Homework 5

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5.1 A Simple Poisson Equation

a

The Laplace operator, in cartesian coordinates, is given by

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}.$$

One can use a change of variables to find how the Laplace operator acts on a function that only depends on the distance from the origin, $f(r)$. The distance from the origin is $r = \sqrt{x^2+y^2+z^2}$. Taking the second derivative with respect to each of the cartesian coordinates is the same process:

$$\begin{aligned}
\frac{\partial^2}{\partial x^2} &= \frac{\partial}{\partial x}\left(\frac{\partial r}{\partial x}\frac{\partial}{\partial r}\right), \\
&= \frac{\partial^2 r}{\partial x^2}\frac{\partial}{\partial r} + \left(\frac{\partial r}{\partial x}\right)^2 \frac{\partial^2}{\partial r^2}, \\
&= \left(\frac{1}{r} - \frac{x^2}{r^3}\right)\frac{\partial}{\partial r} + \frac{x^2}{r^2}\frac{\partial^2}{\partial r^2}, \\
&= \frac{1}{r}\left(\left(1 - \frac{x^2}{r^2}\right)\frac{\partial}{\partial r} + \frac{x^2}{r}\frac{\partial^2}{\partial r^2}\right)
\end{aligned}$$

Since the processes is the same for each variable,

$$\begin{aligned}
\nabla^2 &= \frac{1}{r}\left(\left(3 - \frac{r^2}{r^2}\right)\frac{\partial}{\partial r} + \frac{r^2}{r}\frac{\partial^2}{\partial r^2}\right), \\
&= \frac{1}{r}\left(2\frac{\partial}{\partial r} + r\frac{\partial^2}{\partial r^2}\right), \\
\nabla^2 f(r) &= \frac{1}{r}\frac{\partial^2}{\partial r^2}\big(rf(r)\big).
\end{aligned}$$

b

The possion equation for this charge distribution is given by

$$\nabla^2\phi(\mathbf{r}) = \frac{\rho_0}{\epsilon_0}e^{-\mu r}. \tag{5.1.1}$$

Since the charge distribution is radially symmetric, the potential must also be radially symmetric, since rotating the system in any way doesn't change the problem. From part a, 5.1.1 becomes

$$\frac{1}{r}\frac{\partial^2}{\partial r^2}\big(r\phi(r)\big) = \frac{\rho_0}{\epsilon_0}e^{-\mu r}.$$

One can easily find the solution to this through integration[1]:

$$\begin{aligned}
\frac{\partial}{\partial r}\big(r\phi(r)\big) &= \frac{\rho_0}{\epsilon_0}\int re^{-\mu r}\,\mathrm{d}r\,, \\
&= c_0 - \frac{e^{-\mu r}(\mu r + 1)}{\mu^2}, \\
r\phi(r) &= \int c_0 - \frac{e^{-\mu r}(\mu r+1)}{\mu^2}\,\mathrm{d}r\,, \\
&= c_1 + c_0 r + \frac{\rho_0 e^{-\mu r}\left(\frac{2}{\mu} + r\right)}{\mu^2\epsilon_0}, \\
\phi(r) &= c_0 + \frac{c_1}{r} + \frac{\rho_0 e^{-\mu r}\left(\frac{2}{\mu}+r\right)}{\mu^2\epsilon_0 r}.
\end{aligned}$$

c

Since there is no charge density at infinity, the potential can be set to zero. Hence $c_0 = 0\mathrm{V}$. Something weird is going on. Using the integral form of Gauss' law gives an electric field and potential that do not blow up at $r = 0\mathrm{m}$, yet solving Poisson's equation gave a solution that does blow up at $r = 0\mathrm{m}$.

5.2 Potential for a Box

Being that there is no charge in the box, Gauss' law becomes

$$\nabla^2\phi = 0. \tag{5.2.1}$$

The boundary conditions are

$$\begin{aligned}
\phi(0,y,z) = \phi(a,y,z) &= 0, && (5.2.2)\\
\phi(x,0,z) = \phi(x,b,z) &= 0, && (5.2.3)\\
\phi(x,y,0) &= 0, && (5.2.4)\\
\phi(x,y,c) &= c_0 x, && (5.2.5)
\end{aligned}$$

where c_0 is some constant.

a

Make the assumption that the solution to 5.2.1 is seperable:

$$\phi(x,y,z) = X(x)Y(y)Z(z).$$

Then 5.2.1 becomes

$$\frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} = 0.$$

[1] These integrals were done with the help of Mathematica.

For this to be the case, each term must be a constant, such that their sum vanishes:

$$\begin{aligned}\frac{X''}{X} &= -k_x^2,\\ \frac{Y''}{Y} &= -k_y^2,\\ \frac{Z''}{Z} &= k_z^2,\\ k_z^2 &= k_x^2 + k_y^2.\end{aligned}$$

These ODEs have the following solutions:

$$\begin{aligned}X(x) &= X_0 \sin(k_x x) + X_1 \cos(k_x x),\\ Y(y) &= Y_0 \sin(k_y y) + Y_1 \cos(k_y y),\\ Z(z) &= Z_0 \sinh(k_z z) + Z_1 \cosh(k_z z).\end{aligned}$$

b

Using 5.2.2-5.2.4,

$$\begin{aligned}X_1 = 0,\ k_x &= \frac{m\pi}{a},\\ Y_1 = 0,\ k_y &= \frac{n\pi}{b},\\ Z_1 &= 0.\end{aligned}$$

Thus,

$$\begin{aligned}X(x) &= X_0 \sin\left(\frac{m\pi}{a}x\right),\\ Y(y) &= Y_0 \sin\left(\frac{n\pi}{b}y\right),\\ Z(z) &= Z_0 \sinh(k_{mn}\pi z),\end{aligned}$$

where

$$k_{mn} = \sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}}.$$

Using 5.2.5 gives

$$V(x) = \sum_{m,n}^{\infty} A_{mn} \sin\left(\frac{m\pi}{a}x\right) \sin\left(\frac{n\pi}{b}y\right) \sinh(k_{mn}c\pi).$$

In order to find the coefficients, A_{mn}, one can exploit the inner product

$$\int_0^\tau \sin\left(\frac{m\pi}{\tau}s\right) \sin\left(\frac{n\pi}{\tau}s\right) \mathrm{d}s = \frac{\tau}{2}\delta_{mn}.$$

Multiplying both sides by

$$\sin\left(\frac{m\pi}{a}x\right) \sin\left(\frac{n\pi}{b}y\right),$$

and using the identity

$$\int_0^b \sin^2\left(\frac{n\pi}{b}\eta\right) \mathrm{d}\eta = \frac{b}{2},$$

one can integrate over x and y to get[2]

$$\begin{aligned}A_{mn} &= \frac{bc_0}{2\sinh(k_{mn}c\pi)} \int_0^a \xi \sin^2\left(\frac{m\pi}{a}\xi\right) \mathrm{d}\xi\,,\\ &= \frac{a^2 bc_0}{8\sinh(k_{mn}c\pi)}.\end{aligned}$$

This gives the final solution to be

$$\phi(x,y,z) = \sum_{m,n}^{\infty} A_{mn} \sin\left(\frac{m\pi}{a}x\right) \sin\left(\frac{n\pi}{b}y\right) \sinh(k_{mn}z\pi),$$

where

$$A_{mn} = \frac{a^2 bc_0}{8\sinh(k_{mn}c\pi)} \quad \text{and} \quad k_{mn} = \sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}}.$$

[2] This integral was done with the assistance of Mathematica.