LAB 3: Radio Interferometry at X-Band

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1. Introduction

The purpose of this lab is, broadly speaking, to learn radio interferometry—the basis for most of modern radio astronomy and the method by which the event horizon of a black hole has been imaged. We'll cover the basic principles, the interferometric fringe, the response to a point source, and the response to an extended source (which is the basis of high-angular resolution mapping with interferometry). We will employ least-squares fitting and Fourier transforms to solve for the positions of our antennas to high precision, and measure the angular diameters of the Sun using Bessel functions. This lab runs for four weeks, but with robotic telescopes to control, threaded data collection to code, and fringes to tame, this is perhaps the hardest lab of the course. Don't fall behind!

2. Goals and Instructions for Your Report

You should now be familiar with the process of writing a report for this class. Below we list the specific goals for this lab that your report should address.

- Learn how diffraction theory applies to a radio interferometer. Fringes, their amplitudes and phases. The fringe as a 'giant sine wave in the sky; the fringe as a 'giant DSB mixer in the sky'; and the fringe as the correlation function of the interferometer pair.
- Fourier transforming the fringe; Fourier filtering the fringe.
- Obtain horizon-to-horizon data on the Sun.
- VLBI nonlinear least-squares fringe fitting to determine the most accurate source declination and/or baseline projected lengths.
- Use nonlinear least-squares fitting to obtain an accurate diameter for the Sun. This illustrates the general Fourier transform relationship between interferometer response and sky brightness.

3. Schedule

In contrast to previous labs, this lab spans four weeks (and spring break). However, interferometry is challenging, and this lab requires day-long observations for each group. You will find that it is still easy to fall behind, so try to get ahead early. Below is a suggested schedule.

1. *Week 1:* Each *group* uses rotation matrices to calculate when objects of interest are up. It observes the Sun for a short time to confirm that it sees fringes, and then does a horizon-to-horizon observation, which requires writing an observing script to run automatically.
   Each *person* calculates Fourier power spectra of the Sun data, calculating the range of expected local fringe frequencies (equation 15) and comparing the observed spectra with these expectations. Be ready for show and tell!
2. *Week 2:* Each *group* (re)observes horizon-to-horizon the Sun with the interferometer. Each *person* derives accurate antenna positions from the data using least-squares fitting.
3. *Spring Break*
4. *Week 3:* Finish the Sun observations. Each *person* derives angular diameters for the Sun using least-squares fitting as outlined in §10.
5. *Week 4:* Each *person* finishes all calculations and writes (and hands in) the lab report.

4. Software for Precession, Tracking, and Finding the Sun

As usual, we have attempted to provide you a batteries-included experience with the `ugradio` Python module. In this lab, we will be focusing on the `interf` and `coord` submodules, as well as a second package, `snap_spec`, which is installed on the RPi attached to the SNAP spectrometer board. You can work directly on this RPi via the provided monitor and keyboard, but you may also log in remotely:

```
ssh <username>@ugastro.berkeley.edu
ssh pi@10.32.92.201
```

- `ifm = ugradio.interf.Interferometer()` provides an interface for controlling the pointing of the two telescopes that constitute the interferometer.

- `ifm.stow()` points the telescopes to the zenith. This minimizes wind drag to help keep the telescope from being destroyed by strong winds. When no-one is using the telescope, it should be stowed. Please stow the telescope when you are not using it!
- `ifm.maintenance()` points the telescope to 'maintenance position' to facilitate working on the feed. `alt=20, az=180`.
- `ifm.get_pointing()` returns both telescopes' current `alt, az`.
- `ifm.point(alt,az)` points telescopes to specified `alt, az` It can take bit for the telescopes to move, so be patient until this command returns.
- `spec = snap_spec.snap.UGRadioSnap()` provides an interface for reading the visibility spectra from of the digital correlator. After instantiating this interface (by creating the object), you'll want to call `spec.initialize(mode='corr')` to configure the SNAP to cross-correlate our dish signals. (Don't worry about warnings printed out; they're normal).
- `spec.read_data()` reads a spectrum from the SNAP Spectrometer. You'll want to work this into a (preferably threaded, see the `threading` module) loop that buffers integrated correlations and periodically writes them to disk, just in case your observing script errors out. Integrations are enumerated by `acc_cnt`, so you can make sure you don't miss one. If you provide the previous `acc_cnt` to `spec.read_data()`, it will wait until the next `acc_cnt` before reading and returning data, to ensure you don't re-read data.
- `ugradio.coord.sunpos(jd)` returns the ra/dec of the Sun at the specified jd.
- `ugradio.coord.moonpos(jd,lat,lon,alt)` returns the ra/dec of the Moon at the specified jd relative to the specified observing lat, lon, and alt. Note that because the Moon is close to the Earth, it has a noticeable parallax that makes its ra/dec depend on where you are. This takes that into account.
- `ugradio.coord.get_altaz(ra, dec, jd, lat, lon, alt)` returns the alt/az of the specified ra/dec coordinate for the specified lat/lon/alt and observing time. Use the 'equinox' variable to specify the coordinate system on the input ra/dec parameters.
- `ugradio.coord.precess(ra, dec, jd, equinox)` returns the precessed ra/dec for the ra/dec catalog coordinate with given 'equinox' specifing the coordinate system on the input ra/dec parameters.

5. Our X-Band Interferometer

With our interferometer, which operates at about 10.5 GHz, our attention focuses on small sources. Our interferometer has a relatively short baseline and the fringe spacing is larger than the size of all sources except the Sun and Moon. So for our interferometer all of these sources look like "point sources", for which all the radiation appears to come from a single point—just like a star in optical astronomy. The output of the interferometer is a sinusoidal-like signal called the "fringe". All of the information resides in the frequency, amplitude and phase of the fringe.

If you know the baseline, the fringe properties are direct indicators of the point-source declination. With our $\sim$ 20-m baseline B we can get a fringe spacing $\frac{\lambda}{B} \sim 5'$, and with a horizon-to-horizon measurement we can measure the declination 100 times more accurately (depending...). In addition, we can achieve partial angular resolution on the Sun and measure its diameter to a fraction of a percent—this turns out to be more interesting than it sounds!

To compute interferometric visibilities, we have an FX (Fourier transform, cross-multiply) correlator implemented on a SNAP signal processing board. This board (designed here at UCB!) is versatile and reprogrammable. For this lab, it has been configured to digitize two signals ($v_1(t)$ and $v_2(t)$) at 500 Msps each, compute the real Fourier transform of 2048 time samples into 1024 channels for each input, and to cross-multiply the resulting complex voltage spectra ($\tilde{v}_1(\nu)$ and $\tilde{v}_2(\nu)$). This spectral product is averaged for 305200 spectra, corresponding to $305200 \times 2048 \times 2$ ns = 1.25 s of integration in $\langle V_{12}(\nu, t) = \tilde{v}_1(\nu)\tilde{v}_2^*(\nu)\rangle$. Each frequency channel of these visibility spectra (inside the bandpass filters) is a complex number that will exhibit an oscillating fringe pattern in both real and imaginary components versus time. The phase slope of these visibility spectra versus frequency tell us about the relative time delay between the correlated signals passing through each dish to arrive at the SNAP correlator inputs.

6. Continuum (Point) Sources

Although our sensitivity is limited, it is sometimes possible to measure 'point' sources that lie outside the Solar system. Our telescopes are small, so there are only a few sources that are powerful enough for us; they are listed below. You'll need to precess the coordinates to the current equinox. If you don't, the incorrect RAs will affect your fits. To do the precession, use `ugradio.coord.precess`.

name	**RA (J2000)**	**DEC (J2000)**	**S_{Jy}**
3C144 (Crab Nebula)	$05^h34^m31.95s$	$+22°00'52.1''$	~ 496
Orion Nebula	$05^h35^m17.3s$	$-05°23'28''$	~ 340
M17	18^h20^m26s	$-16°10.6'$	~ 500
3C405 (Cygnus A)	$19^h59^m28.357s$	$+40°44'02.10''$	~ 120
3C461 (Cas A)	23^h23^m24s	$+58°48.9'$	~ 320
SUN	varies	varies	
MOON	varies	varies	

As astronomers, our goal is to measure the absolute declinations as accurately as possible. As geologists, our goal is to measure the interferometer baseline as accurately as possible (to see, for example, the displacement of tectonic plates by earthquake activity). If you look at equations 11 and 12, you see that we don't measure these quantities separably, but rather the products $b_{ew}\cos(\delta)$ and $b_{ns}\cos(\delta)$, where b_{ew} and b_{ns} are the east-west and north-south components of the interferometer baseline, $\vec{b}$. These products can be measured on an absolute basis with interferometry by least-squares fitting the fringe phase to the hour angle; see §9 and equation 12 below. We already know the source positions accurately, from previous interferometric measurements; that's where the positions in the above table come from. So, for this lab, we'll emphasize the geologic aspect and measure baselines. We should be able to measure baseline components to better than 1%.

Coverage for southern sources may be interrupted by the Campanile—check (by going up to the roof periodically and looking)! And coverage for northern sources, especially Cas A, is interrupted by Evans Hall.

We'll want *each group* to pick a source and obtain a horizon-to-horizon observation of the fringes. Each group should pick a different source. Observing a source takes at least half a day. With four groups and less than a week to do the measurements, you need to cooperate in scheduling the interferometer. Organize yourselves!

We'll want *each person* to measure the source's declination by least-squares fitting the horizon-to-horizon track of fringe phase and amplitude. Compare your results with other members of your group. Help each other out, but *each person should write their own software.*

7. The Sun (and Moon)

The Sun and Moon are special cases, for two reasons. First, their positions change from day to day—and for the Moon, the change in just an hour is significant[1]. Second, they are both extended sources, not point sources. This causes their fringe amplitudes and phases to change with time, in a manner that depends on their brightness distribution[2]; this makes the determination of accurate positions a bit tricky, but we'll ignore that detail for now.

You can get the Sun and Moon positions from `ugradio.coord.sunpos` and `moonpos`. For the Moon, you should be aware that it is nearby so that you have to correct for your location on the Earth's surface. The parallax effect is far greater for the Moon than the Sun—so large that unless you correct for it, the Moon will probably lie outside the telescope main beam! `moonpos` takes care of this automatically.

8. Some Astrophysics

Here's a bit of physical information. M17 ("M" for Messier) and Orion are HII regions—places where hot stars have produced warm ($T \sim 10^4$ K) ionized gas, where the electrons flying past the protons get deflected and produce *free-free (bremsstrahlung)* radiation. The Sun also emits free-free radiation, just like the HII regions.

The other sources, and sources paired with some of the HII regions, radiate in *synchrotron* radiation—relativistic electrons gyrating in a magnetic field. For these sources, the source designation 3CXYZ designate source number XYZ from the third Cambridge (England) catalog; in the early 1960's, Cambridge radio astronomers produced the first reliable comprehensive catalog of strong radio sources in the Northern hemisphere. The Crab Nebula (also called Taurus A) is powered by the Crab pulsar, and is a $\sim$ 1000 yr-old supernova remnant in the Galaxy about 1 kpc distant. Cas A is another supernova remnant, *not* powered by a pulsar; rather, the relativistic electrons are produced behind the fast shock wave produced by the explosion. Cas A is $\sim$ 300 yr old and about 2.5 kpc distant. Both of these supernova remnants are expanding rapidly, as befits their young ages, and Cas A (in which a pulsar does not constantly replenish the electrons) is gradually getting dimmer.

In external galaxies, the ultimate source of the electrons involves acceleration of electrons near the black hole at the center; the electrons are then spewed out to extragalactic space in narrow collimated jets, and produce large "emission lobes" at the end of the jets. Cygnus A is a giant elliptical galaxy 220 Mpc distant. Cyg A is a powerful radio source—it's 10^5 times further than Cas A and just as bright! The study of the mechanism by which this enormous power is generated, which implies enormous energies, has led to the current awareness of and interest in *high-energy*

[1] Use what you already know to make an order-of-magnitude estimate of how far the Sun and Moon move in one day!

[2] These changes in fringe properties are exactly what's necessary to map the sources!

astrophysics. For information on both types of source and some beautiful pictures, see chapters 10 and 13 in *Galactic and Extragalactic Radio Astronomy, second edition* (1988, ed. G.L. Verschuur and K.I. Kellermann).

The Moon is a completely different story. Contrary to what you might expect, at radio wavelengths it *doesn't* shine by reflected sunlight. Rather, its emission is blackbody radiation from its solid surface. Its surface is heated by sunlight, and at short wavelengths (but not at long ones) there's a big difference between the temperature of the sunlit and dark parts of the Lunar surface. You can tell a lot about its surface properties from the polarization of the radiation and also from its time variability as the surface heats up from sunlight and cools off from darkness—just like the Sahara.

9. Measuring $b\cos\delta$

We measure the product $b\cos\delta$ from the fringe frequency, which depends on the baseline orientation, baseline length, declination, and hour angle (all these go into the *projected baseline*), as well as the observing frequency (from which you have many to choose from). If we observe a source horizon-to-horizon, the projected baseline changes a lot, and so does the fringe frequency. We first do a 'sanity check', taking the Fourier transform of the data to see if we see the fringe and measure its range of frequencies. If it all looks good, we take those data, together with the known declination, and do a least-squares fit to derive the most accurate baseline components from our data, first one frequency at a time, and then across the whole band.

9.1. Getting the Data

Before doing the weak sources in the Table, do the Sun for a much shorter time, say an hour. This will give you confidence that the system works (or so we hope). There should be an easily-recognizable signal that you can look at visually, think about, and derive the approximate declination with pencil and paper. Then later you can write software to do the same, and make sure you get the right answer. Also, during this first week, do the horizon-to-horizon track of one of the sources from the Table.

9.2. The Fringe

The two interferometer telescopes have different distances from the source. The difference can range from zero (if the source is overhead) to nearly the full baseline (if the source is near the horizon). This distance difference is tiny compared to the distance to the source, but it's important!

It's convenient to think of the different distances in terms of relative path delay in *time* units for the two telescopes; we call this the *geometrical* path delay τ_g. But don't forget! The signals travel through a lot of electronics before they get multiplied and the two paths aren't of equal length, so there is an additional relative delay from the difference in cable length τ_c. The total relative delay is the sum of the two,

$$\tau_{tot} = \tau_g(h_s) + \tau_c \ . \tag{1}$$

We explicitly include the fact that τ_g is a function of time—that is, the hour angle of the source

h_s. In contrast, τ_c is independent of time (unless somebody changes the cable setup. . .).

We don't know τ_c (but the least-squares process can tell us what it is). However, we do know τ_g because it's just geometry—the geometry discussed in the reading. For the east-west baseline component with length b_{ew}, we have

$$\tau_{g,ew}(h_s) = \left[\frac{b_{ew}}{c}\cos\delta\right]\sin h_s \ . \tag{2}$$

and for the north-south component

$$\tau_{g,ns}(h_s) = \left[\frac{b_{ns}}{c}\sin L\cos\delta\right]\cos h_s - \left[\frac{b_{ns}}{c}\cos L\sin\delta\right] \ , \tag{3}$$

where L is the terrestrial latitude. This has two terms. The first is the delay perpendicular to the Earth's axis, which changes as $\cos h$. The second is the delay parallel to the Earth's axis, which has no h dependence; since it is independent of h, we'll lump it into the cable delay τ_c. Accordingly, we define

$$\tau_g'(h_s) = \left[\frac{b_{ew}}{c}\cos\delta\right]\sin h_s + \left[\frac{b_{ns}}{c}\sin L\cos\delta\right]\cos h_s \tag{4}$$

and the b_{ns}-modified cable delay as

$$\tau_c' = \tau_c - \left[\frac{b_{ns}}{c}\cos L\sin\delta\right] \ . \tag{5}$$

The output of the interferometer is the product of what the two telescopes see. If they are looking at a monochromatic source then the voltages for the two telescopes are

$$E_1(t) = \cos(2\pi\nu t) \tag{6}$$

$$E_2(t) = \cos(2\pi\nu[t + \tau_{tot}]) \ . \tag{7}$$

and the product is the interferometer fringe output

$$F(t) = \cos(2\pi\nu t)\cos(2\pi\nu[t + \tau_{tot}]) \ . \tag{8}$$

There's a trigonometric identity that allows us to write this in terms of the sum and difference of the two arguments[3]. The sum term varies rapidly with time and averages to zero; it's the difference term we want, so if we exclude the sum term (and forget about the factor $\frac{1}{2}$) we get

$$F(h_s) = \cos(2\pi\nu[\tau_g'(h_s) + \tau_c']) \ . \tag{9}$$

We want to use a least-squares fit to find the values of quantities that comprise the argument of the cosine. If you have done least-squares fitting before (and if you remember anything about it!) you'll realize that the straightforward least-squares fitting technique won't work on this type of problem, because the unknown quantities appear inside nonlinear functions. We can simplify

[3] $\cos(A)\cos(B) = \frac{1}{2}[\cos(A - B) + \cos(A + B)]$

things for the fitting process by using another trig identity[4] and writing this as the sum of two trig functions

$$F(h_s) = \cos(2\pi\nu\tau_c') \cos\left(2\pi\nu\tau_g'\right) - \sin(2\pi\nu\tau_c') \sin\left(2\pi\nu\tau_g'\right) \ . \tag{10}$$

This may not look simpler! But it is, because for the purposes of least-squares it involves only the *single* variable τ_g' in the trig function arguments (we are assuming that we know the right ascension well enough to get a good value for h_s, which appears in τ_g').

To proceed with least-squares, replace $\cos(2\pi\nu\tau_c')$ and $-\sin(2\pi\nu\tau_c')$ by two "unknown constants" A and B, respectively; assume that they are unrelated and solve for them using the standard least-squares process. Also, it's convenient and intuitive to make the substitution

$$\nu\tau_g'(b_{ew}, b_{ns}, \delta, h_s) = \left[\frac{b_{ew}}{\lambda} \cos\delta\right] \sin h_s + \left[\frac{b_{ns}}{\lambda} \sin L \cos\delta\right] \cos h_s \tag{11}$$

which expresses the delay in units of wavelength, and thus the "number of turns" or phase. These substitutions give the *Fringe Amplitude for a point source*

$$\boxed{F(h_s) = A\cos\left(2\pi\nu\tau_g'\right) + B\sin\left(2\pi\nu\tau_g'\right) \ .} \tag{12}$$

9.3. The Local Fringe Frequency

We want to develop the concept of a *local fringe frequency* f_f. In equation 12, the argument of the trig functions depends on h_s, which is the hour angle and increases monotonically with time, so we can regard it as time[5]. Now it's not h_s itself in the equation, but rather $\sin(h_s)$ and $\cos(h_s)$, which are nonlinear functions of time. They multiply the baseline in units of wavelength, which is large, so the product rapidly oscillates back and forth, but at a frequency that depends on time as h_s changes. This is the local fringe frequency.

To see this, expand the hour angle terms $\sin(h_s)$ and $\cos(h_s)$ each into its own Taylor series, centered on the current hour angle of the source $h_{s,0}$. For example, for the $\sin(h_s)$ term we have

$$\sin(h_s) = \sin(h_{s,0}) + \Delta h_s \left.\frac{d\sin(h_s)}{dh}\right|_{h_{s,0}} = \sin(h_{s,0}) + \Delta h_s \cos(h_{s,0}) \tag{13}$$

The local fringe frequency is contained in the second term because, for a small region around $h_{s,0}$ in equation 12, $F(h_s)$ varies as $f_f\ \Delta h_s$, where the local fringe frequency $f_{f,rad}$ is

$$f_{f,rad} = \left[\frac{b_{ew}}{\lambda} \cos\delta\right] \cos h_{s,0} - \left[\frac{b_{ns}}{\lambda} \sin L \cos\delta\right] \sin h_{s,0} \tag{14}$$

This is the local fringe frequency in cycles per radian on the sky. If you want to turn it into cycles per second coming out of the interferometer's multiplier, multiply by $\frac{dh_s}{dt} = \omega_\oplus = \frac{2\pi}{24\times 60\times 60}$.

$$\boxed{f_{f,Hz}/\omega_\oplus = \left[\tfrac{b_{ew}}{\lambda} \cos\delta\right] \cos h_{s,0} - \left[\tfrac{b_{ns}}{\lambda} \sin L \cos\delta\right] \sin h_{s,0}} \tag{15}$$

---

[4] $\cos(A+B) = \cos(A)\cos(B) - \sin(A)\sin(B)$

[5] Except that its units are radians. This is no problem: 24 hours is 2π radians.

Our interferometer is almost (but not quite) east-west with $b_{ew} \approx 20$ m, $b_{ns} \sim 0$, and $\lambda \approx 2.5$ cm. At the meridian ($h_{s,0} = 0$), the period is $\frac{1}{f_f} \approx \frac{18}{\cos\delta}$ seconds. ; *check these numbers* to make sure you understand this calculation!)

Use the above to calculate the range of local fringe frequencies that you should see in your data. Does it work? Look at the Fourier transform of your data to check! And try Fourier filtering!

9.4. Least-Squares Fringe Fitting

Now let's return to the least-squares solution of equation 12 for the unknown quantities $Q_{ew} = \left[\frac{b_{ew}}{\lambda}\cos\delta\right]$ and $Q_{ns} = \left[\frac{b_{ns}}{\lambda}\sin L\cos\delta\right]$. This is a nonlinear least-squares problem because these quantities sit inside nonlinear functions. There are two ways to attack this: the conventional way and the 'brute-force' way. We'll do the latter first.

9.4.1. The Brute-Force Technique

We'll first assume that $b_{ns} = 0$, which is not unreasonable because our interferometer baseline is close to east-west, and solve for Q_{ew}. Make a guess at the proper value of Q_{ew} and, if you don't know the right ascension α accurately, adopt a reasonably accurate value for it (required so that you can compute h_s from the local sidereal time LST). With this, you know the arguments of the trig functions and this means that you can solve for A and B using the standard least-squares process; be sure and *save the sum of the squares of the residuals*, which we will call $\mathcal{S}$. Then change the guessed-at value of Q_{ew} and do it again. Do this a number of times and plot the sum-of-squares $\mathcal{S}$ versus the guessed-at value of Q_{ew}. The best value of Q_{ew} is where the sum-of-squares is a minimum. What is this? *It's brute-force least-squares!*

Now do the same for Q_{ns}. The best value of Q_{ew} depends on the value of Q_{ns}, so you'll need to develop a 2-d array for $\mathcal{S}$ versus the assumed values of Q_{ew} and Q_{ns} so that you can find the true global minimum.

How about the uncertainties? You can get these from the curvature matrix $[\alpha]$, which is easily derived from the behavior of $\mathcal{S}$ around the minimum. The covariance matrix α^{-1} is the inverse of the curvature matrix. See §2 and §3 of the 'Lite' version of the *Aficionado's* handout.

9.4.2. The Nonlinear Least-Squares Technique

The usual technique for nonlinear least-square fitting is this: you guess values for unknowns Q_{ew} and Q_{ns}, expand the equation for $F(h_s)$ in a one-term Taylor series about these guesses, and solve for the corrections. The procedure is discussed in detail in §7 of the 'Lite' version of the *Aficionado's* handout.

Our particular problem is not well-suited to the 'proper' technique because the initial guessed values need to be quite accurate. If they are not accurate enough, then the proper technique converges to a local, non-global minimum.

The advantage of using the proper technique is that it provides the uncertainties, and in particular the covariance matrix. If you have time, try both methods–the 'brute-force' and 'proper' methods, using the 'brute-force' result as input to the 'proper' method, and compare the results, in particular the uncertainties.

10. Measuring Brightness Distributions: the First Step of Making Maps

Disclaimer *This section assumes an east-west interferometer. Generalizing to the case of* $b_{ns} \neq 0$ *can be regarded either as an 'exercise for the student' or as an unnecessary complication.*

When we think of a time-variable signal, we think of frequency as being cycles per second—and its inverse, the period, is in seconds, the number of seconds that separates adjacent peaks of the sine wave.

The interferometer projects a giant sine wave on the sky. Its frequency, which changes with position, is measured in cycles per radian—and its inverse, the period, is the angular separation of adjacent peaks, measured in the angular units of radians. You can, of course, also think of frequency in terms of cycles per degree or cycles per arcminute, with the corresponding periods ("fringe separation") in units of degrees or arcminutes.

When we observe with a range of baseline lengths and orientations, the giant sine waves in the sky have corresponding ranges of frequencies and orientations. We sample brightness of the sky in *Fourier* space. The fringes at each baseline length and orientation have amplitudes and phases. To recover the brightness of the sky in *real, angular* space, we measure as many Fourier components as we can and take their Fourier transform. If we had complete sampling in Fourier space, we would recover the true brightness distribution. In real life, we have *in*complete sampling, so we recover a distorted representation of the true distribution. There is a whole literature of techniques for minimizing this distortion, the most prominent being "cleaning" and "maximum entropy". Full-fledged research arrays, such as the Very Large Array (VLA) in New Mexico, rely on these techniques to map the sky.

In our case we have just two dishes along an east-west line. The effective baseline length changes as the source rises higher in the sky, and if the source is away from declination $\delta = 0^\circ$ the orientation of the baseline also changes, at least to some degree. We will map the Sun, which never gets very far from $\delta = 0^\circ$, so effectively we have only a 1-d sampling of the source with a range of baseline lengths.

Figure 1 illustrates this 1-d concept. It presents a generic circular source of uniform brightness in the sky, which we call the MUN—a bastardization of the MOON and the SUN[6]. The top panel shows the MUN in the sky as it really is: the fringes cover the 2-d object. At the bottom, we integrate along vertical strips to get the 1-d brightness distribution—the vertically-integrated 1-d equivalent, in which both the brightness distribution and the fringes depend on only one coordinate.

This one coordinate is the horizontal direction in the bottom panel of the Figure. In real life this is hour angle because we have an east-west interferometer and our source is at low declination—meaning that the baseline projected on the sky is mainly east-west, the direction of hour angle. Thus we denote this direction by the letter h.

As the Earth rotates, the source moves through the fringe pattern to give the fringe response $R(h_s)$. For a point source, $R(h_s) = F(h_s)$ (see equation 12); for an extended source, we have to integrate over the extent of the source. Let $I(h - h_s)$ be the 1-d intensity distribution in the sky. The source center is the intensity-weighted mean of the position, i.e.

$$h_s = \frac{\int I(h) h dh}{\int I(h) dh} \ . \tag{16}$$

[6] In truth, it represents neither, because neither the Sun does not have uniform surface brightness.

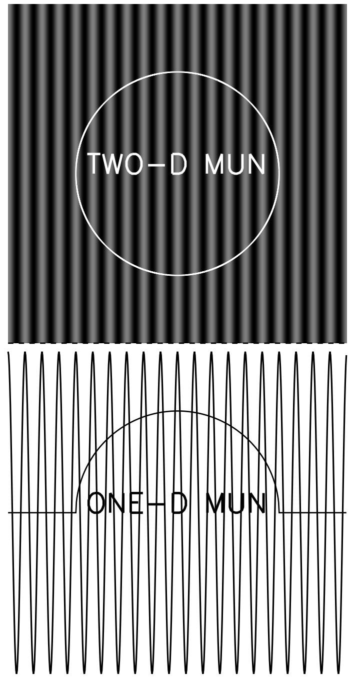

Fig. 1.— The 2-d and 1-d MUN. At the top, we see the situation in the sky as it really is: the fringes cover the 2-d object. At the bottom, we integrate along vertical strips to get the 1-d brightness distribution and, also, the fringe amplitude (which goes from –1 to +1).

That is, on the bottom panel of Figure 1, $I(h-h_s)$ is the intensity of the source (vertical direction) and $\Delta h = h - h_s$ the horizontal coordinate—the hour angle h relative to the hour angle of the source center h_s.

We can express the interferometer response $R(h_s)$ using equation 12, which is for a point source; for an extended source, we imagine $I(\Delta h)$ as being composed of little slices in hour angle, with each little slice of the source characterized by its position offset Δh and its intensity $I(\Delta h)$, so we just integrate:

$$R(h_s) = A \int I(\Delta h) \cos\left[2\pi \left(\frac{B}{\lambda} \cos\delta\right) \sin h\right] d\Delta h + B \int I(\Delta h) \sin\left[2\pi \left(\frac{B}{\lambda} \cos\delta\right) \sin h\right] d\Delta h \quad (17)$$

Now express $\cos\left[2\pi \left(\frac{B}{\lambda}\cos\delta\right)\sin h\right]$ in terms of the local fringe frequency (equations 13 and 15), which replaces $\cos\left[2\pi \left(\frac{B}{\lambda}\cos\delta\right)\sin h\right]$ by a sum of two terms, which we temporarily denote α and β[7]. Now, as usual, we use trig identities $[\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$ and $\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)]$. The first (cosine) term of equation 17 becomes

$$R^{\cos}(h_s) = A\ \cos(\alpha) \int I(\Delta h) \cos(2\pi f_f \Delta h)\ d\Delta h - A\ \sin(\alpha) \int I(\Delta h) \sin(2\pi f_f \Delta h)\ d\Delta h \qquad (18)$$

[7]where $\alpha = 2\pi\left(\frac{B}{\lambda}\cos\delta\right)\sin h_s$ and $\beta = 2\pi f_f \Delta h$.

with an equivalent, similar expression for $R^{\sin}(h_s)$.

For our source (the MUN), *we assume that* $I(\Delta h)$ *is symmetric* (This also retains our algebraic sanity.). This means that in the above equation the second term, which is antisymmetric, integrates to zero. Similarly, the antisymmetric term in the equivalent equation $R^{\sin}(h_s)$ also integrates to zero, so we end up with $R(h_s) = R^{\cos}(h_s) + R^{\sin}(h_s)$, or

$$R(h_s) = \underbrace{F(h_s)}_{Point-source\ Fringe} \times \underbrace{\int I(\Delta h)\cos(2\pi f_f \Delta h)\ d\Delta h}_{Fringe\ Modulator} \tag{19}$$

Note the structure of equation 19. It consists of two factors. The first "Point-source Fringe" term is identical to equation 12—it's the response to a point source located at $\Delta h = 0$. The other modulates (multiplies) this function.

Generally, *the modulating function is the Fourier transform of the source intensity distribution on the sky.* Here, we assumed a one-dimensional symmetric source, which means that the sine portion of the Fourier transform is zero; this is why equation 19 is only a cosine Fourier transform instead of a full one. More generally, the Fringe Modulator depends on the two-dimensional map of intensity on the sky, so it's a double integral instead of a single one.

Figure 2 (top panel) shows two examples of 1-d brightness distributions, a flat and a cosine distribution. The bottom panel shows the Fourier transforms. Both of the modulating functions are trig functions. In particular, for the flat distribution the modulating function is $\frac{\sin(2\pi f_f R)}{2\pi f_f R}$. It can (and does!) go through zero. *The locations of these zero points provide crucial information about the source structure.* The zeros occur for $f_f = \frac{n}{2R}$. It's more intuitive to express the zeros in terms of fringe *period* (equal to $\frac{1}{f_f}$): the zeros occur at $Period = \frac{2R}{n}$. There's a zero whenever there's an integral number of fringe periods over the source width. This makes perfect sense, because then the source contributes equally to the negative and positive portions of the fringe and the net integral is zero.

11. Measuring the Diameter of a Circular Source

Our goal is to measure the diameter of the Sun. We'll make the assumption that the sources are uniformly-bright disks of radius R, which means

$$I(\Delta h) = \frac{(R^2 - \Delta h^2)^{1/2}}{R} \tag{20}$$

To obtain the theoretical modulating function MF_{theory}, you use the integral in equation 19, which is

$$MF_{theory} = \frac{1}{R}\int_{-R}^{R}(R^2 - \Delta h^2)^{1/2}\cos(2\pi f_f \Delta h) d\Delta h \tag{21}$$

If you want, you can do this analytically by substituting $\Delta(R\cos(\theta)$ for Δh; you end up with a Bessel function.

We are running a lab class, not a math class, so let's proceed by doing the integral *numerically*! To accomplish this, split $I(\Delta h)$ into $2N+1$ tiny little slices (the total number is odd, which makes the slices symmetric about $\Delta h = 0$). Each slice has width $\delta h = \frac{R}{N}$, and $\Delta h_n = n\delta h$, where n runs from $-N$ to $+N$. Then the integral becomes a sum:

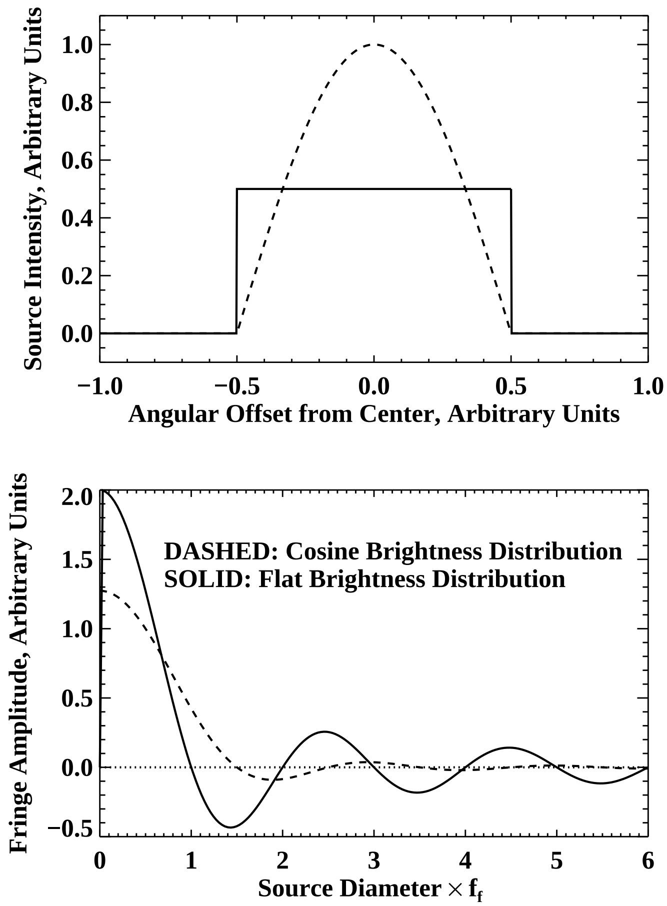

Fig. 2.— Examples of 1-d brightness distributions and their Fourier transforms. Top panel: the brightness distributions. Bottom: the Fourier transforms (Fringe Amplitude vs. $\frac{Source\ Diameter}{FringePeriod}$). In both, panels, the solid line is for a flat brightness distribution and the dashed one for a cosine distribution.

$$MF_{theory} \approx \frac{1}{R} \sum_{n=-N}^{n=+N} [R^2 - (n\delta h)^2]^{1/2} \cos(2\pi f_f n \delta h) \delta h \tag{22}$$

which we rewrite as

$$MF_{theory} \approx \delta h \sum_{n=-N}^{n=+N} \left[1 - \left(\frac{n}{N} \right)^2 \right]^{1/2} \cos \left(\frac{2\pi f_f R n}{N} \right) \tag{23}$$

Note the *first important point* that $MF_{theory} \ldots$

- is a function *only* of the combination $f_f R$, and
- in particular, has zero crossings that occur at specific values of $f_f R$.

Note the *second important point* that for $MF_{observed} \ldots$

- The zero crossings occur at specific measured values of f_f.

Thus, by comparing the zero crossing numbers for MF_{theory} and $MF_{observed}$, you get the radius R.

12. Reference Reading

The appropriate reference for our purposes is the article *Interferometry and Aperture Synthesis*, which is chapter 10 of the book *Galactic and Extragalactic Radio Astronomy, First Edition.* The authors are Fomalont and Wright; Melvyn Wright is a research scientist in our radio lab here at Berkeley and is a real expert. This chapter is excellent, providing the basics without excessive detail (although it has more than we need). If you want more depth than we provide here, we suggest the following sections of this chapter: **(1)** §10.1.3, which describes a two-element interferometer; section *e* of this chapter is on polarization and you can skip it; **(2)** §10.2.1 and §10.2.2, which describe "a working interferometer"; and **(3)** Appendix II, which describes the geometrical details.

There is a scaling mistake in their equation for the fringe frequency ν_f: their equation needs to be multiplied by the rotation rate of the Earth in radians per second. For example, for an east-west baseline of 343.8 wavelengths looking at declination zero on the meridian, the fringe frequency on the sky is 343.8 fringes per radian. This means that the fringe spacing on the sky is $\frac{1}{343.8}$ radians or 10 arcmin; it takes the Earth 40 seconds of time to turn through 10 arcmin, so the fringe period in this case is 40 seconds and $\nu_f = .025$ Hz. More generally, for this east-west interferometer the fringe frequency is $\nu_f = .025 \cos\delta \cos h$ Hz, where δ is the declination and h the hour angle.

The book *Interferometry and Synthesis in Radio Astronomy* by Thompson, Moran, and Swenson goes into even more detail, but such detail is more than we want and can be overwhelming. Our main interest is the geometry—how the baseline projects on the *uv* plane. This is in chapter 4, and the most important sections are chapter 4.2 and 4.4.

With the longer baselines of research-class interferometers/arrays comes increased angular resolution, and all of our sources become finite in angular size, which means that the telescope arrays can map the sources using Fourier techniques. Some types of source are so small that mapping them requires interferometers with baseline lengths comparable to the Earth's diameter, a technique called "Very Long Baseline Interferometry" (VLBI). And some sources, such as pulsars, are even too small for VLBI!