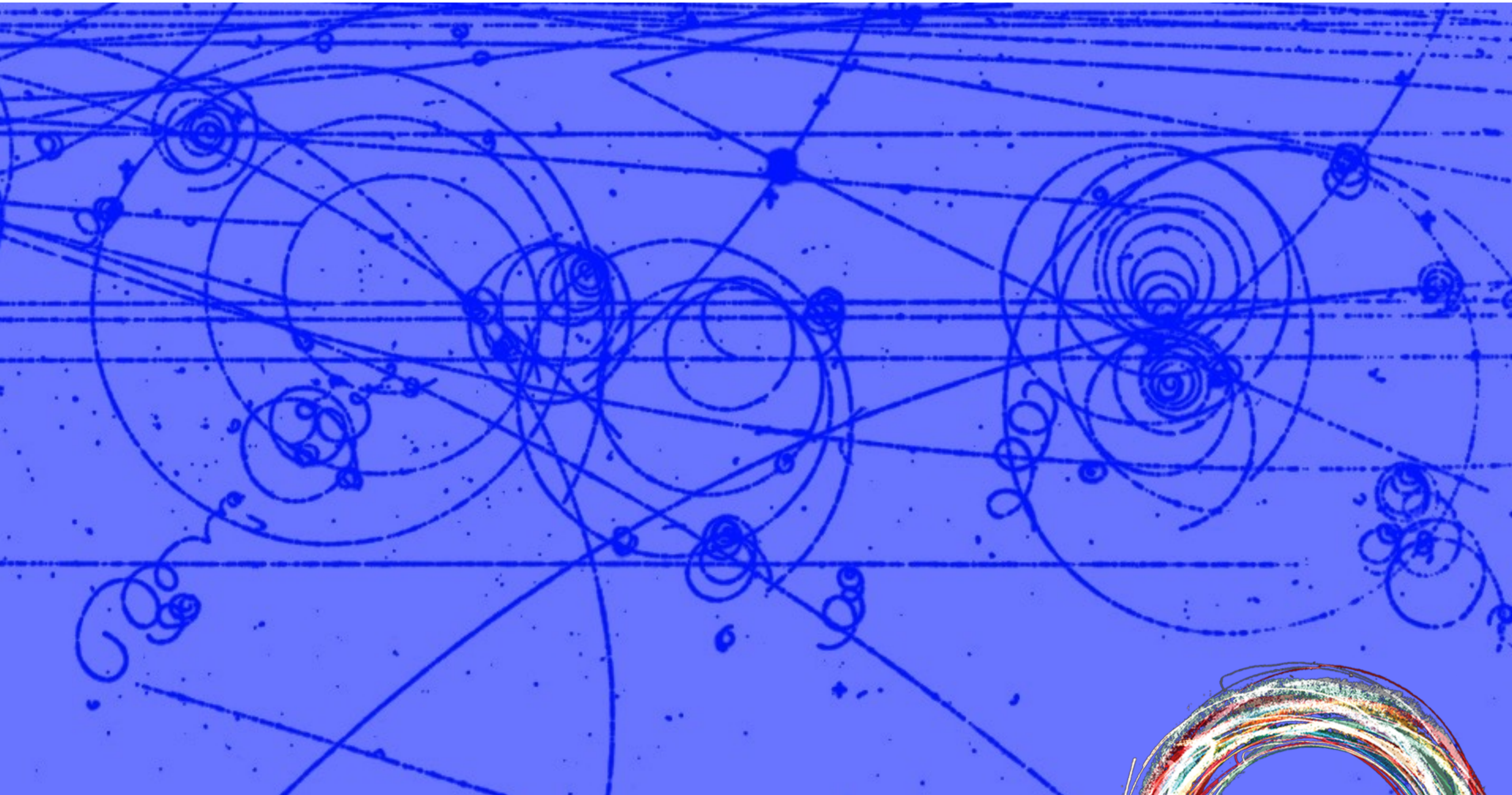


SPIN, TRACE RULES

PHY 493/803

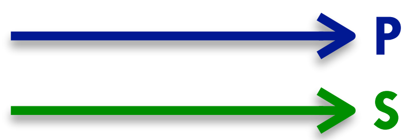
Reminder!

	Fermions	Anti-Fermions
Spinors satisfy the Dirac Equation:	$(\gamma^\mu p_\mu - m)u = 0$	$(\gamma^\mu p_\mu + m)v = 0$
Adjoint spinors:	$\bar{u} = u^\dagger \gamma^0$	$\bar{v} = v^\dagger \gamma^0$
Adjoints satisfy the Dirac Eqn:	$\bar{u}(\gamma^\mu p_\mu - m) = 0$	$\bar{v}(\gamma^\mu p_\mu + m) = 0$
Orthogonality:	$\bar{u}^{(1)} u^{(2)} = 0$	$\bar{v}^{(1)} v^{(2)} = 0$
Normalization:	$\bar{u}u = 2m$	$\bar{v}v = -2m$
Completeness:	$\sum_{s=1,2} u^{(s)} \bar{u}^{(s)} = (\gamma^\mu p_\mu + m)$	$\sum_{s=1,2} v^{(s)} \bar{v}^{(s)} = (\gamma^\mu p_\mu - m)$

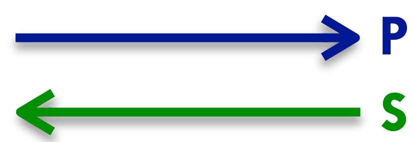
Aside: Chirality vs Helicity

When we say "spin", do we mean helicity? Or chirality?

Helicity is just the relative orientation of the spin and momentum.



Right-handed Helicity



Left-handed Helicity

Chirality is an intrinsic property of a wave function

The chirality of a particle is determined by whether the wave function transforms in a right- or left-handed representation of the Poincare group.

Dirac spinors have both right- and left-handed components.

For massless fermions (which travel at the speed of light), helicity and chirality are the same.

For massive fermions, we can always transform to a reference frame in which the helicity switches. Thus, helicity states are not generally chirality states.

Dealing with Spin

A typical QED amplitude might look something like

$$-\frac{g_e^2}{(p_1+p_2)^2}\left[\bar{v}(p_1)\gamma^\mu u(p_2)\right]\left[\bar{u}(p_4)\gamma_\mu v(p_3)\right]$$

The Feynman rules won't take us any further, but to get a number for $\mathscr{M}$ we will need to substitute explicit forms for the wavefunctions of the external particles

If all external particles have a known polarization, this might be a reasonable way to calculate things. More often, though, we are interested in unpolarized particles.

If we do not care about the polarizations of the particles then we need to

1. Average over the polarizations of the initial-state particles
2. Sum over the polarizations of the final-state particles in the squared amplitude $|\mathscr{M}|^2$.

We call this the spin-averaged amplitude squared (or simply spin-averaged amplitude) and we denote it by $\left\langle |\mathcal{M}|^2 \right\rangle$

*Note that the averaging over initial state polarizations involves summing over all polarizations and then dividing by the number of independent polarizations, so the spin-averaging involves a sum over the polarizations of **all** external particles.*

Dealing with Spin

Let's simplify things and suppose that we have:

$$\mathcal{M} \sim [\bar{u}_1 \Gamma u_2]$$

Then we have:

$$\bar{u} = u^\dagger \gamma^0$$

$$(\gamma^0)^2 = 1$$

$$\gamma^0 \Gamma^\dagger \gamma^0 = \bar{\Gamma}$$

$$\begin{aligned}
|\mathcal{M}|^2 &\sim [\bar{u}_1 \Gamma u_2]\,[\bar{u}_1 \Gamma u_2]^* \\
&\sim [\bar{u}_1 \Gamma u_2]\left[u_1^\dagger \gamma^0 \Gamma u_2\right]^\dagger \\
&\sim [\bar{u}_1 \Gamma u_2]\left[u_2^\dagger \Gamma^\dagger \gamma^{0\dagger} u_1\right] \\
&\sim [\bar{u}_1 \Gamma u_2]\left[u_2^\dagger \gamma^0 \gamma^0 \Gamma^\dagger \gamma^0 u_1\right] \\
&\sim [\bar{u}_1 \Gamma u_2]\,[\bar{u}_2 \bar{\Gamma} u_1]
\end{aligned}$$

Dealing with Spin

We have worked up to:

$$|\mathcal{M}|^2 \sim \quad [\bar{u}_1 \Gamma u_2] \, [\bar{u}_2 \bar{\Gamma} u_1]$$

We can simplify by applying the completeness relation to the 2nd particle (u_2):

$$\sum_{s_i=1,2} u_i^{s_i} \bar{u}_i^{s_i} = (\not{p}_i + m_i)$$

Then we get:

$$\begin{aligned} \sum_{s_2} |\mathcal{M}|^2 \quad &\sim \quad [\bar{u}_1 \Gamma (\not{p}_2 + m_2) \bar{\Gamma} u_1] \\ &\sim \quad [\bar{u}_1 Q u_1] \end{aligned}$$

Dealing with Spin

We have worked up to:

$$\sum_{s_2} |\mathcal{M}|^2 \sim [\bar{u}_1 \Gamma (\not{p}_2 + m_2) \bar{\Gamma} u_1]$$
$$\sim [\bar{u}_1 Q u_1]$$

The RHS is just a number, but we can rewrite the matrix multiplication with summations over indices and simplify:

$$\begin{aligned}
[\bar{u}_1 Q u_1] &= (\bar{u}_1)_i Q_{ij} (u_1)_j \\
&= Q_{ij} (u_1 \bar{u}_1)_{ji} \\
&= [Q (u_1 \bar{u}_1)]_{ii} \\
&= \mathrm{Tr}\,[Q(u_1 \bar{u}_1)]
\end{aligned}$$

Multiplication expanded

$$\bar{u}_i u_j = (u^\dagger \gamma^0)_i u_j = \left\{ \begin{pmatrix} u_1^* & u_2^* & u_3^* & u_4^* \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \right\}_i u_j$$

$$u\bar{u} = u u^\dagger \gamma^0 = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} \begin{pmatrix} u_1^* & u_2^* & u_3^* & u_4^* \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$\bar{u}_i u_j = \begin{array}{rl} u_i^* u_j & (i = 1, 2) \\ -u_i^* u_j & (i = 3, 4) \end{array} \qquad (u\bar{u})_{ij} = \begin{array}{rl} u_i u_j^* & (j = 1, 2) \\ -u_i u_j^* & (j = 3, 4) \end{array}$$

Dealing with Spin

We have worked up to:

$$\begin{aligned}[\bar{u}_1 Q u_1] &= (\bar{u}_1)_i Q_{ij} (u_1)_j \\ &= Q_{ij} (u_1 \bar{u}_1)_{ji} \\ &= [Q (u_1 \bar{u}_1)]_{ii} \\ &= \mathrm{Tr}\,[Q(u_1 \bar{u}_1)]\end{aligned}$$

Next, we apply the completeness relation once again, so that we get

$$\sum_{s_1} |\mathcal{M}|^2 \sim \mathrm{Tr}\,[Q(\not{p}_1 + m_1)]$$

Thus in total we have:

$$\left\langle |\mathcal{M}|^2 \right\rangle \sim F \cdot \mathrm{Tr}\left[\Gamma(\not{p}_2 + m_2)\bar{\Gamma}(\not{p}_1 + m_1)\right]$$

F= 1/4 (2 initial state fermions)
1/2 (1 initial state fermion)
1 (2 initial state photons)

Casimir's Trick

The procedure of calculating spin-averaged amplitudes in terms of traces is known as Casimir's Trick. We typically get two such traces.

$$\sum_{\text{all spins}} [\bar{u}_a \Gamma_1 u_b] [\bar{u}_a \Gamma_2 u_b]^* = \mathrm{Tr}\left[\Gamma_1 (\not{p}_b + m_b) \bar{\Gamma}_2 (\not{p}_a + m_a)\right]$$

Where Γ_1 and Γ_2 are 4x4 matrices (combinations of gamma matrices)
$\bar{\Gamma} = \gamma^0 \Gamma^\dagger \gamma^0$

The Γ matrices we are encountering in QED are just γ^μ or γ^ν, and
$\bar{\gamma}_\mu = \gamma_0 \gamma_\mu^\dagger \gamma_0 = \gamma_\mu$

If antiparticle spinors (v) are present in the spin sum, we use the corresponding completeness relation

$$\sum_{s_i=1,2} v_i^{s_i} \bar{v}_i^{s_i} = (\not{p}_i - m_i)$$

Casimir's Trick

For one typical t-channel line (i.e. one vertex), connecting initial to final state particles, we get

$$\sum_{\text{all spins}} [\bar{u}_a \Gamma_1 u_b] [\bar{u}_a \Gamma_2 u_b]^* = \mathrm{Tr}\left[\Gamma_1 (\not{p}_b + m_b) \bar{\Gamma}_2 (\not{p}_a + m_a)\right]$$

The cross-section will be proportional to the average matrix element squared,

$$\left\langle |M|^2 \right\rangle \propto F \cdot Tr[] \cdot Tr[]$$

$F=$ 1/4 (2 initial state fermions)
1/2 (1 initial state fermion)
1 (2 initial state photons)

Useful Trace Theorems

Because of Casimir's Trick, we're going to find ourselves calculating a lot of traces involving γ-matrices. Some general identities about traces:

$$Tr\left(A+B\right)=Tr\left(A\right)+Tr\left(B\right)$$

$$Tr\left(\alpha A\right)=\alpha Tr\left(A\right)$$

$$Tr(\gamma^5)=0$$

The trace of the product of an odd number of γ matrices is 0

The trace of the product of γ^5 and an odd number of γ matrices is 0

$$Tr\left(\gamma^\mu\gamma^\nu\right)=4g^{\mu\nu}$$

$$Tr(\gamma^5\gamma^\mu\gamma^\nu)=0$$

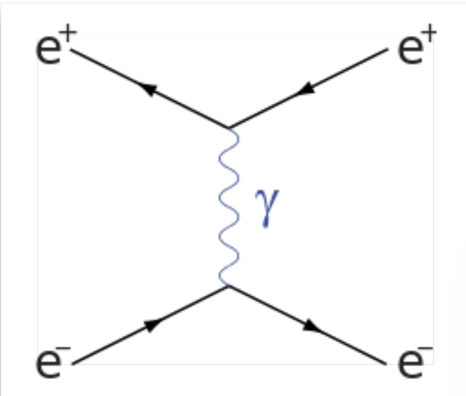
$$Tr\left(\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\sigma\right)=4\left(g^{\mu\nu}g^{\lambda\sigma}-g^{\mu\lambda}g^{\nu\sigma}+g^{\mu\sigma}g^{\nu\lambda}\right)$$

$$Tr(\gamma^5\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\sigma)=4i\epsilon^{\mu\nu\lambda\sigma}$$

Trace to cross-section

- The cross-section calculation usually has two of these traces
- Thus, there will be products of the form $p_i^{\mu} p_{j,\mu} = p_i \cdot p_j$
- In the CM system, we can use relationships
  - $p_1 \cdot p_2 = p_3 \cdot p_4 = E^2 + p^2 \approx 2E^2$
  - $p_1 \cdot p_3 = p_2 \cdot p_4 = E^2 - p_1 p_3 \cos\theta$
  - $p_1 \cdot p_4 = p_2 \cdot p_3 = E^2 + p_1 p_4 \cos\theta$
  - $s = (p_1 + p_2)^2 \approx 4E$

Trace Example



Consider Bhabha scattering. For the lower current line, we have

$[\bar{u}(3)\gamma^{\mu}u(1)][[\bar{u}(3)\gamma^{\nu}u(1)]^{*}$, and we get:

$$T = \mathrm{Tr}\left[\gamma^{\mu}(\not{p}_1 + m)\gamma^{\nu}(\not{p}_3 + m)\right]$$

We can expand this out to create 4 terms, but 2 of these terms (the ones linear in m) will involve 3 γ-matrices, and are therefore zero. Thus, we have:

$$\begin{aligned} T &= \mathrm{Tr}(\gamma^{\mu}\not{p}_1\gamma^{\nu}\not{p}_3) + m^2\mathrm{Tr}(\gamma^{\mu}\gamma^{\nu}) \\ &= 4\left(p_1^{\mu}p_3^{\nu} + p_3^{\mu}p_1^{\nu} - (p_1 \cdot p_3)g^{\mu\nu}\right) + 4m^2 g^{\mu\nu} \end{aligned}$$

Similarly for the upper current line, e+ to e+, but replace +m by -m and u by v. This will give a Trace resulting in covariant vectors, $p_{i,\mu}$