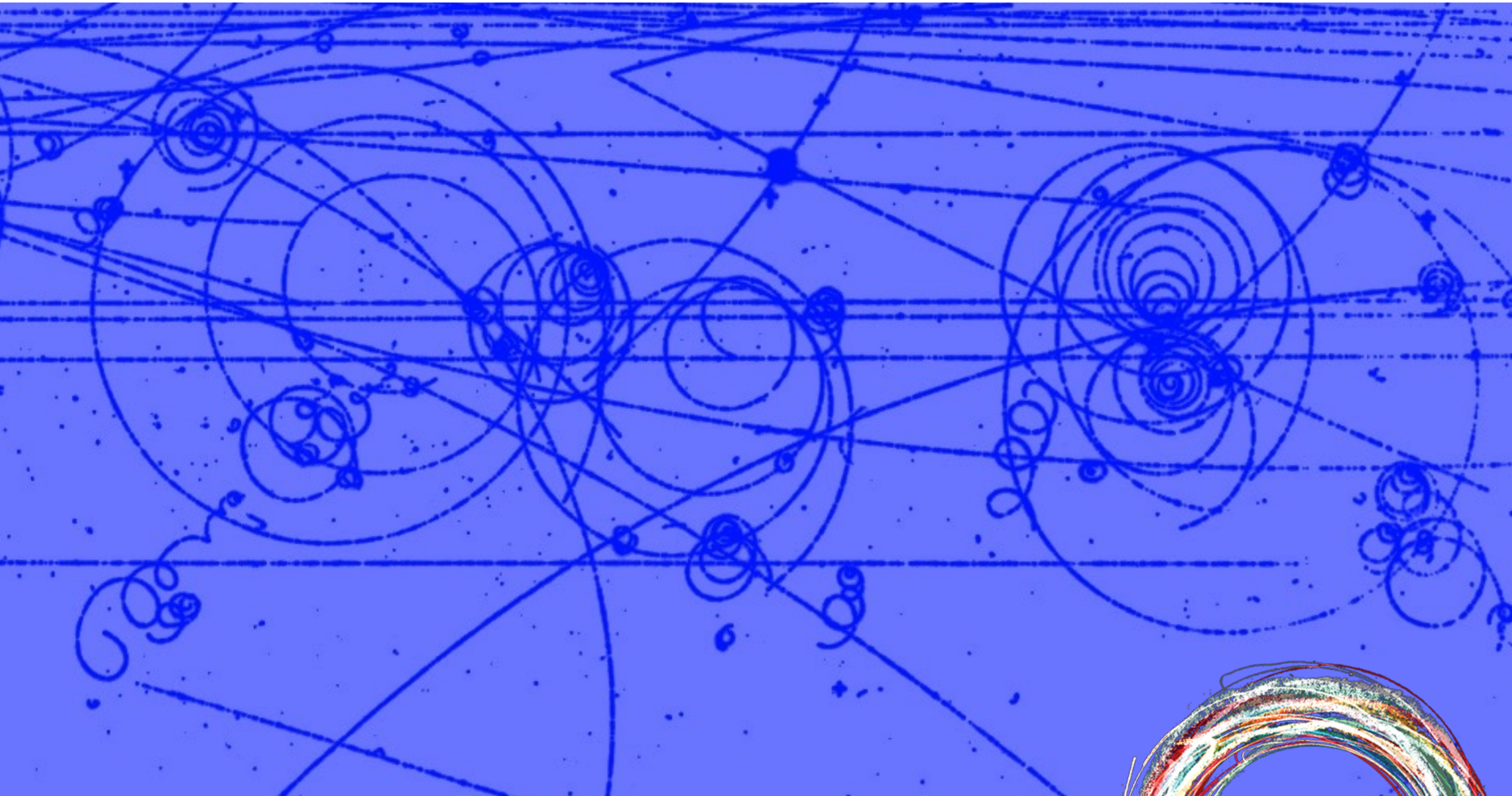


FERMI'S GOLDEN RULE FOR DECAYS AND SCATTERING

PHY 493/803

Observables

To learn anything about particle interactions, we have to observe something.

There are two basic categories:

A) Bound states & their spectra (this was chapter 5)

B)

When particles are "Left Alone", they can:

1) Do nothing
2) Decay
3) Eventually find another particle and go to category C.

C)

When particles encounter another particle, they can:

1) Do nothing
2) Scatter off the other particle.
3) Annihilate on the other particle

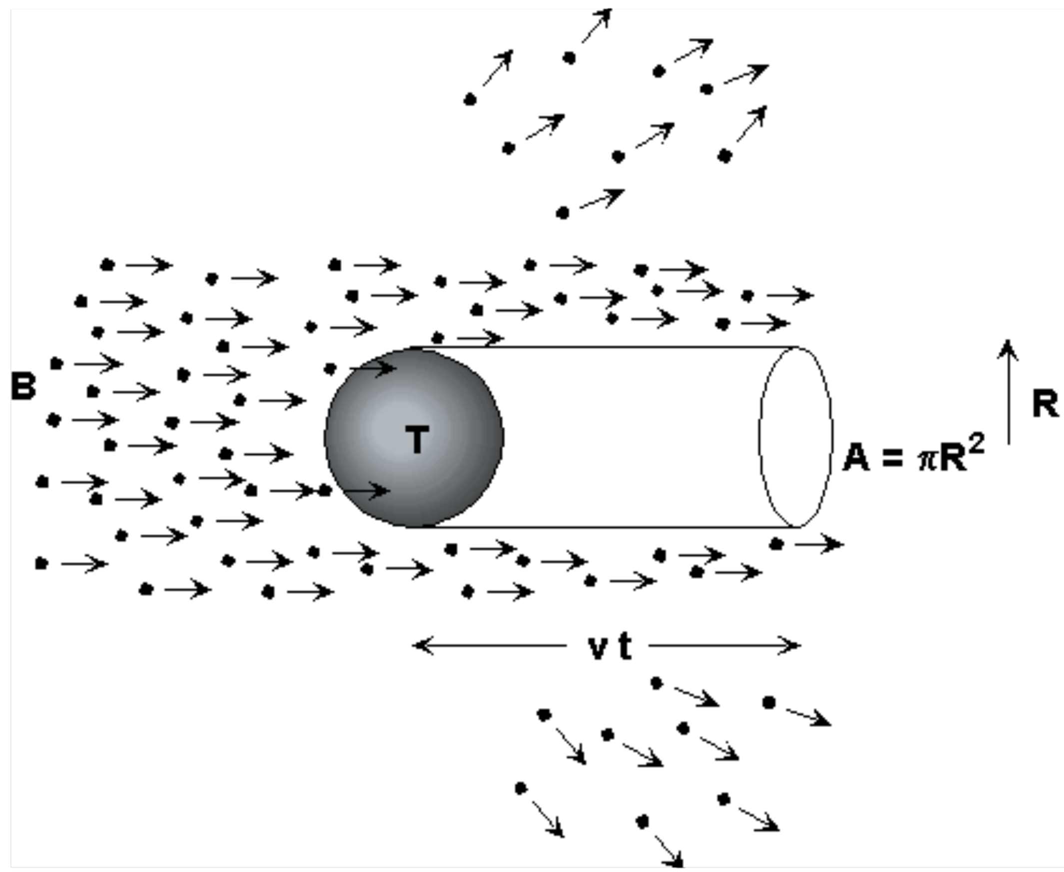
Cross section (σ): Describes the probability for an interaction to occur.

Differential cross section ($\mathrm{d}\sigma/\mathrm{d}X$): Describes the probability for an interaction with a particular final state.

Cross Section

We need to connect the idea of a unit area (classical) and the probability for an interaction to occur.

Classically, the cross section is inherently related to the size of an object.



In the classical sense, scattering occurs when the particles in a beam overlap their path with the target's cross section

Scattering in quantum mechanics

- Particles interact with each other - matrix element for each vertex
- Together with kinematical information - available final state phase space, allowed momenta, energies of particles
- Calculating interaction rate from these two ingredients follows Fermi's golden rule

The Golden Rule

Fermi's Golden Rule states:

The transition rate for a system depends on two fundamental quantities

1) The "Matrix Element" or "Transition Amplitude"
2) The final state phase space

$$W = \frac{2\pi}{\hbar} |\mathcal{M}_{if}|^2 \rho(E)$$

The transition rate (number of decays or interactions per time interval)

The Matrix Element. We will calculate this for some examples

Phase Space or Density of States

Phase space, density of states

Density of states

Also known as the available phase space or just phase space

Describes how many equivalent ways a final state can be configured

1) Assume we have a particle with quantized momentum confined to volume V

$$p = \hbar k = h/\lambda$$

Smallest element of phase space in any coordinate is **h**!

2) The number of equivalent states, N_i, can be calculated by dividing the total phase space volume by the elemental phase space volume:

$$N_i = \frac{1}{(2\pi\hbar)^3}\int dx\,dy\,dz\,dp_x\,dp_y\,dp_z = \frac{V}{(2\pi\hbar)^3}\int d^3p$$

Density of States

Density of states

Also known as the available phase space or just phase space

Describes how many equivalent ways a final state can be configured

3) Simplify life by calculating over a unit volume (V=1 of your favorite unit). Also consider the option to have **n** particles in the final state:

$$N_n = \frac{1}{(2\pi\hbar)^{3n}} \int \prod_{i=1}^{n} d^3p_i$$

4) Impose conservation of momentum in the final state, which means there is one less degree of freedom in the distribution of momenta:

$$N_n = \frac{1}{(2\pi\hbar)^{3n}} \int \prod_{i=1}^{n} d^3p_i \; \delta\left[p_n - (p_0 - \sum_{i=1}^{n-1} p_i)\right]$$

$$= \frac{1}{(2\pi\hbar)^{3n}} \int \prod_{i=1}^{n-1} d^3p_i$$

Dirac Delta Function

I hope this is a review. If not, go study up on this!

The Dirac delta function δ(x) is an infinitesimally small peak at zero

$$\delta(x) = \begin{cases} 0, & (x \neq 0) \\ \infty, & (x = 0) \end{cases}$$

It is the derivative of the Heaviside step function θ(x)

$$\theta(x) = \begin{cases} 0, & (x < 0) \\ 1, & (x > 0) \end{cases}$$

The integral of the delta function has a valuable property

The delta function selects out the zeros of its argument

$$\int_{-\infty}^{+\infty} \delta(x) = 1 \longrightarrow \int_{-\infty}^{+\infty} f(x)\delta(x-a) = f(a)$$

Density of States

Density of states

Also known as the available phase space or just phase space

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$$= \frac{1}{(2\pi\hbar)^{3n}} \int \prod_{i=1}^{n-1} d^3p_i$$

Lorentz Invariant Phase Space

What we did is just fine, but it's not Lorentz Invariant

But we can fix that by taking into account what we've learned

Express our density of states as a 4-momentum

1) Simplify life by calculating over a unit volume (V=1 of your favorite unit). Also consider the option to have **n** particles in the final state:

$$N_n = \frac{1}{(2\pi\hbar)^{3n}} \int \prod_{i=1}^{n} d^3 p_i \longrightarrow N_n = \frac{1}{(2\pi\hbar)^{4n}} \int \prod_{i=1}^{n} d^4 p_i$$

2) Impose Lorentz invariance by fixing the Lorentz invariant inner product for each final state particle:

$$N_n = \frac{1}{(2\pi\hbar)^{4n}} \int \prod_{i=1}^{n} d^4 p_i \, \underline{\delta \left[(p^\mu p_\mu)_i - m_i^2 c^2 \right]}$$

Delta function imposes the requirement that final-state particles are on-shell

Lorentz Invariant Phase Space

While we're at it, we may as well introduce "sanity checks"

1) Energy & momentum are conserved
2) There can be no negative energy states

3) Introduce a delta function for 4-momentum conservation (energy & momentum):

$$N_n = \frac{1}{(2\pi\hbar)^{4n}} \int \underline{\delta^4(p_i - \sum p_f)} \prod_{j=1}^{n} d^4 p_j \, \delta\left[(p^\mu p_\mu)_j - m_j^2 c^2\right]$$

4) Introduce a Heaviside function to remove negative final state energies:

$$N_n = \frac{1}{(2\pi\hbar)^{4n}} \int \delta^4(p_i - \sum p_f) \prod_{j=1}^{n} d^4 p_j \, \delta\left[(p^\mu p_\mu)_j - m_j^2 c^2\right] \, \underline{\theta(E_j)}$$

Lorentz Invariant Phase Space

We're almost there!! One more thing to do…

Recall, our integrals were not truly over continuous momentum space.
Momentum is quantized, so we're actually integrating integer k space!

5) Fourier transform from p-space back to k-space (or vice-versa):

$$\phi(p) = \frac{1}{2\pi}\int \psi(k) e^{ipk} dk$$

Every delta function gets a 2π in the integral.

6) We done! We call the energy derivative the "differential Lorentz Invariant phase Space" or just dLIPS.

$$\rho(E) = \frac{\partial N_n}{\partial E} = \frac{(2\pi)^{n+4}}{(2\pi\hbar)^{4n}}\,\delta^4\left(p_i - \sum p_f\right)\prod_{j=1}^{n} d^4p_j\,\delta\left[(p^\mu p_\mu)_j - m_j^2c^2\right]\,\theta(E_j)$$

Solve this once for 2-particle final states, then derive the rest recursively

dLIPS

- dLIPS depends only on the number of particles in the final state
- Calculate once, re-use result
- 2-particle final states:

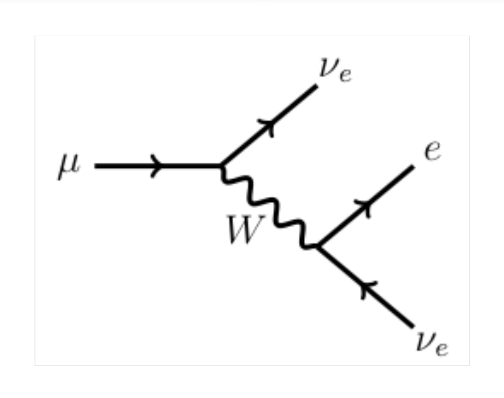
$$\rho(E) = \frac{\pi}{(2\pi)^6} \frac{p_1}{E}$$

where p_1 is the magnitude of the momentum of one of the outgoing particles ($= p_2$ in the CM frame) and E is the total energy of the decaying particle ($= M$ in the CM frame, i.e. the rest frame of the decaying particle

Golden Rule for Decays

Assume we have a particle at rest decaying to n particles: 1→2,3,4…n
Here we assume the form for the decay rate and insert elements as needed

$$\Gamma = \frac{S}{2\hbar m_1} \int |\mathcal{M}|^2 \, \rho(E)$$



The factor *S* factor takes into account final state particle interchange

If the final state has N identical particles, we include a factor of N!

Thus S = 1/N!

The integral is over the phase space $\rho(E)$, which corresponds to the allowed momenta of the outgoing particles

The matrix element *M* depends on momenta of incoming and outgoing particles

Golden Rule for Decays

We can now build the form of the decay rate

Assume we have a particle at rest decaying to n particles: 1→2,3,4…n

Here we assume the form for the decay rate and insert elements as needed

$$\Gamma = \frac{S}{2\hbar m_1} \int |\mathcal{M}|^2 \, \rho(E)$$

$$\Gamma = \frac{S}{2\hbar m_1} \int |\mathcal{M}|^2 \, (2\pi)^4 \, \delta^4(p_1 - p_2 - p_3 \cdots - p_n)$$
$$\times \prod_{j=2}^{n} 2\pi \, \delta(p_j^2 - m_j^2 c^2) \, \theta(E_j) \, \frac{d^4 p_i}{(2\pi)^4}$$

For 2-particle decays (1→2,3):

$$\Gamma = \frac{S \, |p|}{8\pi \hbar \, m_1^2 c} |\mathcal{M}|^2$$

The Golden Rule reminder

Fermi's Golden Rule states:

The transition rate for a system depends on two fundamental quantities

1) The "Matrix Element" or "Transition Amplitude"
2) The final state phase space

$$W = \frac{2\pi}{\hbar} |\mathcal{M}_{if}|^2 \rho(E)$$

The transition rate (number of decays or interactions per time interval)

The Matrix Element. We may not actually know this.

Phase Space or Density of States

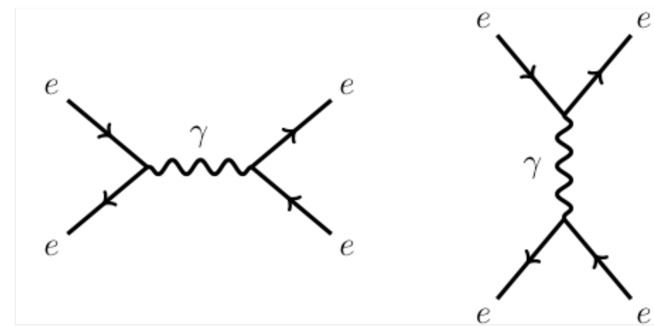
Golden Rule for Scattering

We can build the form for particle scattering

Suppose we have two particle scattering: 1,2 → 3,4,…,n

Don't worry about the inertial frame just yet.

$$\sigma = \frac{S\,\hbar^2}{4\sqrt{(p_1 \cdot p_2)^2 - (m_1 m_2 c^2)^2}} \int |\mathcal{M}|^2 \rho(E)$$



Following the book's derivation for 2-body scattering (1,2→3,4 in CM frame) (In natural units):

$$\frac{d\sigma}{d\Omega} = \left(\frac{1}{8\pi}\right)^2 \frac{S|M|^2}{(E_1 + E_2)^2} \frac{|\vec{p}_f|}{|\vec{p}_i|}$$