Lipkin Model

Brandon Henke[1]

[1]*Michigan State University - Department of Physics & Astronomy*

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I INTRODUCTION

Consider a two level system with the lower energy state ($\sigma = -1$) having an energy of $-\epsilon/2$ and the upper energy state ($\sigma = 1$) having an energy of $\epsilon/2$. Now, consider a another system with $\Omega \in \mathbb{Z}^+$ copies of this two level system.

This gives a system with two quantum numbers, $\sigma \in \{-1, 1\}$, and $m \in \{1, 2, \ldots, \Omega\}$. In Fock space, the Hamiltonian for this system is

$$\hat{H} = \frac{1}{2}\epsilon \sum_{m\sigma} \sigma a^\dagger_{m\sigma} a_{m\sigma} - \frac{1}{2} V \sum_{mm'\sigma} a^\dagger_{m\sigma} a^\dagger_{m'\sigma} a_{m'-\sigma} a_{m-\sigma}. \quad \text{(I.1)}$$

The V term here is the two particle interaction term between adjacent particles.

II QUASI-SPIN OPERATORS

One can consider the operators

$$\hat{K}_3 = \frac{1}{2} \sum_{m=1}^{\Omega} (a^\dagger_{m+} a_{m+} - a^\dagger_{m-} a_{m-}), \quad \text{(II.1)}$$

$$\hat{K}_+ = \sum_{m=1}^{\Omega} a^\dagger_{m+} a_{m-}, \quad \text{(II.2)}$$

$$\hat{K}_- = (\hat{K}_+)^\dagger, \quad \text{(II.3)}$$

$$= \sum_{m=1}^{\Omega} a^\dagger_{m-} a_{m+}. \quad \text{(II.4)}$$

Using these "quasi-spin" operators, one can write I.1 as

$$\hat{H} = \epsilon \hat{K}_3 - \frac{1}{2} V (\hat{K}^2_+ + \hat{K}^2_-). \quad \text{(II.5)}$$

Furthermore, there are two additional "quasi-spin" operators given by the equations:

$$\hat{K}_1 = \frac{1}{2}(\hat{K}_+ + \hat{K}_-), \quad \text{(II.6)}$$

$$\hat{K}_2 = \frac{1}{2i}(\hat{K}_+ - \hat{K}_-). \quad \text{(II.7)}$$

These create a "quasi-spin" space, similar to the angular momentum space with three components representing the three spatial dimensions. One can show that these operators are $SU(2)$ generators, just like the angular momentum operators, by showing that the following commutation relations hold (see the appendix for the algebra):

$$[\hat{K}_+, \hat{K}_-] = 2\hat{K}_3, \quad \text{(II.8)}$$

$$[\hat{K}_3, \hat{K}_\pm] = \pm \hat{K}_\pm. \quad \text{(II.9)}$$

A Signature Operator

Consider the quasi-spin vector, $\boldsymbol{K}$, where

$$\boldsymbol{K} \cdot k_i = \hat{K}_i, \quad \text{(II.10)}$$

where k_i are basis vectors in the quasi-spin space. One can consider a rotation in this space by π in some plane of this space. Let the signature operator be given by

$$\hat{R} = e^{i\pi \hat{K}_3}. \quad \text{(II.11)}$$

This operator has the effect of rotating by π in the k_{12} plane.

The signature operator commutes with the Hamiltonian ($[\hat{R}, \hat{H}] = 0$), so an energy eigenstate is also an eigenstate of the signature operator, with eigenvalue $r \in \{-1, 1, -i, i\}$. If there are an even number of particles, $r \in \{-1, 1\}$. if there are an odd number of particles, $r \in \{-i, i\}$.

III EXAMPLE

Consider a system with 12 particles ($N = 12$), $\epsilon = 1$, $\Omega = 12$. If one imagines "turning on" the interaction term by increasing V from zero, then the energy eigenvalues of the Hamiltonian will begin to diverge, as shown in figure III.1.

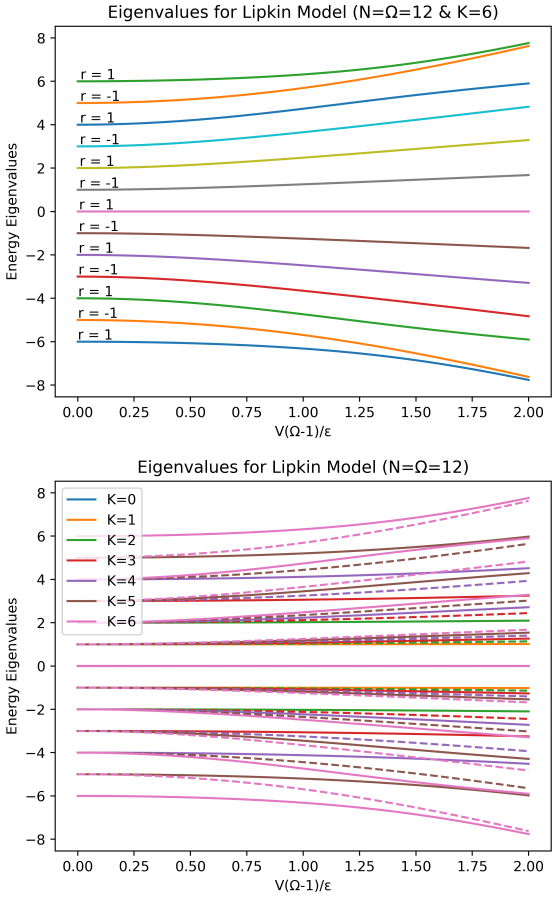

Figure III.1: Each line in this figure corresponds to an eigenvalue of the Hamiltonian, as the interaction term is "switched on". Each line is labeled with the signature eigenvalue, r. As the interaction term is increased, the eigenvalues begin to diverge. The eigenvalues start ($V = 0$) evenly spaced by a value of one, but separate as V increases. In the second figure, all values of K are plotted in different colors, and dotted lines are $r = -1$.

At a value of $V(\Omega-1)/\epsilon = 1$, the two smallest and two largest eigenvalues start becoming noticably closer together, and the difference between them continues to decrease as this V increases, seemingly becoming arbitraily close.

IV TRANSITION OPERATOR

The $\hat{K}_1$ operator is sometimes called the "transition" operator. The reason for this is unknown to me.

For the ground state of the Hamiltonian, regardless of the number of particles, value of ϵ, or value of Ω, the expectation value of this operator is zero. The argument for this comes from the angular momentum argument. The ground state is when all particles have $\sigma = -1$, and this is an eigenstate of the $\hat{K}_3$. Essentially, this state is anti-aligned with the k_3 axis, so measuring a positive value along the k_1 is just as likely as measuring a negative value. Hence, the expectation value (with the ground state) for this operator is zero, always.

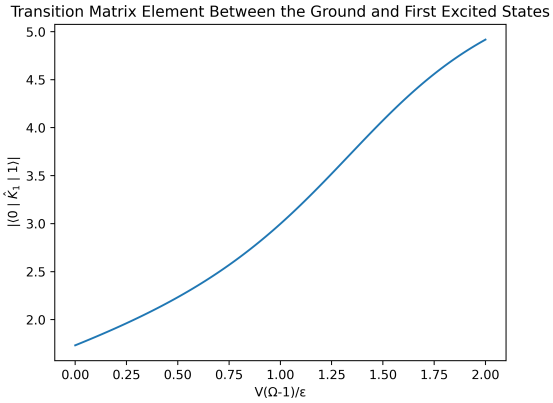

Figure IV.1: Here, the matrix element for the transition operator of the ground and first excited states is shown as a function of the $V(\Omega-1)/\epsilon$.

V COMPARISON WITH THE OCCUPATION BASIS

In the occupation basis, one considers each possible state that the system can be in by considering how many ways one can place N particles in 2Ω states, accounting for the restricting one particle per state.

In the case of $N = \Omega = 2$, there are six allowed states:

$$|1010\rangle\,,$$
$$|1001\rangle\,,$$
$$|0110\rangle\,,$$
$$|0101\rangle\,,$$
$$|1100\rangle\,,$$
$$|0011\rangle\,.$$

In the case of $N = 4$ and $\Omega = 2$, there is only one allowed state:

$$|1111\rangle\,.$$

Consider the $N = 4$ case, first. In this system, all possible states are occupied, and there is no ability to move particles around. Hence, the Hamiltonian in this basis is just the scalar 0 (eigenvalue of zero (obviously)).

In the system where $N = 2$, however, things are a little more interesting. Being that there are more states, one can move between them by use of the $\hat{K}_\pm$ operators. In this basis, the $\hat{K}$ matrices are given by

$$\hat{K}_3 = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\hat{K}_+ = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\hat{K}_- = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

One now can use these to calculate the Hamiltonian and to find the energy eigenvalues of this Hamiltonian. Doing this, the eigenvalues (found numerically) are

$$\lambda \in \{\sim -1.005, 0, 0, 0, 0, \sim 1.005\}. \tag{V.1}$$

Compare that with the method described in the rest of this paper. There are three states with total $K = 1$, and one state with total $K = 0$. The eigenvalues for the $K = 1$ states are

$$\lambda \in \{\sim -1.005, 0, \sim 1.005\}, \tag{V.2}$$

and the eigenvalue for the $K = 1$ state is $\lambda = 0$.

Notably, the occupation basis has two additional states that do not appear in the quasi-spin basis. Those two states are $|1100\rangle$ and $|0011\rangle$: the states with two particles with the same value for m. In these states, the total energy is zero, because there are the same number of particles with $\epsilon/2$ energy as there are with $-\epsilon/2$ energy, and there is no ability to move particles from σ to $\sigma' \neq \sigma$. One can argue that states in the k basis which have an energy of zero are accounting for all states with such energy, i.e., ($|KK_0\rangle$)

$$|10\rangle = \frac{1}{\sqrt{2}}(|1001\rangle + |1100\rangle), \tag{V.3}$$
$$|00\rangle = \frac{1}{\sqrt{2}}(|0110\rangle + |0011\rangle), \tag{V.4}$$

or something along these lines.

VI APPENDIX

A Commutation Relations

$$[\hat{K}_+, \hat{K}_-] = \sum_{m,n=1}^{\Omega} a^\dagger_{m+} a_{m-} a^\dagger_{n-} a_{n+} - a^\dagger_{n-} a_{n+} a^\dagger_{m+} a_{m-}, \tag{VI.1}$$

$$= \sum_{m,n=1}^{\Omega} a^\dagger_{m+} (\delta_{m-n-} - a^\dagger_{n-} a_{m-}) a_{n+}$$

$$- \sum_{m,n=1}^{\Omega} a^\dagger_{n-} (\delta_{m+n+} - a^\dagger_{m+} a_{n+}) a_{m-}, \tag{VI.2}$$

$$= \sum_{m=1}^{\Omega} a^\dagger_{m+} a_{m+} - a^\dagger_{m-} a_{m-}$$

$$+ \sum_{m,n=1}^{\Omega} a^\dagger_{n-} a^\dagger_{m+} a_{n+} a_{m-} - a^\dagger_{m+} a^\dagger_{n-} a_{m-} a_{n+}, \tag{VI.3}$$

$$= 2\hat{K}_3 + 0, \tag{VI.4}$$

$$= 2\hat{K}_3. \tag{VI.5}$$

$$[\hat{K}_3, \hat{K}_\pm] = \frac{1}{2} \sum_{m,n=1}^{\Omega} (a^\dagger_{m+} a_{m+} - a^\dagger_{m-} a_{m-}) a^\dagger_{n\pm} a_{n\mp}$$

$$- \frac{1}{2} \sum_{m,n=1}^{\Omega} a^\dagger_{n\pm} a_{n\mp} (a^\dagger_{m+} a_{m+} - a^\dagger_{m-} a_{m-}), \tag{VI.6}$$

$$= \frac{1}{2} \sum_{m,n=1}^{\Omega} (a^\dagger_{m+} a_{m+} a^\dagger_{n\pm} a_{n\mp} - a^\dagger_{m-} a_{m-} a^\dagger_{n\pm} a_{n\mp})$$

$$- \frac{1}{2} \sum_{m,n=1}^{\Omega} (a^\dagger_{n\pm} a_{n\mp} a^\dagger_{m+} a_{m+} - a^\dagger_{n\pm} a_{n\mp} a^\dagger_{m-} a_{m-}) \tag{VI.7}$$

$$= \frac{1}{2} \sum_{m,n=1}^{\Omega} (a^\dagger_{m+} \delta_{m+n\pm} a_{n\mp} - a^\dagger_{m-} \delta_{m-n\pm} a_{n\mp}$$

$$- a^\dagger_{m+} a^\dagger_{n\pm} a_{m+} a_{n\mp} + a^\dagger_{m-} a^\dagger_{n\pm} a_{m-} a_{n\mp})$$

$$- \frac{1}{2} \sum_{m,n=1}^{\Omega} (a^\dagger_{n\pm} \delta_{n\mp m+} a_{m+} - a^\dagger_{n\pm} \delta_{n\mp m-} a_{m-}$$

$$- a^\dagger_{n\pm} a^\dagger_{m+} a_{n\mp} a_{m+} + a^\dagger_{n\pm} a^\dagger_{m-} a_{n\mp} a_{m-}), \tag{VI.8}$$

$$= \pm \sum_{m=1}^{\Omega} a^\dagger_{m\pm} a_{\mp}$$

$$+ \frac{1}{2} \sum_{m,n=1}^{\Omega} (a^\dagger_{m-} a^\dagger_{n\pm} a_{m-} a_{n\mp} - a^\dagger_{m+} a^\dagger_{n\pm} a_{m+} a_{n\mp})$$

$$+ \frac{1}{2} \sum_{m,n=1}^{\Omega} (a^\dagger_{n\pm} a^\dagger_{m+} a_{n\mp} a_{m+} - a^\dagger_{n\pm} a^\dagger_{m-} a_{n\mp} a_{m-}), \tag{VI.9}$$

$$= \pm \hat{K}_\pm. \tag{VI.10}$$