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COMP 360: Assignment 4

1. Let $X = Opt(LP) \geq r$ for $LP = \max c^T \cdot \vec{x}$ and $r \in \mathbb{R}$

$$\text{s.t. } A\vec{x} \leq \vec{b}$$
$$\vec{x} \geq 0$$

Want to show $X \in NP$ and $X \in CoNP$

$X \in CoNP \Rightarrow \overline{X} \in NP$ with $\overline{X} = Opt(LP)$ not $\geq r$
$= Opt(LP) < r$

So we need to show an efficient certifier for both X and $\overline{X}$.

Efficient Certifier for X (ECX):

Take $(LP, r, \vec{x}')$ where $\vec{x}'$ is a given input vector of size n.
If $c^T\vec{x}' \geq r$ AND $A\vec{x}' \leq \vec{b}$
  then Output Yes
Else
  then Output No.

Assuming vector multiplication takes 1 time unit, this certifier would run in $O(m)$ to check all m constraints. If vector multiplication takes $|\vec{x}|$ time units then this certifier would run in $O(mn)$.
Regardless it runs in polynomial time.

Efficient Certifier for $\overline{X}$ ($EC\overline{X}$):
Take $(LP, r, \vec{x}')$ where $\vec{x}'$ is a given input vector of size n.
If $c^T\vec{x}' < r$ AND $A\vec{x}' \leq \vec{b}$
  then output yes
else
  then output No.

ECX shows that $X \in NP$ and $EC\overline{X}$ shows that $\overline{X} \in NP$
so X is in both NP and CoNP.

2. a) $\max \sum_{i=1}^{n} r_i$

s.t. $r_i + r_j \leq dist(p_i, p_j) \quad \forall\, i,j \in \{1,\dots,n\}$ s.t. $i \neq j$.

$r_i \geq 0 \quad \forall\, i \in \{1,\dots,n\}$

b) $\min \sum_{i=1}^{n} \sum_{j=1}^{n} y_{ij}\, dist(p_i, p_j)$ s.t. $i \neq j$

s.t. $\sum_{j=1}^{n} y_{ij} \geq 1 \quad \forall\, i \in \{1,\dots,n\}$ s.t. $i \neq j$

$y_{ij} \geq 0 \quad \forall\, i,j \in \{1,\dots,n\}$ s.t. $i \neq j$.

$y_{ij} \in \mathbb{N}$

The dual of the no-overlapping disks problem is exactly defining the problem of minimizing the total distance traveled subject to the restriction that we must pass trough every ~~vertex~~ vertex at least once.

A result of strong Duality: If we can show that this problem is at most half the solution to covering the vertices with cycles then we have shown that covering the vertices with cycles is at least twice the solution to the non-overlapping disk problem.

An additional constraint forced by the cycle problem would be that for every outgoing edge y_{ij} there must be at least one incoming edge y_{ki}. This means that the sum of the minimization of the dual will be multiplied by at least 2 to convert it to the cycles problem.

$\Rightarrow$ dual is at most half the solution of the cycles problem

$\Rightarrow$ The cycles problem is at least twice the no-overlapping disks problem

3.

Favourite things / students	Food	Book	etc...
s_1			
s_2			
$\vdots$			
s_n			

Let X be the problem:

Let F_i = Set of Favourite things for s_i student s_i

Want to show $F_i \cap F_j$ contains at least 1 element. $i \neq j$ $i, j \in A$, $A \subseteq \{1, \dots, n\}$ $|A| = K$.

- Showing NP:
  - Certifier takes (Table: T, $K \in \mathbb{N}$) and $B \subseteq \{s_1, s_2, \dots, s_n\}$.
  - If $|B| = K$ and $\forall B_i\ \exists B_j$, $i \neq j$, with at least 1 favourite thing in common then output yes

  else

  then output No. $O((n-1)nm) = O(n^2m + nm) = O(n^2 m)$

- Showing Complete:
  - First attempt: Reducing CLIQUE to our problem X.

Given an input to CLIQUE $(G, K \in \mathbb{N})$, construct a graph H such that a solution to X in H is also a solution to CLIQUE.

Graph G of sizes $|V| = n$, $|E| \geq \frac{K(K-1)}{2}$ where subgraph C with K nodes is complete (ie. C has K choose 2 edges)

Add edges between each pair (S_i, S_j) iff $i \neq j$ and, S_i and S_j share at least one favourite thing and, edge (S_i, S_j) ~~does~~ does not already exist. Remove other edges from all other pairs.

This solution is under the assumption that "every two of them share at least one favourite thing" means that for a given S_i in subset size K all other S_j need to share at least one favourite thing with S_i.

Oracle Alg: - Construct H as above
- If H has a solution to X
  then yes
- else No. $\Rightarrow$ CLIQUE $\leq_p X$.

4. <u>Showing NP:</u>

- Certifier takes (N: Set of n real numbers, $K \in \mathbb{N}$) and $B \subseteq N$
- If $|B| = K$ and $\sum_{i=0}^{|B|} B_i$ has no 2's in the result
  then output Yes

  else
  then output No

$O(m)$ where m is the string size of the sum B

<u>Showing NP-Complete:</u>

- I want to be able to define a set of finite conditions that if those conditions are met, then the problem can be reduced from an SAT problem.

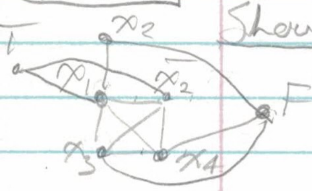
5. Showing NP:

- Certifier takes $(\emptyset, \vec{x})$,
- Pass $\vec{x}$ to $\emptyset$.
- Traverse $\emptyset$ from left to right keeping track of counts T_i and F_i for each clause i in $\emptyset$.
- If $T_i = F_i \; \forall i \in$ Clauses of predicate $\emptyset$.
  then output yes
  Else
  then output No.

$O(n)$ where n is the number of terms in predicate $\emptyset$ in $\emptyset$

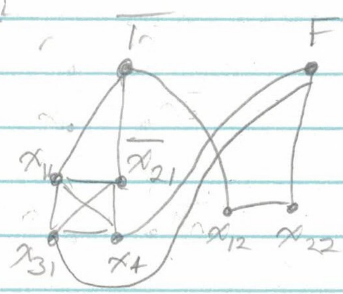
Showing Completeness:

idea 1



For every clause have a complete subgraph connecting all terms of the clause.
Then connect each term to it's assigned T or F node.
Problem evaluates to Yes iff $\forall$complete subgraphs K_i the number of nodes connecting to true = number of nodes connecting to false.

idea 2



For every clause i have a complete subgraph i disjoint from any other subgraph j. Add True and False nodes. Connect each term of each clause to it's corresponding truth evaluation.
Problem evaluates yes iff edges connected to T = edges connected to F.

The construction of this is representative of the prob. but the NP-Complete reduction is unclear.

problem because for the above example $x_1\, \overline{x_2}$ is a complete subgraph not corresponding to any clause.

Idea 1 is incorrect due to , so I will try to continue with idea 2.

If we construct the graph as above, then invert it such that every edge becomes a vertex and every vertex becomes an edge, this problem can be compared directly with a Hamiltonian cycle, which is NP-complete.