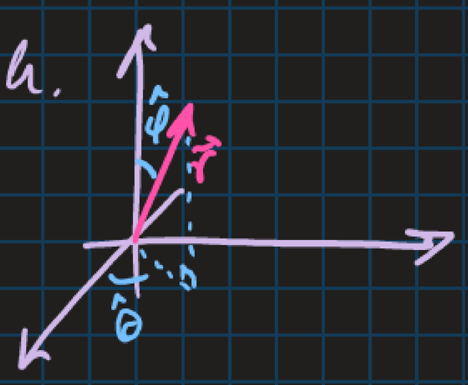
HW1

Thursday, August 20, 2020 1:05 AM

1. a.



$\vec{r} = \rho\hat{\rho}$

$\dot{\vec{r}} = \dot{\rho}\hat{\rho} + \rho\dot{\hat{\rho}}$

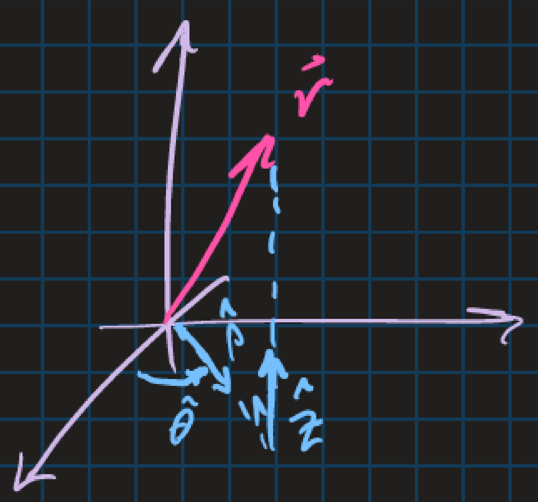
$= \dot{\rho}\hat{\rho} + \rho\left[\dot{\theta}\hat{\theta} + \dot{\varphi}\sin\theta\hat{\varphi}\right]$

$= \dot{\rho}\hat{\rho} + \rho\dot{\theta}\hat{\theta} + \rho\dot{\varphi}\sin\theta\hat{\varphi}$

$\hat{\rho} = \begin{pmatrix} \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta \end{pmatrix}$

$\dot{\hat{\rho}} = \begin{pmatrix} \dot{\theta}\cos\theta\cos\varphi - \sin\theta\sin\varphi\,\dot{\varphi} \\ \dot{\theta}\cos\theta\sin\varphi + \sin\theta\cos\varphi\,\dot{\varphi} \\ -\dot{\theta}\sin\theta \end{pmatrix} = \dot{\theta}\begin{pmatrix} \cos\theta\cos\varphi \\ \cos\theta\sin\varphi \\ -\sin\theta \end{pmatrix} + \dot{\varphi}\begin{pmatrix} -\sin\theta\sin\varphi \\ \sin\theta\cos\varphi \\ 0 \end{pmatrix}$

$= \dot{\theta}\hat{\theta} + \dot{\varphi}\sin\theta\hat{\varphi}$



$\vec{r} = \rho\hat{\rho} + z\hat{z}$

$\dot{\vec{r}} = \dot{\rho}\hat{\rho} + \rho\dot{\hat{\rho}} + \dot{z}\hat{z}$

$= \dot{\rho}\hat{\rho} + \rho\dot{\theta}\hat{\theta} + \dot{z}\hat{z}$

$\hat{\rho} = \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix} \rightarrow \dot{\hat{\rho}} = \dot{\theta}\begin{pmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{pmatrix} = \dot{\theta}\hat{\theta}$

b. $T = \frac{1}{2}m(\dot{\vec{r}}\cdot\dot{\vec{r}})$

$= \frac{1}{2}m\left(\dot{\rho}^2 + \rho^2\dot{\theta}^2 + \rho^2\dot{\varphi}^2\sin^2\theta\right)$

$= \frac{1}{2}m\left(\dot{\rho}^2 + \rho^2\dot{\theta}^2 + \dot{z}^2\right)$

c. Let $V(\vec{r}) = f(\rho)$

$F(\vec{r}) = -\vec{\nabla}V(\vec{r}) = -\vec{\nabla}f(\rho)$

$= -\left(\frac{\partial f}{\partial\rho}\hat{\rho} + \cancel{\frac{1}{\rho}\frac{\partial f}{\partial\theta}}\hat{\theta} + \cancel{\frac{1}{\rho\sin\theta}\frac{\partial f}{\partial\varphi}}\hat{\varphi}\right)$

$= -\frac{\partial f}{\partial\rho}\hat{\rho}$ ✓

2. $\vec{F}(\vec{r}) = V_0\dfrac{e^{-(\rho-b)/a}}{a\left(1+e^{-(\rho-b)/a}\right)^2}\hat{\rho}$

a. Since $\vec{F}$ only has a radial component, only two terms show up in the curl. However, those two terms are taking derivatives with respect to φ and θ, so they vanish. Thus $\vec{\nabla}\times\vec{F} = \vec{0}$.

$$\frac{1}{r\sin\theta}\left(\frac{\partial}{\partial\theta}\left(A_\varphi\sin\theta\right) - \frac{\partial A_\theta}{\partial\varphi}\right)\hat{\mathbf{r}} + \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\varphi} - \frac{\partial}{\partial r}\left(rA_\varphi\right)\right)\hat{\boldsymbol{\theta}} + \frac{1}{r}\left(\frac{\partial}{\partial r}\left(rA_\theta\right) - \frac{\partial A_r}{\partial\theta}\right)\hat{\boldsymbol{\varphi}}$$

$\vec{F}(\vec{r}) = -\vec{\nabla}V(\vec{r})$

$$\frac{\partial f}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial f}{\partial\theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial\varphi}\hat{\boldsymbol{\varphi}}$$

$-\partial_\rho V = V_0\dfrac{e^{-(\rho-b)/a}}{\left(1+e^{-(\rho-b)/a}\right)^2}$

$V = -V_0\displaystyle\int\frac{e^{-(\rho-b)/a}}{\left(1+e^{-(\rho-b)/a}\right)^2}d\rho$

$u = 1 + e^{-(\rho-b)/a}$

$du = -\frac{1}{a}e^{-(\rho-b)/a}d\rho$

$= aV_0\displaystyle\int\frac{du}{u^2}$

$= -aV_0\frac{1}{u}$

$V(\vec{r}) = -\dfrac{aV_0}{1+e^{-(\rho-b)/a}}$

b. $W = \displaystyle\int_0^r\vec{F}(\vec{s})\cdot d\vec{s} = \int_0^r\frac{e^{-(s-b)/a}}{\left(1+e^{-(s-b)/a}\right)^2}ds = \frac{aV_0}{1+e^{b/a}} - \frac{aV_0}{1+e^{-(r-b)/a}}$

The second integral is the same because one first integrates to $(0,0,r)$, which is the same integral), and then adds a path integral where the path is perpendicular to the vector field, which means it vanishes.

3. cylindrical coords: $\vec{F}_1(\vec{r}) = \frac{a_0}{\rho}\hat{\varphi}$ spherical coords: $\vec{F}_2 = -\frac{k}{\rho^2}\hat{\rho}$

a.

$$\left(\frac{1}{\rho}\frac{\partial A_z}{\partial\varphi} - \frac{\partial A_\varphi}{\partial z}\right)\hat{\boldsymbol{\rho}} + \left(\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial\rho}\right)\hat{\boldsymbol{\varphi}} + \frac{1}{\rho}\left(\frac{\partial\left(\rho A_\varphi\right)}{\partial\rho} - \frac{\partial A_\rho}{\partial\varphi}\right)\hat{\mathbf{z}}$$

$$\frac{1}{r\sin\theta}\left(\frac{\partial}{\partial\theta}\left(A_\varphi\sin\theta\right) - \frac{\partial A_\theta}{\partial\varphi}\right)\hat{\mathbf{r}} + \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\varphi} - \frac{\partial}{\partial r}\left(rA_\varphi\right)\right)\hat{\boldsymbol{\theta}} + \frac{1}{r}\left(\frac{\partial}{\partial r}\left(rA_\theta\right) - \frac{\partial A_r}{\partial\theta}\right)\hat{\boldsymbol{\varphi}}$$

$\vec{\nabla}\times\vec{F}_1 = \frac{1}{\rho}\frac{\partial}{\partial\rho}(a_0)\hat{z}$

$= 0$

$\vec{\nabla}\times F_2 = \frac{1}{\rho\sin\theta}\frac{\partial}{\partial\varphi}\left(-\frac{k}{\rho^2}\right)\hat{\theta} - \frac{1}{\rho}\frac{\partial}{\partial\theta}\left(-\frac{k}{\rho^2}\right)\hat{\varphi}$

$= 0$

b. $\varepsilon\displaystyle\int_0^{2\pi}\vec{F}_1\cdot d\vec{\varphi} = \int_0^{2\pi}a_0\,d\varphi = 2\pi a_0$

$\varepsilon\displaystyle\int_0^{2\pi}\vec{F}_2\cdot d\vec{\varphi} = \int 0 = 0$ because $d\vec{\varphi}\cdot\hat{\rho} = 0$.

c. $\displaystyle\int_0^{2\pi}a\vec{F}_1\cdot d\vec{\varphi} + \int_a^b\vec{F}_1\cdot d\vec{r} + \int_{2\pi}^0 b\vec{F}_1\cdot d\vec{\varphi} + \int_b^a\vec{F}_1\cdot d\vec{r}$

$= 2\pi a_0 + 0 - 2\pi a_0 + 0$

$= 0$

$\displaystyle\int_0^{2\pi}a\vec{F}_2\cdot d\vec{\varphi} + \int_a^b\vec{F}_2\cdot d\vec{r} + \int_{2\pi}^0 b\vec{F}_2\cdot d\vec{\varphi} + \int_b^a\vec{F}_2\cdot d\vec{r}$

$= 0 + \left(\frac{k}{b} - \frac{k}{a}\right) + 0 + \left(\frac{k}{a} - \frac{k}{b}\right)$

$= 0$