PHY 831: Statistical Mechanics
Homework 6

Due Friday Nov 12, 2021

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1. Consider a gas of massless identical particles with spin degeneracy g in three dimensions. Assuming $\mu = 0$, derive $P = AgT^4$ and $E/V = BgT^4$ and evaluate A and B for a) Fermions, and b) Bosons.

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2. **Stoner ferromagnetism problem.** In a metal, the conduction electrons can be treated as a gas with an effective spin-spin interaction energy that mocks up the effects of the Coulomb interactions. More precisely, the Coulomb repulsion amongst the conduction electrons favors a *spatial* wave function that is antisymmetric in the coordinates, so that the electrons are kept from overlapping strongly. Since the *full* wave function (spatial and spin variables) is antisymmetric, this interaction can be approximated by an effective spin-spin coupling that favors states with parallel spins. In this simple approximation, the net effect is described by an interaction energy

$$U = \alpha \frac{N_+ N_-}{V} ,$$

where $N_\pm$ represent the number of spin-up and spin-down electrons, and α is a parameter related to the zero-energy scattering amplitude between two electrons.

5 (a) The $T = 0$ ground state consists of two fermi seas filled by the spin-up and spin-down electrons. Express the corresponding Fermi wave vectors $k_F^\pm$ in terms of the densities $n_\pm = N_\pm/V$.

5 (b) Calculate the kinetic energy density of the ground state as a function of $n_\pm$ and various fundamental constants.

5 (c) Assuming small deviations $n_\pm = n/2 \pm \delta$ from the symmetric state, expand the kinetic energy to fourth-order in the small parameter δ.

5 (d) Express the spin-spin interaction density U/V in terms of n and δ. Find the critical value of α_c such that for $\alpha > \alpha_c$, the electron gas can lower its total energy by developing a magnetization. (This is know as the Stoner instability.)

5 (e) Explain qualitatively and sketch the behavior of the spontaneous magnetization as a function of α.

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3. **A generalized quantum ideal gas.** Consider a gas of non-interacting identical spinless quantum particles in d-dimensions with a weird single particle energy dispersion $\epsilon_p = \alpha p^s$, where α is some constant so that ϵ_p has dimensions of energy.

10 (a) Calculate the grand potential and density at a chemical potential of μ, expressing your answer in terms of s, d, and $f_m^\eta(z)$ where $z = e^{\beta\mu}$.

5 (b) Find the ratio of PV/E.

5 (c) For fermions, calculate the dependence of E/N and P on the density $n = N/V$ at zero temperature. (Hint: Recall that $f_m^-(z) \to (\log z)^m/m!$ as $z \to \infty$.)

5 (d) For Bosons, find the dimension $d(s)$ below which there is no Bose-Einstein condensation. Is there condensation at $d = 2$ for $s = 2$? Why or why not?

$\boxed{25}$ 4. Consider an ultrarelativistic gas of bosons in d-dimensions. By mimicking what we did in class for the non-relativistic gas, find the lower critical dimension– i.e., highest dimension d_c for which N diverges as you take $\mu \to 0^-$ indicating no macroscopic condensation in the lowest single particle level. For $d > d_c$, there is condensation. Find an expression for the critical temperature $T_c(N, V)$.

$\boxed{1}$

$$\frac{PV}{k_BT} = \log Z = g\sum_{\vec{p}} \log\left[\left(1-\eta e^{-\beta(\epsilon_p-\mu)}\right)^{-\eta}\right] \qquad \eta = \begin{matrix} +1 \text{ Bosons} \\ -1 \text{ Fermions}\end{matrix}$$

$$\therefore P = -\eta\frac{k_BTg}{V}\sum_{\vec{p}}\log\left(1-\eta e^{-\beta(\epsilon_p-\mu)}\right) \qquad \epsilon_p = pc$$

$$= -\eta g k_BT\int\frac{d^3p}{(2\pi\hbar)^3}\log\left(1-\eta e^{-\beta(pc-\mu)}\right)$$

$$= -\eta g k_BT\frac{4\pi}{(2\pi\hbar)^3}\int dp\, p^2\log\left(1-\eta e^{-\beta(pc-\mu)}\right)$$

$$\underset{\text{parts}}{\overset{\text{int. by}}{=}} -\eta\, gkT\frac{4\pi}{(2\pi\hbar)^3}\cdot\left[\cancelto{0}{\frac{p^3}{3}\log\left(1-\eta e^{-\beta(pc-\mu)}\right)\Big|_0^\infty} - \int dp\,\frac{p^3}{3}\frac{d}{dp}\log\left(1-\eta e^{-\beta(pc-\mu)}\right)\right.$$

$$= -\eta\frac{g\cancel{kT}4\pi}{(2\pi\hbar)^3}\left(-\int dp\,\frac{p^3\cancel{\beta}c\eta\, e^{-\beta(pc-\mu)}}{3\left(1-\eta e^{-\beta(pc-\mu)}\right)}\right.$$

$$= \frac{4\pi g}{(2\pi\hbar)^3}\cdot\frac{1}{3}\int dp\,\frac{p^3e^{-\beta(pc-\mu)}}{\left(1-\eta e^{-\beta(pc-\mu)}\right)}$$

*Now use that we're told $\mu = 0$

$$\therefore P = \frac{4\pi g}{3(2\pi\hbar)^3}\int dp\,\frac{p^3e^{-\beta pc}}{1-\eta e^{-\beta pc}}$$

let $\beta pc = x$; $dp = \frac{1}{\beta c}dx$

$$\therefore\ P = \frac{4\pi g}{3(2\pi\hbar)^3} \int dp\, \frac{p^3\, e^{-\beta p c}}{1 - \eta e^{-\beta p c}}$$

let $\beta p c = x$; $dp = \frac{1}{\beta c} dx$

$$\Rightarrow P = \frac{4\pi g}{3(2\pi\hbar)^3} \cdot \frac{1}{(\beta c)^4} \underbrace{\int_0^\infty dx\, \frac{x^3 e^{-x}}{1 - \eta e^{-x}}}_{\equiv I_\eta}$$

$= \frac{\pi^4}{15}$ (for $\eta = +1$)

$= \frac{7\pi^4}{120}$ (for $\eta = -1$)

Note $I_{Fermion} = \frac{7}{8} I_{Boson}$

$\therefore$ Collecting terms

$$P_\eta = \frac{4\pi g\, k_B^4 T^4}{24\pi^3 \hbar^3 c^4} I_\eta$$

$$= \frac{g(kT)^4}{6\pi^2 \hbar^3 c^4} \cdot \frac{\pi^4}{15} \quad \text{(Bosons)}$$

$$= \frac{g(kT)^4}{6\pi^2 \hbar^3 c^4} \cdot \frac{7\pi^4}{120} \quad \text{(Fermions}$$

$$\therefore \; P \simeq A g T^4$$

where $$A_{Boson} = \frac{k_B^4 \pi^2}{90 \hbar^3 c^4}$$

$$A_{Fermion} = \frac{7}{8} A_{Boson}$$

* For $\frac{E}{V}$, note

$$E = g \sum_{\vec{p}} \epsilon_{\vec{p}} \langle n_{\vec{p}} \rangle_\eta$$

$$\therefore \frac{E}{V} = g \frac{1}{V} \sum_{\vec{p}} \epsilon_p \langle n_p \rangle_\eta$$

$$\Rightarrow g \int \frac{d^3p}{(2\pi\hbar)^3} \epsilon_p \frac{1}{e^{\beta(\epsilon_p - \mu)} - \eta}$$ set $\mu = 0$

$$\frac{E}{V} = \frac{g\, 4\pi c}{(2\pi\hbar)^3} \int dp\, p^3 \frac{e^{-\beta(pc)}}{1 - \eta e^{-\beta pc}}$$

let $x = \beta p c$

$$\left(\frac{E}{V}\right) = \frac{g(4\pi)}{(2\pi\hbar)^3} \cdot \frac{1}{c^3} (k_B T)^4 \cdot \int dx \frac{x^3 e^{-x}}{1 - \eta e^{-x}}$$

$$\therefore \frac{E}{V} = BT^4$$

where $B = 3AC$ where A is from before.

$\boxed{2.}$ Stoner Magnetism

a.) $N_\pm = \sum_{\vec{k}} \Theta(k_F^{+/-} - |\vec{k}|)$

$$= V \int \frac{d^3k}{(2\pi)^3} \Theta(k_F^\pm - k)$$

$$= \frac{V}{2\pi^2} \frac{(k_F^\pm)^3}{3}$$

$$\therefore \frac{N_\pm}{V} = n_\pm = \frac{(k_F^\pm)^3}{6\pi^2}$$

$$\Rightarrow \boxed{k_F^\pm = (6\pi^2 n_\pm)^{1/3}}$$

b.) $\langle KE \rangle = \sum_{\vec{k}} \frac{\hbar^2 k^2}{2m} \Theta(k_F^+ - k) + \sum_{\vec{k}} \frac{\hbar^2 k^2}{2m} \Theta(k_F^- - k)$

$$= V \int \frac{d^3k}{(2\pi)^3} \frac{\hbar^2 k^2}{2m} \Theta(k_F^+ - k) + V \int \frac{d^3k}{(2\pi)^3} \frac{\hbar^2 k^2}{2m} \Theta(k_F^- - k)$$

$$= \frac{V}{2\pi^2} \cdot \frac{\hbar^2}{2m} \frac{(k_F^+)^5}{5} + \frac{V}{2\pi^2} \frac{\hbar^2}{2m} \frac{(k_F^-)^5}{5}$$

$\therefore$ KE density $= \dfrac{\langle KE \rangle}{V}$

$$= \frac{\hbar^2}{2m} \frac{1}{10\pi^2} \left(k_{F_+}^5 + k_{F_-}^5 \right) \qquad k_{F_\pm} = \left(6\pi^2 n_\pm\right)^{1/3}$$

$$= \frac{\hbar^2}{2m} \cdot \frac{1}{10\pi^2} \cdot (6\pi^2)^{5/3} \cdot \left(n_+^{5/3} + n_-^{5/3} \right)$$

$$\frac{\langle KE \rangle}{V} = \frac{\hbar^2}{2m} \cdot \frac{1}{10\pi^2} \cdot (6\pi^2)^{5/3} \cdot \left(n_+^{5/3} + n_-^{5/3} \right)$$

$$= \frac{\hbar^2}{2m} \cdot \frac{6\cancel{\pi^2}}{10\cancel{\pi^2}} (6\pi^2)^{2/3} \left(n_+^{5/3} + n_-^{5/3} \right)$$

$$\boxed{\frac{\langle KE \rangle}{V} = \frac{3}{5} \cdot \frac{\hbar^2}{2m} (6\pi^2)^{2/3} \left(n_+^{5/3} + n_-^{5/3} \right)}$$

c) Let $n_\pm = \frac{n}{2} \pm \delta$ where $\delta << \frac{n}{2}$

* use Mathematica to expand $\frac{KE}{V}$ thru $\mathcal{O}(\delta^4)$

$$\Rightarrow \boxed{\frac{\langle KE \rangle}{V} \simeq \frac{\hbar^2}{2m} \cdot \frac{6}{5} (6\pi^2)^{2/3} \left[\left(\frac{n}{2}\right)^{5/3} + \frac{5}{9} \left(\frac{n}{2}\right)^{-1/3} \delta^2 + \frac{5}{243} \left(\frac{n}{2}\right)^{-7/3} \delta^4 \right]}$$

d) $$\frac{U}{V} = \alpha n_+ n_- = \alpha \left(\frac{n}{2} + \delta\right)\left(\frac{n}{2} - \delta\right)$$

$$= \alpha \frac{n^2}{4} - \alpha \delta^2$$

$$\therefore \frac{E}{V} = \frac{\langle KE \rangle}{V} + \frac{U}{V} = \left(\frac{E}{V}\right)_{\delta=0} + \left[\frac{4}{3} (3\pi^2)^{2/3} \frac{\hbar^2}{2m} n^{-1/3} - \alpha \right] \delta^2$$

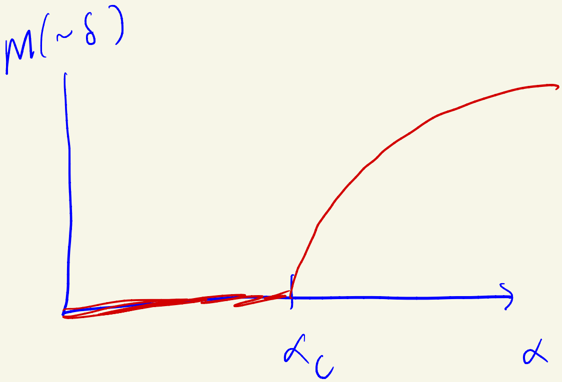
$$+ \mathcal{O}(\delta^4)$$

* We see the energy density can be lowered relative to the non-magnetic ($\delta=0$) case when the coeff. of the δ^2 term is <0.

$\Downarrow$

i.e., $\boxed{\text{for } \alpha > \alpha_c = \frac{4}{3}(3\pi^2)^{2/3}\frac{\hbar^2}{2m} n^{-1/3}}$

e)



* For $\alpha > \alpha_c$, optimal value δ obtained from minimizing $\frac{E}{V}$.

* Since coeff. of $O(\delta^4)$ term in $\frac{E}{V} > 0$, and since optimal $\delta \to 0$ as $\alpha \to \alpha_c$, the energy is minimized for $\delta^2 \propto \alpha - \alpha_c$, so δ goes as $\delta \sim \sqrt{\alpha - \alpha_c}$

$\boxed{3}$ Generalized Ideal gas $\epsilon_p = \alpha p^s$ in d-dimensions

a) Find $\mathcal{Q} = -k_B T \log Z$ & $n = \frac{N}{V}$ in terms of relevant $f_m^\eta(z)$

$$\mathcal{Q} = -kT \log Z = -kT(-\eta) \sum_{\vec{p}} \log\left(1 - \eta e^{-\beta(\epsilon_p - \mu)}\right)$$

$$= \eta k_B T V \int \frac{d^D p}{(2\pi\hbar)^D} \log\left(1 - \eta e^{-\beta(\epsilon_p - \mu)}\right)$$

$$= \frac{\eta k_B T V}{(2\pi\hbar)^D} S_D \int p^{D-1} dp \log\left(1 - \eta e^{-\beta(\epsilon_p - \mu)}\right)$$

where $S_D = \int d\Omega_D$

* integrate by parts =>

0 provided $s > 0$ for $\epsilon_p = \alpha p^s$

$$\mathcal{Q} = \frac{\cancel{\eta} \cancel{k_B T} V}{(2\pi\hbar)^D} S_D \Big[\cancel{\frac{p^D}{D} \log\left(1 - \eta e^{-\beta(\epsilon_p - \mu)}\right)\Big|_0^\infty}$$

$$- \int dp \frac{p^D}{D} \frac{\cancel{\beta}\cancel{\eta} e^{-\beta(\epsilon_p - \mu)}}{1 - \eta e^{-\beta(\epsilon_p - \mu)}} \frac{d\epsilon_p}{dp}$$

$$= -\frac{V S_D}{(2\pi\hbar)^D D} \int dp\, p^D \underbrace{\frac{d\epsilon_p}{dp}}_{\alpha s p^{s-1}} \frac{e^{-\beta(\epsilon_p - \mu)}}{1 - \eta e^{-\beta(\epsilon_p - \mu)}}$$

$$= -\frac{V S_D \alpha s}{(2\pi\hbar)^D D} \int dp\, p^{D+s-1} \frac{e^{-\beta(\alpha p^s - \mu)}}{1 - \eta e^{-\beta(\alpha p^s - \mu)}}$$

let $\beta \alpha p^s = x$

$s \beta \alpha p^{s-1} dp = dx$

$$\mathcal{G} = -\frac{V S_D \alpha s}{(2\pi\hbar)^D D} \int dp\, p^{D+s-1} \frac{e^{-\beta(\alpha p^s - \mu)}}{1-\eta e^{-\beta(\alpha p^s-\mu)}}$$

let $\beta\alpha p^s = x$ $\qquad p^s = \frac{1}{\beta\alpha} x \Rightarrow p = \left(\frac{1}{\beta\alpha}\right)^{1/s} \cdot x^{1/s}$

$s\beta\alpha p^{s-1} dp = dx$

$$dp = \frac{1}{s\beta\alpha} \frac{1}{p^{s-1}} dx = \frac{1}{s\beta\alpha} p^{1-s} dx$$

$$= \frac{1}{s\beta\alpha} \cdot \left(\frac{1}{\beta\alpha}\right)^{\frac{1-s}{s}} \cdot x^{\frac{1-s}{s}} dx$$

$$\Rightarrow \mathcal{G} = -\frac{V S_D \alpha s}{(2\pi\hbar)^D D} \cdot \frac{1}{s\beta\alpha} \cdot \left(\frac{1}{\beta\alpha}\right)^{\frac{1}{s}-1} \cdot \left(\frac{1}{\beta\alpha}\right)^{\frac{D+s-1}{s}} \int dx \frac{x^{\frac{1-s}{s}} \cdot x^{\frac{D+s-1}{s}} z e^{-x}}{1-\eta z e^{-x}}$$

$$= -\frac{V S_D \alpha s}{(2\pi\hbar)^D D} \cdot \frac{1}{s\beta\alpha} \cdot \left(\frac{1}{\beta\alpha}\right)^{\frac{1-s}{s}} \left(\frac{1}{\beta\alpha}\right)^{\frac{D+s-1}{s}} \cdot \int \frac{dx\, x^{D/s}}{z^{-1}e^x - \eta}$$

$$= -\frac{V S_D \alpha s}{(2\pi\hbar)^D D} \cdot \frac{1}{s\beta\alpha} \cdot \left(\frac{1}{\beta\alpha}\right)^{D/s} \int \frac{dx\, x^{D/s}}{z^{-1}e^x - \eta}$$

Now, recall $f_m^\eta(z) = \frac{1}{\Gamma(m)} \int \frac{dx\, x^{m-1}}{z^{-1}e^x - \eta}$

$\therefore \frac{D}{s} = m-1$

$\Rightarrow m = 1 + \frac{D}{s}$

$$\Rightarrow \Omega = \frac{-V S_D \alpha s}{(2\pi\hbar)^D D} \cdot \frac{1}{s} \left(\frac{1}{\beta\alpha}\right)^{\frac{D}{s}+1} \Gamma\left(\frac{D}{s}+1\right) f^{\eta}_{\frac{D}{s}+1}(z)$$

$$\Rightarrow \boxed{\Omega = \frac{-V S_D \alpha}{(2\pi\hbar)^D D} \left(\frac{1}{\beta\alpha}\right)^{\frac{D}{s}+1} \Gamma\left(\frac{D}{s}+1\right) f^{\eta}_{\frac{D}{s}+1}(z)}$$

Note to grader: This was a lot more tedious than I thought it would be. Don't take off if students are off by a prefactor. The main thing is that they should have

$$\Omega \propto f^{\eta}_{\frac{D}{s}+1}(z)$$

For densities, we have

$$\frac{N}{V} = n = \frac{1}{V} \sum_{\vec{p}} \frac{1}{z e^{\beta \epsilon_p} - \eta} \qquad \text{where } V = L^D$$

$$= \int \frac{d^D p}{(2\pi\hbar)^D} \frac{1}{z e^{\beta\alpha p^s} - \eta}$$

$$n = \frac{S_D}{(2\pi\hbar)^D} \int dp\, p^{D-1} \frac{1}{z e^{\beta\alpha p^s} - \eta}$$

$$n = \frac{S_D}{(2\pi\hbar)^D} \int dp\, p^{D-1} \frac{1}{\zeta e^{\beta\alpha p^s} - \eta}$$

let $\beta\alpha p^s = x$ $\qquad p^s = \frac{1}{\beta\alpha} x \Rightarrow p = \left(\frac{1}{\beta\alpha}\right)^{1/s} \cdot x^{1/s}$

$$s\beta\alpha p^{s-1} dp = dx$$

$$dp = \frac{1}{s\beta\alpha} \frac{1}{p^{s-1}} dx = \frac{1}{s\beta\alpha} p^{1-s} dx$$

$$= \frac{1}{s\beta\alpha} \cdot \left(\frac{1}{\beta\alpha}\right)^{\frac{1-s}{s}} \cdot x^{\frac{1-s}{s}} dx$$

$$n = \frac{S_D}{(2\pi\hbar)^D} \cdot \frac{1}{s\beta\alpha} \cdot \left(\frac{1}{\beta\alpha}\right)^{\frac{1-s}{s}} \left(\frac{1}{\beta\alpha}\right)^{\frac{D-1}{s}} \int \frac{dx\, x^{\frac{1-s}{s}}\, x^{\frac{D-1}{s}}}{\zeta e^x - \eta}$$

$$= \frac{S_D}{(2\pi\hbar)^D} \cdot \frac{1}{s\beta\alpha} \cdot \left(\frac{1}{\beta\alpha}\right)^{\frac{D-s}{s}} \cdot \int \frac{dx\, x^{\frac{D-s}{s}}}{\zeta e^x - \eta}$$

$$= \frac{S_D}{(2\pi\hbar)^D} \cdot \frac{1}{s} \left(\frac{1}{\beta\alpha}\right)^{\frac{D}{s}} \int \frac{dx\, x^{\frac{D}{s}-1}}{\zeta e^x - \eta} = \Gamma\left(\tfrac{D}{s}\right) \cdot f^{\eta}_{\frac{D}{s}}(\zeta)$$

* As w/ 9, I don't care if people are off by a prefactor. The main thing is that

$$n \propto f^{\eta}_{D/s}(\zeta)$$

b.) Find $\frac{PV}{E}$

Since $PV = -\Omega = \frac{V S_D \alpha}{(2\pi\hbar)^D D} \left(\frac{1}{\beta\alpha}\right)^{\frac{D}{s}+1} \Gamma\left(\frac{D}{s}+1\right) f^{\eta}_{\frac{D}{s}+1}(z)$,

and $E = -\frac{\partial}{\partial\beta} \log Z \Big|_{z=\text{const}} = -\frac{\partial}{\partial\beta}\left[-\beta\Omega\right]\Big|_z$

$= \Omega + \beta \frac{\partial}{\partial\beta}\Omega\Big|_z$

$E = \left(\cancel{1} - \frac{D}{s} - \cancel{1}\right)\Omega$

$$\therefore \boxed{\frac{PV}{E} = \frac{-\Omega}{-\frac{D}{s}\Omega} = \frac{s}{D}}$$

c.) Find $\frac{E}{N}$ + P as function of n as $T \to 0$

Just use that $\frac{1}{e^{\beta(\epsilon-\mu)}+1} \longrightarrow \Theta(\epsilon_F - \epsilon)$

$\therefore E = \sum_{\vec{p}} \alpha p^s \, \Theta(p_F - p)$

$\frac{E}{V} = \int \frac{d^D p}{(2\pi\hbar)^D} \alpha p^s = \frac{S_D}{(2\pi\hbar)^D} \alpha \int_0^{p_F} p^{D+s-1} \, dp$

$= \frac{S_D \alpha}{(2\pi\hbar)^D} \frac{p_F^{D+s}}{D+s}$ need to express p_F in terms of density n

* But $\frac{N}{V} = \int \frac{d^D p}{(2\pi\hbar)^D} \Theta(p_F - p)$

$$= \frac{S_D}{(2\pi\hbar)^D} \int p^{D-1} \Theta(p_F - p)$$

$$n = \frac{S_D}{(2\pi\hbar)^D} \cdot \frac{1}{D} p_F^D$$

$$\therefore \quad p_F = \left[\frac{(2\pi\hbar)^D D}{S_D} n\right]^{1/D}$$

As in the previous parts, here I don't worry about getting the exact prefactors right. I just care about the dependence on n

$$\boxed{\frac{E}{V} \propto p_F^{D+s} \propto n^{1+\frac{s}{D}}}$$

* Likewise, from the general relation we derived

$$\frac{PV}{E} = \frac{s}{D}$$

$$\boxed{\Rightarrow P = \frac{s}{D} \cdot \frac{E}{V} \propto n^{1+\frac{s}{D}} \quad \text{too}}$$

d) Find $d(s)$ where no BEC

* Since $n \propto f^{+}_{D/s}(z)$, we follow the procedure we did in class and see if $f_{D/s}(z \to 1)$ diverges.

If $f_{\frac{D}{s}}(z \to 1) = \infty$, then no BEC.

If $f_{\frac{D}{s}}(z \to 1) = \text{finite}$, then there is BEC.

Since $f_{\frac{D}{s}}(z=1) = \frac{1}{\Gamma(\frac{D}{s})} \int_0^\infty \frac{dx \; x^{\frac{D}{s}-1}}{e^x - 1}$

integrand can diverge for $x \sim \varepsilon \to 0$

$$\int_0^\varepsilon \frac{dx \; x^{\frac{D}{s}-1}}{e^x - 1} \sim \int \frac{dx \; x^{\frac{D}{s}-1}}{x + \frac{x^2}{2!} \cdots} \sim \int_0^\varepsilon dx \; x^{\frac{D}{s}-2}$$

∴ if $\frac{D}{s} - 2 > -1$ integral converges

* Therefore, there is BEC for $D > s$ (& no BEC for $D \leq s$)
* For $D = 2 = s$, there is no BEC since $f_1(z=1) = \infty$.

$\boxed{4.}$ Ultra-Relativistic bosons in d-dim

Lower critical dimension:

* From 3d.), there is no BEC for $D \leq S$ which in our case means $d_c = 1$ since $S=1$
* Meanwhile, there is BEC for $D = 2, 3, \ldots$

Critical temperature:

From problem 3.), we have

$$n = \frac{S_D}{(2\pi\hbar)^D} \cdot \frac{1}{S} \left(\frac{1}{\beta\alpha}\right)^{\frac{D}{S}} \int \frac{dx\, x^{\frac{D}{S}-1}}{z e^x - \eta} = \Gamma\left(\tfrac{D}{S}\right) \cdot f^{\eta}_{\frac{D}{S}}(z)$$

* Setting $\left.\begin{matrix} S = 1 \\ \alpha = c \end{matrix}\right\}$ since $\epsilon_p = pc$, $\eta = 1$ (Bosons)

$$\Rightarrow \frac{N}{V} = \frac{S_D}{(2\pi\hbar)^D} \left(\frac{1}{\beta c}\right)^D \Gamma(D)\, f^{(+)}_D(z)$$

Riemman Zeta

Now, get $T_c(N,V)$ by setting $z=1$ & using $f^{(+)}_D(1) = \zeta(D)$

$$\frac{N}{V} = \frac{S_D}{(2\pi\hbar)^D} \left(\frac{k_B T_c}{c}\right)^D \Gamma(D)\, \zeta(D)$$

$$\frac{N}{V} = \frac{S_D}{(2\pi\hbar)^D} \left(\frac{k_B T_c}{c}\right)^D \Gamma(D)\, \zeta(D)$$

$$\Rightarrow \boxed{T_c = \frac{2\pi\hbar c}{L} \left(\frac{N}{\Gamma(D)\,\zeta(D)\, S_D}\right)^{1/D}}$$