Solutions

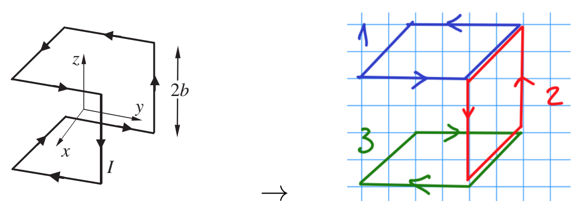
1. **A current loop:**

   For a “flat” current loop in a plane we can calculate the magnetic dipole moment as follows:

$$\vec{m} = \frac{1}{2}\int d^3r'\vec{r}' \times \vec{j} = I\int \underbrace{\frac{1}{2}\vec{r}' \times \vec{dl}}_{dA\,\hat{n}} = IA\,\hat{n}\,, \tag{1}$$

   where $\hat{n}$ is the normal vector of the plane with its direction determined by the right-hand rule.

   The current loop given in the problem is not in a plane. However, we can connect vertices with wires that have 0 current, and then consider the 0 current as a sum of two currents going in opposite directions.



   Each current loop is flat and its magnetic dipole moment can bet calculated with eq. (1). Thus we have

$$\vec{m}_1 = I(2b)^2\hat{z}, \tag{2}$$
$$\vec{m}_2 = I(2b)^2\hat{y}, \tag{3}$$
$$\vec{m}_3 = I(2b)^2(-\hat{z}), \tag{4}$$
$$\vec{m} = \vec{m}_1 + \vec{m}_3 + \vec{m}_3 = 4b^2I\hat{y}\,. \tag{5}$$

2. **Rotating square loop:**

   (a) In this problem, the current density is time dependent and therefore does not immediately permit a treatment as in magnetostatics. We are interested in the static field approximation, where we average the currents and fields over a large time. Due to the linearity of our expressions in $\vec{j}$ we may perform this average right away or after we computed the (in general time-dependent) contribution to the derived quantitites.

   A detailed discussion for rotating charges around a fixed axis was given in the solutions to homework 10 in terms of cylindrical coordinates. Here, we take a slightly sloppy approach in Cartesian coordinates. For a specific point in time, place the square loop in the xz plane, and place the side about which the loop is rotating on the z axis. We can consider separately the contribution of each of the four sides of the loop to the magnetic moment. The side around which the loop rotates has $\vec{j} = 0$, so it doesn't contribute. The far side of the loop generates an average current density on the side of a cylinder as follows:

$$\vec{j}\, d^3r' \;=\; \lambda \vec{v}\, dz' \;=\; \lambda(\hat{y} a\omega)\, dz' \,.$$

The only contribution to the magnetic moment which doesn't average to zero is in the $\hat{z}$ direction:

$$\begin{aligned} \hat{z} \cdot \vec{m}_1 &= \frac{1}{2} \int \hat{z} \cdot \left(\vec{r'} \times \vec{j}(\vec{r'}) \right) d^3r' \\ &= \frac{1}{2} \int_0^a a(\lambda a \omega)\, dz' \\ &= \frac{\lambda a^3 \omega}{2} \,. \end{aligned}$$

The two horizontal sides contribute equal amounts. The current element for the horizontal sides is given by

$$\vec{j}\, d^3r' \;=\; \lambda \vec{v}\, dx' \;=\; \lambda(\hat{y} x' \omega)\, dx' \,.$$

Each of the horizontal sides contribute

$$\begin{aligned} \hat{z} \cdot \vec{m}_2 &= \frac{1}{2} \int \hat{z} \cdot \left(\vec{r'} \times \vec{j}(\vec{r'}) \right) d^3r' \\ &= \frac{1}{2} \int_0^a x'(\lambda x' \omega)\, dx' \\ &= \frac{\lambda a^3 \omega}{6} \,. \end{aligned}$$

The total magnetic moment averaged over time is then

$$\begin{aligned} \vec{m} &= \vec{m}_1 + 2\vec{m}_2 \\ &= \hat{z} \frac{5\lambda a^3 \omega}{6} \,. \end{aligned}$$

Alternatively, one can determine the total angular momentum for the rotating rods as in classical mechanics and derive the magnetic moment from that.

3. **Non-relativistic particle in magnetic field:**

(a) The equation of motion is obtained from the Lorentz force with $\vec{E} = 0$ and $\vec{B} = B_0 \hat{z}$:

$$\frac{d\vec{p}}{dt} = q\vec{v} \times \vec{B} \tag{6}$$

$$\dot{\vec{v}} = \omega\, \vec{v} \times \hat{z} \qquad \text{with} \;\; \omega \equiv \frac{qB_0}{m}. \tag{7}$$

In components we have

$$\dot{v}_x = \omega v_y, \quad \dot{v}_y = -\omega v_x, \quad \dot{v}_z = 0 \tag{8}$$

Therefore we have a constant velocity in z direction, $v_z(t) = v_{z,0}$. To determine the motion in the xy plane we could take another derivative with respect to time to decouple the system of differential equations. Note that the structure of the differential equation for $\vec{v}$ resembles the one we encountered for $\vec{m}$ when we discussed Larmor precession of magnetic moments in our class meeting, and it was exactly this decoupling strategy that we used there (see also the notes on D2L).

Alternatively, we can use the following trick to solve this type of differential equation. We see that

$$\frac{d}{dt}(v_x + iv_y) = -i\omega(v_x + iv_y) \tag{9}$$

and thus

$$v_x + iv_y = a\, e^{-i\omega t} \tag{10}$$

with constant (complex) amplitude a. We have therefore

$$v_x(t) = v_0 \cos(\omega t + \phi_0) \tag{11}$$

$$v_y(t) = -v_0 \sin(\omega t + \phi_0) \tag{12}$$

and, after an integration over time,

$$x(t) = x_0 + R \sin(\omega t + \phi_0) \tag{13}$$

$$y(t) = x_0 + R \cos(\omega t + \phi_0) \tag{14}$$

$$z(t) = z_0 + v_{z,0} t \tag{15}$$

where $R \equiv v_0/\omega$. The trajectory is thus a rotation in the xy plane with radius R, cyclotron frequency ω and constant velocity v_0, plus a motion with constant velocity in z direction, the result of which is a helix. The helix will be left-handed or right-handed depending on the sign of q. The kinetic energy is

$$\mathcal{E} = \frac{1}{2}m|\vec{v}|^2 = \frac{1}{2}m\left(v_0^2 + v_{z,0}^2\right) \tag{16}$$

and thus constant in time, which is expected since the force due to a magnetic field is perpendicular to the motion. In general one would expect the charge to lose energy since it is accelerated. As a consequence, its velocity and radius of rotation would decrease over time such that the charge spirals in.