Lipkin Model

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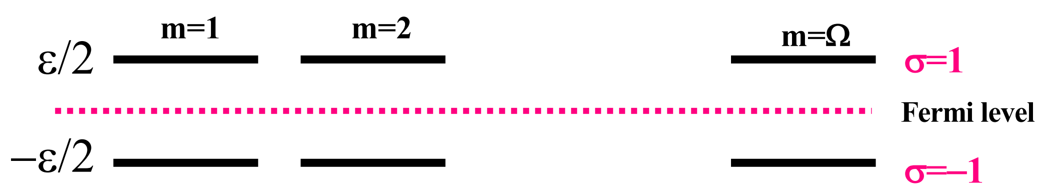
Ring & Schuck, *The Nuclear Many-Body Problem*, Sec. 5.4
A.P. Severyukhin, M. Bender and P.-H. Heenen, Phys. Rev. C **74**, 024311 (2006)



$$\hat{H} = \frac{1}{2}\varepsilon \sum_{m\sigma} \sigma a^+_{m\sigma} a_{m\sigma} - \frac{1}{2} V \sum_{mm'\sigma} a^+_{m\sigma} a^+_{m'\sigma} a_{m'-\sigma} a_{m-\sigma}$$

quasi-spin operators:

$$\hat{K}_0 = \frac{1}{2}\sum_{m=1}^{\Omega}\left(a^+_{m+}a_{m+} - a^+_{m-}a_{m-}\right)$$

$$\hat{K}_+ = \sum_{m=1}^{\Omega} a^+_{m+}a_{m-}, \quad \hat{K}_- = \left(\hat{K}_+\right)^+$$

$$\hat{K}_\pm = \hat{K}_x \pm i\hat{K}_y$$

$$\left[\hat{K}_+, \hat{K}_-\right] = 2\hat{K}_0, \quad \left[\hat{K}_0, \hat{K}_\pm\right] = \pm\hat{K}_\pm$$

commutation relation of angular momenta: SU(2)

The LM Hamiltonian can be written as:

$$\hat{H} = \varepsilon \hat{K}_0 - \frac{1}{2} V \left(\hat{K}_+ \hat{K}_+ + \hat{K}_- \hat{K}_- \right)$$

As the Hamiltonian is written explicitly in terms of SU(2) generators, it commutes with the total quasi-spin operator

$$\hat{K}^2 = \frac{1}{2}\left(\hat{K}_+ \hat{K}_- + \hat{K}_- \hat{K}_+ \right) + \hat{K}_0^2$$

$$\left[\hat{K}^2, \hat{H} \right] = 0 \quad \Rightarrow$$

eigenstates of the LM Hamiltonian conserve the total quasi-spin

The LM Hamiltonian conserves the total number of particles

$$\hat{N} = \sum_{m=1}^{\Omega} \left(a_{m+}^{+} a_{m+} + a_{m-}^{+} a_{m-} \right), \quad \hat{N} \Psi_{\alpha K} = N \Psi_{\alpha K}$$

$$0 \le N \le 2\Omega$$

$$\hat{N} = \hat{N}_+ + \hat{N}_-, \quad \hat{K}_0 = \frac{1}{2}\left(\hat{N}_+ - \hat{N}_- \right)$$

The third component of quasi-spin measures half of the difference between the number of particles in the upper and the lower levels.

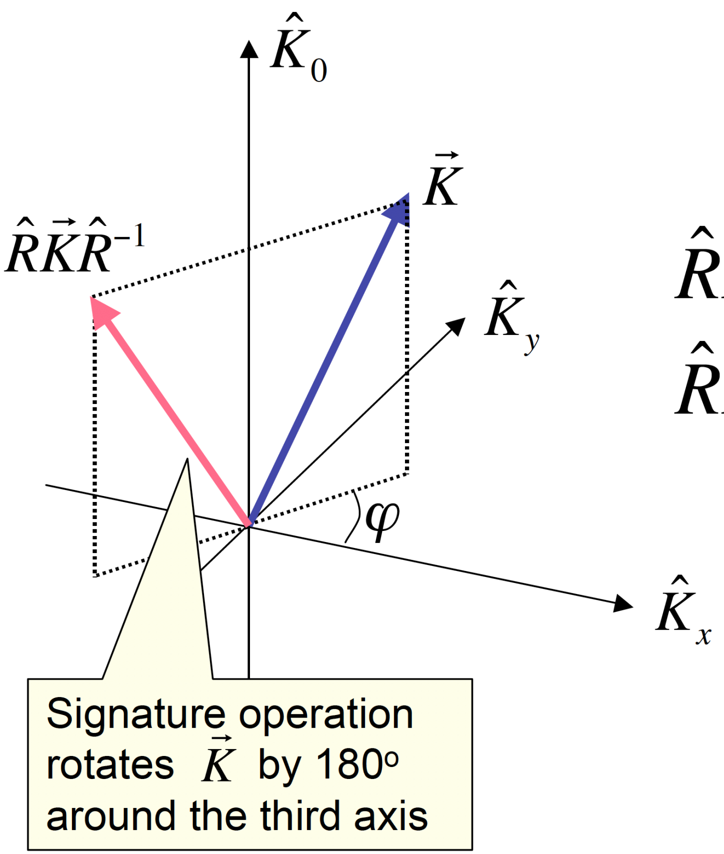
$$\hat{R} = e^{i\pi \hat{K}_o}$$

signature operator

$$r = \begin{cases} +1, & K_0 = 0, \pm 2, \\ -1, & K_0 = \pm 1, \pm 3, \\ +i, & K_0 = 1/2, -3/2, +5/2 \ldots \\ -i, & K_0 = -1/2, 3/2, -5/2 \ldots \end{cases}$$

even number of particles

odd number of particles



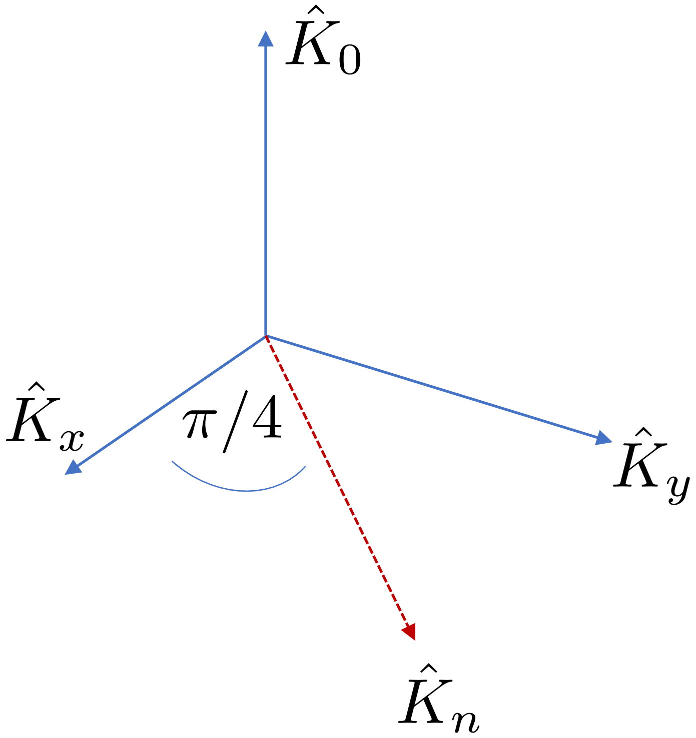
$$\hat{R}\hat{K}_0\hat{R}^{-1} = \hat{K}_0$$

$$\hat{R}\hat{K}_\pm\hat{R}^{-1} = -\hat{K}_\pm$$

$$\left[\hat{R}, \hat{H}\right] = 0 \Rightarrow$$

signature q.n. is conserved

$$\hat{R}\Psi_{\alpha K} = r\Psi_{\alpha K}$$



$$\hat{U} = e^{i\pi \hat{K}_n}$$

$$\hat{U}\hat{K}_0\hat{U}^{-1} = -\hat{K}_0$$

$$\hat{U}\hat{K}_x\hat{U}^{-1} = \hat{K}_y$$

$$\hat{U}\hat{K}_y\hat{U}^{-1} = \hat{K}_x$$

$$\hat{U}\hat{K}_+\hat{U}^{-1} = i\hat{K}_-$$

$$\hat{U}\hat{K}_-\hat{U}^{-1} = -i\hat{K}_+$$

$$\hat{U}\hat{H}\hat{U}^{-1} = -\hat{H}$$

$$\hat{H}\Psi_{\alpha K} = E_{\alpha K}\Psi_{\alpha K} \Rightarrow \hat{H}\hat{U}\Psi_{\alpha K} = -E_{\alpha K}\hat{U}\Psi_{\alpha K}$$

We are now ready to calculate LM Hamiltonian matrix

$$\hat{H} = \varepsilon \hat{K}_0 - \frac{1}{2} V \left(\hat{K}_+ \hat{K}_+ + \hat{K}_- \hat{K}_- \right)$$

$$\hat{K}_0 \left| K K_0 \right\rangle = K_0 \left| K K_0 \right\rangle$$

$$\hat{K}_\pm \left| K K_0 \right\rangle = \sqrt{K(K+1) - K_0 (K_0 \pm 1)} \left| K K_0 \pm 1 \right\rangle$$

basis state

Dimension of the basis (representation) is *2K+1*

$$\left\langle K K_0 \middle| \varepsilon \hat{K}_0 \middle| K K_0 ' \right\rangle = \varepsilon K_0 \delta_{K_0 K_0 '}$$

The single-particle term is diagonal in K_0

$$\hat{K}_\pm^2 \left| K K_0 \right\rangle = \sim \left| K K_0 \pm 2 \right\rangle$$

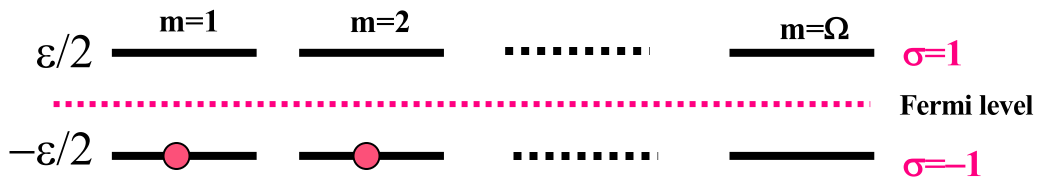
The interaction term changes K_0 by two

$$\hat{R} \left| K K_0 \right\rangle = e^{i\pi \hat{K}_o} \left| K K_0 \right\rangle = e^{i\pi K_o} \left| K K_0 \right\rangle$$

$$\hat{R} \left| K K_0 \pm 2 \right\rangle = e^{i\pi (K_o \pm 2)} \left| K K_0 \pm 2 \right\rangle = e^{i\pi K_o} \left| K K_0 \pm 2 \right\rangle$$

Hamiltonian conserves quasi-spin and signature but mixes K_0. For a given K, the size of the Hamiltonian matrix is *2K+1*. It splits into two blocks with dimensions K and $K+1$ having different signature.

What is quasi-spin of the ground state?



Consider $N \leq \Omega$

If interaction is weak, the lowest energy of the system corresponds to occupying the lower level by *N* particles. What is the corresponding *K*?

$$\hat{K}_0 = \frac{1}{2}\sum_{m=1}^{\Omega}\left(a^{+}_{m+}a_{m+} - a^{+}_{m-}a_{m-}\right) \Rightarrow K_0 = -N/2$$

$$\boxed{K_{g.s.} = N/2}$$

Maximum quasi-spin for given *N*

$$\boxed{r_{g.s.} = e^{-i\pi N/2}}$$

Excited states have other values of *K*

The solution of the Lipkin model by means of the quasi-spin operators represents the symmetry-dictated strategy.

- Identify the symmetry group of operations which leave the Hamiltonian invariant
- Many-body states can be identified through the quantum numbers labeling the *irreducible representations* of the symmetry group
- The Hamiltonian matrix is block-diagonal (one block for every *irrep*)
- If the Hamiltonian is the Casimir operator, the spectrum can be obtained analytically (e.g., seniority model)

In the case of the Lipkin model:

- The underlying symmetries are
  - Quasi-spin SU(2)
  - Signature operation
  - Particle number
- The quantum numbers are *K, r, N*
- The number of single-particle states is 2Ω
- The dimension of the Fock space is $2^{2\Omega}$

Bonus: Imaginary time method (used in Quantum Monte Carlo)

$$\hat{H}|\Psi_m\rangle = E_m|\Psi_m\rangle$$ *m*=0, 1, 2…. labels energy states (*m*=0 is the g.s.)

- Define the trial wave function $|\Psi\rangle$
- Expend it in the set of *true* eigenstates

$$|\Psi\rangle = \sum_m c_m|\Psi_m\rangle$$

$$|\Psi(\tau)\rangle \equiv \mathcal{N}(\tau)\exp(-\hat{H}\tau)|\Psi\rangle$$ imaginary-time propagation

$$|\Psi(\tau)\rangle = \mathcal{N}(\tau)\sum_m \exp(-E_m\tau)c_m|\Psi_m\rangle$$

- Take the limit $\tau \to \infty$ $\quad |\Psi(0)\rangle = \Psi, \quad |\Psi(\infty)\rangle = \Psi_0$
- In practice $\quad \tau = n\Delta\tau$

$$|\Psi(\tau)\rangle \equiv \mathcal{N}(\tau)\left[\exp(-\hat{H}\Delta\tau)\right]^n|\Psi\rangle$$

$$\exp(-\hat{H}\Delta\tau) \approx 1 - \Delta\tau\hat{H} + \frac{1}{2}\Delta\tau^2\hat{H}^2 + \cdots$$

One needs to define $\hat{H}^n|\Psi\rangle$ (recurrence relations are helpful!)

To compute the excited states, use the the Gram–Schmidt process. For instance, for the first excited state

$$|\Psi\rangle = |\Phi\rangle - \langle\Psi_0|\Phi\rangle|\Psi_0\rangle$$

Which guarantees that $\langle\Psi|\Psi_0\rangle = 0$