f01

Wednesday, December 16, 2020 8:02 AM

$$V = mgh = mgl\cos\vartheta \qquad r = \begin{pmatrix} x + l\sin\vartheta \\ l\cos\vartheta \end{pmatrix}$$

$$T = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m\left((\dot{x} + l\cos\vartheta\,\dot{\vartheta})^2 + l^2\sin^2\vartheta\,\dot{\vartheta}^2\right)$$

$$\mathcal{L} = \frac{1}{2}(M+m)\dot{x}^2 + m\dot{x}l\cos\vartheta\,\dot{\vartheta} + \frac{1}{2}ml^2\dot{\vartheta}^2 - mgl\cos\vartheta$$

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{x}} - \frac{\partial\mathcal{L}}{\partial x} = 0 = (M+m)\ddot{x} - ml\sin\vartheta\,\dot{\vartheta}^2 + ml\cos\vartheta\,\ddot{\vartheta} \overset{\text{small angle}}{\approx} (M+m)\ddot{x} - ml\vartheta\dot{\vartheta}^2 + ml\ddot{\vartheta}$$

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\vartheta}} - \frac{\partial\mathcal{L}}{\partial\vartheta} = 0 = \frac{d}{dt}\left(m\dot{x}l\cos\vartheta + ml^2\dot{\vartheta}\right) + m\dot{x}l\sin\vartheta\,\dot{\vartheta} - mgl\sin\vartheta$$

$$= m\ddot{x}l\cos\vartheta - \cancel{m\dot{x}l\sin\vartheta\,\dot{\vartheta}} + ml^2\ddot{\vartheta} + \cancel{m\dot{x}l\sin\vartheta\,\dot{\vartheta}} - mgl\sin\vartheta$$

$$= m\ddot{x}l\cos\vartheta + ml^2\ddot{\vartheta} - mgl\sin\vartheta$$

$$\approx m\ddot{x}l + ml^2\ddot{\vartheta} - mgl\vartheta \quad \text{(small angle)}$$

$$(M+m)\ddot{x} - ml\vartheta\dot{\vartheta}^2 + ml\ddot{\vartheta} = 0 \rightarrow \ddot{x} = \frac{1}{(M+m)}\left(ml\vartheta\dot{\vartheta}^2 - ml\ddot{\vartheta}\right)$$

$$0 = \frac{ml}{(M+m)}\left(ml\vartheta\dot{\vartheta}^2 - ml\ddot{\vartheta}\right) + ml^2\ddot{\vartheta} - mgl\vartheta$$

$$\left(ml^2 - \frac{(ml)^2}{M+m}\right)\ddot{\vartheta} + ml\left(\frac{ml}{M+m}\dot{\vartheta}^2 - g\right)\vartheta = 0$$

This is more complicated than I think it's supposed to be... Something went wrong, but I don't have time to fix it.

f02

Wednesday, December 16, 2020 8:29 AM

$$\vec{\nabla} \times \vec{F} = -\frac{nk}{r^{n+1}} \vec{\nabla} \times \hat{r}$$
$$= \vec{0}$$

Since the curl is zero, $\vec{F}$ is conservative.

$$V(r) = \int_r^\infty \frac{nk}{r^{n+1}} \, dr$$
$$= \left[-\frac{n^2 k}{r^n} \right|_r^\infty$$
$$= -\frac{n^2 k}{r^n}$$

$$V_{eff}(r) = \frac{l^2}{2mr^2} - \frac{n^2 k}{r^n}$$

If $\lim_{r\to 0} V_{eff}(r) < +\infty$, the mass can reach the center:

$$\lim_{r\to 0} V_{eff}(r) = \lim_{r\to 0} \left(\frac{l^2}{2mr^2} - \frac{n^2 k}{r^n} \right)$$
$$= \lim_{r\to 0} \left(\frac{l^2 r^n - 2mr^2 n^2 k}{2m r^{n+2}} \right)$$
$$= \lim_{r\to 0} \left(\frac{n(n-1) l r^{n-2} - 4mn^2 k}{(n+2)(n+1) 2m r^n} \right)$$
$$= \lim_{r\to 0} \left(\frac{n(n-1) l}{(n+2)(n+1) 2mr^2} - \frac{4mn^2 k}{(n+2)(n+1) 2mr^n} \right)$$

I don't know...

For circular orbits:

$$V_{eff}'(R) = 0 = -\frac{l^2}{mR^3} - \frac{n^2 k}{R^{n+1}}$$
$$\frac{l^2}{m} R^{n-2} = -n^2 k$$
$$R = \left(-\frac{n^2 k m}{l^2} \right)^{\frac{1}{n-2}}$$ if n is odd, these are stable.

f03

Wednesday, December 16, 2020 8:47 AM

$$V = \frac{1}{2} k \eta_1^2 + \frac{1}{2} k \eta_2^2 + \frac{1}{2}(2k)(\eta_2 - \eta_1)^2$$

$$\eta_1^2 + \eta_2^2 - 2\eta_1\eta_2 = \sum_{ij} \delta_{ij}\eta_i\eta_j - (1-\delta_{ij})\eta_i\eta_j$$

$$= 2\delta_{ij}\eta_i\eta_j - \eta_i\eta_j$$

$$= \eta_i(2\delta_{ij} - 1)\eta_j$$

$$= \frac{1}{2}\vec{\eta}^T \tilde{V} \vec{\eta}$$

$$\tilde{V} = \begin{pmatrix} 5 & -2 \\ -2 & 5 \end{pmatrix} k$$

$$T = \frac{1}{2} m(\dot{\eta}_1^2 + \dot{\eta}_2^2) = \frac{1}{2}\dot{\vec{\eta}}^T \tilde{T} \dot{\vec{\eta}}$$

$$\tilde{T} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} m$$

$$\mathcal{L} = \frac{1}{2}\dot{\vec{\eta}}^T \tilde{T} \dot{\vec{\eta}} - \frac{1}{2}\vec{\eta}^T \tilde{V} \vec{\eta}$$

$\downarrow$

Normal modes: $\det(\tilde{V} - \omega^2 \tilde{T}) = 0$

$$\begin{vmatrix} 5k - \omega^2 m & -2k \\ -2k & 5k - \omega^2 m \end{vmatrix} = 0$$

$$(5k - \omega^2 m)^2 - 4k^2 = 0$$

$$\omega^2 = (\mp\sqrt{4k^2} + 5k)\frac{1}{m}$$

Normal frequencies $\rightarrow$ $\omega_\pm = (\sqrt{5k \pm 2k})\frac{1}{\sqrt{m}}$

$$\begin{pmatrix} 5k - (5k \pm 2k) & -2k \\ -2k & 5k - (5k \pm 2k) \end{pmatrix} \vec{\rho}_\pm = 0$$

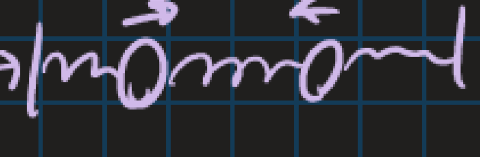
$$[5k - (5 \pm 2)k]\rho_1 - 2k\rho_2 = 0$$

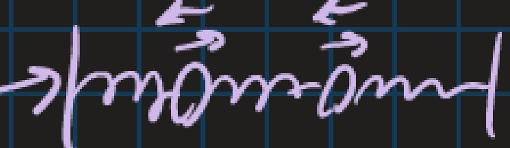
$$-2k\rho_1 + [5k - (5 \pm 2)k]\rho_2 = 0$$

$$\rho_2 = \frac{5 - (5 \pm 2)}{2}\rho_1$$

$\downarrow$

Normal vectors $\rightarrow$ $\vec{\rho}_\pm = \begin{pmatrix} 1 \\ \mp 1 \end{pmatrix}$

$\vec{\rho}_+ = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \rightarrow$  (motion in opposite directions)

$\vec{\rho}_- = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \rightarrow$  (motion in same direction) f04

Wednesday, December 16, 2020 9:36 AM

$$I_{ij} = \int d^3r\, \rho(\vec{r}) (\delta_{ij} r^2 - r_i r_j) \quad \xleftarrow{} r^2 = x^2+y^2+z^2$$

$$\rho(\vec{r}) = H(-x) H(l-x)\, \delta(y)\delta(z) \frac{l}{M} + H(\tfrac{l}{2}-y) H(\tfrac{l}{2}-y)\delta(x-l)\,\delta(z) \frac{l}{M}$$

$$I_{xx} = \frac{l}{M}\int_0^l dx\, (0+0) + \frac{l}{M}\int_{-\frac{l}{2}}^{l/2} dy\, y^2$$

$$= \frac{2}{M}\frac{l^3/8}{3} l$$

$$= \frac{l^4}{12M}$$

$$I_{yy} = \frac{l}{M}\int_0^l dx\,(x^2) + \frac{l}{M}\int_{-\frac{l}{2}}^{l/2} dy\,(l^2)$$

$$= \frac{l}{M}\frac{l^4}{3} + \frac{l^3}{M} l$$

$$= \frac{4l^4}{3M}$$

$$I_{zz} = \frac{l}{M}\int_0^l dx\, x^2 + \frac{l}{M}\int_{-l/2}^{l/2} dy\,(l^2+y^2)$$

$$= \frac{l}{M}\frac{l^4}{3} + \frac{l}{M}\left(l^2 l + \frac{2l^3/8}{3}\right)$$

$$= \frac{l}{M}\frac{l^3}{3} + \frac{l}{M}\left(\frac{13 l^3}{12}\right)$$

$$= \frac{l}{M} l^4 \frac{17}{12}$$

$$I_{xy} = \frac{l}{M}\int_0^l dx\,(-x\cdot 0) + \frac{l}{M}\int_{-\frac{l}{2}}^{l/2} dy\,(-ly)$$

$$= 0$$

$$I_{xz} = \frac{l}{M}\int_0^l dx\,(-x\cdot 0) + \frac{l}{M}\int_{-\frac{l}{2}}^{l/2} dy\,(-l\cdot 0)$$

$$= 0$$

$$I_{yz} = \frac{l}{M}\int_0^l dx\, 0 + \frac{l}{M}\int_{-\frac{l}{2}}^{l/2} dy\,(y\cdot 0)$$

$$= 0$$

$$\tilde{I} = \frac{l^4}{12M}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 16 & 0 \\ 0 & 0 & 17 \end{pmatrix}$$

Center of mass, for the pendulum is at $\vec{p} = \frac{3l}{4}\begin{pmatrix}\cos\vartheta \\ \sin\vartheta\end{pmatrix}$.

$$\mathcal{L} = \frac{1}{2}\dot{\vec{\vartheta}}^T \tilde{I}\dot{\vec{\vartheta}} + \underset{m=2M}{Mg}\frac{3}{2}l\cos\vartheta$$

$$\dot{\vec{\vartheta}} = \begin{pmatrix} 0 \\ 0 \\ \dot{\vartheta}\end{pmatrix}$$

$$= \frac{17}{24}\frac{l^4}{M}\dot{\vartheta}^2 + mg\frac{3}{4}l\cos\vartheta$$

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\vartheta}} - \frac{\partial\mathcal{L}}{\partial\vartheta} = 0 = \frac{17}{12}\frac{l^4}{M}\ddot{\vartheta} + \frac{3}{2}Mgl\sin\vartheta$$

$$\approx \frac{17}{12}\frac{l^4}{M}\ddot{\vartheta} + \frac{3}{2}Mgl\vartheta$$

$$\omega^2 = \frac{18}{17}\frac{M^2 g}{l^3}$$

I don't think this is right. It shouldn't depend on mass. Not sure what's wrong...

f05

Wednesday, December 16, 2020 10:30 AM

$$\frac{\partial F}{\partial q} = p \qquad \frac{\partial F}{\partial P} = q - \frac{eE}{m\omega^2}$$

$$p \to P \qquad q \to q - \frac{eE}{m\omega^2} = Q$$

$$H(Q,P,t) = \frac{P^2}{2m} + \frac{1}{2}m\omega^2\left(Q + \frac{eE}{m\omega^2}\right)^2 - eE\left(Q + \frac{eE}{m\omega^2}\right)$$

$$= \frac{P^2}{2m} + \frac{1}{2}m\omega^2 Q^2 \cancel{+ \frac{1}{2}\frac{e^2E^2}{m\omega^2}} \cancel{+ QeE} \cancel{- eEQ} - \frac{1}{2}\frac{e^2E^2}{m\omega^2}$$

$$= \frac{P^2}{2m} + \frac{1}{2}m\omega^2 Q^2 - \frac{e^2E^2}{2m\omega^2}$$

$$\dot{Q} = \frac{\partial H}{\partial P} = \frac{P}{m}$$

$$\dot{P} = -\frac{\partial H}{\partial Q} = m\omega^2 Q$$

$$m\ddot{Q} = m\omega^2 Q$$

$$\ddot{Q} - \omega^2 Q = 0$$

$$Q(t) = A\cos(\omega t + \delta)$$

$$q(t) = A\cos(\omega t + \delta) + \frac{eE}{m\omega^2}$$

$$p(t) = -\omega A\sin(\omega t + \delta)\, m$$