1. Let x,y both in U and V;

   1.0 is both in U and V;

   2.x+y in U; x+y in V ➔ x+y both in U and V;

   3.c is scalar in R, cx in U and cx in y ,so cx in both U and V.

2.

Let U and V be subspaces of the vector space $\mathbb{R}^n$.
If neither U nor V is a subset of the other, then prove that the union $U \cup V$ is not a subspace of $\mathbb{R}^n$.

Since U is not contained in V, there exists a vector $\mathbf{u} \in U$ but $\mathbf{u} \notin V$.
Similarly, since V is not contained in U, there exists a vector $\mathbf{v} \in V$ but $\mathbf{v} \notin U$.

Seeking a contradiction, let us assume that the union is $U \cup V$ is a subspace of $\mathbb{R}^n$.
The vectors $\mathbf{u}, \mathbf{v}$ lie in the vector space $U \cup V$.
Thus their sum $\mathbf{u} + \mathbf{v}$ is also in $U \cup V$.
This implies that we have either

$$\mathbf{u} + \mathbf{v} \in U \text{ or } \mathbf{u} + \mathbf{v} \in V.$$

If $\mathbf{u} + \mathbf{v} \in U$, then there exists $\mathbf{u}' \in U$ such that

$$\mathbf{u} + \mathbf{v} = \mathbf{u}'.$$

Since the vectors $\mathbf{u}$ and $\mathbf{u}'$ are both in the subspace U, their difference $\mathbf{u}' - \mathbf{u}$ is also in U. Hence we have

$$\mathbf{v} = \mathbf{u}' - \mathbf{u} \in U.$$

However, this contradicts the choice of the vector $\mathbf{v} \notin U$.

Thus, we must have $\mathbf{u} + \mathbf{v} \in V$.
In this case, there exists $\mathbf{v}' \in V$ such that

$$\mathbf{u} + \mathbf{v} = \mathbf{v}'.$$

Since both $\mathbf{v}, \mathbf{v}'$ are vectors of V, it follows that

$$\mathbf{u} = \mathbf{v}' - \mathbf{v} \in V,$$

which contradicts the choice of $\mathbf{u} \notin V$.

Therefore, we have reached a contradiction. Thus, the union $U \cup V$ cannot be a subspace of $\mathbb{R}^n$.

Problem. Let W_1, W_2 be subspaces of a vector space V. Then prove that $W_1 \cup W_2$ is a subspace of V if and only if $W_1 \subset W_2$ or $W_2 \subset W_1$.

3.

We prove the following subspace criteria:

1. The zero vector $\mathbf{0}$ of V is in $W_1 + W_2$.
2. For any $\mathbf{u}, \mathbf{v} \in W_1 + W_2$, we have $\mathbf{u} + \mathbf{v} \in W_1 + W_2$.
3. For any $\mathbf{v} \in W_1 + W_2$ and $r \in K$, we have $r\mathbf{v} \in W_1 + W_2$.

Since W_1 and W_2 are subspaces of V, the zero vector $\mathbf{0}$ of V is in both W_1 and W_2.
Thus we have

$$\mathbf{0} = \mathbf{0} + \mathbf{0} \in W_1 + W_2.$$

for some $\mathbf{x} \in W_1$ and $\mathbf{y} \in W_2$.
Similarly, we write

$$\mathbf{v} = \mathbf{x}' + \mathbf{y}'$$

for some $\mathbf{x}' \in W_1$ and $\mathbf{y}' \in W_2$.

Then we have

$$\begin{aligned}\mathbf{u} + \mathbf{v} &= (\mathbf{x} + \mathbf{y}) + (\mathbf{x}' + \mathbf{y}') \\ &= (\mathbf{x} + \mathbf{x}') + (\mathbf{y} + \mathbf{y}').\end{aligned}$$

Since $\mathbf{x}$ and $\mathbf{x}'$ are both in the vector space W_1, their sum $\mathbf{x} + \mathbf{x}'$ is also in W_1.
Similarly we have $\mathbf{y} + \mathbf{y}' \in W_2$ since $\mathbf{y}, \mathbf{y}' \in W_2$.

Thus from the expression above, we see that

$$\mathbf{u} + \mathbf{v} \in W_1 + W_2,$$

hence condition 2 is met.

Finally, let $\mathbf{v} \in W_1 + W_2$ and $r \in K$.
Then there exist $\mathbf{x} \in W_1$ and $\mathbf{y} \in W_2$ such that

$$\mathbf{v} = \mathbf{x} + \mathbf{y}.$$

Since W_1 is a subspace, it is closed under scalar multiplication. Hence we have $r\mathbf{x} \in W_1$.
Similarly, we have $r\mathbf{y} \in W_2$.

It follows from this observation that

$$\begin{aligned}r\mathbf{v} &= r(\mathbf{x} + \mathbf{y}) \\ &= r\mathbf{x} + r\mathbf{y} \in W_1 + W_2,\end{aligned}$$

and thus condition 3 is met.

Therefore, by the subspace criteria $W_1 + W_2$ is a subspace of V.

4.

Proof. $\dim(W_1 \cap W_2) \le \dim(V)$
$\Rightarrow W_1 \cap W_2$ has a finite basis $\beta = \{u_1, u_2, \cdots, u_k\}$.
We can extend β to a basis $\beta_1 = \{u_1, u_2, \cdots, u_k, v_1, v_2, \cdots, v_m\}$ for W_1 and to a basis $\beta_2 = \{u_1, u_2, \cdots, u_k, k_1, k_2, \cdots, k_p\}$ for W_2.
Let $\alpha = \{u_1, u_2, \cdots, u_k, v_1, v_2, \cdots, v_m, w_1, w_2, \cdots, w_p\}$.
We claim that α is a basis for $W_1 + W_2$.

To prove the claim, we need to check that

1. α is linearly independent.

Let $a_1u_1 + \cdots + a_ku_k + b_1v_1 + \cdots + b_mv_m + c_1w_1 + \cdots + c_pw_p = 0$, for some scalars $a_1, \cdots, a_k, b_1, \cdots, b_m, c_1, \cdots, c_p$.
Then $(-b_1)v_1 + \cdots + (-b_m)v_m = a_1u_1 + \cdots + a_ku_k + c_1w_1 + \cdots + c_pw_p \in W_1 \cap W_2$.
Since β is a basis for $W_1 \cap W_2$, we have $(-b_1)v_1 + \cdots + (-b_m)v_m = d_1u_1 + \cdots + d_ku_k$ for some scalars $d_1, \cdots, d_k$.
$\Rightarrow d_1u_1 + \cdots + d_ku_k + b_1v_1 + \cdots + b_mv_m = 0$
$\Rightarrow d_1 = \cdots = d_k = b_1 = \cdots = b_m = 0$, since β_1 is a basis for W_1.
$\Rightarrow a_1u_1 + \cdots + a_ku_k + c_1w_1 + \cdots + c_pw_p = 0$
$\Rightarrow a_1 = \cdots = a_k = c_1 = \cdots = c_p = 0$, since β_2 is a basis for W_2.
Hence α is linearly independent.

2. $W_1 + W_2 = \mathrm{span}(\alpha)$.

Let $u = v + w \in W_1 + W_2$, where $v \in W_1$ and $w \in W_2$, be any vector in $W_1 + W_2$.
Since β_1 is a basis for W_1 and β_2 is a basis for W_2, we can find some scalars $x_1, \cdots, x_k, y_1, \cdots, y_m, z_1, \cdots, z_k$ such that

$$\begin{aligned} u &= (x_1u_1 + \cdots + x_ku_k + y_1v_1 + \cdots y_mv_m) + (z_1u_1 + \cdots + z_ku_k + t_1w_1 + \cdots + t_pw_p) \\ &= \big((x_1 + z_1)u_1 + \cdots + (x_k + z_k)u_k + y_1v_1 + \cdots y_mv_m + t_1w_1 + \cdots + t_pw_p\big) \end{aligned}$$

That is, $W_1 + W_2 \subseteq \mathrm{span}(\alpha)$.
It is easy to see that $\mathrm{span}(\alpha) \subseteq W_1 + W_2$.
Hence $W_1 + W_2 = \mathrm{span}(\alpha)$.
Therefore, α is a basis for $W_1 + W_2$.

Finally, we have

$$\dim(W_1 + W_2) = k + m + p = (k + m) + (k + p) - k = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2).$$

$\square$