Lipkin Model

Brandon Henke[1]

[1]*Michigan State University - Department of Physics & Astronomy*

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I INTRODUCTION

Consider a two level system with the lower energy state ($\sigma = -1$) having an energy of $-\epsilon/2$ and the upper energy state ($\sigma = 1$) having an energy of $\epsilon/2$. Now, consider a another system with $\Omega \in \mathbb{Z}^+$ copies of this two level system.

This gives a system with two quantum numbers, $\sigma \in \{-1, 1\}$, and $m \in \{1, 2, \ldots, \Omega\}$. In Fock space, the Hamiltonian for this system is

$$\hat{H} = \frac{1}{2}\epsilon \sum_{m\sigma} \sigma a^\dagger_{m\sigma} a_{m\sigma} - \frac{1}{2} V \sum_{mm'\sigma} a^\dagger_{m\sigma} a^\dagger_{m'\sigma} a_{m'-\sigma} a_{m-\sigma}. \tag{I.1}$$

The V term here is the two particle interaction term between adjacent particles.

A Quasi-spin Operators

One can consider the operators

$$\hat{K}_3 = \frac{1}{2}\sum_{m=1}^{\Omega} (a^\dagger_{m+} a_{m+} - a^\dagger_{m-} a_{m-}), \tag{I.2}$$

$$\hat{K}_+ = \sum_{m=1}^{\Omega} a^\dagger_{m+} a_{m-}, \tag{I.3}$$

$$\hat{K}_- = (\hat{K}_+)^\dagger, \tag{I.4}$$

$$= \sum_{m=1}^{\Omega} a^\dagger_{m-} a_{m+}. \tag{I.5}$$

Using these "quasi-spin" operators, one can write I.1 as

$$\hat{H} = \epsilon \hat{K}_3 - \frac{1}{2} V (\hat{K}_+^2 - \hat{K}_-^2). \tag{I.6}$$

1 Commutation Relations

$$[\hat{K}_+, \hat{K}_-] = \sum_{m=1}^{\Omega}\sum_{n=1}^{\Omega} a^\dagger_{m+} a_{m-} a^\dagger_{n-} a_{n+} - a^\dagger_{n-} a_{n+} a^\dagger_{m+} a_{m-}, \tag{I.7}$$

$$= \sum_{m=1}^{\Omega}\sum_{n=1}^{\Omega} a^\dagger_{m+} (\delta_{m-n-} - a^\dagger_{n-} a_{m-}) a_{n+} - \sum_{m=1}^{\Omega}\sum_{n=1}^{\Omega} a^\dagger_{n-} (\delta_{m+n+} - a^\dagger_{m+} a_{n+}) a_{m-}, \tag{I.8}$$

$$= \sum_{m=1}^{\Omega} a^\dagger_{m+} a_{m+} - a^\dagger_{m-} a_{m-} + \sum_{m=1}^{\Omega}\sum_{n=1}^{\Omega} a^\dagger_{n-} a^\dagger_{m+} a_{n+} a_{m-} - a^\dagger_{m+} a^\dagger_{n-} a_{m-} a_{n+}, \tag{I.9}$$

$$= 2\hat{K}_3 + 0, \tag{I.10}$$

$$= 2\hat{K}_3. \tag{I.11}$$

$$[\hat{K}_3, \hat{K}_\pm] = \frac{1}{2}\sum_{m=1}^{\Omega}\sum_{n=1}^{\Omega} (a^\dagger_{m+} a_{m+} - a^\dagger_{m-} a_{m-}) a^\dagger_{n\pm} a_{n\mp} - \frac{1}{2}\sum_{m=1}^{\Omega}\sum_{n=1}^{\Omega} a^\dagger_{n\pm} a_{n\mp} (a^\dagger_{m+} a_{m+} - a^\dagger_{m-} a_{m-}), \tag{I.12}$$

$$= \frac{1}{2}\sum_{m=1}^{\Omega}\sum_{n=1}^{\Omega} (a^\dagger_{m+} a_{m+} a^\dagger_{n\pm} a_{n\mp} - a^\dagger_{m-} a_{m-} a^\dagger_{n\pm} a_{n\mp}) - \frac{1}{2}\sum_{m=1}^{\Omega}\sum_{n=1}^{\Omega} (a^\dagger_{n\pm} a_{n\mp} a^\dagger_{m+} a_{m+} - a^\dagger_{n\pm} a_{n\mp} a^\dagger_{m-} a_{m-}) \tag{I.13}$$

$$= \frac{1}{2}\sum_{m=1}^{\Omega}\sum_{n=1}^{\Omega} (a^\dagger_{m+} \delta_{m+n\pm} a_{n\mp} - a^\dagger_{m-} \delta_{m-n\pm} a_{n\mp} - a^\dagger_{m+} a^\dagger_{n\pm} a_{m+} a_{n\mp} + a^\dagger_{m-} a^\dagger_{n\pm} a_{m-} a_{n\mp}) - \frac{1}{2}\sum_{m=1}^{\Omega}\sum_{n=1}^{\Omega} (a^\dagger_{n\pm} \delta_{n\mp m+} a_{m+} - a^\dagger_{n\pm} \delta_{n\mp m-} a_{m-} - a^\dagger_{n\pm} a^\dagger_{m+} a_{n\mp} a_{m+} + a^\dagger_{n\pm} a^\dagger_{m-} a_{n\mp} a_{m-}), \tag{I.14}$$

$$= \sum_{m=1}^{\Omega} (a^\dagger_{m+} a_{m-} - a^\dagger_{m-} a_{m+}) + \frac{1}{2}\sum_{m=1}^{\Omega}\sum_{n=1}^{\Omega} (a^\dagger_{m-} a^\dagger_{n\pm} a_{m-} a_{n\mp} - a^\dagger_{m+} a^\dagger_{n\pm} a_{m+} a_{n\mp}) + \frac{1}{2}\sum_{m=1}^{\Omega}\sum_{n=1}^{\Omega} (a^\dagger_{n\pm} a^\dagger_{m+} a_{n\mp} a_{m+} - a^\dagger_{n\pm} a^\dagger_{m-} a_{n\mp} a_{m-}), \tag{I.15}$$