Problems

1. **Potential of point charge and conducting sphere:** Consider a grounded conducting sphere of radius R (centered at the origin) in presence of a point charge q located outside of the sphere (at position $(0, 0, a)$ with $a > R$).

   (a) (5 pts) Write the potential as a sum of two terms: (i) the potential of the point charge and (ii) a general solution of the azimuthally symmetric Laplace equation (using Legendre polynomials).

   (b) (10 pts) Determine the unknown coefficients in term (ii) using suitable boundary conditions.

   (c) (10 pts) In this example, the term (ii) can be rewritten in a closed and suggestive form. Please perform this resummation and interpret your result in terms of (image) point charges.

   (d) (10 pts) Derive the surface charge density on the sphere in terms of q, R, a and $x = \cos\theta$. [Optional: discuss the limits $a \to R$ and $a \to \infty$.]

   (e) (10 pts) Calculate the induced charge.

   (f) (10 pts) Plot (or draw qualitatively) the following quantities in dependence of the distance from the center of the sphere (along a line from $(0, 0, 0)$ to $(0, 0, a)$): term (i), term (ii), the sum of (i) and (ii), the actual potential.

   (g) (optional) How is the potential outside of the sphere modified if the sphere is held at a fixed potential ?

2. **Green's function for sphere:** Consider a sphere of radius R. On its surface, the electrostatic potential is prescribed by $\phi(\vec{r})|_{r=R} = V(\theta, \varphi)$ with

   $$V(\theta, \varphi) = V_0 \cos(3\varphi) \sin^3(\theta). \tag{1}$$

   Determine the potential *outside* of the sphere.

   (a) (10 pts) Draw a figure of the sphere. Indicate the volume V you consider for Green's method, *all* elements of the complete surface S surrounding V, and indicate normal vectors $\hat{n}'$ for S.

   (b) (10 pts) Write down a suitable Green's function $G(\vec{r}, \vec{r}')$ and state all required properties of $G(\vec{r}, \vec{r}')$ (indicate for which region each property holds).

   (c) (15 pts) Derive a simple equation for $\phi(\vec{r})$ outside of the sphere in terms of $V(\theta, \varphi)$ and spherical harmonics.

   (d) (10 pts) Evaluate the expression you obtained in the previous step for the explicit expression of $V(\theta, \varphi)$ given above.