f(x) = x + 1$.
 - (b) $X = \mathbb{Z}$, $Y = \mathbb{Z}$, $g(x) = x + 1$.
 - (c) $X = \mathbb{Z}$, $Y = \mathbb{Z}$, $h(x) = x^2 + 5$.
 - (d) $X = \mathbb{Z}$, $Y = \mathbb{Z}$, $p(x) = x^3 + 1$.
 - (e) X is a nonempty set, $Y = \mathcal{P}(X)$ (the set of all subsets of X), $h(x) = \{x\}$.

- *3. Given finite sets A and B , let E be a subset of $A \times B$. For $a \in A$, let

$$E(a) = \{b \in B \mid (a, b) \in E\}$$

and for $b \in B$, let

$$E^\vee(b) = \{a \in A \mid (a, b) \in E\}.$$

Prove that

$$\sum_{a \in A} |E(a)| = \sum_{b \in B} |E^\vee(b)|.$$

4. If there are 40 students in the class, their surnames can't all start with different letters (because there are only 26 letters). What can be said if there are 80 students in the class?

5. How many six-digit numbers are there (not starting with 0)? How many of these have six different digits? Explain your answers.
6. A permutation of $\{1, 2, 3, 4, 5\}$ can be thought of as a number with these five digits in some order, such as 21453 or 41352. There are $5! = 120$ such numbers.
- (a) How many of these numbers start with 5 and end with 2?
 - (b) How many have a first digit which is larger than the second digit?
 - (c) How many have the property that the first digit is larger than the second, which is smaller than the third, which is larger than the fourth, which is smaller than the fifth? (*Hint*: what positions can the digit 1 occupy in such a number?)
- *7. Use the Pigeonhole Principle to prove that for any positive integer n , there is a multiple of n which has no digits other than 0's and 1's (in its usual decimal expression).

Selected numerical answers:

5. 900000, 136080. 6. 6, 60, 16.