Administration notes

1) HW5 (due <u>Monday</u>) is posted

2) I left the "mostly" graded exams in Ohio by mistake. I'll finish those when I return home tomorrow. (Sorry about the delay!)

3) Remaining topics (tentative)

- Non-int fermion systems
- Non-int bosonic systems

} ~ 4-5 lectures

- Kinetic theory/Boltzmann Eqn./Hydrodynamics
- approximate methods for interacting systems

} ~ 4-5 lectures

- Phase transitions/Renormalization Group ~ 6 lectures

- 13 W,F lectures + 6 monday sessions remain
- might either trim a bit of material (Kinetic theory/Approx. methods) to preserve the monday group sessions if possible

Recap

N-identical particles in the GCE

* Still working w/ the ideal gas $\hat{H} = \sum_{a=1}^{N} \frac{-\hbar^2 \vec{\nabla}_a^2}{2m}$ for definiteness,

$$Z_\eta^{GCE}(T,\mu,V) = \sum_{N=0}^{\infty} \sum_{\vec{k}_1 \vec{k}_2 \cdots}{}' e^{-\beta\left(\sum_{a=1}^{N} \frac{\hbar^2 k_a^2}{2m} - \mu N\right)}$$

* use

(1.) $\sum_{\vec{k}_1 \vec{k}_2 \cdots}{}' = \sum_{\{n_{\vec{k}}\}}{}'$

(2.) $\sum_{a=1}^{N} \epsilon(\vec{k}_a) = \sum_{\vec{k}} \epsilon(\vec{k}) n_{\vec{k}}$

(3.) $\sum_{\vec{k}} n_{\vec{k}} = N$

$$\boxed{Z_\eta^{GCE} = \sum_{N=0}^{\infty} \sum_{\{n_{\vec{k}}\}}{}' e^{-\beta \sum_{\vec{k}} (\epsilon(\vec{k}) - \mu) n_{\vec{k}}}}$$

* Now, $\sum_{N=0}^{\infty} \sum_{\{n_{\vec{k}}\}}{}' = \sum_{\{n_{\vec{k}}\}}$

constrained to $\sum_{\vec{k}} n_{\vec{k}} = N$

unconstrained apart from

$n_{\vec{k}} = 0,1,2,3,\ldots$ bosons

$n_{\vec{k}} = 0,1$ fermions

$$\Rightarrow Z_\eta^{GCE} = \sum_{\{n_{\vec{k}}\}} e^{-\beta \sum_{\vec{k}} (\epsilon(\vec{k}) - \mu) n_{\vec{k}}}$$

$$= \sum_{\{n_{\vec{k}}\}} \prod_{\vec{k}} e^{-\beta(\epsilon(\vec{k}) - \mu) n_{\vec{k}}}$$

$$= \prod_{\vec{k}} \left(\sum_{n_{\vec{k}}=0}^{1\,F,\ \infty\,B} e^{-\beta(\epsilon(\vec{k}) - \mu) n_{\vec{k}}} \right)$$

<u>Fermions</u>: $n_{\vec{k}} = 0, 1$ only

$$\Rightarrow \boxed{Z_F^{GCE} = \prod_{\vec{k}} \left(1 + e^{-\beta(\epsilon(\vec{k}) - \mu)} \right)}$$

<u>Bosons</u>: $n_{\vec{k}} = 0, 1, 2, \ldots \infty$

$$Z_B^{GCE} = \prod_{\vec{k}} \left(\sum_{n_{\vec{k}}=0}^{\infty} \left[e^{-\beta(\epsilon(\vec{k}) - \mu)} \right]^{n_{\vec{k}}} \right)$$

$$\boxed{Z_B^{GCE} = \prod_{\vec{k}} \left(\frac{1}{1 - e^{-\beta(\epsilon(\vec{k}) - \mu)}} \right)}$$

On combining into 1 single equation

$$\boxed{Z_\eta^{GCE} = \prod_{\vec{k}} \left(1 - \eta e^{-\beta(\epsilon_{\vec{k}} - \mu)}\right)^{-\eta}}$$

* This is written for the ideal gas in momentum basis. [i.e., the s.p. states are $|\vec{k}\rangle$]

* In general, for N-noninteracting Fermions/Bosons

where $\hat{H} = \sum_{a=1}^{N} \hat{h}_a$

$$\hat{h}|\alpha\rangle = \epsilon_\alpha |\alpha\rangle$$

$$\Downarrow$$

$$E = \sum_\alpha \epsilon_\alpha n_\alpha$$

$$\Downarrow$$

$$\sum_\alpha n_\alpha = N$$

$$\therefore \boxed{\begin{array}{c} Z_\eta^{GCE} = \prod_\alpha \left(1 - \eta e^{-\beta(\epsilon_\alpha - \mu)}\right)^{-\eta} \\ \text{for general non-int. Bosons/Fermions} \end{array}}$$

*Recall, in GCE

$$Z^{GCE} = e^{-\beta \Omega(T,\mu,V)} \qquad \Omega = \text{grand potential}$$

$$d\Omega = -S\,dT - P\,dV - N\,d\mu$$

$$\therefore \Omega_\eta = -\frac{1}{\beta} \log Z_\eta^{GCE}$$

$$= -\frac{1}{\beta} \log \left[\prod_{\vec{k}} \left(1 - \eta e^{-\beta(\epsilon_{\vec{k}} - \mu)}\right)^{-\eta} \right]$$

$$\boxed{\Omega_\eta = \frac{\eta}{\beta} \sum_{\vec{k}} \log\left(1 - \eta e^{-\beta(\epsilon_k - \mu)}\right)}$$

avg. particle # :

$$\langle N \rangle_\eta = -\left(\frac{\partial \Omega}{\partial \mu}\right)_{V,T}$$

$$= \frac{\cancel{-}\eta}{\cancel{\beta}} \sum_{\vec{k}} \frac{(\cancel{-\eta})(\cancel{\beta})\, e^{-\beta(\epsilon_k - \mu)}}{1 - \eta e^{-\beta(\epsilon_k - \mu)}}$$

$$\Rightarrow \boxed{\langle N \rangle_\eta = \sum_{\vec{k}} \frac{e^{-\beta(\epsilon_k - \mu)}}{1 - \eta e^{-\beta(\epsilon_k - \mu)}} = \sum_{\vec{k}} \frac{1}{e^{\beta(\epsilon_k - \mu)} - \eta}}$$

<u>Note that</u>

$$\langle n_{\vec{q}} \rangle_\eta = \frac{1}{Z_\eta^{GCE}} \sum_{\{n_k\}} n_{\vec{q}}\, e^{-\beta \sum_{\vec{k}} (\epsilon_k - \mu) n_{\vec{k}}}$$

$$= \frac{1}{Z_\eta^{GCE}} \left(\frac{1}{\beta} \frac{\partial}{\partial \epsilon_{\vec{q}}} \right) \sum_{\{n_{\vec{k}}\}} e^{-\beta \sum_{\vec{k}} (\epsilon_{\cancel{\mu}} - \mu) n_{\vec{k}}}$$

$$= \frac{1}{\beta} \frac{\partial}{\partial \epsilon_{\vec{q}}} \log Z_\eta^{GCE} = -\frac{\partial}{\partial \epsilon_{\vec{q}}} \Omega_\eta$$

$$\Downarrow$$

$$\boxed{\langle n_{\vec{q}} \rangle_\eta = \frac{1}{e^{\beta(\epsilon_q - \mu)} - \eta}}$$

$$\boxed{\therefore \langle N \rangle_\eta = \sum_{\vec{k}} \langle n_{\vec{k}} \rangle_\eta}$$

Non-relativistic gas - General results

* We want to get eqns. relating $E, P, n = \frac{N}{V}$ (equation of state)

* First, let's be more careful + account for the intrinsic spin (or any intrinsic quantum #'s that endow the s.p. energies w/ some g-fold degeneracy)

* In all of the above, the s.p. states are really

$$|\vec{k}\, m_s\rangle \qquad m_s = -S, -S+1, \ldots +S \quad (2S+1 \text{ values})$$

S_z eigenvalue

* If we are not in an external $\vec{B}$-field, the s.p. energies indep. of m_s, $\epsilon(\vec{p}, m_s) = \epsilon(\vec{p})$

$$\therefore \sum_{\vec{k}, m_s} (\text{indep. of } m_s) = (2S+1) \sum_{\vec{k}} (\ldots)$$

def. $2S+1 = g$ "spin degeneracy factor"

$$\Rightarrow \mathcal{G}_\eta = \frac{\eta}{\beta}\, g \sum_{\vec{k}} \log\left(1 - \eta e^{-\beta(\epsilon_{\vec{k}} - \mu)}\right)$$

$$\Rightarrow \langle N \rangle_\eta = g \sum_{\vec{k}} \frac{1}{e^{\beta(\epsilon_{\vec{k}} - \mu)} - \eta} \qquad \langle E \rangle_\eta = g \sum_{\vec{k}} \epsilon(\vec{k}) \langle n_{\vec{k}, m_s} \rangle$$

$$\langle n_{\vec{k}, m_s} \rangle = \frac{1}{e^{\beta(\epsilon_{\vec{k}} - \mu)} - \eta} \qquad \text{etc...}$$

* To derive general EOS relations, let's first review some basic TD:

$$\mathcal{G}(T,V,\mu) = E - TS - \mu N$$

* but recall way back when we used <u>extensivity</u> ($E(\lambda S, \lambda V, \lambda N) = \lambda E(S,V,N)$) to derive

$$E = TS - PV + \mu N$$

$$\therefore \boxed{\mathcal{G} = -PV = -\frac{1}{\beta}\log Z_{GCE}}$$

$$\therefore \beta P_\eta = \frac{1}{V}\log Z_{GCE,\eta} = -\frac{g\eta}{V}\sum_{\vec{k}} \log\left(1-\eta e^{-\beta(\epsilon_k-\mu)}\right)$$

def. fugacity $z = e^{\beta\mu}$

$$\Rightarrow \beta P_\eta = -\eta \frac{g}{V}\sum_{\vec{k}} \log\left(1-\eta z e^{-\beta\frac{\hbar^2 k^2}{2m}}\right)$$

$$\boxed{\beta P_\eta = -\eta g \int \frac{d^3k}{(2\pi)^3} \log\left(1-\eta z e^{-\beta\frac{\hbar^2 k^2}{2m}}\right)}$$

density: $n_\eta \equiv \frac{\langle N_\eta \rangle}{V} = \frac{g}{V} \sum_{k} \frac{1}{z^{-1} e^{\beta \epsilon_k} - \eta}$

$$\boxed{n_\eta = g \int \frac{d^3k}{(2\pi)^3} \frac{1}{z^{-1} e^{\beta \frac{\hbar^2 k^2}{2m}} - \eta}}$$

Energy/Vol: $$\boxed{\mathcal{E}_\eta = \frac{\langle E \rangle_\eta}{V} = g \int \frac{d^3k}{(2\pi)^3} \frac{\frac{\hbar^2 k^2}{2m}}{z^{-1} e^{\beta \frac{\hbar^2 k^2}{2m}} - \eta}}$$

* Now clean these up w/ the following change of variables

$\beta \frac{\hbar^2 k^2}{2m} = x \implies$ let $k = \frac{2\sqrt{\pi}}{\ell_Q} x^{1/2}$ $\qquad \ell_Q = \sqrt{\frac{2\pi \hbar^2}{m k_B T}}$

$$\implies dk = \frac{\sqrt{\pi}}{\ell_Q} x^{-1/2} dx$$

a bit of algebra + doing trivial $\ast$ integrals

$$\implies \boxed{\begin{aligned} \beta \mathcal{E}_\eta &= \frac{g}{\ell_Q^3} \frac{2}{\sqrt{\pi}} \int_0^\infty \frac{dx\, x^{3/2}}{z^{-1} e^x - \eta} \\ n_\eta &= \frac{g}{\ell_Q^3} \cdot \frac{2}{\sqrt{\pi}} \int_0^\infty \frac{dx\, x^{1/2}}{z^{-1} e^x - \eta} \end{aligned}}$$

hmm. These look kind of similar ...

$$\beta P_\eta = -\eta \frac{g}{2\pi^2} \frac{4\pi^{3/2}}{l_Q^3} \int_0^\infty dx\, x^{1/2} \log\left(1-\eta z e^{-x}\right)$$

* looks a bit different from the other 2, but we can integrate by parts

$$\int_0^\infty dx\, x^{1/2} \log\left(1-\eta z e^{-x}\right)$$

$$= \cancelto{0}{\frac{x^{3/2}}{3/2} \log\left(1-\eta z e^{-x}\right)\Big|_0^\infty} - \int_0^\infty \frac{x^{3/2}}{3/2} \cdot \frac{d}{dx} \log(\quad)$$

$$= -\frac{1}{3/2} \int_0^\infty x^{3/2} \frac{(\eta z e^{-x})}{1-\eta z e^{-x}}$$

$$= -\frac{\eta}{3/2} \int_0^\infty \frac{x^{3/2}}{z^{-1} e^{x} - \eta}\, dx$$

$\therefore$

$$\boxed{\begin{aligned} \beta P_\eta &= \frac{g}{l_Q^3} \cdot \frac{4}{3\sqrt{\pi}} \int_0^\infty \frac{dx\, x^{3/2}}{z^{-1} e^{x} - \eta} \\ n_\eta &= \frac{g}{l_Q^3} \cdot \frac{2}{\sqrt{\pi}} \int_0^\infty \frac{dx\, x^{1/2}}{z^{-1} e^{x} - \eta} \\ \beta \mathcal{E}_\eta &= \frac{g}{l_Q^3} \frac{2}{\sqrt{\pi}} \int_0^\infty \frac{dx\, x^{3/2}}{z^{-1} e^{x} - \eta} \end{aligned}}$$

$\Rightarrow$ suggests defining

$$\boxed{f_m^\eta(z) \equiv \frac{1}{\Gamma(m)} \int_0^\infty \frac{dx\, x^{m-1}}{z^{-1} e^{x} - \eta} \qquad \Gamma(m+1) = m!}$$

$$\Gamma(m) = \int dx\, x^{m-1} e^{-x}$$

$$\Gamma\left(\tfrac{3}{2}\right) = \frac{\sqrt{\pi}}{2} \qquad \text{etc...}$$

$$\Gamma\left(\tfrac{5}{2}\right) = \frac{3}{2} \frac{\sqrt{\pi}}{2}$$

∴ We have

$$\boxed{\begin{aligned} \beta P_\eta &= \frac{g}{l_Q^3} f^\eta_{5/2}(z) \\ n_\eta &= \frac{g}{l_Q^3} f^\eta_{3/2}(z) \\ \mathcal{E}_\eta &= \frac{3}{2}\frac{1}{\beta}\frac{g}{l_Q^3} f^\eta_{5/2}(z) = \frac{3}{2} P_\eta \end{aligned}} \qquad (**)$$

* In a sense, (**) completely solves all such problems of non-int identical particles.

* But it's a bit inconvenient because

1.) all complexity "hidden" in the $f^\eta_m(z)$

2.) We'd really rather have $P = P(n,T)$ instead of $P(T, e^{\beta\mu})$

* To make progress, we need to look at high T & low T limits separately, where we can invert $n_\eta = \frac{g}{l_Q^3} f^\eta_{3/2}(z)$ to get

$\mu = \mu(n,T)$

High T / low density limit ($z = e^{\beta\mu} \ll 1$)

$$f_m^\eta(z) = \frac{1}{(m-1)!}\int_0^\infty \frac{dx\ x^{m-1}}{z^{-1}e^x - \eta}$$

$$= \frac{1}{(m-1)!}\int_0^\infty dx\ \frac{x^{m-1}\ z e^{-x}}{1 - \eta z e^{-x}}$$

Now since $z \ll 1$, $\frac{1}{1-\eta z e^x} = \sum_{\alpha=0}^{\infty} (\eta z e^{-x})^\alpha$

$$\Rightarrow f_m^\eta(z) = \frac{\eta}{(m-1)!}\int dx\ x^{m-1}\ \eta z e^{-x} \sum_{\alpha=0}^{\infty} (\eta z e^{-x})^\alpha$$

$$= \frac{\eta}{(m-1)!}\int dx\ x^{m-1} \sum_{\alpha=0}^{\infty} (\eta z e^{-x})^{\alpha+1}$$

$$= \frac{\eta}{(m-1)!}\int dx\ x^{m-1} \sum_{\alpha=1}^{\infty} (\eta z e^{-x})^{\alpha}$$

$$= \frac{1}{(m-1)!}\sum_\alpha \eta^{\alpha+1} z^\alpha \int dx\ x^{m-1} e^{-\alpha x}$$

$\alpha x = y,\ dx = \frac{1}{\alpha} dy$

$$= \frac{1}{\cancel{(m-1)!}}\sum_{\alpha=1}^{\infty} \eta^{\alpha+1} z^\alpha \frac{1}{\alpha^m} \cancel{\int dy\ y^{m-1} e^{-y}}$$

" $\Gamma(m) = (m-1)!$

$$\therefore \quad f_m^\eta(z \ll 1) \simeq \sum_{\alpha=1}^{\infty} \eta^{\alpha+1} \frac{z^\alpha}{\alpha^m}$$

$$\boxed{f_m^\eta(z \ll 1) = z + \eta \frac{z^2}{2^m} + \frac{z^3}{3^m} + \eta \frac{z^4}{4^m} + \cdots}$$

Makes sense! We find for z small ($e^{\beta\mu} \ll 1$), then $f_m^n(z)$ is small, + so is βP_η, n_η, + $\mathcal{E}_\eta$, so it really does correspond to high T / low-density limit as claimed.

$$\therefore \quad \frac{n_\eta l_\alpha^3}{g} = f_{3/2}^\eta(z) \simeq z + \eta \frac{z^2}{2^{3/2}} + \frac{z^3}{3^{3/2}} + \cdots \qquad (A)$$

$$\frac{\beta P_\eta l_\alpha^3}{g} = f_{5/2}^\eta(z) \simeq z + \eta \frac{z^2}{2^{5/2}} + \frac{z^3}{3^{5/2}} + \cdots \qquad (B)$$

From (A), $\quad z \simeq \frac{n_\eta l_\alpha^3}{g} - \eta \frac{z^2}{2^{3/2}} - \frac{z^3}{3^{3/2}} \cdots$

Can be solved iteratively/perturbatively since $z \ll 1$, then so is the "driving term $\frac{n_\eta l_\alpha^3}{g}$

$$\Rightarrow \quad e^{\beta\mu} = z \simeq \frac{n_\eta l_\alpha^3}{g} - \frac{\eta}{2^{3/2}} \left(\frac{n_\eta l_\alpha^3}{g} \right)^2 + \cdots$$

∴ substituting this expression for z ⇒

$$\beta P_\eta = n_\eta \left[1 - \frac{\eta}{2^{5/2}} \left(\frac{n_\eta l_\alpha^3}{g} \right) + O\left(\frac{n_\eta l_\alpha^3}{g} \right)^2 \cdots \right]$$

Natural expansion parameter $= \dfrac{n_\eta l_\alpha^3}{g}$

* When $n_\eta l_\alpha^3 \gtrsim g$, then this expansion fails (so-called Quantum degenerate limit)

∴ We again see what we derived earlier (via Canonical ensemble), that high T / low-n is the classical limit

* Here, however, the ideal gas correction factor (due to QM exchange) is much easier to calculate, & it's relatively easy to go to higher order terms.