

# Homework 6

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## 6.1

$$PV\beta = \log(Z) = g \sum_{\mathbf{p}} \log \left( \left( 1 - \eta e^{-\beta(\epsilon_p - \mu)} \right)^{-\eta} \right), \quad (6.1.1)$$

$$P = -\eta \frac{g}{V\beta} \sum_{\mathbf{p}} \log \left( 1 - \eta e^{-\beta(\epsilon_p - \mu)} \right), \quad (\epsilon_p = pc) \quad (6.1.2)$$

$$\approx -\eta \frac{g}{\beta} \int \frac{d^3p}{(2\pi\hbar)^3} \log \left( 1 - \eta e^{-\beta(pc - \mu)} \right), \quad (6.1.3)$$

$$= \frac{4\pi g}{3(2\pi\hbar)^3} \int dp \frac{p^3 e^{-\beta(pc - \mu)}}{1 - \eta e^{-\beta(pc - \mu)}}, \quad (6.1.4)$$

$$\mu = 0 : \quad (6.1.5)$$

$$P = \frac{4\pi g}{3(2\pi\hbar)^3} \int_0^\infty dp \frac{p^3 e^{-\beta pc}}{1 - \eta e^{-\beta pc}}, \quad (6.1.6)$$

$$= \begin{cases} \frac{4\pi g}{3(2\pi\hbar)^3} \frac{1}{(\beta c)^4} \frac{\pi^4}{15}, & \eta = +1 \quad (\text{bosons}) \\ \frac{4\pi g}{3(2\pi\hbar)^3} \frac{1}{(\beta c)^4} \frac{7\pi^4}{120}, & \eta = -1 \quad (\text{fermions}) \end{cases}. \quad (6.1.7)$$

Hence:

$$P = AgT^4, \quad (6.1.8)$$

where

$$A = \begin{cases} \frac{4\pi}{3(2\pi\hbar)^3} \frac{k^4}{c^4} \frac{\pi^4}{15}, & \eta = +1 \quad (\text{bosons}) \\ \frac{4\pi}{3(2\pi\hbar)^3} \frac{k^4}{c^4} \frac{7\pi^4}{120}, & \eta = -1 \quad (\text{fermions}) \end{cases}. \quad (6.1.9)$$

$$\frac{E}{V} = \frac{g}{V} \sum_{\mathbf{p}} \epsilon_p \langle n_p \rangle_\eta, \quad (6.1.10)$$

$$\approx g \int \frac{d^3p}{(2\pi\hbar)^3} \frac{\epsilon_p}{e^{\beta(\epsilon_p - \mu)} - \eta}, \quad (6.1.11)$$

$$\mu = 0 : \quad (6.1.12)$$

$$\frac{E}{V} = \frac{g4\pi c}{(2\pi\hbar)^3} \int dp \frac{p^3 e^{-\beta pc}}{1 - \eta e^{-\beta pc}}, \quad (6.1.13)$$

$$= \begin{cases} \frac{g4\pi c}{(2\pi\hbar)^3} \frac{1}{(\beta c)^4} \frac{\pi^4}{15}, & \eta = +1 \quad (\text{bosons}) \\ \frac{g4\pi c}{(2\pi\hbar)^3} \frac{1}{(\beta c)^4} \frac{7\pi^4}{120}, & \eta = -1 \quad (\text{fermions}) \end{cases}. \quad (6.1.14)$$

Hence:

$$\frac{E}{V} = BgT^4, \quad (6.1.15)$$

where  $B = 3Ac$ .

## 6.2

### 6.2.1

$$N_{\pm} = \sum_{\mathbf{k}} \Theta(k_F^{\pm} - |\mathbf{k}|), \quad (6.2.1)$$

$$\approx V \int \frac{d^3k}{(2\pi)^3} \Theta(k_F^{\pm} - k), \quad (6.2.2)$$

$$= \frac{V}{2\pi} \frac{(k_F^{\pm})^3}{6\pi^2}. \quad (6.2.3)$$

$$\therefore k_F^{\pm} = (6\pi^2 n_{\pm})^{1/3}, \quad (6.2.4)$$

where  $n_{\pm} = N_{\pm}/V$ .

### 6.2.2

$$\langle KE \rangle = \sum_{\mathbf{k}} \frac{\hbar^2 k^2}{2m} \Theta(k_F^+ - |\mathbf{k}|) + \sum_{\mathbf{k}} \frac{\hbar^2 k^2}{2m} \Theta(k_F^- - |\mathbf{k}|), \quad (6.2.5)$$

$$\approx V \int \frac{d^3k}{(2\pi)^3} \frac{\hbar^2 k^2}{2m} \Theta(k_F^+ - |\mathbf{k}|) + V \int \frac{d^3k}{(2\pi)^3} \frac{\hbar^2 k^2}{2m} \Theta(k_F^- - |\mathbf{k}|), \quad (6.2.6)$$

$$= \frac{V}{2\pi^2} \frac{\hbar^2}{2m} \frac{((k_F^+)^5 + (k_F^-)^5)}{5}. \quad (6.2.7)$$

$$\rightarrow \frac{\langle KE \rangle}{V} = \frac{\hbar^2}{4m\pi^2} \frac{((k_F^+)^5 + (k_F^-)^5)}{5}. \quad (6.2.8)$$

$$k_F^{\pm} = (6\pi^2 n_{\pm})^{1/3}: \quad (6.2.9)$$

$$\frac{\langle KE \rangle}{V} = \frac{(6\pi^2)^{5/3} \hbar^2}{20m\pi^2} (n_+^5 + n_-^5), \quad (6.2.10)$$

$$= \frac{3}{10m} \hbar^2 (6\pi^2)^{2/3} (n_+^{5/3} + n_-^{5/3}). \quad (6.2.11)$$

### 6.2.3

For small deviations from the symmetric state ( $n_{\pm} = n/2 \pm \delta$ ),

$$\frac{\langle KE \rangle}{V} = \frac{3}{10m} \hbar^2 (6\pi^2)^{2/3} \left( \left( \frac{n}{2} + \delta \right)^{5/3} + \left( \frac{n}{2} - \delta \right)^{5/3} \right), \quad (6.2.12)$$

$$\approx \frac{6}{10m} \hbar^2 (6\pi^2)^{2/3} \left( \left( \frac{n}{2} \right)^{5/3} + \frac{5}{9} \left( \frac{n}{2} \right)^{-1/3} \delta^2 + \frac{5}{243} \left( \frac{n}{2} \right)^{-7/3} \delta^4 \right). \quad (6.2.13)$$

## 6.2.4

$$\frac{U}{V} = \alpha \left( \frac{n}{2} + \delta \right) \left( \frac{n}{2} - \delta \right), \quad (6.2.14)$$

$$= \alpha \frac{n^2}{4} - \alpha \delta^2. \quad (6.2.15)$$

Since  $E = \langle KE \rangle + U$ ,

$$\begin{aligned} \frac{E}{V} &= \alpha \frac{n^2}{4} - \alpha \delta^2 \\ &+ \frac{6}{10m} \hbar^2 (6\pi^2)^{2/3} \left( \left( \frac{n}{2} \right)^{5/3} + \frac{5}{9} \left( \frac{n}{2} \right)^{-1/3} \delta^2 + \frac{5}{243} \left( \frac{n}{2} \right)^{-7/3} \delta^4 \right), \end{aligned} \quad (6.2.16)$$

$$= \left( \frac{E}{V} \right)_{\delta=0} + \left( \frac{4}{3} (3\pi^2)^{2/3} \frac{\hbar^2}{2m} n^{-1/3} - \alpha \right) \delta^2 + \mathcal{O}(\delta^4). \quad (6.2.17)$$

$$\therefore \alpha > \alpha_c = \frac{4}{3} (3\pi^2)^{2/3} \frac{\hbar^2}{2m} n^{-1/3} \quad (6.2.18)$$

causes the electron gas to have the ability to lower its energy by developing a magnetization.

## 6.2.5

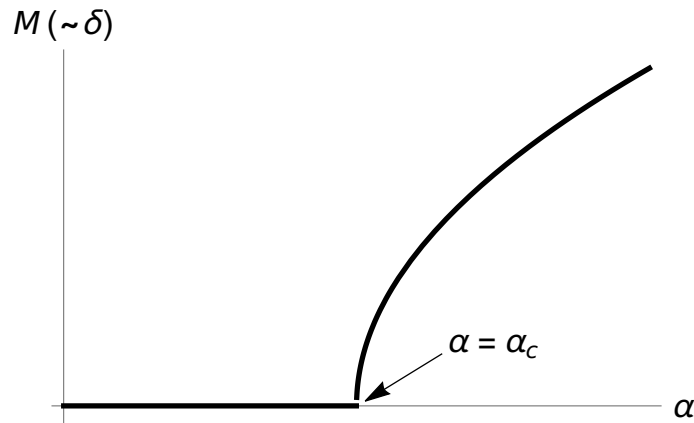


Figure 6.2.1

The magnetization starts at 0 for  $\alpha < \alpha_c$ , but, for  $\alpha \geq \alpha_c$ , the magnetization (which is essentially the quantity  $\delta$ ) grows as  $\sqrt{\alpha - \alpha_c}$ .

## 6.3

### 6.3.1

$$\mathcal{G} = -\frac{1}{\beta} \log(\mathcal{Z}), \quad (6.3.1)$$

$$= \frac{\eta}{\beta} \sum_{\mathbf{p}} \log\left(1 - \eta e^{-\beta(\epsilon_p - \mu)}\right), \quad (6.3.2)$$

$$= \frac{\eta}{\beta} V \int \frac{d^D p}{(2\pi\hbar)^D} \log\left(1 - \eta e^{-\beta(\epsilon_p - \mu)}\right), \quad (6.3.3)$$

$$= \frac{\eta V}{(2\pi\hbar)^D \beta} S_D \int dp p^{D-1} \log\left(1 - \eta e^{-\beta(\epsilon_p - \mu)}\right), \quad \left(S_D = \int d\Omega_D\right) \quad (6.3.4)$$

⋮ (Mathematica)

$$\mathcal{G} = -\frac{V S_D \alpha s}{(2\pi\hbar)^D D} \int dp p^{D+s-1} \frac{e^{-\beta(\alpha p^s - \mu)}}{1 - \eta e^{-\beta(\alpha p^s - \mu)}}. \quad (6.3.5)$$

Let  $x = \beta \alpha p^s$ :

$$\mathcal{G} = -\frac{V S_D \alpha s}{(2\pi\hbar)^D D} \frac{1}{s \beta \alpha} \left(\frac{1}{\beta \alpha}\right)^{D/s} \int dx \frac{x^{D/s}}{x^{-1} e^x - \eta}, \quad (6.3.6)$$

$$= -\frac{V S_D \alpha}{(2\pi\hbar)^D D} \left(\frac{1}{\beta \alpha}\right)^{D/s+1} \Gamma\left(\frac{D}{s} + 1\right) f_{\frac{D}{s}+1}^\eta(z). \quad (6.3.7)$$

For the following,  $V = L^D$ .

$$n = \frac{N}{V}, \quad (6.3.8)$$

$$= \frac{1}{V} \sum_{\mathbf{p}} \frac{1}{z e^{\beta \epsilon_p} - \eta}, \quad (6.3.9)$$

$$= \int \frac{d^D p}{(2\pi\hbar)^D} \frac{1}{z e^{\beta \alpha p^s} - \eta}, \quad (6.3.10)$$

$$= \frac{S_D}{(2\pi\hbar)^D} \int dp p^{D-1} \frac{1}{z e^{\beta \alpha p^s} - \eta}, \quad (6.3.11)$$

$$\vdots \quad (\text{Same integral as before}) \quad (6.3.12)$$

$$= \frac{S_D}{(2\pi\hbar)^D} \frac{1}{s} \frac{1}{\beta \alpha} \Gamma\left(\frac{D}{s}\right) f_{\frac{D}{s}}^\eta(z). \quad (6.3.13)$$

### 6.3.2

$$PV = -\mathcal{G}. \quad (6.3.14)$$

$$E = -\frac{\partial}{\partial \beta} \log \mathcal{Z} \Big|_{z=\text{const.}}, \quad (6.3.15)$$

$$= \frac{D}{s} \mathcal{G}. \quad (6.3.16)$$

$$\frac{PV}{E} = \frac{s}{D}. \quad (6.3.17)$$

### 6.3.3

As  $T \rightarrow 0$ ,

$$\lim_{T \rightarrow 0} \frac{1}{e^{\beta(\epsilon_p - \mu)} + 1} = \Theta(\epsilon_p - \mu). \quad (6.3.18)$$

Itaque,

$$\frac{E}{V} = \frac{1}{V} \sum_{\mathbf{p}} \alpha p^s \Theta(p_f - p), \quad (6.3.19)$$

$$= \int \frac{d^D p}{(2\pi\hbar)^D} \alpha p^s, \quad (6.3.20)$$

$$= \frac{S_d \alpha}{(2\pi\hbar)^D} \frac{p_F^{D+s}}{D+s}. \quad (6.3.21)$$

$$n = \int \frac{d^D p}{(2\pi\hbar)^D} \Theta(p_F - p). \quad (6.3.22)$$

$$p_F = \left( \frac{(2\pi\hbar)^D D}{S_D} n \right)^{1/D}. \quad (6.3.23)$$

$$\therefore \frac{E}{V} = \frac{S_D \alpha}{(2\pi\hbar)^D} \frac{\left( \frac{(2\pi\hbar)^D D}{S_D} n \right)^{1+(s/D)}}{D+s}, \quad (6.3.24)$$

$$\frac{E}{V} \propto n^{1+(s/D)}. \quad (6.3.25)$$

$$\frac{PV}{E} = \frac{s}{D}, \quad (6.3.26)$$

$$P = \frac{s}{D} \frac{S_D \alpha}{(2\pi\hbar)^D} \frac{\left( \frac{(2\pi\hbar)^D D}{S_D} n \right)^{1+(s/D)}}{D+s}, \quad (6.3.27)$$

$$P \propto n^{1+(s/D)}. \quad (6.3.28)$$

### 6.3.4

If, in the limit  $\mathcal{Z} \rightarrow 1$ ,  $f_{D/s}^+(\mathcal{Z} \rightarrow 1)$  is not finite, then no Bose-Einstein condensation forms. However, if  $f_{D/s}^+(\mathcal{Z} \rightarrow 1)$  is finite, then Bose-Einstein condensation forms.

The fugacity at  $\mathcal{Z} = 1$  is given by

$$f_{D/s}^+(\mathcal{Z} = 1) = \lim_{\varepsilon \rightarrow \infty} \frac{1}{\Gamma\left(\frac{D}{s}\right)} \int_0^\varepsilon dx \frac{x^{(D/s)-1}}{e^x - 1}, \quad (6.3.29)$$

$$= \lim_{\varepsilon \rightarrow \infty} \frac{1}{\Gamma\left(\frac{D}{s}\right)} \int_0^\varepsilon dx x^{(D/s)-1} \left( x + \frac{x^2}{2!} + \dots \right)^{-1}, \quad (6.3.30)$$

$$\approx \lim_{\varepsilon \rightarrow \infty} \frac{1}{\Gamma\left(\frac{D}{s}\right)} \int_0^\varepsilon dx x^{(D/s)-2}. \quad (6.3.31)$$

This converges for  $0 > (D/s) - 2 > -1$ , so if  $D > s$  a Bose-Einstein condensate can form. Therefore, if  $D = s = 2$  no Bose-Einstein condensate will form.

## 6.4

If  $s = 1$ , then, from problem 3d,  $d_c = 1$ .

With  $s = 1$  and  $\alpha = c$ ,  $\epsilon_p = pc$ . Additionally,  $eta = 1$  since the problem is discussing bosons. From problem 3,

$$n = \frac{S_D}{(2\pi\hbar)^D} \left( \frac{1}{\beta c} \right)^D \Gamma(D) f_D^+(\mathfrak{z}).T = T_c \quad \text{when} \quad \mathfrak{z} = 1. \quad (6.4.1)$$

$$(6.4.2)$$