Introduction to Probability

Administrivia

- Homework 1

Why We Need Probability

Aspects of the world seem random and unpredictable

- Are we tall or short?
- Do we have Mom's eyes or Dad's chin?
- Is the eye of the hurricane going to pass over City X?
- Which team will win a best of seven series?
- How long will it takes us to drive to the airport?
- How long will it be before the next bus comes?

Why We Need Probability

Aspects of the world seem random and unpredictable

Probability is a way of thinking about unpredictable phenomenon as if they were each generated from some **random process**

It turns out that by thinking of phenomena in this way we can **describe these random processes with math**

Basic Definitions

Think of a random process as a trial or an **experiment**

Def: The sample space Ω is the set of all possible outcomes of the experiment

Example: If we flip a fair coin a single time, what is the sample space?

$$\Omega = \{H, T\}$$

Example: If we're doing a poll, and ask a person their birth month, what is the sample space?

$$\Omega = \{JAN, FEB, MAR, \ldots, DEC\}$$

Observation: These are *discrete* sample spaces because there are a finite number of outcomes

Basic Definitions

Def: For each event in Ω the probability is a measure between 0 and 1 of how likely it is for the event to occur

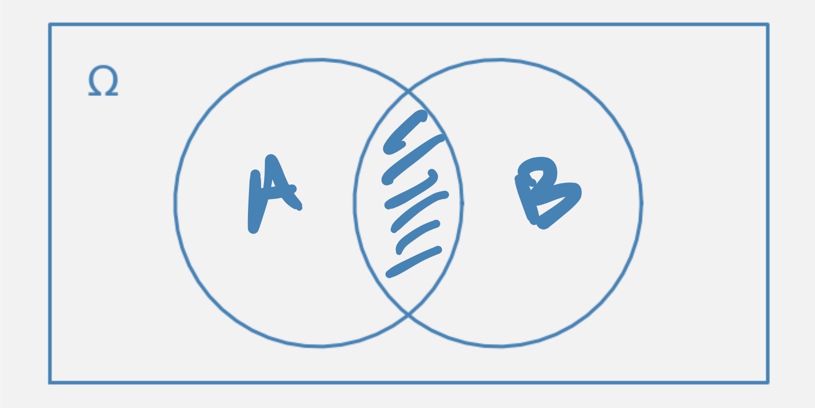
Observation: The sum of the probability of each outcome in Ω is 1. Why?

Set Operations

$DIE = \{1, 2, 3, 4, 5, 6\}$

Def: the **intersection** of two events is the subset of outcomes in **both** events

intersection = "and"



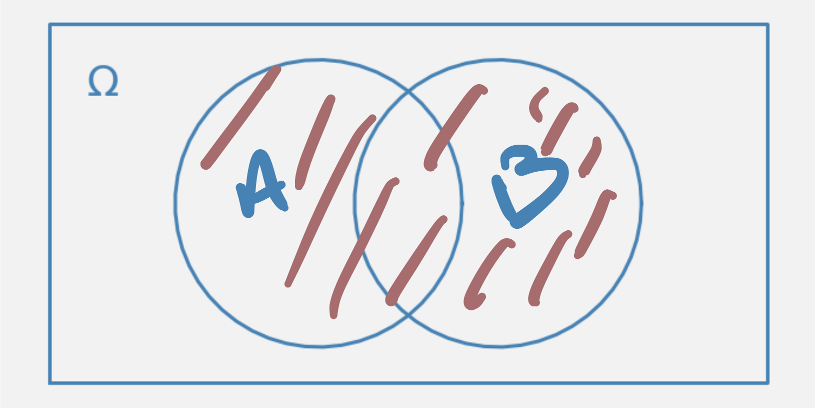
$A = EVEN = \{2, 4, 6\}$

$B = 6 = \{6\}$

$A \cap B = \{6\}$

Def: the **union** of two events is the subset of outcomes in one or **both** events

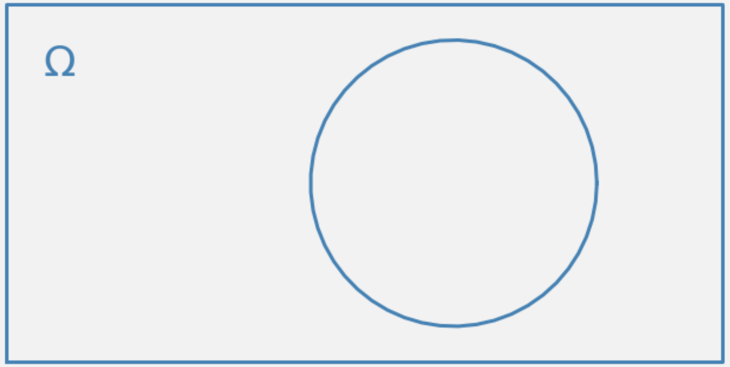
union = "or"



Set Operations

$\Omega = \{1, 2, 3, 4, 5, 6\}$

Def: the **complement** of an event A is the set of outcomes in Ω but **not** in A



$A = \{2, 4, 6\}$

$A^c = \{1, 3, 5\}$

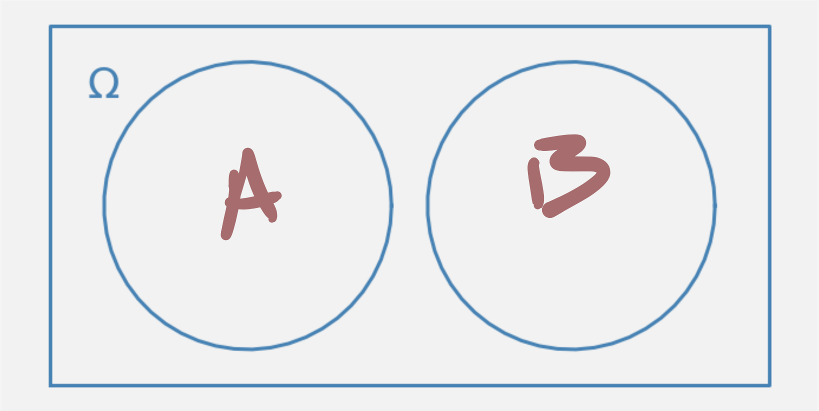
$= \Omega - A$

Notation:

- Complement: A^c
- Intersection: $A \cap B$
- Union: $A \cup B$

Set Operations

Def: when the intersection of two events is empty, we call those two events **disjoint** or **mutually exclusive**



$A = \{2, 4, 6\}$

$B = \{1, 3, 5\}$

$A \cap B = \emptyset$

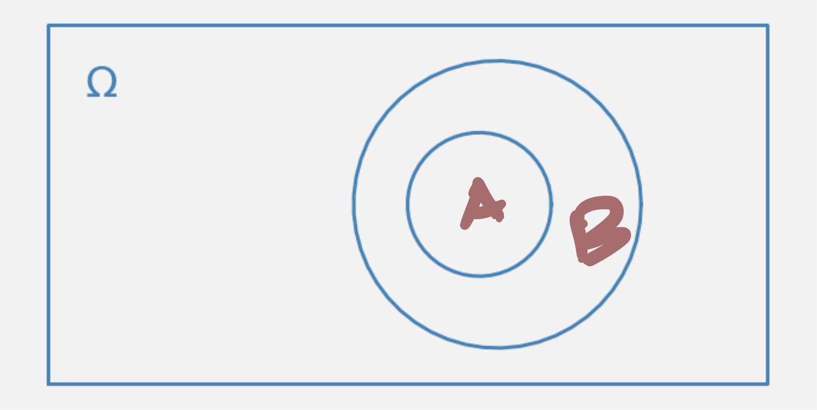
Notation:

- Null set: $\emptyset = \{\}$

Set Operations

Def: If all outcomes of event A are also outcomes of event B, we say A is a subset of B

A = All DS STUDENTS
B = All CU BOULDER



B = WATER TEMP ΔS
A = WATER TEMP ↗

Notation:

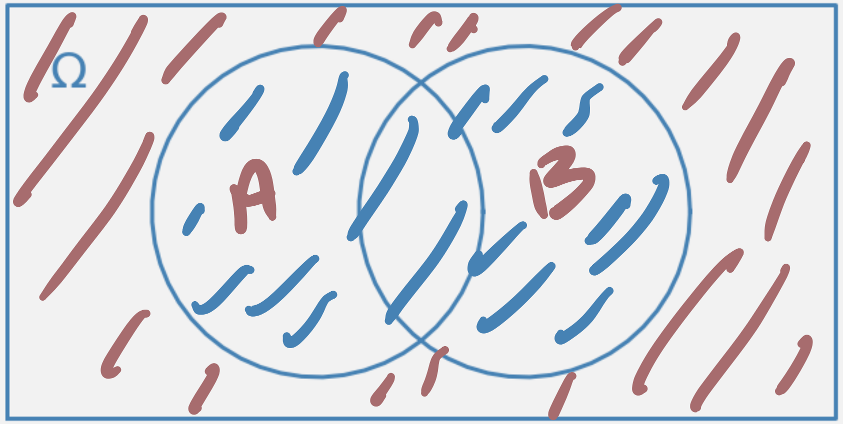
- subset: $A \subset B$

DeMorgan's Laws

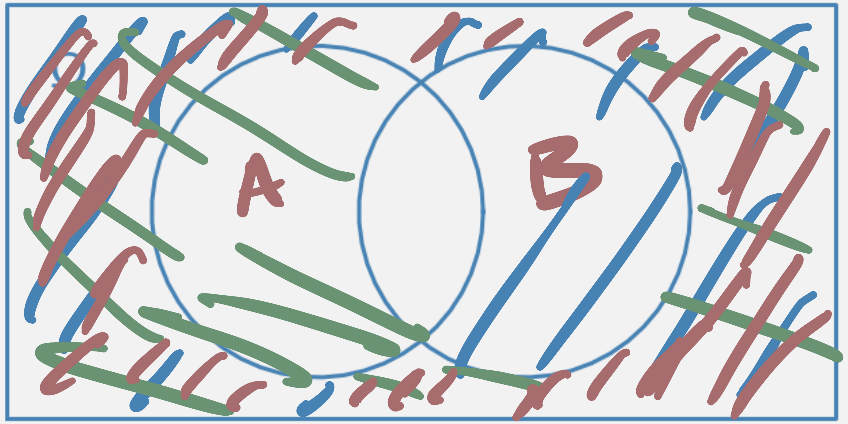
- Complement of an union: $(A \cup B)^c = A^c \cap B^c$
- Complement of an intersection: $(A \cap B)^c = A^c \cup B^c$

Question: Can we do picture proofs of these two facts?

$(A \cup B)^c$



$A^c \cap B^c$



Probability Functions

A **biased coin** is a coin with a modified probability function

Instead of $P(\{H, T\}) = \{\frac{1}{2}, \frac{1}{2}\}$ a biased coin's probability function is $P(\{H, T\}) = \{p, q\}$

Question: What can we say about q ?

$q = 1 - p$

$P(\{H, T\}) = \{p, 1 - p\}$

Looking Ahead: A random process with two outcomes with fixed probabilities assigned to each outcome is called a **Bernoulli Trial**

Probability Functions

Note that a probability function has two key properties:

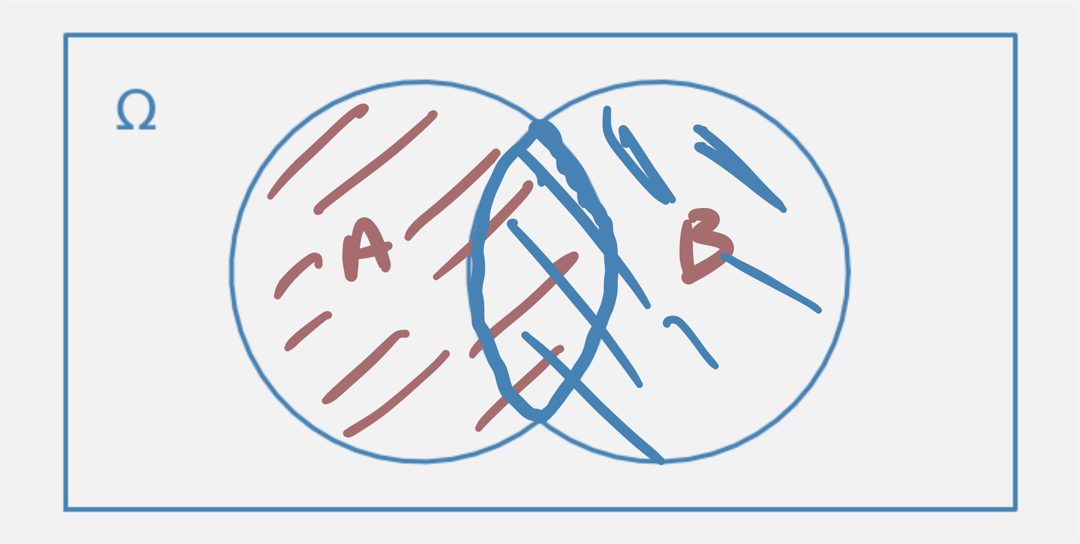
- The probability of the entire sample space is 1
- The probability of the union of disjoint events is the sum of the probability of each event

Formal Def: a probability function P assigns to each event A a number P(A) in [0,1] s.t.:

- $P(\Omega) = 1$
- $P(A \cup B) = P(A) + P(B)$ if A and B are disjoint events

Probability of Non-Disjoint Events

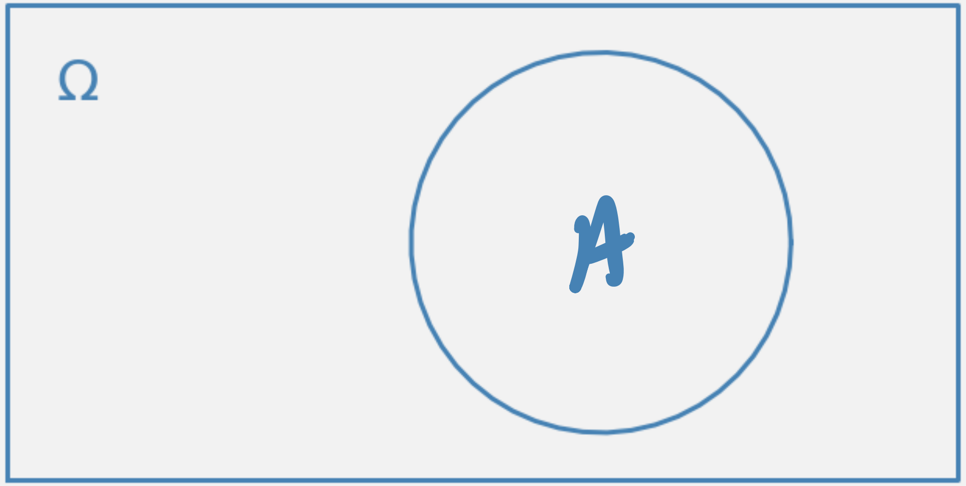
Question: What is the probability of the union of events A and B if A and B are not disjoint?



$$P(A \cup B) = \underline{P(A)} + \underline{P(B)} - P(A \cap B)$$

Probability of the Complement

Question: What is the probability of the complement of an event A?



$$P(A^c) = 1 - P(A)$$

More Complicated Coins

Question: What is the probability that I flip a biased coin twice and both flips come heads?

$P(H) = p \quad P(T) = 1-p$

More Complicated Coins

Question: What is the probability that I flip a biased coin twice and both flips come heads?

The sample space for a single coin flip is $\Omega = \{H, T\}$

The sample space for two coin flips is $\Omega = \{H, T\} \times \{H, T\} = \{(H, H), (H, T), (T, H), (T, T)\}$

This is an example of the a **product** of sample spaces:

$P((H,H)) = \frac{1}{4}$

$= \frac{1}{2} \cdot \frac{1}{2} = P(H)\,P(H)$

More Complicated Coins

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Intuition Check: Does the result of the first flip affect the result of the second flip?

More Complicated Coins

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Intuition Check: Does the result of the first flip affect the result of the second flip?

Def: When two trials do not affect each other, we say they are **independent**

Fact: When two events are independent we can **multiply** their probabilities:

$$P((H,H)) = P(H)\,P(H) = p \cdot p = p^2$$

More Complicated Coins

Question: What is the probability that I flip a biased coin twice and get one H and one T?

We want to know the probability of events (H, T) OR (T, H)

If the outcomes are independent then OR means addition:

$$P((H, T) \ or \ (T, H)) = P((H,T) \cup (T,H)) = P(H,T) + P(T,H)$$

$$= P(H)P(T) + P(T)P(H)$$

$$= p(1-p) + (1-p)p = 2p(1-p)$$

More Complicated Coins

Question: What is the probability that I flip 5 coins and get exactly one H?

$$P(\{HTTTT, THTTT, TTHTT, TTTHT, TTTTH\})$$

$$p(1-p)^4 \quad (1-p)p(1-p)^3 \quad + \cdots + (1-p)^4 p$$

$$= 5p(1-p)^4$$

An Empirical Experiment

Suppose that we know we have a biased coin, but don't know what the probabilities are

What could we do?

1000 FLIPS $\Rightarrow$ 797 HEADS

$$\frac{797}{1000} = 0.797 \approx p$$

MAXIMUM LIKELIHOOD ESTIMATION