PHY 831: Statistical Mechanics
Homework 5

Due Monday Nov 1, 2021

1. Quantum rotor: consider a rotor in two dimensions with

$$\widehat{H} = -\frac{\hbar^2}{2I}\frac{d^2}{d\theta^2} \qquad \text{where } 0 \le \theta \le 2\pi\,.$$

   (a) Find the energy eigenfunctions $\psi_n(\theta)$ and eigenvalues E_n for this Hamiltonian.

   (b) Write the expression for the density matrix $\langle\theta'|\hat{\rho}|\theta\rangle$ in the canonical ensemble, and evaluate the low- and high-T limits (hint: for one of these limits it's appropriate to convert a sum to an integral.)

2. As shown in class, the general treatment of N identical particles in the canonical ensemble is rather tedious. However, for simple toy problems where N is small and/or the number of single particle states is highly limited, calculations can be carried out more readily. In this problem, you will carry out an explicit calculation in the canonical ensemble for a system of $N = 2$ identical particles. Let $Z_1(m)$ denote the partition function for a single quantum particle of mass m in a volume V.

   (a) Show by explicit calculation that the $N = 2$ partition function can be written as

$$Z_2 = \frac{1}{2}\Bigg[Z_1^2(m) \pm Z_1(m/2)\Bigg]\,,$$

   for the bosonic/fermionic cases, respectively. Note that the first term is the classical Gibbs expression, and the 2nd term corresponds to quantum corrections from particle exchange.

   (b) Using that $Z_1(m) = V/l_Q^3$, where l_Q is the de Broglie thermal wavelength, find the corrections to the average energy E and C_V from the particle exchange terms, assuming they are small. (Hint: Assume the Gibbs form is the dominant term in Z_2, and when taking the Logarithm treat the exchange correction as a small perturbation.) Express your final answers in terms of T,l_Q and V.

   (c) Derive a criterion for the temperature T for which the assumption that the exchange corrections are small breaks down.

3. Show that in the grand canonical ensemble, for a three-dimensional gas of spin S particles with relativistic single-particle energies $\epsilon(p) = \sqrt{p^2 + m^2}$,

the pressure can be written as

$$P = (2S+1)\int \frac{d^3p}{(2\pi\hbar)^3}\frac{p^2}{3\epsilon(p)} f_{\mp}(\epsilon(\vec{p})), \tag{1}$$

where $f_{\mp}(\epsilon) = 1/(e^{\beta(\epsilon-\mu)} \mp 1)$ are the Fermi-Dirac (f_+) and Bose-Einstein (f_-) distribution factors.

4. What is the density of single-particle states $g(\epsilon)$ for particles in 3-dimensions trapped in an isotropic harmonic oscillator potential $V(r) = \frac{1}{2}m\omega^2 r^2$? How does the energy-dependence of $g(\epsilon)$ compare to that of a gas of ultra-relativistic particles $\epsilon_p = |\vec{p}|c$ trapped in a 3d box with side lengths L?

5. For a 3-dimensional non-relativistic gas of spin-1/2 fermions, find the change in chemical potential $\delta\mu(T,\rho)$ needed to maintain a constant density while the temperature is raised from 0 to some finite but small T. Give your answer to order T^2.