*Recap

* $T \ll \mu$ limit for Fermions (Sommerfeld expansion)

* Pretty much all quantities of interest take the form

$$I = \int_0^\infty \phi(\epsilon) f(\epsilon)\, d\epsilon \qquad \text{where} \quad f(\epsilon) = \frac{1}{e^{\beta(\epsilon-\mu)}+1}$$

e.g.

$$N = \int_0^\infty d\epsilon\, g(\epsilon) f(\epsilon)$$

$\phi(\epsilon)$ depends on qty looking at

$$g(\epsilon) = \underset{\text{of States}}{\text{density}} = \frac{d}{d\epsilon}\Sigma(\epsilon) = \frac{d}{d\epsilon}\sum_{\alpha}^{\text{sp states}} \Theta(\epsilon - \epsilon_\alpha)$$

$$= \sum_\alpha \delta(\epsilon - \epsilon_\alpha)$$

e.g., for a gas $\alpha = \vec{p}, m_s$ $\quad \epsilon(\vec{p}, m_s) = \epsilon(|\vec{p}|)$ (if no B-field)

$$\sum_{\vec{p}, m_s} \delta(\epsilon - \epsilon(p)) = (2s+1) \sum_{\vec{p}} \delta(\epsilon - \epsilon(p))$$

Usual way ... $\displaystyle \frac{1}{L^D} \sum_{\vec{p}} \longrightarrow \int \frac{d^D\vec{p}}{(2\pi\hbar)^D}$

$$\therefore\ g(\epsilon) = \left(\frac{L}{2\pi\hbar}\right)^D (2s+1) \oint d\Omega_D \int p^{D-1} dp\, \delta(\epsilon - \epsilon(p))$$

$$\delta(\epsilon - \epsilon(p)) = \frac{\delta(p - p_0)}{\left|\frac{\partial \epsilon(p)}{\partial p}\right|_{p_0}}$$

where $\epsilon(p_0) = \epsilon$

$$g(\epsilon) = \left(\frac{L}{2\pi\hbar}\right)^D (2S+1) \oint d\Omega_D \int p^{D-1} dp \; \frac{\delta(p-p_0)}{\left|\frac{\partial \epsilon(p)}{\partial p}\Big|_{p_0}\right|}$$

where $\epsilon(p_0) = \epsilon$ defines $p_0(\epsilon)$

e.g. $\epsilon(p) = \frac{p^2}{2m} \Rightarrow \frac{\partial \epsilon(p)}{\partial p}\Big|_{p_0} = \frac{p_0}{m}$; $\frac{p_0^2}{2m} = \epsilon$

$\therefore p_0 = \sqrt{2m\epsilon}$

$$\Rightarrow g_{NR,D}(\epsilon) = \left(\frac{L}{2\pi\hbar}\right)^D (2S+1) \oint d\Omega_D \frac{p_0^{D-1}}{\frac{p_0}{m}}$$

$$= \left(\frac{L}{2\pi\hbar}\right)^D (2S+1) \oint d\Omega_D \; p_0^{D-2} m$$

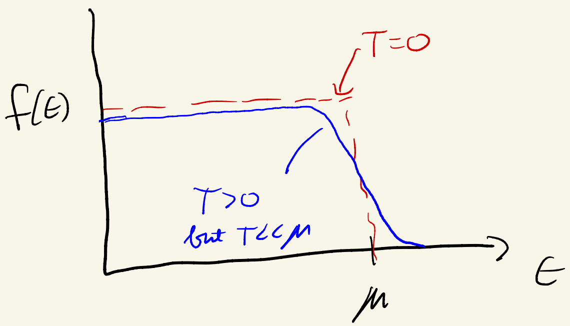
$$\boxed{g_{NR,D}(\epsilon) = \left(\frac{L}{2\pi\hbar}\right)^D (2S+1) \oint d\Omega_D \; (2m\epsilon)^{\frac{D-2}{2}} m}$$

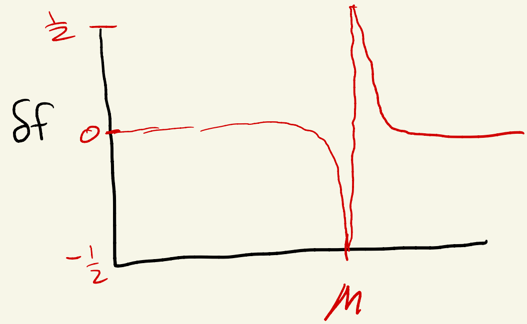
<u>e.g. D=3:</u> $g_{NR,3}(\epsilon) = \frac{V}{8\pi^3\hbar^3} (2S+1) \, 4\pi \sqrt{2m\epsilon} \; m$

$$= \frac{V}{\sqrt{2}\pi^2\hbar^3} (2S+1) \, m^{3/2} \sqrt{\epsilon}$$

Key idea: for $T \ll \mu$, $f(\epsilon)$ "almost" a step function

$$f(\epsilon) = \Theta(\mu - \epsilon) + \delta f(\epsilon)$$





① δf sharply peaked around $\xi = \epsilon - \mu = 0$

② $\delta f(\xi) = -\delta f(-\xi)$ (odd in ξ)

$$\Rightarrow I(T,\mu) = \int_0^\infty d\epsilon\, \phi(\epsilon) f(\epsilon)$$

$$= \int_0^\mu d\epsilon\, \phi(\epsilon) + \int_0^\infty d\epsilon\, \phi(\epsilon)\, \delta f(\epsilon)$$

$$= I(0,\mu) + \delta I$$

$$\delta I = \int_0^\infty d\epsilon\, \phi(\epsilon)\, \delta f(\epsilon) \overset{\text{let } \xi = \epsilon - \mu}{=} \int_{-\mu}^\infty d\xi\, \phi(\mu + \xi)\, \delta f(\xi)$$

$$\delta I = \int_{-\mu}^{\infty} d\xi \, \phi(\mu+\xi) \, \delta f(\xi)$$

$$\approx \int_{-\infty}^{\infty} d\xi \, \phi(\mu+\xi) \, \delta f(\xi)$$ (since δf so sharply peaked about $\xi = 0$

$$\approx \int_{-\infty}^{\infty} d\xi \left(\cancel{\phi(\mu)} + \xi \phi'(\mu) + \cdots \right) \delta f(\xi)$$

0 since $\delta f(\xi) = -\delta f(-\xi)$

$$\therefore \delta I = I(\mu, T) - I(\mu, 0)$$

$$= \phi'(\mu) \int_{-\infty}^{\infty} d\xi \, \xi \, \delta f(\xi)$$

$$= 2\phi'(\mu) \int_{0}^{\infty} d\xi \, \xi \, \delta f(\xi)$$

*putting in $\delta f(\xi > 0) = \dfrac{1}{e^{\beta\xi} + 1}$ + letting $x = \beta\xi$

$$\delta I = \frac{2\phi'(\mu)}{\beta^2} \int_0^\infty \frac{dx \; x}{e^x + 1}$$ $= \frac{\pi^2}{12}$

∴ The general form for some TD qty $I(\mu,T)$ at low $T \ll \mu$ is

$$\boxed{\begin{aligned} I(\mu,T) &\simeq \int_0^\infty d\epsilon\, \phi(\epsilon) f(\epsilon) \\ &\simeq I(\mu, T=0) + \frac{\pi^2}{6}(k_B T)^2 \phi'(\mu) \end{aligned}}$$

* It would be tempting to call the 1st term the value of the TD qty. at $T=0$. But this is not quite right since $\mu = \mu(T,N)$.
(recall, you must dial in μ to give the desired N)

* To better understand this, it's useful to take a step back to the pure $T=0$ case.

* First, we treat $T=0$ in QM. We know $P_{gs} = 1$ whereas all $P_{excited} = 0$. I.e., the system is known to be in the unique gs

$|\Psi_{gs}\rangle = |\vec{k}_1 m_{s_1} \cdots \vec{k}_N m_{s_N}\rangle_-$ where the N-lowest energy s.p. levels are filled

$$N = \sum_{\vec{k}\, m_s}^{\text{lowest } N \text{ energy sp states}}$$

$$= g \sum_{\vec{k}}^{\text{lowest } \frac{N}{g} \text{ energy wave vectors}}$$

$$= gV \int^{k_F} \frac{d^3k}{(2\pi)^3}$$

k_F = wave vector of highest filled sp state
"Fermi wave vector"

$$= \frac{gV\, 4\pi}{(2\pi)^3} \cdot \frac{k_F^3}{3}$$

$$\boxed{N = \frac{gV}{6\pi^2\hbar^3} p_F^3}$$

$p_F = \hbar k_F$ = "Fermi Momentum"

let $\epsilon_F = \frac{p_F^2}{2m}$ "Fermi Energy"
(energy of highest occupied state)

$$\Rightarrow p_F = (2m\epsilon_F)^{1/2}$$

$$\Rightarrow N = \frac{gV}{6\pi^2} \frac{(2m\epsilon_F)^{3/2}}{\hbar^3} = \frac{gV}{6\pi^2}\left(\frac{2m\epsilon_F}{\hbar^2}\right)^{3/2}$$

or

$$\epsilon_F = \frac{\hbar^2}{2m}\left(\frac{6\pi^2}{g}\frac{N}{V}\right)^{2/3}$$

Comparing with

$$\boxed{N = \int_0^{\mu(T=0)} d\varepsilon\, g(\varepsilon), \text{ we see} \quad \epsilon_F = \mu(T=0)}$$

* We can verify other $T=0$ results

$$E = F = \int_0^{\epsilon_F} g(\epsilon)\epsilon\, d\epsilon = \frac{3}{5} N \epsilon_F$$

$$P = -\left(\frac{\partial F}{\partial V}\right)_{T,N} = \frac{2}{3}\frac{E}{V} = \frac{2}{5} n \epsilon_F \quad \text{etc...}$$

So now we can derive a Sommerfeld relation

Writing $\mu(T) = \epsilon_F$ + correction, which in turn will let us write

$$I(\mu(T),T) \approx I(\epsilon_F, 0) + \text{corrections due to finite } T$$

* to do this, start with $I = N$

$$\Rightarrow N \approx \int_0^{\mu} d\epsilon\, g(\epsilon) + g'(\mu)\frac{\pi^2 T^2}{6}$$

but at $T=0$, $N = \int_0^{\epsilon_F} d\epsilon\, g(\epsilon)$

Combine these 2 equations $\Rightarrow$

$$\Rightarrow \int_0^{\epsilon_F} d\epsilon\, g(\epsilon) \simeq \int_0^{\mu} d\epsilon\, g(\epsilon) + \frac{\pi^2 (k_B T)^2}{6} g'(\mu)$$

$$\Rightarrow \int_{\mu}^{\epsilon_F} d\epsilon\, g(\epsilon) \simeq \frac{\pi^2 (k_B T)^2}{6} g'(\mu)$$

Now, expand (Since at low T, dont expect $\epsilon_F - \mu$ to be big)

$$g(\epsilon) \approx g(\epsilon_F) + (\epsilon - \epsilon_F)\, g'(\epsilon_F) \dots$$ + plug in to LHS above

$$\Rightarrow g(\epsilon_F)(\epsilon_F - \mu) + g'(\epsilon_F)\tfrac{1}{2}(\epsilon_F - \mu)^2 \dots = \frac{\pi^2}{6}(k_B T)^2 g'(\mu)$$

$$\therefore (\epsilon_F - \mu) \text{ is } \mathcal{O}((k_B T)^2)$$

$$(\epsilon_F - \mu)^2 \text{ is } \mathcal{O}((k_B T)^4) \quad \text{(Neglect)}$$

$$\Rightarrow (\epsilon_F - \mu) \simeq \frac{\pi^2}{6}(k_B T)^2 \frac{g'(\mu)}{g(\epsilon_F)} \simeq \frac{\pi^2}{6}(k_B T)^2 \frac{g'(\epsilon_F)}{g(\epsilon_F)}$$

$$\therefore \boxed{\mu(T) \simeq \epsilon_F - \frac{d}{d\epsilon}\log g\Big|_{\epsilon_F} \frac{\pi^2}{6}(k_B T)^2 + \mathcal{O}(T^4)}$$

Therefore, back to our general expression

$$I(\mu,T) \approx \int_0^{\mu} d\epsilon\, \phi(\epsilon) + \phi'(\mu)\frac{\pi^2 (kT)^2}{6}$$

Now use $\mu \simeq \epsilon_F + \left(\frac{g'}{g}\right)_{\epsilon_F} \frac{\pi^2}{6}(k_B T)^2$

+ expand keeping terms thru $O((kT)^2)$

$$\boxed{\begin{aligned} \Rightarrow I(\mu,T) &\approx \int_0^{\epsilon_F} d\epsilon\, \phi(\epsilon) + \left[\phi'(\epsilon_F) - \phi(\epsilon_F)\left(\frac{g'}{g}\right)_{\epsilon_F}\right]\frac{\pi^2}{6}(k_B T)^2 \\ &= \underbrace{\int_0^{\epsilon_F} d\epsilon\, \phi(\epsilon)}_{I(\epsilon_F,\, T=0)} + \underbrace{g(\epsilon_F)\frac{d}{d\epsilon}\left(\frac{\phi(\epsilon)}{g(\epsilon)}\right)_{\epsilon_F=\epsilon} \frac{\pi^2}{6}(k_B T)^2}_{\text{correction due to finite } T} \end{aligned}}$$

Now we can apply this general formula to get finite T corrections to various quantities (E, C_V, P, etc.)

e.g.: Say $g(\epsilon) = a\epsilon^n$ (as it usually is!)

Then applying to E $(\phi(\epsilon) = \epsilon g(\epsilon))$

$$\boxed{\begin{aligned} &\Rightarrow E = \left(\frac{n+1}{n+2}\right)\epsilon_F N \left[1 + \frac{(n+2)\pi^2}{6}\left(\frac{k_B T}{\epsilon_F}\right)^2 + \cdots\right] \\ &\Rightarrow C_V = \left(\frac{\partial E}{\partial T}\right)_{N,V} = \frac{(n+1)\pi^2}{3} N \frac{k_B^2 T}{\epsilon_F} \end{aligned}}$$

e.g., $D=3$ NR $g \propto \sqrt{\epsilon}$

$$\Rightarrow E = \frac{3}{5}\epsilon_F N \left[1 + \frac{5\pi^2}{12}\left(\frac{k_B T}{\epsilon_F}\right)^2 + \cdots\right]$$

$$C_V = \frac{\pi^2}{2} N \frac{k_B^2 T}{\epsilon_F}$$

*Useful to define Fermi Temperature $k_B T_F = \epsilon_F$

$$\boxed{\begin{aligned} &\Rightarrow E = \frac{3}{5}(\epsilon_F N)\left[1 + \frac{5\pi^2}{12}\left(\frac{T}{T_F}\right)^2 + \cdots\right] \\ &C_V = \frac{\pi^2}{2} N k_B \left(\frac{T}{T_F}\right) \end{aligned}}$$

*Simple physical interp of linear T-dep of C_v

Since $f(\epsilon)$ "almost" a step function, only particles within a narrow range $\epsilon_F - k_BT \lesssim \epsilon \lesssim \epsilon_F + k_BT$ can be excited thermally (Those further away deep in the Fermi Sea are Pauli Blocked)

* fraction of particles in this range $\sim N\left(\frac{T}{T_F}\right)$

* each of these particles excited by $\mathcal{O}(k_BT)$

$$\therefore \Delta E \sim N\left(\frac{T}{T_F}\right) k_BT$$

$$C_v \sim N k_B \frac{T}{T_F}$$