COMP2823

Lecture 8: Shortest Paths and Minimum Spanning Trees

[GT 14.1-2, 15.1-3]

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Some content is taken from material provided by the textbook publisher Wiley.

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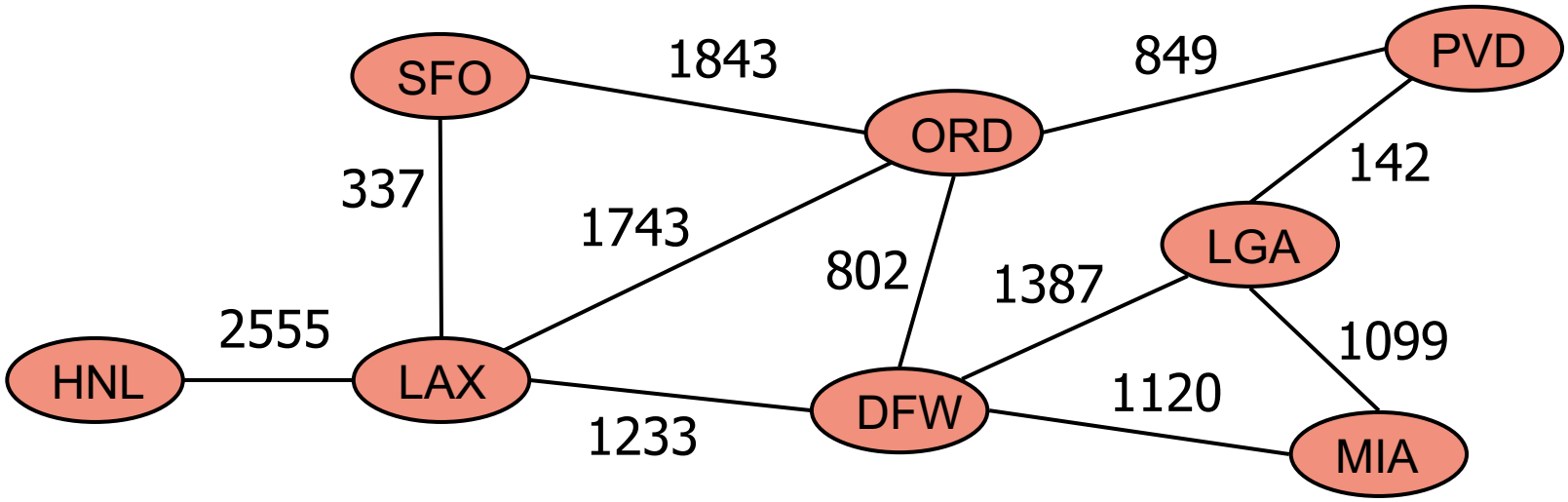
Weighted Graphs

- In a weighted graph, each edge has an associated numerical value, called the weight of the edge
- Edge weights may represent, distances, costs, etc.

Example: In a flight route graph, the weight of an edge represents the distance in miles between the endpoint airports

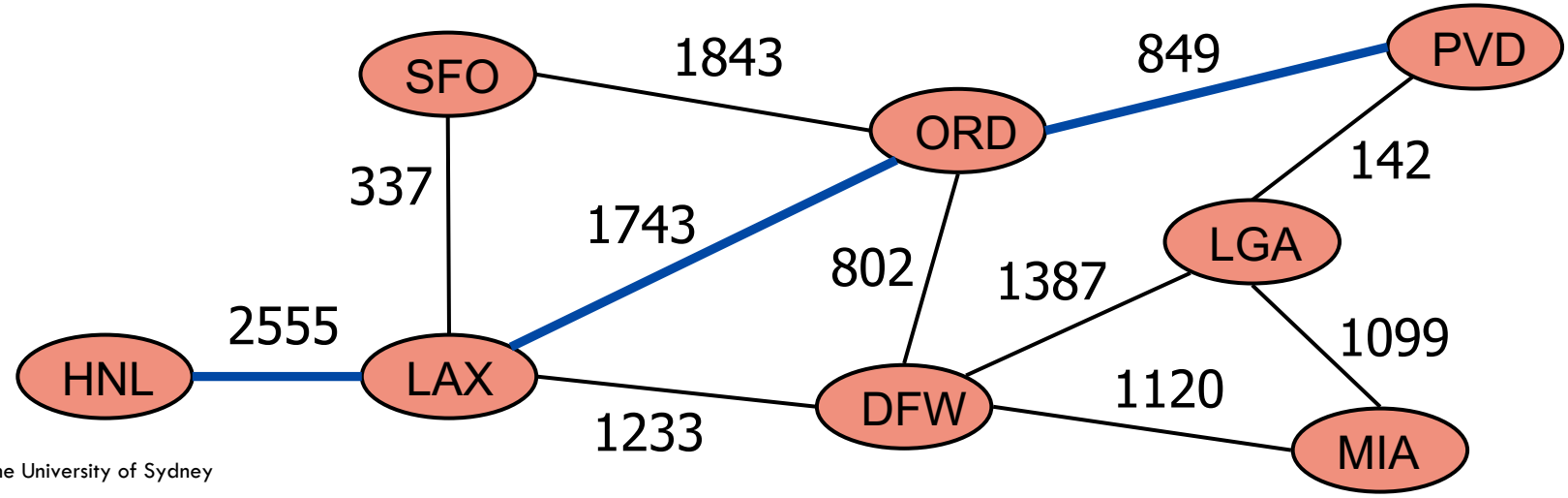


Shortest Paths

Given an edge weighted graph and two vertices u and v, we want to find a path of minimum total weight between u and v, where the weight of a path is the sum of the weights of its edges.

Applications: Internet packet routing, flight reservations and driving directions.

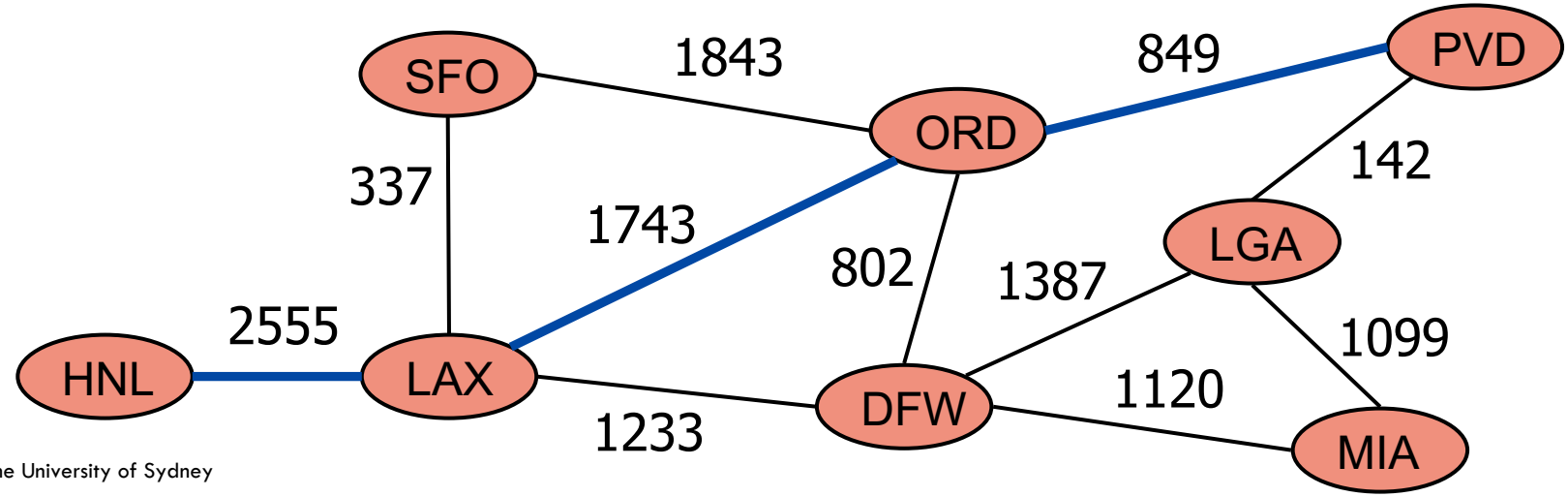
Example: Shortest path between Providence (PVD) and Honolulu (HNL)



Shortest Path Properties

Property: A subpath of a shortest path is itself a shortest path

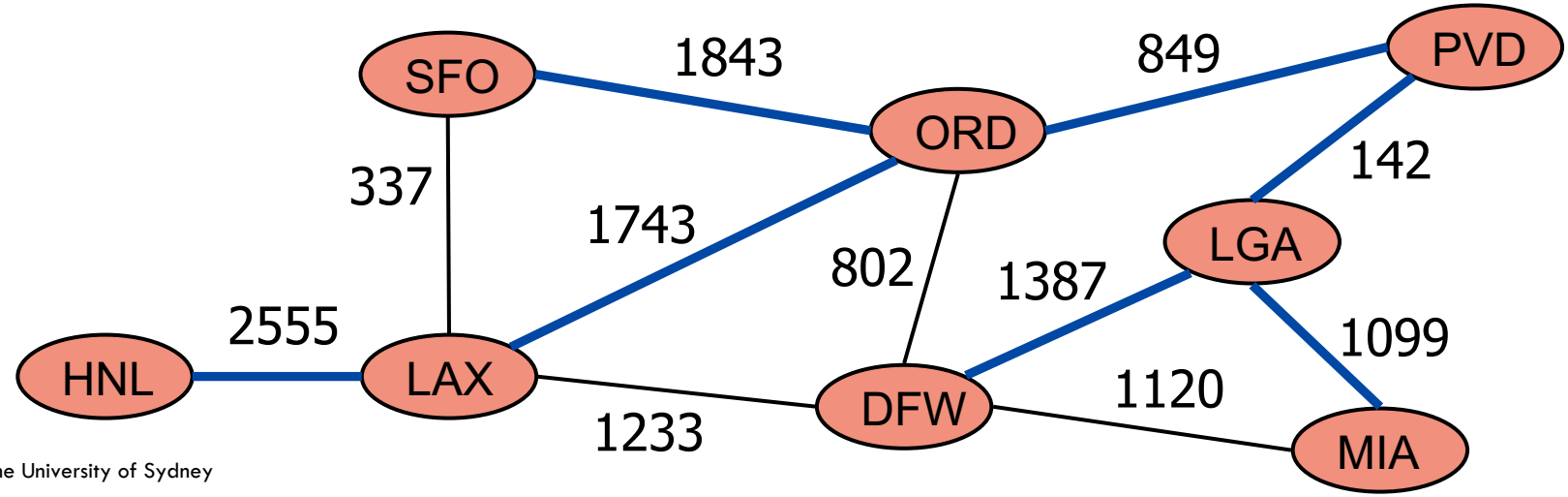
Example: Shortest path from Providence (PVD) to Honolulu (HNL) also contains a shortest path from Providence (PVD) to Los Angeles (LAX)



Shortest Path Properties

Property: There is a tree of shortest paths from a start vertex to all the other vertices (shortest path tree).

Example: Tree of shortest paths from Providence (PVD)



Dijkstra's Algorithm

Input:

- Graph $G = (V, E)$
- Edges weights $w : E \rightarrow R_+$
- Start vertex s

Output:

- Distance from s to all v in V
- Shortest path tree rooted at s

Assumptions:

- G is connected and undirected
- edge weights are nonnegative

High level idea:

- Maintain a distance estimate
  $D[v] \geq dist_w(s, v)$ for all v in V
- Keep track of a subset S of V s.t.
  $D[v] = dist_w(s, v)$ for all v in S

Initially:

- $D[s] = 0$
- $D[v] = \infty$ for all v in $V - s$

In each iteration we:

- add to S vertex u in $V \setminus S$ with smallest $D[u]$
- update D-values for vertices adjacent to u

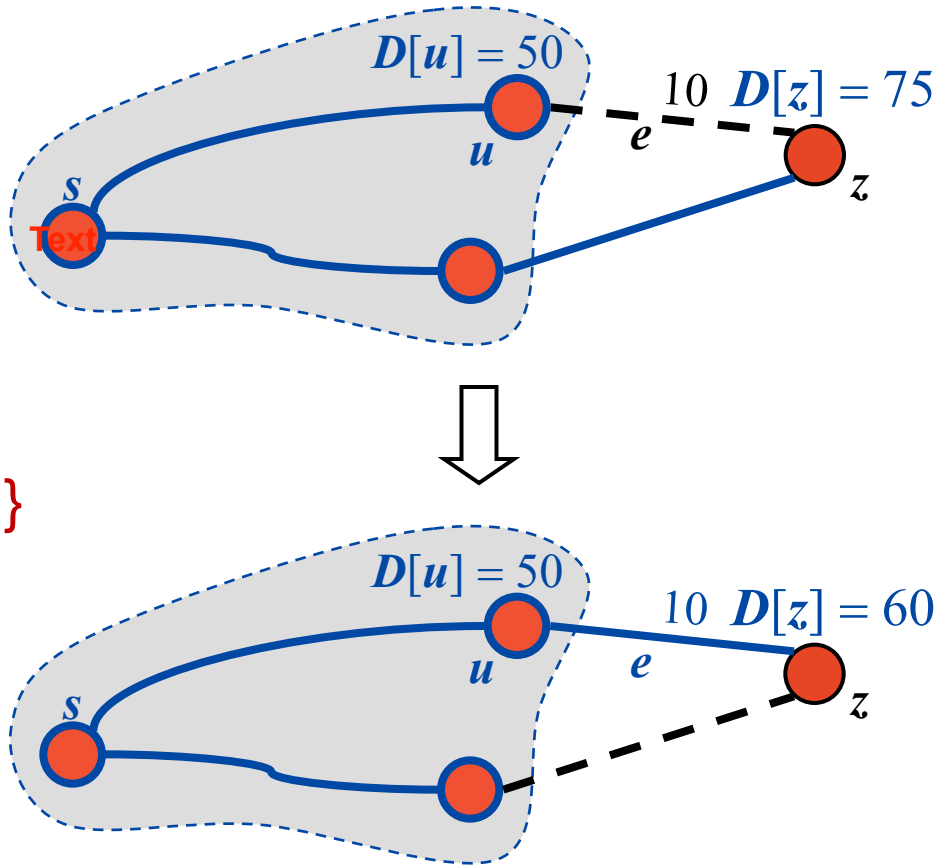
Edge Relaxation

Consider edge e = (u, z) such that:

- u is the last vertex added to S
- z is not in S

The relaxation of edge (u, z) updates D[z] as follows:

$D[z] \leftarrow \min\{D[z], D[u] + w(u, z)\}$



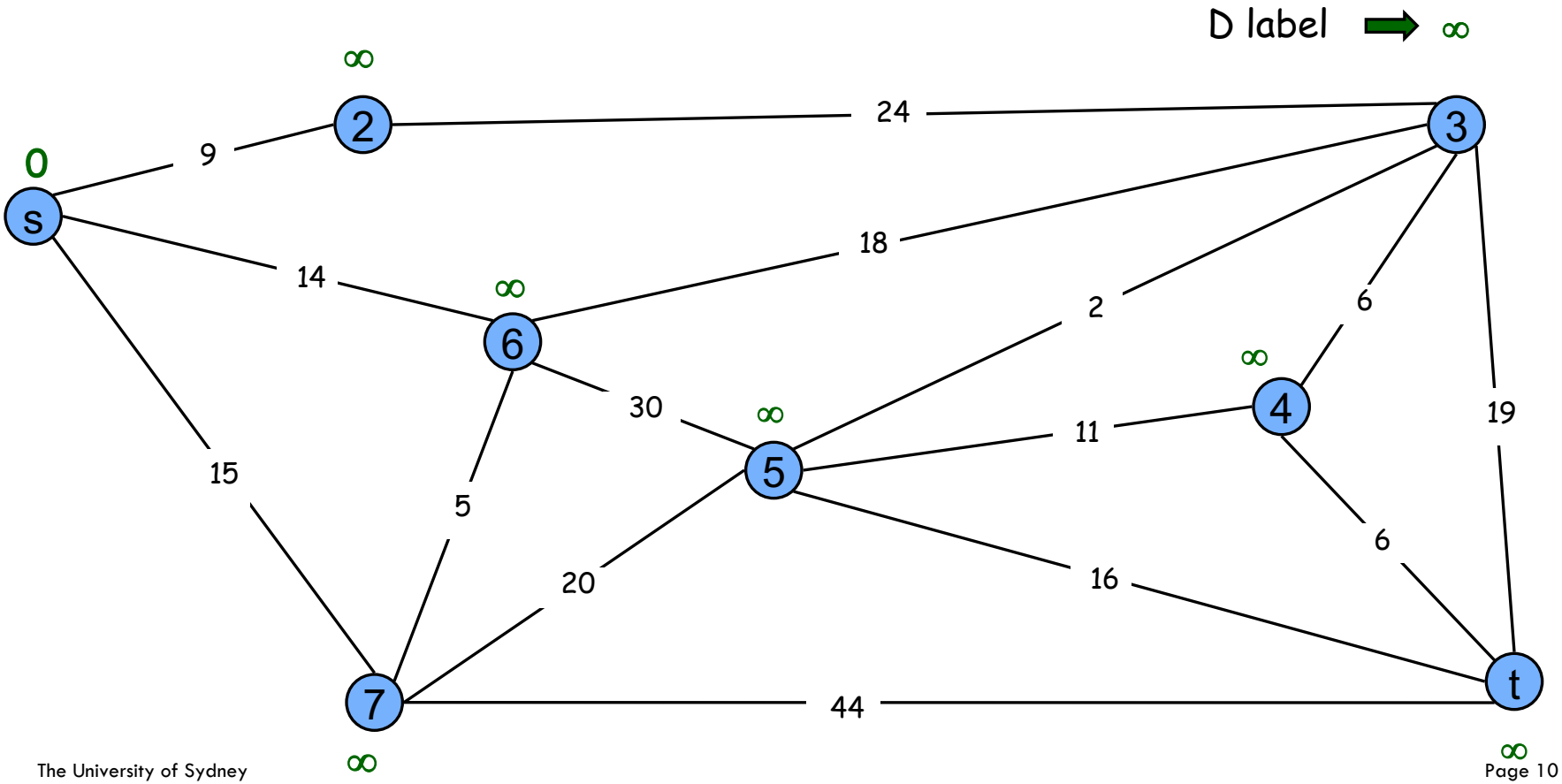
Dijkstra's Algorithm pseudocode

```
def Dijkstra(G, w, s):

  # initialize algorithm
  for v in V do
    D[v] ← ∞
    parent[v] ← ∅
  D[s] ← 0
  Q ← new priority queue for { (v, D[v]) : v in V }

  # iteratively add vertices to S
  while Q is not empty do
    u ← Q.remove_min()
    for z in G.neighbors(u) do
      if D[u] + w[u, z] < D[z] then
        D[z] ← D[u] + w[u, z]
        Q.update_priority(z, D[z])
        parent[z] ← u
  return D, parent
```

Dijkstra's Shortest Path Algorithm

S = { }
PQ = { s, 2, 3, 4, 5, 6, 7, t }

We maintain a priority queue for it



Dijkstra's Shortest Path Algorithm

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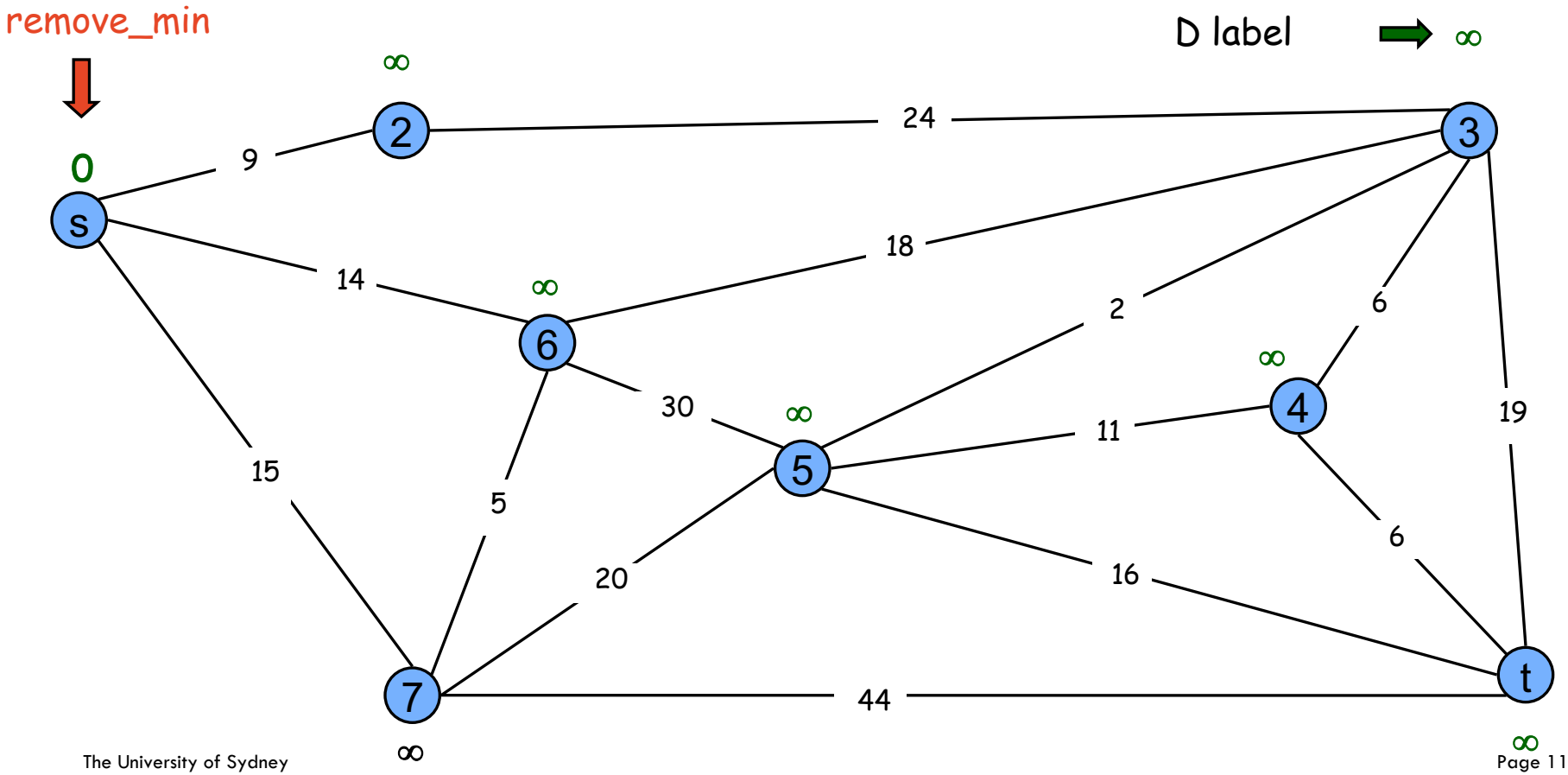

Dijkstra's Shortest Path Algorithm

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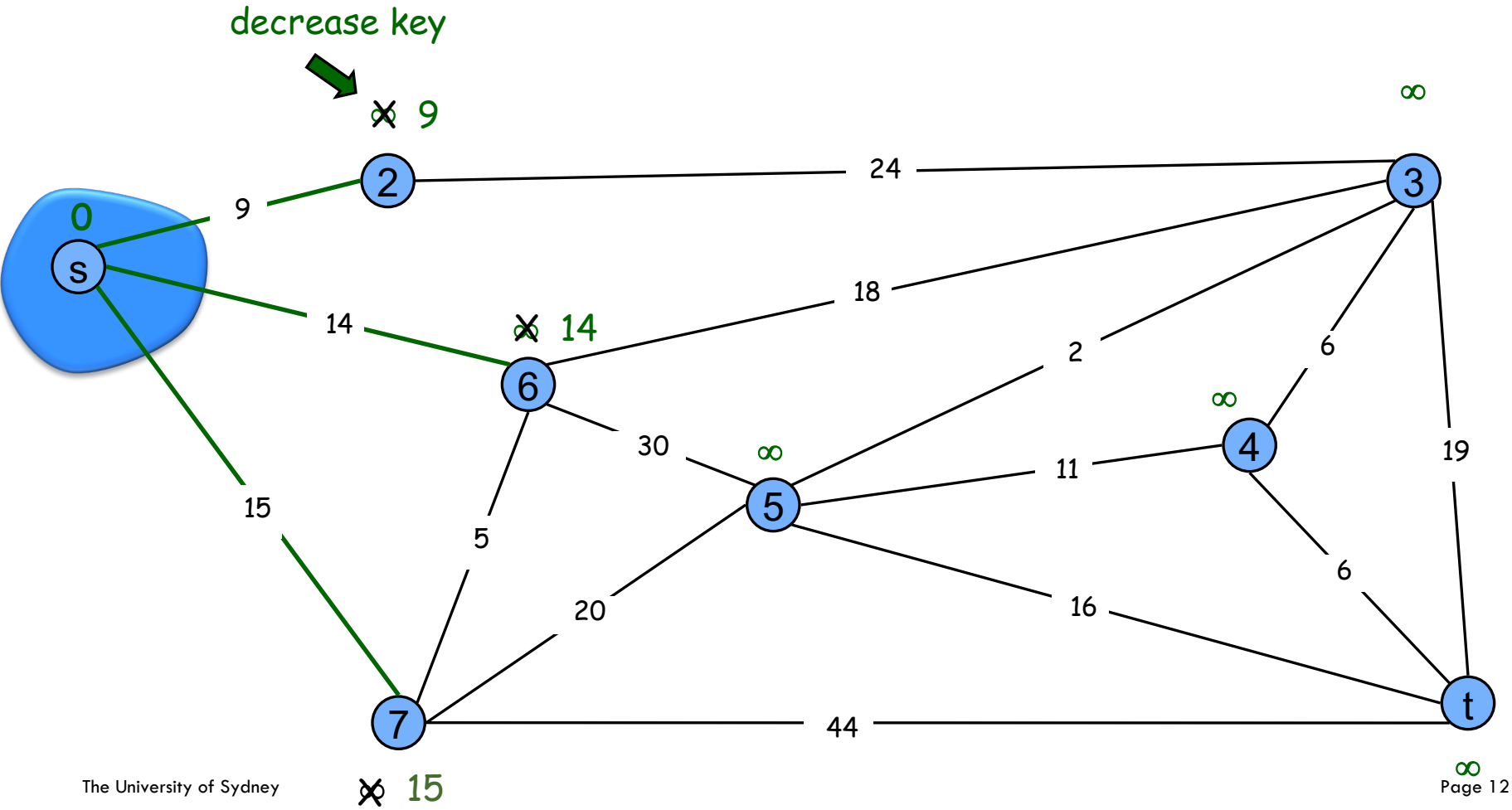

Dijkstra's Shortest Path Algorithm

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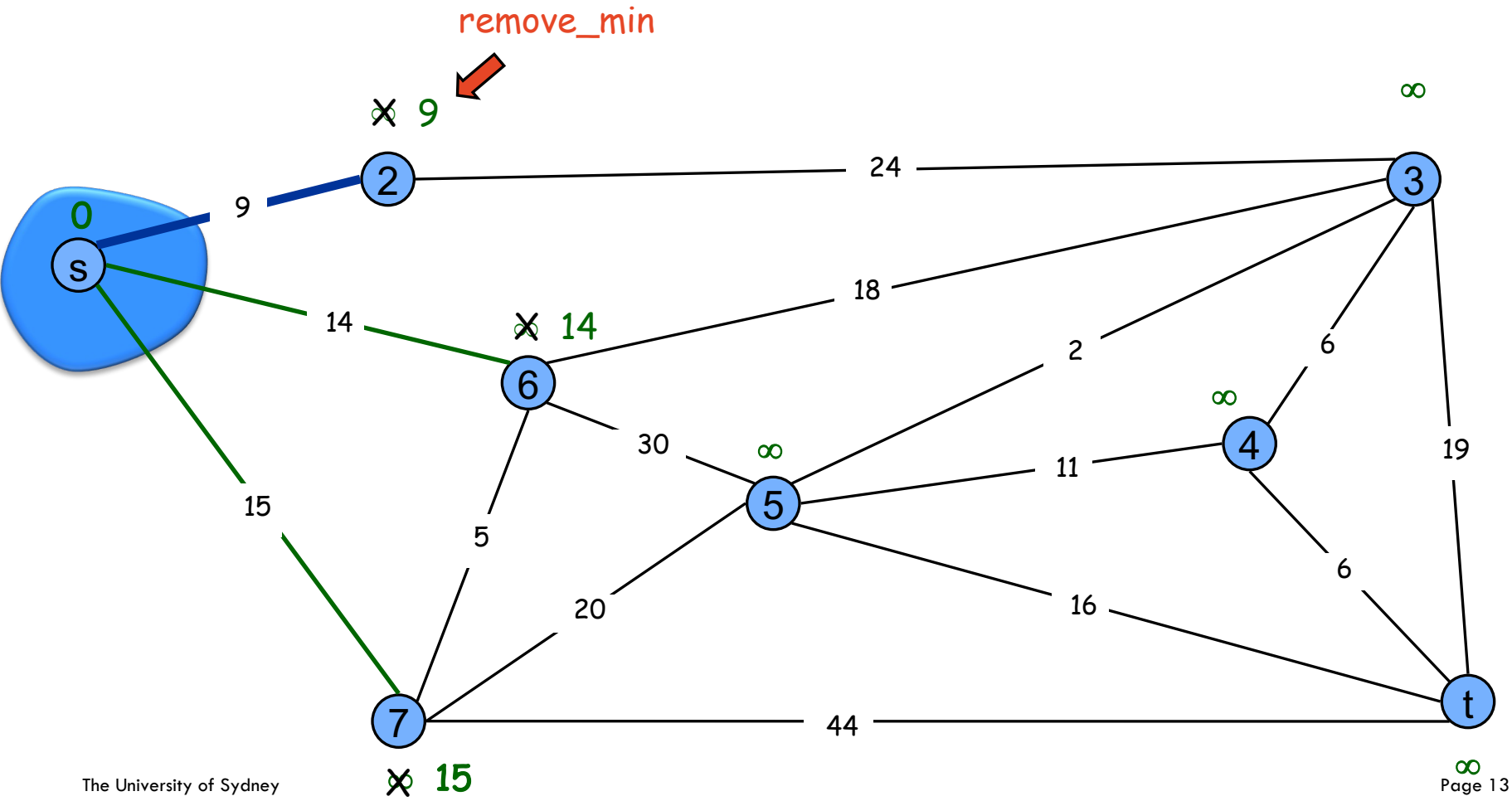
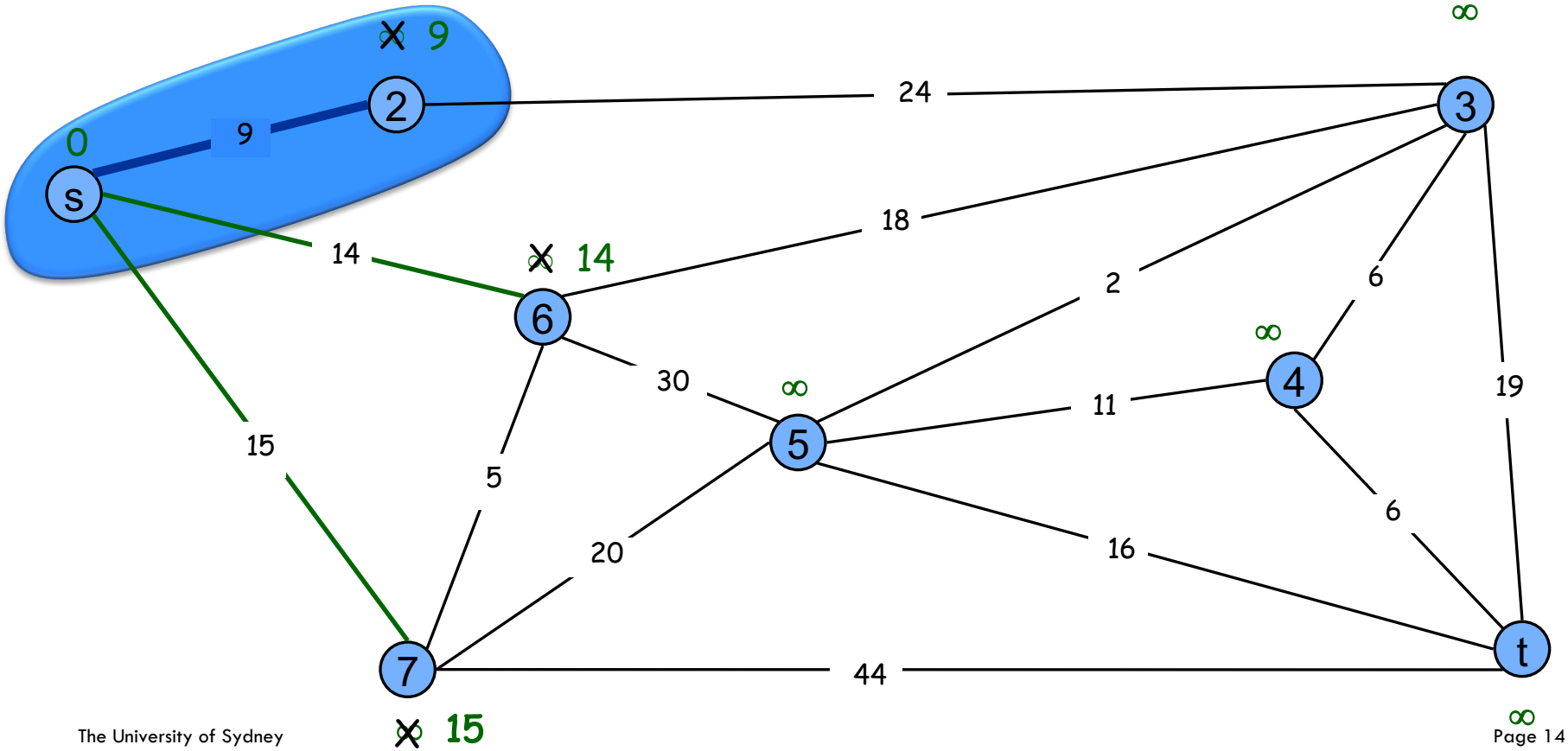


Dijkstra's Shortest Path Algorithm

S = { s, 2 }
PQ = { 3, 4, 5, 6, 7, t }



Dijkstra's Shortest Path Algorithm

S = { s, 2 }
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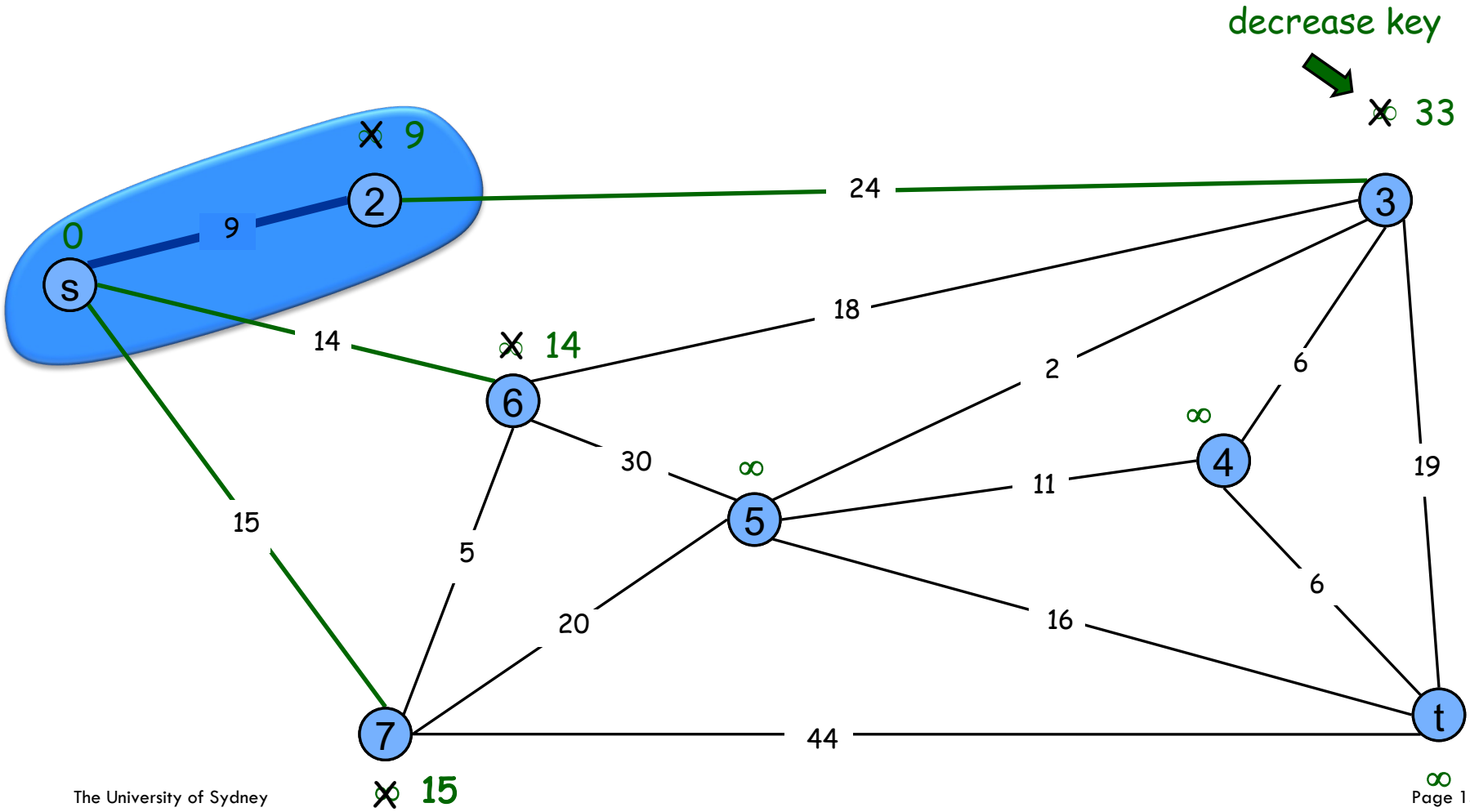

Dijkstra's Shortest Path Algorithm

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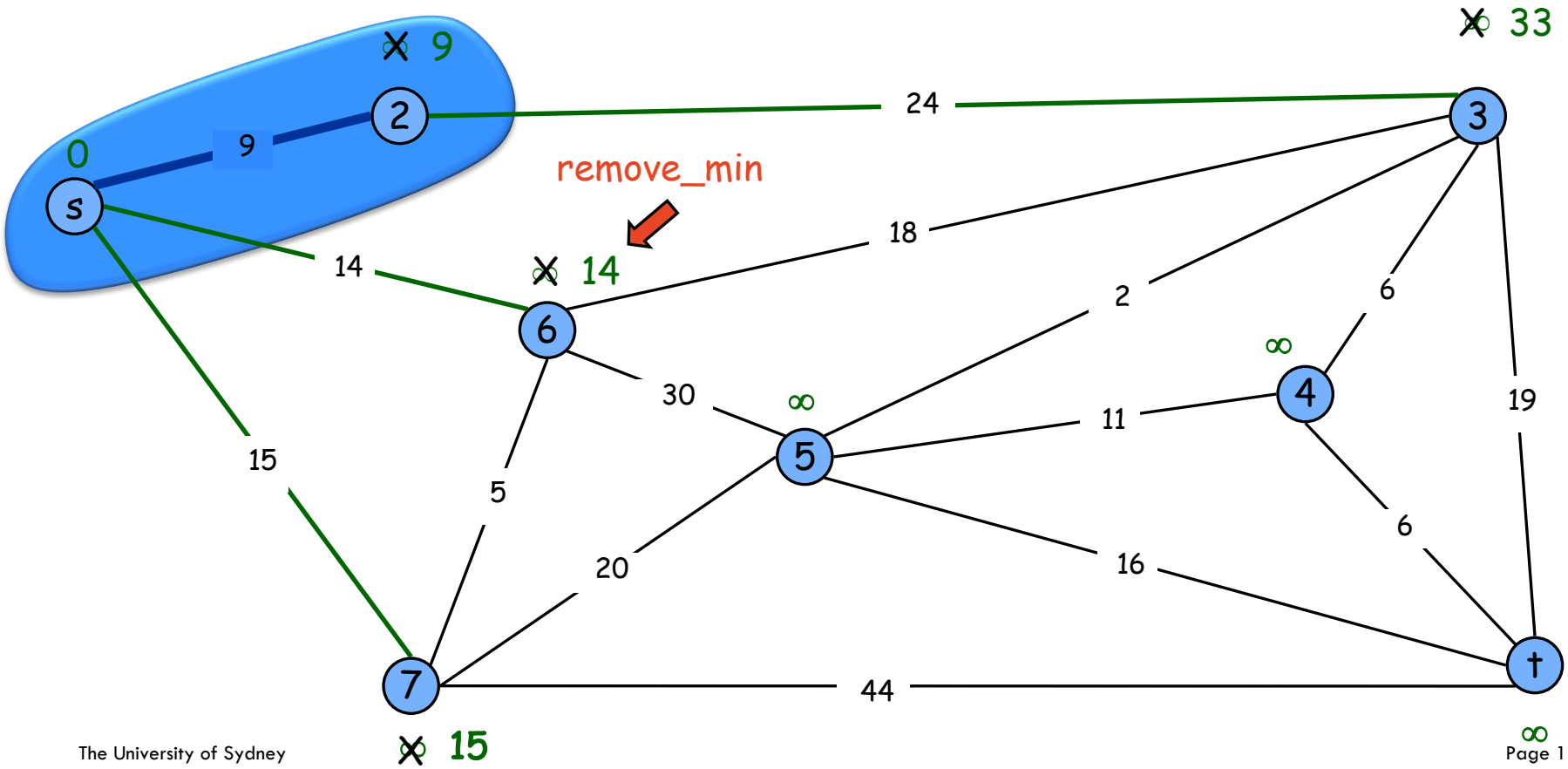

Dijkstra's Shortest Path Algorithm

S = { s, 2, 6 }
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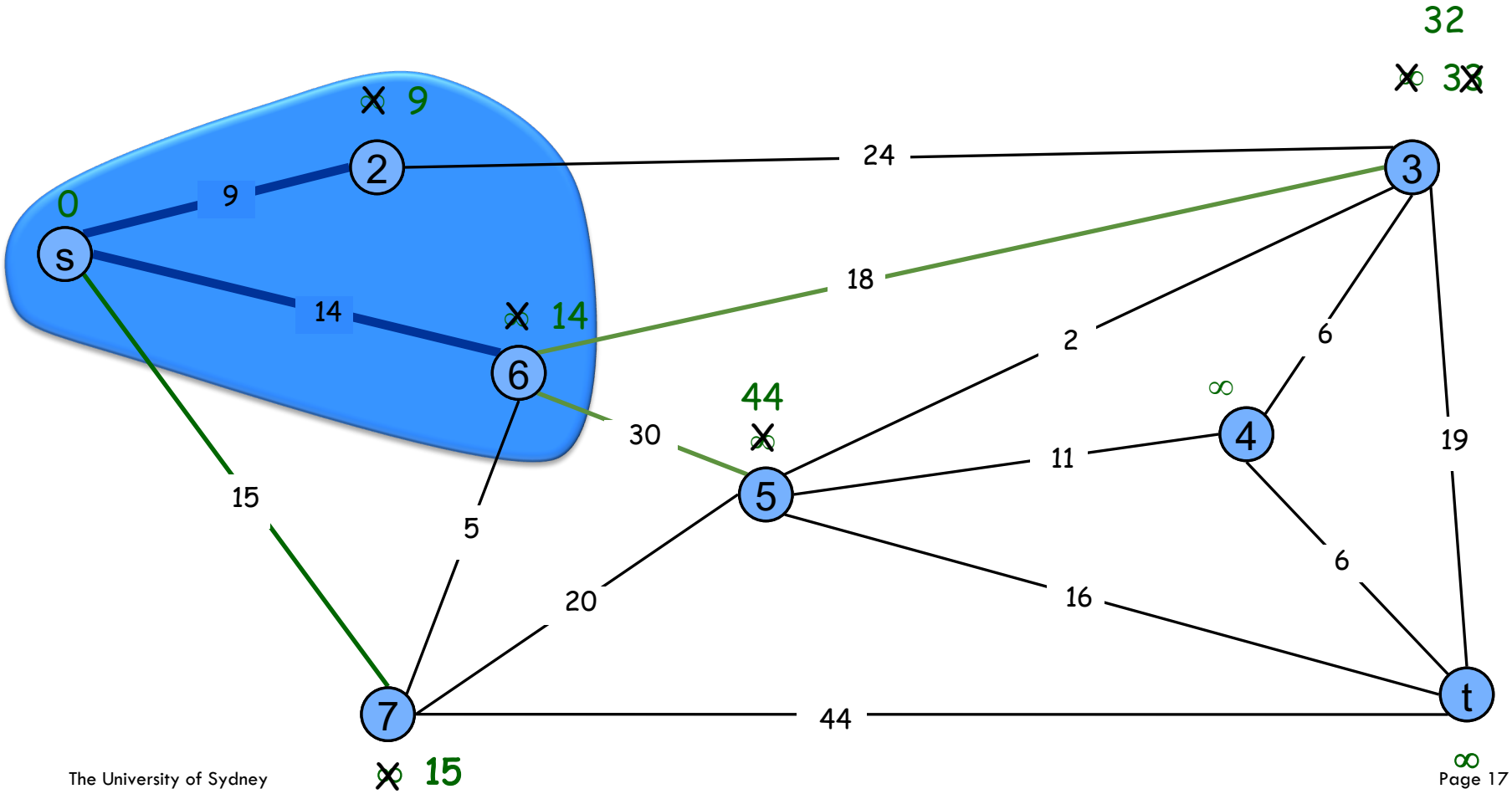

Dijkstra's Shortest Path Algorithm

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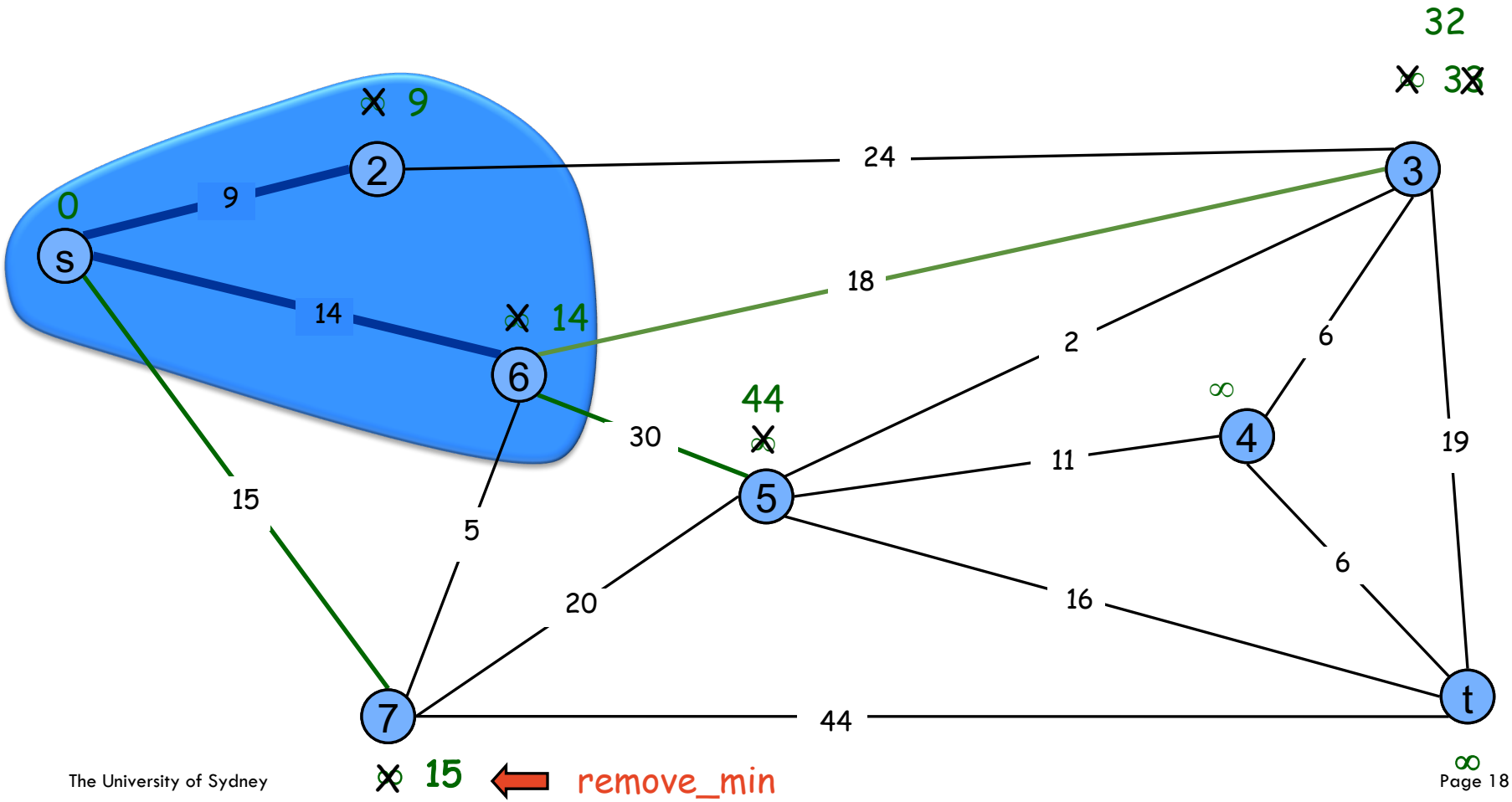

Dijkstra's Shortest Path Algorithm

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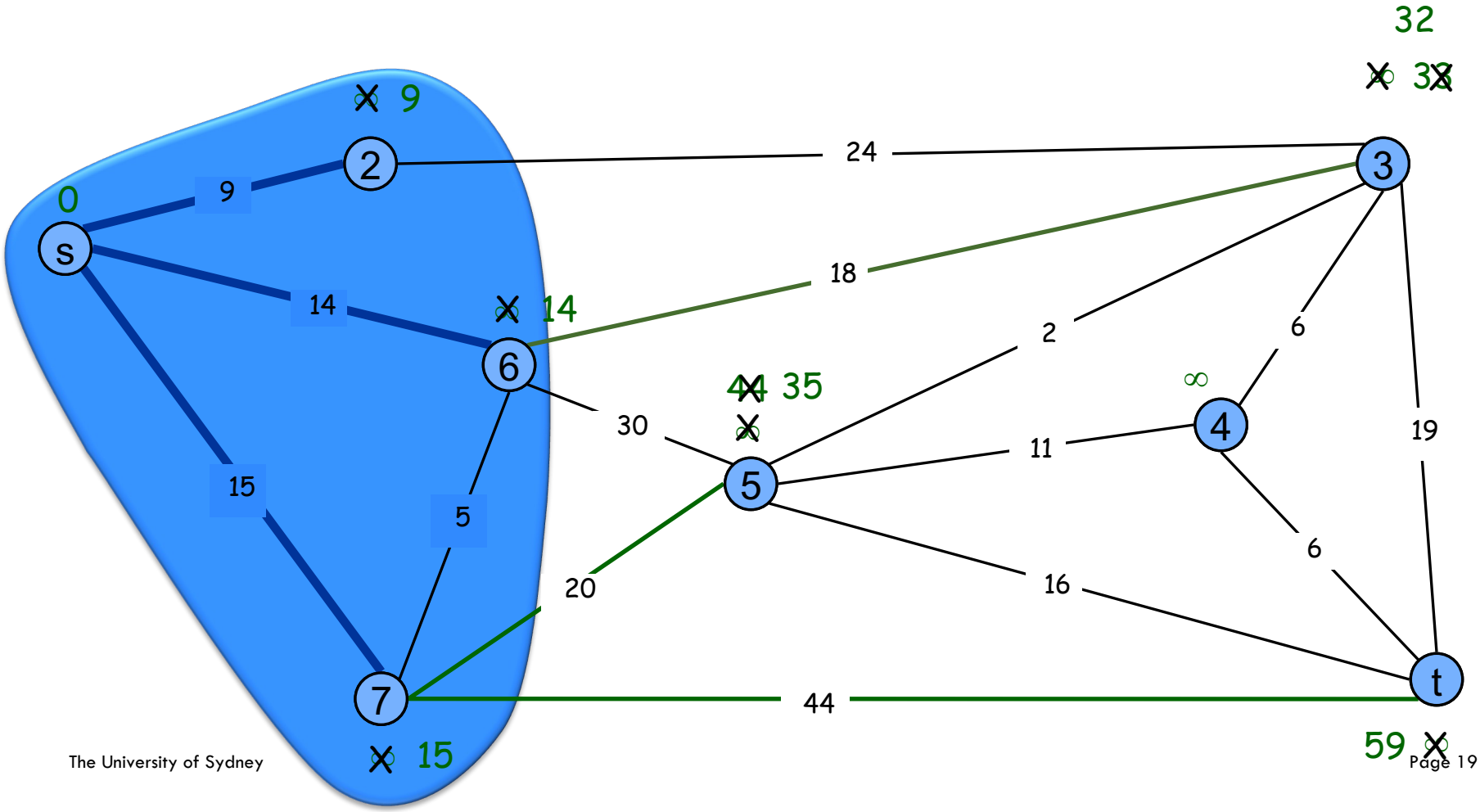

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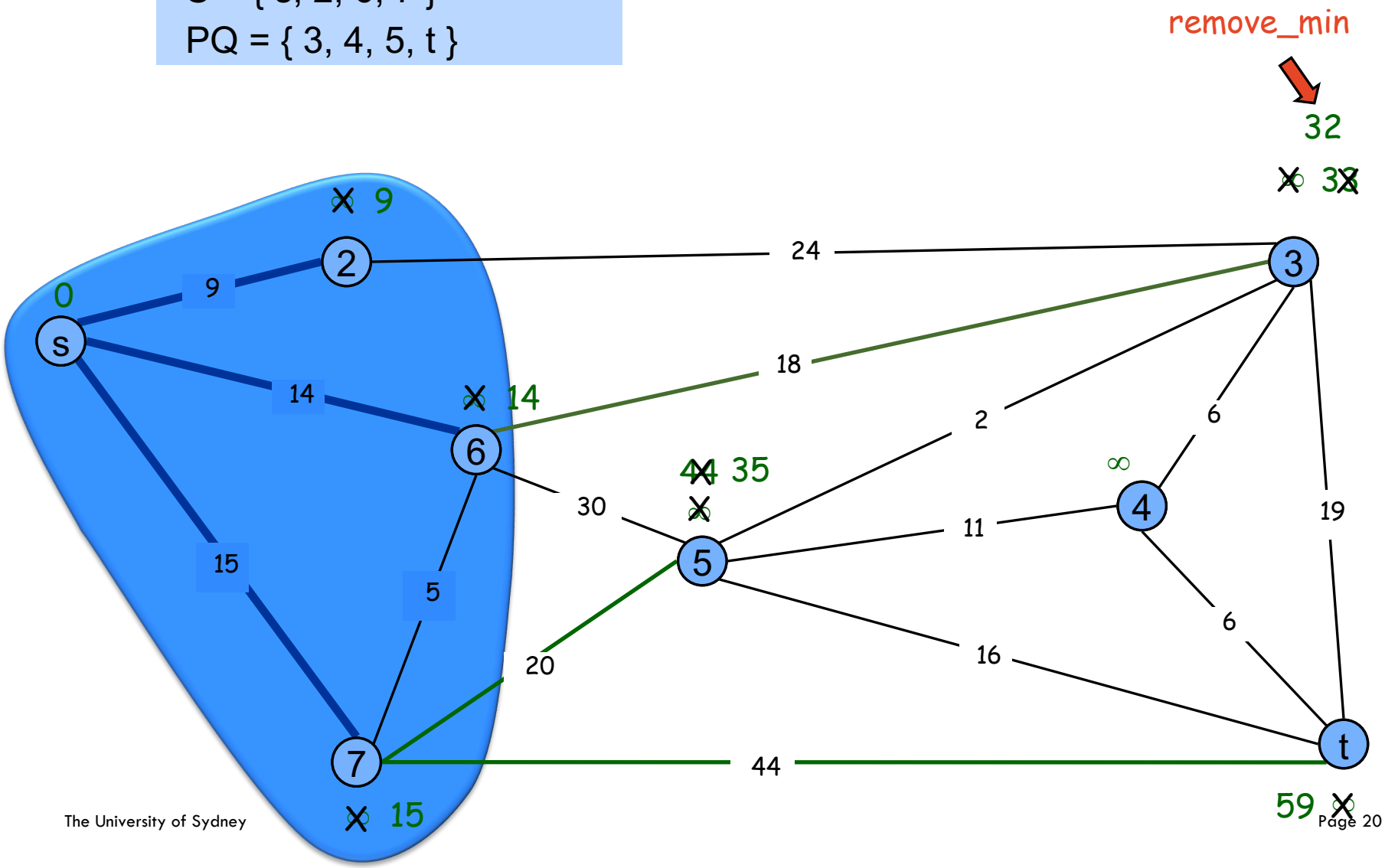

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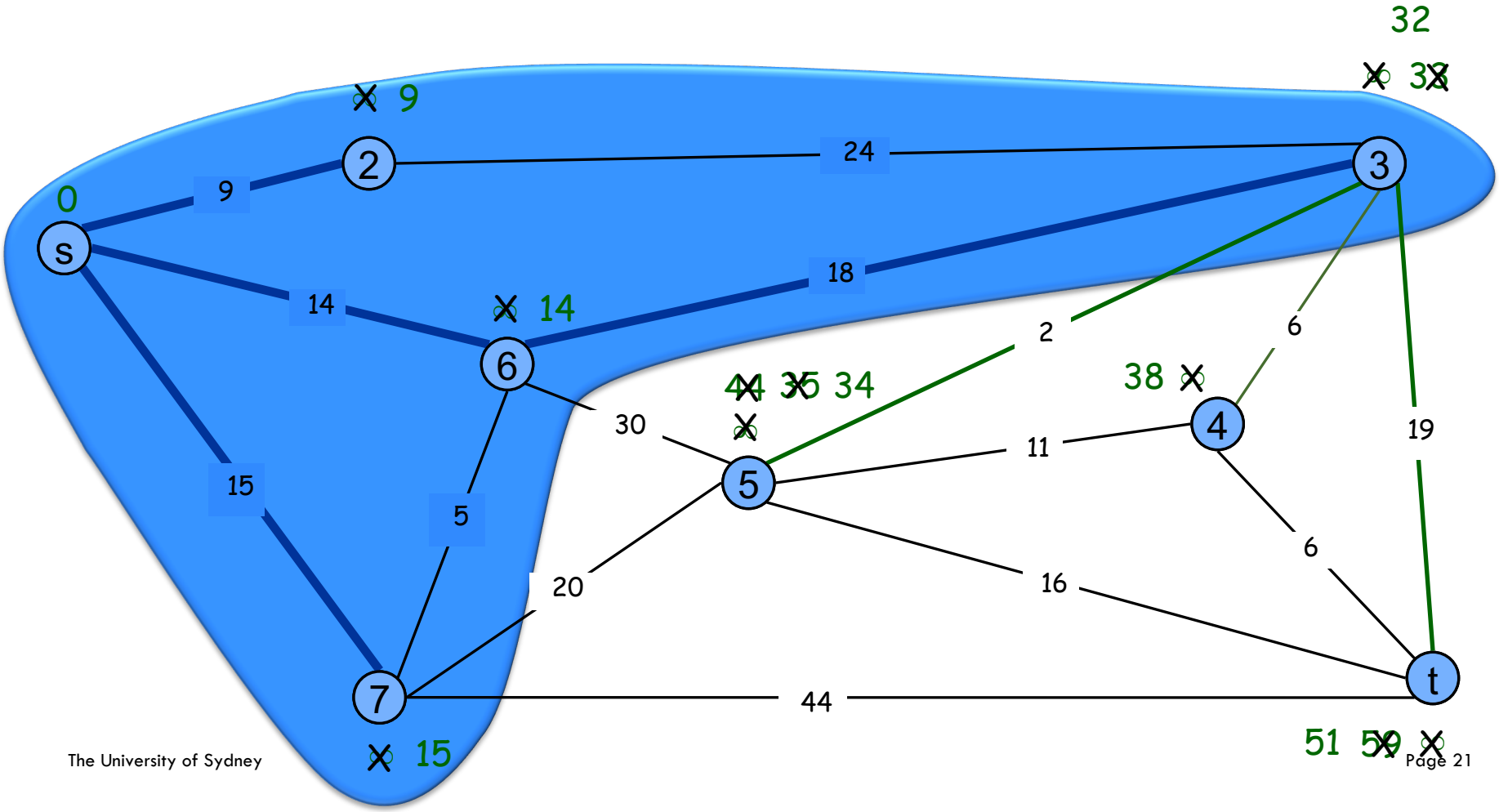

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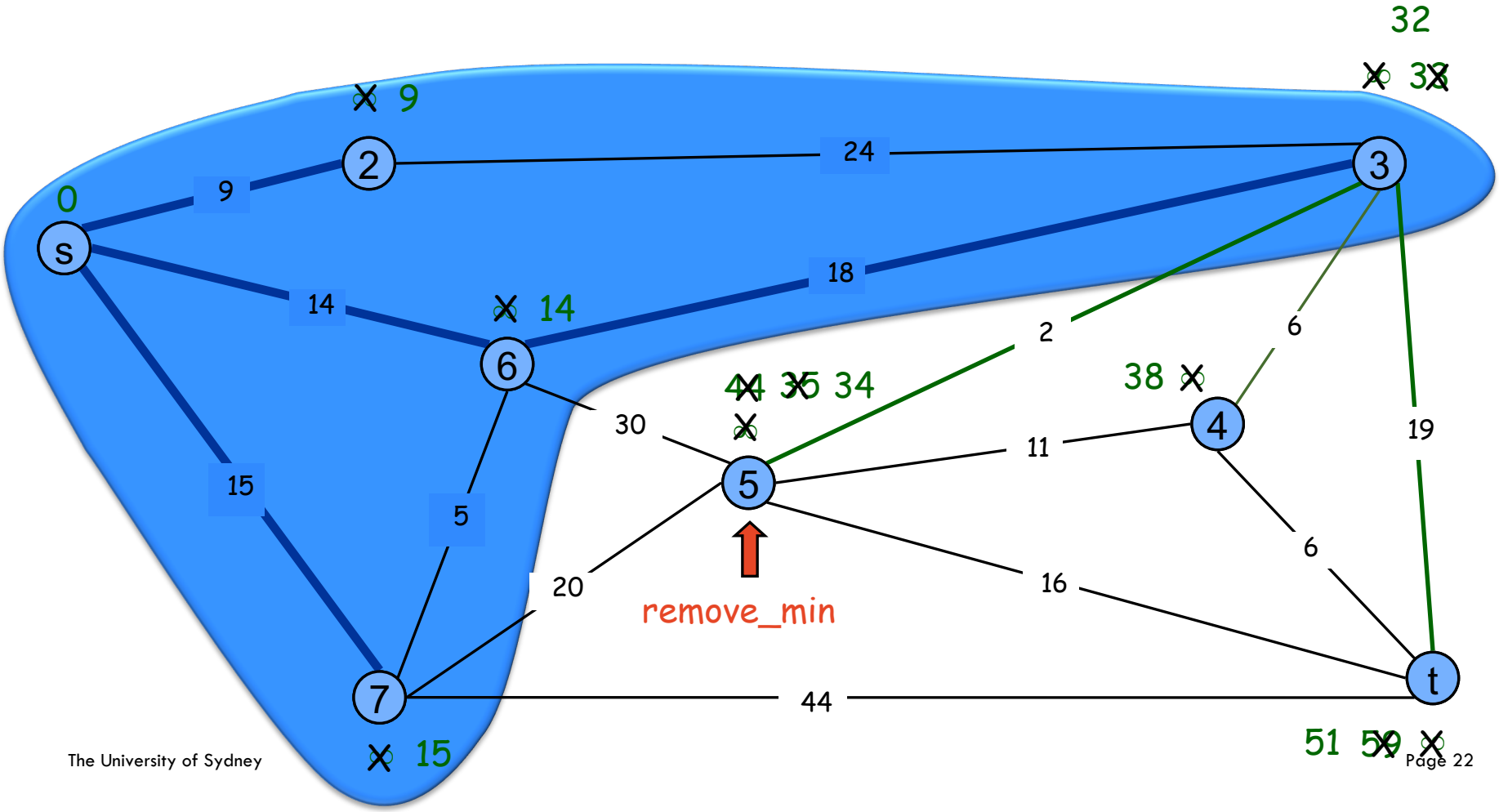

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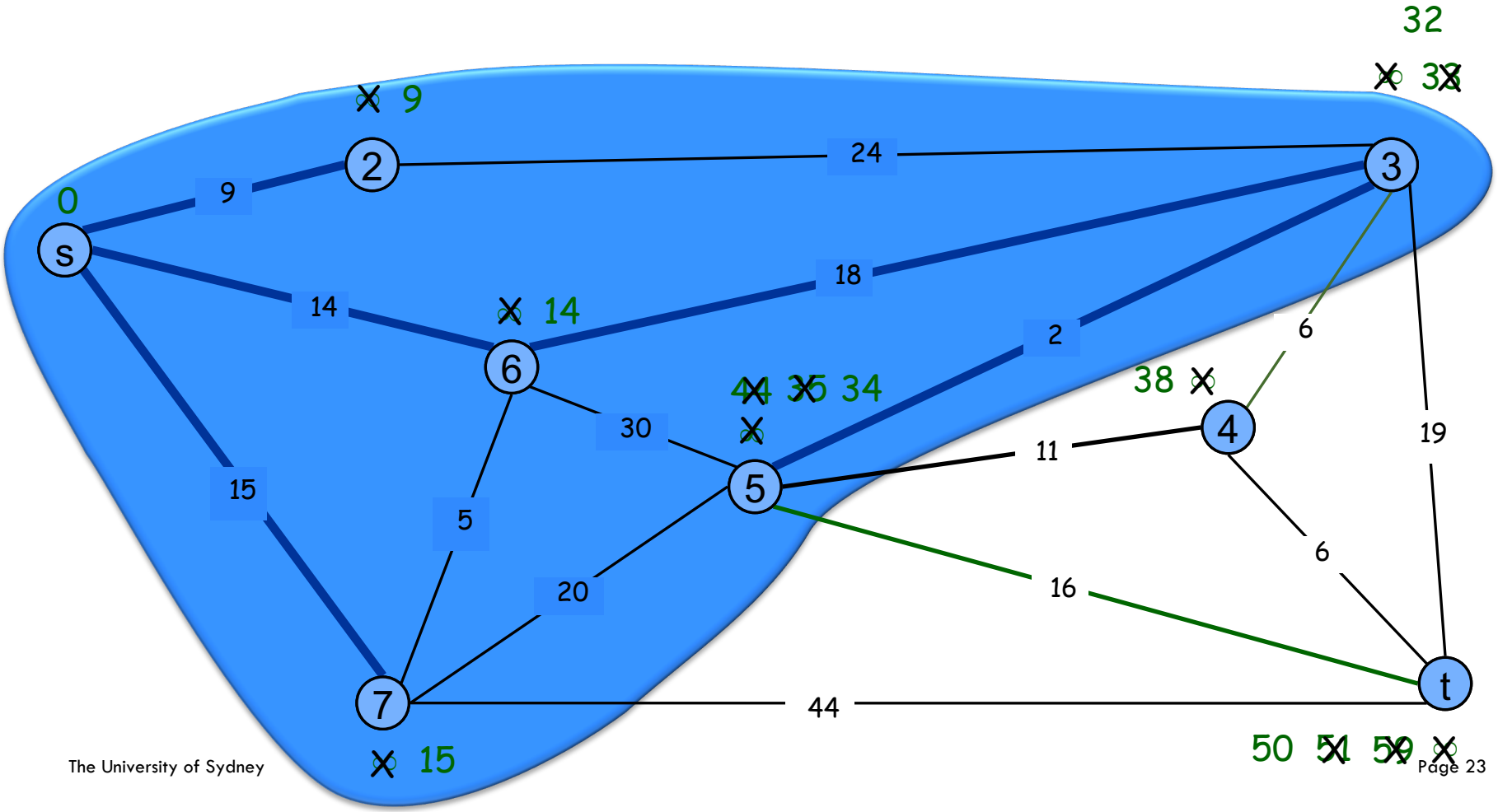

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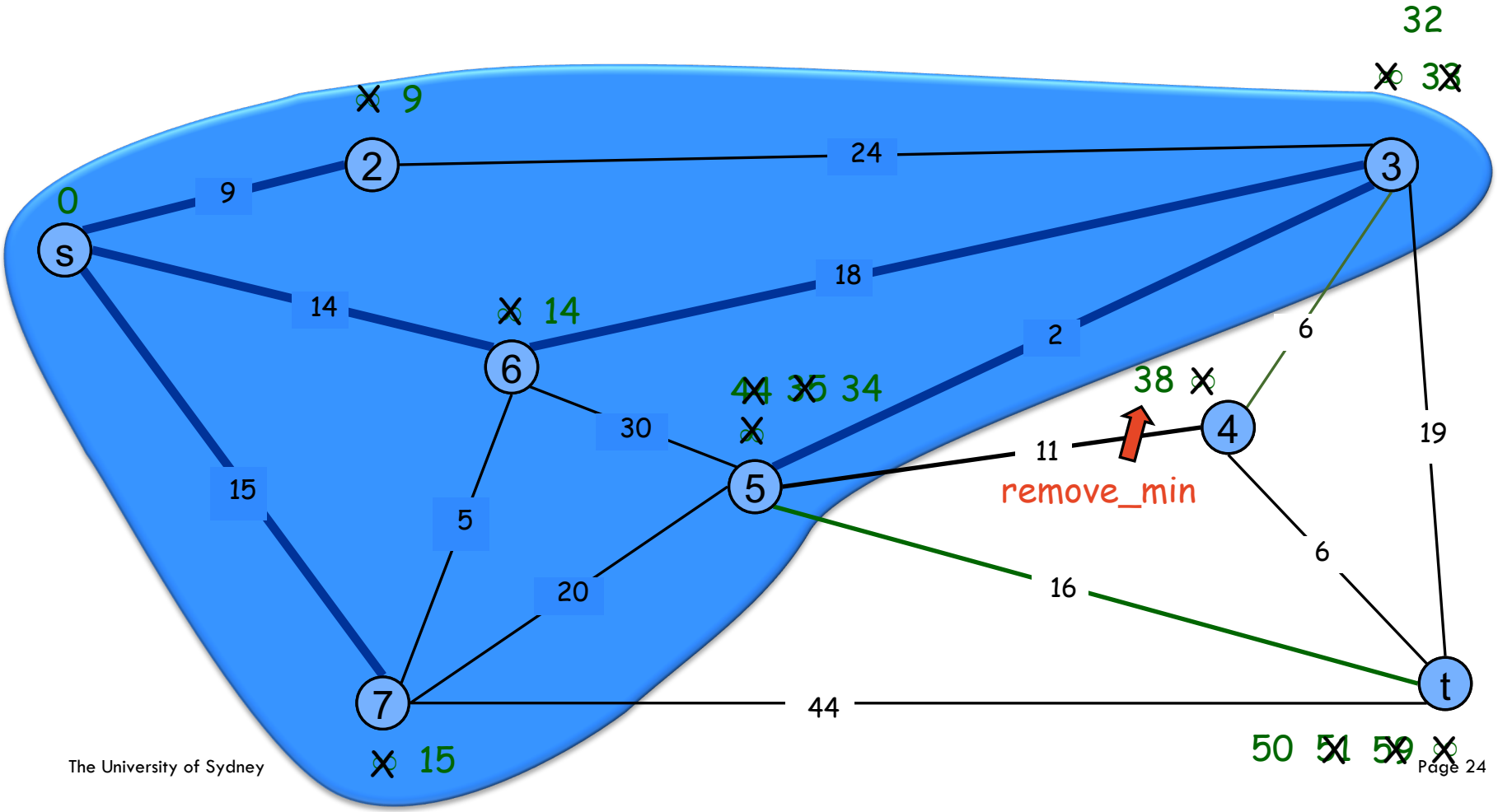

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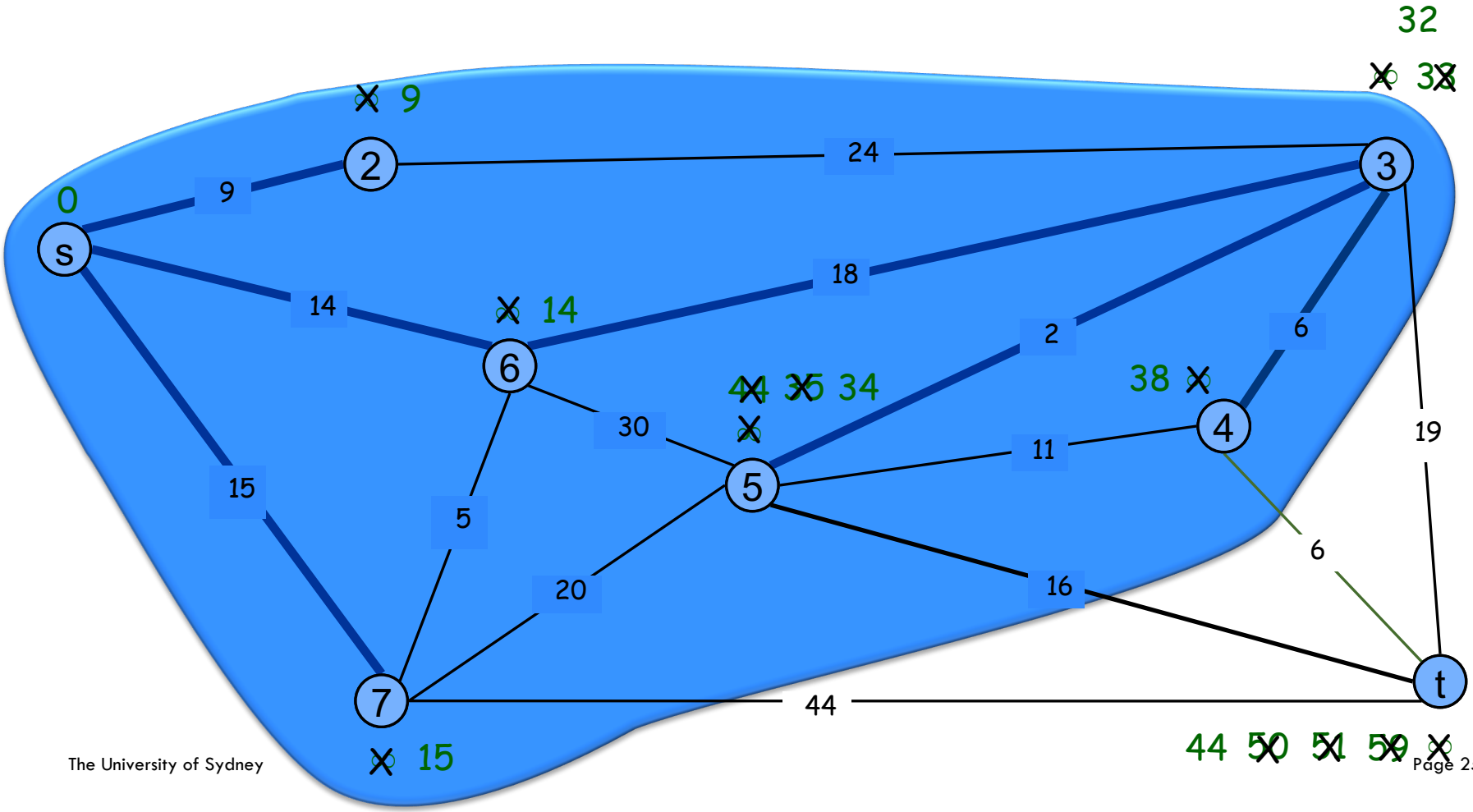

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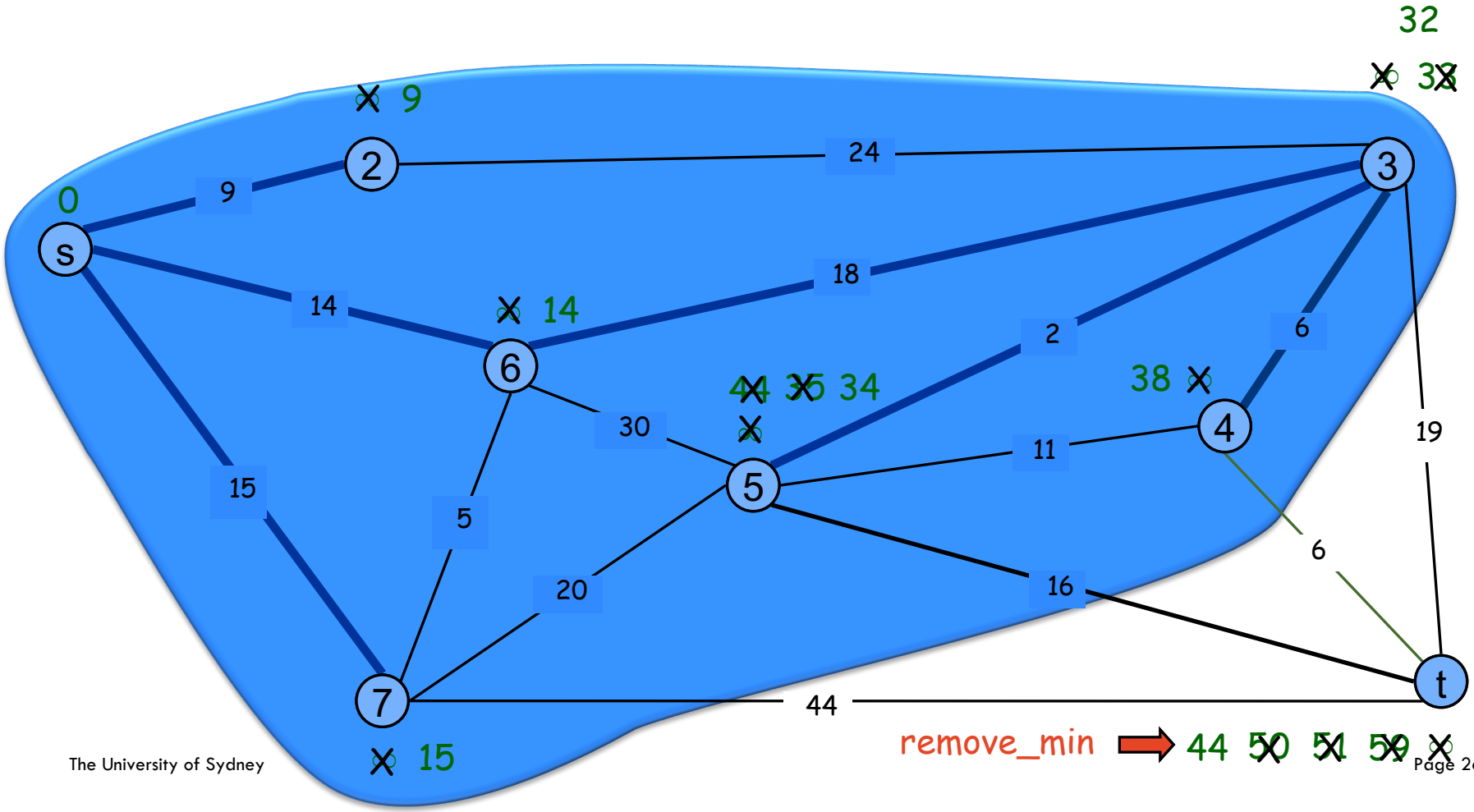

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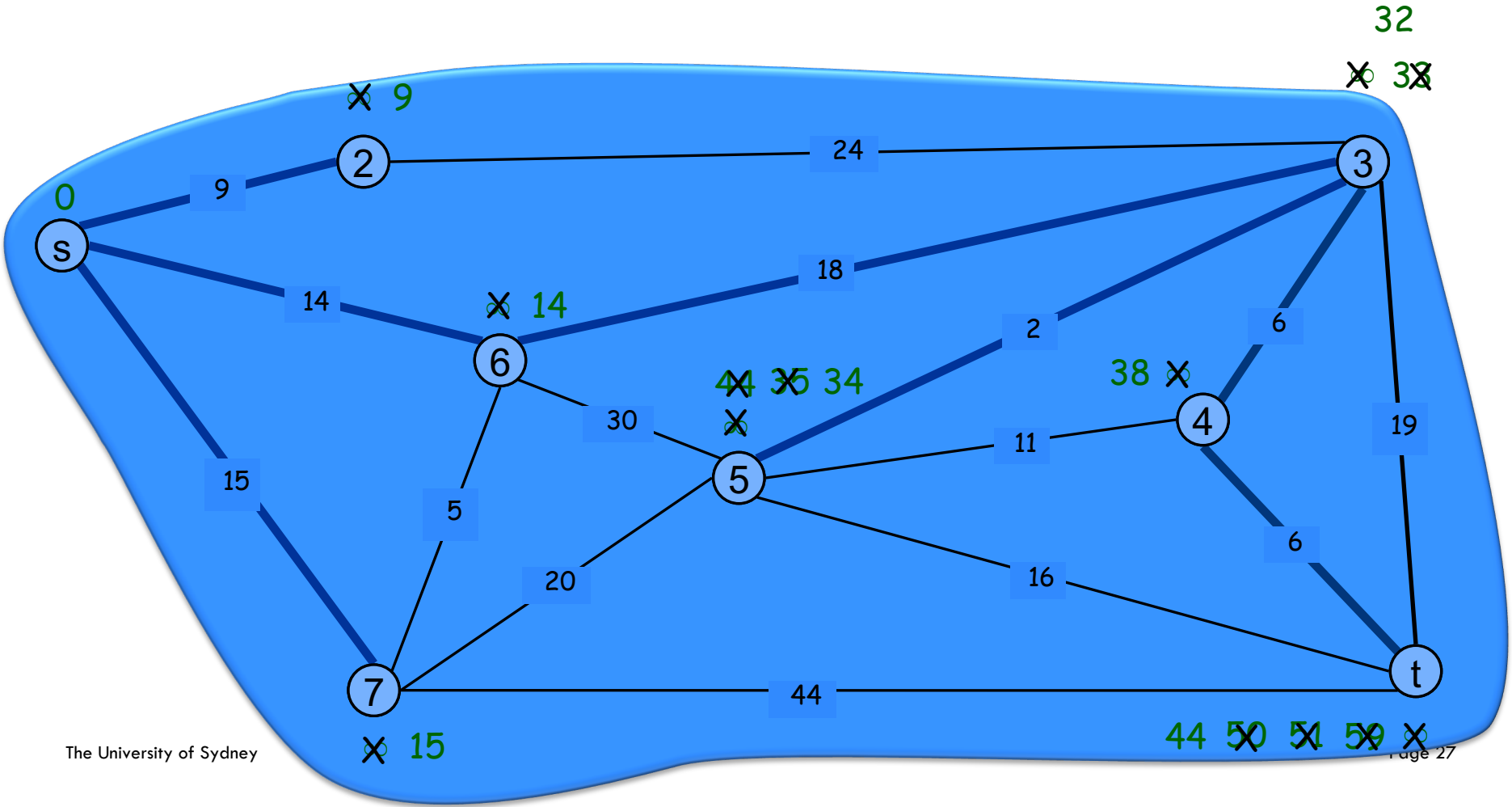

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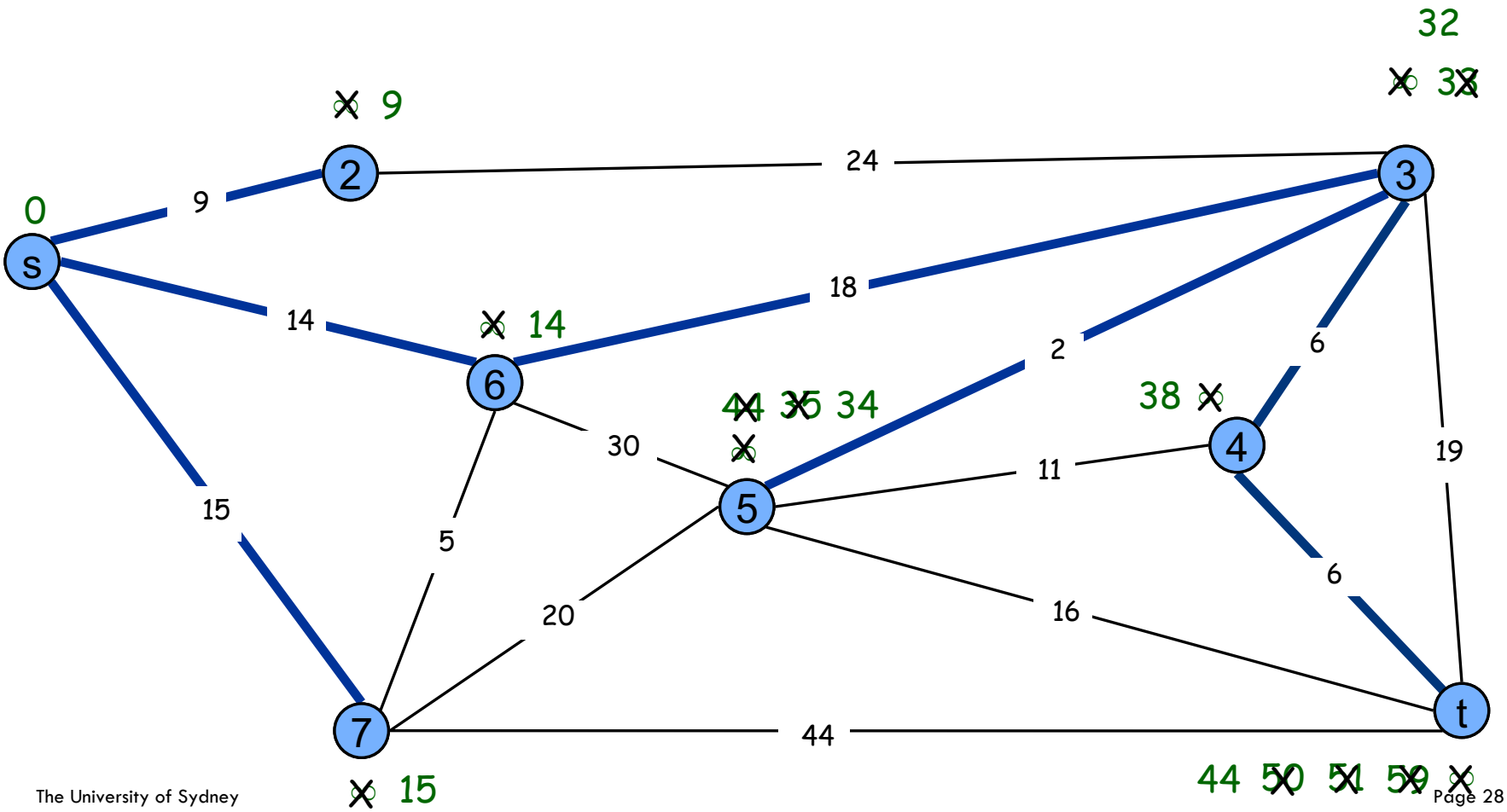


Dijkstra complexity analysis except PQ ops

```
def Dijkstra(G, w, s):

  # initialize algorithm
  for v in V do
    D[v] ← ∞
    parent[v] ← ∅
  D[s] ← 0
  Q ← new priority queue for { (v, D[v]) : v in V }

  # iteratively add vertices to S
  while Q is not empty do
    u ← Q.remove_min()
    for z in G.neighbors(u) do
      if D[u] + w[u, z] < D[z] then
        D[z] ← D[u] + w[u,z]
        Q.update_priority(z, D[z])
        parent[z] ← u
  return D, parent
```

O(n)

O(deg(u)) for each u in V
plus update_priority work

Dijkstra's Algorithm complexity analysis

m is #edges;

n is #vertices

Assuming the graph is connected (so $m \geq n-1$), the algorithm spends O(m) time on everything except PQ operations

Recall that:

Priority queue operation counts:

- insert: n
- decrease_key: m
- remove_min: n

Fact: Using a heap for PQ, Dijkstra runs in O(m log n) time

Fibonacci heaps is a PQ that can carry out decrease key in O(1) amortized time. Using that instead we get O(m + n log n) time.

Dijkstra's Algorithm Correctness

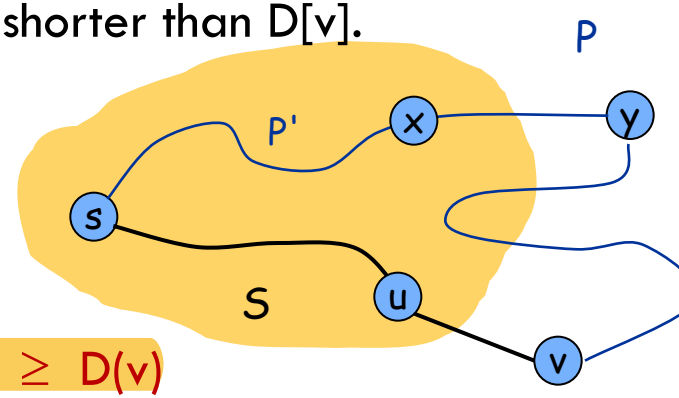
Invariant: For each $u \in S = V \setminus Q$, we have $D[u] = dist_w(s, u)$

Proof: (by induction on $|S|$)

Base case: $|S| = 1$ is trivial since $D[s] = 0$

Inductive hypothesis: Assume true for $|S| = k \geq 1$.

- Let v be next node added to S and u=parent[v]
- The shortest s-u path plus (u, v) is an s-v path of length D[v]
- Consider any s-v path P. We'll see that it's no shorter than D[v].
- Let x-y be the first edge in P that leaves S, and let P' be the subpath to x.
- P is already too long as soon as it leaves S:



$$w(P) \geq w(P') + w(x,y) \geq D(x) + w(x,y) \geq D(y) \geq D(v)$$

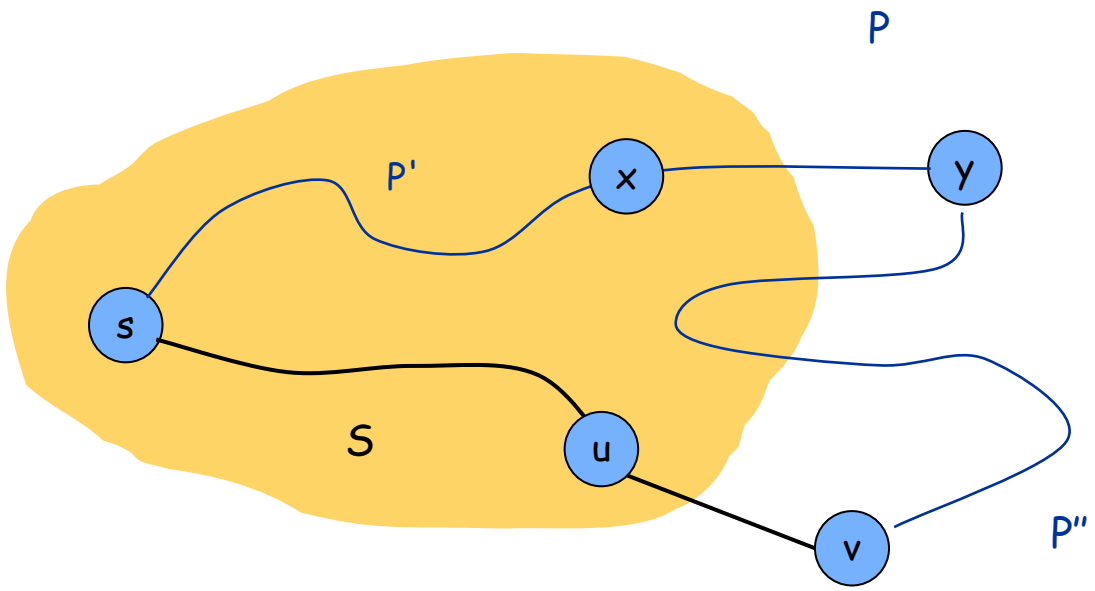
inductive hypothesis

Def of D(y)

Dijkstra chose v instead of y

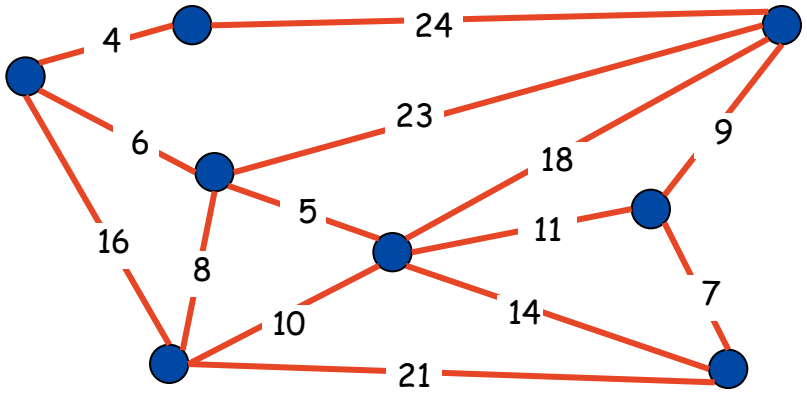
Warning: Dijkstra may not work for negative-weight edges

In the proof of correctness, even if D[v] is the smallest label, it may be that $dist_w(s, v) < D[v]$ if $w(P'') < 0$

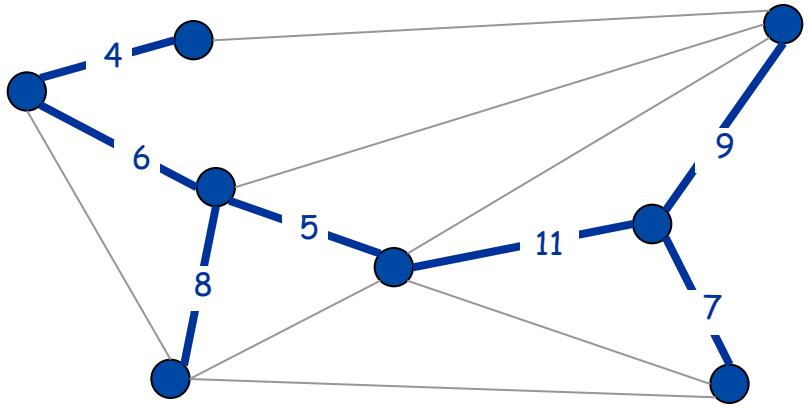


Minimum Spanning Tree

Given a connected graph G = (V, E) with real-valued edge weights c_e, an MST is a subset of the edges $T \subseteq E$ such that T is a spanning tree whose sum of edge weights is minimized.



G = (V, E)



T with $\Sigma_{e \in T} c_e = 50$

Applications

MST is fundamental problem with diverse applications.

Network design: Telephone, electrical, hydraulic, TV cable, computer, road

Approximation algorithms for NP-hard problems: traveling salesperson problem, Steiner tree

Indirect applications.

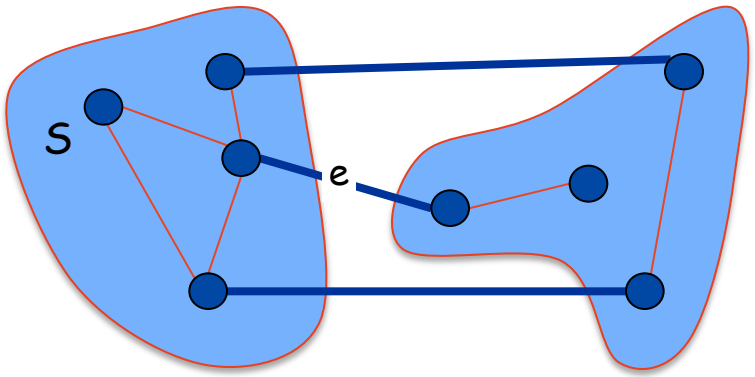
- max bottleneck paths
- LDPC codes for error correction
- image registration with Renyi entropy
- learning salient features for real-time face verification
- reducing data storage in sequencing amino acids in a protein
- …

MST properties

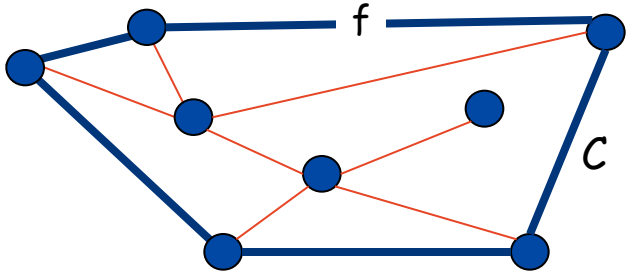
Simplifying assumption. All edge costs c_e are distinct.

Cut property. Let S be any subset of nodes, and let e be the min cost edge with exactly one endpoint in S. Then the MST contains e.

Cycle property. Let C be any cycle, and let f be the max cost edge belonging to C. Then the MST does not contain f.



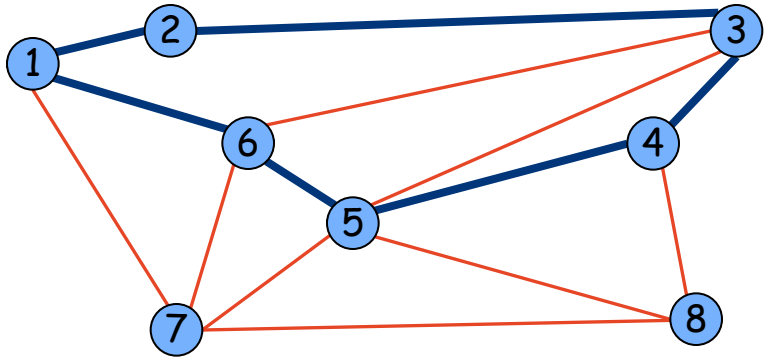
e is in the MST



f is not in the MST

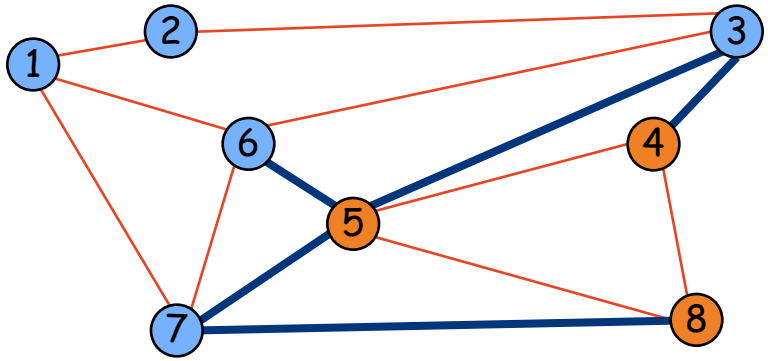
Cycles and Cuts

Cycle. Set of edges of the form a-b, b-c, c-d, …, y-z, z-a.



Cycle C = 1-2, 2-3, 3-4, 4-5, 5-6, 6-1

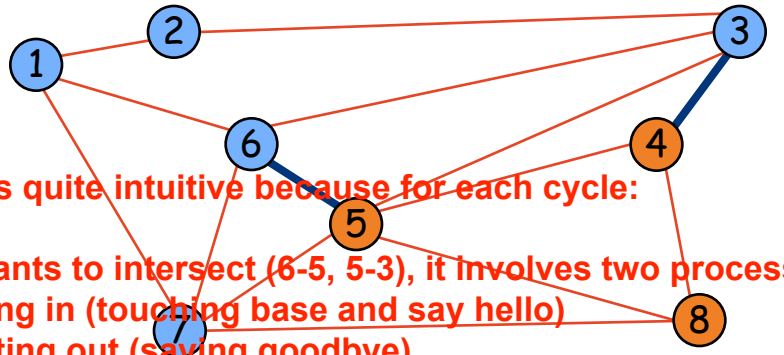
Cutset. A cut is a subset of nodes S. The corresponding cutset D is the subset of edges with exactly one endpoint in S.



Cut S = { 4, 5, 8 }
Cutset D = 5-6, 5-7, 3-4, 3-5, 7-8

Cycle-Cut Intersection

Claim. A cycle and a cutset intersect in an even number of edges.



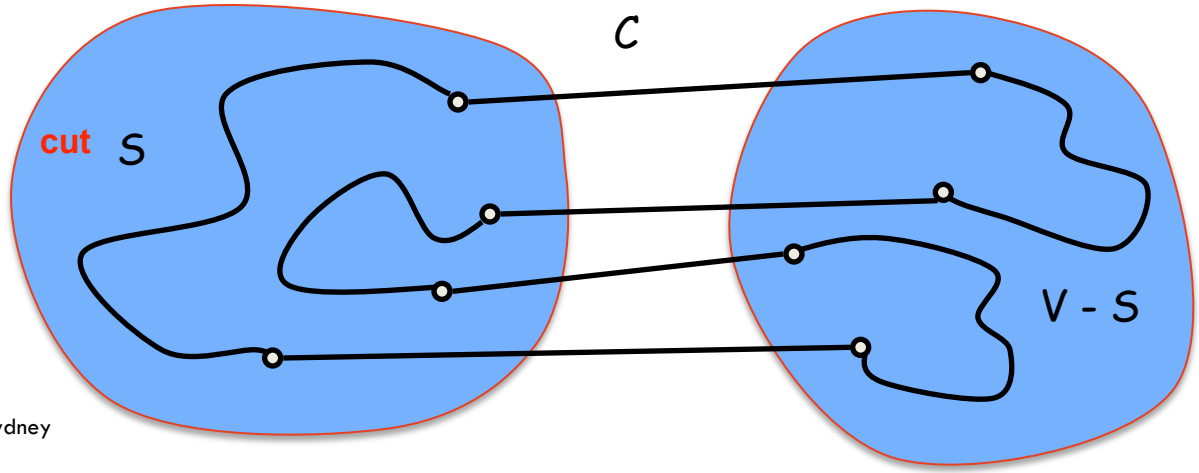
Cycle C = 1-2, 2-3, 3-4, 4-5, 5-6, 6-1
Cutset D = 3-4, 3-5, 5-6, 5-7, 7-8
Intersection = 3-4, 5-6

This is quite intuitive because for each cycle:

if it wants to intersect (6-5, 5-3), it involves two processes:
1. going in (touching base and say hello)
2. getting out (saying goodbye)

If however, it wants to stay with the cutset D, e.g., 6-5, 5-4, 4-3,
then notice that 5-4 doesn't count. So eventually you can even think of it as "squashing 5 and 4 into one"

Proof:



Proving the Cut Property

Simplifying assumption. All edge costs c_e are distinct.

Note that S can be ANY subset of nodes

Cut property. Let S be any subset of nodes, and let e be the min cost edge with exactly one endpoint in S. Then the MST contains e.

Proof: (exchange argument)

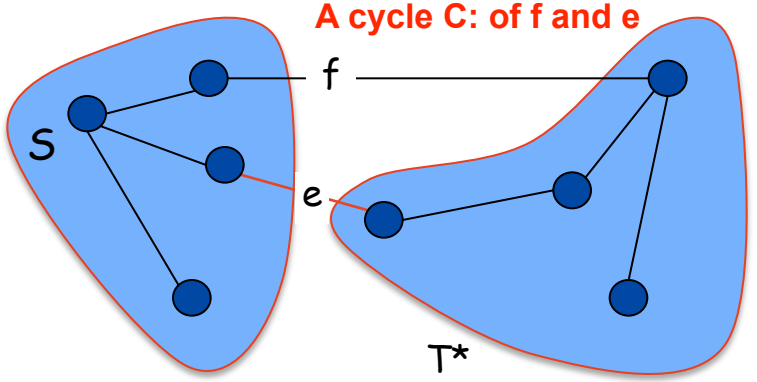
- Let T* be MST and suppose e does not belong to T*
- Adding e to T* creates a cycle C in T*
- Edge e is both in the cycle C and in the cutset D corresponding to S $\Rightarrow$ there exists another edge, say f, that is in both C and D.

This is by the cycle-cut intersection property

- $T' = T^* \cup \{ e \} - \{ f \}$ is also a spanning tree.
- Since $c_e < c_f$, cost(T') < cost(T*).
- A contradiction, so e must belong in T*

Here we just showed that if the assumption is correct, T^{'} is EVEN SMALLER THAN T*

Thus we dispprove the assumption statement



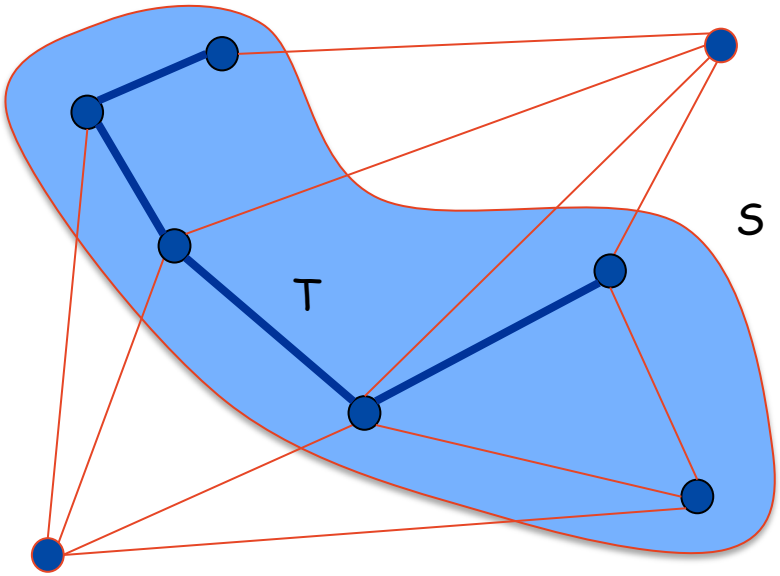
Prim's Algorithm

```
def prim(G, c):
  u ← arbitrary vertex in V
  S ← { u }
  T ← ∅
  while |S| < |V| do
    (u, v) ← min cost edge s.t. u in S and v not in S
    add (u, v) to T
    add v to S
  return T
```

Prim's Algorithm: Correctness

```
def prim(G, c):
  u ← arbitrary vertex in V
  S ← { u }
  T ← ∅
  while |S| < |V| do
    (u, v) ← min cost edge s.t. u in S and v not in S
    add (u, v) to T
    add v to S
  return T
```

Every time we add an edge we follow cut property!



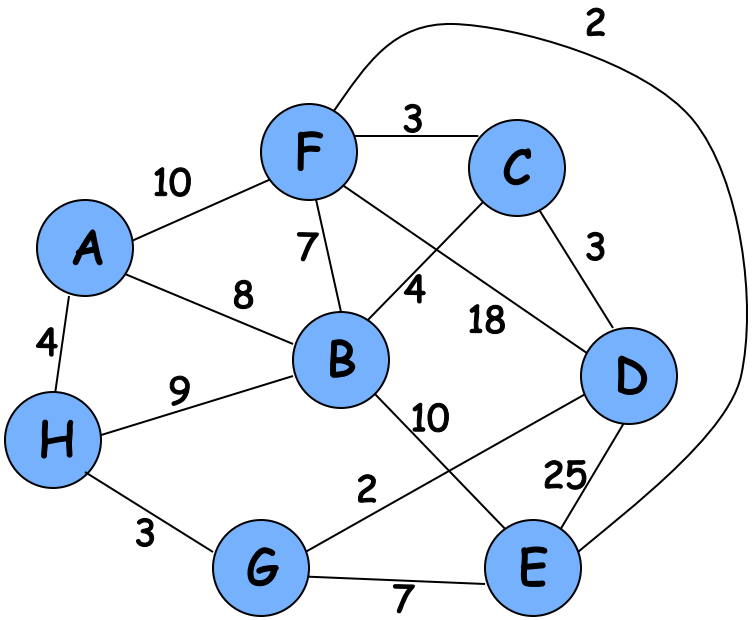
Implementation: Prim's Algorithm

```
def prim(G, c) {

  for v in V do
    d[v] ← ∞
    parent[v] ← ∅
  u ← arbitrary vertex in V
  d[u] ← 0
  Q ← new PQ with items { (v, d[v]) for v in V }
  S ← ∅

  while Q is not empty do
    u ← delete min element from Q
    add u to S
    for (u, v) incident to u do
      if v ∉ S and c_{u,v} < d[v] then
        parent[v] ← u
        decrease priority d[v] to c_{u,v}
  return parent
```

Main idea: for every v in V \ S we keep

- d[v] = distance to closest neighbor in S
- parent[v] = closest neighbor in S

Walk-Through



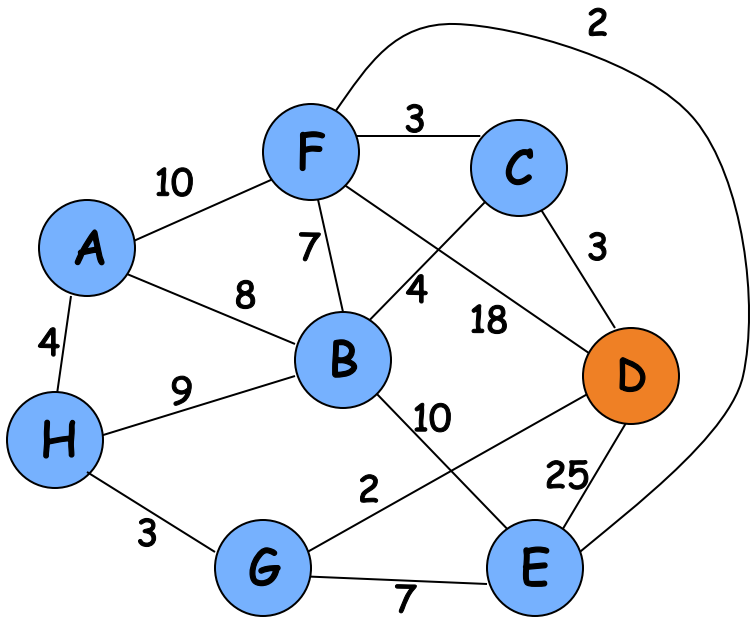
Initialize array

	S	d_v	p_v
A	F	∞	–
B	F	∞	–
C	F	∞	–
D	F	∞	–
E	F	∞	–
F	F	∞	–
G	F	∞	–
H	F	∞	–

Set S

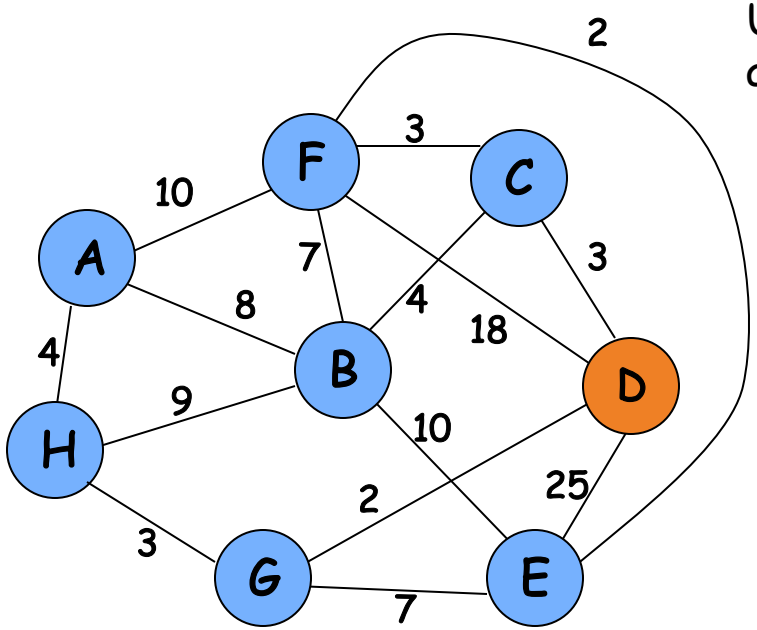
Min distance to S

Closest vertex in S



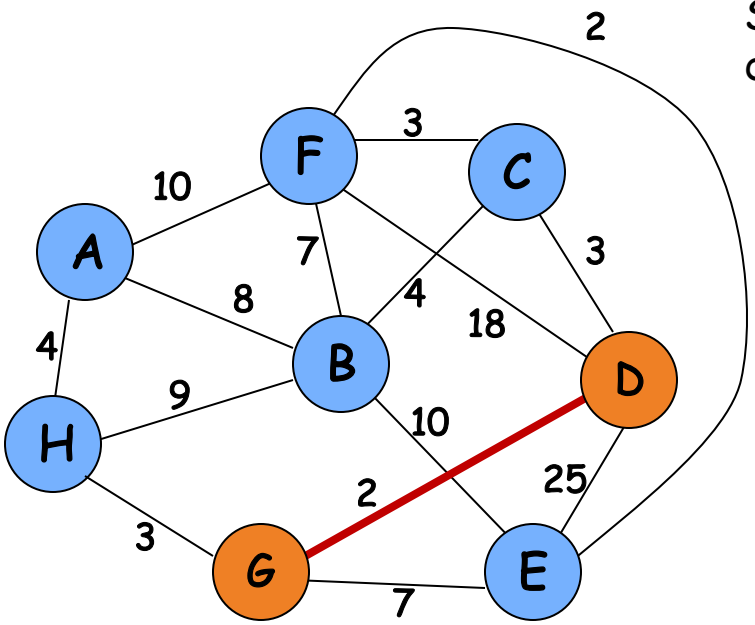
Start with any node, say D

	S	d_v	p_v
A			
B			
C			
D	T	0	–
E			
F			
G			
H			



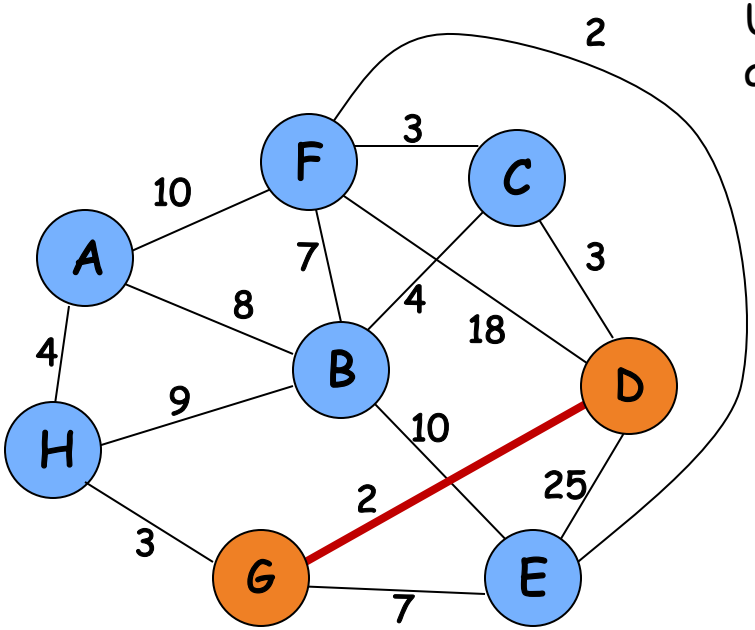
Update distances of adjacent, unselected nodes

	S	d_v	p_v
A			
B			
C		3	D
D	T	0	–
E		25	D
F		18	D
G		2	D
H			



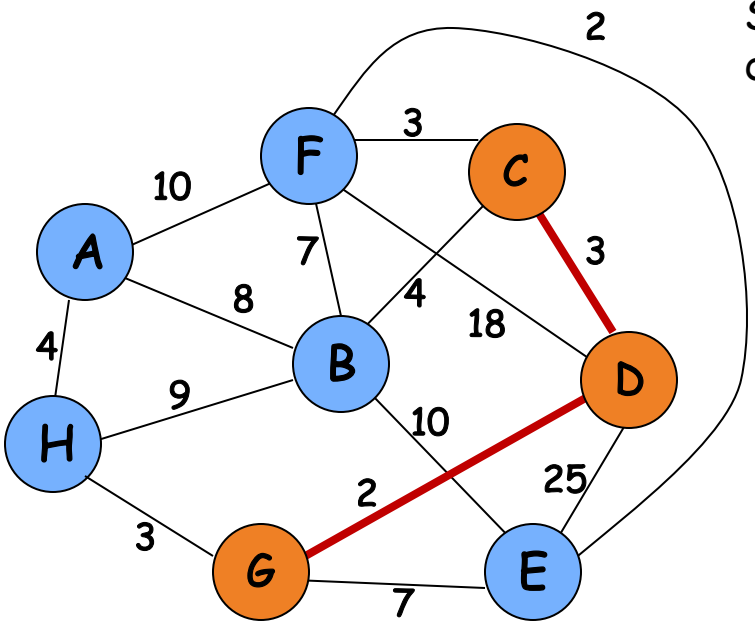
Select node with minimum distance

	S	d_v	p_v
A			
B			
C		3	D
D	T	0	–
E		25	D
F		18	D
G	T	2	D
H			



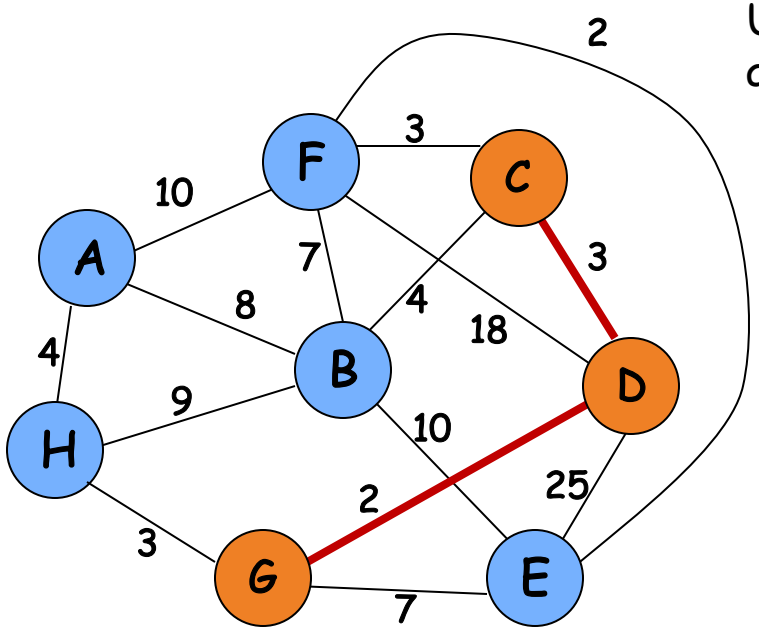
Update distances of adjacent, unselected nodes

	S	d_v	p_v
A			
B			
C		3	D
D	T	0	–
E		7	G
F		18	D
G	T	2	D
H		3	G



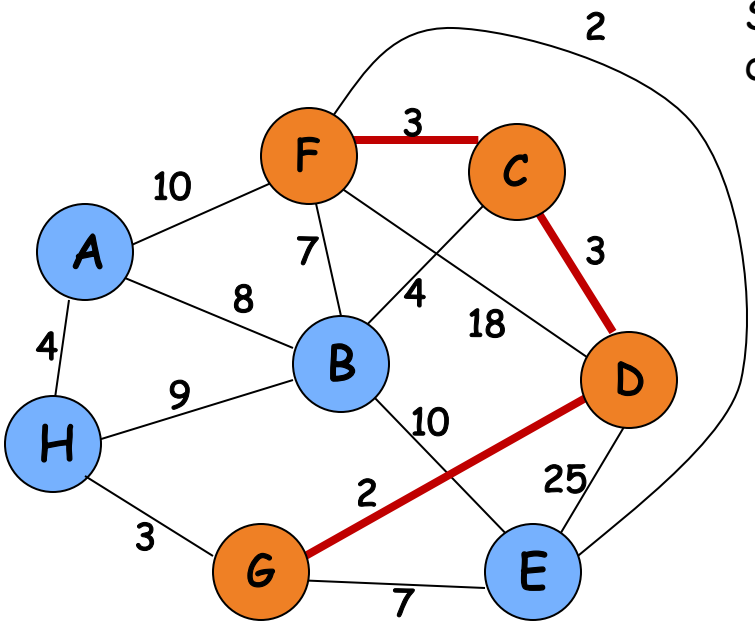
Select node with minimum distance

	S	d_v	p_v
A			
B			
C	T	3	D
D	T	0	–
E		7	G
F		18	D
G	T	2	D
H		3	G



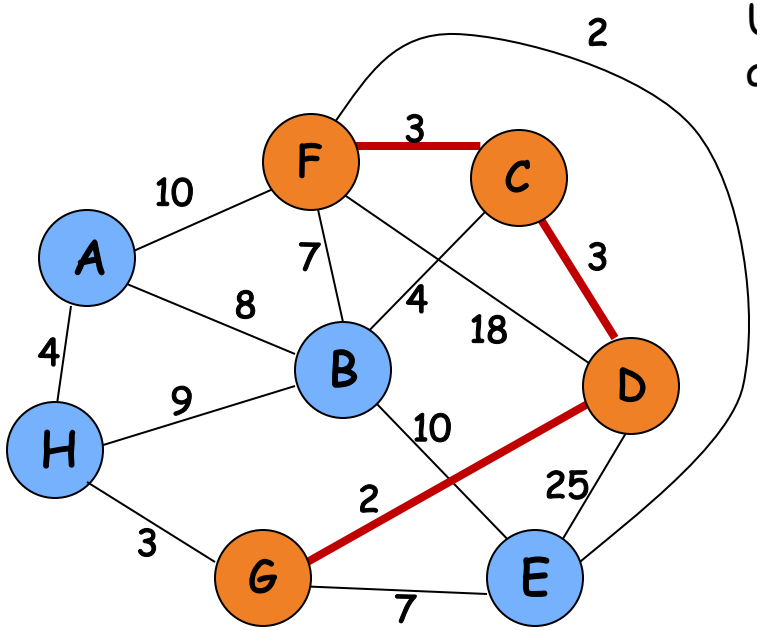
Update distances of adjacent, unselected nodes

	S	d_v	p_v
A			
B		4	C
C	T	3	D
D	T	0	–
E		7	G
F		3	C
G	T	2	D
H		3	G



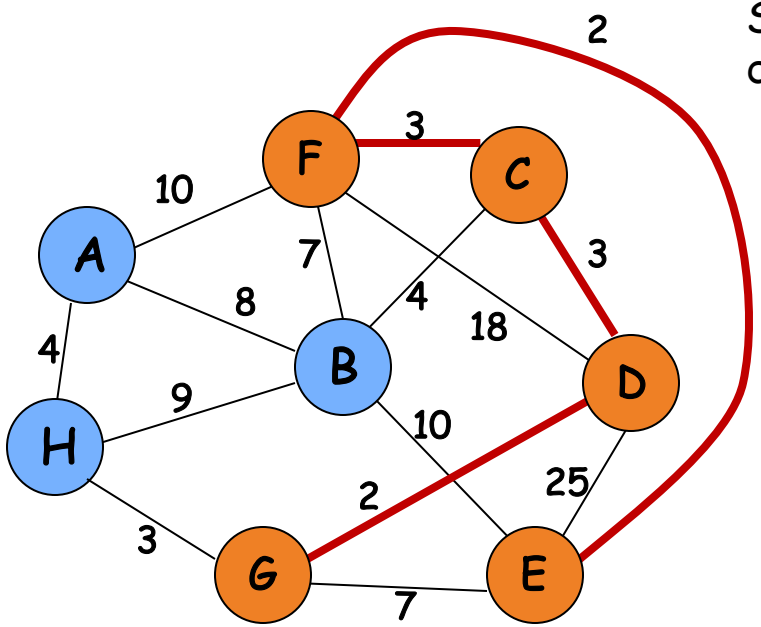
Select node with minimum distance

	S	d_v	p_v
A			
B		4	C
C	T	3	D
D	T	0	–
E		7	G
F	T	3	C
G	T	2	D
H		3	G



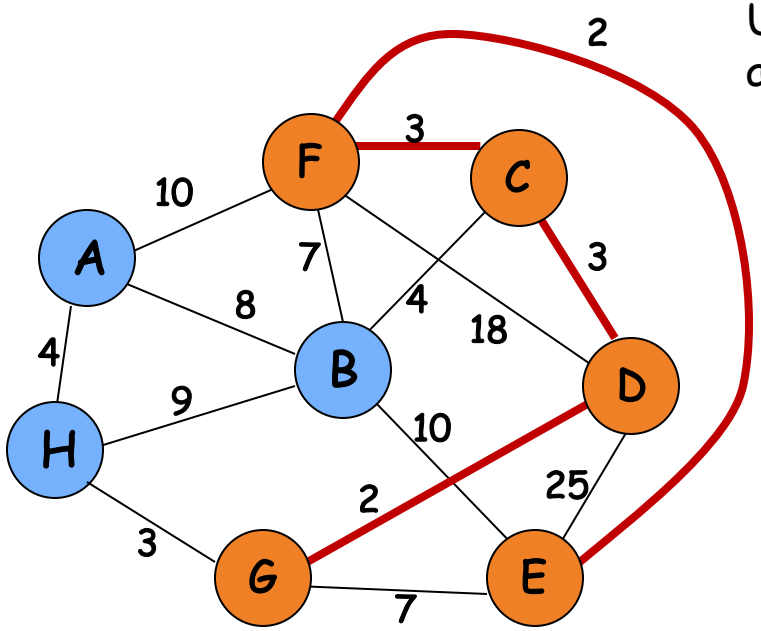
Update distances of adjacent, unselected nodes

	S	d_v	p_v
A		10	F
B		4	C
C	T	3	D
D	T	0	–
E		2	F
F	T	3	C
G	T	2	D
H		3	G



Select node with minimum distance

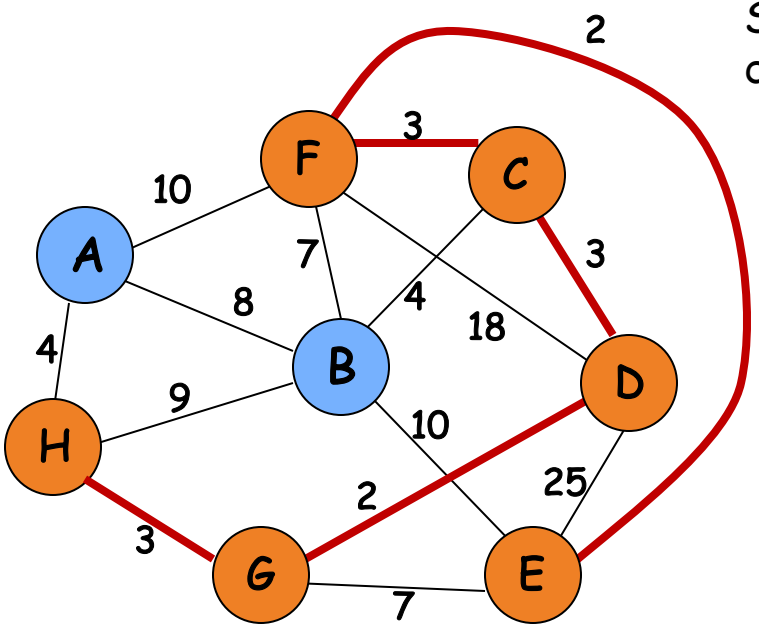
	S	d_v	p_v
A		10	F
B		4	C
C	T	3	D
D	T	0	–
E	T	2	F
F	T	3	C
G	T	2	D
H		3	G



Update distances of adjacent, unselected nodes

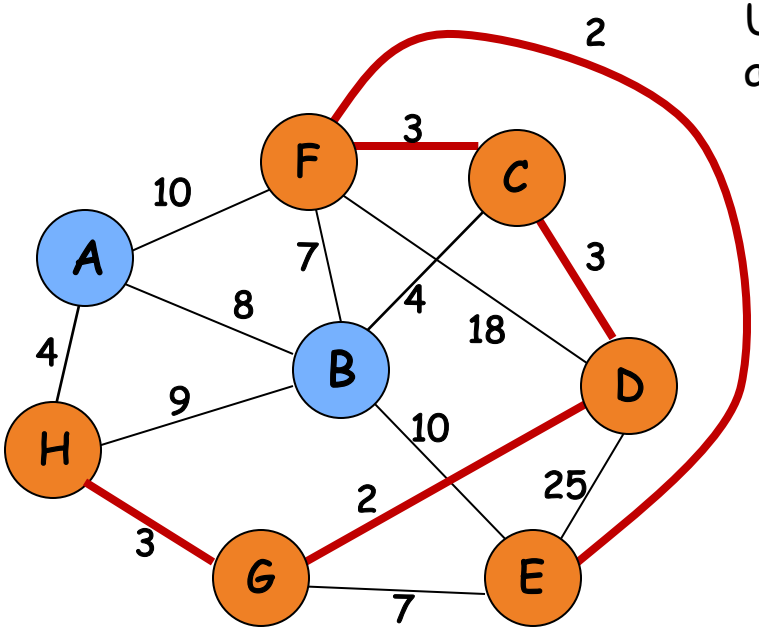
	S	d_v	p_v
A		10	F
B		4	C
C	T	3	D
D	T	0	–
E	T	2	F
F	T	3	C
G	T	2	D
H		3	G

Table entries unchanged



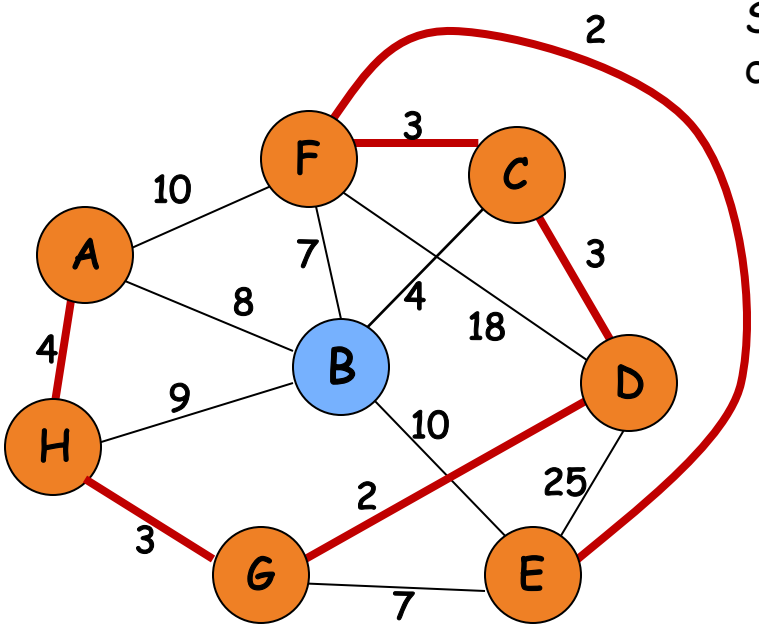
Select node with minimum distance

	S	d_v	p_v
A		10	F
B		4	C
C	T	3	D
D	T	0	–
E	T	2	F
F	T	3	C
G	T	2	D
H	T	3	G



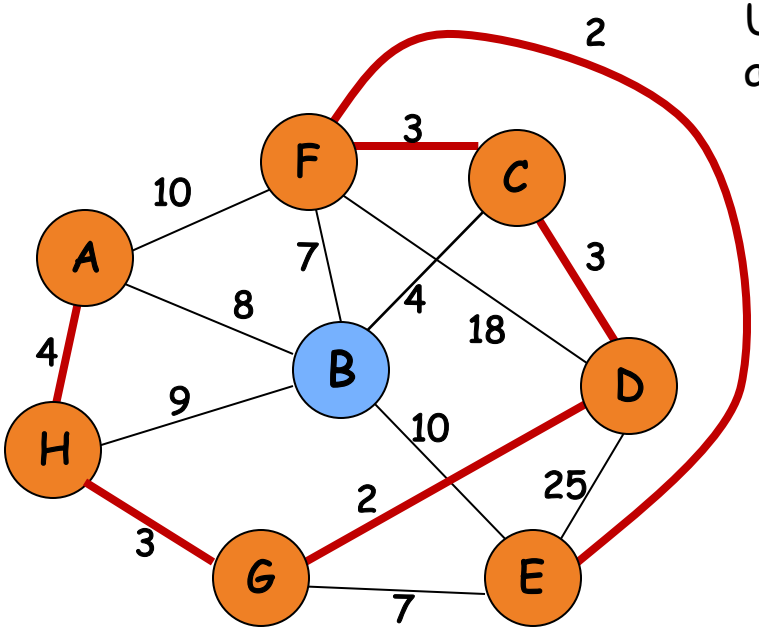
Update distances of adjacent, unselected nodes

	S	d_v	p_v
A		4	H
B		4	C
C	T	3	D
D	T	0	–
E	T	2	F
F	T	3	C
G	T	2	D
H	T	3	G



Select node with minimum distance

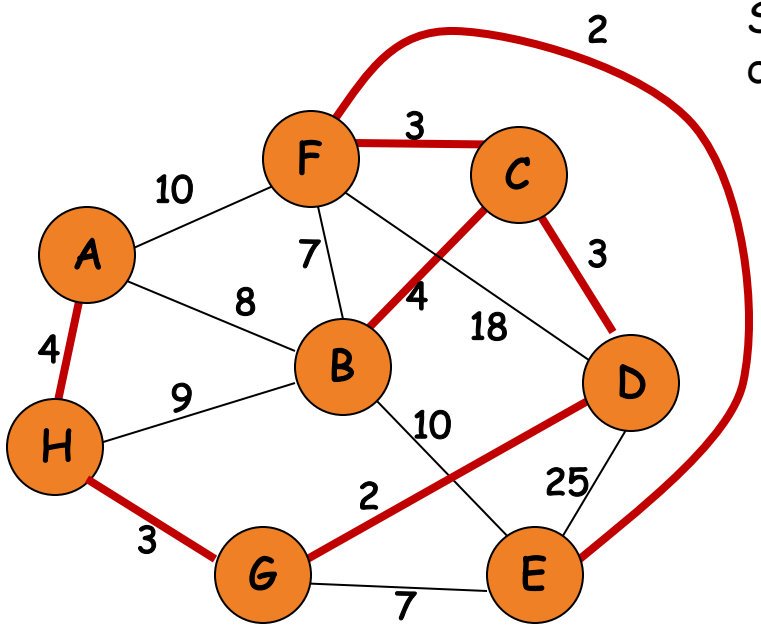
	S	d_v	p_v
A	T	4	H
B		4	C
C	T	3	D
D	T	0	–
E	T	2	F
F	T	3	C
G	T	2	D
H	T	3	G



Update distances of adjacent, unselected nodes

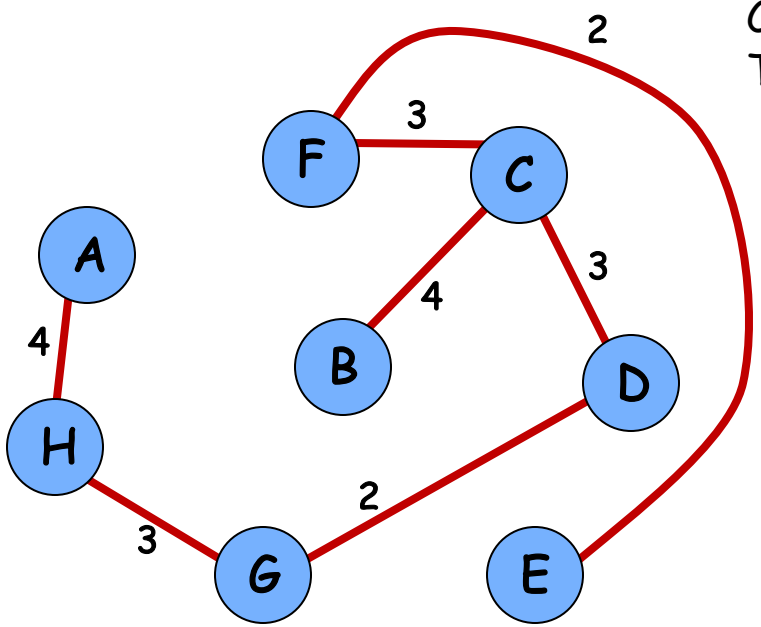
	S	d_v	p_v
A	T	4	H
B		4	C
C	T	3	D
D	T	0	–
E	T	2	F
F	T	3	C
G	T	2	D
H	T	3	G

Table entries unchanged



Select node with minimum distance

	S	d_v	p_v
A	T	4	H
B	T	4	C
C	T	3	D
D	T	0	–
E	T	2	F
F	T	3	C
G	T	2	D
H	T	3	G



Cost of Minimum Spanning Tree = $\Sigma\, d_v$ = 21

	S	d_v	p_v
A	T	4	H
B	T	4	C
C	T	3	D
D	T	0	–
E	T	2	F
F	T	3	C
G	T	2	D
H	T	3	G

Done!

Prim's Algorithm complexity

Big question: are the Prim's and Dj… algorithms equivalent?

```
def prim(G, c) {

  for v in V do
    d[v] ← ∞
    parent[v] ← ∅
  u ← arbitrary vertex in V
  d[u] ← 0
  Q ← new PQ with items { (v, d[v]) for v in V }
  S ← ∅

  while Q is not empty do
    u ← delete min element from Q
    S ← S ∪ { u }
    for (u, v) incident to u do
      if v ∉ S and c_{u,v} < d[v] then
        parent[v] ← u
        decrease priority d[v] to c_{u,v}
  return parent
```

Similar analysis to Dijkstra's algorithm:

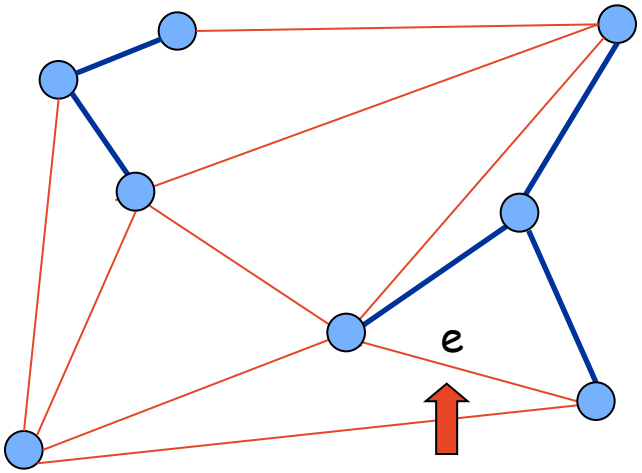
- O(m log n) using a heap
- O(m + n log n) using Fibonacci heap

Kruskal's Algorithm

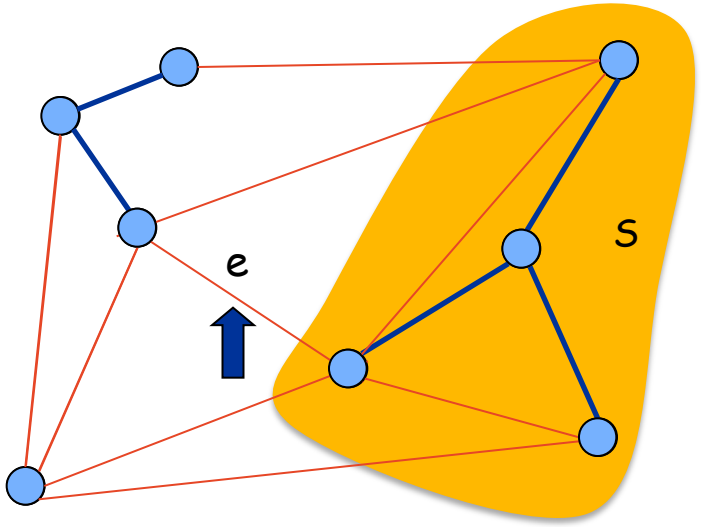
Consider edges in ascending order of weight.

Case 1: If adding e to T creates a cycle, discard e according to cycle property.

Case 2: Otherwise, insert e = (u, v) into T according to cut property where S = set of nodes in u's connected component.

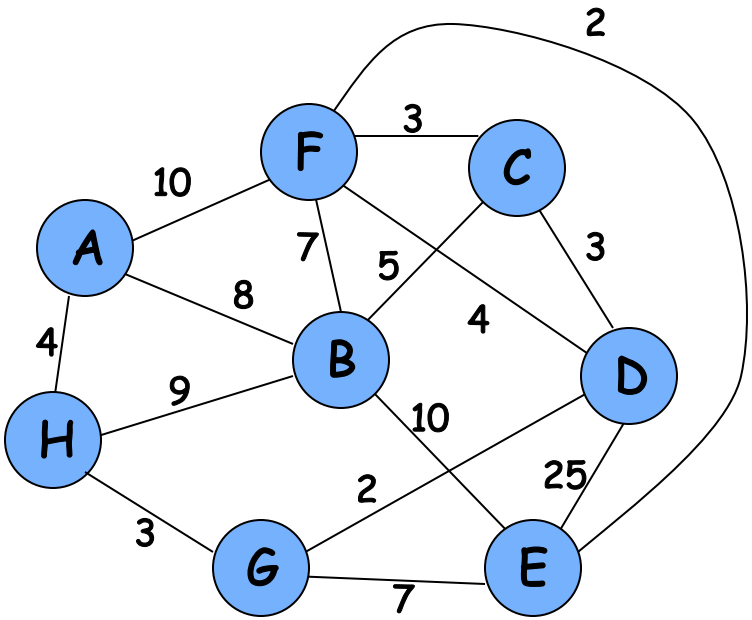


Case 1

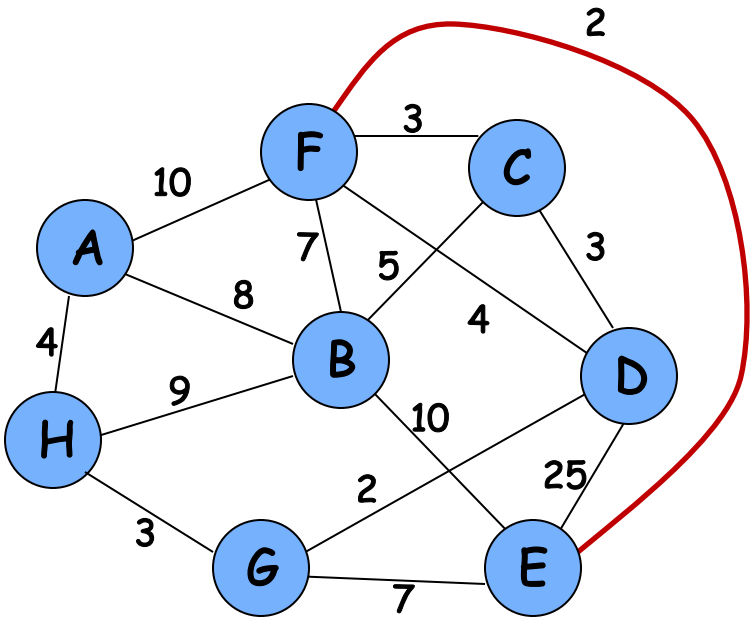


Case 2

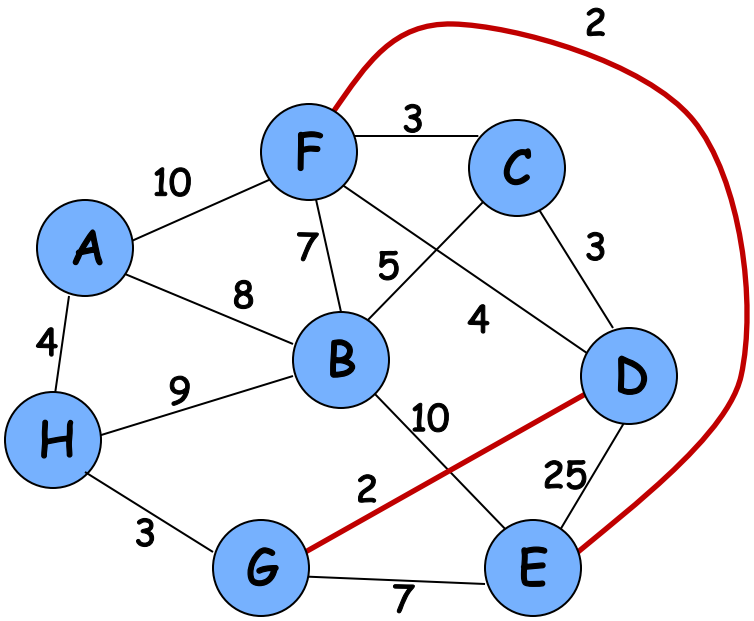
Walk-Through



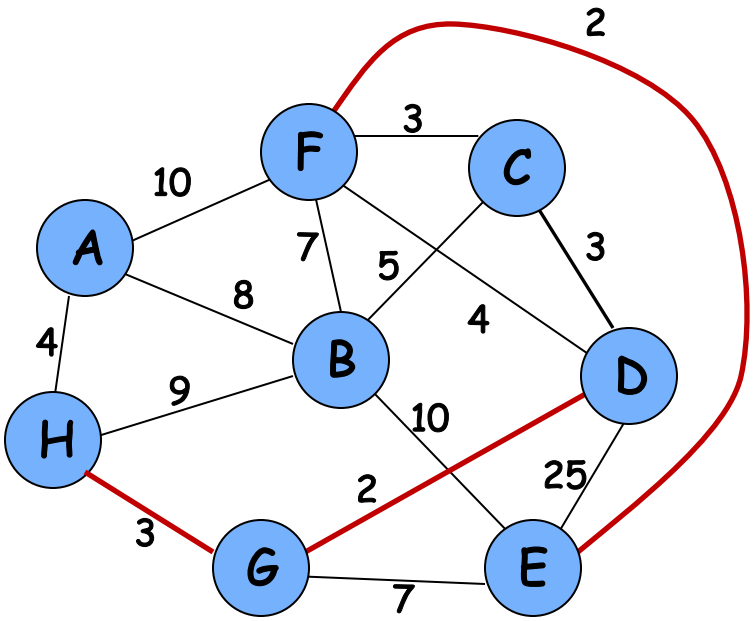
Walk-Through



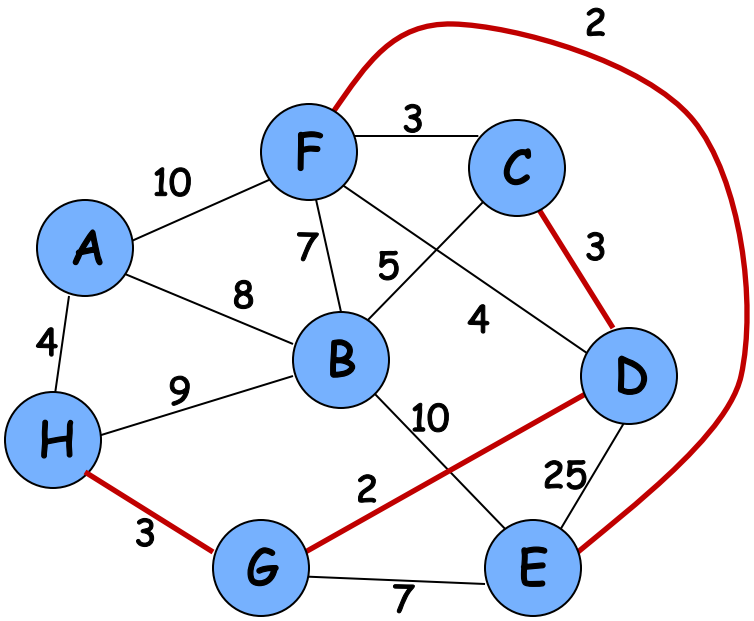
Walk-Through



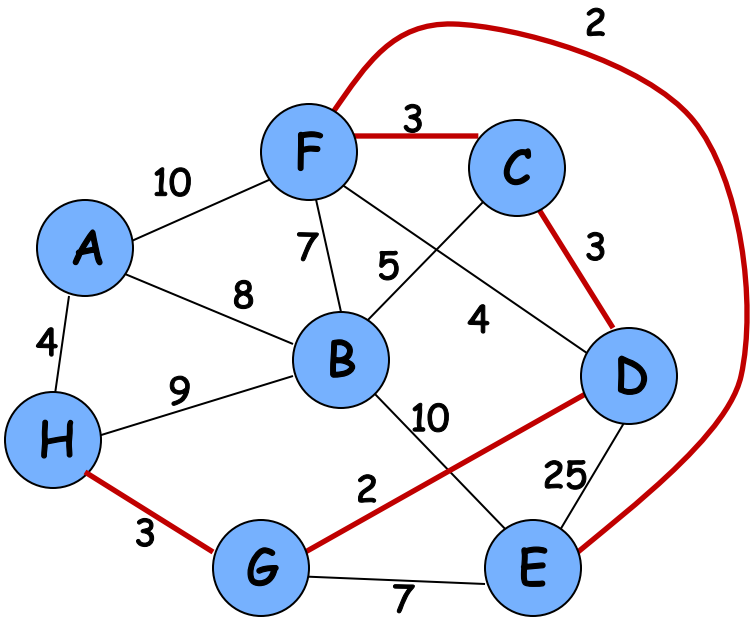
Walk-Through



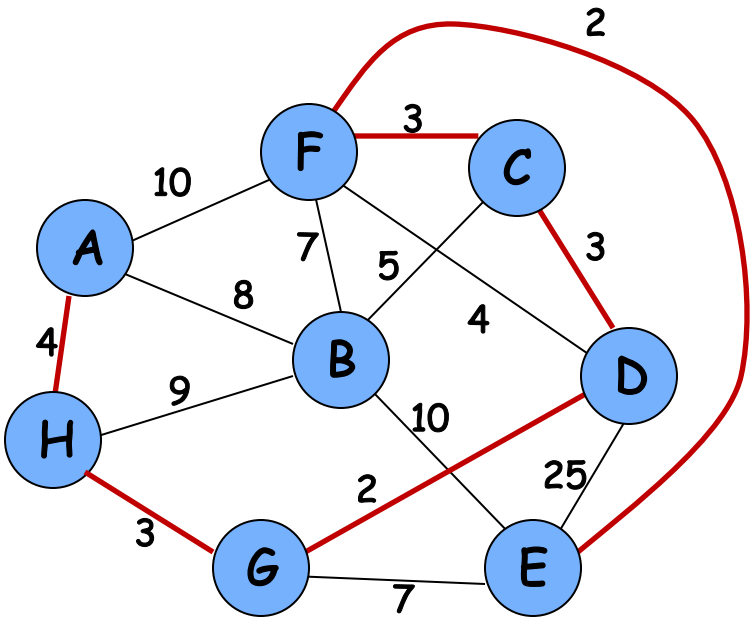
Walk-Through



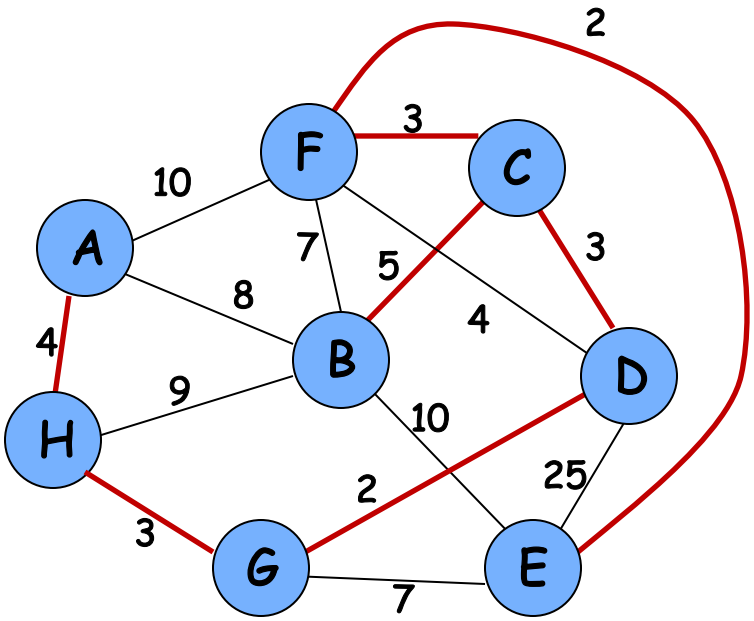
Walk-Through



Walk-Through



Walk-Through



Kruskal's Algorithm: Time complexity

Sorting edges takes O(m log m) time

We need to be able to test if adding a new edge creates a cycle, in which case we skip the edge

One option is to run DFS in each iteration to see if the number of connect components stays the same. This leads to O(m n) time for the main loop

Can we do better? ? ?

Yes, keep track of the connected components with a data structure

Union Find ADT

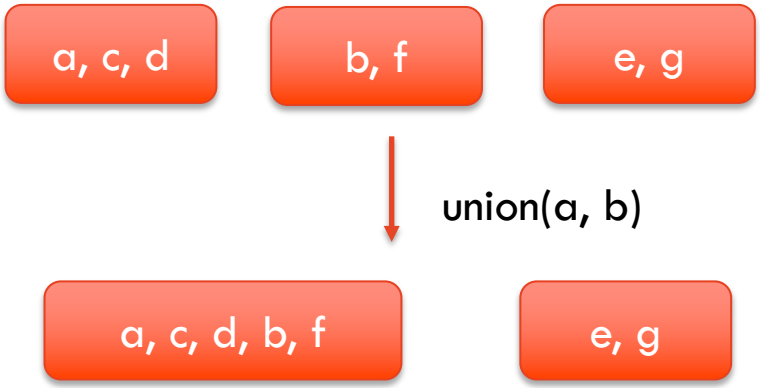
Data structure defined on a ground set of elements A

Used to keep track of an evolving partition of A

The Union-Find ADT is a data structure that represents a collection of disjoint sets. It supports two main operations: union and find. The union operation merges two sets into a single set, while the find operation determines which set an element belongs to.

Supported operations:

- make_sets(A) : makes |A| singleton sets with elements in A
- find(a) : returns an id for the set element a belongs to
- union(a,b) : union the sets elements a and b belong to



Kurskal's algorithm implementation

```
def Kruskal(G,c):

  sort E in increasing c-value
  answer ← [ ]
  comp ← make_sets(V)
  for (u,v) in E do
    if comp.find(u) ≠ comp.find(v) then
      answer.append( (u,v) )
      comp.union(u, v)
  return answer
```

Union find operations:

- make_sets(A) : one call with |A| = |V|
- find(a) : 2m calls
- union(a,b) : n-1 calls

Simple union-find implementation

Sets are represented with lists. And we keep an array mapping elements to the set they belongs to

- make_sets(A) creates and initialized the array
- find(u) is a simple lookup in the array
- union(u,v) adds elements in u's set to v's set

Time complexity:

- make_sets(A) takes O(n) time, where n = |A|
- find(u) takes O(1) time
- union(u,v) take O(n) time

Kruskal's algorithm would run in $O(n^2)$ time after the edge weights are sorted.

Better union-find implementation

Keep track of cardinality of each set. When taking the union of two sets change the smallest.

That way the union of two sets A and B take $O(\min(|A|, |B|))$

This way an element can change sets at most $O(\log n)$ time. So a sequence of n union operations takes at most $O(n \log n)$ time.

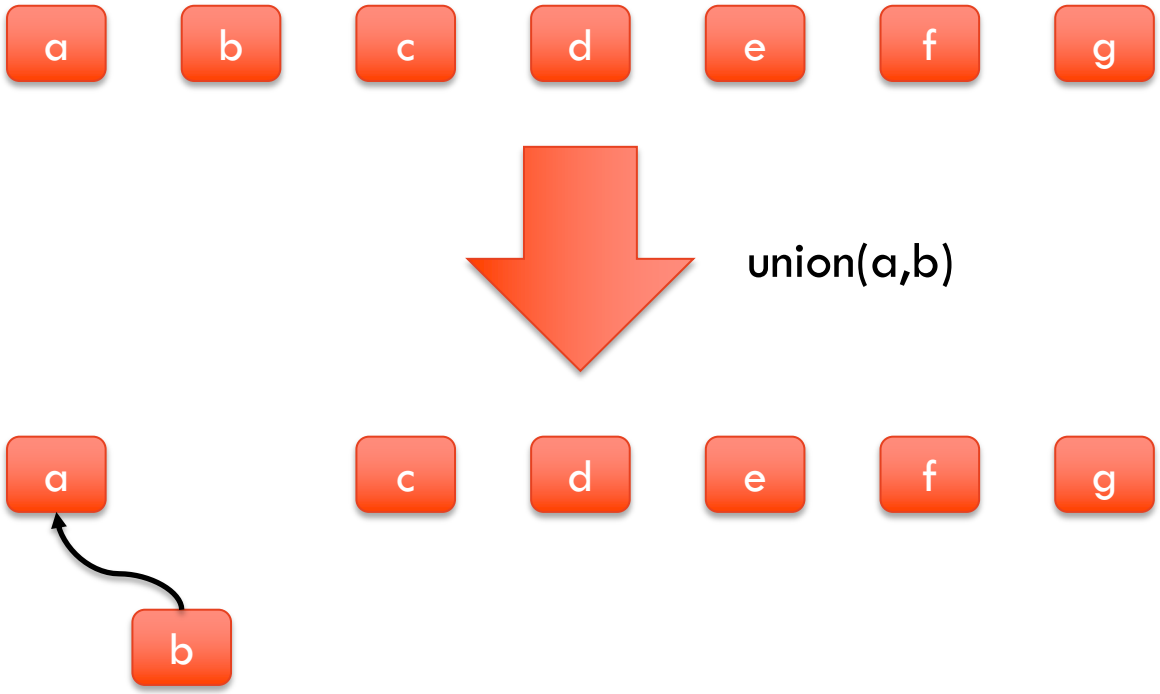
With this implementation, Kruskal's algorithm would run in $O(m \log n)$ time after the edge weights are sorted.

Even better union-find implementation!

Keep track of sets using trees: For every node u in the graph we keep parent[u]

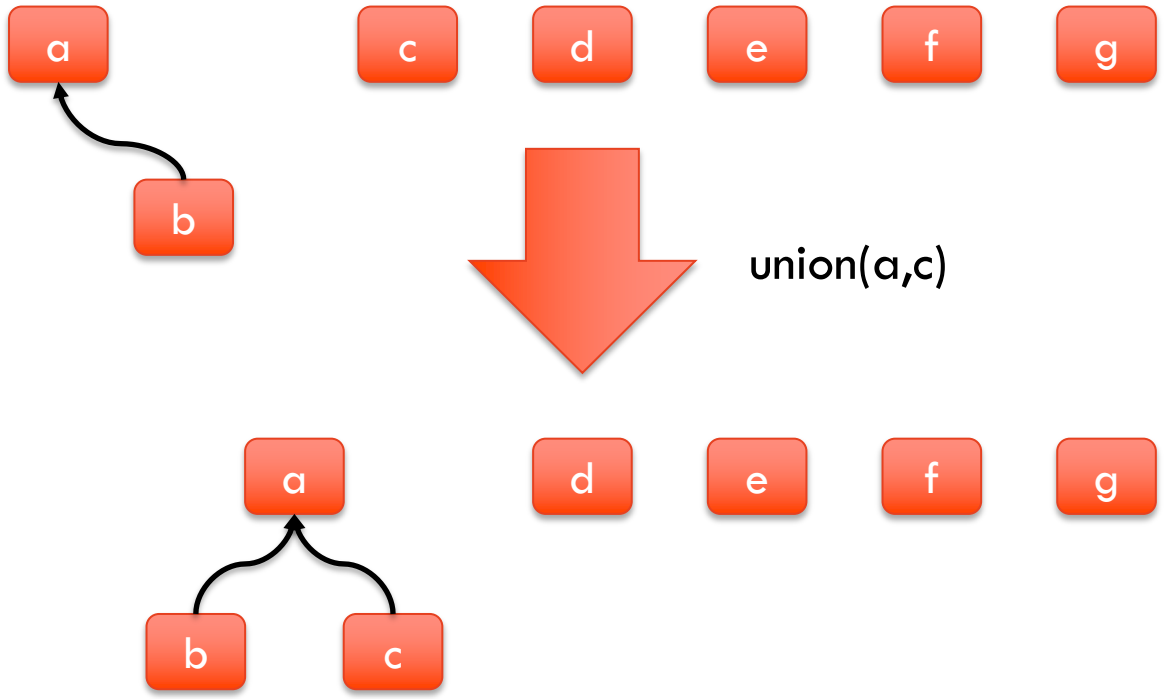
Even better union-find implementation!

Taking the union of two roots u and v we set parent[u] = v or parent[v] = u, depending on who has the largest tree.



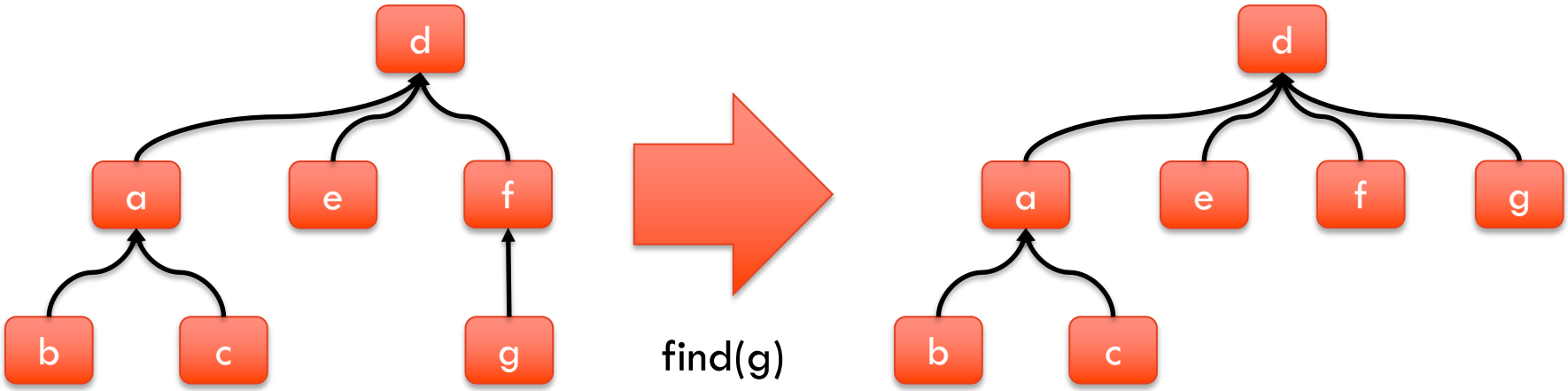
Even better union-find implementation!

Taking the union of two roots u and v we set parent[u] = v or parent[v] = u, depending on who has the largest tree.



Even better union-find implementation!

Perform find(u) by following parent[u] until reaching root r of tree.
Set parent[v] to r for every node found along the way



Even better union-find implementation!

It can be shown that a sequence of n union and $m > n$ find operations takes at most $O(m\ \alpha(n))$ time, where $\alpha(n)$ is a **very slow** growing function called the *Inverse Ackerman function*.

How slow?

For $\alpha(n)$ to be larger than 4, you need n to be much larger than the number of subatomic particles in the universe.

So not a constant from a theoretical perspective, but for all practical (and impractical) purposes, you can treat it as 4.

With this implementation, Kruskal's algorithm would run in $O(m\ \alpha(n))$ time after the edge weights are sorted.

Lexicographic Tiebreaking

To remove the assumption that all edge costs are distinct: perturb all edge costs by tiny amounts to break any ties.

Impact. Kruskal and Prim only interact with costs via pairwise comparisons. If perturbations are sufficiently small, MST with perturbed costs is MST with original costs.

For example, assuming all costs are integral, if we add i/n^2 to each edge e_i then any MST under the perturbed weights is still an MST under the original weights.

Implementation. Can handle arbitrarily small perturbations implicitly by breaking ties lexicographically, according to index.