Midterm

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1.True or false (20%)

a.) If an elementary row operation is applied to a matrix that is in row echelon form, the resulting matrix will still be in row echelon form. F

b.) If a linear system has more unknowns than equations, then it must have infinitely many solutions. F

c.) An expression of the invertible matrix A as a product of elementary matrices is unique. F

d.) If A and B are row equivalent matrices, then the linear systems $\mathbf{Ax} = \mathbf{0}$ and $\mathbf{Bx} = \mathbf{0}$ have the same solution set. T

e.) If A and B are $n * n$ square matrices, then $\det(A^T B) = \det(B) / \det(A^{-1})$.T

f.) If A is a square matrix, then $(A + A^T)$ is a symmetric matrix. T

g.) For all vectors u, v and w in R^n, we have

$$\|u + v + w\| \leq \|u\| + \|v\| + \|w\| \quad T$$

h.) Let A and B be 2×2 invertible matrices. If $AB = B$ then $A = I$. T

i.) Let A is an $m \times m$ matrix and $n > m$. If $Ax = b$ is consistent then x has infinite number of solutions. T

j.) If u, v and w are vectors in R^3, where u is nonzero and $u \times v = u \times w$, then $v = w$. F

2.(10%)

Determine conditions on the $b_i{'s}$ in order to guarantee that the linear system is consistent.

$$x - 2y + 5z = b_1$$
$$4x - 5y + 8z = b_2$$
$$-3x + 3y - 3z = b_3$$

$$\begin{bmatrix} 1 & -2 & 5 & b_1 \\ 4 & -5 & 8 & b_2 \\ -3 & 3 & -3 & b_3 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 5 & b_1 \\ 0 & 3 & -12 & b_2 - 4b_1 \\ 0 & -3 & 12 & b_3 + 3b_1 \end{bmatrix} =$$

$$\begin{bmatrix} 1 & -2 & 5 & b_1 \\ 0 & 3 & -12 & b_2 - 4b_1 \\ 0 & 0 & 0 & b_2 + b_3 - b_1 \end{bmatrix}$$

We observe the above formula, when $b_2 + b_3 - b_1 = 0$ *so we have infinite solution*

3.(10%)

Evaluate the determinant, given that $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = -$

a.) (5%) $\begin{vmatrix} -a & -b & -c \\ 2d & 2e & 2f \\ 5g & 5h & 5i \end{vmatrix}$

b.) (5%) $\begin{vmatrix} a & b & c \\ 2d & 2e & 2f \\ g+3a & h+3b & i+3c \end{vmatrix}$

a.) $\begin{vmatrix} -a & -b & -c \\ 2d & 2e & 2f \\ 5g & 5h & 5i \end{vmatrix} = (-1) \begin{vmatrix} a & b & c \\ 2d & 2e & 2f \\ 5g & 5h & 5i \end{vmatrix} = (-1) \times 2 \begin{vmatrix} a & b & c \\ d & e & f \\ 5g & 5h & 5i \end{vmatrix} =$

$(-1) \times 2 \times 5 \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = (-1) \times 2 \times 5 \times (-6) = 60$

b.) $\begin{vmatrix} a & b & c \\ 2d & 2e & 2f \\ g+3a & h+3b & i+3c \end{vmatrix} = 2 \begin{vmatrix} a & b & c \\ d & e & f \\ g+3a & h+3b & i+3c \end{vmatrix} = 2 \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} =$

$2 \times (-6) = -12$

4.(10%)

Consider the following 3 vectors in R^3 :

$$\mathbf{u} = (1,2,3), \mathbf{v} = (2,0,1), \mathbf{w} = (3,1,0)$$

a.)(5%)Find $\mathrm{proj}_{\mathrm{v}}(w)$

b.)(5%)Find $\mathrm{w} - \mathrm{proj}_{\mathrm{v}}(w)$

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a.) $\mathrm{u} \cdot \mathrm{v} = ||\mathrm{u}|| \times ||\mathrm{v}|| \times \cos\theta = \sqrt{1^2+2^2+3^2} \times \sqrt{2^2+0^2+1^2} \times \cos\theta$

$\cos\theta = \frac{5}{\sqrt{70}} = \frac{\sqrt{70}}{14} \rightarrow \theta = \cos^{-1}(\frac{\sqrt{70}}{14})$

b.) $\mathrm{proj}_{\mathrm{v}}(w) = \frac{v \cdot w}{||v^2||}(v) = \frac{2\times3+0\times1+1\times0}{2^2+0^2+1^2}(2,0,1) = \frac{6}{5}(2,0,1) = \left(\frac{12}{5}, 0, \frac{6}{5}\right)$

c.) $\mathrm{w} - \mathrm{proj}_{\mathrm{v}}(w) = (3,1,0) - \left(\frac{12}{5}, 0, \frac{6}{5}\right) = (\frac{3}{5}, 1, -\frac{6}{5})$

5.(10%)

Consider the following 3 points in R^3 :

$$(1,2,3), (4,0,2), (3,5,0)$$

Find an equation of the plane P that contains the given 3 points. (Hint : Find the normal of plane P first.)

Let $A=(1,2,3)$ $B=(4,0,2)$ $C=(3,5,0)$

$\overrightarrow{AB}=(3,-2,-1)$ $\overrightarrow{AC}=(2,3,-3)\rightarrow\overrightarrow{AB}\times\overrightarrow{AC}=(9,7,13)$ normal vector

$9(x-1)+7(y-2)+13(z-3)=0\rightarrow 9x+7y+13z=62$

or $9(x-4)+7(y-0)+13(z-2)=0$ or $9(x-3)+7(y-5)+13(z-0)=0$

6.(10%)

Calculate the inverse of the following matrix :

$$A=\begin{pmatrix}1 & -1 & 0 & 1\\ 0 & 1 & 1 & 1\\ -1 & 0 & 1 & 0\\ -1 & 1 & 1 & 1\end{pmatrix}$$

$$\left(\begin{array}{cccc|cccc}1 & -1 & 0 & 1 & 1 & 0 & 0 & 0\\ 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0\\ -1 & 0 & 1 & 0 & 0 & 0 & 1 & 0\\ -1 & 1 & 1 & 1 & 0 & 0 & 0 & 1\end{array}\right)\rightarrow \textit{row operation}\rightarrow$$

$$\left(\begin{array}{cccc|cccc}1 & 0 & 0 & 0 & 0 & 1 & 0 & -1\\ 0 & 1 & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & 0\\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & -1\\ 0 & 0 & 0 & 1 & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 1\end{array}\right)$$

7.(10%)

Let A be a $3*3$ invertible matrix. Let the reduced row-echelon form of A be obtained from A by the following sequential row operations:

1.) Swap row 1 and 3.
2.) Add -5 times row 3 to row 2.
3.) Add 2 times row 2 to row 1.
4.) Multiply row 2 by -2.

a.) (6%) Write A^{-1} as the product of elementary matrices.

b.) (4%) Find matrix A.

a.)$\because$ A be a $3*3$ $\therefore$reduced row-echelon form of $A=I_{3\times3}$

$$1.),2.),3.),4.)\ \rightarrow E$$

$$EA=I_{3\times3}\rightarrow E=A^{-1}=\begin{bmatrix}1 & 0 & 0\\ 0 & -2 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 2 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & -5\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}0 & 0 & 1\\ 0 & 1 & 0\\ 1 & 0 & 0\end{bmatrix}$$

$$\text{b.) } E^{-1}=A=\left(\begin{bmatrix}1 & 0 & 0\\ 0 & -2 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 2 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & -5\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}0 & 0 & 1\\ 0 & 1 & 0\\ 1 & 0 & 0\end{bmatrix}\right)^{-1}=$$

$$\begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&5\\0&0&1\end{bmatrix}\begin{bmatrix}1&-2&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&-\frac{1}{2}&0\\0&0&1\end{bmatrix}=\begin{bmatrix}0&0&1\\0&-\frac{1}{2}&5\\1&1&0\end{bmatrix}$$

8.(10%) Prove if A is an $\mathrm{n}\times\mathrm{n}$ matrix, then

$$\det(\mathrm{adj}(\mathrm{A}))=(\det(A))^{n-1}$$

Since $\mathrm{A}^{-1}=\frac{\mathrm{adj}(\mathrm{A})}{det(A)}\rightarrow\mathrm{adj}(\mathrm{A})=\mathrm{A}^{-1}det(A)\rightarrow\det(\mathrm{adj}(\mathrm{A}))=\det(\mathrm{A}^{-1}det(A))\rightarrow$

$\det(\mathrm{adj}(\mathrm{A}))=\det(\mathrm{A}^{-1})\times\det(det(A))=\det(\mathrm{A}^{-1})\times(\det(A))^{n}=(\det(A))^{n-1}$

9 (10%)Show that there do not exist scalars c_1,c_2 and c_3 such that

$$\mathrm{c}_1(-2,9,6)+c_2(-3,2,1)+c_3(1,7,5)=(0,5,4)$$

$$\left(\begin{array}{ccc|c}-2&-3&1&0\\9&2&7&5\\6&1&5&4\end{array}\right)\rightarrow\text{row operation}\rightarrow\left(\begin{array}{ccc|c}-2&-3&1&0\\0&0&0&\frac{43}{4}\\0&-1&1&-\frac{1}{2}\end{array}\right)$$

$\because \mathrm{C}_1 0+\mathrm{C}_2 0+C_3 0=\frac{43}{4}\rightarrow 0=\frac{43}{4}\,(contradiction)$

$\therefore\ \mathrm{c}_1,c_2$ and c_3 don't exist