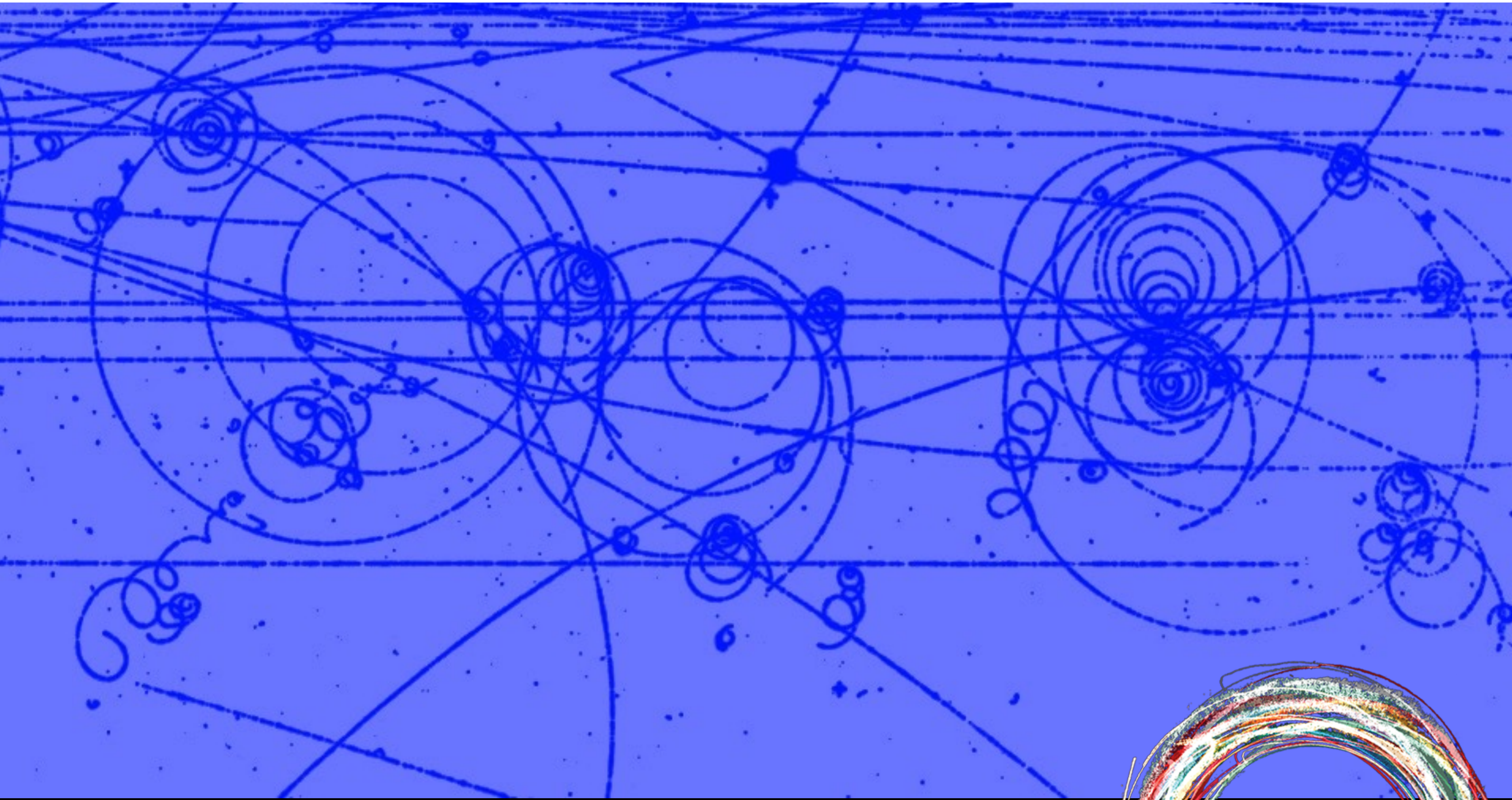


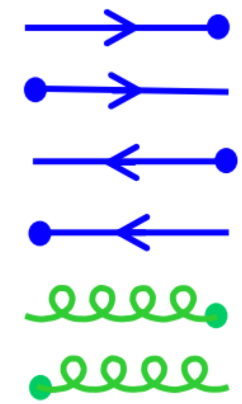
QCD FEYNMAN RULES, COLOR FACTORS

PHY 493/803

Feynman Rules for QCD

External lines:

spin 1/2	incoming quark	$u(p)$	
	outgoing quark	$\overline{u}(p)$	
	incoming anti-quark	$\overline{v}(p)$	
	outgoing anti-quark	$v(p)$	
spin 1	incoming gluon	$\varepsilon^{\mu}(p)$	
	outgoing gluon	$\varepsilon^{\mu}(p)^{*}$	



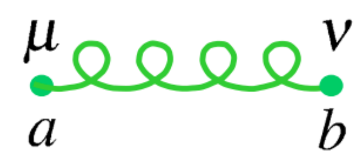
Internal lines (propagators):

spin 1/2 quark

$$\boxed{\frac{i(\not{q}+m)}{q^2-m^2}}$$

spin 1 gluon

$$\boxed{\frac{-ig_{\mu\nu}}{q^2}\delta^{ab}}$$

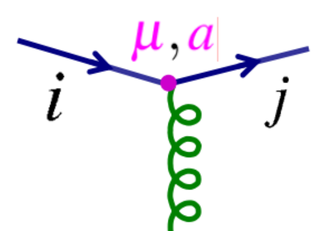


a, b = 1,2,…,8 are gluon color indices

Vertex factors:

spin 1/2 quark

$$\boxed{-ig_s\tfrac{1}{2}\lambda^{a}_{ji}\gamma^{\mu}}$$



i, j = 1,2,3 are quark colors,

λ^a a = 1,2,..8 are the Gell-Mann SU(3) matrices

Feynman Rules for QCD

Color factor:

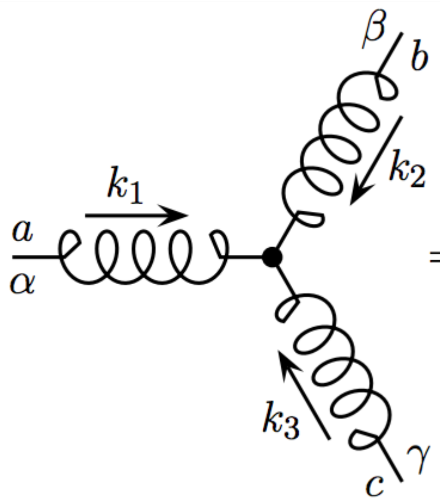
- **Determine possible color flow (color is conserved at each vertex)**
- **Sum over all possible color configurations**
- **Divide by number of possible initial state color states (if averaging).**

- If we start from a specific color configuration (such as the examples in the book do) then just sum
- If we start from an unknown configuration (for example in proton-proton collisions), then average

Feynman Rules for QCD

We also have the 3- and 4-gluon vertex factors.

We will look at these here, but not focus on them in this course.

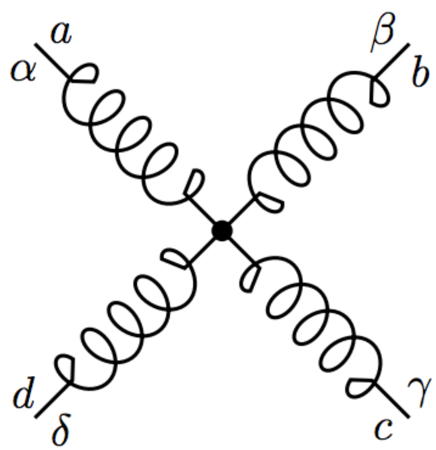


$$= -g f^{abc}\left[g^{\alpha\beta}(k_1-k_2)^{\gamma} + g^{\beta\gamma}(k_2-k_3)^{\alpha} + g^{\gamma\alpha}(k_3-k_1)^{\beta}\right]$$

Proportional to g, at same level as quark-gluon interactions

f^{abc} are numbers (structure constants) obtained via

$$[\lambda^{\alpha}, \lambda^{\beta}] = 2i\, f^{\alpha\beta\gamma}\, \lambda^{\gamma}$$



$$= -ig^2 \left[\begin{array}{c} f^{abe} f^{cde}(g^{\alpha\gamma}g^{\beta\delta} - g^{\alpha\delta}g^{\beta\gamma}) \\ + f^{ace} f^{bde}(g^{\alpha\beta}g^{\gamma\delta} - g^{\alpha\delta}g^{\gamma\beta}) \\ + f^{ade} f^{bce}(g^{\alpha\beta}g^{\delta\gamma} - g^{\alpha\gamma}g^{\delta\beta}) \end{array}\right]$$

Suppressed because proportional to g^2

The Quark-Gluon Interaction

We need to modify the spin-1/2 solutions to the Dirac equation with color vectors:

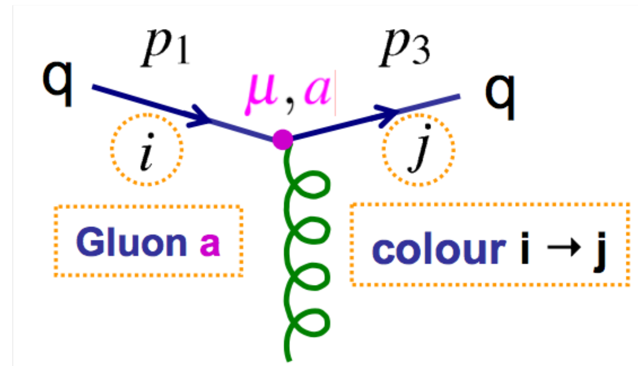
$$r = c_1 = \begin{pmatrix}1\\0\\0\end{pmatrix} \quad g = c_2 = \begin{pmatrix}0\\1\\0\end{pmatrix} \quad b = c_3 = \begin{pmatrix}0\\0\\1\end{pmatrix}$$

$$u(p) \longrightarrow c_i u(p)$$

The QCD qgq vertex is written:

$$\overline{u}(p_3)c_j^\dagger\{-\tfrac{1}{2}ig_s\lambda^a\gamma^\mu\}c_i u(p_1)$$

Only difference w.r.t. QED is the insertion of the 3x3 SU(3) Gell-Mann matrices & color vectors.



Isolating the color part

$$c_j^\dagger\lambda^a c_i = c_j^\dagger \begin{pmatrix}\lambda^a_{1i}\\ \lambda^a_{2i}\\ \lambda^a_{3i}\end{pmatrix} = \lambda^a_{ji}$$

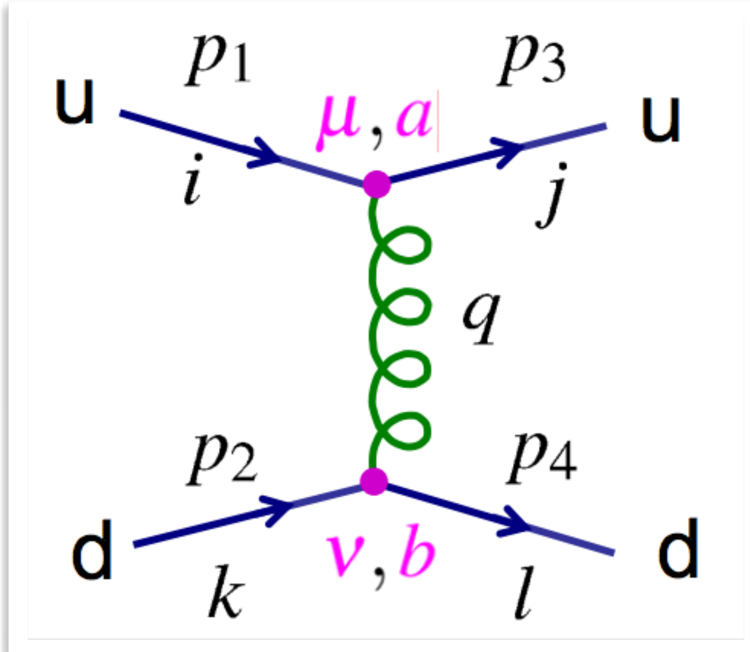
The fundamental quark - gluon QCD interaction can be written as:

$$\overline{u}(p_3)c_j^\dagger\{-\tfrac{1}{2}ig_s\lambda^a\gamma^\mu\}c_i u(p_1) \equiv \overline{u}(p_3)\{-\tfrac{1}{2}ig_s\lambda^a_{ji}\gamma^\mu\}u(p_1)$$

Quark-Quark Scattering

Consider scattering of an up and a down quark.

- The incoming and out-going quark colors are labelled by **i,j,k,l = {1,2,3} or {r,g,b}**
- Thus in terms of color scattering, this is: **ik→jl**
- The 8 possible gluons are accounted for by the color indices: **a,b = 1,2,….,8**
- **NOTE:** The delta function in the propagator ensures **a=b**. The gluon emitted at a is the same as is absorbed at b.



Apply Feynman rules:

$$\mathcal{M} = i[\bar{u}_u c_3^\dagger]\left[-i\frac{g_s}{2}\lambda^a_{ji}\gamma^\mu\right][u_u c_1]\left(\frac{-ig_{\mu\nu}\delta^{ab}}{q^2}\right)[\bar{u}_d c_4^\dagger]\left[-i\frac{g_s}{2}\lambda^b_{lk}\gamma^\nu\right][u_d c_2]$$

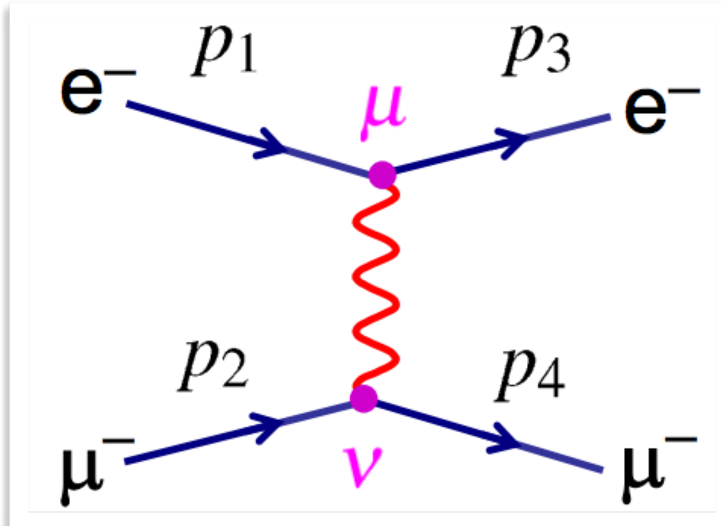
$$\mathcal{M} = -\frac{g_s^2}{4q^2}[\bar{u}_u\gamma^\mu u_u][\bar{u}_d\gamma_\mu u_d](c_j^\dagger\lambda^a c_i)(c_l^\dagger\lambda^a c_k)$$

$$\mathcal{M} = -\frac{g_s^2}{4q^2}[\bar{u}_u\gamma^\mu u_u][\bar{u}_d\gamma_\mu u_d](\lambda^a_{ji}\lambda^a_{lk})$$

Comparing QED and QCD

Lepton scattering:

$$\mathcal{M} = -\frac{g_e^2}{q^2}[\bar{u}_e\gamma^\mu u_e][\bar{u}_\mu\gamma_\mu u_\mu]$$



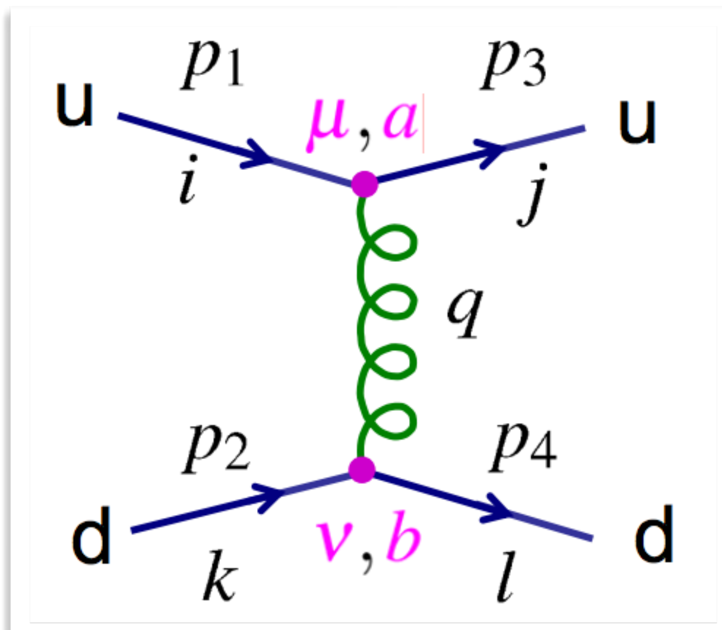
Quark scattering:

$$\mathcal{M} = -\frac{g_s^2}{4q^2}[\bar{u}_u\gamma^\mu u_u][\bar{u}_d\gamma_\mu u_d](\lambda_{ji}^a\lambda_{lk}^a)$$

QCD Matrix Element = QED Matrix Element with:

Strong coupling constant $\alpha = \frac{e^2}{4\pi} \rightarrow \alpha_s = \frac{g_s^2}{4\pi}$

Color Factor $C(ik \rightarrow jl) \equiv \frac{1}{4}\sum_{a=1}^{8}\lambda_{ji}^a\lambda_{lk}^a$



Quark-Gluon Color Factors

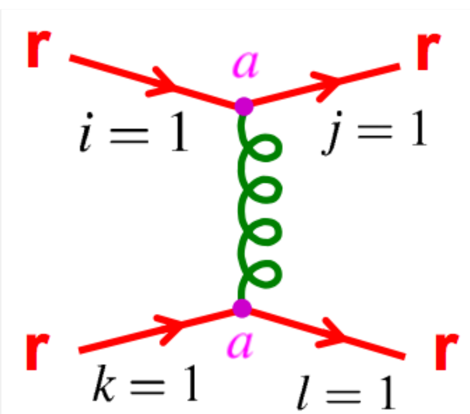
QCD color factors reflect the gluon states that are involved

$$\lambda^1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda^4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \lambda^6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \lambda^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\lambda^2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda^5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \quad \lambda^7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad \lambda^8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

Gluons: $r\bar{g}, g\bar{r}$ $\quad r\bar{b}, b\bar{r}$ $\quad g\bar{b}, b\bar{g}$ $\quad \frac{1}{\sqrt{2}}(r\bar{r} - g\bar{g}) \quad \frac{1}{\sqrt{6}}(r\bar{r} + g\bar{g} - 2b\bar{b})$

Consider a single color



$$C(rr \to rr) = \frac{1}{4}\sum_{a=1}^{8} \lambda^a_{11}\lambda^a_{11} = \frac{1}{4}(\lambda^3_{11}\lambda^3_{11} + \lambda^8_{11}\lambda^8_{11})$$

$$= \frac{1}{4}\left(1 + \frac{1}{3}\right) = \frac{1}{3}$$

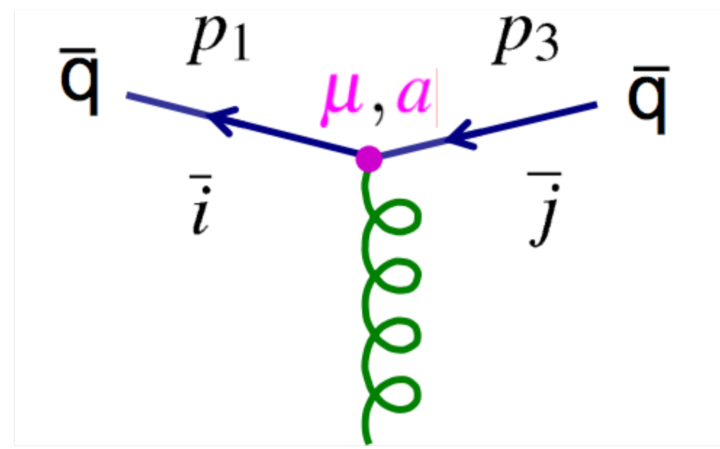
$$C(rr \to rr) = C(gg \to gg) = C(bb \to bb) = \frac{1}{3}$$

Anti-Quark Color Factors

Anti-quarks get somewhat different color factors, giving a subtly different result

Recall the quark-gluon-quark vertex:

$$\bar{u}(p_3)c_j^\dagger\{-\tfrac{1}{2}ig_s\lambda^a\gamma^\mu\}c_i u(p_1)$$



Consider the equivalent anti-quark vertex:

$$\bar{v}(p_1)c_i^\dagger\{-\tfrac{1}{2}ig_s\lambda^a\gamma^\mu\}c_j v(p_3)$$

Note that the incoming anti-particle now enters on the LHS of the expression

Quark vertex:

$$\bar{u}(p_3)c_j^\dagger\{-\tfrac{1}{2}ig_s\lambda^a\gamma^\mu\}c_i u(p_1) \equiv \bar{u}(p_3)\{-\tfrac{1}{2}ig_s\lambda^a_{ji}\gamma^\mu\}u(p_1)$$

Anti-quark vertex:

$$\bar{v}(p_1)c_i^\dagger\{-\tfrac{1}{2}ig_s\lambda^a\gamma^\mu\}c_j v(p_3) \equiv \bar{v}(p_1)\{-\tfrac{1}{2}ig_s\lambda^a_{ij}\gamma^\mu\}v(p_3)$$

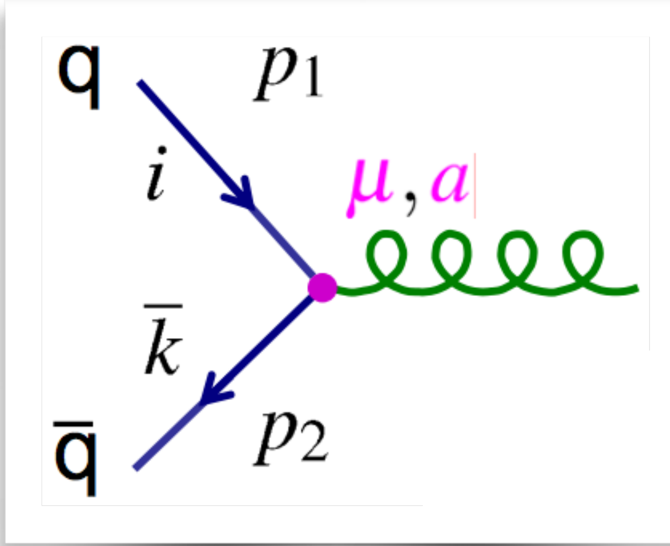
Quark / Anti-Quark Annihilation

Quark-antiquark vertex:

$$\bar{v}(p_2)c_k^\dagger\{-\tfrac{1}{2}ig_s\lambda^a\gamma^\mu\}c_i u(p_1)$$

Color contraction:

$$c_k^\dagger\lambda^a c_i = \lambda_{ki}^a$$



Quark-antiquark vertex:

$$\bar{v}(p_2)c_k^\dagger\{-\tfrac{1}{2}ig_s\lambda^a\gamma^\mu\}c_i u(p_1) \equiv \bar{v}(p_2)\{-\tfrac{1}{2}ig_s\lambda_{ki}^a\gamma^\mu\}u(p_1)$$

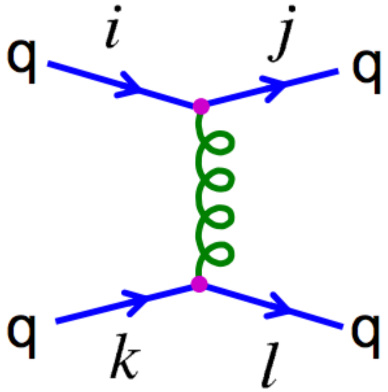
Quark-quark vertex:

$$\bar{u}(p_3)c_j^\dagger\{-\tfrac{1}{2}ig_s\lambda^a\gamma^\mu\}c_i u(p_1) \equiv \bar{u}(p_3)\{-\tfrac{1}{2}ig_s\lambda_{ji}^a\gamma^\mu\}u(p_1)$$

Antiquark-antiquark vertex:

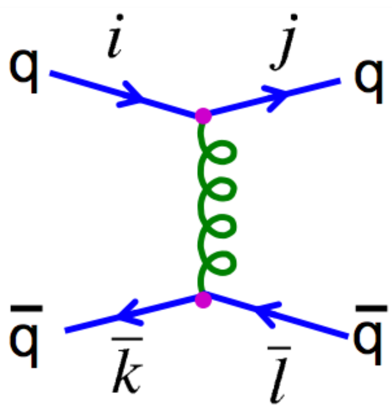
$$\bar{v}(p_1)c_i^\dagger\{-\tfrac{1}{2}ig_s\lambda^a\gamma^\mu\}c_j v(p_3) \equiv \bar{v}(p_1)\{-\tfrac{1}{2}ig_s\lambda_{ij}^a\gamma^\mu\}v(p_3)$$

Color Factor Summary



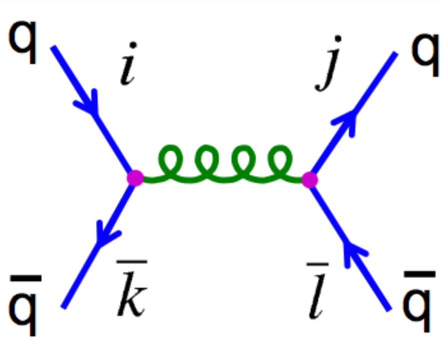
$$\boxed{C(ik \to jl) \equiv \frac{1}{4}\sum_{a=1}^{8} \lambda^a_{ji}\lambda^a_{lk}}$$

$$C(rr \to rr) = \frac{1}{3}$$
$$C(rg \to rg) = -\frac{1}{6}$$
$$C(rg \to gr) = \frac{1}{2}$$



$$\boxed{C(i\bar{k} \to j\bar{l}) \equiv \frac{1}{4}\sum_{a=1}^{8} \lambda^a_{ji}\lambda^a_{kl}}$$

$$C(r\bar{r} \to r\bar{r}) = \frac{1}{3}$$
$$C(r\bar{g} \to r\bar{g}) = -\frac{1}{6}$$
$$C(r\bar{r} \to g\bar{g}) = \frac{1}{2}$$



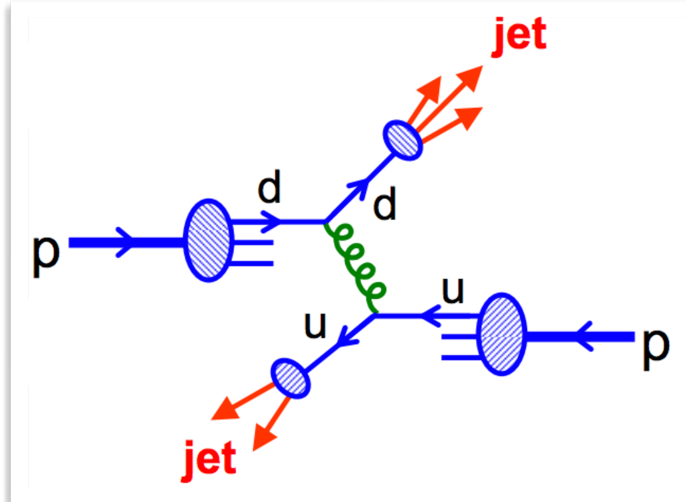
$$\boxed{C(i\bar{k} \to j\bar{l}) \equiv \frac{1}{4}\sum_{a=1}^{8} \lambda^a_{ki}\lambda^a_{jl}}$$

$$C(r\bar{r} \to r\bar{r}) = \frac{1}{3}$$
$$C(r\bar{g} \to r\bar{g}) = \frac{1}{2}$$
$$C(r\bar{r} \to g\bar{g}) = -\frac{1}{6}$$

Quark-quark Scattering

Consider the $u + d \to u + d$ scattering process

- There are nine possible color configurations of the colliding quarks which are all equally likely.
- We need to determine the average matrix element which is the sum over all possible colors divided by the number of possible initial color states



$$\langle |M_{fi}|^2 \rangle = \frac{1}{3} \cdot \frac{1}{3} \sum_{i,j,k,l=1}^{3} |M_{fi}(ij \to kl)|^2$$

$$\langle |C|^2 \rangle = \frac{1}{9} \sum_{i,j,k,l=1}^{3} |C(ij \to kl)|^2$$

$qq \to qq$

rr→rr,.. **rb→rb,..** **rb→br,..**

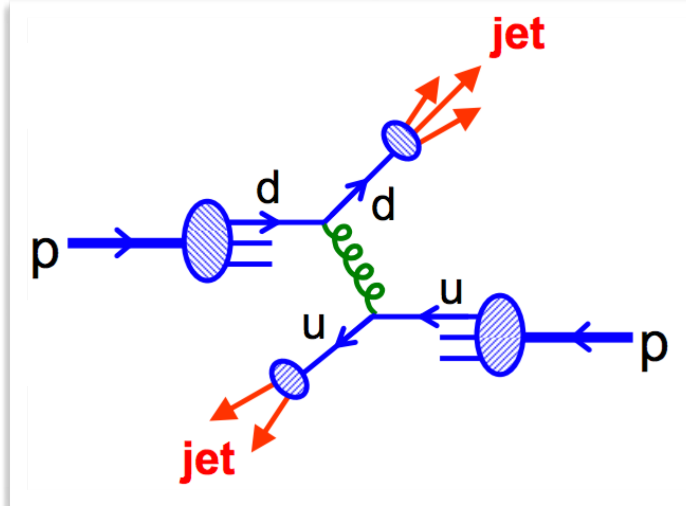
$$\langle |C|^2 \rangle = \frac{1}{9} \left[3 \times \left(\frac{1}{3}\right)^2 + 6 \times \left(-\frac{1}{6}\right)^2 + 6 \times \left(\frac{1}{2}\right)^2 \right] = \frac{2}{9}$$

Quark-quark Scattering

We've seen this before in electron-muon scattering!
- The cross section can thus be recycled

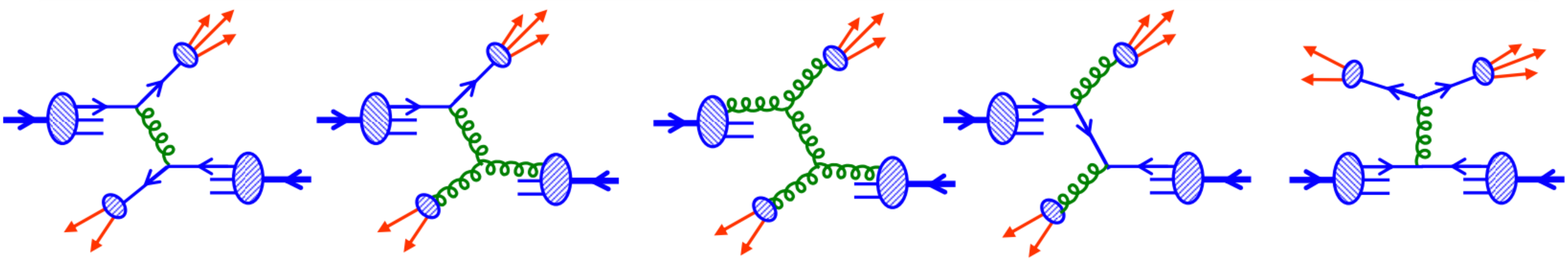
$$\sigma(e^-\mu^- \to e^-\mu^-) = \frac{4\pi\alpha^2}{E^4}$$

$$\sigma(ud \to ud) = \frac{2}{9}\frac{4\pi\alpha_s^2}{E^4}$$



The calculation of hadron-hadron scattering is very involved, need to include parton structure functions and include all possible interactions.

For example, generic two jet production from proton collisions



$q\bar{q} \to q\bar{q}$ $\quad qg \to qg$ $\quad gg \to gg$ $\quad q\bar{q} \to gg$ $\quad q\bar{q} \to q\bar{q}$