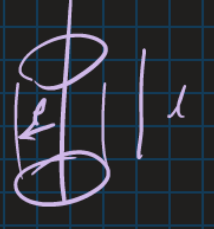
Homework 8

Friday, March 12, 2021 12:35 AM

1.

$$\oint \vec{E}\cdot d\vec{A} = \frac{q}{\varepsilon_0}$$



$$2\pi\rho l E = \frac{\lambda l}{\varepsilon_0}$$

$$E = \frac{\lambda}{2\pi\varepsilon_0\rho}$$

$$\phi = \frac{\lambda}{2\pi\varepsilon_0}\int_R^r \frac{1}{\rho}\,d\rho$$

$$= \frac{\lambda}{4\pi\varepsilon_0}\ln\left(\frac{R^2}{r^2}\right)$$

Let $r^2 = \rho^2 = (x-x_0)^2+(y-y_0)^2$. Then we get what was requested.

Using the method of images, adding three more lines of charge of

$$
\begin{aligned}
&-\lambda \text{ at } (-x_0, y_0), && (\rho_2^2 = (x+x_0)^2+(y-y_0)^2)\\
&-\lambda \text{ at } (x_0, -y_0), && (\rho_3^2 = (x-x_0)^2+(y+y_0)^2)\\
\text{and } &\lambda \text{ at } (-x_0, -y_0). && (\rho_4^2 = (x+x_0)^2+(y+y_0)^2)
\end{aligned}
$$

Then the potential will vanish on the conducting planes and so the potential in the region in question will be the the unique solution. Hence

because each of the distances to each of the planes are the same.

$$\phi = \frac{1}{4\pi\varepsilon_0}\lambda\ln\left(\frac{\rho_2^2\rho_3^2}{\rho_1^2\rho_4^2}\right).$$

The induced charge density on the xz-plane is

$$\sigma(x,z) = \varepsilon_0\frac{\partial\phi}{\partial y}\bigg|_{y=0} = -\frac{\lambda y_0}{\pi}\left[\frac{1}{(x-x_0)^2+y^2} - \frac{1}{(x+x_0)^2+y^2}\right].$$

2. For the potential to vanish on the grounded planes,

$$a_0 = b_0 = c_0 = d_0 = 0.$$

Additionally, the coefficients on the cosine terms must be zero since $\phi(\rho,0)=0$ but $\cos(0)=1$. Next, $\sin\nu_n\varphi = 0$ at $\varphi=\beta$, so

$$\nu_n = n\pi\beta^{-1}.$$

Lastly, there is no charge at $\rho=0$, so

$$\lim_{\rho\to 0}\phi \neq \pm\infty.$$

Donc $c_n = d_n = 0$, leaving only $b_n \neq 0$. Ainsi the general solution for the potential is

$$\phi = \sum_{n=1}^{\infty}\rho^{n\pi\beta^{-1}}b_n\sin(n\pi\beta^{-1}\varphi).$$

As $\rho\to 0$, $\phi\to 0$.

If one only keeps the leading term to find the electric field and the induced charge density one gets

$$\phi \approx \rho^{\pi\beta^{-1}}b_1\sin\pi\beta^{-1}\varphi.$$

$$\vec{E} = \vec{\nabla}\phi \approx \frac{\pi}{\beta}b_1\rho^{\pi\beta^{-1}-1}\left[\sin\pi\beta^{-1}\varphi\,\hat{\rho} + \cos\pi\beta^{-1}\varphi\,\hat{\varphi}\right]$$

$$\sigma_\beta = \varepsilon_0\frac{\partial\phi}{\partial n}\bigg|_{\varphi=\beta} = -\varepsilon_0\rho^{\pi\beta^{-1}-1}\frac{\pi}{\beta}b_1$$

$$\sigma_0 = \varepsilon_0\frac{\partial\phi}{\partial n}\bigg|_{\varphi=0} = -\varepsilon_0\rho^{\pi\beta^{-1}-1}\frac{\pi}{\beta}b_1$$

The values of the field and induced surface charge density, near the origin, are given for a few values of β:

$\beta\approx 0$: Field rapidly approaches 0 at the origin, but becomes very large away from $\rho=0$. Surface density is similar.

$\beta=\pi$: The power $\pi\beta^{-1}-1=0$, so the field and surface densities no longer depend on ρ, so $\rho=0$ isn't special.

$\beta\approx 2\pi$: The power $\pi\beta^{-1}-1=\frac{1}{2}$, so both the field and the surface densities become very large at $\rho=0$.