hw04

Friday, October 2, 2020 12:36 PM

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PHY422/820: Classical Mechanics

FS 2020

Homework #4 (Due: Oct 2)

September 17, 2020

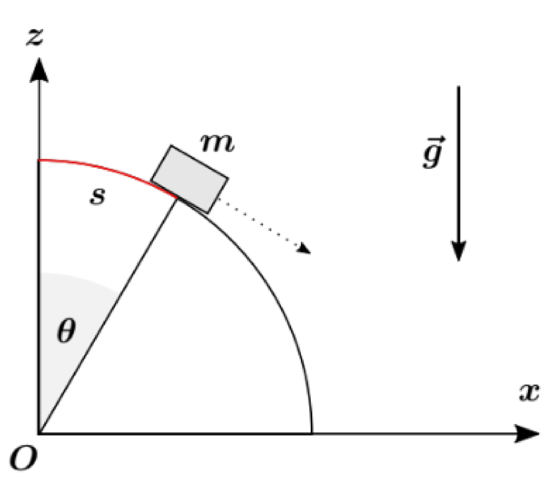
Problem H9 – Mass Sliding Down a Hemisphere

[**15 Points**] A block of mass m is released from rest at the top of a frictionless hemisphere of radius R, and slides down the surface under the influence of gravity until it flies off.

1. Construct the Lagrangian for the initial phase of the block's motion, coupling it to the constraint with a Lagrange multiplier. Use appropriate coordinates.
2. Use the Lagrange equations of the first kind to determine the angle at which the block flies off, and the length of the arc from the top to the point at which it launches.

   HINT: Consider what happens to the constraint when the block flies off. You will also find the following relation useful:

$$\frac{d}{dt}\dot{\theta}^2 = 2\dot{\theta}\ddot{\theta}\,. \tag{1}$$



3. Compute the block's point of impact on the ground.

Problem H10 – Gauge Symmetry for a Particle in an Electromagnetic Field

[**15 Points**] A particle with charge q and mass m is moving in an external electromagnetic field. In SI units, the electric and magnetic fields, $\vec{E}(\vec{r},t)$ and $\vec{B}(\vec{r},t)$, are related to the scalar and vector potentials $\phi(\vec{r},t)$ and $\vec{A}(\vec{r},t)$ via

$$\vec{E}(\vec{r},t) = -\nabla\phi(\vec{r},t) - \frac{\partial}{\partial t}\vec{A}(\vec{r},t), \qquad \vec{B}(\vec{r},t) = \nabla\times\vec{A}(\vec{r},t)\,. \tag{2}$$

The Lagrangian of the particle is given by

$$L(\vec{r},\dot{\vec{r}}) = \frac{1}{2}m\dot{\vec{r}}^2 - q\left(\phi - \dot{\vec{r}}\cdot\vec{A}\right) \tag{3}$$

1. Show explicitly that the Lagrange equations lead to Lorentz's force law,

$$m\ddot{\vec{r}} = q\vec{E} + q\dot{\vec{r}}\times\vec{B}\,. \tag{4}$$

   HINT:

$$\vec{a}\times(\vec{b}\times\vec{c}) = \vec{b}(\vec{a}\cdot\vec{c}) - \vec{c}(\vec{a}\cdot\vec{b}) \tag{5}$$

   (Be careful about the ordering of the terms when any of the vectors is a gradient!)

2. Now consider the **gauge transformation**

$$\phi(\vec{r},t) \longrightarrow \phi(\vec{r},t) - \frac{\partial}{\partial t}\Lambda(\vec{r},t)\,, \tag{6}$$

$$\vec{A}(\vec{r},t) \longrightarrow \vec{A}(\vec{r},t) + \nabla\Lambda(\vec{r},t)\,. \tag{7}$$

   with an arbitray twice differentiable function $\Lambda(\vec{r},t)$.

   How do the electromagnetic fields change? How does the gauge transformation affect the Lagrangian?

$$E = -\frac{\partial\varphi}{\partial\vec{r}} - \frac{\partial\vec{A}}{\partial t} \to -\frac{\partial\varphi}{\partial\vec{r}} + \frac{\partial^2}{\partial\vec{r}\partial t}\Lambda - \frac{\partial}{\partial t}\vec{A} - \frac{\partial^2}{\partial t\partial\vec{r}}\Lambda$$

$$= -\frac{\partial\varphi}{\partial\vec{r}} - \frac{\partial\vec{A}}{\partial t}$$ unchanged

$$B = \vec{\nabla}\times\vec{A} \to \vec{\nabla}\times\vec{A} + \vec{\nabla}\times(\vec{\nabla}\Lambda)$$

$$= \vec{\nabla}\times\vec{A}$$ unchanged

$$\mathcal{L} \to \frac{1}{2}m\dot{\vec{r}}^2 - q\left(\varphi - \frac{\partial}{\partial t}\Lambda - \dot{\vec{r}}\cdot(\vec{A} + \vec{\nabla}\Lambda)\right)$$

$$= \frac{1}{2}m\dot{\vec{r}}^2 - q\left(\varphi - \dot{\vec{r}}\cdot\vec{A} - (\vec{\nabla}\Lambda + \frac{\partial}{\partial t}\Lambda)\right)$$

$$= \frac{1}{2}m\dot{\vec{r}}^2 - q\left(\varphi - \dot{\vec{r}}\cdot\vec{A} - \dot{\Lambda}\right)$$

The Lagrangian gets an extra $q\dot{\Lambda}$ term.

1. $T = \frac{1}{2}m(\dot{\vec{\rho}})^2 \qquad \dot{\vec{\rho}} = \dot{\rho}\hat{\rho} + \rho(\dot{\varphi}\sin\vartheta\,\hat{\varphi} - \dot{\vartheta}\hat{\vartheta}), \quad \dot{\varphi} = 0 \to \dot{\vec{\rho}} = \dot{\rho}\hat{\rho} + \rho\dot{\vartheta}\hat{\vartheta}$

$= \frac{1}{2}m(\dot{\rho}^2 + \rho^2\dot{\vartheta}^2)$

$V = mg\rho\cos\vartheta$

$\mathcal{L} = \frac{1}{2}m(\dot{\rho}^2 + \rho^2\dot{\vartheta}^2) - mg\rho\cos\vartheta + \lambda(R - \rho)$

2. $\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\rho}} - \frac{\partial\mathcal{L}}{\partial\rho} = 0 = m\ddot{\rho} - m\rho\dot{\vartheta}^2 + mg\cos\vartheta - \lambda$

$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\vartheta}} - \frac{\partial\mathcal{L}}{\partial\vartheta} = 0 = m\frac{d}{dt}(\rho^2\dot{\vartheta}) - mg\rho\sin\vartheta$

$= m(2\rho\dot{\rho}\dot{\vartheta} + \rho^2\ddot{\vartheta}) - mg\rho\sin\vartheta$

$-\frac{\partial\mathcal{L}}{\partial\lambda} = 0 = R - \rho \to \rho = R \to \dot{\rho} = 0 \to \ddot{\rho} = 0$

$\lambda = m\ddot{\rho} - m\rho\dot{\vartheta}^2 + mg\cos\vartheta$

$= -mR\dot{\vartheta}^2 + mg\cos\vartheta$

$0 = R^2\ddot{\vartheta} - gR\sin\vartheta$

$\ddot{\vartheta} = \frac{g}{R}\sin\vartheta \longrightarrow \frac{d}{dt}(\dot{\vartheta}^2) = 2\dot{\vartheta}\ddot{\vartheta} = 2\dot{\vartheta}\frac{g}{R}\sin\vartheta$

$\frac{d\dot{\vartheta}^2}{dt} = 2\frac{g}{R}\frac{d\vartheta}{dt}\sin\vartheta$

$\int_0^t \frac{d\dot{\vartheta}^2}{dt}dt = \dot{\vartheta}^2 = \int_0^\vartheta \frac{2g}{R}\sin\vartheta\,d\vartheta$

$= \frac{2g}{R}(1 - \cos\vartheta)$

$\lambda = -mR\left(\frac{2g}{R}(1 - \cos\vartheta)\right) + mg\cos\vartheta$

$= -2mg + 3mg\cos\vartheta$

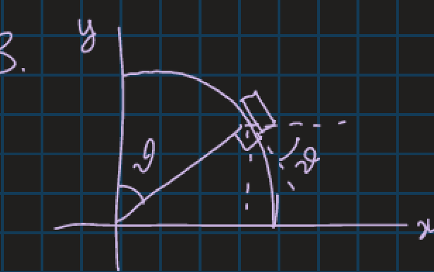
when the block leaves the surface, $\lambda = 0$:

$0 = -2 + 3\cos\vartheta$

$\vartheta = \arccos(2/3)$

$\approx 48.2^\circ$

3.



$v = \omega R \to \omega = \sqrt{\frac{2g}{3R}}$

$r_0 = R\begin{pmatrix}\sin\vartheta \\ \cos\vartheta\end{pmatrix} = R\begin{pmatrix}\sqrt{5}/3 \\ 2/3\end{pmatrix} \qquad \vec{v}_0 = \sqrt{\frac{2}{3}gR}\begin{pmatrix}\cos\vartheta \\ -\sin\vartheta\end{pmatrix} = \sqrt{\frac{2}{3}gR}\begin{pmatrix}2/3 \\ -\sqrt{5}/3\end{pmatrix}$

$0 = y_0 + v_{0y}t - \frac{1}{2}gt^2$

$t = \frac{v_{0y} + \sqrt{v_{0y}^2 + 2gy_0}}{g}$

$x = x_0 + v_{0x}t$

$= x_0 + v_{0x}\left(\frac{v_{0y} + \sqrt{v_{0y}^2 + 2gy_0}}{g}\right)$

$= \frac{\sqrt{5}}{3}R + \frac{2}{3}\sqrt{\frac{2R}{3g}}\left(-\frac{\sqrt{5}}{3}\sqrt{\frac{2}{3}gR} + \sqrt{\frac{10}{27}gR + \frac{4}{3}gR}\right)$

$= \frac{\sqrt{5}}{3}R + \frac{4}{27}(\sqrt{23} - \sqrt{5})R$

$= \frac{1}{54}(19\sqrt{5} + 8\sqrt{23})R$

$\approx 1.5R$

$\mathcal{L} = \frac{1}{2}m\dot{\vec{r}}^2 - q(\varphi - \dot{\vec{r}}\cdot\vec{A})$

$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\vec{r}}} - \frac{\partial\mathcal{L}}{\partial\vec{r}} = \frac{d}{dt}(m\dot{\vec{r}} + \vec{A}) + q\left(\frac{\partial\varphi}{\partial\vec{r}} - \frac{\partial}{\partial\vec{r}}(\vec{A}\cdot\dot{\vec{r}})\right)$

$= m\ddot{\vec{r}} + q\dot{\vec{A}} + q\frac{\partial}{\partial\vec{r}}(\varphi - \vec{A}\cdot\dot{\vec{r}})$

$\dot{\vec{A}} = \frac{\partial\vec{A}}{\partial\vec{r}}\dot{\vec{r}} + \frac{\partial\vec{A}}{\partial t}$

$m\ddot{\vec{r}} = -q\left(\frac{\partial\vec{A}}{\partial\vec{r}}\dot{\vec{r}} - \frac{\partial}{\partial\vec{r}}(\vec{A}\cdot\dot{\vec{r}}) + \frac{\partial\varphi}{\partial\vec{r}} + \frac{\partial\vec{A}}{\partial t}\right)$

$= q\left[\left(\frac{\partial\vec{A}}{\partial\vec{r}}^T - \frac{\partial\vec{A}}{\partial\vec{r}}\right)\dot{\vec{r}} + E\right]$

$m\ddot{\vec{r}} = q(\dot{\vec{r}}\times\vec{B} + E)$ ✓

$\begin{matrix}\partial_1(a_1\dot{x} + a_2\dot{y} + a_3\dot{z}) \\ \partial_2 \quad '' \\ \partial_3 \quad ''\end{matrix} = \begin{pmatrix}\partial_1 a_1 & \partial_1 a_2 & \partial_1 a_3 \\ \partial_2 a_1 & \partial_2 a_2 & \partial_2 a_3 \\ \partial_3 a_1 & \partial_3 a_2 & \partial_3 a_3\end{pmatrix}\begin{pmatrix}\dot{x} \\ \dot{y} \\ \dot{z}\end{pmatrix}$

$\left(\frac{\partial\vec{A}}{\partial\vec{r}}^T - \frac{\partial\vec{A}}{\partial\vec{r}}\right)\dot{\vec{r}} = \begin{pmatrix}0 & \partial_1 a_2 - \partial_2 a_1 & \partial_1 a_3 - \partial_3 a_1 \\ \partial_2 a_1 - \partial_1 a_2 & 0 & \partial_2 a_3 - \partial_3 a_2 \\ \partial_3 a_1 - \partial_1 a_3 & \partial_3 a_2 - \partial_2 a_3 & 0\end{pmatrix}\begin{pmatrix}\dot{x} \\ \dot{y} \\ \dot{z}\end{pmatrix}$

$= X\dot{\vec{r}}$ where $X_{ij} = \partial_i a_j - \partial_j a_i$ (note $X_{ij} = -X_{ji}$)

$= \begin{pmatrix}0 + \dot{y}X_{12} + \dot{z}X_{13} \\ -\dot{x}X_{12} + 0 + \dot{z}X_{23} \\ -\dot{x}X_{13} + \dot{y}X_{23} + 0\end{pmatrix}$

$= \hat{x}(\dot{y}X_{12} + \dot{z}X_{13})$

$+ \hat{y}(\dot{z}X_{23} - \dot{x}X_{12})$

$+ \hat{z}(\dot{y}X_{23} - \dot{x}X_{13})$

$= \dot{\vec{r}}\times\vec{\nabla}\times\vec{A}$

$= \dot{\vec{r}}\times\vec{B}$