

# Homework 01

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## 1.1

**a**

Integrating both sides from  $(V_0, \Theta_0)$  to  $(V, \Theta)$  gives

From the first law,

$$dQ = dE - dW. \quad (1.1.1) \quad \int_{\Theta_0}^{\Theta} \frac{1}{\tau} \frac{\partial E}{\partial \Theta} \Big|_{\tau} d\tau = -Nk_B \ln\left(\frac{V}{V_0}\right), \quad (1.1.10)$$

Hence,

or

$$Q_h = \int \left( \frac{\partial E}{\partial \Theta} d\Theta + P dV \right), \quad (1.1.2) \quad \frac{V}{V_0} = e^{-\frac{1}{Nk_B} \int_{\Theta_0}^{\Theta} \frac{1}{\tau} \frac{\partial E}{\partial \Theta} \Big|_{\tau} d\tau}. \quad (1.1.11)$$

$$= \int \frac{Nk_B \Theta_h}{V} dV, \quad (1.1.3) \quad \mathbf{c}$$

$$= Nk_B \Theta_h \ln\left(\frac{V_2}{V_1}\right). \quad (1.1.4) \quad \text{By definition,}$$

$$Q_c = Nk_B \Theta_c \ln\left(\frac{V_3}{V_4}\right). \quad (1.1.5) \quad \eta \equiv \frac{W}{Q_h} = \frac{Q_h - Q_c}{Q_h} = 1 - \frac{Q_c}{Q_h}. \quad (1.1.12)$$

For 1.1.5, heat is being expelled, hence the minus sign (evident in swapping the bounds of integration).

From equations 1.1.4 and 1.1.5,

$$1 - \frac{Q_c}{Q_h} = 1 - \frac{\Theta_c \ln(V_3/V_4)}{\Theta_h \ln(V_2/V_1)} \quad (1.1.13)$$

**b**

From the first law,

Additionally, from 1.1.11,  $\ln(V_2/V_1) = \ln(V_3/V_4)$ . Thus

$$dE = -P dV, \quad (1.1.6)$$

$$\frac{\partial E}{\partial \Theta} d\Theta = -P dV. \quad (1.1.7) \quad \eta = 1 - \frac{\Theta_c}{\Theta_h}. \quad (1.1.14)$$

Assuming an ideal gas,

## 1.2

$$P = \frac{Nk_B \Theta}{V}. \quad (1.1.8) \quad \mathbf{a}$$

Plugging this into 1.1.7 gives

Take the first law:

$$\frac{1}{\Theta} \frac{\partial E}{\partial \Theta} d\Theta = -Nk_B \frac{dV}{V}. \quad (1.1.9) \quad dQ = dE - dW. \quad (1.2.1)$$

If one uses  $(T, V)$  as the independent variables, then

$$dQ = \frac{\partial E}{\partial T} dT + \left( \frac{\partial E}{\partial V} + P \right) dV, \quad (1.2.2)$$

$$= C_V dT + \left( \frac{\partial E}{\partial V} + P \right) dV. \quad (1.2.3)$$

**b**

Finding an expression for  $dS$ , it follows from the second law that

$$dS = \frac{dQ}{T}, \quad (1.2.4)$$

$$= \frac{C_V}{T} dT + \left( \frac{1}{T} \frac{\partial E}{\partial V} + \frac{P}{T} \right) dV. \quad (1.2.5)$$

By definition of an exact differential,

$$\frac{\partial S}{\partial V} = \frac{1}{T} \frac{\partial E}{\partial V} + \frac{P}{T}, \quad (1.2.6)$$

$$\frac{\partial S}{\partial T} = \frac{C_V}{T}. \quad (1.2.7)$$

Additionally

$$\frac{\partial^2 S}{\partial V \partial T} = \frac{\partial^2 S}{\partial T \partial V}, \quad (1.2.8)$$

so

$$\frac{\partial}{\partial V} \frac{C_V}{T} = \frac{\partial}{\partial T} \left( \frac{1}{T} \frac{\partial E}{\partial V} + \frac{P}{T} \right), \quad (1.2.9)$$

$$0 = -\frac{1}{T^2} \frac{\partial E}{\partial V} - \frac{P}{T^2} + \frac{1}{T} \frac{\partial P}{\partial T}, \quad (1.2.10)$$

$$= \frac{\partial E}{\partial V} + P - T \frac{\partial P}{\partial T}, \quad (1.2.11)$$

$$\frac{\partial E}{\partial V} = T \frac{\partial P}{\partial T} - P. \quad (1.2.12)$$

**c**

For an ideal gas,

$$\frac{\partial E}{\partial V} = \frac{Nk_B T}{V} - P = 0. \quad (1.2.13)$$

Thus the internal energy cannot depend on  $V$ , and is only a function of  $T$ . However, since no consideration was made pertaining to the particle number,  $N$ , the internal energy could still depend on  $N$ . Therefore  $E = E(T, N)$ .

### 1.3

From the first law,

$$dE = -P dV - S dT + \mu dN, \quad (1.3.1)$$

where

$$P \equiv -\frac{\partial E}{\partial V}, \quad (1.3.2)$$

$$S \equiv -\frac{\partial E}{\partial T}, \quad (1.3.3)$$

$$\mu \equiv \frac{\partial E}{\partial N}. \quad (1.3.4)$$

The Maxwell relations mentioned are derived as follows:

$$\frac{\partial \mu}{\partial T} = \frac{\partial^2 E}{\partial T \partial N} = -\frac{\partial S}{\partial N}. \quad (1.3.5)$$

$$\frac{\partial S}{\partial V} = -\frac{\partial^2 E}{\partial T \partial V} = \frac{\partial P}{\partial T}. \quad (1.3.6)$$

Inverting these gives the requested relations:

$$\frac{\partial T}{\partial \mu} = -\frac{\partial N}{\partial S}, \quad (1.3.7)$$

$$\frac{\partial P}{\partial T} = \frac{\partial S}{\partial V}. \quad (1.3.8)$$

### 1.4

**a**

The Helmholtz Free Energy is given by

$$A = T_0 \ln \left( \frac{V}{V_0} \right) - T \frac{V}{V_0} \left( \frac{T}{T_0} \right)^a. \quad (1.4.1)$$

**b**

The equation of state is found by using the Maxwell relation

$$-P = \frac{\partial A}{\partial V} = \frac{T_0}{V} - \frac{T}{V_0} \left( \frac{T}{T_0} \right)^a. \quad (1.4.2)$$

This can be rewritten more simply as

$$PV = ST - T_0. \quad (1.4.3)$$

**c**

From [1.4.3](#),

$$W = - \int_{V_a}^{V_b} P \, dV, \quad (1.4.4)$$

$$= \int_{V_a}^{V_b} \frac{T_0}{V} - \frac{T_a}{V_0} \left( \frac{T_a}{T_0} \right)^a \, dV, \quad (1.4.5)$$

$$= T_0 \ln \left( \frac{V_b}{V_a} \right) - \frac{(V_b - V_a) T_a}{V_0} \left( \frac{T_a}{T_0} \right)^a \quad (1.4.6)$$