1. (20 points) Each correct answer will gain two point and each incorrect answer will deduct one point.

F (a) The union of any two subspaces of a vector space $V$ is a subspace of $V$.

T (b) The span of two vectors in $R^3$ is a plane.

F (c) Let $v_1, v_2, v_3$ be 3 vectors in a vector space. If $v_1, v_2, v_3$ are linearly dependent, then $\text{span}\{v_1, v_2\} = \text{span}\{v_1, v_2, v_3\}$.

F (d) There are only three distinct two-dimensional subspaces of $R^3$.

F (e) If $\text{rank}(A^T) = \text{rank}(A)$, then $A$ is a square matrix.

T (f) The nullity of a square matrix with linearly dependent rows is at least one.

T (g) If the characteristic polynomial of a $4 \times 4$ matrix $A$ is $p(\lambda) = -\lambda(1-\lambda)^2(3-\lambda)$, then $\text{rank}(A) = 3$.

F (h) Distinct eigenvectors are linearly independent.

T (i) There is an inner product space which has a unique unit vector.

F (j) If $A$ is an $n \times n$ matrix with nonzero diagonal elements, $A$ has a $QR$-decomposition.

2. (10 points) Let $V = R^2$, and define addition and scalar multiplication as follows:

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F (j) If $A$ is an $n \times n$ matrix with nonzero diagonal elements, $A$ has a $QR$-decomposition. $= n$

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2. (10 points) Let $V = R^2$, and define addition and scalar multiplication as follows:

$$\mathbf{u} + \mathbf{v} = (u_1, u_2) + (v_1, v_2) = (u_1 v_1, u_2 v_2)$$
$$k\mathbf{u} = k(u_1, u_2) = (u_1^k, u_2^k)$$

Given that $V$ is a vector space,

(a) (3 points) determine the value of $\mathbf{0}$ in $V$. $(1, 1)$

(b) (2 points) verify that $k\mathbf{0} = \mathbf{0}$ holds.

(c) (3 points) let $\mathbf{u} = (u_1, u_2)$ be in $V$. Find the value of $-\mathbf{u}$ such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$.

(d) (2 points) verify that $(-1)\mathbf{u} = -\mathbf{u}$ holds.

(a).
$u + 0 = u$.
let $0 = (0_1, 0_2)$.
$(u_1, u_2) + (0_1, 0_2)$
$= (u_1 \cdot 0_1, u_2 \cdot 0_2) = (u_1, u_2)$
$\therefore (0_1, 0_2) = (1, 1)$ #

(b). $k \cdot (1, 1) = (1, 1)$
$\Rightarrow k \cdot (1, 1) = (1^k, 1^k)$
$= (1, 1)$ #

(c) $u + (-u) = 0$
let $-u = (x_1, x_2)$.
$(u_1, u_2) \cdot (x_1, x_2) = (1, 1)$
$\Rightarrow (u_1 \cdot x_1, u_2 \cdot x_2) = (1, 1)$
$\therefore x_1 = \frac{1}{u_1}, \ x_2 = \frac{1}{u_2}$
$-u = (\frac{1}{u_1}, \frac{1}{u_2})$ #

(d) $-1(u) = (u_1^{-1}, u_2^{-1})$
$= (\frac{1}{u_1}, \frac{1}{u_2}) = -u$. #
得證

3. (10 points) Prove that if $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a linearly independent set of vectors in some vector space $V$, then

$$\{\mathbf{v}_1 + \mathbf{v}_2, \mathbf{v}_1 - \mathbf{v}_2, \mathbf{v}_1 - 2\mathbf{v}_2 + \mathbf{v}_3\}$$

is also a linearly independent set in $V$.

Since $\{v_1, v_2, v_3\}$ is LI. so $av_1 + bv_2 + cv_3 = 0$ iff $a = b = c = 0$.

the $\{v_1 + v_2, v_1 - v_2, v_1 - 2v_2 + v_3\}$

let $c_1(v_1 + v_2) + c_2(v_1 - v_2) + c_3(v_1 - 2v_2 + v_3) = 0$.

$\Rightarrow c_1 v_1 + c_1 v_2 + c_2 v_1 - c_2 v_2 + c_3 v_1 - 2c_3 v_2 + c_3 v_3 = 0$.

$\Rightarrow (c_1 + c_2 + c_3) v_1 + (c_1 - c_2 - 2c_3) v_2 + (c_3) v_3 = 0$ $\because \{v_1, v_2, v_3\}$ is LI

$\therefore \begin{cases} c_1 + c_2 + c_3 = 0 \\ c_1 - c_2 - 2c_3 = 0 \\ c_3 = 0 \end{cases} \Rightarrow \begin{cases} c_1 = 0 \\ c_2 = 0 \\ c_3 = 0 \end{cases} \Rightarrow$ LI. 得證. #

4. (10 points) In each part, find a basis for the given subspace of $M_{22}$, and state its dimension.
   (a) (5 points) The subspace of all symmetric matrices.
   (b) (5 points) The subspace of all matrices in which the sum of the diagonal elements

4. (10 points) In each part, find a basis for the given subspace of $M_{22}$, and state its dimension.
   (a) (5 points) The subspace of all symmetric matrices.
   (b) (5 points) The subspace of all matrices in which the sum of the diagonal elements are 0.

(a) $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\}$ dim = 3. #

(b). $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right\}$ dim = 3. #

5. (10 points) Let $V$ be the subspace of $M_{22}$ spanned by

$$\left\{ \begin{bmatrix} 1 & 5 \\ -3 & -11 \end{bmatrix}, \begin{bmatrix} 7 & 4 \\ -1 & 2 \end{bmatrix}, \begin{bmatrix} 5 & 1 \\ 9 & 2 \end{bmatrix}, \begin{bmatrix} 3 & -1 \\ 7 & 5 \end{bmatrix} \right\}$$

(a) (5 points) Find a basis $S$ for $V$.

(b) (5 points) Find the coordinate vector of $\begin{bmatrix} 19 & 18 \\ -13 & -10 \end{bmatrix}$ relative to the basis $S$ you obtained in part (a).

(a) check LI ⇒

$$\begin{bmatrix} 1 & 5 & -3 & -11 \\ 7 & 4 & -1 & 2 \\ 5 & 1 & 9 & 2 \\ 3 & -1 & 7 & 5 \end{bmatrix} \begin{matrix} -7 \\ -5 \\ -3 \end{matrix} \sim \begin{bmatrix} 1 & 5 & -3 & -11 \\ 0 & -31 & 20 & 79 \\ 0 & -24 & 24 & 57 \\ 0 & -16 & 16 & 38 \end{bmatrix} \sim$$

$$\begin{bmatrix} 1 & 5 & -3 & -11 \\ 0 & -31 & 20 & 79 \\ 0 & -1 & 1 & 57/24 \\ 0 & -1 & 1 & 38/16 \end{bmatrix} \sim \begin{bmatrix} 1 & 5 & -3 & -11 \\ 0 & -31 & 20 & 79 \\ 0 & -1 & 1 & 19/8 \\ 0 & -1 & 1 & 19/8 \end{bmatrix} \sim \begin{bmatrix} 1 & 5 & -3 & -11 \\ 0 & -31 & 20 & 79 \\ 0 & -1 & 1 & 19/8 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

So, we know basis is $\left\{ \begin{bmatrix} 1 & 5 \\ -3 & -11 \end{bmatrix}, \begin{bmatrix} 7 & 4 \\ -1 & 2 \end{bmatrix}, \begin{bmatrix} 5 & 1 \\ 9 & 2 \end{bmatrix} \right\}$. #.

(b).

$$a\begin{bmatrix} 1 & 5 \\ -3 & -11 \end{bmatrix} + b\begin{bmatrix} 7 & 4 \\ -1 & 2 \end{bmatrix} + c\begin{bmatrix} 5 & 1 \\ 9 & 2 \end{bmatrix} = \begin{bmatrix} 19 & 18 \\ -13 & -10 \end{bmatrix}$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 7 & 5 & 19 \\ 5 & 4 & 1 & 18 \\ -3 & -1 & 9 & -13 \\ -11 & 2 & 2 & -10 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 7 & 5 & 19 \\ 0 & -31 & -24 & -77 \\ 0 & 20 & 24 & 44 \\ 0 & 79 & 57 & 199 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 7 & 5 & 19 \\ 0 & -31 & -24 & -77 \\ 0 & -11 & 0 & -33 \\ 0 & 79 & 57 & * \end{array}\right]$$

$\therefore -11b = -33 \Rightarrow b = 3$.

$-31 \times 3 - 24c = -77 \Rightarrow c = \frac{-16}{-24} = \frac{2}{3} \Rightarrow a + 21 - \frac{10}{3} = 19 \Rightarrow a = \frac{4}{3}$

$$\begin{pmatrix} \frac{4}{3} \\ 3 \\ \frac{2}{3} \end{pmatrix}$$ #.

6. (10 points) Let $P_3$ be the vector space of all polynomials $p(x)$ of degree at most 3. Let $W$ be the subspace of $P_3$ consisting of those polynomials satisfying the condition:

$$p(0) = p(-1) = 0.$$

Find a basis of $W$.

$P(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$

$P(0) = 0 \Rightarrow a_0 = 0$

$P(-1) = 0 \Rightarrow a_0 - a_1 + a_2 - a_3 = 0$

$\Rightarrow a_2 = a_1 + a_3.$

$P(t) = a_1 t + (a_1 + a_3) t^2 + a_3 t^3$

$= a_1 t + a_1 t^2 + a_3 t^2 + a_3 t^3$

$= a_1 (t + t^2) + a_3 (t^2 + t^3).$

$\because (t + t^2), (t^2 + t^3)$ are linearly independent.

$\Rightarrow \{(t+t^2), (t^2+t^3)\}$ is a basis of $W$.

7. (10 points) Consider the matrix

$$A = \begin{bmatrix} 1 & 1 & 17 \\ 0 & -1 & 101 \\ 0 & 0 & \sqrt{2} \end{bmatrix}$$

and let $B$ be a matrix similar to $A^2 + A^{10}$. Compute the determinants of $B$.

$\because$ A is ~~a~~ upper triangular matrix.

$\therefore$ det (A) we just need to consider the diagonal.

$\det(A) = -\sqrt{2}$.

$$A^2 = \begin{bmatrix} 1 & 1 & 17 \\ 0 & -1 & 101 \\ 0 & 0 & \sqrt{2} \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 17 \\ 0 & -1 & 101 \\ 0 & 0 & \sqrt{2} \end{bmatrix} = \begin{bmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 2 \end{bmatrix}$$

$$A^4 = A^2 \cdot A^2 = \begin{bmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 4 \end{bmatrix}, \quad A^8 = A^4 \cdot A^4 = \begin{bmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 16 \end{bmatrix}$$

$$A^{10} = A^2 \cdot A^8 = \begin{bmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 16 \end{bmatrix} = \begin{bmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 32 \end{bmatrix}$$

$$B = P(A^2 + A^{10})P^{-1} \Rightarrow \det(B) = \cancel{\det(P)} \cdot \det(A^2 + A^{10}) \cdot \cancel{\det(P^{-1})}$$

$$= \det(A^2 + A^{10}).$$

$$= \det\left(\begin{bmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 2 \end{bmatrix} + \begin{bmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 32 \end{bmatrix}\right)$$

$$= \det\left(\begin{bmatrix} 2 & * & * \\ 0 & 2 & * \\ 0 & 0 & 34 \end{bmatrix}\right)$$

$$= \underline{136} \#$$

8. (10 points) Let $A$ be a $2 \times 2$ matrix such that $\begin{bmatrix}1\\1\end{bmatrix}$ is an eigenvector of $A$ with eigenvalue 2 and $\begin{bmatrix}2\\3\end{bmatrix}$ is an eigenvector of $A$ with eigenvalue 1. If $\mathbf{v} = \begin{bmatrix}3\\4\end{bmatrix}$,

(a) (4 points) compute $A\mathbf{v}$

(b) (6 points) compute $A^3\mathbf{v}$

(a) $v = \begin{bmatrix}3\\4\end{bmatrix} = \begin{bmatrix}1\\1\end{bmatrix} + \begin{bmatrix}2\\3\end{bmatrix}$

$\therefore A v = A\left(\begin{bmatrix}1\\1\end{bmatrix} + \begin{bmatrix}2\\3\end{bmatrix}\right).$

$= A\begin{bmatrix}1\\1\end{bmatrix} + A\begin{bmatrix}2\\3\end{bmatrix}.$ $\boxed{Au = \lambda u}$

$= \lambda_1 \begin{bmatrix}1\\1\end{bmatrix} + \lambda_2 \begin{bmatrix}2\\3\end{bmatrix}$

$= \begin{bmatrix}2\\2\end{bmatrix} + \begin{bmatrix}2\\3\end{bmatrix} = \begin{bmatrix}4\\5\end{bmatrix}.$ #

(b). $\because Au = \lambda u \Rightarrow A^2 u = A \cdot (Au) = A(\lambda u) = \lambda(Au) = \lambda^2 u.$

$\therefore A^3(v) = A^3\left(\begin{bmatrix}1\\1\end{bmatrix} + \begin{bmatrix}2\\3\end{bmatrix}\right) = A^3\begin{bmatrix}1\\1\end{bmatrix} + A^3\begin{bmatrix}2\\3\end{bmatrix}$

$= \lambda_1^3 \begin{bmatrix}1\\1\end{bmatrix} + \lambda_2^3 \begin{bmatrix}2\\3\end{bmatrix}$

$= \begin{bmatrix}8\\8\end{bmatrix} + \begin{bmatrix}2\\3\end{bmatrix} = \begin{bmatrix}10\\11\end{bmatrix}$ #

9. (10 points) Let $W \subset R^4$ be the subspace of vectors $(x_1, x_2, x_3, x_4)^T$ satisfying

$$2x_1 - x_3 + x_4 = 0.$$

Find an orthonormal basis of $W$.

$[2 \quad 0 \quad -1 \quad 1]$

$[1 \quad 0 \quad -\frac{1}{2} \quad \frac{1}{2}]$

we can easily find basis.

$$\left\{ \begin{pmatrix} 0\\1\\0\\0 \end{pmatrix}, \begin{pmatrix} 1\\0\\2\\0 \end{pmatrix}, \begin{pmatrix} -1\\0\\0\\2 \end{pmatrix} \right\}$$ (⇒ 不一定要是這了.)

let $(u_1, u_2, u_3)$.

$$v_1 = u_1 = \begin{pmatrix} 0\\1\\0\\0 \end{pmatrix}.$$

$$v_2 = u_2 - \frac{u_2 \cdot v_1}{\|v_1\|^2} \cdot v_1 = \begin{pmatrix} 1\\0\\2\\0 \end{pmatrix} - \frac{0}{1} v_1 = \begin{pmatrix} 1\\0\\2\\0 \end{pmatrix} \Rightarrow \frac{1}{\sqrt{5}} \begin{pmatrix} 1\\0\\2\\0 \end{pmatrix}$$

$$v_3 = u_3 - \frac{u_3 \cdot v_1}{\|v_1\|^2} \cdot v_1 - \frac{u_3 \cdot v_2}{\|v_2\|^2} v_2 \qquad u_3 \cdot v_2 = -1$$

$$= \begin{pmatrix} -1\\0\\0\\2 \end{pmatrix} - \frac{-1}{(\sqrt{5})^2} \begin{pmatrix} 1\\0\\2\\0 \end{pmatrix} = \begin{pmatrix} -1\\0\\0\\2 \end{pmatrix} + \begin{pmatrix} \frac{1}{5}\\0\\\frac{2}{5}\\0 \end{pmatrix} = \begin{pmatrix} -\frac{4}{5}\\0\\\frac{2}{5}\\2 \end{pmatrix} \Rightarrow \frac{2}{5} \begin{pmatrix} -2\\0\\1\\5 \end{pmatrix} = \frac{2}{5\sqrt{30}} \begin{pmatrix} -2\\0\\1\\5 \end{pmatrix}$$

#