Solutions

1. **Potential of point charge and conducting sphere:** This calculation was sketched in the lecture, but no details were given. Some are as follows.

   (a) Using $\vec{a} = (0, 0, a)$, we use the ansatz

   $$\phi(\vec{r}) = \underbrace{\frac{1}{4\pi\epsilon_0}\frac{q}{|\vec{r}-\vec{a}|}}_{\text{term (i)}} + \underbrace{\frac{1}{4\pi\epsilon_0}\sum_{l=0}^{\infty}\left(\frac{A_l}{r^{l+1}} + B_l r^l\right) P_l(\cos\theta)}_{\text{term (ii)}} \tag{1}$$

   which fulfills the required Poisson equation in the region outside of the sphere.

   (b) The boundary condition $\phi \to 0$ for $r \to \infty$ requires for all l

   $$B_l = 0\,. \tag{2}$$

   On the sphere we want $\phi = 0$. We expand term (i) also in Legendre polynomials

   $$\left.\frac{1}{|\vec{r}-\vec{a}|}\right|_{r=R} = \frac{1}{a}\sum_{l=0}^{\infty}\left(\frac{R}{a}\right)^l P_l(\cos\theta) \tag{3}$$

   and find by comparison of coefficients for all l

   $$A_l = -qR^{2l+1}/a^{l+1} \tag{4}$$

   (c) We can use the formula for the generating function for the Legendre polynomials to write the infinity sum over l in a closed form. Let's try to map our result to an expression of this form:

   $$\frac{1}{|\vec{r}-\vec{a}'|} = \sum_{l=0}^{\infty}\frac{(a')^l}{r^{l+1}} P_l(\cos\theta) \tag{5}$$

   In order to match this to our expression, we write A_l such that it is an expression to the power l times some l independent prefactor:

   $$A_l = (-qR/a)(R^2/a)^l\,. \tag{6}$$

   Now it's easy to see that

   $$\phi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{q}{|\vec{r}-\vec{a}|} + \frac{1}{4\pi\epsilon_0}\frac{q'}{|\vec{r}-\vec{a}\,'|} \tag{7}$$

   with

   $$q' = -q\frac{R}{a}, \qquad \vec{a}\,' = \frac{R^2}{a^2}\vec{a}\,. \tag{8}$$

   The second term in the potential looks like the potential of a "virtual" charge of strength q' located at $\vec{a}'$ inside the sphere (outside of the observer region). This so-called "image charge" exactly compensates the potential of the physical point charge at $\vec{a}$ for points on the sphere. We saw in the lecture that we can use $G_D = \phi\, 4\pi\epsilon_0/q$ as a Green's function to implement Dirichlet boundary conditions on the sphere.

(d) The surface charge density on the sphere induced by the point charge is

$$\sigma(R,\theta) = -\epsilon_0 \frac{\partial \phi}{\partial r}\bigg|_{r=R} = -\frac{q}{4\pi R}\frac{(a^2 - R^2)}{(a^2 + R^2 - 2aRx)^{3/2}} \tag{9}$$

with $x = \cos\theta$. For $a \to R$ we have $\sigma = -q\delta(\cos\theta)/(2\pi R^2)$, i.e. all induced surface charge is concentrated close to the physical charge. For $a \to \infty$ we have $\sigma = 0$, i.e. no induced charge on the sphere.

(e) The induced charge is

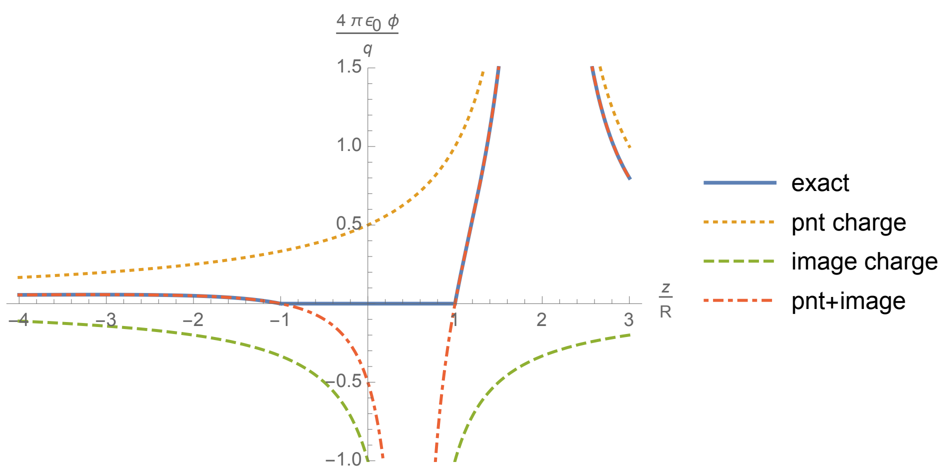
$$Q_{\text{induced}} = \int_S dA\, \sigma(\vec{r}) = \int_S R^2 d\Omega\, \sigma(R,\theta) \tag{10}$$

$$= \int_0^{2\pi} \mathrm{d}\varphi \int_{-1}^{+1} \mathrm{d}x\, R^2 \left(-\frac{q}{4\pi R}\right) \frac{(a^2 - R^2)}{(a^2 + R^2 - 2aRx)^{3/2}} \tag{11}$$

$$= -\frac{R}{a} q = q' \tag{12}$$

and therefore equal to the strength of the image charge.

(f) We choose $a = 2R$ and plot the different terms in (7) and their sum:



One can see how the image charge compensates the potential of the actual point charge on the surface of the sphere (at $z = R$ but also at $z = -R$!). Note that (7) is valid only *outside* of the sphere. In fact, inside the conducting sphere we must have $\phi = 0$ (solid curve). While the potential is continuous, it has kinks for $z = \pm R$. This discontinuity in the first derivative (normal component of the electric field) is due to the surface charge.

(g) A fixed potential different from 0 on the sphere requires a different surface charge density. It can be simulated by adding the potential of a point charge at the center of the sphere. The potential is

$$\phi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{q}{|\vec{r} - \vec{a}|} + \frac{1}{4\pi\epsilon_0}\frac{q'}{|\vec{r} - \vec{a}\,'|} + \frac{1}{4\pi\epsilon_0}\frac{q''}{|\vec{r}|} \tag{13}$$

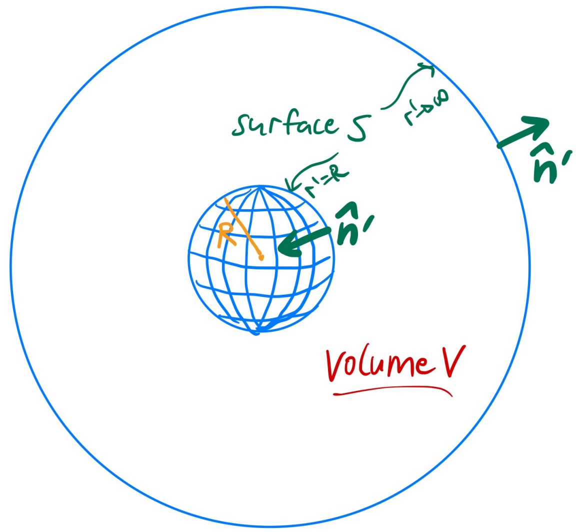
We can satisfy our boundary condition at $r = R$ by setting

$$q'' = 4\pi\epsilon_0\, \phi_0 R. \tag{14}$$

Alternatively, we could have used the Green's function obtained above and apply the "master formula" for Dirichlet problems discussed in the lecture.

2. **Green's function for sphere:**

(a) We want to solve the potential in the volume V outside of the sphere. This volume is bounded by the surface of the sphere and a surface at infinity.



(b) We know the value of the potential on the sphere and can assume that it vanishes at infinity. This is called a Dirichlet boundary problem and a suitable Green's function $G_D(\vec{r}, \vec{r}\,')$ should fulfill

$$G_D(\vec{r}, \vec{r}\,')|_{r'=R} = 0 \qquad \forall\, \vec{r} \in V, \tag{15}$$

$$G_D(\vec{r}, \vec{r}\,')|_{r'\to\infty} = 0 \qquad \forall\, \vec{r} \in V, \tag{16}$$

$$\Delta G_D(\vec{r}, \vec{r}\,') = -4\pi\delta(\vec{r} - \vec{r}\,'), \quad \forall\, \vec{r}, \vec{r}\,' \in V\,. \tag{17}$$

We will also have $G_D(\vec{r}, \vec{r}\,') = G_D(\vec{r}\,', \vec{r})$, but this property does not need to be enforced explicitly. We can use the method of image charges to construct

$$G_D(\vec{r}, \vec{r}\,') = \frac{1}{|\vec{r} - \vec{r}\,'|} - \frac{R/r'}{|\vec{r} - (R^2/r'^2)\vec{r}\,'|} \tag{18}$$

which fulfills all of these requirements. Notice that

$$\Delta G_D(\vec{r}, \vec{r}\,') = -4\pi\delta(\vec{r} - \vec{r}\,') + 4\pi(R/r')\delta(\vec{r} - (R^2/r'^2)\vec{r}\,') \tag{19}$$

but the second term does not contribute in V since the position of the image charge, $(R^2/r'^2)\vec{r}\,'$, is inside the sphere, i.e. outside V.

(c) Green's method gives the potential as

$$\phi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int_V \mathrm{d}^3r'\, \underbrace{\rho(\vec{r}\,')}_{0}\, G_D(\vec{r}, \vec{r}\,') - \frac{1}{4\pi}\int_S \mathrm{d}A'\, \phi(\vec{r}\,') \frac{\partial G_D(\vec{r}, \vec{r}\,')}{\partial n'} \tag{20}$$

for the potential. For the surface integral we get no contribution from the surface at $r' \to \infty$ and we focus on the other boundary of V, which is the surface of the sphere. Since the normal vector points to the outside of V we have $\hat{n}' = -\hat{r}'$ and thus

$$\partial/\partial n' = -\partial/\partial r' \tag{21}$$

As discussed in the lecture, it is convenient to use an expansion in spherical harmonics to arrive at a simple integral over $\vec{r}\,'$. We write

$$G_D(\vec{r}, \vec{r}\,') = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{4\pi}{2l+1} \left[\frac{r_<^l}{r_>^{l+1}} - \frac{1}{R} \left(\frac{R}{rr'} \right)^{l+1} \right] Y_{lm}^*(\theta', \varphi') Y_{lm}(\theta, \varphi) \tag{22}$$

We really need this expression only for $\vec{r}\,'$ on or close-to the surface of the sphere, $r' \approx R$, where $r_< = r'$ and $r_> = r$. We obtain

$$\frac{\partial G_D(\vec{r}, \vec{r}\,')}{\partial n'} = -\frac{\partial G_D(\vec{r}, \vec{r}\,')}{\partial r'} = -\frac{4\pi}{R^2} \sum_{l,m} \left(\frac{R}{r} \right)^{l+1} Y_{lm}^*(\theta', \varphi') Y_{lm}(\theta, \varphi) \tag{23}$$

and

$$\phi(\vec{r}) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left(\frac{R}{r} \right)^{l+1} \left[\int \mathrm{d}\Omega' V(\theta', \varphi') Y_{lm}^*(\theta', \varphi') \right] Y_{lm}(\theta, \varphi) \,. \tag{24}$$

Only the integral over the surface of the unit sphere in the bracket depends on the specific boundary values of the problem, everything else depends only on the geometry of the problem.

(d) We write the boundary values in terms of spherical harmonics,

$$V(\theta, \varphi) = V_0 \cos(3\varphi) \sin^3(\theta) = -V_0 \sqrt{16\pi/35}\, [Y_{3,3}(\theta, \varphi) - Y_{3,-3}(\theta, \varphi)] \tag{25}$$

such that it is easy to obtain the last missing piece in equation (24),

$$\int \mathrm{d}\Omega' V(\theta', \varphi') Y_{lm}^*(\theta', \varphi') = -V_0 \sqrt{16\pi/35}\, [\delta_{l,3}\delta_{m,3} - \delta_{l,3}\delta_{m,-3}] \,, \tag{26}$$

using orthonormality of the spherical harmonics. This results in

$$\phi(\vec{r}) = -V_0 \sqrt{16\pi/35} \left(\frac{R}{r} \right)^4 (Y_{3,3}(\theta, \varphi) - Y_{3,-3}(\theta, \varphi)) \,. \tag{27}$$

or just

$$\phi(\vec{r}) = V_0 \left(\frac{R}{r} \right)^4 \cos(3\varphi) \sin^3(\theta) \,. \tag{28}$$