Please read all of the following before starting the exam:

- You then have 2 hours to solve the problems.
- Please stay connected to Zoom during the entire duration of your exam and switch on your camera.
- To solve the problems, you may use a simple calculator, but no computer algebra systems, external notes, books, etc.
- All problems are in S.I. units unless stated otherwise. Please give your answers in terms of the given variables and units.
- A complete answer usually includes a derivation of the result (unless stated otherwise). Show all work as neatly and logically as possible to maximize your credit. State clearly which equations were used. Circle or otherwise indicate your final answers.
- Please ask if the problem description is unclear.
- *Good luck !*

$$
\begin{aligned}
\vec{a}\times(\vec{b}\times\vec{c}) &= \vec{b}(\vec{a}\cdot\vec{c})-\vec{c}(\vec{a}\cdot\vec{b}),\\
\vec{a}\cdot(\vec{b}\times\vec{c}) &= \vec{b}\cdot(\vec{c}\times\vec{a})=\vec{c}\cdot(\vec{a}\times\vec{b}),\\
(\vec{a}\times\vec{b})\cdot(\vec{c}\times\vec{d}) &= (\vec{a}\cdot\vec{c})(\vec{b}\cdot\vec{d})-(\vec{a}\cdot\vec{d})(\vec{b}\cdot\vec{c}),\\
\vec{\nabla}\times(\vec{\nabla}\psi) &= 0,\\
\vec{\nabla}\cdot(\vec{\nabla}\times\vec{a}) &= 0,\\
\vec{\nabla}\times(\vec{\nabla}\times\vec{a}) &= \vec{\nabla}(\vec{\nabla}\cdot\vec{a})-\nabla^2\vec{a},\\
\vec{\nabla}\cdot(\psi\vec{a}) &= \vec{a}\cdot\vec{\nabla}\psi+\psi\vec{\nabla}\cdot\vec{a},\\
\vec{\nabla}\times(\psi\vec{a}) &= \vec{\nabla}\psi\times\vec{a}+\psi\vec{\nabla}\times\vec{a},\\
\vec{\nabla}(\vec{a}\cdot\vec{b}) &= (\vec{a}\cdot\vec{\nabla})\vec{b}+(\vec{b}\cdot\vec{\nabla})\vec{a}+\vec{a}\times(\vec{\nabla}\times\vec{b})+\vec{b}\times(\vec{\nabla}\times\vec{a}),\\
\vec{\nabla}\cdot(\vec{a}\times\vec{b}) &= \vec{b}\cdot(\vec{\nabla}\times\vec{a})-\vec{a}\cdot(\vec{\nabla}\times\vec{b}),\\
\vec{\nabla}\times(\vec{a}\times\vec{b}) &= \vec{a}(\vec{\nabla}\times\vec{b})-\vec{b}(\vec{\nabla}\times\vec{a})+(\vec{b}\cdot\vec{\nabla})\vec{a}-(\vec{a}\cdot\vec{\nabla})\vec{b},\\
\vec{\nabla}\cdot\vec{r} &= 3,\\
\vec{\nabla}\times\vec{r} &= 0,\\
\vec{\nabla}\cdot\hat{r} &= 2/r,\\
\vec{\nabla}\times\hat{r} &= 0,\\
\vec{\nabla}r &= \hat{r},\\
\vec{\nabla}\frac{1}{r} &= -\frac{\hat{r}}{r^2},\\
\vec{\nabla}\cdot(\hat{r}f(r)) &= \frac{2}{r}f+\frac{df}{dr},\\
(\vec{a}\cdot\vec{\nabla})\hat{r} &= \frac{1}{r}[\vec{a}-\hat{r}(\vec{a}\cdot\hat{r})]=\frac{\vec{a}_\perp}{r},\\
\vec{\nabla}^2\left(\frac{1}{r}\right) &= -4\pi\delta(\vec{r}),\\
\int_V d^3r\ \vec{\nabla}\cdot\vec{A} &= \int_S d\vec{S}\cdot\vec{A},\\
\int_V d^3r\ \vec{\nabla}\psi &= \int_S \psi\ d\vec{S},\\
\int_V d^3r\ \vec{\nabla}\times\vec{A} &= \int_S d\vec{S}\times\vec{A},\\
\int_V d^3r\ (\phi\nabla^2\psi+\vec{\nabla}\phi\cdot\vec{\nabla}\psi) &= \int_S \phi\ d\vec{S}\cdot\vec{\nabla}\psi,\\
\int_V d^3r\ (\phi\nabla^2\psi-\psi\nabla^2\phi) &= \int_S (\phi\vec{\nabla}\psi-\psi\vec{\nabla}\phi)\cdot d\vec{S},\\
\int_S (\vec{\nabla}\times\vec{A})\cdot d\vec{S} &= \oint d\vec{\ell}\cdot\vec{A},\\
\int_S d\vec{S}\times\vec{\nabla}\psi &= \oint_C d\vec{\ell}\ \psi.
\end{aligned}
$$

$$
\begin{aligned}
\mathscr{L} &= -\frac{1}{4\mu_0} F_{\mu\nu} F^{\mu\nu} - J_\mu A^\mu \\
L &= \frac{1}{\gamma}(-mc^2 - q A_\mu u^\mu) \\
\frac{dp^\mu}{d\tau} &= q F^{\mu\nu} u_\nu \\
\partial_\mu F^{\mu\nu} &= \mu_0 J^\nu \\
\partial_\mu \tilde{F}^{\mu\nu} &= 0 \\
\partial_\mu J^\mu &= 0 \\
(g_{\mu\nu}) &= \mathrm{diag}(1, -1, -1, -1), \\
\vec{\beta} &= \vec{v}/c, \\
\gamma &= 1/\sqrt{1-\beta^2}, \\
x'^\mu &= \Lambda^\mu{}_\nu x^\nu \\
(\Lambda^\mu{}_\nu) &= \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},
\end{aligned}
$$

$$
\begin{aligned}
x^\mu &= (ct, x, y, z), \\
\partial^\mu &= ((1/c)\partial/\partial t, -\vec{\nabla}) \\
k^\mu &= (\omega/c, \vec{k}), \\
u^\mu &= (\gamma c, \gamma \vec{v}), \\
p^\mu &= (E/c, \vec{p}), \\
A^\mu &= (\phi/c, \vec{A}), \\
J^\mu &= (c\rho, \vec{j}), \\
F^{\mu\nu} &= \partial^\mu A^\nu - \partial^\nu A^\mu, \\
\tilde{F}^{\mu\nu} &= \frac{1}{2}\epsilon^{\mu\nu\alpha\beta} F_{\alpha\beta} \\
\vec{E}'_\parallel &= \vec{E}_\parallel \\
\vec{B}'_\parallel &= \vec{B}_\parallel \\
\vec{E}'_\perp &= \gamma(\vec{E}_\perp + c\vec{\beta} \times \vec{B}) \\
\vec{B}'_\perp &= \gamma(\vec{B}_\perp - \frac{1}{c}\vec{\beta} \times \vec{E})
\end{aligned}
$$

$$
\begin{aligned}
\vec{\nabla} \cdot \vec{E} &= \frac{1}{\epsilon_0}\rho, & \vec{\nabla} \cdot \vec{B} &= 0, \\
\vec{\nabla} \times \vec{B} - \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t} &= \mu_0 \vec{j}, & \vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} &= 0,
\end{aligned}
$$

$$
\begin{aligned}
\frac{d\vec{p}}{dt} &= q(\vec{E} + \vec{v} \times \vec{B}) \\
\vec{E} &= -\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t} \\
\vec{B} &= \vec{\nabla} \times \vec{A}
\end{aligned}
$$

$$
\begin{aligned}
(\vec{E}_{\text{out}} - \vec{E}_{\text{in}}) \cdot \hat{n} &= \sigma/\epsilon_0 \\
\phi(\vec{r}) &= \frac{1}{4\pi\epsilon_0}\int_V d^3r'\, \rho(\vec{r}')G(\vec{r}, \vec{r}') - \frac{1}{4\pi}\int_S dA' \phi(\vec{r}')\frac{\partial G(\vec{r}, \vec{r}')}{\partial n'} \\
\phi(\vec{r}) &= \frac{1}{4\pi\epsilon_0}\int_V d^3r'\, \rho(\vec{r}')G(\vec{r}, \vec{r}') + \frac{1}{4\pi}\int_S dA' \frac{\partial \phi(\vec{r}')}{\partial n'} G(\vec{r}, \vec{r}') + \langle\phi\rangle_S \\
\phi &= \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left[A_{lm} r^l + B_{lm}/r^{l+1}\right] Y_{lm}(\theta, \varphi) \\
\phi &= a_0 + b_0 \ln s + \sum_{n=1}^{\infty}\left[s^n(a_n \cos(n\varphi) + b_n \sin(n\varphi)) + s^{-n}(c_n \cos(n\varphi) + d_n \sin(n\varphi))\right]
\end{aligned}
$$

$$P_0(\cos\theta) = 1$$
$$P_1(\cos\theta) = \cos\theta$$
$$P_2(\cos\theta) = \frac{1}{2}(3\cos^2\theta - 1)$$
$$Y_{0,0} = \frac{1}{\sqrt{4\pi}}$$
$$Y_{1,0} = \sqrt{\frac{3}{4\pi}}\cos\theta, \quad Y_{1,\pm1} = \mp\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{\pm i\varphi}$$
$$Y_{2,0} = \sqrt{\frac{5}{16\pi}}\left(3\cos^2\theta - 1\right), \quad Y_{2,\pm1} = \mp\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\, e^{\pm i\varphi}, \quad Y_{2,\pm2} = \sqrt{\frac{15}{32\pi}}\sin^2\theta\, e^{\pm 2i\varphi}$$

$$P_l^m(x) = \frac{(-1)^m}{2^l l!}(1-x^2)^{m/2}\frac{d^{l+m}}{dx^{l+m}}(x^2-1)^l$$
$$Y_{lm}(\theta,\varphi) = \sqrt{\frac{(2l+1)(l-m)!}{4\pi(l+m)!}}e^{im\varphi}P_l^m(\cos\theta)$$
$$\frac{1}{|\vec{r}-\vec{r}'|} = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left(\frac{4\pi}{2l+1}\right)\frac{r_<^l}{r_>^{l+1}}Y_{lm}^*(\theta',\varphi')Y_{lm}(\theta,\varphi) = \sum_{l=0}^{\infty}\frac{r_<^l}{r_>^{l+1}}P_l(\cos\gamma)$$
$$\phi = \sum_{l=0}^{\infty}\left(a_l r^l + b_l/r^{l+1}\right)P_l(\cos\theta)$$
$$\phi = \frac{1}{4\pi\epsilon_0}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left(\frac{4\pi}{2l+1}\right)\frac{q_{lm}}{r^{l+1}}Y_{lm}(\theta,\varphi)$$
$$q_{lm} = \int d^3r'\,\rho(\vec{r}')r'^l Y_{lm}^*(\theta,\varphi)$$

$$\phi = \frac{1}{4\pi\epsilon_0}\left(\frac{Q_{\rm tot}}{r} + \frac{p_i\hat{r}_i}{r^2} + \frac{1}{2!}Q_{ij}\frac{\hat{r}_i\hat{r}_j}{r^3} + \dots\right)$$
$$p_i = \int d^3\,r'\rho(\vec{r}')\,r'_i$$
$$Q_{ij} = \int d^3r'\,\rho(\vec{r}')\,(3r'_i r'_j - \delta_{ij}r'^2)$$
$$U = Q_{\rm tot}\phi(\vec{r}) - p_i E_i(\vec{r}) - \frac{1}{6}Q_{ij}\partial_i E_j + \dots$$

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\left(\frac{\vec{m}\times\vec{r}}{r^3} + \dots\right)$$
$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\left(\frac{3(\vec{m}\cdot\hat{r})\hat{r} - \vec{m}}{r^3} + \dots\right)$$
$$\vec{m} = \frac{1}{2}\int d^3r'\,\vec{r}'\times\vec{j}(\vec{r}')$$
$$U = -m_i B_i(\vec{r}) + \dots$$
$$\vec{\tau} = \vec{m}\times\vec{B} + \vec{r}\times\vec{F} + \dots$$

Spherical Multipole Moments

Consider three point charges: one charge q at position $(x, y, z) = (0, 0, -a)$, one charge q at position $(0, 0, a)$ and one charge $-2q$ at position $(0, 0, 0)$ where $a > 0$. The potential of these charges is observed at a point $\vec{r}$ far away from the origin, $r = |\vec{r}| \gg a$.

1. (10 pts) Write the leading term of the potential in the $1/r$ expansion using spherical harmonics (symbolically) and explain (using Cartesian or spherical coordinates) why lower terms vanish.

2. (15 pts) Determine the components of the leading *spherical* multipole moment.

Cartesian Boundary Value Problem

Consider an electrostatic problem, where the potential is given on two half-planes. On the xz-halfplane with $x \geq 0$ the potential is $\phi(x, 0, z) = 0$, on the yz-halfplane with $y \geq 0$ the potential is $\phi(0, y, z) = \phi_0(y, z)$. You may assume that the potential is zero far away from the origin. The goal is to determine the potential in the region $x > 0$, $y > 0$.

3. (15 pts) Find an appropriate Green's function and describe all the required properties it fulfills.

4. (10 pts) Determine a solution for the potential in the region $x > 0$, $y > 0$ using the function ϕ_0. Try to be as explicit as possible.

Long Hollow Cylinder

A very long, hollow, cylindrical shell of radius R is centered on the z-axis and has a surface charge density that is uniform along its length z, but depends on the azimuthal angle φ by the formula

$$\sigma(\varphi, z) = \sigma_0 \cos^2 \varphi = \frac{\sigma_0}{2}(1 + \cos(2\varphi)) \, . \tag{1}$$

5. (25 pts) Obtain the electrostatic potential $\phi(\vec{r})$ inside and outside the cylindrical shell.

Rotating Discs

A total charge $Q > 0$ is uniformly distributed over a thin disc of radius a. The disc is rotating around its symmetry axis with constant angular velocity. Another, identical disc is placed at a large distance b between the centers ($b \gg a$), such that the symmetry axes of the two discs are parallel, but the discs rotate in opposite directions. The centers of the discs are at $\vec{r}_1 = (0, 0, 0)$ and $\vec{r}_2 = (b, 0, 0)$, respectively, and the angular velocities are $\vec{\omega}_1 = \omega\hat{z}$ and $\vec{\omega}_2 = -\omega\hat{z}$, respectively (with $\omega > 0$).

6. (5 pts) Calculate the surface charge density σ of one of the discs in terms of Q and a.

7. (10 pts) Calculate the magnitude of the magnetic moment of one of the discs, $m = |\vec{m}|$, in terms of σ, ω and a.

8. (10 pts) Calculate the interaction energy in terms of m and b.

Solutions of Laplace Equation

9. [Bonus: 5 pts] Consider a general product ansatz to solve the Laplace equation in spherical coordinates. For the azimuthal dependence one obtains solutions $e^{\pm im\varphi}$, where m is a complex number introduced as a separation constant. Discuss if (or under which circumstances) there are any restrictions for the values of m and explain the relevance for electrostatic problems.