Homework 01

Brandon Henke
PHY831
Scott Bogner

September 17, 2021

1.1

a

From the first law,

$$dQ = \mathrm{d}E - dW. \tag{1.1.1}$$

Hence,

$$Q_h = \int \left(\frac{\partial E}{\partial \Theta} \,\mathrm{d}\Theta + P \,\mathrm{d}V \right), \tag{1.1.2}$$

$$= \int \frac{N k_B \Theta_h}{V} \,\mathrm{d}V\,, \tag{1.1.3}$$

$$= N k_B \Theta_h \ln\left(\frac{V_2}{V_1} \right). \tag{1.1.4}$$

$$Q_c = N k_B \Theta_c \ln\left(\frac{V_3}{V_4} \right). \tag{1.1.5}$$

For 1.1.5, heat is being expelled, hence the minus sign (evident in swapping the bounds of integration).

b

From the first law,

$$\mathrm{d}E = -P \,\mathrm{d}V\,, \tag{1.1.6}$$

$$\frac{\partial E}{\partial \Theta} \,\mathrm{d}\Theta = -P \,\mathrm{d}V\,. \tag{1.1.7}$$

Assuming an ideal gas,

$$P = \frac{N k_B \Theta}{V}. \tag{1.1.8}$$

Plugging this into 1.1.7 gives

$$\frac{1}{\Theta} \frac{\partial E}{\partial \Theta} \,\mathrm{d}\Theta = -N k_B \frac{\mathrm{d}V}{V}. \tag{1.1.9}$$

Integrating both sides from (V_0, Θ_0) to (V, Θ) gives

$$\int_{\Theta_0}^{\Theta} \frac{1}{\tau} \left.\frac{\partial E}{\partial \Theta}\right|_{\tau} \mathrm{d}\tau = -N k_B \ln\left(\frac{V}{V_0} \right), \tag{1.1.10}$$

or

$$\frac{V}{V_0} = e^{-\frac{1}{N k_B} \int_{\Theta_0}^{\Theta} \frac{1}{\tau} \left.\frac{\partial E}{\partial \Theta}\right|_{\tau} \mathrm{d}\tau}. \tag{1.1.11}$$

c

By definition,

$$\eta \equiv \frac{W}{Q_h} = \frac{Q_h - Q_c}{Q_h} = 1 - \frac{Q_c}{Q_h}. \tag{1.1.12}$$

From equations 1.1.4 and 1.1.5,

$$1 - \frac{Q_c}{Q_h} = 1 - \frac{\Theta_c \ln(V_3/V_4)}{\Theta_h \ln(V_2/V_1)} \tag{1.1.13}$$

Additionally, from 1.1.11, $\ln(V_2/V_1) = \ln(V_3/V_4)$. Thus

$$\eta = 1 - \frac{\Theta_c}{\Theta_h}. \tag{1.1.14}$$

1.2

a

Take the first law:

$$dQ = \mathrm{d}E - dW. \tag{1.2.1}$$

If one uses (T, V) as the independent variables, then

$$dQ = \frac{\partial E}{\partial T}\, \mathrm{d}T + \left(\frac{\partial E}{\partial V} + P\right) \mathrm{d}V\,, \tag{1.2.2}$$

$$= C_V\, \mathrm{d}T + \left(\frac{\partial E}{\partial V} + P\right) \mathrm{d}V\,. \tag{1.2.3}$$

b

Finding an expression for dS, if follows from the second law that

$$\mathrm{d}S = \frac{dQ}{T}, \tag{1.2.4}$$

$$= \frac{C_V}{T}\, \mathrm{d}T + \left(\frac{1}{T}\frac{\partial E}{\partial V} + \frac{P}{T}\right) \mathrm{d}V\,. \tag{1.2.5}$$

By definition of an exact differential,

$$\frac{\partial S}{\partial V} = \frac{1}{T}\frac{\partial E}{\partial V} + \frac{P}{T}, \tag{1.2.6}$$

$$\frac{\partial S}{\partial T} = \frac{C_V}{T}. \tag{1.2.7}$$

Additionally

$$\frac{\partial^2 S}{\partial V \partial T} = \frac{\partial^2 S}{\partial T \partial V}, \tag{1.2.8}$$

so

$$\frac{\partial}{\partial V}\frac{C_V}{T} = \frac{\partial}{\partial T}\left(\frac{1}{T}\frac{\partial E}{\partial V} + \frac{P}{T}\right), \tag{1.2.9}$$

$$0 = -\frac{1}{T^2}\frac{\partial E}{\partial V} - \frac{P}{T^2} + \frac{1}{T}\frac{\partial P}{\partial T}, \tag{1.2.10}$$

$$= \frac{\partial E}{\partial V} + P - T\frac{\partial P}{\partial T}, \tag{1.2.11}$$

$$\frac{\partial E}{\partial V} = T\frac{\partial P}{\partial T} - P. \tag{1.2.12}$$

c

For an ideal gas,

$$\frac{\partial E}{\partial V} = \frac{N k_B T}{V} - P = 0. \tag{1.2.13}$$

Thus the internal energy cannot depend on V, and is only a function of T. However, since no consideration was made pertaining to the particle number, N, the internal energy could still depend on N. Therefore $E = E(T, N)$.

1.3

From the first law,

$$\mathrm{d}E = -P\, \mathrm{d}V - S\, \mathrm{d}T + \mu\, \mathrm{d}N\,, \tag{1.3.1}$$

where

$$P \equiv -\frac{\partial E}{\partial V}, \tag{1.3.2}$$

$$S \equiv -\frac{\partial E}{\partial T}, \tag{1.3.3}$$

$$\mu \equiv \frac{\partial E}{\partial N}. \tag{1.3.4}$$

The Maxwell relations mentioned are derived as follows:

$$\frac{\partial \mu}{\partial T} = \frac{\partial^2 E}{\partial T \partial N} = -\frac{\partial S}{\partial N}. \tag{1.3.5}$$

$$\frac{\partial S}{\partial V} = -\frac{\partial^2 E}{\partial T \partial V} = \frac{\partial P}{\partial T}. \tag{1.3.6}$$

Inverting these gives the requested relations:

$$\frac{\partial T}{\partial \mu} = -\frac{\partial N}{\partial S}, \tag{1.3.7}$$

$$\frac{\partial P}{\partial T} = \frac{\partial S}{\partial V}. \tag{1.3.8}$$

1.4

a

The Helmholtz Free Energy is given by

$$A - A_0 = T_0 \ln\left(\frac{V}{V_0}\right) - \frac{1}{a+1}\frac{V}{V_0}\left(\frac{T^{a+1}}{T_0^a} - T_0\right). \tag{1.4.1}$$

b

The equation of state is found by using the Maxwell relation

$$-P = \frac{\partial A}{\partial V}, \tag{1.4.2}$$

$$= \frac{T_0}{V} - \frac{1}{a+1}\frac{1}{V_0}\left(\frac{T^{a+1}}{T_0^a} - T_0\right). \tag{1.4.3}$$

c

From 1.4.3,

$$
\begin{aligned}
W &= -\int_{V_a}^{V_b} P \, \mathrm{d}V \,, && (1.4.4) \\
&= \int_{V_a}^{V_b} \frac{T_0}{V} \, \mathrm{d}V \\
&\quad - \int_{V_a}^{V_b} \frac{1}{a+1} \frac{1}{V_0} \left(\frac{T_a^{a+1}}{T_0^a} - T_0 \right) \mathrm{d}V \,, \\
& && (1.4.5) \\
&= T_0 \ln\left(\frac{V_b}{V_a} \right) \\
&\quad - \frac{1}{a+1} \frac{(V_b - V_a) T_a}{V_0} \left(\frac{T_a^{a+1}}{T_0^a} - T_0 \right) . \\
& && (1.4.6)
\end{aligned}
$$