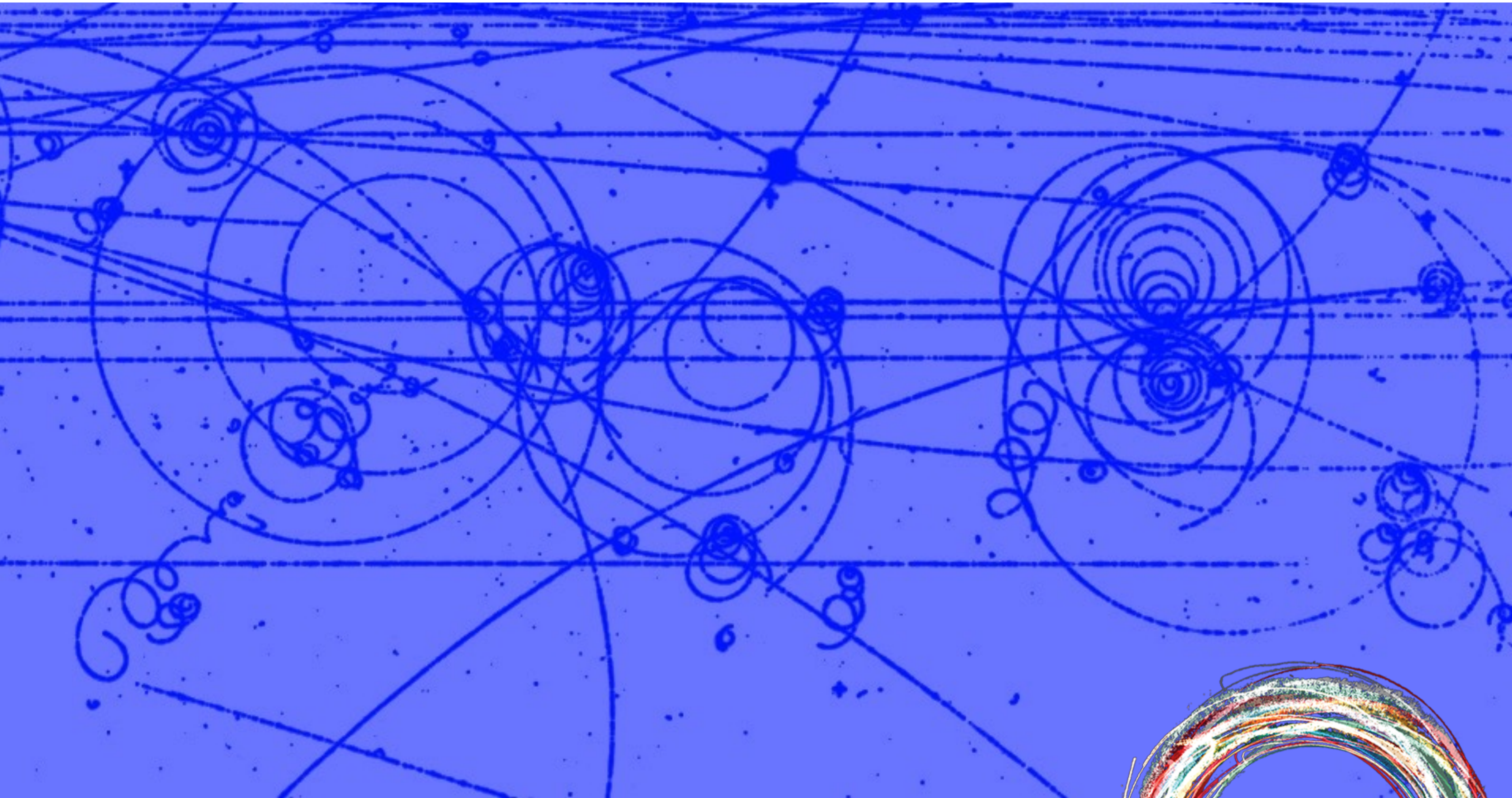

GAMMA MATRICES

PHY 493/803

Dirac equation

The Dirac equation written with gamma matrices

$$i\left(\gamma^0\frac{\partial}{\partial t}+\vec{\gamma}\cdot\nabla\right)\psi=m\psi$$

$$\vec{\gamma}=\begin{pmatrix}\gamma^1\\ \gamma^2\\ \gamma^3\end{pmatrix}$$

The gamma's are coefficients that make sure the relativistic energy-momentum relationship is satisfied. Those coefficients are actually matrices!

Gamma Matrices

The four gamma matrices fulfill the equations:

(1) $$(\gamma^0)^2 = 1$$

(2) $$(\gamma^{1,2,3})^2 = -1$$

(3) $$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 0$$ For $\mu \neq \nu$

The γ^μ are unitary and anti-commute.

The Dirac Equation

$$i\left(\gamma^0\frac{\partial}{\partial t}+\vec{\gamma}\cdot\nabla\right)\psi=m\psi$$

$$i\gamma^\mu\partial_\mu\psi=m\psi$$

$$(i\gamma^\mu\partial_\mu-m)\psi=0$$

Slash Notation

When we contract γ^μ with a four-vector q_μ, we can abbreviate this using the Feynman slash notation

$$\gamma^\mu q_\mu = \not{q}$$

With the slash notation, the Dirac equation becomes

$$(i\gamma^\mu \partial_\mu - m)\psi = 0$$

$$\not{p} - m = 0$$

$$(i\not{\partial} - m)\psi = 0$$

Gamma Matrices

We had 3 conditions, including an anti-commutation relation.

Can't do this with numbers since they commute (AB=BA always) but we can do it with matrices (which do not, in general, commute).

$$(\gamma^0)^2 = 1 \quad (\gamma^{1,2,3})^2 = -1 \quad \{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 0$$ For $\mu \neq \nu$

Dirac's clever idea was to let γ represent a set of 4x4 matrices

$$\gamma^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \qquad \gamma^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

$$\gamma^2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix} \qquad \gamma^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

Gamma matrices and Pauli matrices

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$

$$\gamma^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \qquad \gamma^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

$$\gamma^2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix} \qquad \gamma^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

Gamma matrices and Pauli matrices

The "Bjorken & Drell" convention is often used to reduce the notation.
Clearly the physics is independent of the representation.

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$

$$1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Pauli spin matrices:

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Gamma matrices and Pauli matrices

The "Bjorken & Drell" convention is often used to reduce the notation. Clearly the physics is independent of the representation.

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We'll use the anti-commutation relation later, so it's worthwhile noting a feature of it:

$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 0 \quad \text{For } \mu \neq \nu$$

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$$

Vectors and metric tensor

- We handle gamma matrices analogously to vectors, where we defined the a vector p^μ and the covariant vector $p_\mu = g_{\mu\nu} p^\nu$
  - The metric tensor $g_{\mu\nu} = g^{\mu\nu}$ has 1, -1, -1, -1 on the diagonal and zeroes elsewhere
- Then the inner product is $p_\mu p^\mu = p_0^2 - p_1^2 - p_2^2 - p_3^2$ where p_i are the components of the vector
- We apply the same to the gamma matrices

Gamma matrix upper and lower indices

- We can multiply two gamma matrices together to get another gamma matrix, and the matrices anti-commute:
  $\gamma^{\mu}\gamma^{\nu} + \gamma^{\nu}\gamma^{\mu} = 2g^{\mu\nu}$
- The index of a gamma matrix can be lowered, $\gamma_{\mu} = g_{\mu\nu}\gamma^{\nu}$, where having the same label upper and lower implies a sum and the transpose of the matrix,

$$g_{\mu\nu}\gamma^{\nu} = \sum_{i=0}^{3} g_{\mu,i}\,\gamma^{i}$$

Gamma matrix inner product

- The inner product $\gamma^\mu \gamma_\mu$ of a gamma matrix is $\gamma^\mu g_{\mu\nu} \gamma^\nu$, so
  $\gamma^\mu \gamma_\mu = \gamma_0^2 - \gamma_1^2 - \gamma_2^2 - \gamma_3^2$
  - And γ_0^2 is the identity matrix, while each of the other squares is -1 times the identity matrix
  - Thus, $\gamma^\mu \gamma_\mu = \gamma_\mu \gamma^\mu = 4$, i.e. 4 times the identity matrix

Gamma matrix products

- A combination of upper and lower indices implies summation over gamma matrices
- For example $\gamma_\mu \gamma^\nu \gamma^\mu$ is a sum over the first and last gamma matrix, $\gamma_\mu \gamma^\nu \gamma^\mu = g_{\mu\lambda} \gamma^\lambda \gamma^\nu \gamma^\mu$
  - To flip the order of two gamma matrices, use the anti-commutator, $\gamma^\nu \gamma^\mu = 2g^{\mu\nu} - \gamma^\mu \gamma^\nu$
  - Then $\gamma_\mu \gamma^\nu \gamma^\mu = \gamma_\mu (2g^{\mu\nu} - \gamma^\mu \gamma^\nu) = 2\gamma^\nu - 4\gamma^\nu = -2\gamma^\nu$