p01

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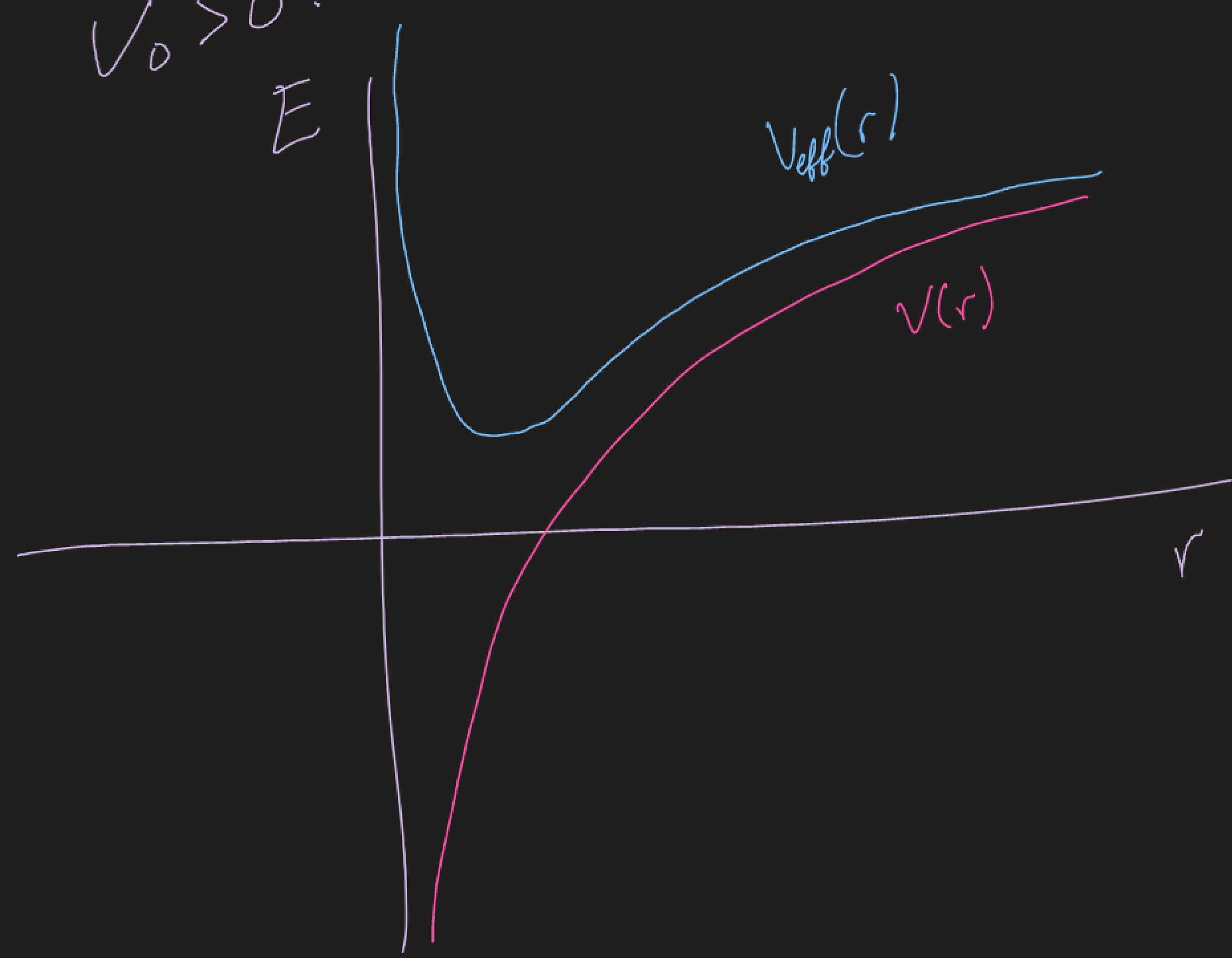
$$V(r) = V_0 \ln(r/r_0)$$

$$\vec{F}(r) = -\frac{\partial}{\partial r} V(r)\,\hat{r} = -V_0 \frac{r_0}{r}\frac{1}{r_0}\hat{r}$$

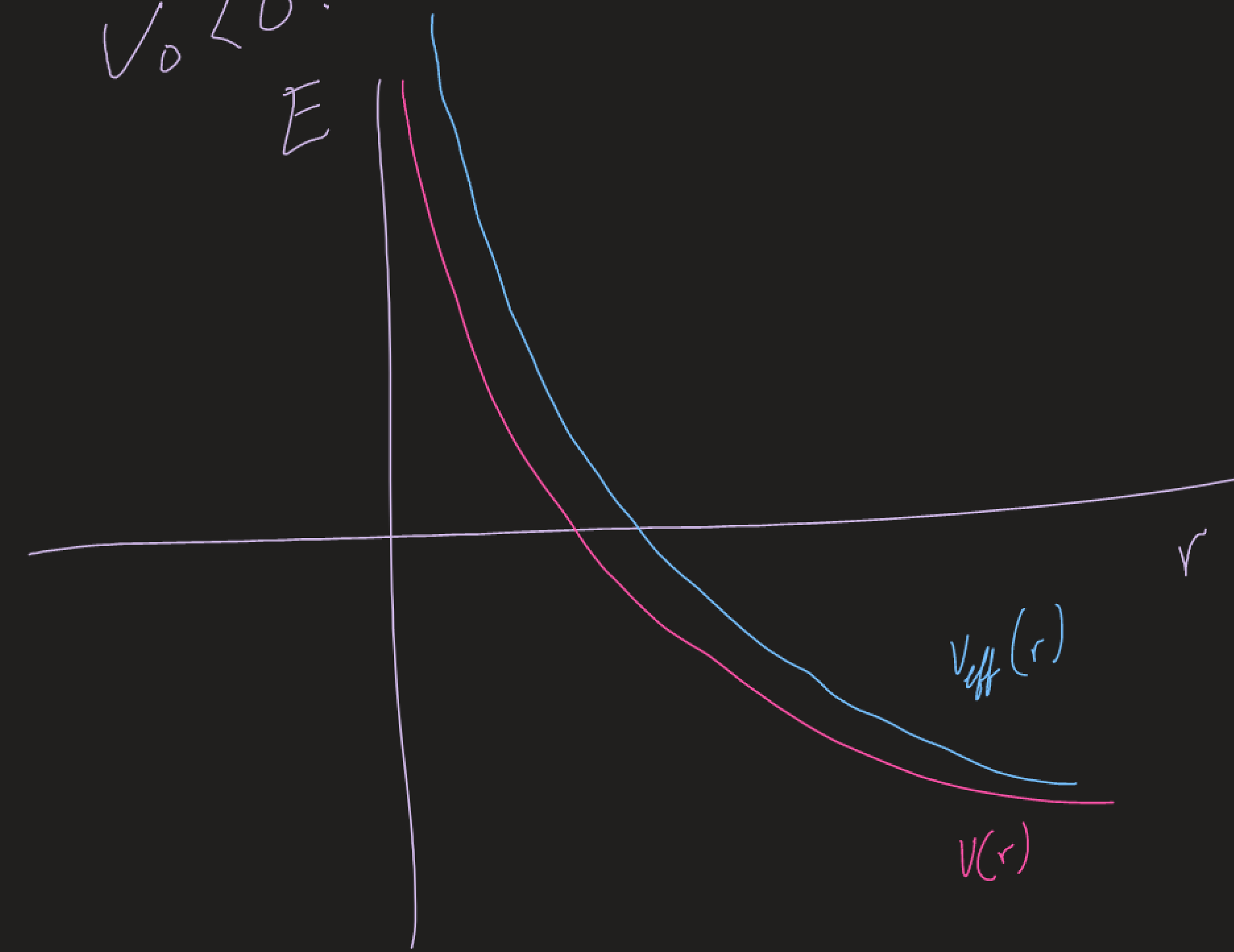
$$= -\frac{V_0}{r}\hat{r}$$

$V_0 > 0$:



allowed trajectories:
- circular
- bound orbit

$V_0 < 0$:



allowed trajectories:
- scattering

$$V_{eff}(r) = \frac{l^2}{2mr^2} + V_0 \ln(r/r_0)$$

Bounded orbits are possible if $V_0 > 0$.

This is true for all orbit energies, since $\lim_{x\to\infty} \ln(x) = \infty$.

$$\mathcal{L} = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\varphi}^2) - V(r)$$

$$\hookrightarrow \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{r}} - \frac{\partial\mathcal{L}}{\partial r} = 0 = m\ddot{r} - mr\dot{\varphi}^2 + V'(r)$$

$$\hookrightarrow \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\varphi}} - \frac{\partial\mathcal{L}}{\partial\varphi} = 0 = \frac{d}{dt}(mr^2\dot{\varphi})$$

$\therefore\ mr^2\dot{\varphi} = l$ is a constant.

$$\dot{\varphi} = \frac{l}{mr^2} \rightarrow m\ddot{r} = \frac{l^2}{mr^3} - V'(r)$$

In a circular orbit, $\dot{r} = 0$ and $r = R$.

$$m\ddot{R} = 0 = \frac{l^2}{mR^3} - \frac{V_0}{R}$$

$$\hookrightarrow mR^2 V_0 = l^2$$

$$R = \frac{l}{\sqrt{mV_0}}$$

If $r(t) = R + \varepsilon(t)$:

$$m\ddot{\varepsilon} = \frac{l^2}{m(R+\varepsilon)^3} - \frac{V_0}{R+\varepsilon}$$

$$\approx \frac{l^2}{mR^3}\left(1 - 3\frac{\varepsilon}{R}\right) - \frac{V_0}{R}\left(1 - \frac{\varepsilon}{R}\right)$$

$$= \underbrace{\cancel{\frac{l^2}{mR^3} - \frac{V_0}{R}}}_{0 \text{ (from before)}} - \underbrace{\left(\frac{3l^2}{mR^4} - \frac{V_0}{R^2}\right)}_{m\omega^2}\varepsilon$$

So,

$$\ddot{\varepsilon} + \omega^2\varepsilon = 0,$$

where $\omega^2 = \frac{1}{m}\left(\frac{3l^2}{mR^4} - \frac{V_0}{R^2}\right) \geq 0.$

p02

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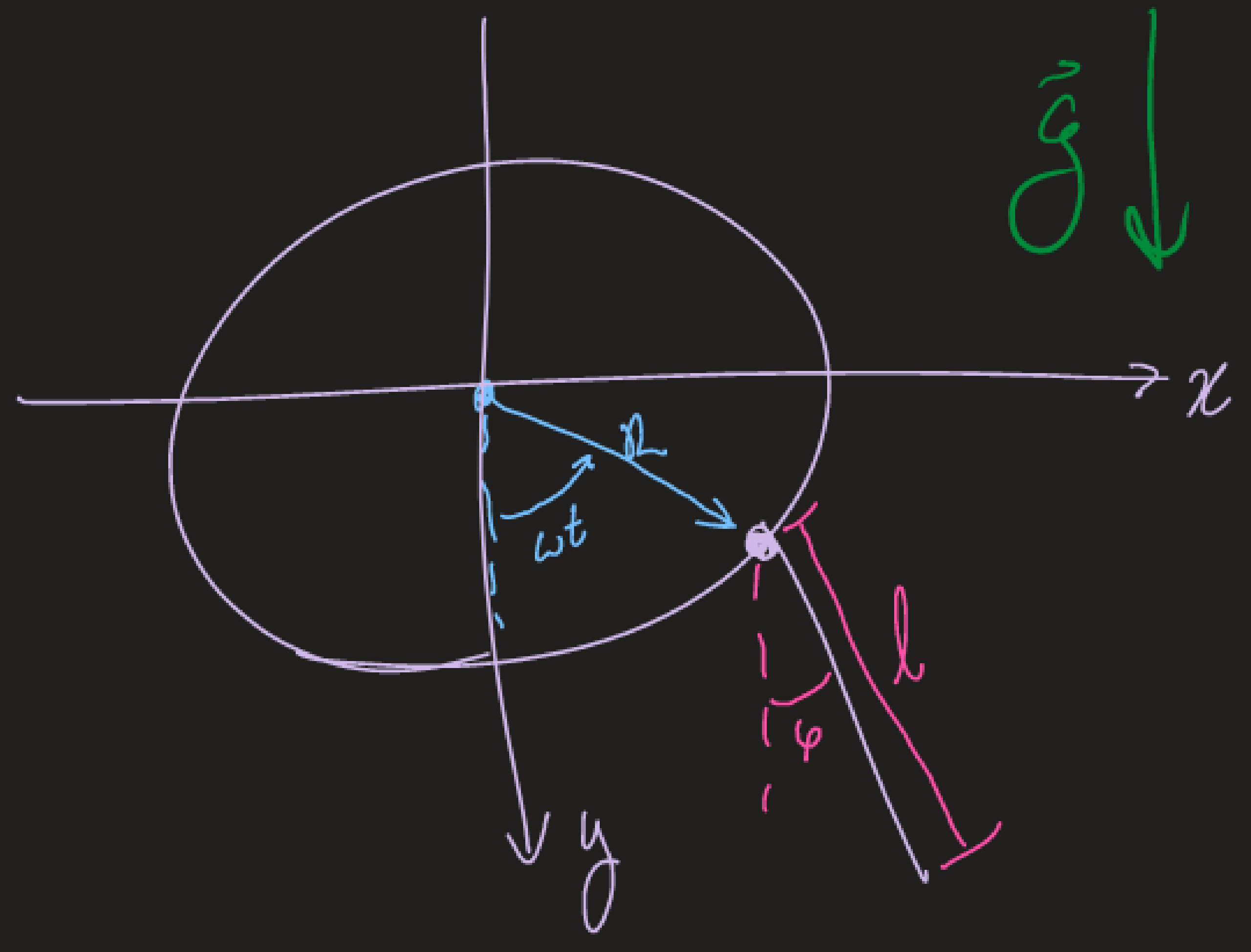
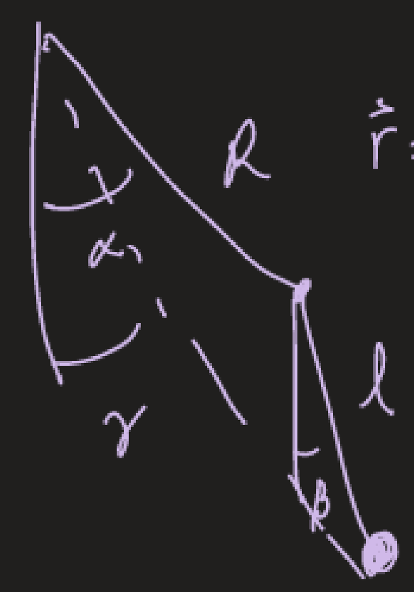


$$V = -mgh$$
$$= -mg\left(R\cos\omega t + \tfrac{1}{2}l\cos\varphi\right)$$

$$T = \frac{1}{2}m\dot{\vec{r}}^2$$
$$= \frac{1}{2}m\left(\omega^2R^2 + \frac{l^2}{4}\dot{\varphi}^2 + \omega Rl\dot{\varphi}\cos(\varphi-\omega t)\right)$$



$$\vec{r} = \begin{pmatrix} R\sin\alpha + \frac{l}{2}\sin\beta \\ R\cos\alpha + \frac{l}{2}\cos\beta \end{pmatrix} \rightarrow \dot{\vec{r}} = \begin{pmatrix} \omega R\cos\omega t + \frac{l}{2}\cos\varphi\,\dot{\varphi} \\ -\omega R\sin\omega t - \frac{l}{2}\sin\varphi\,\dot{\varphi} \end{pmatrix}$$

$$\dot{\vec{r}}^2 = \left(\omega R\cos\omega t + \tfrac{l}{2}\cos\varphi\,\dot{\varphi}\right)^2 + \left(\omega R\sin\omega t + \tfrac{l}{2}\sin\varphi\,\dot{\varphi}\right)^2$$
$$= \omega^2R^2\cos^2\omega t + \frac{l^2}{4}\cos^2\varphi\,\dot{\varphi}^2 + \omega Rl\cos\omega t\cos\varphi\,\dot{\varphi} + \omega^2R^2\sin^2\omega t + \frac{l^2}{4}\sin^2\varphi\,\dot{\varphi}^2 + \omega Rl\sin\omega t\sin\varphi\,\dot{\varphi}$$
$$= \omega^2R^2 + \frac{l^2}{4}\dot{\varphi}^2 + \omega Rl\dot{\varphi}\cos(\varphi-\omega t)$$

$$\mathcal{L} = \frac{l^2}{8}m\dot{\varphi}^2 + \underbrace{\cancel{\frac{m\omega^2R^2}{2}}}_{\text{"irrelevant"}} + \frac{1}{2}m\omega Rl\dot{\varphi}\cos(\varphi-\omega t) + \underbrace{\cancel{mgR\cos\omega t}}_{\text{"irrelevant"}} + \frac{1}{2}mgl\cos\varphi$$

$$= \frac{1}{8}ml^2\dot{\varphi}^2 + \frac{1}{2}m\omega Rl\dot{\varphi}\cos(\varphi-\omega t) + \frac{1}{2}mgl\cos\varphi$$

Supposed to be a 6? How?

From here on, I use eq. 2 from the exam.

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\varphi}} - \frac{\partial\mathcal{L}}{\partial\varphi} = 0 = \frac{d}{dt}\left[\frac{1}{3}ml^2\dot{\varphi} + \frac{1}{2}mRl\omega\cos(\varphi-\omega t)\right] + \frac{1}{2}mRl\omega\dot{\varphi}\sin(\varphi-\omega t) + \frac{1}{2}mgl\sin\varphi$$
$$= \frac{d}{dt}\left[\frac{1}{3}ml^2\dot{\varphi}\right] - \frac{1}{2}mRl\omega\sin(\varphi-\omega t)(\dot{\varphi}-\omega) + \frac{1}{2}mRl\omega\dot{\varphi}\sin(\varphi-\omega t) + \frac{1}{2}mgl\sin\varphi$$
$$= \frac{1}{3}ml^2\ddot{\varphi} - \frac{1}{2}mRl\omega\sin(\varphi-\omega t)\ldots$$

For $R\omega^2 \ll g$: I

p03

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$$\vec{N} = \frac{d}{dt}\vec{L} + \vec{\omega}\times\vec{L}, \quad \text{where } \vec{L} = I\vec{\omega} = \begin{pmatrix} A\omega_{x'} \\ B\omega_{y'} \\ C\omega_{z'} \end{pmatrix}$$

$$\frac{d}{dt}\vec{L} = \begin{pmatrix} \dot{A}\omega_{x'} + A\dot{\omega}_{x'} \\ \dot{B}\omega_{y'} + B\dot{\omega}_{y'} \\ \dot{C}\omega_{z'} + C\dot{\omega}_{z'} \end{pmatrix}$$

$$\vec{N}_z = \dot{C}\omega_{z'} + C\dot{\omega}_{z'} + \left(\omega_{x'}B\omega_{y'} - A\omega_{x'}\omega_{y'}\right)$$

$$= \dot{C}\omega_{z'} + C\dot{\omega}_{z'} + \underline{(B-A)\cancel{\omega_{x'}\omega_{y'}}} \rightarrow 0$$

$$= \dot{C}\omega_{z'} + C\dot{\omega}_{z'}$$

$$C'(t) = -\frac{2}{5}Ma^2\varepsilon\cos\Omega t\,\Omega$$

$$N_{z'} = -\frac{2}{5}Ma^2\varepsilon\Omega\cos(\Omega t)\omega_{z'} + \frac{2}{5}Ma^2(1+\varepsilon\sin\Omega t)\dot{\omega}_{z'}$$

Star in space, so $\vec{N} = \vec{0}$:

$$\dot{\omega}_{z'} = \frac{\varepsilon\Omega\cos(\Omega t)\omega_{z'}}{1+\varepsilon\sin(\Omega t)}$$

For $\varepsilon \ll 1$: $\dot{\omega}_{z'} \approx 0$. ✓

In the case where derivatives of A, B, and C can be ignored ($\Omega \ll \omega_{z'}$),

$$A\dot{\omega}_{x'} + (A-C)\omega_{y'}\omega_{z'} = 0,$$
$$A\dot{\omega}_{y'} + (C-A)\omega_{z'}\omega_{x'} = 0,$$
$$C\dot{\omega}_{z'} = 0.$$

Then the value Ω_p can be defined as follows:

$$\Omega_p = \frac{C-A}{A}\omega_{z'}.$$

From the Euler equations above,

$$\dot{\omega}_{x'} = -\Omega_p\omega_{y'}, \quad \dot{\omega}_{y'} = \Omega_p\omega_{x'}.$$

This gives

$$\ddot{\omega}_{x'} + \Omega_p\omega_{x'} = 0,$$
$$\omega_{x'} = \Gamma\cos\Omega_p t,$$
$$\omega_{y'} = \Gamma\sin\Omega_p t,$$

where Γ is just some constant.

So the angular velocity is precessing around the z' axis with a frequency of Ω_p.

$$\Omega_p = \frac{\frac{2}{5}Ma^2(1+\varepsilon\sin\Omega t) - \frac{1}{5}M(a^2+b^2)(1-\frac{\varepsilon}{2}\sin\Omega t)}{\frac{1}{5}M(a^2+b^2)(1-\frac{\varepsilon}{2}\sin\Omega t)}\omega_{z'}$$

$$= \left[\frac{2a^2(1+\varepsilon\sin\Omega t)}{(a^2+b^2)(1-\frac{\varepsilon}{2}\sin\Omega t)} - 1\right]\omega_{z'}(t)$$

$$\approx \left[\frac{a^2-b^2}{a^2+b^2} + \frac{3a^2\sin\Omega t\,\varepsilon}{a^2+b^2} + O(\varepsilon^2)\right]\omega_{z'}(t)$$

If $a=b=R$ the star is spherical (?), the $O(1)$ term vanishes, and the $O(\varepsilon)$ term simplifies to,

$$\frac{3}{2}\sin\Omega t\,\varepsilon.$$

The first term dictates precession in non ellipsoidal bodies, and the second term is a correction for more spherical bodies.