Recap

$$I(\mu,T) \simeq \int_0^{\mu} d\epsilon\, \phi(\epsilon) + \phi'(\mu)\frac{\pi^2 (kT)^2}{6}$$

Now use $\mu \simeq \epsilon_F + \left(\frac{g'}{g}\right)_{\epsilon_F} \frac{\pi^2}{6}(k_B T)^2$

+ expand keeping terms thru $\mathcal{O}((kT)^2)$

$$\boxed{\begin{aligned} \Rightarrow I(\mu,T) &\simeq \int_0^{\epsilon_F} d\epsilon\, \phi(\epsilon) + \left[\phi'(\epsilon_F) - \phi(\epsilon_F)\left(\frac{g'}{g}\right)_{\epsilon_F}\right]\frac{\pi^2}{6}(k_B T)^2 \\ &= \underbrace{\int_0^{\epsilon_F} d\epsilon\, \phi(\epsilon)}_{I(\epsilon_F,\, T=0)} + \underbrace{g(\epsilon_F)\frac{d}{d\epsilon}\left(\frac{\phi(\epsilon)}{g(\epsilon)}\right)_{\epsilon_F=\epsilon} \frac{\pi^2}{6}(k_B T)^2}_{\text{correction due to finite } T} \end{aligned}}$$

Now we can apply this general formula to get finite T corrections to various quantities (E, C_V, P, etc.)

e.g.: Say $g(\epsilon) = a\epsilon^n$ (as it usually is!)

Then applying to E $\left(\phi(\epsilon) = \epsilon g(\epsilon)\right)$

$$\boxed{\begin{aligned} &\Rightarrow E = \left(\frac{n+1}{n+2}\right)\epsilon_F N\left[1 + \frac{(n+2)\pi^2}{6}\left(\frac{k_B T}{\epsilon_F}\right)^2 + \cdots\right] \\ &\Rightarrow C_V = \left(\frac{\partial E}{\partial T}\right)_{N,V} = \frac{(n+1)\pi^2}{3} N \frac{k_B^2 T}{\epsilon_F} \end{aligned}}$$

e.g., $D=3$ NR $g \propto \sqrt{\epsilon}$

$$\Rightarrow E = \frac{3}{5}\epsilon_F N\left[1 + \frac{5\pi^2}{12}\left(\frac{k_B T}{\epsilon_F}\right)^2 + \cdots\right]$$

$$C_V = \frac{\pi^2}{2} N \frac{k_B^2 T}{\epsilon_F}$$

*Useful to define Fermi Temperature $k_B T_F = \epsilon_F$

$$\boxed{\begin{aligned} &\Rightarrow E = \frac{3}{5}(\epsilon_F N)\left[1 + \frac{5\pi^2}{12}\left(\frac{T}{T_F}\right)^2 + \cdots\right] \\ &C_V = \frac{\pi^2}{2} N k_B \left(\frac{T}{T_F}\right) \end{aligned}}$$

* Simple physical interp of linear T-dep of C_v

Since $f(\epsilon)$ "almost" a step function,
only particles within a narrow range
$\epsilon_F - k_B T \lesssim \epsilon \lesssim \epsilon_F + k_B T$ can be excited
thermally (Those further away deep in the Fermi Sea
are Pauli Blocked)

* fraction of particles in this range $\sim N\left(\frac{T}{T_F}\right)$

* each of these particles excited by $\mathcal{O}(k_B T)$

$$\therefore \Delta E \sim N\left(\frac{T}{T_F}\right) k_B T$$

$$C_v \sim N k_B \frac{T}{T_F}$$

Bosons at Low T

* Recall a few lectures ago, we wrote down general formulas for Fermi/Bose gases

$$n_\eta = \frac{g}{\ell_Q^3} f^\eta_{3/2}(z)$$

$$\beta P_\eta = \frac{g}{\ell_Q^3} f^\eta_{5/2}(z)$$

$$\varepsilon_\eta = \frac{E_\eta}{V} = \frac{3}{2} P_\eta$$

$$f^\eta_m(z) = \frac{1}{(m-1)!}\int_0^\infty \frac{dx\, x^{m-1}}{z^{-1}e^x - \eta}$$

* For either case (Bosonic/Fermionic), the small z (high T/low density → classical) limit followed from

$$f^\eta_m(z \ll 1) = z + \eta\frac{z^2}{2^m} + \frac{z^3}{3^m} + \eta\frac{z^4}{4^m} + \cdots$$

* For the opposite limit (low T/high density → QM), we have to separately treat Fermionic/Bosonic cases because they're so different.

* Having done Fermions (Sommerfeld expansion etc.), let's now turn to Bosons

* 1^{st}, we note there was no restriction on μ for the Fermionic case; It was positive for low T (roughly $0 \le T \le T_F$) and negative for $T \gtrsim T_F$.

* But for Bosons, we note that μ has an upper bound:

$$Z = \prod_{\vec{k}, m_s} \sum_{n_{\vec{k}}=0}^{\infty} e^{-\beta(\epsilon_k - \mu) n_{\vec{k}}}$$

$$\longrightarrow \quad = \prod_{\vec{k}, m_s} \frac{1}{1 - e^{-\beta(\epsilon_k - \mu)}}$$

only if $e^{-\beta(\epsilon_k - \mu)} \le 1$ for all $\epsilon_{\vec{k}}$

$$\boxed{\Rightarrow \quad \mu \le \epsilon_{\vec{k}, min}}$$

Or, equivalently look at

$$\langle n_{\vec{k}} \rangle = \frac{1}{e^{\beta(\epsilon_{\vec{k}} - \mu)} - 1}$$

* becomes negative (nonsensical!) if $e^{\beta(\epsilon_k - \mu)} < 1$

* and blows up for $\mu = \epsilon_{\vec{k}}$

* Knowing that bosons can clump together in the same s.p. state tells us that the blowing up of $\langle n_{\vec{k}=0} \rangle$ for $\mu \to 0^-$ is "real" + needs to be understood (this is the Bose-Einstein Condensation) + interpreted (and treated with mathematical care!)

* For definiteness, consider $S=0$ NR bosons in d-dim.

$$g(\epsilon) = \left(\frac{L}{2\pi\hbar}\right)^D S_D\, m (2m\epsilon)^{D/2-1}$$

$$\Downarrow$$

$$N \approx \left(\frac{L}{2\pi\hbar}\right)^D S_D \frac{(2m)^{D/2}}{2} \int_0^\infty \frac{d\epsilon\, \epsilon^{D/2-1}}{e^{\beta(\epsilon-\mu)}-1}$$

Putting $\approx$ rather than $=$ to remind you that we converted sums to integrals

* In the usual way, let $x=\beta\epsilon$ and $z=e^{\beta\mu}$

$$\Rightarrow \boxed{N \approx \frac{L^D}{\ell_Q^D} f^{(+)}_{D/2}(z)}$$

* Consider now the behavior of N as $\mu \to 0^-$

$$f_{D/2}(z) \xrightarrow[\mu\to 0]{} \frac{1}{(D/2-1)!} \int_0^\infty \frac{dx\, x^{D/2-1}}{e^x - 1}$$

$$= \frac{1}{\Gamma(\frac{D}{2})} \int_0^\infty dx\, \frac{x^{D/2-1} e^{-x}}{1-e^{-x}}$$

$$f^{(+)}_{D/2}(z\to 1) = \frac{1}{\Gamma(D/2)} \sum_{n=0}^{\infty} \int_0^{\infty} dx\, x^{D/2-1} \left(e^{-x}\right)^{n+1}$$

let $y = (n+1)x$

$dy = (n+1)dx$

$$f^{(+)}_{D/2}(z\to 1) = \frac{1}{\cancel{\Gamma(D/2)}} \sum_{n=0}^{\infty} (n+1)^{-D/2} \times \underbrace{\int_0^{\infty} dy\, y^{D/2-1} e^{-y}}_{\cancel{\Gamma(D/2)}}$$

$$\boxed{\therefore f^{(+)}_{D/2}(z\to 1) = \sum_{n=0}^{\infty} \frac{1}{(n+1)^{D/2}} = \sum_{m=1}^{\infty} \frac{1}{m^{D/2}}}$$

This is the so called Riemann Zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} = \frac{1}{\Gamma(s)} \int_0^{\infty} dx \frac{x^{s-1}}{e^x - 1}$$

$\zeta(1) = \infty$ (divergent series)

$\zeta(\frac{3}{2}) = 2.6123.....$

$\zeta(2) = \frac{\pi^2}{6}$

$\zeta(4) = \frac{\pi^4}{90}$

$\therefore$ For $D=1,2$ at $\mu \to 0$

$$N = \frac{L^D}{\ell_Q^D} f^{(+)}_{D/2}(z=1) = \infty$$

$\boxed{\text{So there is no problem in tuning } \mu \text{ from } \mu=-\infty \text{ to } \mu=0 \text{ to dial in any desired value of } N \text{ in } D=1,2}$

* But, for $D \geq 3$, $N(\mu=0^-)$ has a finite maximum allowable value.

e.g. $D=3$: $N(\mu=0^-) = \frac{V}{\ell_Q^3}\zeta(3/2) \simeq (2.61\ldots)\frac{V}{\ell_Q^3}$

This poses a conundrum since we want our theory to be able to accomodate any N from 0 to ∞.

What's going on?!

The issue is converting sums to integrals via the DOS

$$g_{NR}(\epsilon) \propto \epsilon^{D/2-1} = 0 \text{ at } \epsilon=0$$
$$\text{for } D \geq 3$$

$\therefore$ Converting $\sum_\alpha \longrightarrow \int d\epsilon\, g(\epsilon)$ artificially zeros out the most important $\epsilon = 0$ contribution!

$\Rightarrow$ I.e, the $\int d\epsilon\, g(\epsilon) \cdots$ spuriously suppresses the contributions from the lowest s.p. level, which can have a macroscopically large fraction of N occupying it!

Solution: Explicitly separate out the $\epsilon_{min} \simeq 0$ contribution in the discrete sum, and then convert the remaining $\sum_\alpha \rightarrow \int d\epsilon\, g(\epsilon)$

$$\log Z = \sum_{\vec{k}} -\log\left(1 - e^{-\beta(\epsilon_k - \mu)}\right)$$

$$= -\log\left(1 - e^{-\beta(\epsilon_{min} - \mu)}\right) + \sum_{\vec{k} > k_{min}}{}' -\log\left(1 - e^{-\beta(\epsilon_k - \mu)}\right)$$

$$\simeq -\log\left(1 - e^{\beta\mu}\right) - \int_0^\infty d\epsilon\, g(\epsilon) \log\left(1 - e^{-\beta(\epsilon - \mu)}\right)$$

$\left(\text{take } \epsilon_{min} = \left(\frac{2\pi}{L}\right)^2 \frac{\hbar^2}{2m} \simeq 0\right)$

$\therefore \log Z \approx -\log(1-e^{\beta\mu}) - \int d\epsilon\, g(\epsilon) \log(1-e^{-\beta(\epsilon-\mu)})$

$$N = \frac{\partial}{\partial(\beta\mu)} \log Z = \frac{e^{\beta\mu}}{1-e^{\beta\mu}} + \int d\epsilon\, g(\epsilon) \frac{e^{-\beta(\epsilon-\mu)}}{1-e^{-\beta(\epsilon-\mu)}}$$

$$\Rightarrow \boxed{N = \underbrace{\frac{e^{\beta\mu}}{1-e^{\beta\mu}}}_{N_0 = \#\text{ bosons occupying lowest s.p. energy } \epsilon_{min} \simeq 0} + \underbrace{\int d\epsilon\, g(\epsilon) \frac{1}{e^{\beta(\epsilon-\mu)}-1}}_{N_x = \#\text{ of bosons in the other sp levels.}}}$$

Now, if we sweep thru $\mu = -\infty$ thru $\mu = 0^-$, we can get any desired N from 0 to ∞.

* but from before

$$N_x(\mu = 0^-) = \zeta\left(\tfrac{3}{2}\right) \frac{V}{\ell_Q^3} \simeq 2.61 \frac{V}{\ell_Q^3} \Rightarrow n_x(\mu=0^-) = \frac{2.61}{\ell_Q^3}$$

$\therefore$ For $n > \frac{2.61}{\ell_Q^3}$, must have finite fraction of N occupying the lowest level

* This macroscopic occupation of the lowest level is the so-called BEC condensate.

$$N_0 = \frac{e^{\beta\mu}}{1 - e^{\beta\mu}} = \frac{z}{1-z}$$

* as $\mu \to 0^-$ at fixed T, we have

$$\frac{1}{N_0} = e^{-\beta\mu} - 1 \approx -\beta\mu$$

$\therefore$ For $\left(\frac{N_0}{N}\right)$ to be finite fraction even as $N \to \infty$, this means $N_0 \to \infty$ too so we have $\mu \sim 0$.

* For fixed T, there must be a critical density above which BEC occurs

$$\frac{N_c}{V} = n_c = \frac{\zeta(3/2)}{\ell_Q^3}$$

Since if $N > N_c$, then $N_0 = N - N_c$ becomes a sizable fraction of N.

Explicitly, $\frac{N_c}{V} \simeq 2.61 \left(\frac{m k_B T}{2\pi\hbar^2}\right)^{3/2}$

$\therefore$ $N_c(T)$ decreases w/ decreasing T.

Critical temperature

* Say we have fixed N and initially are at high T where $N_c(T) > N$. Then there is no macroscopic occupation of the ϵ_{min} state.

* Now lower T. Eventually, $N_c(T_c) = N$. This is when condensation begins!

$$\frac{N}{V} = \frac{N_c}{V} = 2.61 \left(\frac{m k_B T_c}{2\pi\hbar^2}\right)^{3/2}$$

$$\Rightarrow \boxed{T_c(N,V) = \frac{2\pi\hbar^2}{m k_B}\left(\frac{N}{V\,\zeta(\frac{3}{2})}\right)^{2/3}}$$

$$\frac{N}{V} = \frac{N_c}{V} = 2.61 \left(\frac{mk_BT_c}{2\pi\hbar^2}\right)^{3/2}$$

$$\Rightarrow \boxed{T_c(N,V) = \frac{2\pi\hbar^2}{mk_B}\left(\frac{N}{V\zeta(\frac{3}{2})}\right)^{2/3}}$$

* Now we can express N_0 via T & T_c as:

$$N = N_0 + N_c$$

$$\Rightarrow \frac{N_0}{N} = 1 - \frac{N_c}{N} = 1 - \left(\frac{T}{T_c}\right)^{3/2}$$

Since

$$N = V\zeta(\tfrac{3}{2})\left(\frac{mkT_c}{2\pi\hbar^2}\right)^{3/2}$$

$$N_c = V\zeta(\tfrac{3}{2})\left(\frac{mkT}{2\pi\hbar^2}\right)^{3/2}$$

$$\Rightarrow \boxed{\frac{N_0}{N} = 1 - \left(\frac{T}{T_c}\right)^{3/2}}$$

EOS of the BEC gas

$$\frac{PV}{k_BT} = \log Z = \underbrace{-\log(1-e^{\beta\mu})} + \underbrace{\frac{V}{\ell_Q^3} f_{5/2}^{(+)}(z = e^{\beta\mu})}$$

$\epsilon_{min} = 0$ term from the discrete sum

what we had before we realized need to separate out the $\epsilon = 0$ term before converting to integral

$\Rightarrow$ using that $N_0 = \frac{e^{\beta\mu}}{1-e^{\beta\mu}} = \frac{z}{1-z} \Rightarrow z = \frac{N_0}{1+N_0}$

$$\therefore -\log(1-z) = -\log\left(\frac{1}{1+N_0}\right) = \log(1+N_0)$$

$$\boxed{\Rightarrow \frac{PV}{K_B T} = \log(1+N_0) + \frac{V}{l_Q^3} f_{5/2}^{(+)}(z)}$$

Since $\log(1+N_0) \sim \log(N_0) \lesssim \log N$, this is negligible compared to the 2nd term which $\propto N$

$$\therefore \frac{P\cancel{V}}{K_B T} \approx \frac{\cancel{V}}{l_Q^3} f_{5/2}^{(+)}(z)$$

$$\Rightarrow P = \frac{K_B T}{l_Q^3} f_{5/2}^{(+)}(z)$$

$\Rightarrow$ 1.) Condensate doesn't contribute to pressure (makes sense – $\vec{k}=0$ sp states)

2.) $\frac{\partial P}{\partial V} = 0$ when $\mu = 0$, so system is infinitely compressible

$$K = V\frac{\partial V}{\partial P} = \infty$$