Counting principles

<u>Thm</u> (Sum Principle).

Suppose a finite set X is the <u>disjoint union</u> of two subsets,

$$X = A \cup B \text{ and } A \cap B = \emptyset.$$

Then $|X| = |A| + |B|$.

<u>Notation</u>: We write $X = A \sqcup B$ for a disjoint union.

More generally, if

$$X = A_1 \sqcup A_2 \sqcup \cdots \sqcup A_n$$

with

$$A_i \cap A_j = \emptyset \text{ for } i \neq j,$$

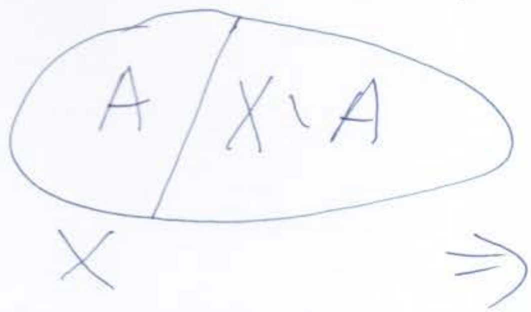
then

$$|X| = |A_1| + |A_2| + \cdots + |A_n|.$$

Thm (Difference Principle).
For any subset A of a set X we have $|X \setminus A| = |X| - |A|$.

We call $X \setminus A$ the *complement* of A in X.

For the proof note that

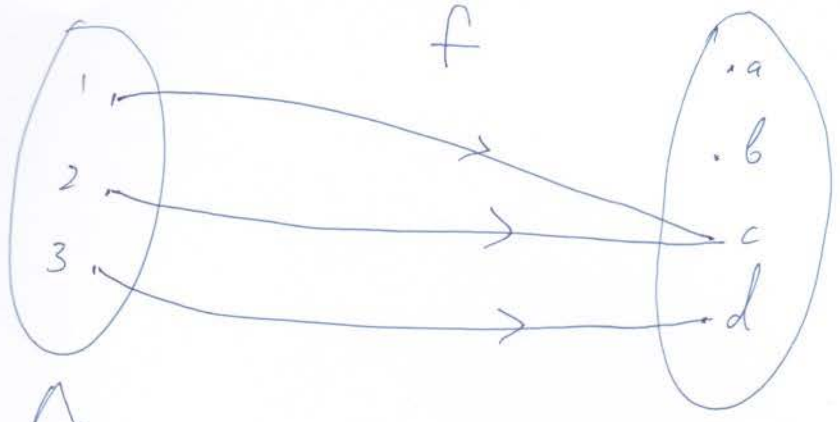


$$X = A \sqcup (X \setminus A).$$

$$\Rightarrow |X| = |A| + |X \setminus A|. \quad \square$$

Functions

Given two sets A and B a *function* $f: A \to B$ is a rule which assigns to *every* element of A a *unique* (*only one*) element of B.

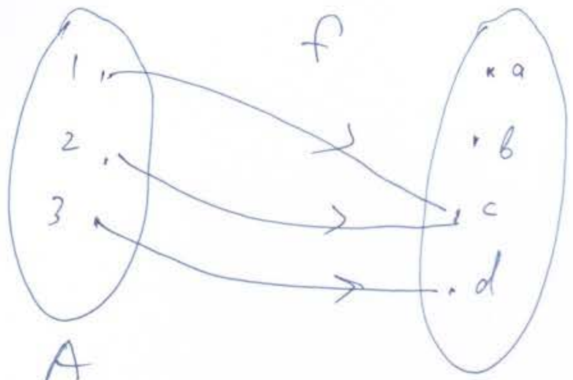


A (domain of f)

B (codomain of f)

We have $f(1) = c$, $f(2) = c$, $f(3) = d$.

A function $f: A \to B$ is <u>injective</u> if for every $b \in B$ there is at most one $a \in A$ s.t. $f(a) = b$.



A

B

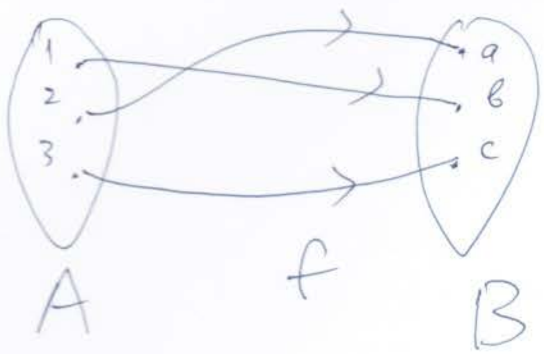
not injective

not surjective

A function $f: A \to B$ is <u>surjective</u> if for every $b \in B$ there is at least one $a \in A$ s.t. $f(a) = b$.

A function f is <u>bijective</u> (f is a <u>bijection</u>) if f is both injective and surjective.

We also say that f is <u>one-to-one correspondence</u>.



is a bijection

If $f: A \to B$ is a bijection, then we can define the <u>inverse function</u>

$$f^{-1}: B \longrightarrow A \quad \text{by}$$

$$f^{-1}(b) = a \quad \text{if} \quad f(a) = b.$$

Suppose that X and Y are finite sets.

1) If $f: X \to Y$ is

an injective function, then
$$|X| \leq |Y|.$$

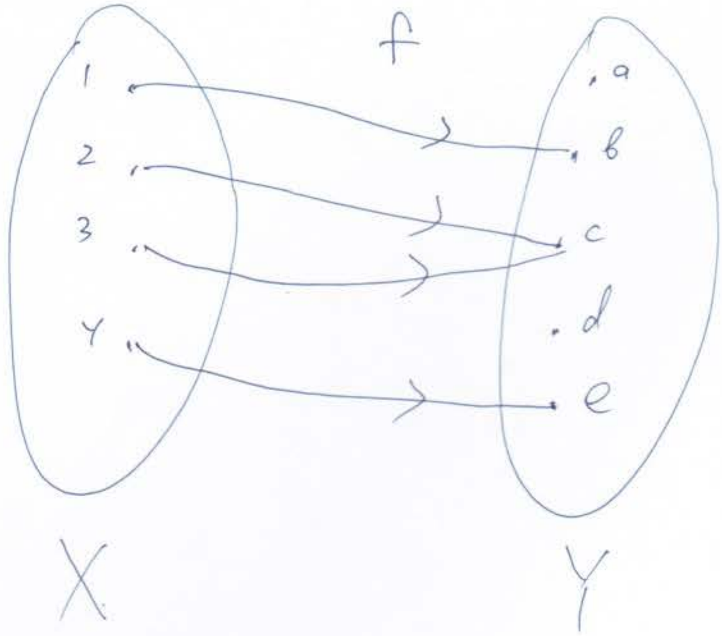
2) If $f: X \to Y$ is a surjective function, then
$$|X| \geq |Y|.$$

<u>Thm</u> (Bijection Principle).
If there is a bijection
$f: X \to Y$, then
$$|X| = |Y|.$$

<u>Pigeonhole Principle</u>

For any function $f: X \to Y$ and any $y \in Y$ the <u>preimage</u> $f^{-1}(y)$ is defined by
$$f^{-1}(y) = \{x \in X \mid f(x) = y\}.$$

Ex.

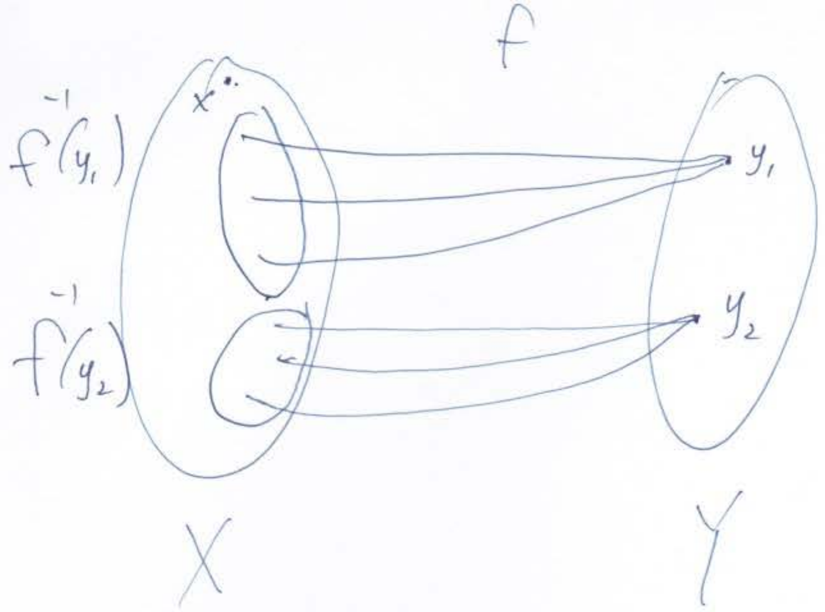


Then $f^{-1}(a) = \emptyset$, $f^{-1}(b) = \{1\}$,
$f^{-1}(c) = \{2, 3\}$, $f^{-1}(d) = \emptyset$, $f^{-1}(e) = \{4\}$.

We have the property:

X is the disjoint union of the sets $f^{-1}(y)$ as y runs over Y.

Hence
$$|X| = \sum_{y \in Y} |f^{-1}(y)|$$
by the Sum Principle.


f
x
$f^{-1}(y_1)$
$f^{-1}(y_2)$
y_1
y_2
X
Y