Homework 5

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5.1

5.1.1

$$-\frac{\hbar^2}{2I}\psi_n''(\theta) = E_n\psi_n(\theta), \tag{5.1.1}$$
$$\therefore \psi_n(\theta) = c_{1,n}e^{ik_n\theta} + c_{2,n}e^{-ik_n\theta}, \tag{5.1.2}$$

where $k_n = \sqrt{2IE_n}/\hbar$. Since $[H, L_z] = 0$ the solution to Shrödinger's equation must also be an eigenstate of the L_z operator:

$$\hat{L}_z\left|\psi_n\right\rangle = l_n\left|\psi_n\right\rangle, \tag{5.1.3}$$
$$-i\hbar\frac{\partial\psi_n}{\partial\theta} = l_n\psi_n, \tag{5.1.4}$$
$$\psi_n(\theta) = d_n e^{il_n\theta/\hbar}. \tag{5.1.5}$$

Therefore, in equation 5.1.2, either $c_{1,n} = 0$ or $c_{2,n} = 0$, so, let $c_{2,n} = 0$:

$$\psi_n(\theta) = c_{1,n}e^{ik_n\theta}. \tag{5.1.6}$$

Since the boundary conditions are periodic, with a period of 2π,

$$\frac{\sqrt{2IE_n}}{\hbar} \in \mathbb{Z} \to n = \frac{\sqrt{2IE_n}}{\hbar}, \tag{5.1.7}$$
$$E_n = \frac{n^2\hbar^2}{2I}. \tag{5.1.8}$$

Additionally, $|\psi_n|^2 = 1$, so

$$1 = \int_0^{2\pi} c_n^2 \,\mathrm{d}\theta\,, \tag{5.1.9}$$
$$c_n = \sqrt{\frac{1}{2\pi}}. \tag{5.1.10}$$

Therefore,

$$\psi_n(\theta) = \frac{1}{\sqrt{2\pi}}e^{in\theta}, \tag{5.1.11}$$
$$E_n = \frac{n^2\hbar^2}{2I}. \tag{5.1.12}$$

5.1.2

Let $\rho = \frac{1}{Z} e^{-\beta \hat{H}}$, where Z is the canonical partition function, and $\hat{H}$ is the Hamiltonian. Then

$$\begin{aligned}
\langle \theta' | \rho | \theta \rangle =& \frac{1}{Z} \langle \theta' | e^{-\beta \hat{H}} | \theta \rangle , && (5.1.13)\\
=& \frac{\sum_n e^{-\beta E_n} \langle \theta' | \psi_n \rangle \langle \psi_n | \theta \rangle}{\mathrm{Tr}\left(e^{-\beta \hat{H}} \right)}, && (5.1.14)\\
=& \frac{\sum_n e^{-\beta E_n} e^{in(\theta' - \theta)}}{2\pi \, \mathrm{Tr}\left(e^{-\beta \hat{H}} \right)}. && (5.1.15)
\end{aligned}$$

In the high temperature limit, $\beta \to 0$, and the sum can be closely approximated by an integral:

$$\begin{aligned}
\langle \theta' | \rho | \theta \rangle =& \sqrt{\frac{\beta \hbar^2}{2\pi I}} \frac{1}{2\pi} \int e^{-\beta E_n} e^{in(\theta' - \theta)} \, \mathrm{d}n \, , && (5.1.16)\\
=& \sqrt{\frac{\beta \hbar^2}{2\pi I}} \frac{1}{2\pi} \int e^{-\beta \frac{n^2 \hbar^2}{2I} + in\Delta\theta} \, \mathrm{d}n \, , && (5.1.17)\\
=& \sqrt{\frac{\beta \hbar^2}{2\pi I}} \frac{1}{2\pi} \int e^{-\beta \frac{n^2 \hbar^2}{2I} + in\Delta\theta} \, \mathrm{d}n \, , && (5.1.18)\\
=& \sqrt{\frac{\beta \hbar^2}{2\pi I}} \frac{1}{2\pi} \sqrt{\frac{2\pi I}{\beta \hbar^2}} e^{-\frac{\Delta\theta^2 I}{2\beta\hbar^2}}, && (5.1.19)\\
=& \frac{1}{2\pi} e^{-\frac{\Delta\theta^2 I}{2\beta\hbar^2}}. && (5.1.20)
\end{aligned}$$

In the low temperature limit, $\beta \to \infty$, so the lower energy states dominate:

$$\begin{aligned}
\langle \theta' | \rho | \theta \rangle =& \frac{\sum_n e^{-\beta E_n} e^{in(\theta' - \theta)}}{2\pi \, \mathrm{Tr}\left(e^{-\beta \hat{H}} \right)}, && (5.1.21)\\
\approx& \frac{1}{2\pi} \frac{1}{1}, && (5.1.22)\\
=& \frac{1}{2\pi}. && (5.1.23)
\end{aligned}$$

5.2

For N-identical particles, the partition function is given by

$$Z_N(V,T) = \frac{1}{N!} \left(\frac{V^N}{l_Q^{3N}} + \int \prod_{a=1}^{N} \mathrm{d}^3 \mathbf{x}_a \sum_{p=1}^{N!-1} \eta^p e^{-\frac{\pi}{l_Q^2} \sum_{a=1}^{N} (\mathbf{x}_a - \mathbf{x}_{p_a})^2} \right), \tag{5.2.1}$$

where $\eta = 1$ for bosons and $\eta = -1$ for fermions.

5.2.1

For $N = 2$,

$$Z_2 = \frac{1}{2}\left(\frac{V^2}{l_Q^6} + \int \prod_{a=1}^{2} \mathrm{d}^3 x_a \sum_{p=1}^{1} \eta^p e^{-\frac{\pi}{l_Q^2}\sum_{a=1}^{2}(\mathbf{x}_a - \mathbf{x}_{p_a})^2}\right), \tag{5.2.2}$$

$$= \frac{1}{2}\left(\frac{V^2}{l_Q^6} + \eta \int \mathrm{d}^3 x_1\, \mathrm{d}^3 x_2\, e^{-\frac{2\pi}{l_Q^2}(\mathbf{x}_1 - \mathbf{x}_2)^2}\right), \tag{5.2.3}$$

$$= \frac{1}{2}\left(\frac{V^2}{l_Q^6} \pm \frac{V}{2^{3/2} l_Q^3}\right). \qquad \eta = \begin{cases} +1, & \text{for bosons} \\ -1, & \text{for fermions} \end{cases} \tag{5.2.4}$$

Since $Z_1(m) = V/l_Q^3$, then $Z_1(m/2) = V/(2^{3/2} l_Q^3)$. Thus

$$Z_2(m) = \frac{1}{2}\left(Z_1^2(m) \pm Z_1(m/2)\right). \tag{5.2.5}$$

5.2.2

The average energy is given by

$$\langle E \rangle = \frac{1}{Z_N} \sum_n \epsilon_n e^{-\beta \epsilon_n}. \tag{5.2.6}$$

One can see that one can acheive this result by taking the negative of the derivative of the partition function with respect to β, then dividing by the partition function. Thus, for this problem, the average energy is given by

$$\langle E \rangle = -\frac{1}{Z_2}\frac{\partial Z_2}{\partial \beta}, \tag{5.2.7}$$

$$= -\frac{\partial}{\partial \beta}\log(Z_2), \tag{5.2.8}$$

$$= -\frac{\partial}{\partial \beta}\log\left(\frac{Z_1(m)^2}{2}\right) \pm \frac{Z_1(m/2)}{Z_1^2(m)}, \tag{5.2.9}$$

$$= \langle E \rangle_\sim + \delta \langle E \rangle, \tag{5.2.10}$$

where

$$\delta \langle E \rangle = \mp \frac{\partial}{\partial \beta}\frac{Z_1(m/2)}{Z_1^2(m)} = \mp \frac{3}{2^{5/2}}\frac{l_Q^3}{V} k_B T. \tag{5.2.11}$$

Additionally

$$\delta C_v = \frac{\partial}{\partial T}\delta \langle E \rangle = \mp \frac{3}{2^{5/2}}\frac{l_Q^3}{V} k_B. \tag{5.2.12}$$

5.2.3

For the corrections to be small,

$$Z_1^2(m) \gg Z_1(m/2) \to V \gg \frac{l_Q^3}{2^{3/2}}. \tag{5.2.13}$$

If this is not the case, $l_Q \to V^{1/3} = L$. Hence $T < \frac{2\pi\hbar^2}{m k_B L^2}$.

5.3

$$\begin{aligned}
PV\beta = \log(\mathscr{L}) =& -\eta(2S+1)V \int \frac{\mathrm{d}^3 p}{(2\pi\hbar)^3} \log\Big(1 - \eta e^{-\beta(\epsilon(p)-\mu)}\Big), && (5.3.1)\\
=& -\frac{\eta(2S+1)V4\pi}{(2\pi\hbar)^3} \int \mathrm{d}p\, p^2 \log\Big(1 - \eta e^{-\beta(\epsilon(p)-\mu)}\Big), && (5.3.2)\\
=& \frac{\eta(2S+1)V4\pi\beta}{(2\pi\hbar)^3} \int \mathrm{d}p\, \frac{p^4}{3\epsilon(p)} \frac{1}{\eta - e^{-\beta(\epsilon(p)-\mu)}}, && (5.3.3)\\
P =& (2S+1) \int \frac{\mathrm{d}^3 p}{(2\pi\hbar)^3} \frac{p^2}{3\epsilon(p)} \frac{1}{e^{-\beta(\epsilon(p)-\mu)} - \eta}. && (5.3.4)
\end{aligned}$$

5.4

Density of states is given by

$$g(\epsilon) = \frac{\partial \Sigma(\epsilon)}{\partial \epsilon}, \tag{5.4.1}$$

where

$$\Sigma(\epsilon) = \sum_{n_x,n_y,n_z} \Theta(\epsilon - \hbar\omega(n_x + n_y + n_z)). \tag{5.4.2}$$

Thus

$$\begin{aligned}
g(\epsilon) =& \sum_{n_x,n_y,n_z} \delta(\epsilon - \hbar\omega(n_x + n_y + n_z)), && (5.4.3)\\
\approx& \int \mathrm{d}n_x\, \mathrm{d}n_y\, \mathrm{d}n_z\, \delta(\epsilon - \hbar\omega(n_x + n_y + n_z)), && (5.4.4)\\
\approx& \int \mathrm{d}n_x\, \mathrm{d}n_y\, \mathrm{d}n_z \int_{-\infty}^{\infty} \frac{1}{2\pi}\, \mathrm{d}\alpha\, e^{i\alpha(\epsilon - \hbar\omega(n_x+n_y+n_z))}, && (5.4.5)\\
=& \int_{-\infty}^{\infty} \frac{\mathrm{d}\alpha}{2\pi} e^{i\alpha\epsilon} \int_0^\infty \mathrm{d}n_x\, e^{-i\alpha\hbar\omega n_x} \int_0^\infty \mathrm{d}n_y\, e^{-i\alpha\hbar\omega n_y} \int_0^\infty \mathrm{d}n_z\, e^{-i\alpha\hbar\omega n_z}, && (5.4.6)\\
=& \lim_{\delta\to 0} \int_{-\infty}^{\infty} \frac{\mathrm{d}\alpha}{2\pi} e^{i\alpha\epsilon} \left(\frac{1}{i\alpha\hbar\omega + \delta}\right)^3, && (5.4.7)\\
=& \frac{1}{4|\epsilon|} \left(\frac{\epsilon}{\hbar\omega}\right)^3. \quad \text{(Mathematica)} && (5.4.8)
\end{aligned}$$

This has the same ϵ dependence as UR gas in $3D$ box ($\sim \epsilon^2$).

5.5

$$N(\mu, T) = g'(\mu)\frac{\pi^2}{6\beta^2} + \int_0^\mu \mathrm{d}\epsilon\, g(\epsilon). \tag{5.5.1}$$

Take $\mu \to \mu + \delta\mu$:

$$N(\mu, T) = g'(\mu)\frac{\pi^2}{6\beta^2} + \int_0^{\mu+\delta\mu} \mathrm{d}\epsilon\, g(\epsilon), \tag{5.5.2}$$

$$= g'(\mu)\frac{\pi^2}{6\beta^2} + \delta\mu g(\mu) + \int_0^{\mu} \mathrm{d}\epsilon\, g(\epsilon), \tag{5.5.3}$$

$$\text{(want)} \quad = \int_0^{\mu} \mathrm{d}\epsilon\, g(\epsilon). \tag{5.5.4}$$

$$\therefore \delta\mu = -\frac{g'(\mu)}{g(\mu)}\frac{\pi^2}{6\beta^2}. \tag{5.5.5}$$

For a $3D$, non-relativistic, gas of spin $\frac{1}{2}$ particles,

$$g(\epsilon) = \frac{2V}{\sqrt{2}\pi^2\hbar^3} m^{3/2}\sqrt{\epsilon}. \tag{5.5.6}$$

Itaque,

$$\delta\mu = -\frac{1}{\mu}\frac{\pi^2}{12\beta^2}. \tag{5.5.7}$$

For an answer on the order of $\mathcal{O}(T^2)$, $\mu \to \epsilon_F$, so

$$\delta\mu(\rho, T) = -\frac{\pi^2}{6\beta}\frac{m}{\hbar^2(3\pi^2\rho)^{2/3}}. \tag{5.5.8}$$