Solutions

1. **Cylindrical conductor with current:**

   (a) This problem is most conveniently solved using Ampère's law. Due to the geometry there is no z dependence. We consider a circular path C in the (x, y) plane around the z axis and parametrize it using cylindrical coordinates (ρ, φ, z). The distance from the z axis is ρ and the circle is clockwise when looking into the positive z direction, i.e. the path is always in the $\hat{\varphi}$ direction. We have

$$\int_C d\vec{l}\vec{B} = \mu_0 \int_S d\vec{A} \cdot \vec{j} \tag{1}$$

$$B_\varphi 2\pi\rho = \mu_0 I_C \qquad \text{with } I_C(\rho) = \begin{cases} j\pi\rho^2 & \rho \leq R \\ j\pi R^2 & \rho > R \end{cases} \tag{2}$$

$$\Rightarrow \vec{B}(\rho, \varphi) = \begin{cases} \frac{\mu_0 j\rho}{2}\hat{\varphi} & \rho \leq R \\ \frac{\mu_0 j R^2}{2\rho}\hat{\varphi} & \rho > R \end{cases} \tag{3}$$

   (b) We use the general transformation law

$$\vec{E}'_\parallel = \vec{E}_\parallel \qquad \vec{E}'_\perp = \gamma(\vec{E}_\perp + c\vec{\beta} \times \vec{B}) \tag{4}$$

$$\vec{B}'_\parallel = \vec{B}_\parallel \qquad \vec{B}'_\perp = \gamma(\vec{B}_\perp - \frac{1}{c}\vec{\beta} \times \vec{E}) \tag{5}$$

   to boost into the moving frame F'. In the initial frame F there is no net charge and consequently no electric field. For a boost in the z direction we find

$$E'_z = E_z = B'_z = B_z = 0 \tag{6}$$

$$\vec{E}' = -\hat{\rho}\gamma c\beta|\vec{B}| \tag{7}$$

$$\vec{B}' = \gamma\vec{B} \tag{8}$$

   There is an electric field in the moving frame, which can be understood by observing that the zero net charge is a frame-dependent statement: charge and current densities transform as components of a four-vector under boosts.

2. **Rotating cylinder:**

   (a) We compute the magnetic dipole moment according to

$$\vec{m} = \frac{1}{2}\int d^3r'\, \vec{r}' \times \vec{j}(\vec{r}') \tag{9}$$

   for a rotation with angular velocity $\vec{\omega} = \omega\hat{z}$ with $\omega > 0$. For a region of uniform charge density ρ_0 moving with $\vec{v}$ the current density is

$$\vec{j} = \rho_0\vec{v} = \rho_0\rho\omega\hat{\varphi} \tag{10}$$

We use $\vec{r} = \rho\hat{\rho} + z\hat{z}$ and $\vec{r} \times \hat{\varphi} = \rho\hat{z} - z\hat{\rho}$. The $\hat{\rho}$ component will cancel out after integration, while the $\hat{z}$ contributions will accumulate and result in

$$\vec{m} = m\hat{z} \,. \tag{11}$$

where $m = |\vec{m}|$. We find

$$\begin{aligned} m &= \frac{\rho_0\omega}{2} \int_{-h/2}^{h/2} dz \int_0^{2\pi} d\varphi \int_0^R \rho d\rho \; \rho\rho & (12) \\ &= 2\pi\rho_0\omega \int_0^{h/2} dz \int_0^R d\rho \; \rho^3 & (13) \\ &= 2\pi\rho_0\omega \frac{hR^4}{8} & (14) \\ &= \frac{1}{4} Q\omega R^2 & (15) \end{aligned}$$

with the total charge $Q = \rho_0 \pi R^2 h$.

3. At a large distance, the magnetic field due to the rotating cylinder can be approximated by

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \left[\frac{3(\vec{m} \cdot \hat{r})\hat{r} - \vec{m}}{r^3} \right] + \ldots \tag{16}$$

Since we are interested in the field at $\vec{r} = \vec{r}_D = z_D\hat{z}$ and $\vec{m} = m\hat{z}$, this evaluates to

$$\vec{B}(\vec{r}_D) \approx \frac{\mu_0}{4\pi} \frac{2m\hat{z}}{z_D^3} \,. \tag{17}$$

The interaction energy of the second dipole with this magnetic field of the rotating cylinder is

$$U_{\text{orig}} = -\vec{m}_D \cdot \vec{B}(\vec{r}_D) = -\frac{\mu_0}{4\pi} \frac{2mm_D}{z_D^3} \tag{18}$$

in the original configuration. Reversing the direction of the dipole $\vec{m}_D$ leads to a higher energy

$$U_{\text{flipped}} = -(-\vec{m}_D) \cdot \vec{B}(\vec{r}_D) = \frac{\mu_0}{4\pi} \frac{2mm_D}{z_D^3} \tag{19}$$

of the dipole-cylinder system. The increase in energy of the system is

$$\Delta U = U_{\text{flipped}} - U_{\text{orig}} = \frac{\mu_0}{4\pi} \frac{4mm_D}{z_D^3} > 0 \,. \tag{20}$$