Solutions

1. **Inhomogeneously charged sphere:**

(a) The charge enclosed by a "sub-sphere" V of radius $r \leq R$ is

$$Q_{\rm enc}(r) = \int_V d^3r' \rho(\vec{r}') = \int_0^r dr'\, r'^2 \int d\phi'\, d\cos\theta'\, \rho_0 \left(\frac{r'}{R}\right)^2 \tag{1}$$

$$= \frac{4\pi\rho_0}{R^2}\int_0^r dr'\, r'^4 = \frac{4\pi\rho_0 r^5}{5R^2} \qquad (r \leq R)\,. \tag{2}$$

and the total charge is therefore

$$Q_{\rm tot} = Q_{\rm enc}(R) = \frac{4\pi\rho_0 R^3}{5}\,. \tag{3}$$

If we enlarge the radius r to $r > R$, the charge does not increase anymore and we can summarize

$$Q_{\rm enc}(r) = \begin{cases} Q_{\rm tot}\frac{r^5}{R^5} & \text{for } r \leq R, \\ Q_{\rm tot} & \text{for } r > R, \end{cases} \tag{4}$$

as a preparation for the calculation of the electric field in the next question.

(b) Due to the symmetry of the problem, we must have for the electric field

$$\vec{E}(\vec{r}) = E(r)\hat{r} \tag{5}$$

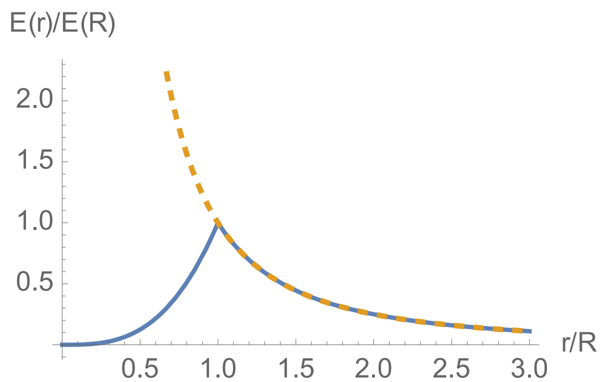
where $\hat{r} \equiv \vec{r}/r$. By applying Gauss's law to a sphere of arbitrary radius r we have

$$\int_V \vec{E} d\vec{A} = \int E(r) dA = E(r) 4\pi r^2 = \frac{Q_{\rm enc}(r)}{\epsilon_0} \tag{6}$$

and thus, with $Q_{\rm enc}(r)$ given by equation (4),

$$E(r) = \frac{Q_{\rm enc}(r)}{4\pi\epsilon_0 r^2} = \begin{cases} \frac{\rho_0 r^3}{5\epsilon_0 R^2} & \text{for } r \leq R, \\ \frac{\rho_0 R^3}{5\epsilon_0 r^2} & \text{for } r > R\,. \end{cases} \tag{7}$$

(c) We see from equation (7) that $|\vec{E}(\vec{r})| = E(r)$ rises $\propto r^3$ for $r \leq R$ and falls off like $\propto 1/r^2$. The following figure illustrates this behavior (solid blue curve):



Outside of the sphere, the electric field looks like that of a point charge at the origin with $Q_{\rm t}$ as the total charge (orange dashed curve).

2. **Spherical cavity:**

   (a) This problem can be solved using the superposition principle. The sphere, radius R and charge density ρ_0 with a spherical cavity of radius b, can be considered as the combination of two complete spheres, one with radius R and charge density ρ_0 and a second with radius b and charge density $-\rho_0$. The superposition of the $-\rho_0$ charged sphere inside the larger $+\rho_0$ charged sphere is equivalent to the cavity in the larger sphere. The electric field from the entire system is just the superposition of the fields from the two uniform spheres. If the big sphere is centered at the origin, then Gauss's law gives

$$(4\pi r^2)|\vec{E}_1| = (4\pi r^3/3)\frac{\rho_0}{\epsilon_0} \tag{8}$$

   or

$$\vec{E}_1(\vec{r}) \;=\; \frac{\rho_0}{3\epsilon_0}\vec{r} \tag{9}$$

   for the field inside this sphere (also see lecture). Similarly, for the field inside the negatively-charged sphere, we obtain

$$\vec{E}_2(\vec{r}) \;=\; -\frac{\rho_0}{3\epsilon_0}\vec{r}\,' \,, \tag{10}$$

   where $\vec{r}\,' = \vec{r} - \vec{a}$ is the vector from the center of the cavity at $\vec{a}$ to the point $\vec{r}$. The sum of these gives the field inside the cavity:

$$\vec{E}(\vec{r}) \;= \vec{E}_1(\vec{r}) + \vec{E}_2(\vec{r}) \;=\; \frac{\rho_0}{3\epsilon_0}\vec{a} \,. \tag{11}$$

   Note that the field is uniform in magnitude and direction, independent of the point inside the cavity.

3. **Charge density of a special field:**

   (a) To find the charge density, use the Maxwell equation

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}. \tag{12}$$

   Since the electric field points radially and only depends on r, it is most natural to use spherical coordinates. Then

$$\rho \;=\; \epsilon_0\, \vec{\nabla} \cdot \vec{E} \;=\; \epsilon_0 \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 E_r) \;=\; \frac{q}{4\pi r^2}\left(-\frac{4r^2}{a^3}e^{-2r/a}\right) = -\frac{q}{\pi a^3}e^{-2r/a} \,. \tag{13}$$

   This is valid everywhere except $r = 0$, where the field is singular. Near $r = 0$, we note that

$$\vec{E} \sim \frac{q}{4\pi\epsilon_0}\vec{r}/r^3, \tag{14}$$

   which is the field of a point source at the origin, with charge q. Thus, we can write the total charge density for the system as

$$\rho(\vec{r}) \;=\; q\left[\delta(\vec{r}) - \frac{1}{\pi a^3}e^{-2r/a}\right] \,. \tag{15}$$

(b) The total charge of the system is

$$Q_{\rm tot} \;=\; \int d^3r\,\rho(\vec{r}) \;=\; q\left[1-\frac{4}{a^3}\int_0^\infty dr\,r^2 e^{-2r/a}\right] \;=\; 0\,. \tag{16}$$

(c) The system has a positive point charge of $+q$ at the origin, surrounded by a isotropic negative cloud with total charge $-q$. This is reminiscent of the Hydrogen atom (with $q = e$). Noting that the wave function for the $1s$ electron state of the Hydrogen atom is $\psi_{1s} = e^{-r/a_0}/(\pi a_0^3)^{1/2}$, we find that it has a mean-field charge density of

$$\rho_{1s} \;=\; (-e)|\psi_{1s}|^2 \;=\; -\frac{e}{\pi a_0^3}e^{-2r/a_0} \tag{17}$$

Identifying $a = a_0$ with the Bohr radius of the atom, and adding in the field from the proton, we recognize our field as that due to the mean-field of the Hydrogen atom in a $1s$ state.

4. **Field of a thin disc:**

(a) The charge density in cylindrical coordinates $\{s, \phi, z\}$ can be written

$$\rho(\vec{r}\,') \;=\; \frac{q}{\pi R^2}\Theta(R-s')\delta(z')\,. \tag{18}$$

The Cartesian surface charge density $\sigma(x', y')$ can be identified by comparing

$$\rho(\vec{r}) = \sigma(x', y')\,\delta(z') \tag{19}$$

to the expression (18). We obtain

$$\sigma(x', y') = \frac{q}{\pi R^2}\Theta(R-s') \tag{20}$$

$$= \frac{q}{\pi R^2}\,\Theta\left(R-\sqrt{x'^2+y'^2}\right). \tag{21}$$

(b) By symmetry considerations, the electric field on the z-axis must point in the z direction. We obtain

$$\begin{aligned}
\vec{E}(\vec{r}) \;&=\; \frac{1}{4\pi\epsilon_0}\int d^3r'\,\frac{\rho(\vec{r}\,')(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3} \\
&=\; \frac{1}{4\pi\epsilon_0}\int_0^{2\pi} d\phi' \int_0^R ds'\,s'\,\frac{q}{\pi R^2}\,\frac{(-s'\cos\phi', -s'\sin\phi', z)}{(z^2+s'^2)^{3/2}} \\
&=\; \frac{1}{4\pi\epsilon_0}\int_0^{R^2} d(s'^2)\,\frac{q}{R^2}\,\frac{(0,0,z)}{(z^2+s'^2)^{3/2}} \\
&=\; \frac{1}{4\pi\epsilon_0}\cdot\hat{\mathbf{z}}\,\frac{2q}{R^2}\left(1-\frac{|z|}{\sqrt{z^2+R^2}}\right)\,.
\end{aligned}$$

(c) In the limit $z \gg R$ we obtain

$$\vec{E} \approx \frac{1}{4\pi\epsilon_0}\cdot\frac{q\hat{z}}{z^2} \qquad (z \gg R) \tag{22}$$

which is the same as for a point charge q at large distance z. In the limit $z \ll R$ we obtain

$$\vec{E} \approx \frac{1}{4\pi\epsilon_0} \cdot \frac{2q\hat{z}}{R^2} \qquad (z \ll R). \tag{23}$$

Very close to the center of the disk, the edges become irrelevant and it should look like the field from an infinite plane charge. Using Gauss's law gives $E_z(2A) = (\sigma A)/\epsilon_0$ or $E_z = \sigma/(2\epsilon_0) = 2q/(4\pi\epsilon R^2)$, exactly as we found.

Note: This problem could also be done in spherical coordinates $\{r, \theta, \phi\}$, in which case the charge density would be

$$\rho(\vec{r}\,') \;=\; \frac{q}{\pi R^2}\Theta(R - r')\delta(r' \cos\theta') \;=\; \frac{q}{\pi R^2 r'}\Theta(R - r')\delta(\cos\theta')\,. \tag{24}$$

After integrating over the $\delta(\cos\theta')$ the integrals become exactly the same as in cylindrical coordinates.