Homework 4

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4.1 Cartesian Multipole Moments of a Charged Ring

a

The dipole moment of a charge distribution is given by

$$\mathbf{p} = \int \rho(\mathbf{r})\mathbf{r}\,\mathrm{d}r\,. \tag{4.1.1}$$

The given density is

$$\rho(\mathbf{r}) = \begin{cases} \lambda_0\delta(r-R)\delta(z) & 0 \le \phi < \pi \\ -\lambda_0\delta(r-R)\delta(z) & \pi \le \phi < 2\pi \end{cases}.$$

Plugging this into equation 4.1.1 gives

$$\begin{aligned} \mathbf{p} &= \int_0^\pi R^2 \begin{pmatrix}\cos\phi\\ \sin\phi\\ 0\end{pmatrix} \lambda_0\,\mathrm{d}\phi - \int_\pi^{2\pi} R^2 \begin{pmatrix}\cos\phi\\ \sin\phi\\ 0\end{pmatrix} \lambda_0\,\mathrm{d}\phi\,, \\ &= R^2\lambda_0 \begin{pmatrix}0\\4\\0\end{pmatrix} \end{aligned}$$

b

The components of the quadrupole tensor are

$$Q_{ij} = Q_{ji} = \int \rho(\mathbf{r})(3r_ir_j - \mathbf{r}\cdot\mathbf{r}\delta_{ij})\mathrm{d}^3r. \tag{4.1.2}$$

Since there is no charge density outside of the xy-plane, any component with a z must vanish. That just leaves Q_{xx}, Q_{yy}, and Q_{xy}:

$$\begin{aligned} Q_{xx} &= \int \rho(\mathbf{r})(2x^2 - y^2)\,\mathrm{d}^3r\,, \\ &= \lambda_0R^2\left[\int_0^\pi 2\cos^2\phi - \sin^2\phi\,\mathrm{d}\phi \right.\\ &\qquad\left. - \int_\pi^{2\pi} 2\cos^2\phi - \sin^2\phi\,\mathrm{d}\phi\right], \\ &= 0. \\ Q_{yy} &= \int \rho(\mathbf{r})(2y^2 - x^2)\,\mathrm{d}^3r\,, \\ &= 0. \qquad \text{(similar to } Q_{xx}\text{)} \\ Q_{xy} &= \int \rho(\mathbf{r})(3xy)\,\mathrm{d}^3r\,, \\ &= 3\lambda_0R^3\left[\int_0^\pi \cos\phi\sin\phi\,\mathrm{d}\phi - \int_\pi^{2\pi}\cos\phi\sin\phi\,\mathrm{d}\phi\right], \\ &= 0. \end{aligned}$$

Thus

$$Q_{ij} = 0.$$

c

Since there is no net charge, these multipole moments are independent of the choice of origin.

d

The torque on the dipole, $\mathbf{p}_2$ in an electric field is

$$\boldsymbol{\tau} = \mathbf{p}_2 \times \mathbf{E}.$$

In the case of this problem, the electric field produced can be approximated by a dipole, so

$$\mathbf{E} = \frac{3\hat{\mathbf{r}}(\mathbf{p}\cdot\hat{\mathbf{r}}) - \mathbf{p}}{4\pi\epsilon_0 r^3}.$$

Furthermore, $\mathbf{p}_2$ is on the z-axis, so $\mathbf{r} = z\hat{\mathbf{z}}$:

$$\begin{aligned} \boldsymbol{\tau} &= \frac{1}{4\pi\epsilon_0 z^3}\left[3\mathbf{p}_2 \times \hat{\mathbf{z}}(\mathbf{p}\cdot\hat{\mathbf{z}}) - \mathbf{p}_2 \times \mathbf{p}\right], \\ &= \frac{-\mathbf{p}_2 \times \mathbf{p}}{4\pi\epsilon_0 z^3}. \end{aligned}$$

4.2 Summary Of Course Topics

4.2.1 Lorentz Transformations

Light moves at the same speed in all reference frames, which has some weird implications for how length and time work in the universe. Rather than a Galilean transformation between reference frames, which is the classical limit, one needs to use a Lorentz transformation to get the most accurate transition to another inertial reference frame.

4.2.2 Consequences of Lorentz Transformations

Some consequences of these transformations are length contraction and time dilation. Objects that are moving relative to an inertial reference frame have their length in the direction of that motion contracted or shortened. Additionally, that motion causes those clocks in motion to tick slower. What feels like one second to the rest frame, feels like less than one second to any frames in motion, relative to the rest frame. Depending on the situation, two events can be seen as having very different locations and times.

4.2.3 Minkowski Space

Minkowski space is a way of visualizing and thinking about spacetime. A Minkowski diagram is a graph where one axis represents time, and the other axes represent the spatial coordinates we're familiar with. The time axis is understood to be the rest frame of the diagram, and events are placed as points in spacetime. Different inertial reference frames can be represented as straight lines, and their slopes represent their relative velocity to the rest frame. In all frames, light travels at the same speed, c.

4.2.4 Relativistic Kinematics

Let

$$\gamma = \frac{1}{\sqrt{1-\beta^2}}, \qquad \beta = \frac{v}{c}.$$

The following changes are made for relativistic kinematics:

$$\begin{aligned}
\mathrm{d}t \rightarrow \mathrm{d}\tau &= \frac{\mathrm{d}t}{\gamma}, \quad \text{(proper time)} \\
\mathbf{v} = \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} \rightarrow \mathbf{u} &= \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}\tau} = \gamma_v \mathbf{v}, \\
\mathbf{p} &\rightarrow \gamma m \mathbf{v}, \\
T = \frac{1}{2}mv^2 \rightarrow T &= E_{\text{tot}} - mc^2, \qquad E_{\text{tot}} = \gamma mc^2.
\end{aligned}$$

Everything else follows from these, such as the famous

$$E = \sqrt{p^2c^2 + m^2c^4}.$$

4.2.5 Poincare Group, Lorentz Group, Tensors

Define

$$a_\mu b^\mu \equiv a_0 b_0 - \sum_{n=1}^{3} a_n b_n.$$

Then

$$s^2 = s_\mu s^\mu = x_0^2 - \sum_{n=1}^{3} x_n^2.$$

The group of all transformations that leave this quantity invariant is called the *homogeneous Lorentz group* or just the *Lorentz group*, and it contains ordinary rotations and Lorentz transformations. Additionally, we have

$$(s_x - s_y)^2 = (x_0 - y_0)^2 - \sum_{n=1}^{3}(x_n - y_n)^2.$$

The group of transformations which leaves this quantity invariant is called the *Poincaré group*, and it contains translations and reflections of both space and time, as well as all transformations of the homogeneous Lorentz group.

Tensors are a mathematical construct, and they help with doing the physics here. The details will be omitted, and the important details will just be defined here:

$$\begin{aligned}
g_{\alpha\beta} = g^{\alpha\beta} &= \begin{cases} 1 & \alpha = \beta = 0 \\ -1 & \alpha = \beta = 1,2,3 \\ 0 & \alpha \neq \beta \end{cases}, \\
g_{\alpha\beta} g^{\alpha\beta} = \delta_\alpha^\beta &= \begin{cases} 1 & \alpha = \beta = 0,1,2,3 \\ 0 & \alpha \neq \beta \end{cases}, \\
F^{..\alpha..}_{..} &= g^{\alpha\beta} F^{....}_{..\beta}, \\
G^{...}_{..\alpha.} &= g_{\alpha\beta} G^{..\beta}_{....}, \\
A^\alpha = (A^0, \mathbf{A}), &\qquad A_\alpha = (A^0, -\mathbf{A}), \\
A_\alpha B^\alpha &= A^0 B^0 - \mathbf{A} \cdot \mathbf{B}, \\
\partial^\alpha \equiv \frac{\partial}{\partial x_\alpha} &= \left(\frac{\partial}{\partial x^0}, -\boldsymbol{\nabla} \right), \\
\partial_\alpha \equiv \frac{\partial}{\partial x^\alpha} &= \left(\frac{\partial}{\partial x^0}, \boldsymbol{\nabla} \right).
\end{aligned}$$