Solutions

1. **Line charge and conducting half-planes:**

(a) A convenient way to calculate the potential of a long thin wire uses the law of Gauss in its integrated form. We enclose a segment of the wire with length l by a cylinder with radius ρ. Gauss' law gives

$$(2\pi\rho) l E = \frac{1}{\epsilon_0}\lambda l \tag{1}$$

where we employed the cylindrical symmetry of the problem and introduced the line charge λ. This gives us the electric field

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{2\lambda}{\rho}\hat{\rho} \tag{2}$$

and thus the potential $\phi = \frac{1}{4\pi\epsilon_0}(-2\lambda\ln\rho) + \text{const}$. In order to render the argument of the logarithm dimensionless we introduce a dimensionful constant R to adjust the "offset" of the potential and write

$$\phi = \frac{1}{4\pi\epsilon_0}\lambda\ln\frac{R^2}{\rho^2} \tag{3}$$

Alternatively, one can directly determine the potential for a segment of the known charge distribution with our basic integration formula and then take the limit of the length of the segment to infinity.

(b) The potential for an isolated line charge at (x_0, y_0) is $\phi(x,y,z) = (1/(4\pi\epsilon_0))\lambda\ln(R^2/\rho^2)$ with $\rho^2 = (x-x_0)^2 + (y-y_0)^2$ and a constant R. The potential in the presence of the grounded half-planes can be obtained by image charges. Reflecting across the yz-plane and inverting the charge suggests an image line-charge of $-\lambda$ at $(-x_0, y_0)$. Then reflecting across the xz-plane and inverting charges suggests image line-charges of $-\lambda$ at $(x_0, -y_0)$ and $+\lambda$ at $(-x_0, -y_0)$. Using superposition for these four charges gives

$$\phi(x,y,z) \; = \frac{1}{4\pi\epsilon_0}\lambda\ln\frac{\rho_2^2\rho_4^2}{\rho_1^2\rho_3^2} \; , \tag{4}$$

where

$$\begin{aligned}
\rho_1^2 &= (x-x_0)^2 + (y-y_0)^2 & (5)\\
\rho_2^2 &= (x+x_0)^2 + (y-y_0)^2 & (6)\\
\rho_3^2 &= (x+x_0)^2 + (y+y_0)^2 & (7)\\
\rho_4^2 &= (x-x_0)^2 + (y+y_0)^2 \; . & (8)
\end{aligned}$$

It is easy to see that the potential vanishes on the surfaces. For example, in the xz half-plane, $y = 0$, so $\rho_1 = \rho_4$ and $\rho_2 = \rho_3$, so the argument of the logarithm is 1 and ϕ vanishes. Similar arguments hold in the yz half-plane. As for the tangential electric field, it is clearly zero in the z-direction, since ϕ is independent of z. In the x direction, on the xz half-plane, we have

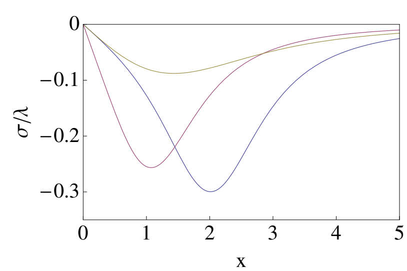
$$\begin{aligned} E_x(x,0,z) &= -\left.\frac{\partial\phi}{\partial x}\right|_{y=0} \\ &= -\frac{1}{4\pi\epsilon_0}\lambda\left[-\frac{2(x-x_0)}{\rho_1^2}+\frac{2(x+x_0)}{\rho_2^2}-\frac{2(x+x_0)}{\rho_3^2}+\frac{2(x-x_0)}{\rho_4^2}\right]_{y=0} \\ &= 0\ , \end{aligned} \tag{9}$$

and similarly in the xz half-plane.

(c) The charge density is given by $E_n = 4\pi\sigma$ on the surface of a conductor. Therefore on the xz surface we obtain

$$\begin{aligned} \sigma(x,z) &= -\epsilon_0\left.\frac{\partial\phi}{\partial y}\right|_{y=0} \\ &= -\frac{\lambda}{4\pi}\left[-\frac{2(y-y_0)}{\rho_1^2}+\frac{2(y-y_0)}{\rho_2^2}-\frac{2(y+y_0)}{\rho_3^2}+\frac{2(y+y_0)}{\rho_4^2}\right]_{y=0} \\ &= -\frac{\lambda\, y_0}{\pi}\left[\frac{1}{(x-x_0)^2+y_0^2}-\frac{1}{(x+x_0)^2+y_0^2}\right]\ . \end{aligned}$$

A plot of σ/λ versus x is shown at right for various choices of (x_0, y_0). For $(x_0, y_0) = (1, 1)$ the distribution has a peak magnitude near $x = 1$, while for $(x_0, y_0) = (2, 1)$ the peak magnitude is near $x = 2$. For $(x_0, y_0) = (1, 2)$ the peak magnitude is smaller since the line charge is farther from the plane.



[Remark:] The total charge per unit length z in the xz half plane is

$$\frac{dq}{dz} = \int_0^\infty \sigma(x,z)\,dx\ .$$

Using the change of variable, $x = \pm x_0 + y_0\tan\theta$, $dx = (y_0/\cos^2\theta)d\theta$ in the two terms of the integrand gives

$$\begin{aligned} \frac{dq}{dz} &= -\frac{\lambda}{\pi}\left[\tan^{-1}(\infty)-\tan^{-1}\left(\frac{-x_0}{y_0}\right)-\tan^{-1}(\infty)+\tan^{-1}\left(\frac{x_0}{y_0}\right)\right] \\ &= -\frac{2\lambda}{\pi}\tan^{-1}\left(\frac{x_0}{y_0}\right)\ . \end{aligned}$$

As the line charge moves far from the xz half-plane (or close to the yz half-plane), the induced charge per unit length on the xz half-plane goes to zero. As the line charge moves close to the xz half-plane (or far from the yz half-plane), the induced charge per unit length on the xz half-plane goes to $-\lambda$.

2. **A corner:**

(a) The general solution to the Laplace equation in cylindrical coordinates, without assuming periodicity in φ is

$$\begin{aligned}\phi(\rho,\varphi) &= a_0 + b_0\varphi + (c_0 + d_0\varphi)\ln\rho + \sum_{n=1}^{\infty}\Big[\rho^{\nu_n}\left(a_n\cos(\nu_n\varphi) + b_n\sin(\nu_n\varphi)\right) \\ &\qquad + \rho^{-\nu_n}\left(c_n\cos(\nu_n\varphi) + d_n\sin(\nu_n\varphi)\right)\Big]\ .\end{aligned}$$

Here we are concerned with the solution for $\phi(\rho,\varphi)$ in the angular region $0 \leq \varphi \leq \beta$ with boundary conditions, $\phi(\rho,0) = \phi(\rho,\beta) = 0$. Setting $\varphi = 0$ in the general solution and requiring $\phi(\rho,0) = 0$ for all values of ρ implies that $a_n = c_n = 0$ for all n. Next, setting $\varphi = \beta$ in the general solution and requiring $\phi(\rho,\beta) = 0$ for all values of ρ implies that $b_0 = d_0 = 0$ and $\sin(\nu_n\beta) = 0$, or $\nu_n\beta = n\pi$. Thus, we obtain the most general solution for this case:

$$\phi(\rho,\varphi) = \sum_{n=1}^{\infty}\left(b_n\rho^{(n\pi/\beta)} + d_n\rho^{-(n\pi/\beta)}\right)\sin\frac{n\pi\varphi}{\beta}\ .$$

(b) Requiring that the potential remain finite as $\rho \to 0$ restricts the solution further to

$$\phi(\rho,\varphi) = \sum_{n=1}^{\infty} b_n\rho^{(n\pi/\beta)}\sin\frac{n\pi\varphi}{\beta}\ .$$

(Recall that a term proportional to $\ln\rho$ as $\rho \to 0$ implies a line charge at $\rho = 0$. Any terms with negative powers of ρ in the potential would be even more singular than a line charge.) The leading contribution near $\rho = 0$ is given by the term with the lowest power of ρ in the summation. So, near $\rho = 0$, we have

$$\phi(\rho,\varphi) \approx b_1\rho^{(\pi/\beta)}\sin\frac{\pi\varphi}{\beta}\ .$$

(c) Keeping only this leading term near $\rho = 0$, we obtain for the $\vec{E}$ field:

$$\begin{aligned}\vec{E} &= -\vec{\nabla}\phi = -\frac{\partial\phi}{\partial\rho}\vec{e}_\rho - \frac{1}{\rho}\frac{\partial\phi}{\partial\varphi}\vec{e}_\varphi \\ &\approx -b_1\frac{\pi}{\beta}\rho^{(\pi/\beta-1)}\left(\vec{e}_\rho\sin\frac{\pi\varphi}{\beta} + \vec{e}_\varphi\cos\frac{\pi\varphi}{\beta}\right)\ .\end{aligned}$$

The surface charge density on the conductors is given by $\sigma = \epsilon_0 E_n$. The normal to the surface at $\varphi = 0$ is $\vec{e}_\varphi$ and the normal to the surface at $\varphi = \beta$ is $-\vec{e}_\varphi$. We obtain the same surface charge density on each half-plane (as one would expect):

$$\sigma(\rho,z) \approx -\frac{\pi\epsilon_0 b_1}{\beta}\rho^{(\pi/\beta-1)}\ .$$

For $\beta \approx 0$, we have a very thin slit cut out of a conducting volume. The field and the surface charge density both vary as $\rho^{(\pi/\beta-1)}$, which is a very large power for $\beta \approx 0$. Thus, the fields and the charges in the conductor go rapidly to zero near the origin and are strongly repelled from the region inside the slit. This makes sense since the field inside a conductor is zero and the charges all move to the outer surface.

For $\beta = \pi$, the conductor becomes a half-plane. Now the variation goes as $\rho^{(\pi/\beta-1)} = \rho^0$. Thus, the field and surface charge density is independent of ρ near $\rho = 0$, which just reflects the fact that the point $\rho = 0$ is not special for the half-plane.

For $\beta \approx 2\pi$, we have a very thin conducting volume, which ends at $\rho = 0$. Now, both the field and the surface charge density vary as $\rho^{-1/2}$. The fields become very large near the edge of the sharp conducting volume, and the surface charge diverges (although it is still integrable). This reflects the general fact that the fields and charge densities of a conductor tend to be largest near sharp convex features in the conductor. (This fact is utilized in the lightning rod.)