Solutions

PHY 831: Statistical Mechanics
Homework 5

Due Monday Nov 1, 2021

20 pt

1. Quantum rotor: consider a rotor in two dimensions with

$$\widehat{H} = -\frac{\hbar^2}{2I}\frac{d^2}{d\theta^2} \qquad \text{where } 0 \le \theta \le 2\pi\,.$$

   10 (a) Find the energy eigenfunctions $\psi_n(\theta)$ and eigenvalues E_n for this Hamiltonian.

   10 (b) Write the expression for the density matrix $\langle\theta'|\hat{\rho}|\theta\rangle$ in the canonical ensemble, and evaluate the low- and high-T limits (hint: for one of these limits it's appropriate to convert a sum to an integral.)

20 pt

2. As shown in class, the general treatment of N identical particles in the canonical ensemble is rather tedious. However, for simple toy problems where N is small and/or the number of single particle states is highly limited, calculations can be carried out more readily. In this problem, you will carry out an explicit calculation in the canonical ensemble for a system of $N = 2$ identical particles. Let $Z_1(m)$ denote the partition function for a single quantum particle of mass m in a volume V.

   8 (a) Show by explicit calculation that the $N = 2$ partition function can be written as

$$Z_2 = \frac{1}{2}\Big[Z_1^2(m) \pm Z_1(m/2)\Big]\,,$$

   for the bosonic/fermionic cases, respectively. Note that the first term is the classical Gibbs expression, and the 2nd term corresponds to quantum corrections from particle exchange.

   8 (b) Using that $Z_1(m) = V/l_Q^3$, where l_Q is the de Broglie thermal wavelength, find the corrections to the average energy E and C_V from the particle exchange terms, assuming they are small. (Hint: Assume the Gibbs form is the dominant term in Z_2, and when taking the Logarithm treat the exchange correction as a small perturbation.) Express your final answers in terms of T,l_Q and V.

   4 (c) Derive a criterion for the temperature T for which the assumption that the exchange corrections are small breaks down.

20 pt

3. Show that in the grand canonical ensemble, for a three-dimensional gas of spin S particles with relativistic single-particle energies $\epsilon(p) = \sqrt{p^2 + m^2}$,

the pressure can be written as

$$P = (2S+1) \int \frac{d^3p}{(2\pi\hbar)^3} \frac{p^2}{3\epsilon(p)} f_{\mp}(\epsilon(\vec{p})), \tag{1}$$

where $f_{\mp}(\epsilon) = 1/(e^{\beta(\epsilon-\mu)} \mp 1)$ are the Fermi-Dirac (f_+) and Bose-Einstein (f_-) distribution factors.

20pt

4. What is the density of single-particle states $g(\epsilon)$ for particles in 3-dimensions trapped in an isotropic harmonic oscillator potential $V(r) = \frac{1}{2}m\omega^2 r^2$? How does the energy-dependence of $g(\epsilon)$ compare to that of a gas of ultra-relativistic particles $\epsilon_p = |\vec{p}|c$ trapped in a 3d box with side lengths L?

20pt

5. For a 3-dimensional non-relativistic gas of spin-1/2 fermions, find the change in chemical potential $\delta\mu(T, \rho)$ needed to maintain a constant density while the temperature is raised from 0 to some finite but small T. Give your answer to order T^2.

1 Quantum Rotor $H = -\frac{\hbar^2}{2I}\frac{d^2}{d\theta^2}$ $(0 \le \theta \le 2\pi)$

a.) Find energy eigenvalues / eigenfunctions

Sch. eqn $\Rightarrow$ $-\frac{\hbar^2}{2I}\frac{d^2\psi}{d\theta^2} = E\psi$ let $\frac{2IE}{\hbar^2} = k^2$

$$\Rightarrow -\frac{d^2\psi}{d\theta^2} = k^2\psi$$

$$\therefore \psi(\theta) = A e^{ik\theta}$$

But $\psi(\theta) = \psi(\theta + 2\pi) \Rightarrow e^{i2\pi k} = 1$

$\therefore \cancel{2\pi} k = n \cancel{2\pi} \Rightarrow k = n$ $n = 0, \pm 1, \pm 2, \ldots$

$$\boxed{\therefore \psi_n(\theta) = \frac{1}{\sqrt{2\pi}} e^{in\theta} \quad n = \text{integer}}$$

from normalization $\int_0^{2\pi} d\theta |\psi|^2 = 1$

$$\boxed{E_n = \frac{\hbar^2 n^2}{2I} \quad n = \text{integer}}$$

b.) Find $\langle \theta' | \hat{\rho} | \theta \rangle$ in Canonical Ensemble

$$\langle \theta' | \hat{\rho} | \theta \rangle = \frac{1}{Z} \langle \theta' | e^{-\beta \hat{H}} | \theta \rangle = \frac{1}{Z} \sum_n \langle \theta' | \psi_n \rangle \langle \psi_n | \theta \rangle e^{-\beta E_n}$$

$$\Rightarrow \langle \theta' | \hat{\rho} | \theta \rangle = \frac{1}{Z} \sum_n \frac{e^{in(\theta-\theta')}}{2\pi} e^{-\beta \frac{\hbar^2 n^2}{2I}}$$

and $$Z = \mathrm{Tr}\, e^{-\beta \hat{H}} = \sum_n \langle \psi_n | e^{-\beta \hat{H}} | \psi_n \rangle = \sum_n e^{-\beta \frac{\hbar^2 n^2}{2I}}$$

$$\therefore \boxed{\langle \theta' | \hat{\rho} | \theta \rangle = \frac{1}{2\pi} \frac{\sum_n e^{in(\theta-\theta')} e^{-\beta \frac{\hbar^2 n^2}{2I}}}{\sum_n e^{-\frac{\beta \hbar^2 n^2}{2I}}}}$$

***High T limit ($\beta \to 0$)**

appropriate to convert $\sum_n \to \int dn$

$$\therefore Z = \sum_n e^{-\frac{\beta \hbar^2 n^2}{2I}} \to \int_{-\infty}^{\infty} dn\, e^{-\frac{\beta \hbar^2}{2I} n^2} = \sqrt{\frac{2\pi I}{\beta \hbar^2}}$$

and

$$\sum_n e^{in\Delta\theta} e^{-\beta \frac{\hbar^2 n^2}{2I}} \to \int_{-\infty}^{\infty} dn\, e^{-\frac{\beta \hbar^2}{2I}\left(n^2 - in\Delta\theta \frac{2I}{\beta \hbar^2}\right)}$$

$$= \int_{-\infty}^{\infty} dn\, e^{-\frac{\beta \hbar^2}{2I}\left[\left(n - \frac{i\Delta\theta I}{\beta \hbar^2}\right)^2 + \left(\frac{\Delta\theta I}{\beta \hbar^2}\right)^2\right]}$$

$$= e^{-\frac{(\Delta\theta)^2}{2} \frac{I}{\beta \hbar^2}} \cdot \sqrt{\frac{2\pi I}{\beta \hbar^2}}$$

$$\therefore \boxed{\langle \theta' | \hat{\rho} | \theta \rangle \xrightarrow{T \to \infty} \frac{1}{2\pi} e^{-\frac{I}{2\beta \hbar^2} (\theta' - \theta)^2}}$$

*Low-T Limit ($T \to 0$; $\beta \to \infty$)

as usual in this limit, the ground + low-lying excited states dominate. Here I'll just include the ground state and the 2 degenerate 1st excited states (note to grader: it's fine if students just include the ground state only.)

$\Downarrow$

$$\therefore \frac{1}{2\pi} \sum_n e^{in(\theta-\theta')} e^{-\frac{\beta\hbar^2 n^2}{2I}} \longrightarrow \frac{1}{2\pi}\left(1 + e^{i\Delta\theta} e^{-\frac{\beta\hbar^2}{2I}} + e^{-i\Delta\theta} e^{-\frac{\beta\hbar^2}{2I}} + \cdots\right)$$

and

$$\sum_n e^{-\frac{\beta\hbar^2 n^2}{2I}} \longrightarrow 1 + 2e^{-\frac{\beta\hbar^2}{2I}} + \cdots$$

$$\Rightarrow \boxed{\begin{aligned} \langle\theta'|\hat{\rho}|\theta'\rangle \xrightarrow[\beta\to\infty]{T\to 0} & \frac{\frac{1}{2\pi}\left(1 + 2\cos(\theta'-\theta) e^{-\frac{\beta\hbar^2}{2I}} + \cdots\right)}{1 + 2e^{-\frac{\beta\hbar^2}{2I}}} \\ & \simeq \frac{1}{2\pi}\left(1 - 2e^{-\frac{\beta\hbar^2}{2I}} \cdots\right)\left(1 + 2\cos\Delta\theta\, e^{-\frac{\beta\hbar^2}{2I}} \cdots\right) \\ & = \frac{1}{2\pi}\left(1 - 2e^{-\frac{\beta\hbar^2}{2I}} + 2\cos(\theta-\theta') e^{-\frac{\beta\hbar^2}{2I}}\right) \\ & = \frac{1}{2\pi}\left(1 - 4\sin^2\frac{\theta-\theta'}{2}\, e^{-\frac{\beta\hbar^2}{2I}} + \cdots\right) \end{aligned}}$$

[2] Canonical Ensemble for $N=2$ identical bosons/fermions

a) Show $Z_2 = \frac{1}{2}\left[Z_1^2(m) \pm Z_1\left(\frac{m}{2}\right)\right]$ where $Z_1(m) = \sum_{\vec{k}} e^{-\frac{\hbar^2 k^2}{2m}}$ is the partition function for 1 particle of mass m.

Bosons:

$$|\vec{k}_1\vec{k}_2\rangle_+ = \begin{cases} \frac{1}{\sqrt{2!}}\left[|\vec{k}_1\vec{k}_2) + |\vec{k}_2\vec{k}_1)\right] & (\vec{k}_1 \neq \vec{k}_2) \\ \frac{1}{\sqrt{2!\cdot 2!}}\left[|\vec{k}_1\vec{k}_2) + |\vec{k}_2\vec{k}_1)\right] = |\vec{k}_1\vec{k}_2) & (\vec{k}_1 = \vec{k}_2) \end{cases}$$

$$Z^B_{N=2} = \mathrm{Tr}\left(e^{-\beta\hat{H}}\right) = \sum_{\vec{k}_1\vec{k}_2}{}' \, {}_+\langle\vec{k}_1\vec{k}_2|\, e^{-\beta\hat{H}}\, |\vec{k}_1\vec{k}_2\rangle_+$$

means sum only over the physically distinct $|\vec{k}_1\vec{k}_2\rangle_+$ states. So will include $\vec{k}_1 < \vec{k}_2$ (but not $\vec{k}_1 > \vec{k}_2$ since it's obtained from $P_{12}|\vec{k}_2\vec{k}_1\rangle_+$ where $\vec{k}_2 < \vec{k}_1$) and $\vec{k}_1 = \vec{k}_2$.

$$\Rightarrow Z_2^B = \sum_{\vec{k}_1 > \vec{k}_2} \left(\frac{(\vec{k}_1\vec{k}_2| + (\vec{k}_2\vec{k}_1|}{\sqrt{2}}\right) e^{-\beta H}\left(\frac{|\vec{k}_1\vec{k}_2) + |\vec{k}_2\vec{k}_1)}{\sqrt{2}}\right)$$

$$+ \sum_{\vec{k}} (\vec{k}\vec{k}|\, e^{-\beta H}\, |\vec{k}\vec{k})$$

Now use:

$$(\vec{k}_1\vec{k}_2|\, e^{-\beta\hat{H}}\, |\vec{k}_3\vec{k}_4) = e^{-\beta\left(\frac{\hbar^2 k_1^2}{2m} + \frac{\hbar^2 k_2^2}{2m}\right)} \delta_{\vec{k}_1\vec{k}_3}\delta_{\vec{k}_2\vec{k}_4}$$

$$\Rightarrow Z_2^B = \sum_{\vec{k}_1<\vec{k}_2} e^{-\beta\left(\frac{\hbar^2\vec{k}_1^2}{2m}+\frac{\hbar^2\vec{k}_2^2}{2m}\right)}$$

$$+\sum_{\vec{k}} e^{-\beta\left(\frac{\hbar^2\vec{k}^2}{2m}+\frac{\hbar^2k^2}{2m}\right)}$$

$$= \frac{1}{2}\sum_{k_1\neq k_2} e^{-\beta\left(\frac{\hbar^2k_1^2}{2m}+\frac{\hbar^2k_2^2}{2m}\right)} + \sum_{\vec{k}} e^{-\beta\frac{\hbar^2\vec{k}^2}{2m}\times 2}$$

(used $\displaystyle\sum_{\vec{k}_1<\vec{k}_2} = \frac{1}{2}\sum_{\vec{k}_1\neq\vec{k}_2}$)

$$= \frac{1}{2}\sum_{\vec{k}_1\vec{k}_2} e^{-\beta\left(\frac{\hbar^2k_1^2}{2m}+\frac{\hbar^2\vec{k}_2^2}{2m}\right)} - \frac{1}{2}\sum_{\vec{k}} e^{-\beta\frac{\hbar^2\vec{k}^2}{m}} + \sum_{\vec{k}} e^{-\beta\frac{\hbar^2k^2}{m}}$$

now includes
$\vec{k}_1=\vec{k}_2$
So need to
subtract it off

but $\displaystyle\sum_{\vec{k}_1\vec{k}_2} e^{-\beta\left(\frac{\hbar^2k_1^2}{2m}+\frac{\hbar^2k_2^2}{2m}\right)} = \sum_{k_1} e^{-\beta\frac{\hbar^2k_1^2}{2m}} \times \sum_{k_2} e^{-\beta\frac{\hbar^2k_2^2}{2m}}$

$$= Z_1^2(m)$$

$$\therefore Z_2^B = \frac{1}{2}\left[Z_1^2(m) + Z_1\left(\frac{m}{2}\right)\right] \checkmark$$

Fermions $|\vec{k}_1\vec{k}_2\rangle_- = \frac{1}{\sqrt{2!}}\left(|\vec{k}_1\vec{k}_2) - |\vec{k}_2\vec{k}_1)\right) \quad \vec{k}_1 \neq \vec{k}_2$

(no $\vec{k}_1 = \vec{k}_2$ state since $|\vec{k}\vec{k}\rangle_- = 0$)

$$\therefore Z_2^F = \mathrm{Tr}\, e^{-\beta \hat{H}} = \sum_{\vec{k}_1\vec{k}_2}{}' {}_-\langle \vec{k}_1\vec{k}_2| e^{-\beta\hat{H}} |\vec{k}_1\vec{k}_2\rangle_-$$

as before, Σ' means we only sum over physically distinct states $|\vec{k}_1\vec{k}_2\rangle_-$. Since $|\vec{k}_1\vec{k}_2\rangle_-$ & $|\vec{k}_2\vec{k}_1\rangle_-$ rep. the same physical state (i.e., they only differ by a phase (-1)), we had better only include one of them (say by restricting $\vec{k}_1 < \vec{k}_2$ in the sum)

$$= \frac{1}{2}\sum_{\vec{k}_1<\vec{k}_2}\Big[(\vec{k}_1\vec{k}_2|e^{-\beta\hat{H}}|\vec{k}_1\vec{k}_2) + (\vec{k}_2\vec{k}_1|e^{-\beta\hat{H}}|\vec{k}_2\vec{k}_1) - (\vec{k}_2\vec{k}_1|e^{-\beta\hat{H}}|\vec{k}_1\vec{k}_2) - (\vec{k}_1\vec{k}_2|e^{-\beta\hat{H}}|\vec{k}_2\vec{k}_1)\Big]$$

* as before, use

$$(\vec{k}_1\vec{k}_2|e^{-\beta\hat{H}}|\vec{k}_3\vec{k}_4) = e^{-\beta\left(\frac{\hbar^2 k_1^2}{2m} + \frac{\hbar^2 k_2^2}{2m}\right)}\delta_{\vec{k}_1\vec{k}_3}\delta_{\vec{k}_2\vec{k}_4}$$

$$\Rightarrow Z_2^F = 2\times\frac{1}{2}\sum_{\vec{k}_1<\vec{k}_2} e^{-\beta\left(\frac{\hbar^2 k_1^2}{2m} + \frac{\hbar^2}{2m}k_2^2\right)}$$

$$= \frac{1}{2}\sum_{\vec{k}_1\neq\vec{k}_2} e^{-\beta\left(\frac{\hbar^2 k_1^2}{2m} + \frac{\hbar^2 k_2^2}{2m}\right)}$$

$$Z_2^F = \frac{1}{2}\sum_{\vec{k}_1 \neq \vec{k}_2} e^{-\beta\left(\frac{\hbar^2 k_1^2}{2m} + \frac{\hbar^2 k_2^2}{2m}\right)}$$

$$= \frac{1}{2}\sum_{\vec{k}_1 \vec{k}_2} e^{-\beta\left(\frac{\hbar^2 k_1^2}{2m} + \frac{\hbar^2 k_2^2}{2m}\right)} - \frac{1}{2}\sum_{\vec{k}} e^{-\beta\frac{\hbar^2 k^2}{m}}$$

Now includes spurious $\vec{k}_1 = \vec{k}_2$ terms

So we subtract it off

$$= \frac{1}{2}\left[Z_1^2(m) - Z_1\left(\frac{m}{2}\right)\right] \quad \checkmark$$

$\therefore$ We have shown by explicit calculation that

Bosons

$$\boxed{Z_2 = \frac{1}{2}\left[Z_1^2(m) \pm Z_1\left(\frac{m}{2}\right)\right]}$$

Fermions

b.) Corrections to E & C_V from the $\pm Z_1(\frac{m}{2})$ terms

$$\log Z_2^{B/F} = \log\left[\frac{1}{2}\left\{\underbrace{Z_1^2(m)}_{\text{dominant}} \pm \underbrace{Z_1\left(\tfrac{m}{2}\right)}_{\text{small correction}}\right\}\right]$$

$$\therefore \log Z_2^{B/F} = \log\left[\frac{Z_1^2(m)}{2}\left(1 \pm \frac{Z_1(\frac{m}{2})}{Z_1^2(m)}\right)\right]$$

$$\log Z_2^{B/F} \simeq \log\left(\frac{Z_1^2(m)}{2}\right) \pm \frac{Z_1(\frac{m}{2})}{Z_1^2(m)}$$

Now use: $\langle E\rangle = \frac{-1}{Z_2^{B/F}}\frac{\partial}{\partial\beta} Z_2^{B/F}$

$$= -\frac{\partial}{\partial\beta}\log Z_2^{B/F}$$

$$\simeq -\frac{\partial}{\partial\beta}\log\frac{Z_1^2(m)}{2} \mp \frac{\partial}{\partial\beta}\frac{Z_1(\frac{m}{2})}{Z_1^2(m)}$$

$$= \underbrace{\langle E\rangle_{\text{classical}}}_{\text{w/ Gibbs correction}} + \delta\langle E\rangle^{B,F}$$

$$\therefore \delta\langle E\rangle^{B/F} = \mp\frac{\partial}{\partial\beta}\frac{Z_1(\frac{m}{2})}{Z_1^2(m)}$$

$$\therefore \ \delta \langle E \rangle^{B/F} = \mp \frac{\partial}{\partial \beta} \frac{Z_1(\frac{m}{2})}{Z_1^2(m)}$$

* Using $Z_1^{cl}(m) = \frac{V}{l_Q^3}$ where $l_Q = \sqrt{\frac{2\pi\hbar^2}{m k_B T}} = \sqrt{\frac{2\pi\hbar^2 \beta}{m}}$

and letting Mathematica do the algebra...

Bosons

$$\boxed{\delta \langle E \rangle^{B/F} = \mp \frac{3}{2^{5/2}} \cdot \frac{l_Q^3}{V} \cdot k_B T}$$

fermions

* To get $\delta C_V^{B/F}$, just take $\frac{\partial}{\partial T} \delta \langle E \rangle^{B,F}$

(remembering that l_Q is function of T)

⋮ algebra

$$\boxed{\delta C_V^{B/F} = \pm \frac{3}{2^{7/2}} k_B \left(\frac{l_Q^3}{V} \right)}$$

C.) Criteria where our "perturbative" treatment breaks down

We assumed $Z_1(\frac{m}{2}) \ll Z_1^2(m)$

$$\Rightarrow \cancel{\frac{V}{l_Q^3(\frac{m}{2})}} \ll \cancel{\frac{V}{l_Q^3(m)}} \cdot \frac{V}{l_Q^3(m)}$$

$$\frac{l_Q^6(m)}{l_Q^3(m/2)} \ll V$$

but $l_Q(m) = \frac{1}{\sqrt{2}} l_Q(\frac{m}{2})$

$$\therefore \frac{l_Q^6(m)}{2^{3/2} l_Q^3(m)} \ll V$$

$$\Rightarrow l_Q^3(m) \ll 2^{3/2} L^3$$

$\therefore$ as $l_Q \to L$, our assumption breaks down.

equivalently, breaks down for

$$T \lesssim \frac{2\pi\hbar^2}{m k_B L^2}$$

**

$\boxed{3.}$

Starting from $\frac{PV}{kT} = \log Z \Rightarrow$

$$\frac{PV}{k_B T} = -\eta(2S+1)V \int \frac{d^3p}{(2\pi\hbar)^3} \log\left(1 - \eta e^{-\beta(\epsilon(p)-\mu)}\right)$$

$$= -\eta \frac{(2S+1)V \cdot 4\pi}{8\pi^3\hbar^3} \int p^2 dp \log(\quad)$$

* Now integrate by parts:

$$= \frac{-\eta(2S+1)V\,4\pi}{8\pi^3\hbar^3}\left[\cancel{\frac{p^3}{3}\log\left(1-\eta e^{-\beta(\epsilon(p)-\mu)}\right)\Big|_0^\infty}^{\,0}\right.$$

$$\left. - \int \frac{p^3}{3}\frac{d}{dp}\log\left(1-\eta e^{-\beta(\epsilon_p-\mu)}\right) dp\right]$$

$$= +\eta \frac{(2S+1)V \cdot 4\pi}{(2\pi\hbar)^3} \int dp \cdot \frac{p^3}{3} \cdot \frac{(-\beta)(-\eta)\frac{d\epsilon_p}{dp} e^{-\beta(\epsilon_p-\mu)}}{1-\eta e^{-\beta(\epsilon_p-\mu)}}$$

* $\frac{d\epsilon}{dp} = \frac{p}{\sqrt{p^2+m^2}} = \frac{p}{\epsilon(p)}$

$$= \frac{(2S+1)V \cdot 4\pi}{(2\pi\hbar)^3} \cdot \frac{1}{k_B T} \cdot \int \frac{dp\, p^4}{3\epsilon(p)} \frac{1}{e^{\beta(\epsilon_p-\mu)}-\eta}$$

$$\therefore \frac{P\cancel{V}}{k_B T} = \frac{(2S+1)\cancel{V}}{\cancel{k_B T}} \int \frac{d^3p}{(2\pi\hbar)^3} \frac{p^2}{3E(p)} \frac{1}{e^{\beta(E-\mu)} - \eta}$$

$$\Rightarrow \boxed{P = (2S+1) \int \frac{d^3p}{(2\pi\hbar)^3} \frac{p^2}{3E(p)} \frac{1}{e^{\beta(E(p)-\mu)} - \eta}} \checkmark$$

$\boxed{4.}$ s.p. density of states for 3d oscillator

$$\Sigma(\epsilon) = \sum_{n_x n_y n_z} \Theta(\epsilon - \hbar\omega(n_x+n_y+n_z))$$ (ignore 0-pt energy)

* for $\hbar\omega << k_B T$, can replace $\sum_n \to \int dn$

$$\therefore \Sigma(\epsilon) = \int_0^\infty dn_x \int_0^\infty dn_y \int_0^\infty dn_z \; \Theta(\epsilon - \hbar\omega(n_x+n_y+n_z))$$

$\Rightarrow$ DOS $g(\epsilon) = \dfrac{d\Sigma(\epsilon)}{d\epsilon}$ & use $\dfrac{d}{dx}\Theta(x) = \delta(x)$

$$g(\epsilon) = \int dn_x\, dn_y\, dn_z \; \delta(\epsilon - \hbar\omega(n_x+n_y+n_z))$$

$$= \int dn_x\, dn_y\, dn_z \int_{-\infty}^{\infty} \frac{dt}{2\pi} e^{it(\epsilon - \hbar\omega(n_x+n_y+n_z))}$$

I used $\delta(\epsilon - \hbar\omega(n_x+n_y+n_z)) = \displaystyle\int_{-\infty}^{\infty} \frac{dt}{2\pi} e^{it(\epsilon - \hbar\omega(n_x+n_y+n_z))}$

$$= \int_{-\infty}^{\infty} \frac{dt}{2\pi} e^{it\epsilon} \int_0^\infty dn_x\, e^{-i\hbar\omega t n_x} \int_0^\infty dn_y\, e^{-i\hbar\omega t n_y} \int_0^\infty dn_z\, e^{-i\hbar\omega t n_z}$$

$$= \int_{-\infty}^{\infty} \frac{dt}{2\pi} e^{it\epsilon} I^3(t) \quad \text{where } I(t) = \int_0^\infty dn\, e^{-i\hbar\omega t n}$$

* to give meaning to $I(t)$, I introduce a convergence factor δ that I'll take to zero at the end

i.e., $$I(t) = \lim_{\delta\to 0} \int_0^\infty dn\, e^{-i\hbar\omega t n}\, e^{-n\delta}$$

$$= \lim_{\delta\to 0} \frac{1}{i\hbar\omega t + \delta}$$

$$\therefore\ g(\varepsilon) = \lim_{\delta\to 0} \int_{-\infty}^{\infty} \frac{dt}{2\pi}\, e^{it\varepsilon} \left(\frac{1}{i\hbar\omega t + \delta}\right)^3$$

let $t\varepsilon = x$

$$g(\varepsilon) = \lim_{\delta\to 0} \frac{1}{\varepsilon} \int_{-\infty}^{\infty} \frac{dx}{2\pi}\, e^{ix} \left(\frac{1}{i\frac{\hbar\omega}{\varepsilon}x + \delta}\right)^3$$

$$= \lim_{\delta\to 0} \frac{\varepsilon^2}{\hbar\omega^3} \underbrace{\int_{-\infty}^{\infty} \frac{dx}{2\pi}\, e^{ix} \left(\frac{1}{ix + \delta}\right)^3}_{}$$

evaluate by contour integration (or Mathematica)
$\parallel$
$\pi e^{-i\delta} \to \pi \quad (\delta\to 0)$

$$\boxed{\therefore\ g(\varepsilon) = \frac{\varepsilon^2}{2(\hbar\omega)^3}}$$

This has same ε-dependence as ultrarelativistic gas in 3d box

$g_{UR,3}(\varepsilon) \sim \varepsilon^2$.

$\boxed{5}$

From class, at low $k_BT \ll \mu$,

$$N(\mu, T) = \int_0^{\mu} d\varepsilon\, g(\varepsilon) + \frac{\pi^2}{6}(k_BT)^2 g'(\mu)$$

* want to adjust $\mu(T) \to \mu(T) + \delta\mu$ so $\delta N = 0$
fixed

$\Downarrow$

take $\mu \to \mu + \delta\mu$ in the 1st integral

$$N(\mu, T) = \int_0^{\mu+\delta\mu} d\varepsilon\, g(\varepsilon) + \frac{\pi^2}{6}(kT)^2 g'(\mu)$$

$$= \int_0^{\mu} d\varepsilon\, g(\varepsilon) + \underbrace{\delta\mu\, g(\mu) + \frac{\pi^2}{6}(kT)^2 g'(\mu)}_{\text{want this to add up to zero}}$$

$$\therefore \boxed{\delta\mu = -\frac{\pi^2}{6}(k_BT)^2 \frac{g'(\mu)}{g(\mu)}}$$

Now, for $S=\frac{1}{2}$ 3d NR gas, $g(\varepsilon) = \frac{2V}{\sqrt{2}\pi^2\hbar^3} m^{3/2}\sqrt{\varepsilon}$

$$\therefore \frac{g'(\mu)}{g(\mu)} = \frac{\cancel{\frac{2V}{\sqrt{2}\pi^2\hbar^3} m^{3/2}} \cdot \frac{1}{2}\frac{1}{\sqrt{\mu}}}{\cancel{\frac{2V}{\sqrt{2}\pi^2\hbar^3} m^{3/2}} \sqrt{\mu}} = \frac{1}{2}\frac{1}{\mu}$$

$$\therefore \boxed{\delta\mu = -\frac{\pi^2}{6}(k_BT)^2 \cdot \frac{1}{2\mu}}$$

* lastly, need to express $\delta\mu = \delta\mu(T,\rho)$ where $\rho = \frac{N}{V}$.

* Since we're asked to only calculate term $\mathcal{O}(T^2)$ accuracy, we can replace $\mu(T) \longrightarrow \epsilon_F$ since the difference is higher order in T^2.

$$\therefore \delta\mu = -\frac{\pi^2}{12}(k_BT)^2 \frac{1}{\epsilon_F}$$

but $\epsilon_F = \frac{\hbar^2}{2m}k_F^2$ and $\rho = \frac{2}{6\pi^2}k_F^3 = \frac{1}{3\pi^2}k_F^3$

$$\Rightarrow k_F = (3\pi^2\rho)^{1/3}$$

$$\therefore \boxed{\delta\mu(\rho,T) = -\frac{\pi^2}{12}(k_BT)^2 \frac{2m}{\hbar^2(3\pi^2\rho)^{2/3}}}$$