Problems

1. **Sphere at fixed potential:** Consider a sphere of radius R, which is charged to a potential on its surface given by $\phi(R, \theta, \phi) = \phi_0 \sin^2 \theta$. Find the electrostatic potential and the electric field outside the sphere.

   (a) (20 pts) Derive the electrostatic potential in spherical coordinates.

   (b) (10 pts) Derive the electric field from your result for the potential using the gradient in spherical coordinates.

   (c) (10 pts) Identify the spherical electric multipole moments q_{lm}.

2. **Spherical multipole moments of discrete charges** [see also Jackson 4.1]: Consider the following charge distribution:

   A charge $-q$ at $(-a/2, a/2, 0)$, a charge $+q$ at $(a/2, a/2, 0)$,
   a charge $+q$ at $(-a/2, -a/2, 0)$, a charge $-q$ at $(a/2, -a/2, 0)$.

   (a) (20 pts) Calculate the spherical multipole moments q_{lm} for the charge distributions. Try to start from the general values l and m to see the general pattern. Write explicit solutions up to the *second non-vanishing* multipole.

   (b) (5 pts) Compare the leading term to the Cartesian result and verify that you obtain $Q_{12} = Q_{21} = 3a^2 q$, $Q_{ij} = 0$ else.

   (c) (optional) Plot the expansion up to some value of l and compare it against the exact result generated by the point charges. [Hint: You may want to use a computer algebra tool which has the relevant functions built in. This can also come in handy for he explicit evaluations in the question above.]

3. **Spherical multipole moments of a charged ring:** A thin circular ring of radius R is located in the xy-plane and centered at the z-axis. In cylindrical coordinates (s, φ, z) the charge density is $\rho = \pm\lambda_0 \delta(z)\delta(s - R)$ with $+$ for $0 \leq \varphi \leq \pi$ and $-$ for $\pi < \varphi < 2\pi$.

   (a) (20 pts) Calculate the spherical dipole moments of the ring. Please perform the calculation "from scratch", that is, do *not* translate the Cartesian multipole moments.

   (b) (5 pts) Based on symmetry considerations, analyze possible restrictions on higher order multipoles.

   (c) (10 pts) Assume a different charge density $\rho = \lambda_0 \sin(\varphi)\, \delta(z)\delta(s - R)$ and find the spherical dipole moment.