PHY 831: Statistical Mechanics
Exam 1

October 18, 2021

1. (10 points) Consider a one-dimensional solid of length L at temperature T containing N atoms in a chain. Each atom has one spin-half conduction electron (so the rest of the electrons can effectively be neglected). Model excitations of the lattice using a one-dimensional version of the Debye model, so that the density of states in frequency space is given by $g(\omega) = L/(2\pi c_s)$, where c_s is the sound speed, $k = \omega/c_s$ is the wavenumber, and the energy of a phonon is given by $\epsilon = \hbar\omega$. Since motion is only possible in the x-direction, the waves can have only one polarization. Treat the electrons as a free, non-relativistic gas confined to move in one-dimension.

   (a) What is the electron chemical potential of this system at zero temperature (i.e., what is the Fermi energy ϵ_F, expressed in terms of the electron density)?

   (b) What is the electron contribution to the energy of the system at zero temperature?

   (c) What is the Debye frequency for the lattice, expressed in terms of N, L, and c_s?

   (d) What is the phonon contribution to the energy for T small compared to the Debye temperature?

2. For free bosons in a D-dimensional box with an energy-momentum relation $\epsilon = ap^s$, where a and s are positive constants, what is the dimension at which Bose-Einstein condensation begins to occur at low temperatures, in terms of D and s?

3. Consider a gas of N spin-1/2 non-relativistic electrons of mass m (non-interacting) confined to a two dimensional area A.

   (a) Compute the single particle density of states and derive an expression for the Fermi energy ϵ_F in terms of the matter density.

   (b) At $T = 0$, compute the average energy, expressed in terms of N and the Fermin energy.

   (c) Calculate the force per unit length exerted by the system (i.e., the 2d-analog of pressure) at $T = 0$.

   (d) Now consider $T > 0$ but $k_B T << \epsilon_F$. What is the average energy to $\mathcal{O}(T^2)$ in the Sommerfeld expansion?

4. Consider a gas of spinless bosons with a pairwise hard core interaction $u(r < a) = \infty$.

(a) Treating the system as a classical gas, compute the b_2 coefficient in the virial cluster expansion.

(b) Treating the system as a quantum gas, compute the b_2 coefficient to leading non-vanishing order in a/l_Q.