Physics 841 - Homework 1 due Fri, Jan 29, 10am

Solutions

1. **Charged kaon decay:**

   (a) From the particle data group listing we find the mean lifetime of the K^+ to be

   $$\tau = 1.238 \cdot 10^{-8} \text{ s.} \tag{1}$$

   Note that this is the life expectancy of the kaon *in its rest frame*. In the laboratory frame this time interval appears to be longer:

   $$\tau_{\text{lab}} = \gamma \cdot \tau = 8.78 \cdot 10^{-8} \text{ s} \tag{2}$$

   since $\gamma = 1/\sqrt{1-\beta^2} \approx 7.1$ for $\beta = 0.99$. The distance traveled during this time is

   $$x = \beta c \tau_{\text{lab}} = 26 \text{ m.} \tag{3}$$

2. **Alarm clock:**

   (a) Let the position of your alarm clock be the origin of the coordinate system, with the x axis pointing toward U of M. Also let the time for your alarm clock ringing be $t = 0$. Then event 1 (your alarm clock ringing) would occur at $(ct_1, x_1) = (0, 0)$ and event 2 (UM alarm clock ringing) would occur at $(ct_2, x_2) = (c \cdot 10^{-5}$ s$, 10^5$ m$) = (3 \cdot 10^3, 10^5)$ m. (We suppress the irrelevant coordinates y and z here.) In a Lorentz frame moving with speed v in the x direction relative to yours, the two events would occur at

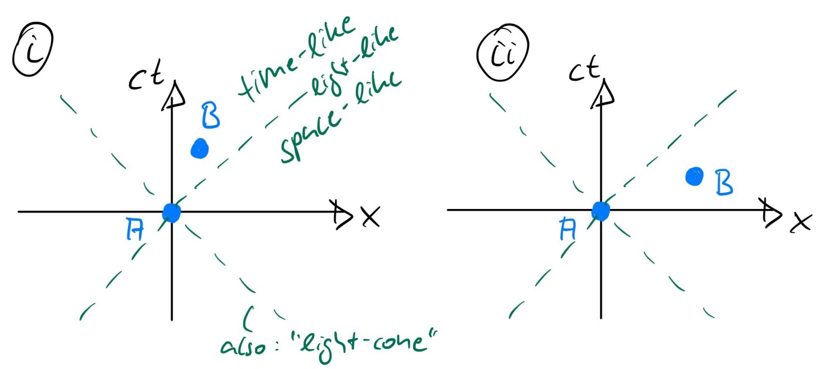
   $$(ct'_1, x'_1) = (0, 0), \tag{4}$$
   $$(ct'_2, x'_2) = \Big(\gamma(ct_2 - \beta x_2), \gamma(x_2 - \beta ct_2)\Big), \tag{5}$$

   where $\beta = v/c$ and $\gamma = (1-\beta^2)^{-1/2}$. In order to have $t'_2 < t'_1$, we need

   $$ct_2 - \beta x_2 < 0 \tag{6}$$

   or $v > (ct_2/x_2)c = 0.03c$. Thus, the UM alarm clock would ring first in any frame moving in the direction from East Lansing towards Ann Arbor at a speed $v > 0.3c$.

   (b) The two situations can be displayed as follows.



(c) From the arguments in part (a), it is only possible to find a frame in which $t'_2 < t'_1$ if $ct_2/x_2 < 1$ or

$$t_2 < x_2/c \approx 3.34 \times 10^{-4}\text{s}. \tag{7}$$

In other words, the time difference must be less than the time for light to travel the 100 km. Equivalently, the interval between the two events must be space-like or $s_{12}^2 = c^2(t_2 - t_1)^2 - (x_1 - x_2)^2 < 0$.

3. **Colliding rockets:**

(a) We can obtain the coordinates of B in A's frame (primed) by a Lorentz transformation from the Earth frame (unprimed):

$$\Delta x' = \gamma(\Delta x - v_A \Delta t) \tag{8}$$

$$\Delta t' = \gamma\left(\Delta t - \frac{v_A}{c^2}\Delta x\right) \tag{9}$$

Using these equations, the correct signs for the different velocities and $|v_B| = |\Delta x/\Delta t|$ in the Earth frame we obtain

$$v'_B = \frac{\Delta x'}{\Delta t'} = \frac{-|v_B| - |v_A|}{1 + |v_A||v_B|/c^2} = -0.95c. \tag{10}$$

The same result but with opposite sign holds for the velocity of A in B's frame.

(b) In the Earth frame it takes

$$\Delta t = \frac{l}{v_A + v_B} = 1\ s \tag{11}$$

until the collision (if Δx_A is the distance from a to the collision point we have $\Delta x_A = v_A t$, $l - \Delta x_A = v_B t$, and thus (**??**)). The clock in frame A goes slower and that time interval is observed as

$$\Delta t_A = \frac{\Delta t}{\gamma_A} = \sqrt{1 - v_A^2/c^2} = 0.6\ s. \tag{12}$$

Similarly, the time interval seen in B's frame is

$$\Delta t_B = \frac{\Delta t}{\gamma_B} = \sqrt{1 - v_B^2/c^2} = 0.8\ s. \tag{13}$$

Note that also the initial distance to the other rocket will look different in the frames of A and B, since the starting point is defined with respect to a measurement in the Earth frame (which breaks the symmetry).

4. **Explicit boost:**

(a) The problem could be solved by rotating the coordinate system to align the x axis with the direction of the boost. A simpler method is as follows. It is only the component of the position vector $\vec{x}$ *in the direction* of the boost, which requires a non-trivial transformation. We therefore decompose the position vector as follows:

$$\vec{x} = \vec{x}_\parallel + \vec{x}_\perp \tag{14}$$

Starting from $\vec{x}$ and $\vec{v}$, the components are determined by

$$\vec{x}_\parallel = x_\parallel \hat{v} \qquad \text{with } x_\parallel = \vec{x} \cdot \hat{v} \text{ and } \hat{v} = \vec{v}/|\vec{v}|, \tag{15}$$
$$\vec{x}_\perp = \vec{x} - \vec{x}_\parallel. \tag{16}$$

Plugging in numbers we have

$$x_\parallel \approx (2.23, 2.79, 3.35)\ m, \tag{17}$$
$$x_\perp \approx (1.77, 0.21, -1.35)\ m \tag{18}$$

We can obtain the vector in the boosted frame by transformation both terms in (**??**) separately, since Lorentz transformations are linear:

$$\vec{x}' = \vec{x}'_\parallel + \vec{x}'_\perp. \tag{19}$$

We use the Lorentz transformation for a boost with $\beta = |\vec{v}|/c = 0.88$ in x direction and replace x with $x_\parallel$ and the y and z components by $\vec{x}_\perp$:

$$ct' = \gamma(ct - \beta x_\parallel) \qquad \approx 2.09(5 - 0.88 \cdot 4.90)\ m \qquad \approx 1.46\ m, \tag{20}$$
$$x'_\parallel = \gamma(x_\parallel - \beta ct) \qquad \approx 2.09(4.90 - 0.88 \cdot 5)\ m \qquad \approx 1.07\ m, \tag{21}$$
$$\vec{x}'_\perp = \vec{x}_\perp \tag{22}$$

For the 3-vectors we have

$$\vec{x}'_\parallel = x'_\parallel \hat{v} \qquad \approx (0.49, 0.61, 0.73)\ m, \tag{23}$$
$$\vec{x}' = \vec{x}'_\parallel + \vec{x}_\perp \qquad \approx (2.25, 0.82, -0.62)\ m \tag{24}$$

and thus for the four-vector in the boosted frame:

$$x'^\mu \approx (2.50, 1.24, -0.20, -1.64)\ m. \tag{25}$$

(b) The rapidity ζ is given by

$$\zeta = \operatorname{arctanh} |\beta| \approx 1.36\,. \tag{26}$$

(c) We consider the Lorentz invariant quantity

$$s^2 = x_\mu x^\mu = (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2 = -4\ \mathrm{m}^2 < 0. \tag{27}$$

With our metric convention, a positive value means that *the four-vector x is space-like*. From the fact that s^2 is Lorentz invariant it is immediately clear that also x'^μ is space-like. Events which differ by a space-like vector are not causally connected.