Lipkin Model

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I INTRODUCTION

Consider a two level system with the lower energy state ($\sigma = -1$) having an energy of $-\epsilon/2$ and the upper energy state ($\sigma = 1$) having an energy of $\epsilon/2$. Now, consider a another system with $\Omega \in \mathbb{Z}^+$ copies of this two level system.

This gives a system with two quantum numbers, $\sigma \in \{-1, 1\}$, and $m \in \{1, 2, \ldots, \Omega\}$. In Fock space, the Hamiltonian for this system is

$$\hat{H} = \frac{1}{2}\epsilon \sum_{m\sigma} \sigma a^\dagger_{m\sigma} a_{m\sigma} - \frac{1}{2} V \sum_{mm'\sigma} a^\dagger_{m\sigma} a^\dagger_{m'\sigma} a_{m'-\sigma} a_{m-\sigma}. \quad \text{(I.1)}$$

The V term here is the two particle interaction term between adjacent particles.

II QUASI-SPIN OPERATORS

One can consider the operators

$$\hat{K}_3 = \frac{1}{2} \sum_{m=1}^{\Omega} (a^\dagger_{m+} a_{m+} - a^\dagger_{m-} a_{m-}), \quad \text{(II.1)}$$

$$\hat{K}_+ = \sum_{m=1}^{\Omega} a^\dagger_{m+} a_{m-}, \quad \text{(II.2)}$$

$$\hat{K}_- = (\hat{K}_+)^\dagger, \quad \text{(II.3)}$$

$$= \sum_{m=1}^{\Omega} a^\dagger_{m-} a_{m+}. \quad \text{(II.4)}$$

Using these "quasi-spin" operators, one can write I.1 as

$$\hat{H} = \epsilon \hat{K}_3 - \frac{1}{2} V (\hat{K}_+^2 + \hat{K}_-^2). \quad \text{(II.5)}$$

Furthermore, there are two additional "quasi-spin" operators given by the equations:

$$\hat{K}_1 = \frac{1}{2}(\hat{K}_+ + \hat{K}_-), \quad \text{(II.6)}$$

$$\hat{K}_2 = \frac{1}{2i}(\hat{K}_+ - \hat{K}_-). \quad \text{(II.7)}$$

These create a "quasi-spin" space, similar to the angular momentum space with three components representing the three spatial dimensions. One can show that these operators are $SU(2)$ generators, just like the angular momentum operators, by showing that the following commutation relations hold (see the appendix for the algebra):

$$[\hat{K}_+, \hat{K}_-] = 2\hat{K}_3, \quad \text{(II.8)}$$

$$[\hat{K}_3, \hat{K}_\pm] = \pm \hat{K}_\pm. \quad \text{(II.9)}$$

A Signature Operator

Consider the quasi-spin vector, $\boldsymbol{K}$, where

$$\boldsymbol{K} \cdot k_i = \langle \hat{K}_i \rangle, \quad \text{(II.10)}$$

where k_i are basis vectors in the quasi-spin space. One can consider a rotation in this space by π in some plane of this space. Let the signature operator be given by

$$\hat{R} = e^{i\pi \hat{K}_3}. \quad \text{(II.11)}$$

This operator has the effect of rotating by π in the k_{12} plane.

The signature operator commutes with the Hamiltonian($[\hat{R}, \hat{H}] = 0$), so an energy eigenstate is also an eigenstate of the signature operator, with eigenvalue $r \in \{-1, 1, -i, i\}$. If there are an even number of particles, $r \in \{-1, 1\}$. if there are an odd number of particles, $r \in \{-i, i\}$.

III EXAMPLE

Consider a system with 12 particles ($N = 12$), $\epsilon = 1$, $\Omega = 12$. If one imagines "turning on" the interaction term by increasing V from zero, then the energy eigenvalues of the Hamiltonian will begin to diverge, as shown in figure III.1.

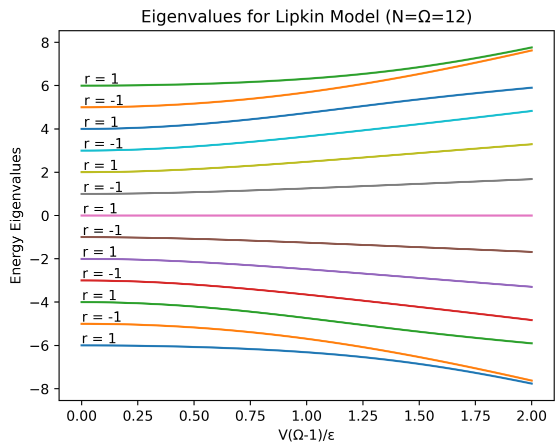

Figure III.1: Each line in this figure corresponds to an eigenvalue of the Hamiltonian, as the interaction term is "switched on". Each line is labeled with the signature eigenvalue, r. As the interaction term is increased, the eigenvalues begin to diverge. The eigenvalues start ($V = 0$) evenly spaced by a value of one, but separate as V increases.

At a value of $V(\Omega - 1)/\epsilon = 1$, the two smallest and two largest eigenvalues start becoming noticably closer together, and the difference between them continues to decrease as this V increases, seemingly becoming arbitraily close.

IV TRANSITION OPERATOR

The $\hat{K}_1$ operator is sometimes called the "transition" operator. The reason for this is unknown to me.

For the ground state of the Hamiltonian, regardless of the number of particles, value of ϵ, or value of Ω, the expectation value of this operator is zero.

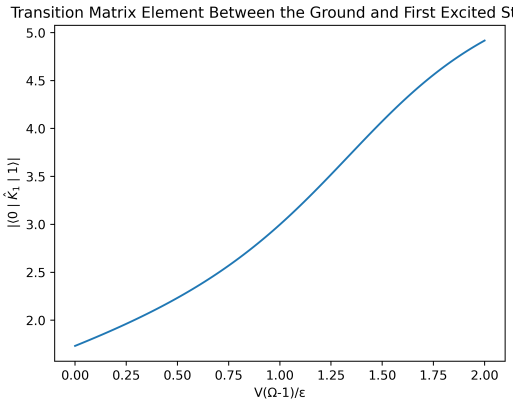

Figure IV.1: Here, the matrix element for the transition operator of the ground and first excited states is shown as a function of the $V(\Omega-1)/\epsilon$.

V APPENDIX

A Commutation Relations

$$[\hat{K}_+, \hat{K}_-] = \sum_{m,n=1}^{\Omega} a^\dagger_{m+}a_{m-}a^\dagger_{n-}a_{n+} - a^\dagger_{n-}a_{n+}a^\dagger_{m+}a_{m-}, \tag{V.1}$$

$$= \sum_{m,n=1}^{\Omega} a^\dagger_{m+}(\delta_{m-n-} - a^\dagger_{n-}a_{m-})a_{n+} - \sum_{m,n=1}^{\Omega} a^\dagger_{n-}(\delta_{m+n+} - a^\dagger_{m+}a_{n+})a_{m-}, \tag{V.2}$$

$$= \sum_{m=1}^{\Omega} a^\dagger_{m+}a_{m+} - a^\dagger_{m-}a_{m-} + \sum_{m,n=1}^{\Omega} a^\dagger_{n-}a^\dagger_{m+}a_{n+}a_{m-} - a^\dagger_{m+}a^\dagger_{n-}a_{m-}a_{n+}, \tag{V.3}$$

$$= 2\hat{K}_3 + 0, \tag{V.4}$$

$$= 2\hat{K}_3. \tag{V.5}$$

$$[\hat{K}_3, \hat{K}_\pm] = \frac{1}{2}\sum_{m,n=1}^{\Omega} (a^\dagger_{m+}a_{m+} - a^\dagger_{m-}a_{m-})a^\dagger_{n\pm}a_{n\mp} - \frac{1}{2}\sum_{m,n=1}^{\Omega} a^\dagger_{n\pm}a_{n\mp}(a^\dagger_{m+}a_{m+} - a^\dagger_{m-}a_{m-}), \tag{V.6}$$

$$= \frac{1}{2}\sum_{m,n=1}^{\Omega} (a^\dagger_{m+}a_{m+}a^\dagger_{n\pm}a_{n\mp} - a^\dagger_{m-}a_{m-}a^\dagger_{n\pm}a_{n\mp}) - \frac{1}{2}\sum_{m,n=1}^{\Omega} (a^\dagger_{n\pm}a_{n\mp}a^\dagger_{m+}a_{m+} - a^\dagger_{n\pm}a_{n\mp}a^\dagger_{m-}a_{m-}) \tag{V.7}$$

$$= \frac{1}{2}\sum_{m,n=1}^{\Omega} (a^\dagger_{m+}\delta_{m+n\pm}a_{n\mp} - a^\dagger_{m-}\delta_{m-n\pm}a_{n\mp} - a^\dagger_{m+}a^\dagger_{n\pm}a_{m+}a_{n\mp} + a^\dagger_{m-}a^\dagger_{n\pm}a_{m-}a_{n\mp}) - \frac{1}{2}\sum_{m,n=1}^{\Omega} (a^\dagger_{n\pm}\delta_{n\mp m+}a_{m+} - a^\dagger_{n\pm}\delta_{n\mp m-}a_{m-} - a^\dagger_{n\pm}a^\dagger_{m+}a_{n\mp}a_{m+} + a^\dagger_{n\pm}a^\dagger_{m-}a_{n\mp}a_{m-}), \tag{V.8}$$

$$= \pm\sum_{m=1}^{\Omega} a^\dagger_{m\pm}a_{\mp} + \frac{1}{2}\sum_{m,n=1}^{\Omega} (a^\dagger_{m-}a^\dagger_{n\pm}a_{m-}a_{n\mp} - a^\dagger_{m+}a^\dagger_{n\pm}a_{m+}a_{n\mp}) + \frac{1}{2}\sum_{m,n=1}^{\Omega} (a^\dagger_{n\pm}a^\dagger_{m+}a_{n\mp}a_{m+} - a^\dagger_{n\pm}a^\dagger_{m-}a_{n\mp}a_{m-}), \tag{V.9}$$

$$= \pm\hat{K}_\pm. \tag{V.10}$$