Summary of *Hadronic-vacuum-polarization contribution to the muon's anomalous magnetic moment from four-flavor lattice QCD*[1]

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I INTRODUCTION

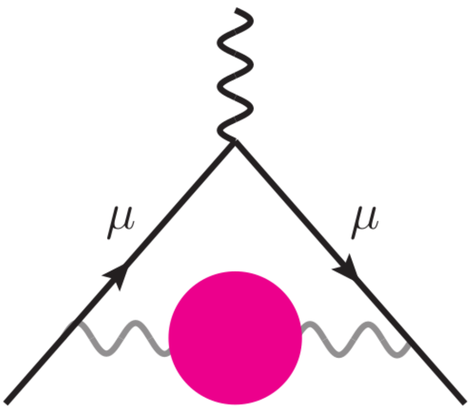

Figure I.1: Figure 1 from [1].

The objective of this paper is to calculate the contribution of the muon anomalous magnetic moment hadronic vacuum polarization from the connected diagrams of up and down quarks, omitting electromagnetism. In order to do this, It employs QCD guage-field configurations with dynamical u, d, s, and c quarks and the physical pion mass, and analyze five ensembles with different lattice spacings ($a \approx 0.06$fm to $a \approx 0.15$fm).

II BACKGROUND AND METHODOLOGY

The relation between the leading-order hadronic-vacuum-polarization contribution to the muon's anomalous magnetic moment and the renormalized quark vacuum polarization function $\hat{\Pi}(Q^2) \equiv \Pi(Q^2) - \Pi(0)$, is given by

$$a_\mu^{HVP,LO} = \left(\frac{\alpha}{2}\right)^2 \int_0^\infty \mathrm{d}Q^2\, K_E(Q^2)\hat{\Pi}(Q^2). \tag{II.1}$$

The current-current correlation function, which is used throughout the paper to calculate the muon's anomalous magnetic moment is given by

$$G(t) = \frac{1}{27}\int \mathrm{d}\mathbf{x} \left[4\left\langle j_i^u(\mathbf{x},t) j_i^u(0,0)\right\rangle + \left\langle j_i^d(\mathbf{x},t), j_i^d(0,0)\right\rangle\right]. \tag{II.2}$$

Using the current-current correlation function, one calculates the quark vacuum polarization function to be

$$\hat{\Pi}(\omega^2) = \frac{4\pi^2}{\omega^2}\int_0^\infty \mathrm{d}t\, G(t)\left[\omega^2 t^2 - 4\sin^2\left(\frac{\omega t}{2}\right)\right]. \tag{II.3}$$

This can then be used to achieve the goal of the paper, which is to calculate equation II.1.

The traditional and currently still most precise determinations of $a_\mu^{\mathrm{HVP,LO}}$ use dispersive methods to obtain the vacuum-polarization function from experimental "R-ratio" data:

$$\hat{\Pi}(Q^2) = \frac{Q^2}{3}\int_0^\infty \mathrm{d}s\, \frac{R_\gamma(s)}{s(s+Q^2)}, \tag{II.4}$$

where

$$R_\gamma(s) \equiv \frac{\sigma(e^+e^- \to \gamma^* \to \text{hadrons})}{4\pi\alpha(s)^2/(3s)}. \tag{II.5}$$

III LATTICE-QCD CALCULATION

A Numerical Simulation

A calculation on QCD gauge-field configurations generated by the MILC Collaboration with four flavors of HISQ quarks was performed. These configurations are isospin-symmetric, i.e., the up and down sea-quark masses are equal with a mass $m_l = (m_u + m_d)/2$.

Two of the ensembles listed in table III.1 were also used in [2]: the $a \approx 0.15$fm ensemble with approximately 1000 configurations and the $a \approx 0.12$fm ensemble. In addition, a new ensemble is included with $a \approx 0.15$fm and parameters identical to the older $a \approx 0.15$fm physical-mass ensemble, except for having better tuned quark masses. Thus, comparing this high-statistics ensemble and the older low-statistics one enables us to test our methods for extracting all a_μ^{ll}(conn.) from noisy data.

On the three newer ensembles analyzed in this work, in addition, a cost-effective variance-reduction technique called the truncated solver method (TSM) was employed, which reduces computational costs by a factor of > 2.

$\approx a$(fm)	$am_l^{\text{sea}}/am_s^{\text{sea}}/am_c^{\text{sea}}$	w_0/a	$Z_{V,\bar{s}s}$	M_{π_5}(MeV)	$E_{2\pi,\min}$(MeV)	$(L/a)^3 \times (T/a)$	$N_{\text{conf.}}$	N_{wall}
0.15	0.00235/0.0647/0.831	1.13670(50)	0.9881(10)	133.04(70)	640.4(3.4)	$32^3 \times 48$	997	16
0.15	0.002426/0.0673/0.8447	1.13215(35)	0.9881(10)	134.73(71)	639.7(3.4)	$32^3 \times 48$	9362	48(TSM)
0.12	0.00184/0.0507/0.628	1.41490(60)	0.99220(40)	132.73(70)	540.8(3.3)	$48^3 \times 64$	998	16
0.09	0.00120/0.0363/0.432	1.95180(70)	0.99400(50)	128.34(68)	524.3(2.8)	$64^3 \times 96$	1557	16(TSM)
0.06	0.0008/0.022/0.260	3.0170(23)	0.9941(11)	134.95(72)	530.8(2.8)	$96^3 \times 192$	1230	16(TSM)

Table III.1: "Table I" from [1].

B Extraction of Muon Anomaly

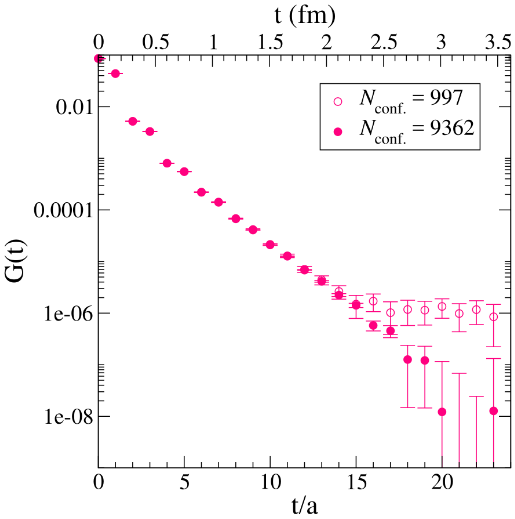

Figure III.1: Figure 2 from [1].

A challenge common to all lattice-QCD calculations of a_μ^{ll} (conn.) is the large statistical noise in the vector- current correlator at the physical light-quark mass. Figure III.1 shows the local-local vector-current correlator $G(t)$ on the two $a \approx 0.15$fm ensembles. The correlator values at times t and $T-t$ are averaged to increase statistics. Beyond this range, the data with low statistics become too noisy to yield a reliable estimate of the correlation function. Several strategies to address the noise problem can used. In this paper, the same strategy is used as in [2].

C Lattice Corrections and Continuum Extrapolation

Before extrapolating the values obtained for a_μ^{ll}(conn.) in the previous section to zero lattice spacing, The data are corrected for the finite lattice spatial volume and for dis cretization effects from the mass splittings between staggered pions of different tastes. Both effects arise from one-loop diagrams with $\pi\pi$ intermediate states.

For staggered quarks, the sea-pion masses are heavier than the taste-Goldstone pion for other representations of the approximate SO(4) taste symmetry. Consequently, the combined finite-volume plus discretization corrections are largest for the coarsest lattices, and decrease toward the continuum.

One can test the estimates of the lattice corrections by comparing our results for the Taylor coefficients of the vacuum-polarization function with phenomenological determinations from R-ratio data. Because the experimental data include all possible diagrammatic contributions, for this test, the full one-loop correction is used, which includes both the connected and disconnected pieces.

IV RESULTS

In this section, the results for a_μ^{ll}(conn.), Π_1^{ll}, Π_2^{ll}, and $a_\mu^{\text{HVP,LO}}$ and the slope and curvature of $\hat{\Pi}(Q^2)$ comprehensive error budgets are presented. There are several types of corrections to the values calculated. These, along with the found values are shown in table IV.1.

A Light-Quark Connected Contribution

Their numerical calculation of a_μ^{ll}(conn.) and the slope and curvature of the renormalized vacuum-polarization function is with equal up- and down-quark masses, and without electromagnetism. The results in the isospin-symmetric limit are

$$a_\mu^{ll}(\text{conn.}) = 637.8(8.8) \times 10^{-10}, \tag{IV.1}$$
$$\Pi_1^{ll}(\text{conn.}) = 0.0932(14)\text{GeV}^2, \tag{IV.2}$$
$$\Pi_2^{ll}(\text{conn.}) = -0.2089(64)\text{GeV}^4. \tag{IV.3}$$

A total uncertainty of 1.4% on the light-quark connected contribution to $a_\mu^{\text{HVP,LO}}$ in the isospin-symmetric limit without electromagnetism is obtained.

B Isospin-Breaking, Electromagnetic, and Quark-Disconnected Contributions

To be able to compare the total summed over all quark flavors with experiment, one needs to correct the result for a_μ^{ll}(conn.) (equation IV.1)

1 $\pi\pi$ Corrections

A large part of the isospin, electromagnetic, and quark-disconnected corrections comes from diagrams in figure I.1 with $\pi\pi$ intermediate states. These corrections can be estimated using the leading term in the chiral model.

Achieved was a total $\pi\pi$ correction to $a_\mu^{\text{HVP,LO}}$ from strong-isospin breaking, electromagnetism, and quark-disconnected diagrams

$$\Delta a_\mu^{\pi\pi}(\text{disc.}) = -12(3) \times 10^{-10}. \tag{IV.1}$$

2 Residual Light-Quark Disconnect Corrections

There are also quark-line disconnected corrections to $a_\mu^{\text{HVP,LO}}$ that have nothing to do with the $\pi\pi$ contribution. One can estimate these by examining the contributions to the anomaly

Correction for	Correction value
$\pi\pi$ corrections	$\Delta a_\mu^{\pi\pi}(\text{disc.}) = -12(3) \times 10^{-10}$
non-$\pi\pi$ corrections	$\Delta a_\mu^{\rho\omega}(\text{disc.}) = -5(5) \times 10^{-10}$
Strong-Isospin Breaking (SIB) corrections	$\Delta a_\mu^{ud}(\text{SIB}) = 10(10) \times 10^{-10}$
QED corrections	$\Delta a_\mu^{ud}(\text{QED}) = 0(5) \times 10^{-10}$

Table IV.1: This table shows each of the light-quark corrections to equation IV.1. Summing these with equation IV.1, and corrections for heavy-quarks (see section IVC), gives the hadron-vacuum-polarization contribution to the muon's anomalous magnetic moment, $a_\mu^{\text{HVP,LO}}$.

from the ρ and ω mesons. These resonances, together, account for almost 80% of the total $a_\mu^{\text{HVP,LO}}$. A disconnected contribution from non-$\pi\pi$ states is achieved:

$$\Delta a_\mu^{\rho\omega}(\text{disc.}) = -5(5) \times 10^{-10}. \tag{IV.2}$$

3 Residual Strong-Isospin Breaking Corrections

The effects from QCD-isospin breaking (i.e., quark-mass differences) and QED are intertwined both in nature and in lattice-QCD simulations. In the paper, the residual strong-isospin correctioon to $a_\mu^{\text{HVP,LO}}$ is defined as the shift relative to the isospin-symmetric value $a_\mu^{ll}(\text{conn.})$ that results when the bare u and d quark masses are retuned separately. A relative correction of $\delta a_\mu^{\text{HVP,LO}}(\text{SIB}) = 1.5(7)\%$ was found, which translates to an absolute correction of $\Delta a_\mu^{\text{HVP,LO}}(\text{SIB}) = 9.5(4.5) \times 10^{-10}$, when commined with $a_\mu^{ll}(\text{conn.})$ from IV.1. It is concluded by increasing the errors on their initial estimate of the total residual correction from strong-isospin breaking to allow for disconnected contributions of a commensurate size, giving

$$\Delta a_\mu^{ud}(\text{SIB}) = 10(10) \times 10^{-10}. \tag{IV.3}$$

4 Residual QED Corrections

The residual corrections from QED is estimated, beyond those accounted for above, via power-counting to be of order $\sim 1\%$. This yields an estimate for the absolute correction to $a_\mu^{\text{HVP,LO}}$ of

$$\Delta a_\mu^{ud}(\text{QED}) = 0(5) \times 10^{-10}. \tag{IV.4}$$

5 Total Contribution from u/d Quarks

Summing the corrections from equations IV.1 and equations IV.2-IV.4 gives the total correction from strong-isospin breaking, QED, and quark-disconnectedcontributions:

$$\Delta a_\mu^{ud}(\text{SIB,QED,disc.}) = -7(13) \times 10^{-10}. \tag{IV.5}$$

Adding this to $a_\mu^{ll}(\text{conn.})$ (equation IV.1) gives the total contribution to $a_\mu^{HVP,LO}$ from light quarks:

$$a_\mu^{ud} = 630.8(8.8)(13) \times 10^{-10}, \tag{IV.6}$$

where the first error is from $a_\mu^{ll}(\text{conn.})$ and the second error is from Δa_μ^{ud}.

C Total Leading-Order Contribution

Finally, to obtain the total leading-order hadronic vacuum polarization contribution to a_μ, the contributions from heavy flavors are added to a_μ^{ud} (equation IV.6):

$$a_\mu^{\text{HVP,LO}} = 699(15)_{u,d}(1)_{s,c,b} \times 10^{-10}. \tag{IV.1}$$

Disconnected contributions from these quarks are expected to be negligible compared with our other uncertainties.

V REFERENCES

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