PHY 831: Statistical Mechanics
Homework 6

Due Friday Nov 12, 2021

1. Consider a gas of massless identical particles with spin degeneracy g in three dimensions. Assuming $\mu = 0$, derive $P = AgT^4$ and $E/V = BgT^4$ and evaluate A and B for a) Fermions, and b) Bosons.

2. **Stoner ferromagnetism problem.** In a metal, the conduction electrons can be treated as a gas with an effective spin-spin interaction energy that mocks up the effects of the Coulomb interactions. More precisely, the Coulomb repulsion amongst the conduction electrons favors a *spatial* wave function that is antisymmetric in the coordinates, so that the electrons are kept from overlapping strongly. Since the *full* wave function (spatial and spin variables) is antisymmetric, this interaction can be approximated by an effective spin-spin coupling that favors states with parallel spins. In this simple approximation, the net effect is described by an interaction energy

$$U = \alpha \frac{N_+ N_-}{V} \,,$$

where $N_\pm$ represent the number of spin-up and spin-down electrons, and α is a parameter related to the zero-energy scattering amplitude between two electrons.

   (a) The $T = 0$ ground state consists of two fermi seas filled by the spin-up and spin-down electrons. Express the corresponding Fermi wave vectors $k_F^\pm$ in terms of the densities $n_\pm = N_\pm/V$.

   (b) Calculate the kinetic energy density of the ground state as a function of $n_\pm$ and various fundamental constants.

   (c) Assuming small deviations $n_\pm = n/2 \pm \delta$ from the symmetric state, expand the kinetic energy to fourth-order in the small parameter δ.

   (d) Express the spin-spin interaction density U/V in terms of n and δ. Find the critical value of α_c such that for $\alpha > \alpha_c$, the electron gas can lower its total energy by developing a magnetization. (This is know as the Stoner instability.)

   (e) Explain qualitatively and sketch the behavior of the spontaneous magnetization as a function of α.

3. **A generalized quantum ideal gas.** Consider a gas of non-interacting identical spinless quantum particles in d-dimensions with a weird single particle energy dispersion $\epsilon_p = \alpha p^s$, where α is some constant so that ϵ_p has dimensions of energy.

(a) Calculate the grand potential and density at a chemical potential of μ, expressing your answer in terms of s, d, and $f_m^\eta(z)$ where $z = e^{\beta\mu}$.

(b) Find the ratio of PV/E.

(c) For fermions, calculate the dependence of E/N and P on the density $n = N/V$ at zero temperature. (Hint: Recall that $f_m^-(z) \to (\log z)^m/m!$ as $z \to \infty$.)

(d) For Bosons, find the dimension $d(s)$ below which there is no Bose-Einstein condensation. Is there condensation at $d = 2$ for $s = 2$? Why or why not?

4. Consider an ultrarelativistic gas of bosons in d-dimensions. By mimicking what we did in class for the non-relativistic gas, find the lower critical dimension– i.e., highest dimension d_c for which N diverges as you take $\mu \to 0^-$ indicating no macroscopic condensation in the lowest single particle level. For $d > d_c$, there is condensation. Find an expression for the critical temperature $T_c(N, V)$.