Solutions

1. **Simple antenna:**

   (a) The formula for the power of electric dipole radiation is

$$P = \frac{\mu_0}{4\pi c}\frac{2}{3}|\ddot{\vec{p}}|^2 \ . \tag{1}$$

We can replace the derivative of the dipole by the integral of the current using,

$$\begin{aligned}
\ddot{\vec{p}} &= \frac{d^2}{dt'^2}\sum_a q_a \vec{r}_a(t') && (2)\\
&= \frac{d}{dt'}\sum_a q_a \vec{v}_a(t') && (3)\\
&= \frac{d}{dt'}\int d^3r'\,\vec{j}(\vec{r}',t') && (4)\\
&= \hat{z}\frac{d}{dt'}\int I(z',t')\,dz' \ . && (5)
\end{aligned}$$

As was shown in the lecture, one can also derive this formula using continuous charge distributions (instead of point charges), applying the continuity equation and integrate by parts. With $I(z',t') = I_0\cos(\omega t')$ over a length l, we get

$$\ddot{\vec{p}} = -\omega I_0 l \sin(\omega t')\,\hat{z} \ . \tag{6}$$

The power radiated is then

$$P = \frac{\mu_0}{4\pi c}\frac{2}{3}\Big(\omega I_0 l \sin(\omega t')\Big)^2 \ . \tag{7}$$

Averaging over one period of the oscillating current gives

$$\begin{aligned}
\langle \sin^2(\omega t')\rangle_{t'} &= \frac{1}{T}\int_0^T dt'\,\sin^2\omega t' && (8)\\
&= \frac{1}{T}\int_0^T dt'\,\frac{1+\cos 2\omega t'}{2} && (9)\\
&= \frac{1}{2}\ , && (10)
\end{aligned}$$

so

$$\langle P\rangle_t = \frac{\mu_0}{4\pi c}\frac{(\omega I_0 l)^2}{3} \ . \tag{11}$$

While this type of current (non-vanishing at the boundaries) seems unphysical, it can be used as a building block for small segments of more realistic current setups, see the comment at the end of the solution for (b).

(b) Just using our standard approximations we would proceed in a similar way as in (a), but this time for $I(t) = I_0 \cos(\pi z/l)\cos(\omega t)$, and obtain

$$\begin{aligned}
\ddot{\vec{p}} &= \hat{z}\frac{d}{dt'}\int_{-l/2}^{l/2} I(z',t')dz' && (12)\\
&= \hat{z}\frac{d}{dt'}I_0\frac{2l}{\pi}\cos(\omega t') && (13)\\
&= -\omega I_0 \frac{2l}{\pi}\sin(\omega t')\hat{z}. && (14)
\end{aligned}$$

From this we have

$$\langle P\rangle_t \;=\; \frac{\mu_0}{4\pi c}\frac{(\omega I_0 2l/\pi)^2}{3}\;. \tag{15}$$

Here, we assumed that $\lambda \ll l$. In reality, we should observe that this antenna design is used for l matching a standing half-wave, where this approximation is not justified. A suitable treatment for this case is as follows: split the current into n small segments of length l/n, such that we have an approximately constant current within one segment and $l/n \ll \lambda$. Then we can calculate a small dipole contribution to the potential or fields using our simple approximations for each segment at its respective position. Summing these contribution leads to interference effects, such that the angular distribution for the power is modified with respect to the $\sin^2\theta$ distribution we obtain for the "global" $\lambda \ll l$ approximation.

(c) In both cases (a) and (b) the dipole has a fixed orientation in $\hat{z}$ direction, such that for a receiver with a given size the receivable power scales like

$$P_{\text{received}} \propto \frac{\sin^2\theta}{r^2}, \tag{16}$$

where θ is the polar angle of the observer and r is the distance from the source. Note that for a given solid angle, there is no decrease with r, but for a fixed size of the receiver at a specific angle the power falls off with $1/r^2$ (since the solid angle reaching the receiver decreases with the distance). We have

$$\begin{aligned}
\sin^2\theta_a &= 1, && (17)\\
\sin^2\theta_b &= 0, && (18)\\
0 < \sin^2\theta_c &< 1, && (19)
\end{aligned}$$

and

$$r_a = r_c = 1\,\text{m}. \tag{20}$$

Therefore, there is no radiation in direction $\vec{b}$ and the receiver at $\vec{a}$ receives the largest signal.

2. **Non-relativistic charge in $\vec{E}$ field:**

(a) The dipole radiation formula applied to a single point charge with

$$\vec{p} = q\vec{r} \tag{21}$$

is

$$P = \frac{\mu_0}{4\pi c}\frac{2}{3}|\ddot{\vec{p}}|^2 = \frac{\mu_0}{6\pi c}|\ddot{\vec{p}}|^2 = \frac{\mu_0 q^2}{6\pi c}|\ddot{\vec{r}}|^2, \tag{22}$$

where we assumed that the particle is non-relativistic. For a particle moving in a constant electric field, the acceleration is constant,

$$\ddot{\vec{r}} = \frac{q}{m}\vec{E} = \frac{qV}{mb}\vec{z}\ , \tag{23}$$

where we take the direction of the electric field to be the z axis. The power radiated is

$$P = \frac{\mu_0 q^4 V^2}{6m^2 b^2 \pi c}. \tag{24}$$

Since the power is constant, we can find the total energy lost during the trajectory by just multiplying by the time to travel across the capacitor. The time to traverse the capacitor is the solution to the quadratic equation

$$b = u_0 \cos\theta t + \frac{qV}{2mb}t^2\ , \tag{25}$$

or

$$t = \frac{mbu_0}{qV}\left[\sqrt{\cos^2\theta + qV/(mu_0^2/2)} - \cos\theta\right]\ . \tag{26}$$

Therefore the energy lost during the time of flight is

$$\mathcal{E} = \frac{\mu_0 q^3 V u_0}{6bm\pi c}\left[\sqrt{\cos^2\theta + qV/(mu_0^2/2)} - \cos\theta\right]\ . \tag{27}$$

(b) The initial kinetic energy is $K_0 = 4$ eV, but the *final* kinetic energy K_f is much larger, $K_f = K_0 + qV \approx qV = 10^4$ eV (V is the voltage, V is the unit Volt). The mass energy of the particle is $mc^2 \approx 0.5 \times 10^6$ eV. Thus, the final velocity-squared is $\beta^2 = v^2/c^2 \approx 2K_f/(mc^2) \approx 0.04 \ll 1$. Thus, the assumption of non-relativistic kinematics is reasonable.
The initial velocity is related to the initial kinetic energy by $u_0/c = \sqrt{2K_0/(mc^2)} \approx 4 \times 10^{-3}$. For $\theta = 0$, we can then obtain

$$\frac{\mathcal{E}_{\text{lost}}}{K_0} = \frac{\mu_0 c^2}{6\pi}\frac{e^2/(mc^2)}{b}\frac{eV}{K_0}\frac{u_0}{c}\left[\sqrt{1 + eV/K_0} - 1\right]\ .$$

The quantity $r_e = e^2/(mc^2) \approx 3 \times 10^{-13}$ cm is the classical electron radius. Plugging in numbers, we find

$$\frac{\mathcal{E}_{\text{lost}}}{K_0} \approx 10^{-10}\ .$$

The energy lost due to radiation is negligible.

3. **Non-relativistic charge in $\vec{B}$ field:**

   (a) The dipole radiation formula applied to a single point charge is

$$P = \frac{\mu_0}{4\pi c}\frac{2|\ddot{\vec{p}}|^2}{3} = \frac{\mu_0}{4\pi c}\frac{2q^2|\ddot{\vec{r}}|^2}{3}. \tag{28}$$

   The acceleration of a particle moving in uniform circular motion can be expressed as $|\ddot{\vec{r}}| = \omega v$, where the cyclotron frequency is given by

$$\omega = \frac{qB}{m} \tag{29}$$

   in the non-relativistic limit. The rate of change of energy of the proton is the negative of the power radiated by the proton, so we obtain

$$\begin{aligned} \frac{d\mathcal{E}}{dt} &= -P = -\frac{\mu_0}{4\pi c}\frac{2q^2}{3}\left(\frac{qBv}{m}\right)^2 && (30)\\ &= -\frac{\mu_0}{4\pi c}\frac{4q^4B^2}{3m^3}\left(\frac{mv^2}{2}\right) && (31)\\ &= -\Gamma\mathcal{E}\ , && (32) \end{aligned}$$

   where

$$\Gamma = \frac{\mu_0}{4\pi c}\frac{4q^4B^2}{3m^3}\ . \tag{33}$$

   Thus, the energy of the proton as a function of time is

$$\mathcal{E} = \mathcal{E}_0 e^{-\Gamma t}\ . \tag{34}$$

4. **Precessing magnetic dipole:**

   (a) The Larmor frequency for precession of a magnetic dipole is

$$\omega_L = \frac{qB}{2m}\ . \tag{35}$$

   (b) The rotating magnetic dipole will radiate with the standard dipole angular distribution:

$$\frac{dP}{d\Omega} = \frac{\mu_0}{(4\pi)^2 c^3}|\ddot{\vec{m}}|^2 \sin^2\alpha(t). \tag{36}$$

   The rotating magnetic dipole can be written

$$\vec{m} = \mu\Big[\hat{x}\cos(\omega_L t) - \hat{y}\sin(\omega_L t)\Big]\ , \tag{37}$$

   where $\mu = |\vec{m}|$. From this we have $\ddot{\vec{m}} = -\omega_L^2\vec{m}$. The angle $\alpha(t)$ is the angle between $\ddot{\vec{m}}$ and the observation point,

$$\hat{r} = \hat{x}\sin\theta\cos\phi + \hat{y}\sin\theta\sin\phi + \hat{z}\cos\theta\ , \tag{38}$$

   so we have

$$\begin{aligned} \cos\alpha(t) &= -\cos(\omega_L t)\sin\theta\cos\phi + \sin(\omega_L t)\sin\theta\sin\phi && (39)\\ &= -\sin\theta\cos(\omega_L t + \phi)\ . && (40) \end{aligned}$$

Therefore, the angular power of the magnetic dipole radiation is

$$\frac{dP}{d\Omega} = \frac{\mu_0}{(4\pi)^2 c^3}\omega_L^4 \mu^2 \left(1 - \sin^2\theta \cos^2(\omega_L t + \phi)\right) . \tag{41}$$

We discussed a very similar case (for electric dipole radiation) in class, there you can also find some drawings.

(c) Just as in problem 1, we find the average of $\cos^2(\omega_L t + \phi)$ over one period to be

$$\langle \cos^2(\omega_L t + \phi)\rangle_t = \frac{1}{2} . \tag{42}$$

Thus, the angular distribution of the radiated power, averaged over one period is

$$\frac{dP}{d\Omega} = \frac{\mu_0}{(4\pi)^2 c^3}\omega_L^4 \mu^2 \left(1 - \frac{1}{2}\sin^2\theta\right) \tag{43}$$

$$= \frac{\mu_0}{(4\pi)^2 c^3}\omega_L^4 \mu^2 \frac{(1 + \cos^2\theta)}{2} . \tag{44}$$

(d) The total time-averaged power is obtained by integrating over the solid angle $d\Omega = d\phi\, d(\cos\theta)$. We find

$$\int d\phi\, d(\cos\theta)\, \frac{1 + \cos^2\theta}{2} = 2\pi \left(1 + \frac{1}{3}\right) = \frac{8\pi}{3} . \tag{45}$$

Thus, we obtain

$$\langle P\rangle_t = \frac{\mu_0}{4\pi c^3}\frac{2\omega_L^4 \mu^2}{3} . \tag{46}$$

We already performed the time average in part (c), so this is our answer for the time-averaged power.