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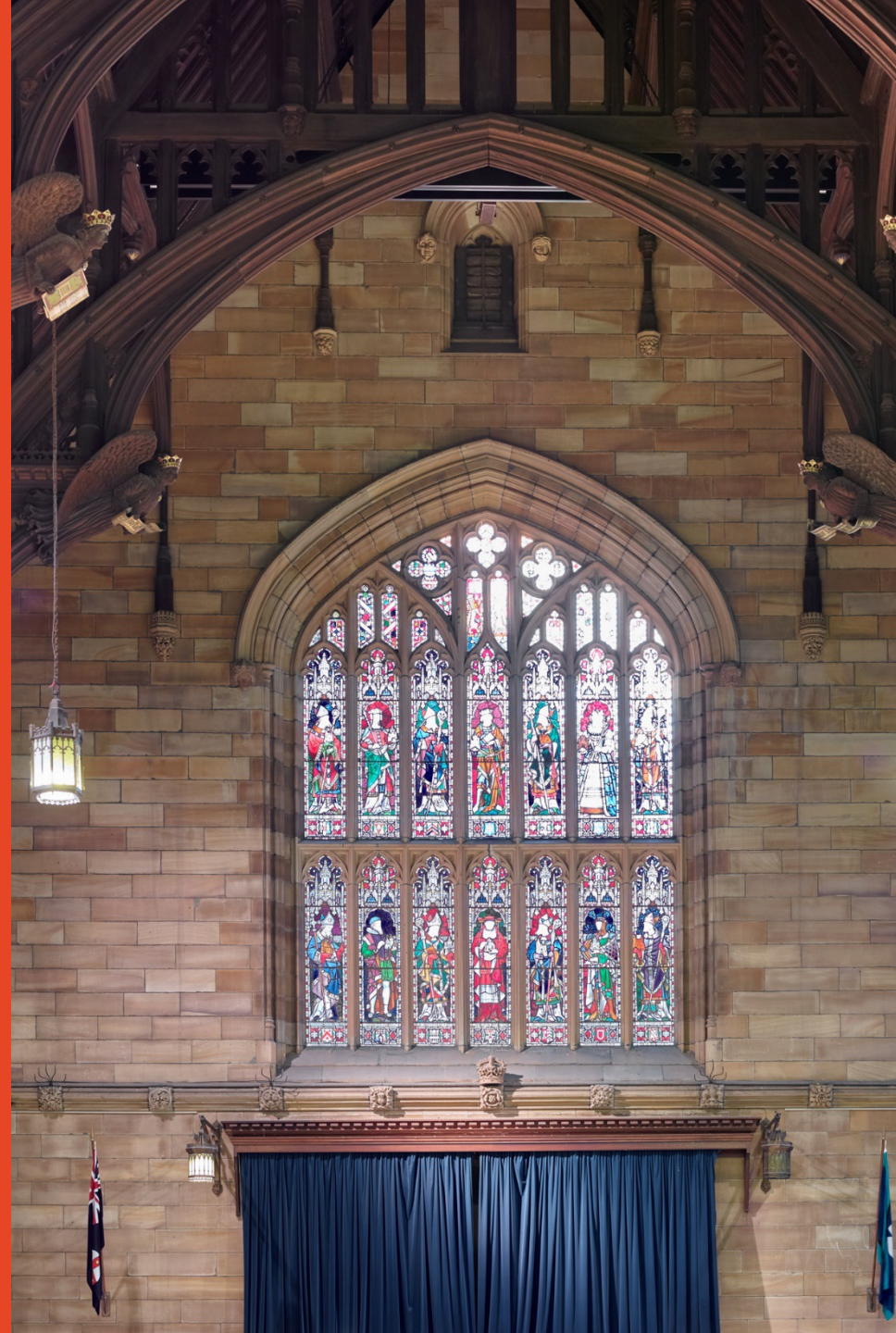
Lecture 11: Divide and Conquer [GT 11, 9, and 25]

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We went through four example problems in this lecture

Divide and Conquer

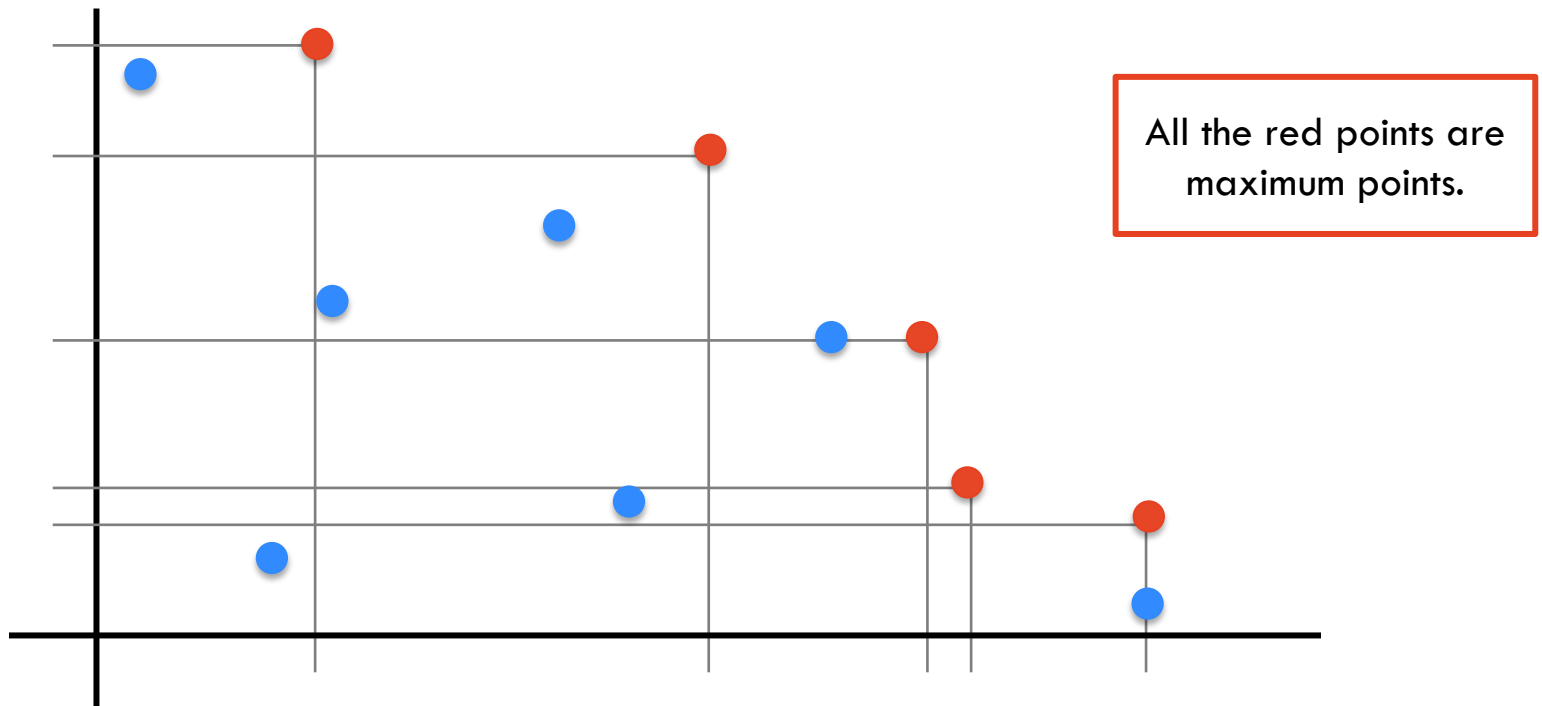
Divide and Conquer algorithms can normally be broken into these three parts:

1. **Divide** If it is a base case, solve directly, otherwise break up the problem into several parts.
2. **Recur/Delegate** Recursively solve each part [each sub-problem].
3. **Conquer** Combine the solutions of each part into the overall solution.

Maxima-Set (Pareto frontier)

Definition A point is maximum in a set if all other points in the set have either a smaller x- or smaller y-coordinate.

Problem Given a set S of n distinct points in the plane (2D), find the set of all maximum points.



Maxima-Set: Naïve Solution

Idea: Check every point (one at a time) to see if it is a maximum point in the set S .

To check if point p is a maximum point in S :

```
for q in S do  
    if  $q \neq p$  and  $q.x \geq p.x$  and  $q.y \geq p.y$  then  
        return "No"  
return "Yes"
```



There is a point q
that dominates p

Maxima-Set: Naïve Solution

Idea: Check every point (one at a time) to see if it is a maximum point in the set S .

To check if point p is a maximum point in S :

```
for q in S do
    if q ≠ p and q.x ≥ p.x and q.y ≥ p.y then
        return "No"
return "Yes"
```

Naïve algorithm to find the maxima-set of S :

```
maximaSet ← empty list
for p in S do
    if p is a maximum point in S then
        add p to the maximaSet
return maximaSet
```

Maxima-Set: Naïve Solution

Idea: Check every point (one at a time) to see if it is a maximum point in the set S .

To check if point p is a maximum point in S :

This is the sub-function that's used in below:
if p is a maximum point in S then:

```
for  $q$  in  $S$  do
    if  $q \neq p$  and  $q.x \geq p.x$  and  $q.y \geq p.y$  then
        return "No"
return "Yes"
```

$O(n)$

Naïve algorithm to find the maxima-set of S :

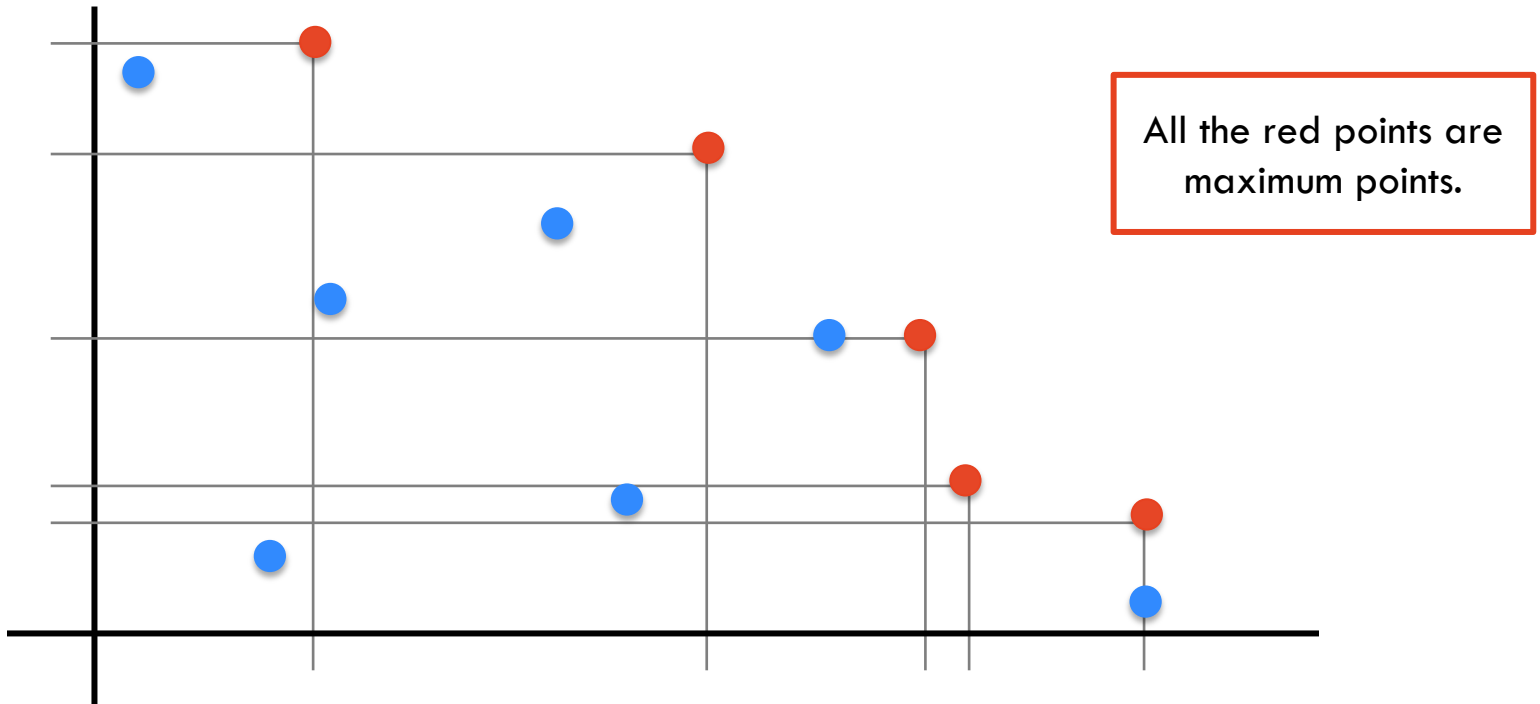
```
maximaSet  $\leftarrow$  empty list
for  $p$  in  $S$  do
    if  $p$  is a maximum point in  $S$  then
        add  $p$  to the maximaSet
return maximaSet
```

$O(n)$ $O(n^2)$

Maxima-Set

Definition A point is maximum in a set if all other points in the set have either a smaller x or smaller y coordinate.

Problem Given a set S of n distinct points in the plane (2D), find the set of all maximum points.



Maxima-Set

Preprocessing Sort the points by increasing x coordinate and store them in an array. Note: we only do this once. Break ties in x by sorting by increasing y coordinate.

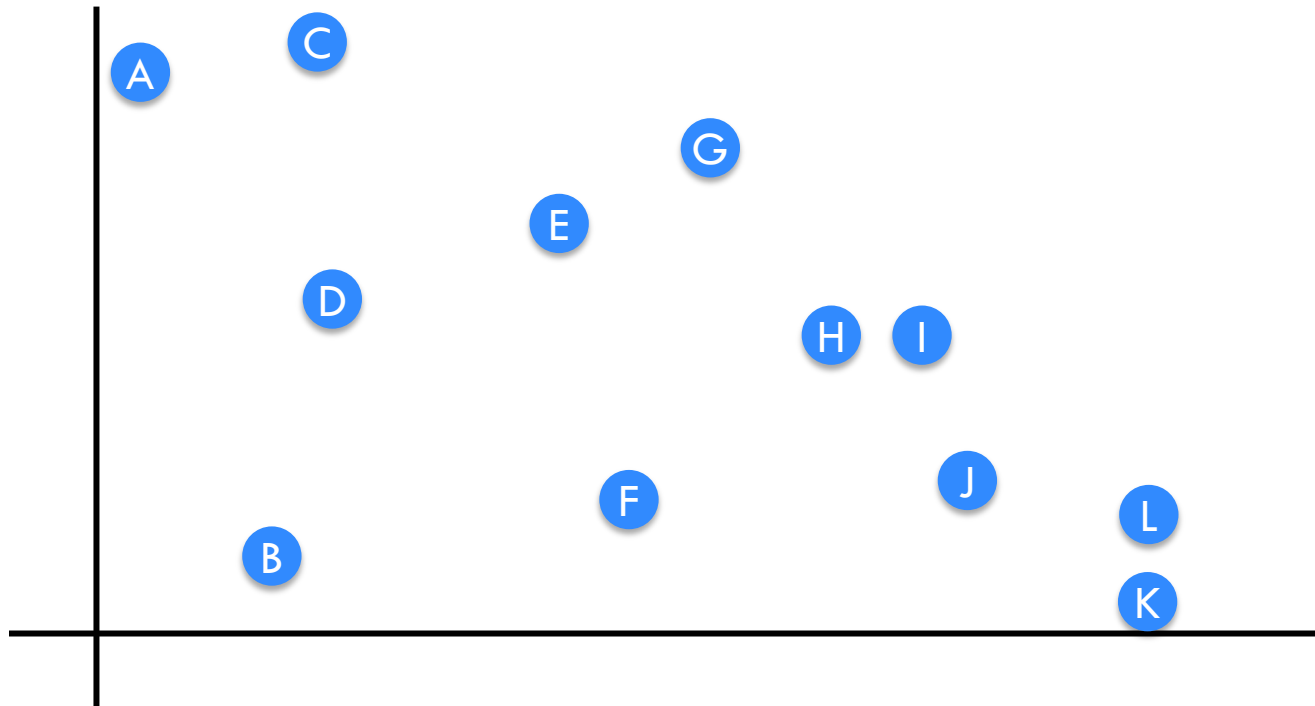
Divide sorted array into two halves.

Recur recursively find the MS of each half.

Conquer compute the MS of the union of Left and Right MS

Maxima-Set

Preprocessing Sort the points by increasing x coordinate and store them in an array. Note: we only do this once. Break ties in x by sorting by increasing y coordinate.

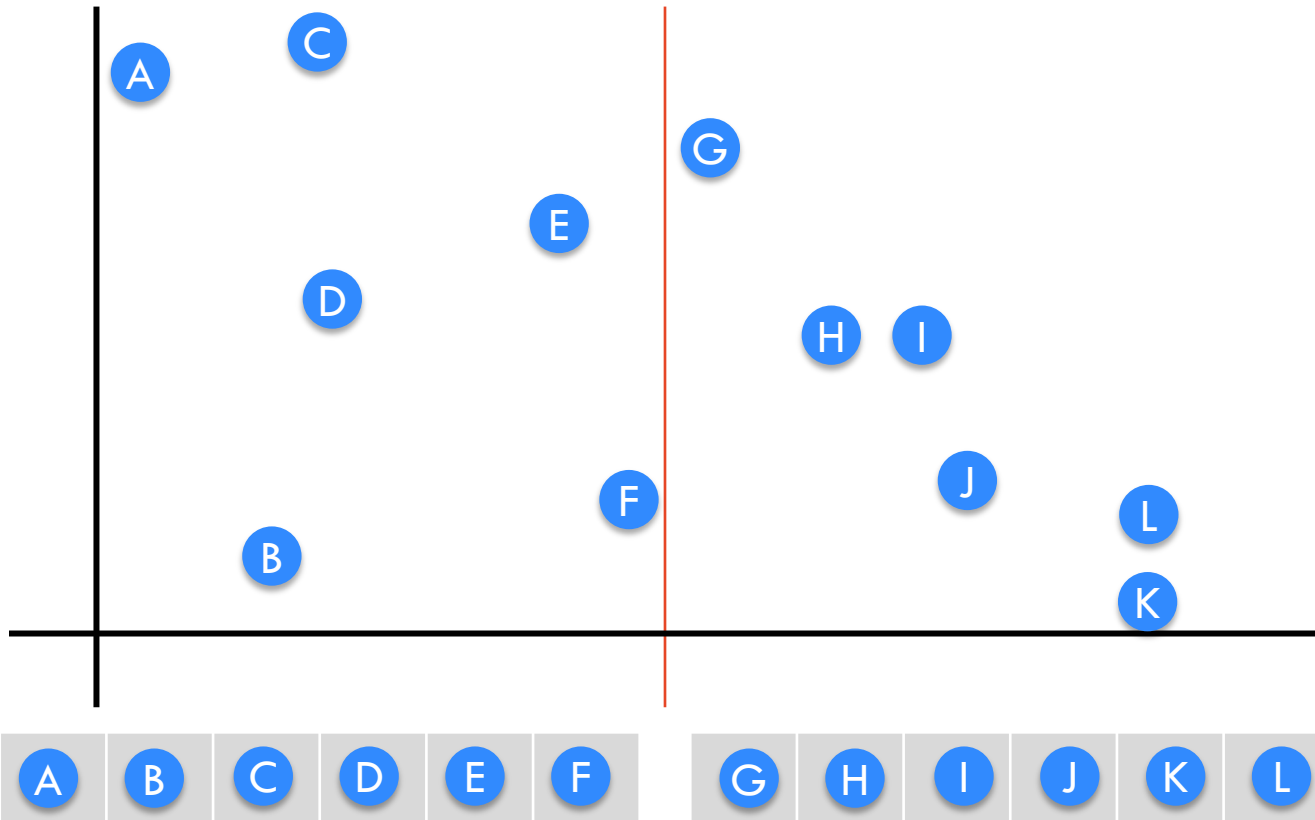


Sorted Points



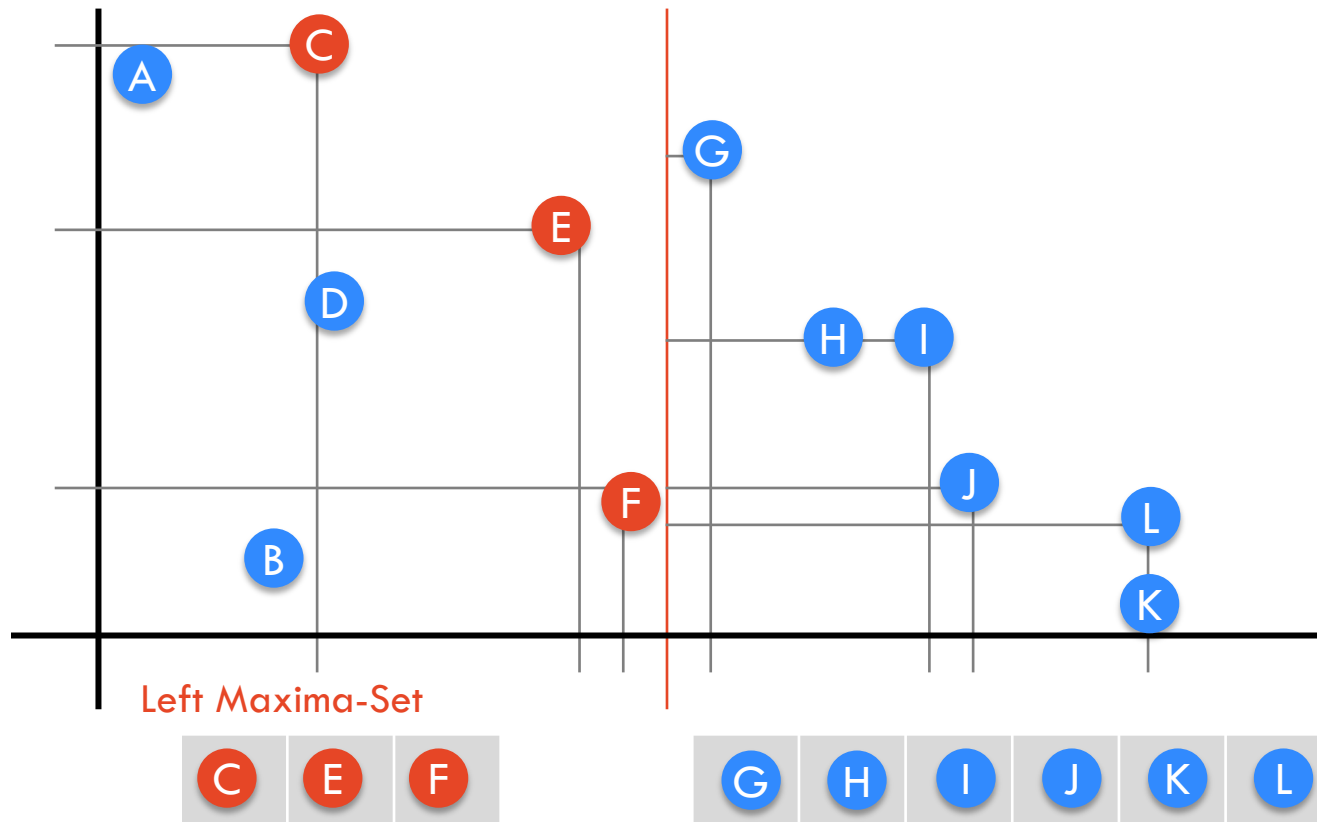
Maxima-Set

Divide array into two halves.



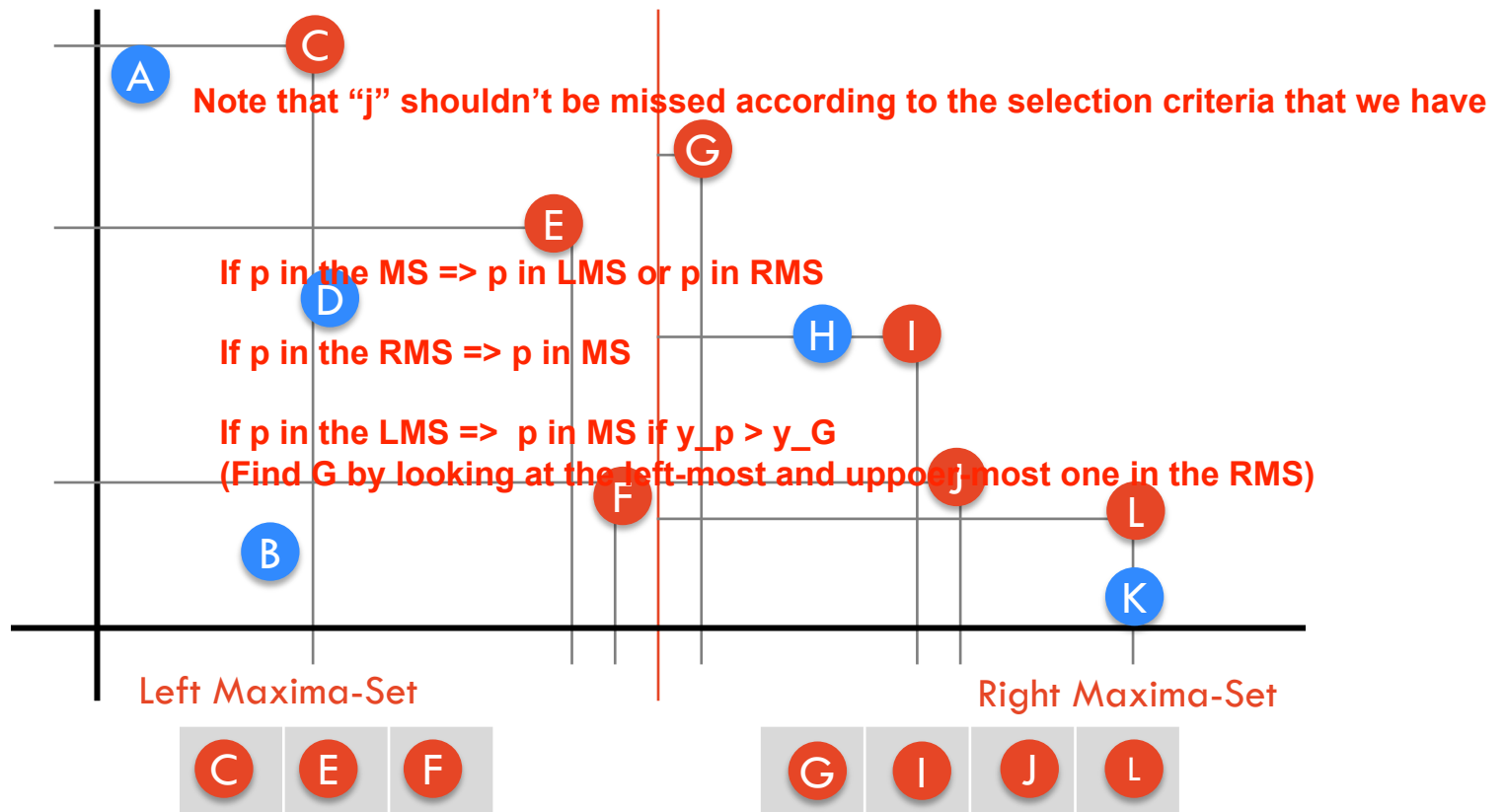
Maxima-Set

Recur recursively find the Maxima-Set of each half.



Maxima-Set

Recur recursively find the Maxima-Set of each half.



Maxima-Set

Conquer

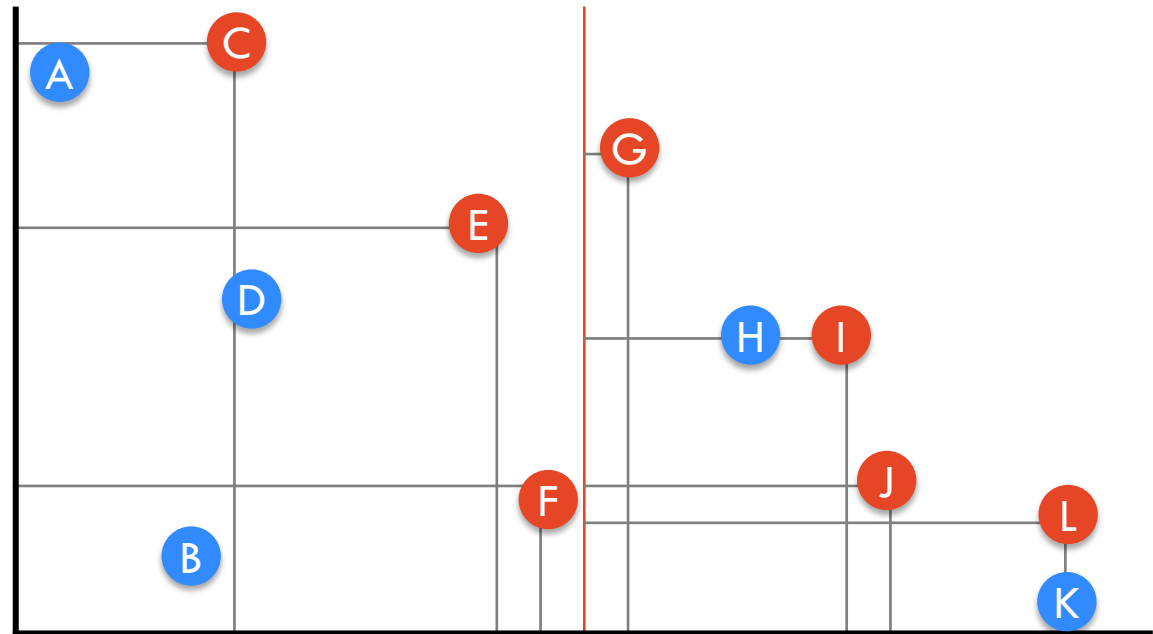
1. Find the highest point p in the Right MS

$$p = \text{G}$$

Observations:

Maxima-set: MS

1. Every point in MS of the whole is in Left MS or Right MS
2. Every point in Right MS is in MS of the whole
3. Every point in Left MS is either in MS of the whole or is dominated by p



Left Maxima-Set

Right Maxima-Set



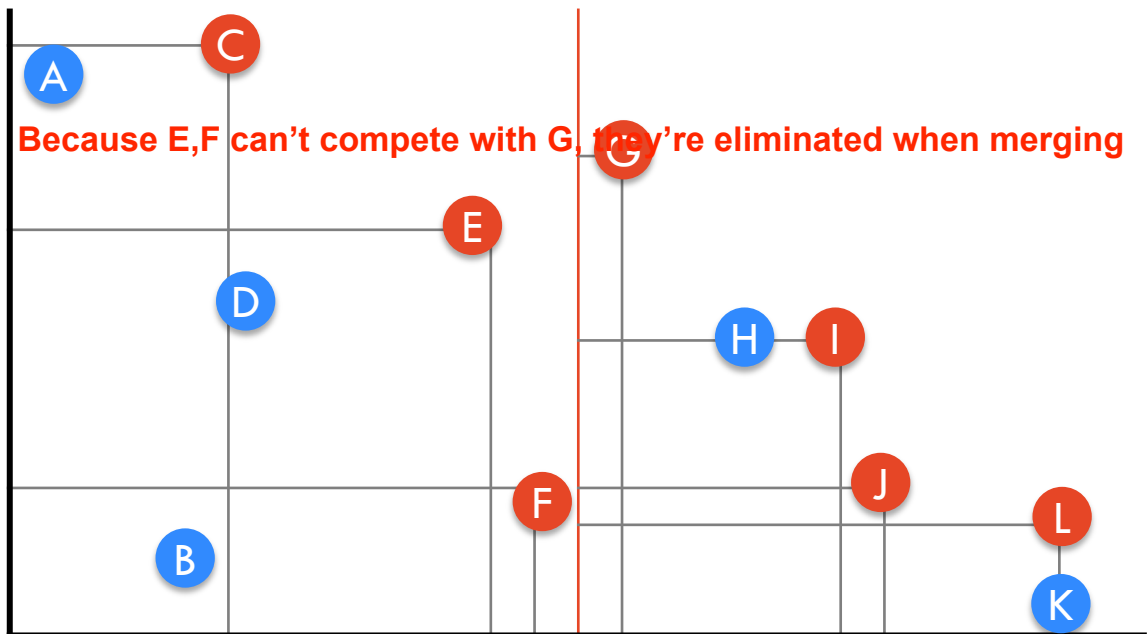
Maxima-Set

Conquer

1. Find the highest point p in the **Right MS**
2. Compare every point q in the **Left MS** to this point.
If $q.y > p.y$, add q to the **Merged MS**
3. Add every point in the **Right MS** to the **Merged MS**

$$p = \text{G}$$

Merged Maxima-Set



Left Maxima-Set



Right Maxima-Set



Maxima-Set

This is the base case when we call on recursion

Base case a single point.

The MS of a single point is the point itself.



Maxima-Set



Maxima-Set: Analysis

Preprocessing Sort the points by increasing x coordinate and store them in an array. Note: we only do this once. Break ties in x by sorting by increasing y coordinate.

$O(n \log n)$

Divide sorted array into two halves.

$O(n)$

Recur recursively find the MS of each half.

$2T(n/2)$

Conquer compute the MS of the union of Left and Right MS

1. Find the highest point p in the **Right MS**
2. Compare every point q in the Left MS to this point.
If $q.y > p.y$, add q to the **Merged MS**
3. Add every point in the **Right MS** to the **Merged MS**

$O(n)$

This is the same as the merge sort

$$T(n) = 2T(n/2) + O(n) = O(n \log n)$$

Overall Running Time: pre-processing + $T(n) = O(n \log n)$

Maxima-Set: Correctness

Preprocessing Sort the points by increasing x coordinate and store them in an array. Note: we only do this once. Break ties in x by sorting by increasing y coordinate.

Divide sorted array into two halves.

Recur recursively find the MS of each half.

Conquer compute MS of union of Left/Right MS

1. Find the highest point p in the **Right MS**
2. Compare every point q in the Left MS to this point.
If $q.y > p.y$, add q to the **Merged MS**
3. Add every point in the **Right MS** to the **Merged MS**

Observations:

1. Every point in MS of the whole is in Left MS or Right MS
2. Every point in Right MS is in MS of the whole
3. Every point in Left MS is either in MS of the whole or is dominated by p

Problem 2

Integer multiplication

Given two n -digit integers x and y

Problem compute the product $x \cdot y$

While this seems like recreational mathematics, it does have real applications: Public key encryption is based on manipulating integers with thousands of bits.

Integer multiplication: Naïve approach

Given two n -digit integers x and y

Problem compute the product $x \cdot y$

Suppose we wanted to do it by hand. We assume that two digits can be multiplied or added in constant time

In primary school we all learn an algorithm for this problem that performs $\Theta(n^2)$ operations

Let's say that we have the binary representation for the inputs x and y

Integer multiplication: Divide and conquer

Let $x = x_1 2^{n/2} + x_0$ and $y = y_1 2^{n/2} + y_0$

Then $x y = x_1 y_1 2^n + x_1 y_0 2^{n/2} + x_0 y_1 2^{n/2} + x_0 y_0$

This 2^n is shifting by n bits

We can compute the product of two n -digit numbers by making 4 recursive calls on $n/2$ -digit numbers and then combining the solutions to the subproblems.

Integer multiplication: Divide and conquer

```
def multiply(x, y):  
    // x and y are positive integers represented in binary  
    if x = 0 or y = 0 then return 0  
    if x = 1 then return y  
    if y = 1 then return x  
  
    // recursive case  
    let  $x_1$  and  $x_0$  be such that  $x = x_1 2^{n/2} + x_0$   
    let  $y_1$  and  $y_0$  be such that  $y = y_1 2^{n/2} + y_0$   
  
    return multiply( $x_1$ ,  $y_1$ )  $2^n$  +  
           (multiply( $x_1$ ,  $y_0$ ) + multiply( $x_0$ ,  $y_1$ ))  $2^{n/2}$  +  
           multiply( $x_0$ ,  $y_0$ )
```

The recursion in essence is:

- get the last digit of the number
- cut it out
- and return the rest with the appropriate $2^{(n/2)}$ coefficient

Integer multiplication: Correctness

Let $x = x_1 2^{n/2} + x_0$ and $y = y_1 2^{n/2} + y_0$

Then $x y = x_1 y_1 2^n + x_1 y_0 2^{n/2} + x_0 y_1 2^{n/2} + x_0 y_0$

Straight forward application of induction to prove
that $\text{multiply}(x, y) = x y$

Integer multiplication: Complexity analysis

Recall $x \cdot y = x_1 \cdot y_1 \cdot 2^n + (x_1 \cdot y_0 + x_0 \cdot y_1) \cdot 2^{n/2} + x_0 \cdot y_0$

Divide step (produce halves) takes $O(n)$

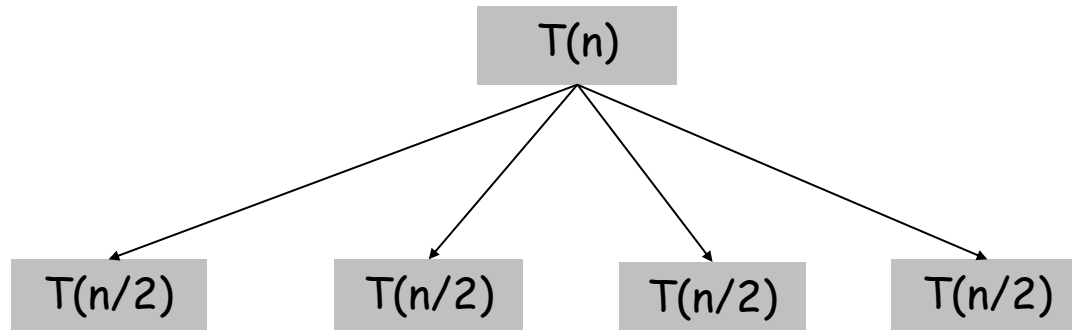
Recur step (solve subproblems) takes $4 T(n/2)$

Conquer step (add up results) takes $O(n)$

$$T(n) = \begin{cases} 4 T(n/2) + O(n) & \text{for } n > 1 \\ O(1) & \text{for } n = 1 \end{cases}$$

This solves to $T(n) = O(n^2)$. No better than naïve!!!

Proof by unrolling



1 (of size n)

4 (of size $n/2$)

...

4^k (of size $n/2^k$)

...

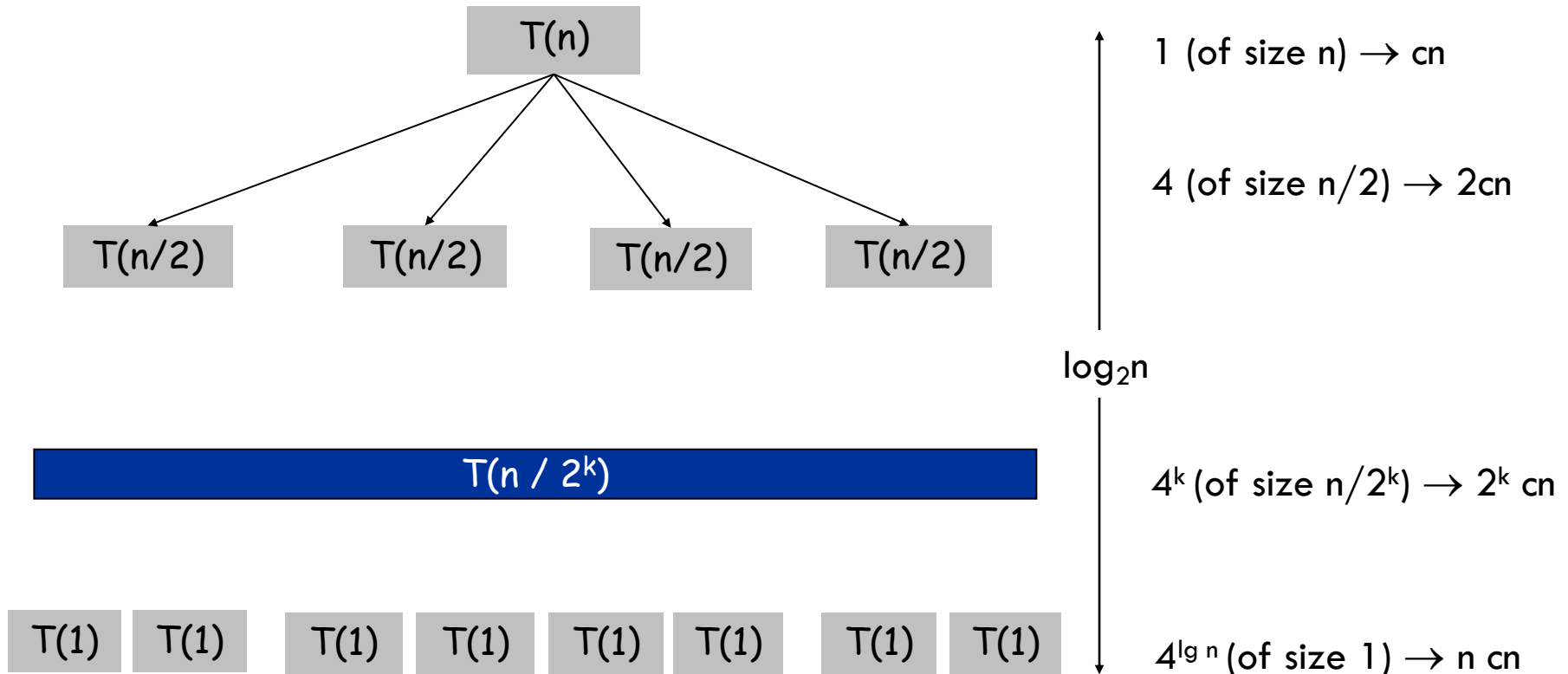
$4^{\lg n}$ (of size 1)

$T(n / 2^k)$

$T(1)$ $T(1)$ $T(1)$ $T(1)$ $T(1)$ $T(1)$ $T(1)$ $T(1)$

Proof by unrolling

$$4^{\log_2 n} = n^{\log_2 4} = n^2$$



Integer multiplication: Divide and conquer v2.0

Let $x = x_1 2^{n/2} + x_0$ and $y = y_1 2^{n/2} + y_0$

$$x y = x_1 y_1 2^n + (x_1 y_0 + x_0 y_1) 2^{n/2} + x_0 y_0$$

$$(x_1 + x_0)(y_1 + y_0) = x_1 y_1 + x_1 y_0 + x_0 y_1 + x_0 y_0$$

This is only three problems, and we can leave the rest to the algorithms

We can compute the product of two n -digit numbers by making 3 recursive calls on $n/2$ -digit numbers and then combining the solutions to the subproblems.

Integer multiplication: Divide and conquer

```
def multiply(x, y):  
    // base case  
    :  
    // recursive case  
    let  $x_1$  and  $x_0$  be such that  $x = x_1 2^{n/2} + x_0$   
    let  $y_1$  and  $y_0$  be such that  $y = y_1 2^{n/2} + y_0$   
  
    first_term  $\leftarrow$  multiply( $x_1$ ,  $y_1$ )  
    last_term  $\leftarrow$  multiply( $x_0$ ,  $y_0$ )  
    other_term  $\leftarrow$  multiply( $x_1 + x_0$ ,  $y_1 + y_0$ )  
  
    return first_term  $2^n$  +  
           (other_term - first_term - last_term)  $2^{n/2}$  +  
           last_term
```

Integer multiplication: Complexity analysis

Divide step (produce halves) takes $O(n)$

Recur step (solve subproblems) takes $3 T(n/2)$

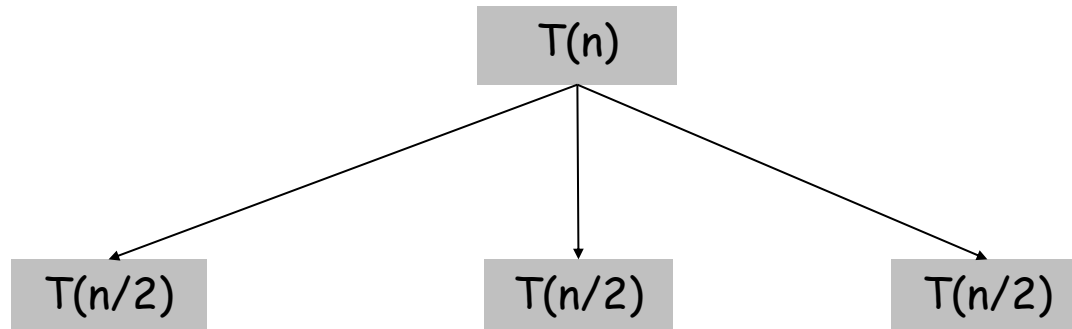
Conquer step (add up results) takes $O(n)$

$$T(n) = \begin{cases} 3 T(n/2) + O(n) & \text{for } n > 1 \\ O(1) & \text{for } n = 1 \end{cases}$$

This solves to $T(n) = O(n^{\log_2 3})$, where $\log_2 3 \approx 1.6$

Better than naïve!!!

Proof by unrolling



1 (of size n)

3 (of size $n/2$)

...

$T(n / 2^k)$

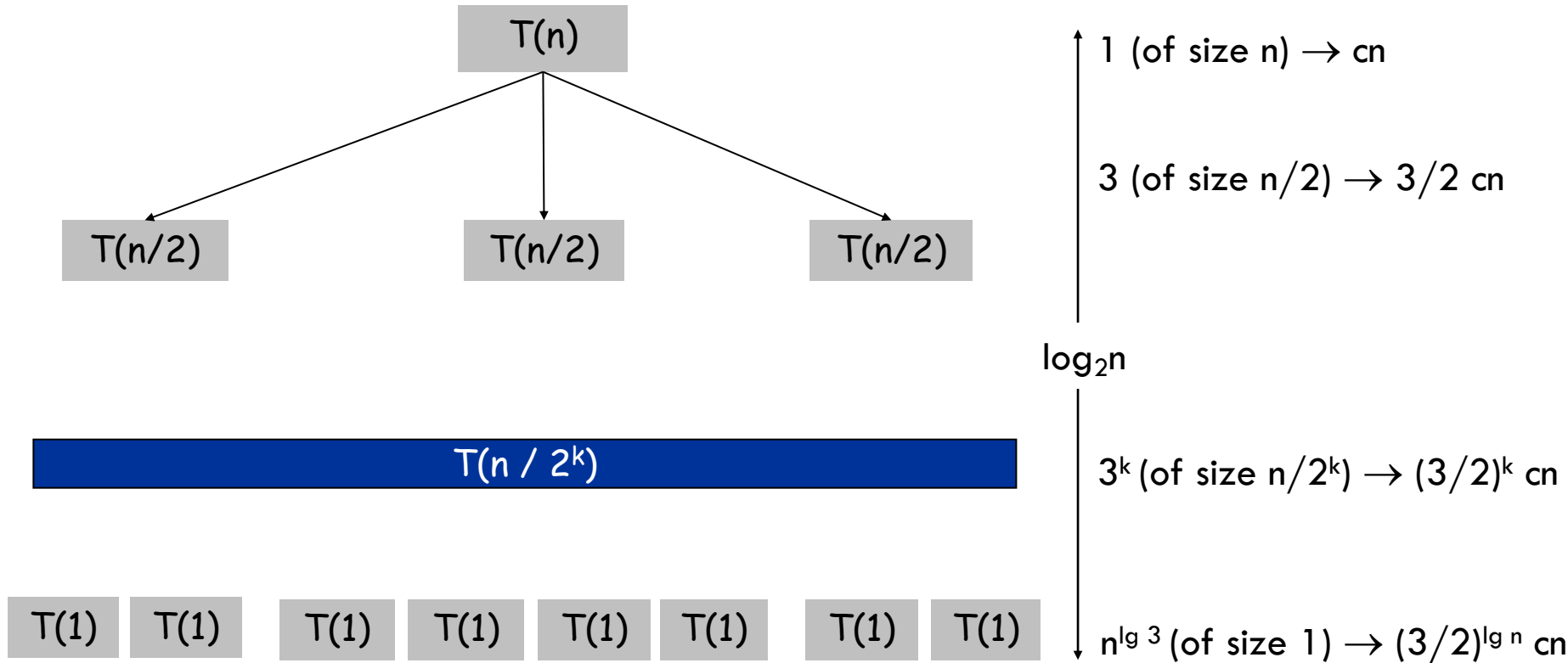
3^k (of size $n/2^k$)

...

$T(1)$ $T(1)$ $T(1)$ $T(1)$ $T(1)$ $T(1)$ $T(1)$ $T(1)$

$n^{\lg 3}$ (of size 1)

Proof by unrolling



Geometric series facts

Let r be a positive real and k a positive integer then

$$1 + r + r^2 + \dots + r^k = (r^{k+1} - 1)/(r-1)$$

Consequently if $r > 1$ then

$$1 + r + r^2 + \dots + r^k < r^{k+1} / (r-1)$$

and if $r < 1$ then

$$1 + r + r^2 + \dots + r^k < 1 / (1-r)$$

Logarithms facts

Base exchange rule:

$$\log_a x = (\log_b x) / (\log_b a)$$

Product rule:

$$\log_a (xy) = (\log_a x) + (\log_a y)$$

Power rule:

$$\log_a x^b = b \log_a x$$

Master Theorem

Let $f(n)$ and $T(n)$ be defined as follows:

$$T(n) = \begin{cases} a T(n/b) + f(n) & \text{for } n \geq d \\ c & \text{for } n < d \end{cases}$$

Depending on a , b and $f(n)$ the recurrence solves to:

1. if $f(n) = O(n^{\log_b a - \varepsilon})$ for $\varepsilon > 0$ then $T(n) = \Theta(n^{\log_b a})$,
2. if $f(n) = \Theta(n^{\log_b a} \log^k n)$ for $k \geq 0$ then $T(n) = \Theta(n^{\log_b a} \log^{k+1} n)$,
3. if $f(n) = \Omega(n^{\log_b a + \varepsilon})$ and $a f(n/b) \leq \delta f(n)$ for $\varepsilon > 0$ and $\delta < 1$ then $T(n) = \Theta(f(n))$,

Note: If $f(n)$ is given as big-O, you can only conclude $T(n)$ as big-O (not Θ).

Note: You should be able to solve all recurrences in this class using unrolling, but if you are comfortable using the Master Theorem, go for it.

The Master Theorem

Examples

1. $T(n) = 8T(n/2) + n^2$

$a=8, b=2, f(n)=n^2, \log_b(a) \rightarrow \log_2(8) = 3$; so $f(n) = O(n^{\log_b a - \epsilon})$ (case 1)

$$T(n) \in \Theta(n^3)$$

2. $T(n) = 2T(n/2) + O(n)$

$a=2, b=2, f(n)=n, \log_b(a) \rightarrow \log_2(2) = 1$; so $f(n) = \Theta(n^{\log_b a} \log^k n)$ (case 2 with $k=0$)

$$T(n) \in O(n \log(n))$$

3. $T(n) = 2T(n/2) + O(n^2)$

$a=2, b=2, f(n)=n^2, \log_b(a) \rightarrow \log_2(2) = 1$; so $f(n) = \Omega(n^{\log_b a + \epsilon})$ (case 3)

$$T(n) \in O(n^2)$$

Selection

Given an unsorted array A holding n numbers and an integer k , find the k th smallest number in A

Trivial solution: Sort the elements and return k th element

Can we do better than $O(n \log n)$?

Yes, with divide and conquer!

First attempt

The array is not sorted, so the median value can't be found

1. **Divide** find the median ($\lfloor n/2 \rfloor$ th element for simplicity) and split array on the halves, \leq and $>$ than the median
2. **Recur** if $k \leq \lfloor n/2 \rfloor$ find k th element on smaller half
if $k > \lfloor n/2 \rfloor$ find $(k - \lfloor n/2 \rfloor)$ th element on larger half
3. **Conquer** return value of the recursive call

1	12	5	16	19	7	23	6	13	20
---	----	---	----	----	---	----	---	----	----

$k = 6$
 $n = 10$

1	12	5	7	6	12	16	19	23	13	20	Divide
---	----	---	---	---	----	----	----	----	----	----	--------

16	19	23	13	20	Recur	$k = 1$
----	----	----	----	----	-------	---------

13 Conquer

Selection time complexity

Divide step (find median and split) takes at least $O(n)$

Recur step (solve left or right subproblem) takes $T(n/2)$

Conquer step (return recursive result) takes $O(1)$

Wrong

If we could compute the median in $O(n)$ time then:

$$T(n) = \begin{cases} T(n/2) + O(n) & \text{for } n > 1 \\ O(1) & \text{for } n = 1 \end{cases}$$

This solves to $T(n) = O(n)$ but only if we can solve the median problem, which is in fact a special case of selection with $k = \lfloor n/2 \rfloor$

Second attempt: Approximating the median

We don't need the exact median. Suppose we could find in $O(n)$ time an element x in A such that

$$|A| / 3 \leq \text{rank}(A, x) \leq 2 |A| / 3$$

Then we get the recurrence

$$T(n) = \begin{cases} T(2n/3) + O(n) & \text{for } n > 1 \\ O(1) & \text{for } n = 1 \end{cases}$$

Which again solves to $T(n) = O(n)$

To approximate the median we can use a recursive call!

Median of 3 medians

Consider the following procedure

- Partition A into $|A| / 3$ groups of 3
- For each group find the median (bruteforce)
- Let x be the median of the medians (computed recursively)

We claim that x has the desired property

$$|A| / 3 \leq \text{rank}(A, x) \leq 2|A| / 3$$

Half of the groups have a median that is smaller/larger than x , and each group has two elements smaller/larger than x , thus

$$\# \text{ elements smaller than } x > 2(|A| / 6) = |A| / 3$$

$$\# \text{ elements greater than } x > 2(|A| / 6) = |A| / 3$$

Median of 3 medians

Let x be the median of the medians, then

$$|A| / 3 \leq \text{rank}(A, x) \leq 2|A| / 3$$

1	12	5	16	19	7	23	6	13
---	----	---	----	----	---	----	---	----

We're finding the median of the median so to speak

1	12	5
---	----	---

16	19	7
----	----	---

23	6	13
----	---	----

1	5	12
---	---	----

7	16	19
---	----	----

6	13	23
---	----	----

1	5	12
---	---	----

6	13	23
---	----	----

7	16	19
---	----	----

elements smaller than $x > 2(|A| / 6) = |A| / 3$

elements greater than $x > 2(|A| / 6) = |A| / 3$

Median of 3 median time complexity

We don't need the exact median. With a recursive call on $n/3$ elements, we can find x in A such that

$$|A|/3 < \text{rank}(A, x) < 2|A|/3$$

Then we get the recurrence

$$T(n) = T(2n/3) + T(n/3) + O(n)$$

Which solves to $T(n) = O(n \log n)$

No better than sorting!

Median of 5 medians

Separate into 5 groups, find the median of each group, and find the median of the medians.
That's $n/5$, then do a $1/2 * n/5$.

We don't need the exact median. With a recursive call on $n/5$ elements, we can find x in A such that

Because we have No.3 as the median of the medians, then $3/10$

Finally, to calculate the rank takes $n/5$ time.

$$3|A|/10 < \text{rank}(A, x) < 7|A|/10$$

Then we get the recurrence

5 elements in a group: $n/5$ groups, x

$$T(n) = T(7n/10) + T(n/5) + O(n)$$

Which solves to $T(n) = O(n)$

We can see that $7/10 + 2/10 < 10/10 = 1$ is less

The amount of work drops exponentially when we drop down.

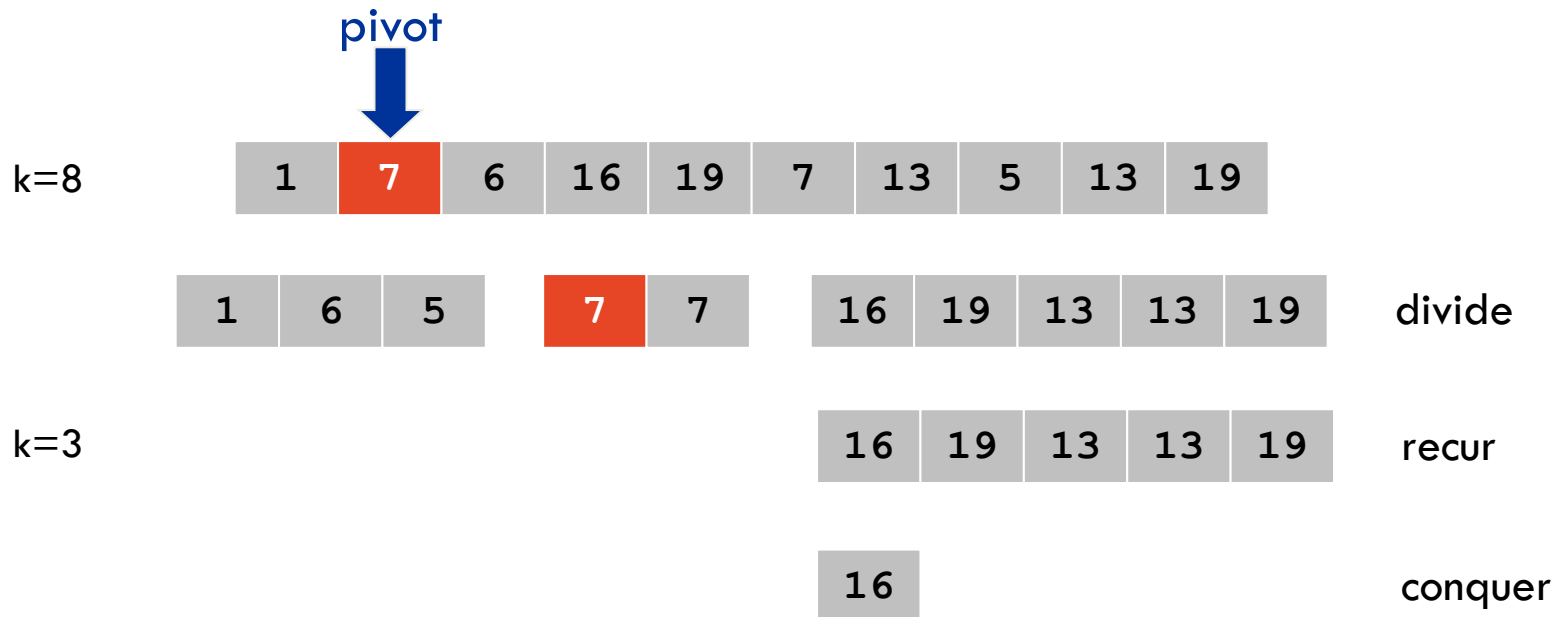
Asymptotically faster than sorting!

I've taken pictures

This is another algorithm

Quick selection

1. **Divide** Choose a random element from the list as the **pivot**
Partition the elements into 3 lists:
(i) less than, (ii) equal to and (iii) greater than the **pivot**
2. **Recur** Recursively select right element from correct list
3. **Conquer** Return solution to recursive problem



Quick selection complexity analysis

Divide step (pick pivot and split) takes $O(n)$

Recur step (solve left and right subproblem) takes $T(n')$

Conquer step (return result) takes $O(1)$

Now we can set up the recurrence for $T(n)$:

$$E[T(n)] = \begin{cases} E[T(n')] + O(n) & \text{for } n > 1 \\ O(1) & \text{for } n = 1 \end{cases}$$

This solves to $E[T(n)] = O(n)$

(details available on the textbook but not examinable)

Convolution

Given two n -degree polynomials $A(x) = \sum_{j=0}^{n-1} a_j x^j$

and $B(x) = \sum_{j=0}^{n-1} b_j x^j$

It's basically multiplying the two polynomials:
 $(3x^2 + 7x + 1) * ()$
 $[3, 7, 1]$ we have coefficients in a list

Problem compute the product $C(x) = A(x) B(x)$, i.e.,

$$C(x) = \sum_{j=0}^{2n-2} c_j x^j, \text{ where } c_i = \sum_{k=0}^i a_k b_{i-k}$$

While this seems like recreational mathematics, it does have real applications: **Signal processing**

Convolution

Different approaches:

- Trivial in $O(n^2)$ time
- Can be done in $O(n^{\log 3})$ using integer multiplication algorithm
- Let's see how to do it in $O(n \log n)$ time

Convolution: Point-value representation

Polynomial $A(x)$ of degree n is a set of n point-value pairs

$$\{(x_0, y_0), \dots, (x_{n-1}, y_{n-1})\}$$

such that all x_k are distinct and $y_k = A(x_k)$.

This is talking about operations in terms of the coefficients

Conveniently, in point-value representation we have:

$$\text{if } C(x) = A(x) B(x), \text{ then } C(x_k) = A(x_k) B(x_k)$$

But $C(x)$ has degree $2n$, so we need to extend the representation of $A(x)$ and $B(x)$ to $2n$ point-value pairs.

We'll look at which point-value pairs to use later.

Convolution: Algorithm

1. Extend coefficient representation of $A(x)$ and $B(x)$ to degree $2n$ by adding n high-order 0 coefficients
2. Compute point-value representations of $A(x)$ and $B(x)$
3. Compute point-value $C(x) = A(x) B(x)$
4. Extract coefficient representation of $C(x)$

Time complexity:

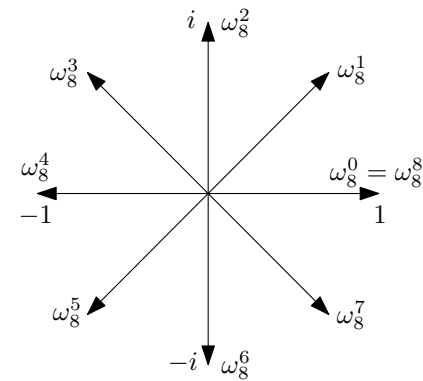
Step 1 and 3 take $O(n)$ time and step 2 and 4 can be done in $O(n \log n)$ time using divide and conquer.

Convolution: Which point-value pairs to use?

Complex n -th root of unity is a complex number ω such that $\omega^n = 1$

There are exactly n complex n -th roots of unity: $\omega(k, 2n) = e^{2\pi i k/n}$ for $k \in \{0, \dots, n-1\}$, where i is imaginary number.

Interpretation: $e^{iu} = \cos(u) + i \sin(u)$



Convolution: Which point-value pairs to use?

Property:

$$\omega^2(k, 2n) = \omega(k, n)$$

Proof:

$$\begin{aligned}\omega^2(k, 2n) &= (e^{2\pi i k / 2n})^2 \\ &= e^{2\pi i k / n} \\ &= \omega(k, n)\end{aligned}$$

This is doubling the angle

Convolution: Evaluating all roots in one go

Observe that $A(x) = A_{\text{even}}(x^2) + x A_{\text{odd}}(x^2)$, where

$$A_{\text{even}}(x) = a_0 + a_2 x + a_4 x^2 + \dots + a_{n-2} x^{(n/2)-1} \text{ and}$$

$$A_{\text{odd}}(x) = a_1 + a_3 x + a_5 x^2 + \dots + a_{n-1} x^{(n/2)-1}$$

To compute $A(\omega(0,2n)) A(\omega(1,2n)) \dots A(\omega(2n-1,2n))$
we recursively compute

$$A_{\text{even}}(\omega(0,n)) A_{\text{even}}(\omega(1,n)) \dots A_{\text{even}}(\omega(n-1,n)) \text{ and}$$

$$A_{\text{odd}}(\omega(0,n)) A_{\text{odd}}(\omega(1,n)) \dots A_{\text{odd}}(\omega(n-1,n))$$

Combine them using the above observation leads to:

$$T(n) = 2 T(n/2) + O(n), \text{ which solves to } O(n \log n)$$

Convolution: Reconstructing $C(x)$

We now have something of the form

$$D(x) = \sum_{j=0}^{2n-1} C(\omega(j, 2n)) x^j$$

Claim:

$$c_i = D(\omega(2n-i, 2n))/2n$$

So to compute C we can evaluate D , like we did for A and B !

Proofs not examined