Homework 6

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6.1

$$PV\beta = \log(Z) = g \sum_{\mathbf{p}} \log\left(\left(1 - \eta e^{-\beta(\epsilon_p - \mu)}\right)^{-\eta}\right), \tag{6.1.1}$$

$$P = -\eta \frac{g}{V\beta} \sum_{\mathbf{p}} \log\left(1 - \eta e^{-\beta(\epsilon - \mu)}\right), \quad (\epsilon_p = pc) \tag{6.1.2}$$

$$\approx -\eta \frac{g}{\beta} \int \frac{\mathrm{d}^3 p}{(2\pi\hbar)^3} \log\left(1 - \eta e^{-\beta(pc - \mu)}\right), \tag{6.1.3}$$

$$= \frac{4\pi g}{3(2\pi\hbar)^3} \int \mathrm{d}p \, \frac{p^3 e^{-\beta(pc-\mu)}}{1 - \eta e^{-\beta(pc-\mu)}}, \tag{6.1.4}$$

$$\mu = 0 : \tag{6.1.5}$$

$$P = \frac{4\pi g}{3(2\pi\hbar)^3} \int_0^\infty \mathrm{d}p \, \frac{p^3 e^{-\beta pc}}{1 - \eta e^{-\beta pc}}, \tag{6.1.6}$$

$$= \begin{cases} \frac{4\pi g}{3(2\pi\hbar)^3} \frac{1}{(\beta c)^4} \frac{\pi^4}{15}, & \eta = +1 \quad \text{(bosons)} \\ \frac{4\pi g}{3(2\pi\hbar)^3} \frac{1}{(\beta c)^4} \frac{7\pi^4}{120}, & \eta = -1 \quad \text{(fermions)} \end{cases} . \tag{6.1.7}$$

Hence:

$$P = AgT^4, \tag{6.1.8}$$

where

$$A = \begin{cases} \frac{4\pi}{3(2\pi\hbar)^3} \frac{k^4}{c^4} \frac{\pi^4}{15}, & \eta = +1 \quad \text{(bosons)} \\ \frac{4\pi}{3(2\pi\hbar)^3} \frac{k^4}{c^4} \frac{7\pi^4}{120}, & \eta = -1 \quad \text{(fermions)} \end{cases} . \tag{6.1.9}$$

$$\frac{E}{V} = \frac{g}{V} \sum_{\mathbf{p}} \epsilon_p \left\langle n_p \right\rangle_\eta, \tag{6.1.10}$$

$$\approx g \int \frac{\mathrm{d}^3 p}{(2\pi\hbar)^3} \frac{\epsilon_p}{e^{\beta(\epsilon_p - \mu)} - \eta}, \tag{6.1.11}$$

$$\mu = 0 : \tag{6.1.12}$$

$$\frac{E}{V} = \frac{g 4\pi c}{(2\pi\hbar)^3} \int \mathrm{d}p \, \frac{p^3 e^{-\beta pc}}{1 - \eta e^{-\beta pc}}, \tag{6.1.13}$$

$$= \begin{cases} \frac{g4\pi c}{(2\pi\hbar)^3} \frac{1}{(\beta c)^4} \frac{\pi^4}{15}, & \eta = +1 \quad \text{(bosons)} \\ \frac{g4\pi c}{(2\pi\hbar)^3} \frac{1}{(\beta c)^4} \frac{7\pi^4}{120}, & \eta = -1 \quad \text{(fermions)} \end{cases} . \tag{6.1.14}$$

Hence:

$$\frac{E}{V} = BgT^4, \tag{6.1.15}$$

where $B = 3Ac$.

6.2

6.2.1

$$\begin{aligned}
N_\pm &= \sum_{\mathbf{k}} \Theta(k_F^\pm - |\mathbf{k}|), &(6.2.1)\\
&\approx V \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \Theta(k_F^\pm - k), &(6.2.2)\\
&= \frac{V}{2\pi} \frac{(k_F^\pm)^3}{6\pi^2}. &(6.2.3)\\
\therefore k_F^\pm &= (6\pi^2 n_\pm)^{1/3}, &(6.2.4)
\end{aligned}$$

where $n_\pm = N_\pm / V$.

6.2.2

$$\begin{aligned}
\langle KE \rangle &= \sum_{\mathbf{k}} \frac{\hbar^2 k^2}{2m} \Theta(k_F^+ - |\mathbf{k}|) + \sum_{\mathbf{k}} \frac{\hbar^2 k^2}{2m} \Theta(k_F^- - |\mathbf{k}|), &(6.2.5)\\
&\approx V \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \frac{\hbar^2 k^2}{2m} \Theta(k_F^+ - |\mathbf{k}|)\\
&\quad + V \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \frac{\hbar^2 k^2}{2m} \Theta(k_F^- - |\mathbf{k}|), &(6.2.6)\\
&= \frac{V}{2\pi^2} \frac{\hbar^2}{2m} \frac{((k_F^+)^5 + (k_F^-)^5)}{5}. &(6.2.7)\\
\to \frac{\langle KE \rangle}{V} &= \frac{\hbar^2}{4m\pi^2} \frac{((k_F^+)^5 + (k_F^-)^5)}{5}. &(6.2.8)\\
k_F^\pm = (6\pi^2 n_\pm)^{1/3} : & &(6.2.9)\\
\frac{\langle KE \rangle}{V} &= \frac{(6\pi^2)^{5/3} \hbar^2}{20m\pi^2} (n_+^5 + n_-^5), &(6.2.10)\\
&= \frac{3}{10m} \hbar^2 (6\pi^2)^{2/3} (n_+^{5/3} + n_-^{5/3}). &(6.2.11)
\end{aligned}$$

6.2.3

For small deviations from the symmetric state ($n_\pm = n/2 \pm \delta$),

$$\begin{aligned}
\frac{\langle KE \rangle}{V} &= \frac{3}{10m} \hbar^2 (6\pi^2)^{2/3} \left(\left(\frac{n}{2} + \delta \right)^{5/3} + \left(\frac{n}{2} - \delta \right)^{5/3} \right), &(6.2.12)\\
&\approx \frac{6}{10m} \hbar^2 (6\pi^2)^{2/3} \left(\left(\frac{n}{2} \right)^{5/3} + \frac{5}{9} \left(\frac{n}{2} \right)^{-1/3} \delta^2 + \frac{5}{243} \left(\frac{n}{2} \right)^{-7/3} \delta^4 \right). &(6.2.13)
\end{aligned}$$

6.2.4

$$\frac{U}{V} = \alpha\left(\frac{n}{2}+\delta\right)\left(\frac{n}{2}-\delta\right), \tag{6.2.14}$$
$$= \alpha\frac{n^2}{4} - \alpha\delta^2. \tag{6.2.15}$$

Since $E = \langle KE \rangle + U$,

$$\begin{aligned}\frac{E}{V} &= \alpha\frac{n^2}{4} - \alpha\delta^2 \\ &\quad + \frac{6}{10m}\hbar^2(6\pi^2)^{2/3}\left(\left(\frac{n}{2}\right)^{5/3} + \frac{5}{9}\left(\frac{n}{2}\right)^{-1/3}\delta^2 + \frac{5}{243}\left(\frac{n}{2}\right)^{-7/3}\delta^4\right), \quad (6.2.16)\\ &= \left(\frac{E}{V}\right)_{\delta=0} + \left(\frac{4}{3}(3\pi^2)^{2/3}\frac{\hbar^2}{2m}n^{-1/3} - \alpha\right)\delta^2 + \mathcal{O}(\delta^4). \quad (6.2.17)\end{aligned}$$

$$\therefore \alpha > \alpha_c = \frac{4}{3}(3\pi^2)^{2/3}\frac{\hbar^2}{2m}n^{-1/3} \tag{6.2.18}$$

causes the electron gas to have the ability to lower its energy by developing a magnetization.

6.2.5

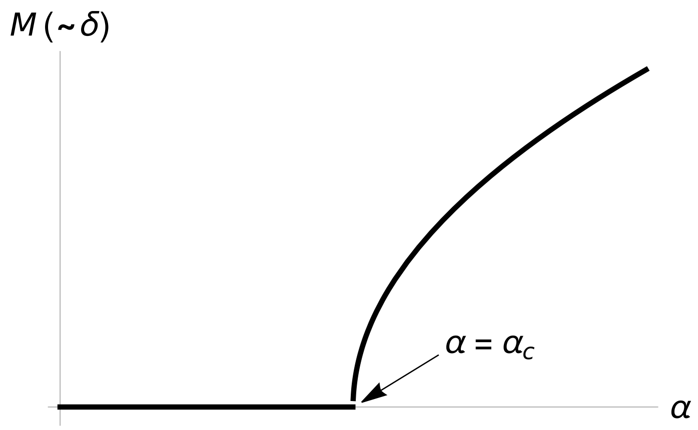

Figure 6.2.1

The magnetization starts at 0 for $\alpha < \alpha_c$, but, for $\alpha \geq \alpha_c$, the magnetization (which is essentially the quantity δ) grows as $\sqrt{a - a_c}$.

6.3

6.3.1

$$
\begin{aligned}
\mathscr{G} &= -\frac{1}{\beta}\log(\mathscr{Z}), && (6.3.1)\\
&= \frac{\eta}{\beta}\sum_{\mathbf{p}}\log\Big(1-\eta e^{-\beta(\epsilon_p-\mu)}\Big), && (6.3.2)\\
&= \frac{\eta}{\beta}V\int\frac{\mathrm{d}^D p}{(2\pi\hbar)^D}\log\Big(1-\eta e^{-\beta(\epsilon_p-\mu)}\Big), && (6.3.3)\\
&= \frac{\eta V}{(2\pi\hbar)^D\beta}S_D\int\mathrm{d}p\,p^{D-1}\log\Big(1-\eta e^{-\beta(\epsilon_p-\mu)}\Big), \qquad \Big(S_D=\int\mathrm{d}\Omega_D\Big) && (6.3.4)\\
&\vdots \qquad \text{(Mathematica)}\\
\mathscr{G} &= -\frac{VS_D\alpha s}{(2\pi\hbar)^D D}\int\mathrm{d}p\,p^{D+s-1}\frac{e^{-\beta(\alpha p^s-\mu)}}{1-\eta e^{-\beta(\alpha p^s-\mu)}}. && (6.3.5)
\end{aligned}
$$

Let $x=\beta\alpha p^s$:

$$
\begin{aligned}
\mathscr{G} &= -\frac{VS_D\alpha s}{(2\pi\hbar)^D D}\frac{1}{s\beta\alpha}\left(\frac{1}{\beta\alpha}\right)^{D/s}\int\mathrm{d}x\,\frac{x^{D/s}}{\mathfrak{z}^{-1}e^x-\eta}, && (6.3.6)\\
&= -\frac{VS_D\alpha}{(2\pi\hbar)^D D}\left(\frac{1}{\beta\alpha}\right)^{D/s+1}\Gamma\left(\frac{D}{s}+1\right)f^{\eta}_{\frac{D}{s}+1}(\mathfrak{z}). && (6.3.7)
\end{aligned}
$$

For the following, $V=L^D$.

$$
\begin{aligned}
n &= \frac{N}{V}, && (6.3.8)\\
&= \frac{1}{V}\sum_{\mathbf{p}}\frac{1}{\mathfrak{z}e^{\beta\epsilon_p}-\eta}, && (6.3.9)\\
&= \int\frac{\mathrm{d}^D p}{(2\pi\hbar)^D}\frac{1}{\mathfrak{z}e^{\beta\alpha p^s}-\eta}, && (6.3.10)\\
&= \frac{S_D}{(2\pi\hbar)^D}\int\mathrm{d}p\,p^{D-1}\frac{1}{\mathfrak{z}e^{\beta\alpha p^s}-\eta}, && (6.3.11)\\
&\vdots \qquad \text{(Same integral as before)} && (6.3.12)\\
&= \frac{S_D}{(2\pi\hbar)^D}\frac{1}{s}\frac{1}{\beta\alpha}^{D/s}\Gamma\left(\frac{D}{s}\right)f^{\eta}_{\frac{D}{s}}(\mathfrak{z}). && (6.3.13)
\end{aligned}
$$

6.3.2

$$
\begin{aligned}
PV &= -\mathscr{G}. && (6.3.14)\\
E &= -\frac{\partial}{\partial\beta}\log\mathscr{Z}\,\big|_{\mathfrak{z}=\text{const.}}, && (6.3.15)\\
&= \frac{D}{s}\mathscr{G}. && (6.3.16)\\
\frac{PV}{E} &= \frac{s}{D}. && (6.3.17)
\end{aligned}
$$

6.3.3

As $T \to 0$,

$$\lim_{T\to 0} \frac{1}{e^{\beta(\epsilon_p-\mu)}+1} = \Theta(\epsilon_p - \mu). \tag{6.3.18}$$

Itaque,

$$\frac{E}{V} = \frac{1}{V}\sum_{\mathbf{p}} \alpha p^s \Theta(p_f - p), \tag{6.3.19}$$

$$= \int \frac{\mathrm{d}^D p}{(2\pi\hbar)^D} \alpha p^s, \tag{6.3.20}$$

$$= \frac{S_d \alpha}{(2\pi\hbar)^D} \frac{p_F^{D+s}}{D+s}. \tag{6.3.21}$$

$$n = \int \frac{\mathrm{d}^D p}{(2\pi\hbar)^D} \Theta(p_F - p). \tag{6.3.22}$$

$$p_F = \left(\frac{(2\pi\hbar)^D D}{S_D} n\right)^{1/D}. \tag{6.3.23}$$

$$\therefore \frac{E}{V} = \frac{S_D \alpha}{(2\pi\hbar)^D} \frac{\left(\frac{(2\pi\hbar)^D D}{S_D} n\right)^{1+(s/D)}}{D+s}, \tag{6.3.24}$$

$$\frac{E}{V} \propto n^{1+(s/D)}. \tag{6.3.25}$$

$$\frac{PV}{E} = \frac{s}{D}, \tag{6.3.26}$$

$$P = \frac{s}{D} \frac{S_D \alpha}{(2\pi\hbar)^D} \frac{\left(\frac{(2\pi\hbar)^D D}{S_D} n\right)^{1+(s/D)}}{D+s}, \tag{6.3.27}$$

$$P \propto n^{1+(s/D)}. \tag{6.3.28}$$

6.3.4

If, in the limit $\mathfrak{z} \to 1$, $f^+_{D/s}(\mathfrak{z} \to 1)$ is not finite, then no Bose-Einstein condensation forms. However, if $f^+_{D/s}(\mathfrak{z} \to 1)$ is finite, then Bose-Einstein condensation forms.

The fugacity at $\mathfrak{z} = 1$ is given by

$$f^+_{D/s}(\mathfrak{z} = 1) = \lim_{\varepsilon\to\infty} \frac{1}{\Gamma\left(\frac{D}{s}\right)} \int_0^\varepsilon \mathrm{d}x \, \frac{x^{(D/s)-1}}{e^x - 1}, \tag{6.3.29}$$

$$= \lim_{\varepsilon\to\infty} \frac{1}{\Gamma\left(\frac{D}{s}\right)} \int_0^\varepsilon \mathrm{d}x \, x^{(D/s)-1} \left(x + \frac{x^2}{2!} + \dots\right)^{-1}, \tag{6.3.30}$$

$$\approx \lim_{\varepsilon\to\infty} \frac{1}{\Gamma\left(\frac{D}{s}\right)} \int_0^\varepsilon \mathrm{d}x \, x^{(D/s)-2}. \tag{6.3.31}$$

This converges for $0 > (D/s) - 2 > -1$, so if $D > s$ a Bose-Einstein condensate can form. Therefore, if $D = s = 2$ no Bose-Einstein condensate will form.

6.4

If $s = 1$, then, from problem 3d, $d_c = 1$.

With $s = 1$ and $\alpha = c$, $\epsilon_p = pc$. Additionally, $\eta = 1$ since the problem is discussing bosons. From problem 3,

$$n = \frac{S_D}{(2\pi\hbar)^D} \left(\frac{1}{\beta c}\right)^D \Gamma(D) f_D^+(\mathfrak{z}). \tag{6.4.1}$$

To get the critical temperature, let $\mathfrak{z} = 1$:

$$n = \frac{S_D}{(2\pi\hbar)^D} \left(\frac{k_B T_c}{c}\right)^D \Gamma(D) f_D^+(1), \tag{6.4.2}$$

$$T_c = \frac{2\pi\hbar c}{L} \left(\frac{N}{\Gamma(D)\zeta(D)S_D}\right)^{1/D} \tag{6.4.3}$$