Recap: Bose-Einstein Condensation

$$\langle n_{\vec{k}} \rangle = \frac{1}{e^{\beta(\epsilon_{\vec{k}}-\mu)}-1}$$

* becomes negative (nonsensical!) if $e^{\beta(\epsilon_k-\mu)} < 1$

* and <u>blows up</u> for $\mu = \epsilon_{\vec{k}}$

$\therefore$ For Bosons (unlike Fermions), must have

$$\mu \lesssim \epsilon_{min} \quad \left(\text{for a gas } \epsilon_{min} \cong 0 \implies \mu \leq 0 \right)$$

What are the implications of this restriction?

* Following the usual approach of introducing s.p. DOS to convert $\sum_{\vec{k}} \cdots \longrightarrow \int d\epsilon\, g(\epsilon) \cdots$

$$g(\epsilon) = \left(\frac{L}{2\pi\hbar}\right)^D S_D\, m (2m\epsilon)^{D/2-1} \quad \text{(spinless NR bosons in D-dim.)}$$

$\Downarrow$

$$N \approx \left(\frac{L}{2\pi\hbar}\right)^D S_D \frac{(2m)^{D/2}}{2} \int_0^\infty \frac{d\epsilon\, \epsilon^{D/2-1}}{e^{\beta(\epsilon-\mu)}-1}$$

note that as you increase μ, N increases.

$$\boxed{\text{Question: Since } \mu \text{ limited to } \leq 0, \text{ can we dial } \mu \text{ to get arbitrary } N \text{ (including } N \to \infty)?}$$

* In the usual way, let $x = \beta\epsilon$ and $z = e^{\beta\mu}$

$$\Rightarrow \boxed{N \approx \frac{L^D}{\ell_\alpha^D} f^{(+)}_{D/2}(z)}$$

study this in the limit $\mu \to 0^-$ (or $z \to 1$)

* Last time, we showed

$$f^{(+)}_{D/2}(z=1) = \sum_{m=1}^{\infty} \frac{1}{m^{D/2}} = \zeta(D/2)$$

"Riemann Zeta function"

Some special cases

$$\zeta(1) = \infty \quad \text{(divergent series)}$$

$$\zeta(\tfrac{3}{2}) = 2.6123\ldots$$

$$\zeta(2) = \frac{\pi^2}{6}$$

$$\zeta(4) = \frac{\pi^4}{90}$$

$\Rightarrow$ For $D < 3$, we can accomodate all $N\ (0, 1, 2, \ldots \infty)$ by sweeping μ over $[-\infty, 0]$.

But for $D \geq 3$, it seems to imply we can only describe systems w/ finite N. Eg., $D = 3$

$$\boxed{N^{D=3}_{max} = \frac{V}{\ell_\alpha^3} \zeta(\tfrac{3}{2}) \simeq 2.61 \frac{V}{\ell_\alpha^3}}$$

What's going on ?!

The issue is converting sums to integrals via the DOS

$$g_{NR}(\epsilon) \propto \epsilon^{D/2-1} = 0 \text{ at } \epsilon = 0$$
$$\text{for } D \geq 3$$

* but we know that $\langle n(k=0) \rangle$ is getting huge!

$\therefore$ Converting $\sum_\alpha \longrightarrow \int d\epsilon\, g(\epsilon)$ artificially zeros out the most important $\epsilon = 0$ contribution!

Solution: Explicitly separate out the $\epsilon_{min} \approx 0$ contribution in the discrete sum, and then convert the remaining $\sum_\alpha \to \int d\epsilon\, g(\epsilon)$

$$\log Z = -\sum_{\vec{k}} \log\left(1 - e^{-\beta(\epsilon_k - \mu)}\right)$$

$$= \underbrace{-\log\left(1 - e^{\beta\mu}\right)}_{\vec{k}=0 \text{ term}} - \underbrace{\int_0^\infty d\epsilon\, g(\epsilon) \log\left(1 - e^{-\beta(\epsilon-\mu)}\right)}_{\vec{k}>0 \text{ converted to integral}}$$

* as before, get $N = \frac{\partial}{\partial(\beta\mu)} \log Z$

$\Downarrow$

$$\boxed{N = \underbrace{\frac{e^{\beta\mu}}{1-e^{\beta\mu}}}_{N_0} + \underbrace{\int d\epsilon\, g(\epsilon) \frac{1}{e^{\beta(\epsilon-\mu)}-1}}_{N_x}}$$

N_0 = # bosons occupying lowest s.p. energy $\epsilon_{min} \simeq 0$

N_x = # of bosons in the other sp levels.

Now, if we sweep thru $\mu = -\infty$ thru $\mu = 0^-$, we can get any desired N from 0 to ∞.

* but from before

$$N_x(\mu=0^-) = \zeta\left(\tfrac{3}{2}\right)\frac{V}{\ell_Q^3} \simeq 2.61 \frac{V}{\ell_Q^3} \Rightarrow n_x(\mu=0^-) = \frac{2.61}{\ell_Q^3}$$

$\boxed{\therefore \text{ For } n > \frac{2.61}{\ell_Q^3}, \text{ must have finite fraction of } N \text{ occupying the lowest level}}$

* For fixed T, there must be a <u>critical density</u> above which BEC occurs

$$\frac{N_c}{V} = n_c = \frac{\zeta(3/2)}{\ell_\alpha^3}$$

Since if $N > N_c$, then $N_0 = N - N_c$ becomes a sizable fraction of N.

Explicitly, $\dfrac{N_c}{V} \simeq 2.61 \left(\dfrac{m k_B T}{2\pi\hbar^2}\right)^{3/2}$

$\therefore$ $N_c(T)$ <u>decreases</u> w/ decreasing T.

Critical temperature

* Say $N < N_c$. If we <u>lower</u> T, $N_c(T)$ will drop eventually reaching $N_c(T_c) = N$.

$$\therefore \quad \boxed{T_c(N,V) = \frac{2\pi\hbar^2}{m k_B}\left(\frac{N}{V\,\zeta(\frac{3}{2})}\right)^{2/3}}$$

Now, $N = N_0 + N_c \qquad N_0 \geq 0$ for $T \leq T_c$

$$\therefore \boxed{\frac{N_0}{N} = 1 - \frac{N_c}{N} = 1 - \left(\frac{T}{T_c}\right)^{3/2}}$$

from prior expression for N_c & N.

EOS of the BEC gas

$$\frac{PV}{k_B T} = \log Z = \underbrace{-\log(1 - e^{\beta\mu})}_{\substack{\epsilon_{min}=0 \\ \text{term from} \\ \text{the discrete} \\ \text{sum}}} + \underbrace{\frac{V}{\ell_Q^3} f_{5/2}^{(+)}(z = e^{\beta\mu})}_{\substack{\text{what we had} \\ \text{before we realized} \\ \text{need to separate} \\ \text{out the } \epsilon=0 \text{ term} \\ \text{before converting to integral}}}$$

$\Rightarrow$ using that $N_0 = \dfrac{e^{\beta\mu}}{1 - e^{\beta\mu}} = \dfrac{z}{1-z} \Rightarrow z = \dfrac{N_0}{1+N_0}$

$$\therefore -\log(1-z) = -\log\left(\frac{1}{1+N_0}\right) = \log(1+N_0)$$

$$\boxed{\Rightarrow \frac{PV}{k_B T} = \log(1+N_0) + \frac{V}{\ell_Q^3} f_{5/2}^{(+)}(z)}$$

Since $\log(1+N_0) \sim \log(N_0) \lesssim \log N$, this is negligible compared to the 2nd term which $\propto N$

$$\therefore \quad \frac{P\cancel{V}}{k_B T} \approx \frac{\cancel{V}}{\ell_\alpha^3} f^{(+)}_{5/2}(z)$$

$$\Rightarrow \quad P = \frac{k_B T}{\ell_\alpha^3} f^{(+)}_{5/2}(z)$$

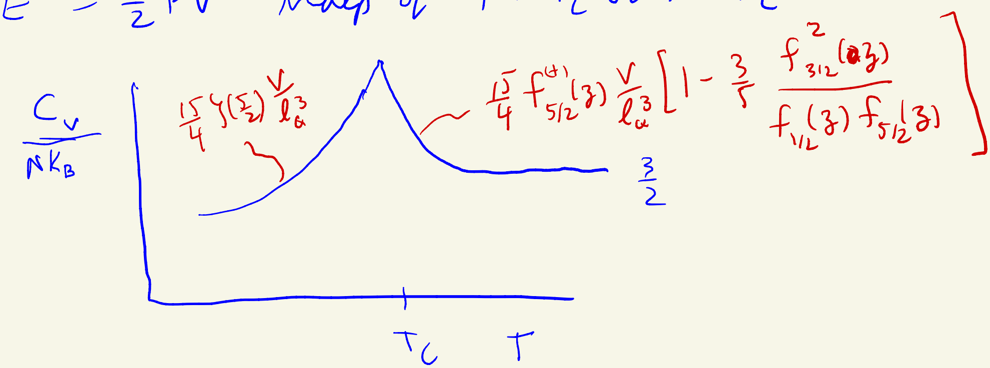
$\Rightarrow$ 1.) Condensate doesn't contribute to pressure (makes sense – $\vec{k}=0$ sp states)

2.) $\frac{\partial P}{\partial V} = 0$ when $\mu = 0$, so system is infinitely compressible

$$K = V \frac{\partial V}{\partial P} = \infty .$$

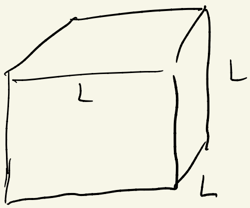
* In your next HW set, you'll show

(1.) $E = \frac{3}{2} PV$ indep of $T > T_c$ or $T < T_c$

(2) 

* 1st example of discontinuous behavior at a phase transition -- much more on this later!

Photons & Blackbody Radiation



* properties of <u>thermalized</u> EM radiation in a cavity

* photons can be created/destroyed by interactions w/ background medium/container walls.

<u>2 ways to treat</u>

- large # of indep HO's for each cavity allowed ω_n
- gas of indistinguishable photons (J=1 Bosons, massless) w/ s.p. energies $\hbar\omega_n$

<u>Way #1</u> (see Monday act. sheet)

$$\vec{A}(\vec{r},t) = \frac{1}{\sqrt{V}} \sum_{\vec{k}} \left(\vec{A}_{\vec{k}} e^{i\vec{k}\cdot\vec{r} - i\omega_k t} + c.c. \right)$$

<u>where</u> $\vec{\nabla}\cdot\vec{A} = 0 \Rightarrow \vec{k}\cdot\vec{A}_k = 0$

$\nabla^2 \vec{A} - \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} = 0 \Rightarrow \omega_k = ck$

periodic BC's $\Rightarrow \vec{k} = \frac{2\pi}{L}(n_x, n_y, n_z)$

$$H_{em} = \int d^3r \, \frac{\vec{E}^2 + \vec{B}^2}{8\pi} \quad ; \quad \vec{E} = -\frac{1}{c}\frac{\partial \vec{A}}{\partial t}$$

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

$\Downarrow$ a bit of dirty work (cf. Monday act. sheet)

$$= \frac{1}{2\pi c^2} \sum_{\vec{k},\lambda} \omega_k^2 \left|A_{\vec{k},\lambda}\right|^2$$

(use $\vec{A}_{\vec{k}} = \sum_{\lambda=1,2} A_{k,\lambda} \hat{\epsilon}_{\vec{k},\lambda}$)

polarizations

+ <u>Clever change of variables</u>...

$$Q_{\vec{k},\lambda} = \sqrt{\frac{1}{4\pi c^2}} \left(A_{\vec{k},\lambda} + A^*_{\vec{k},\lambda} \right)$$

$$P_{\vec{k},\lambda} = -i\omega_k \sqrt{\frac{1}{4\pi c^2}} \left(A_{k,\lambda} - A^*_{k,\lambda} \right)$$

$$\boxed{H_{em} = \sum_{\vec{k},\lambda} \frac{1}{2}\left(P^2_{\vec{k},\lambda} + \omega_k^2 Q^2_{\vec{k},\lambda} \right)}$$

polarizations

i.e., H_{em} => bunch of indep HO's, 2 for each allowed $\vec{k}$ in the cavity

* Since we know how to solve the H.O.,

$$H_{em} = \sum_{\vec{k},\lambda} \left(n_{\vec{k},\lambda} + \frac{1}{2}\right) \hbar\omega_k$$

* as always, it's convenient to redefine E-scale to kill off the 0-pt. energy

$$\Downarrow$$

$$\tilde{H}_{em} = \sum_{\vec{k},\lambda} n_{\vec{k},\lambda} \hbar\omega_k$$

$$\therefore Z = \sum_{\{n_{\vec{k},\lambda}\}} e^{-\beta \sum_{\vec{k},\lambda} n_{\vec{k},\lambda} \hbar\omega_k}$$

$$= \sum_{\{n_{\vec{k},\lambda}\}} \prod_{\vec{k},\lambda} e^{-\beta \hbar\omega_k n_{\vec{k},\lambda}}$$

$$= \prod_{\vec{k},\lambda} \sum_{n_{\vec{k},\lambda}=0}^{\infty} \left(e^{-\beta\hbar\omega_k}\right)^{n_{\vec{k},\lambda}}$$

$$\boxed{Z = \prod_{\vec{k},\lambda} \frac{1}{1 - e^{-\beta\hbar\omega_k}}}$$

Way #2: gas of identical massless bosons w/ 2 polarization states + s.p. energies $\hbar\omega_k$

NOTE: Since photons not conserved, it makes no sense to impose the constraint in the GCE that $\langle N\rangle = N$ is fixed

$\therefore$ the Lagrange multiplier that usually implements the constraint (μ) should be set to zero.

$\therefore$ Use Bose distribution w/ $\mu = 0$.

$$\Rightarrow Z = \prod_{\vec{k},\lambda} \left(\frac{1}{1 - e^{-\beta\hbar\omega_k}} \right)$$ (Same as the above HO picture!)

$$\therefore \log Z = -\sum_{\vec{k},\lambda} \log\left(1 - e^{-\beta\hbar\omega_k}\right)$$

DOS (Conventional to use ω instead of $\varepsilon = \hbar\omega$)

$$g(\omega) = \underset{\substack{\uparrow \\ \text{polarizations}}}{2} \times \frac{L^3}{2\pi^2 c^3}\,\omega^2$$

$$\Rightarrow \log Z = -\frac{V}{\pi^2 c^3}\int_0^\infty d\omega\, \omega^2 \log\left(1 - e^{-\beta\hbar\omega}\right)$$

$\therefore$ Using $\frac{PV}{T} = \log Z$

$$\Rightarrow P_\gamma = -\frac{T}{\pi^2 c^3}\int_0^\infty d\omega\, \omega^2 \log\left(1 - e^{-\beta\hbar\omega}\right)$$

$$\overset{\text{int.}}{\underset{\text{by parts}}{=}} \frac{T^4}{\pi^2 \hbar^3 c^3} \cdot \frac{1}{3}\Gamma(4)\,\zeta(4)$$

$$\boxed{P_\gamma = V\,\frac{\pi^2 T^4}{45\,\hbar^3 c^3}}$$

* likewise, can show

$$E = \frac{V}{\pi^2 \hbar^3 c^3}\int d\epsilon\, \frac{\epsilon^3}{e^{\beta\epsilon} - 1} = 3PV$$