Vibration Hw 1

Brandon Ho, Justin Hoey, Yongjun Kim

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Question 1: EOM & Frequency of Vibration

Problem Statement: Derive the equation of motion of the following systems and their frequency of vibration.

(a) Simple Pendulum
The equation for rotational motion is listed below.

$$\sum T = I\ddot{\theta} \tag{1}$$

where T is the net torque, I is the moment of inertia of the pendulum ($I = ml^2$), m is mass of the pendulum bob, l is the length of the pendulum, and $\ddot{\theta}$ is the angle double derivative. For this problem, the mass of the pendulum rod is ignored, and the pendulum bob is considered as a point particle. Here is the FBD in figure 1,

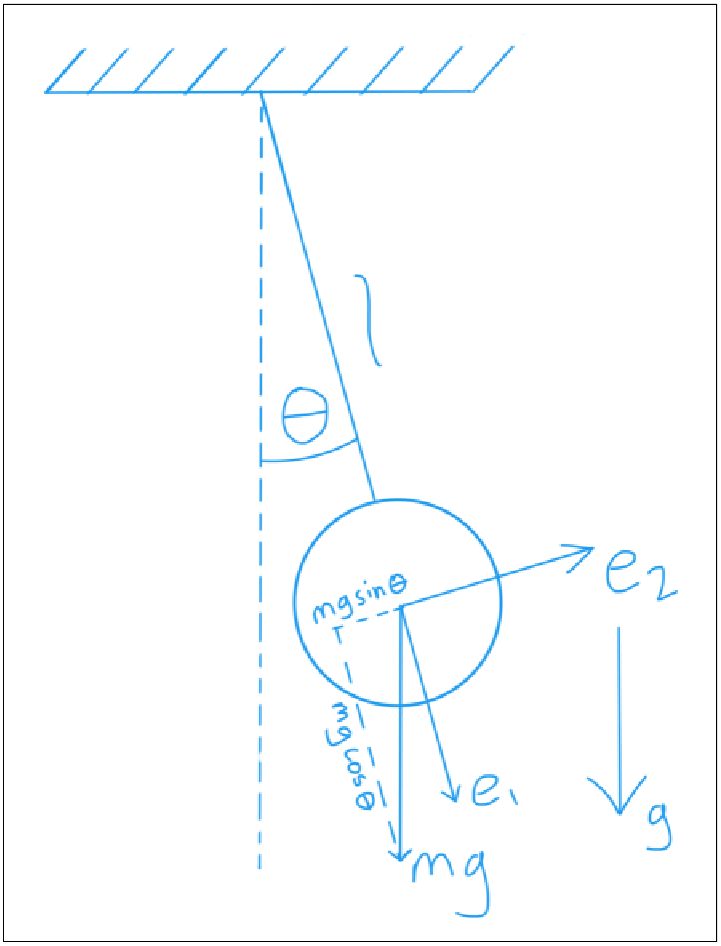

Figure 1: FBD of the simple Pendulum

The only torque and thus net torque applied on this simple pendulum is the torque due to gravity, which is shown in equation below.

$$\sum T = -mgl\sin(\theta) \tag{2}$$

For harmonic motion, θ will be approximately small, so $\sin\theta \approx \theta$. Then bringing equation (2) over to equation (1), the new equation will be

$$I\ddot{\theta} + mgl\theta = 0 \tag{3}$$

Now, the new equation is a second order linear ODE (ordinary differential equation). Using the guess solution and its derivatives,

$$\theta(t) = Ce^{\lambda t} \tag{4}$$

$$\dot{\theta}(t) = \lambda Ce^{\lambda t}$$

$$\ddot{\theta}(t) = \lambda^2 Ce^{\lambda t}$$

where λ and C are constants. The new equation with the guess solution will now be

$$I\lambda^2 Ce^{\lambda t} + mglCe^{\lambda t} = 0 \tag{5}$$

The following steps below show how to find λ from equation (5):

$$I\lambda^2 Ce^{\lambda t} + mglCe^{\lambda t} = 0 \tag{6}$$

$$I\lambda^2 + mgl = 0 \tag{7}$$

$$\lambda^2 = \frac{-mgl}{I} \tag{8}$$

$$\lambda^2 = \frac{-g}{l} \tag{9}$$

$$\lambda = \pm j\sqrt{\frac{g}{l}} \quad \omega_n = \sqrt{\frac{g}{l}} \tag{10}$$

The frequency of vibration, ω_n, is $\sqrt{\frac{g}{l}}$. To increase the frequency of vibration, l must decrease, or g increases . With this relation between ω_n and l, one can measure time with a simple pendulum, such as a grandfather clock.

(b) Shaft and disk The equation for rotational motion is listed below.

$$\sum T = I\ddot{\theta} \tag{11}$$

where T is the net torque, I is the moment of inertia of the disk ($I = \frac{1}{2}mr^2$), m is the mass of the disk, r is the radius of the disk, and $\ddot{\theta}$ is the angle double derivative. For this problem, the mass of the shaft is ignored. Here is the FBD in figure2,

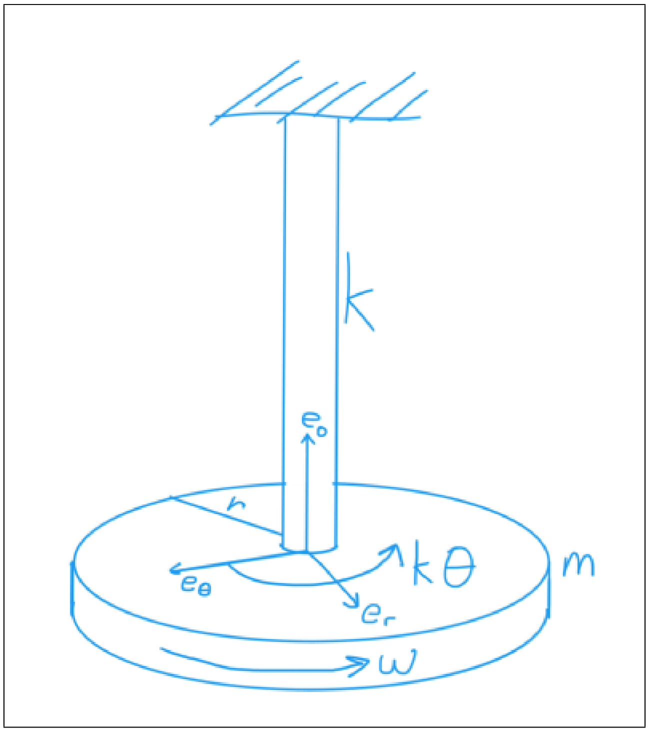

Figure 2: FBD of the Shaft & Disk

The only torque and thus net torque applied on the shaft and disk is the torque due to torsion, which is shown in equation below.

$$\sum T = k\theta \tag{12}$$

Then bringing equation (12) over to equation (11), the new equation will be

$$I\ddot{\theta} + k\theta = 0 \tag{13}$$

Now, the new equation is a second order linear ODE. Using the guess solution and its derivatives from equation (4), the new equation will now be

$$I\lambda^2 Ce^{\lambda t} + kCe^{\lambda t} = 0 \tag{14}$$

The following steps below show how to find λ from equation (14):

$$I\lambda^2 Ce^{\lambda t} + kCe^{\lambda t} = 0 \tag{15}$$

$$I\lambda^2 + k = 0 \tag{16}$$

$$\lambda^2 = \frac{-k}{I} \tag{17}$$

$$\lambda^2 = \frac{-2k}{mr^2} \tag{18}$$

$$\lambda = \pm j\sqrt{\frac{2k}{mr^2}} \quad \omega_n = \sqrt{\frac{2k}{mr^2}} \tag{19}$$

The frequency of vibration, ω_n, is $\sqrt{\frac{2k}{mr^2}}$. To increase ω_n, k must increase, or m or r must decrease. This seems reasonable because having a bigger m or r means a larger I or a larger inertia, lowering ω_n

(c) Inverted Pendulum The equation for rotational motion is listed below.

$$\sum T = I\ddot{\theta} \tag{20}$$

where T is the net torque, I is the moment of inertia of the disk ($I = ml^2$), m is the mass of the disk, l is the length of the inverted pendulum, and $\ddot{\theta}$ is the angle double derivative. For this problem, the mass of the pendulum rod is ignored, and the pendulum bob is considered as a point particle. On the FBD in figure3,

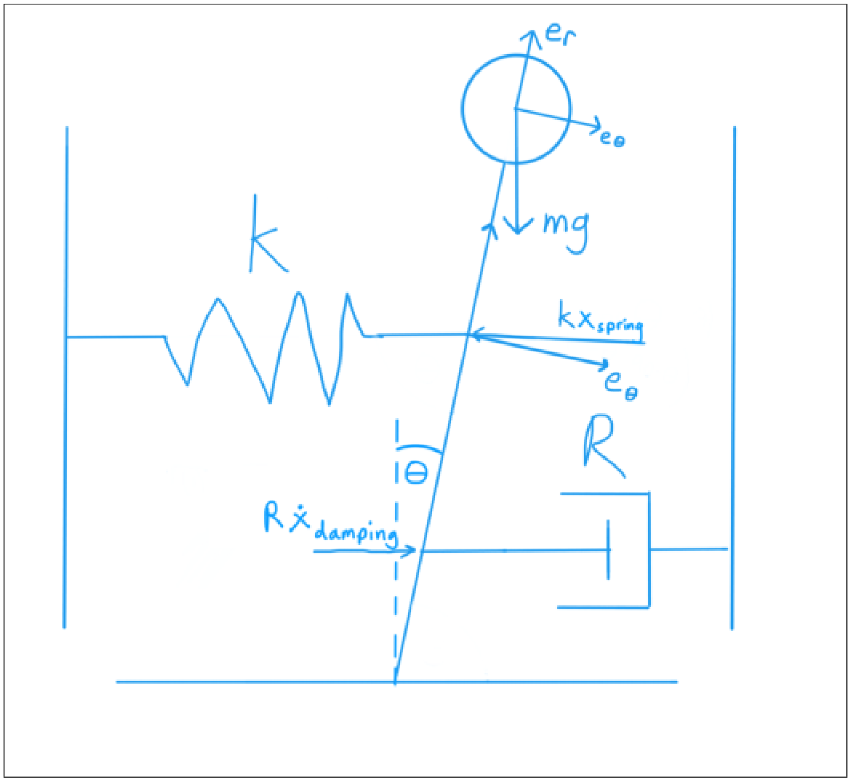

Figure 3: FBD of the Shaft & Disk

The only torques applied on the inverted pendulum is the torque due to damping force, spring force, and gravity.
The displaced spring distance is listed below:

$$x_{spring} = \frac{2l}{3}\sin(\theta) \tag{21}$$

The velocity that the damper feels is listed below:

$$\dot{x}_{damper} = \frac{d}{dt}[\frac{l}{3}\sin(\theta)] \tag{22}$$
$$= \frac{l}{3}\cos(\theta)\dot{\theta} \tag{23}$$

The new torque equation is shown below.

$$\sum T = -(kx_{spring})(\frac{2l}{3}) - (R\dot{x}_{damper})(\frac{l}{3}) + mgl\sin\theta \tag{24}$$

Then bringing equation (24) over to equation (20), the new equation will be

$$I\ddot{\theta} + (R\dot{x}_{damper})(\frac{l}{3}) + (kx_{spring})(\frac{2l}{3}) - mgl\sin\theta = 0 \tag{25}$$

For harmonic motion, θ will be approximately small, so $\sin\theta \approx \theta$ and $\cos(\theta) \approx 1$, the new equation will be the following:

$$I\ddot{\theta} + R(\frac{l}{3})^2\dot{\theta} + (k(\frac{2l}{3})^2 - mgl)\theta = 0 \tag{26}$$

Now, the new equation is a second order linear ODE. Using the guess solution and its derivatives from equation (4), the new equation will now be

$$I\lambda^2 Ce^{\lambda t} + [R(\frac{l}{3})^2]\lambda Ce^{\lambda t} + [k(\frac{2l}{3})^2 - mgl]Ce^{\lambda t} = 0 \tag{27}$$

The following step below shows how to find λ from equation(27):

$$I\lambda^2 Ce^{\lambda t} + [R(\frac{l}{3})^2)]\lambda Ce^{\lambda t} + [k(\frac{2l}{3})^2 - mgl]Ce^{\lambda t} = 0 \tag{28}$$

$$I\lambda^2 + (R(\frac{l}{3})^2))\lambda + (k(\frac{2l}{3})^2 - mgl) = 0 \tag{29}$$

$$\lambda = \frac{-R(\frac{l}{3})^2}{2I} \pm \sqrt{(\frac{R(\frac{l}{3})^2}{2I})^2 - 4I\frac{(k(\frac{2l}{3})^2 - mgl)}{4I^2}} \tag{30}$$

$$\lambda = \frac{-R(\frac{l}{3})^2}{2ml^2} \pm \sqrt{(\frac{R(\frac{l}{3})^2}{2ml^2})^2 - \frac{(k(\frac{2l}{3})^2) - mgl}{ml^2}} \tag{31}$$

$$\lambda = \frac{-R}{18m} \pm j\sqrt{\frac{4k}{9m} - \frac{g}{l} - (\frac{R}{18m})^2} \tag{32}$$

$$\tag{33}$$

The frequency of vibration, ω_n, is $\sqrt{\frac{4k}{9m} - \frac{g}{l}}$. To increase ω_n, l or k, must increase, or m , R, g must decrease. Since torque due to gravity points in the same direction as with angular displacement, that is why there is a negative sign in the square root.

Discussion: All the parts in this problem have a restoring force and inertia, which the frequency of vibration depends on. In the equations for frequency of vibration, the numerator is the restoring forces, and the denominator is the inertia. A larger restoring force means a larger frequency of vibration, and a larger inertia means a smaller frequency of vibration.

Question 2: Formula Front Suspension Vibration

(a) Vibration of Front Suspension
Problem Statement: compute the natural frequency of the assembly. The equation for rotational motion is listed below.

$$\sum T = I\ddot{\theta} \tag{34}$$

where T is the net torque, I is the moment of inertia of the disk ($I = mr^2$), m is the mass of the disk, r is the distance between the wheel and the axis of rotation, and $\ddot{\theta}$ is the angle double derivative. For this problem, the mass of the shaft is ignored, and the wheel is considered as a point particle. Here is the FBD in figure4,

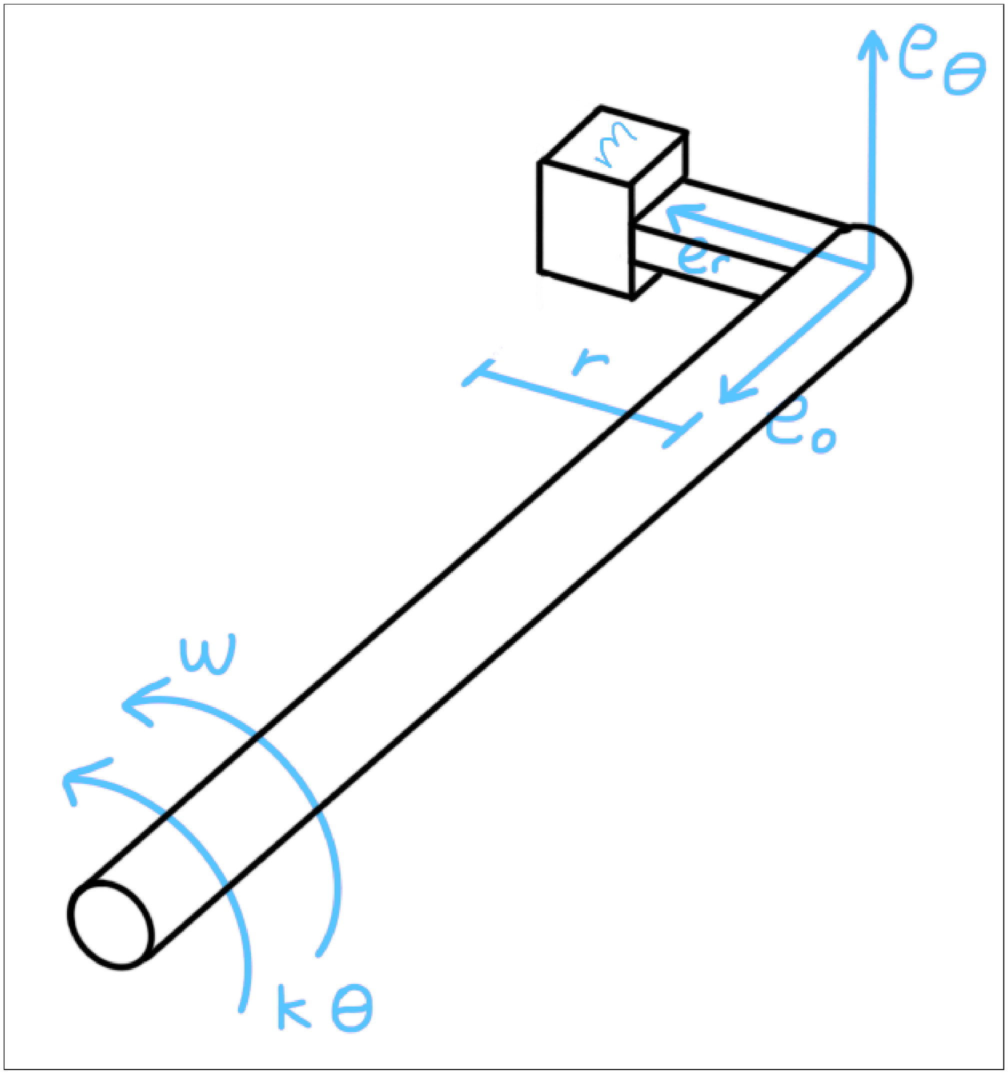

Figure 4: FBD of the Formula Wheel Assembly

The only torque and thus net torque applied on the assembly is the torque due to torsion, which is shown in equation below.

$$\sum T = k\theta \tag{35}$$

Then bringing equation (35) over to equation (34), the new equation will be

$$I\ddot{\theta} + k\theta = 0 \tag{36}$$

Now, the new equation is a second order linear ODE. Using the guess solution and its derivatives from equation (4), the new equation will now be

$$I\lambda^2 Ce^{\lambda t} + kCe^{\lambda t} = 0 \tag{37}$$

The following step below shows how to find λ from equation(37):

$$I\lambda^2 Ce^{\lambda t} + kCe^{\lambda t} = 0 \tag{38}$$

$$I\lambda^2 + k = 0 \tag{39}$$

$$\lambda^2 = \frac{-k}{I} \tag{40}$$

$$\lambda^2 = \frac{-k}{mr^2} \tag{41}$$

$$\lambda = \pm j\sqrt{\frac{k}{mr^2}} \quad \omega_n = \sqrt{\frac{k}{mr^2}} \tag{42}$$

The frequency of vibration, ω_n, is $\sqrt{\frac{k}{mr^2}}$. With $k = 2550\frac{Nm}{rad}$, $m = 47kg$, and $r = 0.3m$, the value of ω_n is $24.31\frac{rad}{s}$. When an moment is applied on the front suspension and then released, the assembly will be vibrating $24.55\frac{rad}{s}$ along the axis of symmetry of the shaft.

(b) Less Weight

Problem Statement: What is the new natural frequency of the assembly if the wheel has a mass of $35kg$?

Since we know the equation for ω_n in equation (42), plug in the new value of ω_n with the new mass. $\omega_n = 28.45\frac{rad}{s}$. The natural frequency increases since ω_n is inversely square root proportional to the mass of the wheel, m. Then a lower mass means a high natural frequency of the assembly.