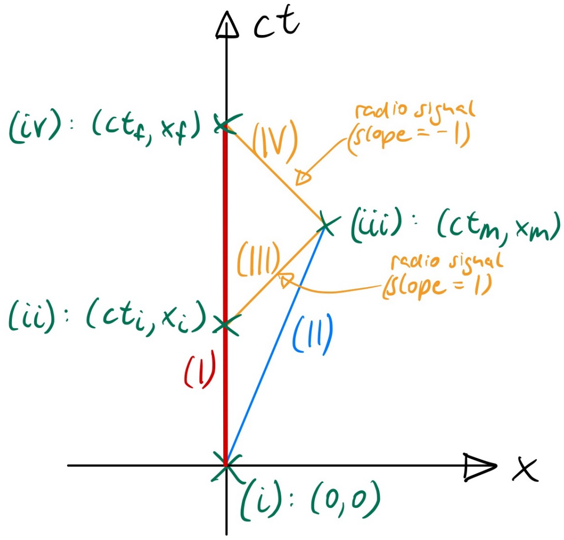
Solutions

1. **Greetings to the traveler:**

(a)



(b) We analyze everything in the Earth frame. Our friend travels with a speed

$$v = \frac{x_m}{t_m} \tag{1}$$

(worldline II). For the radio signals we have, with $x_i = x_f = 0$,

$$c = \frac{x_m}{t_m - t_i}, \qquad c = \frac{x_m}{t_f - t_m} \tag{2}$$

(worldlines III and IV). Therefore $t_m = (t_f + t_i)/2$, $x_m = c(t_f - t_i)/2$ and

$$\beta = \frac{(t_f - t_i)}{(t_f + t_i)} = \frac{1}{21} \tag{3}$$

since we know $t_i = 10h$, $t_f - t_i = 1h$, and thus $t_f = 11h$.

(c) In your friends frame, she will receive the signal at

$$t'_m = \gamma(t_m - \beta x_m/c) = \gamma(t_m - \beta v t_m/c) = \frac{t_m}{\gamma} = \frac{10.5h}{1.00114} = 10.488h. \tag{4}$$

where $\gamma = 1/\sqrt{1-\beta^2}$.

2. **Field tensor and four-potential:**

(a) Considering time-like and space-like components separately, we find

$$\begin{aligned} F^{01} &= \partial^0 A^1 - \partial^i A^0 = (1/c)\partial/\partial t\, A^1 + \partial/\partial x\, A^0 \\ &= -(1/c)(-c\partial/\partial x\, A^0 - \partial/\partial t\, A^1) = -E_x/c, \end{aligned} \tag{5}$$

$$F^{12} = \partial^1 A^2 - \partial^2 A^1 = -\partial/\partial x\, A^2 - \partial/\partial y\, A^1 = -B_z, \tag{6}$$

where we have used

$$\partial^\mu = \frac{\partial}{\partial x_\mu} = \left(\frac{\partial}{c\partial t}, -\frac{\partial}{\partial x}, -\frac{\partial}{\partial y}, -\frac{\partial}{\partial z}\right), \tag{7}$$

$$\vec{E} = -c\vec{\nabla}A^0 - \partial/\partial t\,\vec{A}, \qquad \vec{B} = \vec{\nabla}\times\vec{A}. \tag{8}$$

By anti-symmetry of $F^{\mu\nu}$ we obtain

$$F^{10} = -F^{01} = E_x/c\,, \qquad F^{21} = -F^{12} = B_z\,. \tag{9}$$

The components F^{02}, F^{03}, F^{13}, F^{23} can be derived in a similar way, and the remaining components follow from the anti-symmetry of $F^{\mu\nu}$ again. We find:

$$(F^{\mu\nu}) = \begin{pmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & -B_z & B_y \\ E_y/c & B_z & 0 & -B_x \\ E_z/c & -B_y & B_x & 0 \end{pmatrix}. \tag{10}$$

(b) The four-potential A^μ is *not unique* for given $\vec{E}$ and $\vec{B}$ fields. In fact, any four-potentials related by a *gauge transformation*

$$A^\mu(\vec{x},t) \to A^\mu(\vec{x},t) - \partial^\mu\Lambda(\vec{x},t) \tag{11}$$

for arbitrary (smooth) functions $\Lambda(\vec{x},t)$ give rise to the same electric and magnetic fields, as can be seen from

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \to \partial^\mu(A^\nu - \partial^\nu\Lambda) - \partial^\nu(A^\mu - \partial^\mu\Lambda)) = F^{\mu\nu}. \tag{12}$$

3. **Particle decay:** We set $c = 1$ during the calculation and restore an appropriate power of it for the final result by dimensional analysis. Let the Σ^+ four momentum be

$$p^\mu_\Sigma \;=\; (E_\Sigma, 0, 0, p_\Sigma) \tag{13}$$

where $p_\Sigma = 900$MeV/c. Let the pion four momentum be

$$p^\mu_\pi \;=\; (E_\pi, p_\pi\sin\theta, 0, p_\pi\cos\theta) \tag{14}$$

where $p_\pi = 200$ MeV/c and $\theta = 60°$. The four momentum of the undetected particle is obtained by conservation of four momentum:

$$p^\mu_? \;=\; p^\mu_\Sigma - p^\mu_\pi\,. \tag{15}$$

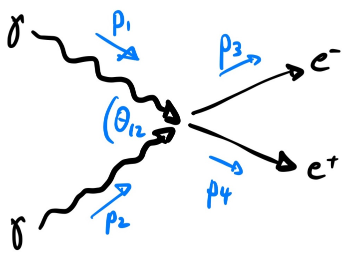
Squaring this gives

$$p^2_? \;=\; (p_\Sigma - p_\pi)^2 \tag{16}$$

$$m^2_? \;=\; m^2_\Sigma + m^2_\pi - 2(E_\Sigma E_\pi - p_\Sigma p_\pi\cos\theta). \tag{17}$$

Using $E_\Sigma = (p^2_\Sigma + m^2_\Sigma)^{1/2}$ and $E_\pi = (p^2_\pi + m^2_\pi)^{1/2}$ and plugging in the data gives $m^2_? = (940\ \mathrm{MeV/c^2})^2$. Since the particle must be neutral and the mass is close to that of a neutron, the particle must be a neutron.

4. **Electron-positron pair creation in photon-photon annihilation:**

(a) 

We set $c = 1$ for convenience. Energy-momentum conservation means

$$p_1^\mu + p_2^\mu = p_3^\mu + p_4^\mu \tag{18}$$

and the norm squared of the total four-momentum is

$$W^2 = (p_1 + p_2)^2 = (p_3 + p_4)^2 \,. \tag{19}$$

In the center-of-mass frame, W gives the total energy. The value of W will influence how fast the final state particles are, but W need to be large enough to at least allow for the production of the electron-positron pair at rest. An electron or positron at rest has four momentum $p_{3,\text{cms}}^\mu = p_{4,\text{cms}}^\mu = (m_e, 0, 0, 0)$ and we see that we need

$$W \geq 2m_e \tag{20}$$

in the center-of-mass frame and thus, since it's a Lorentz invariant quantity, in any (inertial) frame. We thus want

$$\begin{aligned} W^2 &= (p_1 + p_2)^2 = p_1^2 + p_2^2 + 2p_1 \cdot p_2 = 2(E_1 E_2 - |\vec{p}_1|\,|\vec{p}_2|\cos\theta_{12}) && (21)\\ &= 2E_1E_2(1 - \cos\theta_{12}) \geq (2m_e)^2 = 4m_e^2 \,, && (22) \end{aligned}$$

where we have used that photons are massless such that

$$p_1^2 = p_2^2 = 0, \qquad E_1 = |\vec{p}_1|, \;\; E_2 = |\vec{p}_2|. \tag{23}$$

As expected, a head-on collision with $\theta_{12} = \pi$ maximizes the available center-of-mass energy. We conclude that

$$E_2 \geq \frac{2m_e^2}{E_1(1 - \cos\theta_{12})} = \begin{cases} 2.6 \cdot 10^{15}\text{ eV} & \text{for } \theta_{12} = \pi,\; E_1 = 10^{-4}\text{eV}\,, \\ 0.26 \cdot 10^{6}\text{ eV} & \text{for } \theta_{12} = \pi,\; E_1 = 10^{6}\text{eV}\,. \end{cases} \tag{24}$$

Note that the small E_1 value corresponds roughly to the thermal energy of photon from the a relic cosmic microwave background, while the second example involves photon energies for both photons which are easily reachable at colliders.

(b) Let's consider the four-vector $(E, \vec{p}) = (E_1 + E_2, \vec{p}_1 + \vec{p}_2)$ and choose an orientation of our coordinate system such that $\vec{p}$ is in the x direction, $\vec{p} = |\vec{p}|\hat{x}$. We then require that our boost achieves

$$\begin{pmatrix} W \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} E \\ |\vec{p}| \\ 0 \\ 0 \end{pmatrix} . \tag{25}$$

From the x component we read off $0 = -\gamma\beta E + \gamma|\vec{p}|$. Using $\vec{\beta} = \beta\hat{x}$ we see that we can write this independent of a specific orientation of the coordinate system as

$$\vec{\beta} = \frac{\vec{p}}{E} \tag{26}$$

or $\vec{\beta} = (\vec{p}_1 + \vec{p}_2)/(E_1 + E_2)$ for the specific case at hand. The t component of eq. (25) gives $W = \gamma(E - \beta|\vec{p}|) = \gamma(E - |\vec{p}|^2/E)$. Identifying $E^2 - |\vec{p}|^2 = W^2$ we get

$$\gamma = \frac{E}{W}\,. \tag{27}$$