Please read all of the following before starting the exam:

- Please access the exam problems on Gradescope only after we agree to start *in the Zoom meeting*
- You then have 2 hours to solve the problems.
- Please upload your solutions to Gradescope and indicate for each problem where the relevant pages are.
- Please stay connected to Zoom during the entire duration of your exam and switch on your camera.
- To solve the problems, you may use a simple calculator, but no computer algebra systems, external notes, books, etc.
- All problems are in S.I. units unless stated otherwise. Please give your answers in terms of the given variables and units.
- A complete answer usually includes a derivation of the result (unless stated otherwise). Show all work as neatly and logically as possible to maximize your credit. State clearly which equations were used. Circle or otherwise indicate your final answers.
- Please ask if the problem description is unclear.
- *Good luck !*

$$
\begin{aligned}
\vec{a}\times(\vec{b}\times\vec{c}) &= \vec{b}(\vec{a}\cdot\vec{c})-\vec{c}(\vec{a}\cdot\vec{b}),\\
\vec{a}\cdot(\vec{b}\times\vec{c}) &= \vec{b}\cdot(\vec{c}\times\vec{a})=\vec{c}\cdot(\vec{a}\times\vec{b}),\\
(\vec{a}\times\vec{b})\cdot(\vec{c}\times\vec{d}) &= (\vec{a}\cdot\vec{c})(\vec{b}\cdot\vec{d})-(\vec{a}\cdot\vec{d})(\vec{b}\cdot\vec{c}),\\
\vec{\nabla}\times(\vec{\nabla}\psi) &= 0,\\
\vec{\nabla}\cdot(\vec{\nabla}\times\vec{a}) &= 0,\\
\vec{\nabla}\times(\vec{\nabla}\times\vec{a}) &= \vec{\nabla}(\vec{\nabla}\cdot\vec{a})-\nabla^2\vec{a},\\
\vec{\nabla}\cdot(\psi\vec{a}) &= \vec{a}\cdot\vec{\nabla}\psi+\psi\vec{\nabla}\cdot\vec{a},\\
\vec{\nabla}\times(\psi\vec{a}) &= \vec{\nabla}\psi\times\vec{a}+\psi\vec{\nabla}\times\vec{a},\\
\vec{\nabla}(\vec{a}\cdot\vec{b}) &= (\vec{a}\cdot\vec{\nabla})\vec{b}+(\vec{b}\cdot\vec{\nabla})\vec{a}+\vec{a}\times(\vec{\nabla}\times\vec{b})+\vec{b}\times(\vec{\nabla}\times\vec{a}),\\
\vec{\nabla}\cdot(\vec{a}\times\vec{b}) &= \vec{b}\cdot(\vec{\nabla}\times\vec{a})-\vec{a}\cdot(\vec{\nabla}\times\vec{b}),\\
\vec{\nabla}\times(\vec{a}\times\vec{b}) &= \vec{a}(\vec{\nabla}\times\vec{b})-\vec{b}(\vec{\nabla}\times\vec{a})+(\vec{b}\cdot\vec{\nabla})\vec{a}-(\vec{a}\cdot\vec{\nabla})\vec{b},\\
\vec{\nabla}\cdot\vec{r} &= 3,\\
\vec{\nabla}\times\vec{r} &= 0,\\
\vec{\nabla}\cdot\hat{r} &= 2/r,\\
\vec{\nabla}\times\hat{r} &= 0,\\
\vec{\nabla}r &= \hat{r},\\
\vec{\nabla}\frac{1}{r} &= -\frac{\hat{r}}{r^2},\\
\vec{\nabla}\cdot(\hat{r}f(r)) &= \frac{2}{r}f+\frac{df}{dr},\\
(\vec{a}\cdot\vec{\nabla})\hat{r} &= \frac{1}{r}[\vec{a}-\hat{r}(\vec{a}\cdot\hat{r})]=\frac{\vec{a}_\perp}{r},\\
\vec{\nabla}^2\left(\frac{1}{r}\right) &= -4\pi\delta(\vec{r}),\\
\int_V d^3r\ \vec{\nabla}\cdot\vec{A} &= \int_S d\vec{S}\cdot\vec{A},\\
\int_V d^3r\ \vec{\nabla}\psi &= \int_S \psi\ d\vec{S},\\
\int_V d^3r\ \vec{\nabla}\times\vec{A} &= \int_S d\vec{S}\times\vec{A},\\
\int_V d^3r\ (\phi\nabla^2\psi+\vec{\nabla}\phi\cdot\vec{\nabla}\psi) &= \int_S \phi\ d\vec{S}\cdot\vec{\nabla}\psi,\\
\int_V d^3r\ (\phi\nabla^2\psi-\psi\nabla^2\phi) &= \int_S (\phi\vec{\nabla}\psi-\psi\vec{\nabla}\phi)\cdot d\vec{S},\\
\int_S (\vec{\nabla}\times\vec{A})\cdot d\vec{S} &= \oint d\vec{\ell}\cdot\vec{A},\\
\int_S d\vec{S}\times\vec{\nabla}\psi &= \oint_C d\vec{\ell}\ \psi.
\end{aligned}
$$

$$\mathscr{L} = -\frac{1}{4\mu_0} F_{\mu\nu} F^{\mu\nu} - J_\mu A^\mu$$
$$L = \frac{1}{\gamma}(-mc^2 - qA_\mu u^\mu)$$
$$\frac{dp^\mu}{d\tau} = qF^{\mu\nu} u_\nu$$
$$\partial_\mu F^{\mu\nu} = \mu_0 J^\nu$$
$$\partial_\mu \tilde{F}^{\mu\nu} = 0$$
$$\partial_\mu J^\mu = 0$$
$$(g_{\mu\nu}) = \mathrm{diag}(1, -1, -1, -1),$$
$$\vec{\beta} = \vec{v}/c,$$
$$\gamma = 1/\sqrt{1-\beta^2},$$
$$x'^\mu = \Lambda^\mu{}_\nu x^\nu$$
$$(\Lambda^\mu{}_\nu) = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$x^\mu = (ct, x, y, z),$$
$$\partial^\mu = ((1/c)\partial/\partial t, -\vec{\nabla})$$
$$k^\mu = (\omega/c, \vec{k}),$$
$$u^\mu = (\gamma c, \gamma \vec{v}),$$
$$p^\mu = (E/c, \vec{p}),$$
$$A^\mu = (\phi/c, \vec{A}),$$
$$J^\mu = (c\rho, \vec{j}),$$
$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu,$$
$$\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta} F_{\alpha\beta}$$
$$\vec{E}'_\parallel = \vec{E}_\parallel$$
$$\vec{B}'_\parallel = \vec{B}_\parallel$$
$$\vec{E}'_\perp = \gamma(\vec{E}_\perp + c\vec{\beta} \times \vec{B})$$
$$\vec{B}'_\perp = \gamma(\vec{B}_\perp - \frac{1}{c}\vec{\beta} \times \vec{E})$$

$$\vec{\nabla} \cdot \vec{E} = \frac{1}{\epsilon_0}\rho, \qquad \vec{\nabla} \cdot \vec{B} = 0,$$
$$\vec{\nabla} \times \vec{B} - \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t} = \mu_0 \vec{j}, \qquad \vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0,$$

$$\frac{d\vec{p}}{dt} = q(\vec{E} + \vec{v} \times \vec{B})$$
$$\vec{E} = -\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t}$$
$$\vec{B} = \vec{\nabla} \times \vec{A}$$

$$(\vec{E}_{\mathrm{out}} - \vec{E}_{\mathrm{in}}) \cdot \hat{n} = \sigma/\epsilon_0$$
$$\phi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int_V d^3r'\, \rho(\vec{r}')G(\vec{r}, \vec{r}') - \frac{1}{4\pi}\int_S dA' \phi(\vec{r}')\frac{\partial G(\vec{r}, \vec{r}')}{\partial n'}$$
$$\phi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int_V d^3r'\, \rho(\vec{r}')G(\vec{r}, \vec{r}') + \frac{1}{4\pi}\int_S dA' \frac{\partial \phi(\vec{r}')}{\partial n'} G(\vec{r}, \vec{r}') + \langle \phi \rangle_S$$
$$\phi = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left[A_{lm} r^l + B_{lm}/r^{l+1}\right] Y_{lm}(\theta, \varphi)$$
$$\phi = a_0 + b_0 \ln s + \sum_{n=1}^{\infty}\left[s^n(a_n \cos(n\varphi) + b_n \sin(n\varphi)) + s^{-n}(c_n \cos(n\varphi) + d_n \sin(n\varphi))\right]$$

$$P_0(\cos\theta) = 1$$
$$P_1(\cos\theta) = \cos\theta$$
$$P_2(\cos\theta) = \frac{1}{2}(3\cos^2\theta - 1)$$
$$Y_{0,0} = \frac{1}{\sqrt{4\pi}}$$
$$Y_{1,0} = \sqrt{\frac{3}{4\pi}}\cos\theta, \quad Y_{1,\pm 1} = \mp\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{\pm i\varphi}$$
$$Y_{2,0} = \sqrt{\frac{5}{16\pi}}\left(3\cos^2\theta - 1\right), \quad Y_{2,\pm 1} = \mp\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\, e^{\pm i\varphi}, \quad Y_{2,\pm 2} = \sqrt{\frac{15}{32\pi}}\sin^2\theta\, e^{\pm 2i\varphi}$$

$$P_l^m(x) = \frac{(-1)^m}{2^l l!}(1-x^2)^{m/2}\frac{d^{l+m}}{dx^{l+m}}(x^2-1)^l$$
$$Y_{lm}(\theta,\varphi) = \sqrt{\frac{(2l+1)(l-m)!}{4\pi(l+m)!}}e^{im\varphi}P_l^m(\cos\theta)$$
$$\frac{1}{|\vec{r}-\vec{r}\,'|} = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left(\frac{4\pi}{2l+1}\right)\frac{r_<^l}{r_>^{l+1}}Y_{lm}^*(\theta',\varphi')Y_{lm}(\theta,\varphi) = \sum_{l=0}^{\infty}\frac{r_<^l}{r_>^{l+1}}P_l(\cos\gamma)$$
$$\phi = \sum_{l=0}^{\infty}\left(a_l r^l + b_l/r^{l+1}\right)P_l(\cos\theta)$$
$$\phi = \frac{1}{4\pi\epsilon_0}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left(\frac{4\pi}{2l+1}\right)\frac{q_{lm}}{r^{l+1}}Y_{lm}(\theta,\varphi)$$
$$q_{lm} = \int d^3r'\,\rho(\vec{r}\,')r'^l Y_{lm}^*(\theta,\varphi)$$

$$\phi = \frac{1}{4\pi\epsilon_0}\left(\frac{Q_{\rm tot}}{r} + \frac{p_i\hat{r}_i}{r^2} + \frac{1}{2!}Q_{ij}\frac{\hat{r}_i\hat{r}_j}{r^3} + \ldots\right)$$
$$p_i = \int d^3\,r'\rho(\vec{r}\,')\,r_i'$$
$$Q_{ij} = \int d^3r'\,\rho(\vec{r}\,')\,(3r_i'r_j' - \delta_{ij}r'^2)$$
$$U = Q_{\rm tot}\phi(\vec{r}) - p_iE_i(\vec{r}) - \frac{1}{6}Q_{ij}\partial_iE_j + \ldots$$

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\left(\frac{\vec{m}\times\vec{r}}{r^3} + \ldots\right)$$
$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\left(\frac{3(\vec{m}\cdot\hat{r})\hat{r} - \vec{m}}{r^3} + \ldots\right)$$
$$\vec{m} = \frac{1}{2}\int d^3r'\,\vec{r}\,'\times\vec{j}(\vec{r}\,')$$
$$U = -m_iB_i(\vec{r}) + \ldots$$
$$\vec{\tau} = \vec{m}\times\vec{B} + \vec{r}\times\vec{F} + \ldots$$

## Sphere enclosing charges

A sphere of radius $R$ contains charges, the outside region and the surface of the sphere is charge-free. The potential on the surface of the sphere is given by

$$\phi(r, \theta, \varphi)|_{r=R} = V_0(-1 + 2\cos\theta + 3\cos^2\theta), \tag{1}$$

where $V_0$ is constant and $r$, $\theta$, $\varphi$ are the usual spherical coordinates.

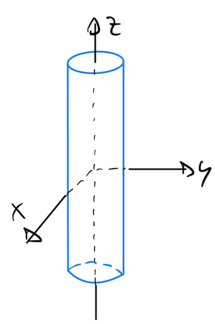

1. (20 pts) At large distance $r$, the electrostatic potential is of the form $\phi \approx F(\theta)/r^n$ and approaches zero for $r \to \infty$. Determine $n$ and $F(\theta)$. [Hint: which terms in the potential could generate this "imprint" on the sphere ?]

## Charged cylindrical shell

An infinitely long cylindrical shell with radius $R$ is centered around the $z$ axis. The shell carries a surface charge density

$$\sigma = \sigma_0 \sin(2\varphi) \tag{2}$$

which is constant along the $z$ axis, but varies with the azimuthal angle $\varphi$.

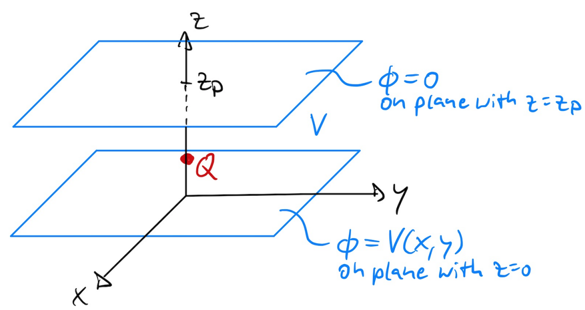



2. (25 pts) *Derive* the electrostatic potential $\phi$ as a function of $s \equiv \sqrt{x^2 + y^2}$ and $\varphi$ both *inside* and *outside* of the shell.

3. (bonus: 5 pts) If the surface charge was $\sigma = \sigma_0 \sin^2(\varphi)$ instead of the one given in Eq. (2), what would be the leading term's dependence on the distance $s$ for large values of $s$ ? Just explain, no calculation is needed.

## Charge between two planes

Consider the region $V$ between two planes, one at $z = 0$ and one at $z = z_P$. The electrostatic potential $\phi(x, y, z)$ on the plane with $z = 0$ is given by $\phi(x, y, 0) = V(x, y)$, while the potential vanishes on the entire plane at $z_P = 0$, $\phi(x, y, z_P) = 0$. There is a point charge $Q$ located between the planes at $\vec{r}_Q = (0, 0, z_Q)$ with $0 < z_Q < z_P$.




4. (25 pts) *Derive* a solution for the electrostatic potential $\phi(x, y, z)$ in the region $V$ using Green's method. [It is fine if your result is given in terms of an infinite sum, where each summand contains an integral, which could directly be integrated numerically for some well-behaved function $V(x, y)$ and numerical values for the components of $\vec{r} = (x, y, z)$, $z_Q$ and $Q$. Please simplify, for example, terms involving $\delta$ functions and evaluate derivatives. You do not need to insert all components explicitly in your final result, but make sure to clearly define all of your expressions.]

## Short answer section

5. (15 pts) *Derive* the magnitude of the magnetic field generated by a current $I$ in a very long, thin wire. To be specific, the current is in the positive $z$ direction and the observation point has distance $s$ from the wire.

6. (10 pts) *Discuss* whether a time-independent current density (no charges) can generate an electric field in a moving frame. Starting from a static charge distribution (no currents), *discuss* whether there is always a frame in which there is only a magnetic, but no electric field.

7. (5 pts) A static magnetic field exerts a force on a slowly moving charge $q$ with velocity $\vec{v}$. How will the speed $\vec{v}$ of the particle change ?