Solutions

1. **Plane waves:**

(a) The direction of propagation of the field is clearly in the $+x$ direction. The polarization is right-handed circular. (This can be seen by the fact that $\alpha = 0$ gives $\vec{E} = E_0\hat{y}$, while at the later time $\alpha = \pi/2$ gives $\vec{E} = E_0\hat{z}$, which indicates a vector of constant magnitude E_0, rotating in the right-handed sense around the $+x$ direction.) We can also see this formally by writing

$$\begin{aligned}\vec{E} &= \mathrm{Re}\left[E_0\,(\hat{y} + i\hat{z})\,e^{-i\alpha}\right] \\ &= \mathrm{Re}\left[(\vec{b}_1 + i\vec{b}_2)\,e^{-i\alpha}\right] ,\end{aligned}$$

where $\vec{b}_1 = b_1\,\hat{e}_1 = E_0\,\hat{y}$ and $\vec{b}_2 = b_2\,\hat{e}_2 = E_0\,\hat{z}$. Since $b_1 = b_2$, it is circular polarization, and since $\hat{e}_1 \times \hat{e}_2 = \hat{x} = +\hat{k}$, it is right-handed. The magnetic field can be obtained from $\vec{B} = \frac{1}{c}\hat{x} \times \vec{E}$, or

$$B_x = 0, \qquad B_y = -\frac{1}{c}E_z = -\frac{1}{c}E_0 \sin\alpha, \qquad B_z = \frac{1}{c}E_y = \frac{1}{c}E_0\cos\alpha\ .$$

(b) Some examples of different polarization patterns are

(i) linear polarization: $\beta = 0$ or $\beta = \pi$ or $a_2 = 0$,

(ii) right-handed circular polarization: $\beta = \pi/2$ and $a_1 = a_2$,

(iii) right-handed elliptical polarization: $a_1 \neq a_2$ and $\beta = \pi/2$.

Let's consider the general superposition $\vec{E} = \vec{E}_1 + \vec{E}_2$ of

$$\begin{aligned}\vec{E}_1 &= a_1\,\hat{\mathbf{x}}\,\cos(\mathbf{k}\cdot\mathbf{r} - \omega t - \alpha_1) \\ \vec{E}_2 &= a_2\,\hat{\mathbf{y}}\,\cos(\mathbf{k}\cdot\mathbf{r} - \omega t - \alpha_2)\ .\end{aligned}$$

Note that we allowed here for some offsets α_1 and α_2 in the phases for both $\vec{E}_1$ and $\vec{E}_2$. For definiteness, we take $(\alpha_1 - \alpha_2)/2$ between 0 and $\pi/2$. We want to prove now, that, in general, the combined field will be elliptically polarized. The combined field can be written as

$$\begin{aligned}\vec{E} &= \mathrm{Re}\left[a_1\,\hat{x}\,e^{i(\vec{k}\cdot\vec{r}-\omega t-\alpha_1)} + a_2\,\hat{y}\,e^{i(\vec{k}\cdot\vec{r}-\omega t-\alpha_2)}\right] \\ &= \mathrm{Re}\left[\vec{b}\,e^{i(\vec{k}\cdot\vec{r}-\omega t-\alpha)}\right] ,\end{aligned}$$

where $\vec{b} = \vec{b}_1 + i\vec{b}_2$, and $\vec{b}_1$ and $\vec{b}_2$ are real vectors which give the main axes of the elliptical polarization. To find $\vec{b}$ and α, we have

$$\begin{aligned}\vec{b}^2 e^{-2i\alpha} &= \left(a_1\,\hat{x}\,e^{-i\alpha_1} + a_2\,\hat{y}\,e^{-i\alpha_2}\right)^2 \\ &= a_1^2 e^{-2i\alpha_1} + a_2^2 e^{-2i\alpha_2} ,\end{aligned}$$

and require that $\vec{b}^2$ be real. Let $\alpha = (\alpha' + \alpha_1 + \alpha_2)/2$ and define $\theta = \alpha_1 - \alpha_2$. Then we have

$$\begin{aligned}\mathrm{Im}\left[\vec{b}^2\right] &= \mathrm{Im}\left[a_1^2 e^{i(\alpha'-\theta)} + a_2^2 e^{i(\alpha'+\theta)}\right] \\ &= a_1^2 \sin\left(\alpha' - \theta\right) + a_2^2 \sin\left(\alpha' + \theta\right) \\ &= 2\left(a_1^2 + a_2^2\right)\sin\alpha'\cos\theta - 2\left(a_1^2 - a_2^2\right)\cos\alpha'\sin\theta \\ &= 0\ .\end{aligned}$$

Solving for α', we obtain

$$\tan\alpha' = \frac{a_1^2 - a_2^2}{a_1^2 + a_2^2}\tan\theta \ . \tag{1}$$

Now we can write

$$\begin{aligned} \vec{b} &= \vec{b}_1 + i\vec{b}_2 \\ &= a_1\,\hat{x}\,e^{i(\alpha'-\theta)/2} + a_2\,\hat{y}\,e^{i(\alpha'+\theta)/2} \ . \end{aligned}$$

Taking the real and imaginary parts gives us the vectors describing our two main axes:

$$\begin{aligned} \vec{b}_1 &= a_1\,\hat{x}\,\cos\frac{\alpha'-\theta}{2} + a_2\,\hat{\mathbf{y}}\,\cos\frac{\alpha'+\theta}{2} \\ \vec{b}_2 &= a_1\,\hat{x}\,\sin\frac{\alpha'-\theta}{2} + a_2\,\hat{y}\,\sin\frac{\alpha'+\theta}{2} \ , \end{aligned}$$

where $\theta = \alpha_1 - \alpha_2$, and α' is determined from equation (2). The squared magnitudes are

$$\begin{aligned} b_1^2 &= a_1^2\cos^2\frac{\alpha'-\theta}{2} + a_2^2\cos^2\frac{\alpha'+\theta}{2} \\ b_2^2 &= a_1^2\sin^2\frac{\alpha'-\theta}{2} + a_2^2\sin^2\frac{\alpha'+\theta}{2} \ , \end{aligned}$$

and the directions are

$$\begin{aligned} \hat{e}_1 &= \vec{b}_1/b_1 \\ \hat{e}_2 &= \vec{b}_2/b_2 \ . \end{aligned}$$

With some algebra it is possible to show that the squared magnitudes can be expressed as

$$\begin{aligned} b_{1,2}^2 &= \frac{a_1^2 + a_2^2}{2}\left[1 \pm \frac{\cos\theta}{\cos\alpha'}\right] \\ &= \frac{a_1^2 + a_2^2}{2} \pm \frac{1}{2}\sqrt{(a_1^2 + a_2^2)^2\cos^2\theta + (a_1^2 - a_2^2)^2\sin^2\theta} \ . \end{aligned}$$

(c) In the special case when $a_1 = a_2 = a$, equation (2) gives $\alpha' = 0$. Then our two vectors become

$$\begin{aligned} \vec{b}_1 &= (\hat{x} + \hat{y})\,a\cos\frac{\theta}{2} = \left(\sqrt{2}a\cos\frac{\theta}{2}\right)\hat{e}_1 \\ \vec{b}_2 &= (\hat{y} - \hat{x})\,a\sin\frac{\theta}{2} = \left(\sqrt{2}a\sin\frac{\theta}{2}\right)\hat{e}_2 \ , \end{aligned}$$

where

$$\begin{aligned} \hat{e}_1 &= \frac{\hat{x} + \hat{y}}{\sqrt{2}} \\ \hat{e}_2 &= \frac{\hat{y} - \hat{x}}{\sqrt{2}} \ . \end{aligned}$$

Since $\hat{e}_1 \times \hat{e}_2 = +\hat{k}$, this corresponds to right-handed polarization. (Here we used the fact that $0 < \theta/2 < \pi/2$ when obtaining the directions $\hat{e}_1$ and $\hat{e}_2$.)

2. **Pulse:**

   (a) One way to immediately see that there is no spread of the wave packet with time would be along the lines of the arguments in section 4.2.1 of this course, when we discussed general solution of the vacuum wave equation. Here, we take a somewhat tedious detour by computing a Fourier transformation to highlight the fact, that the linear relation between ω and k avoids spreading of the wave packet. In more complicated situations, e.g. due to the presence of media, this dispersion relation could be altered, leading to a spreading.

   We know the function $F(z,t)$ only at time $t = 0$ and we want to study its full time dependence. Expand the wave in its Fourier components

$$F(z,t) = \int dk\, \tilde{F}(k) e^{ik(z-ct)} \,,$$

   where we used $\omega = ck$ and the inverse Fourier transform is

$$\begin{aligned}
\tilde{F}(k) &= \frac{1}{2\pi}\int dz\, F(z,0)e^{-ikz} \\
&= \frac{1}{2\pi}\int dz\, \frac{a}{\sqrt{2\pi\sigma^2}} e^{-z^2/(2\sigma^2)} e^{-ikz} \\
&= \frac{1}{2\pi}\frac{a}{\sqrt{2\pi\sigma^2}}\int dz\, e^{-(z+ik\sigma^2)^2/(2\sigma^2)} e^{-k^2\sigma^2/2} \\
&= \frac{1}{2\pi} a e^{-k^2\sigma^2/2} \,.
\end{aligned}$$

   Plugging back into the original Fourier expansion gives

$$\begin{aligned}
F(z,t) &= \int dk \left(\frac{1}{2\pi} a e^{-k^2\sigma^2/2}\right) e^{ik(z-ct)} \\
&= \frac{a}{2\pi}\int dk\, \exp\left[\frac{-\sigma^2\Big(k - i(z-ct)/\sigma^2\Big)^2}{2}\right] e^{-(z-ct)^2/(2\sigma^2)} \\
&= \frac{a}{\sqrt{2\pi\sigma^2}} e^{-(z-ct)^2/(2\sigma^2)} \,.
\end{aligned}$$

   Thus, the wave packet moves at the velocity of light, maintaining its shape with no spreading. This is in contrast to the situation of massive particles in quantum mechanics, which usually do have spreading of their wave packet. For light in vacuum there is no dispersion, so all Fourier components of the packet move with the same velocity. Therefore, the group velocity and the phase velocity are equal.

3. **Plane Wave and Cylinder:**

   (a) There are two ways to obtain the force on the cylinder. The first, more heuristic way, is to imagine the wave moving to the cylinder. After some time Δt, a volume ΔV in front of the cylinder will have crossed the cylinder surface pointing in that direction. The cylinder completely absorbs the momentum $\Delta\vec{p}$ carried by the field in that volume. The momentum density carried by the field is

$$\vec{g} = \epsilon_0 \vec{E}\times\vec{B} = \hat{x}\frac{\epsilon_0}{c}E_0^2(\cos^2\alpha + \sin^2\alpha) = \hat{x}\frac{\epsilon_0}{c}E_0^2$$

(Note that this is time-independent. If the fields were linearly polarized, then the momentum density would oscillate in time with a factor $\cos^2\alpha$. After time-averaging over one period, the result for linear polarization would be reduced by a factor 1/2.) The volume of wave absorbed by the cylinder in time Δt is

$$\Delta V = (2Rh)(c\Delta t) ,$$

where $2Rh$ is the cross-sectional area of the cylinder, and $c\Delta t$ is the distance the wave travelled in the time Δt. Then

$$\Delta\vec{p} = \vec{g}\,\Delta V = \vec{g}\,(2Rh)(c\Delta t) ,$$

and the force applied to the cylinder is

$$\vec{F} = \frac{\Delta\vec{p}}{\Delta t} = \vec{g}\,(2Rhc) = \hat{x}\epsilon_0 2E_0^2 Rh .$$

The more rigorous way to obtain the force on the cylinder is to use the conservation of momentum equation for the field:

$$\frac{dp^i_{\text{field}}}{dt} = \int \sigma^{ij}\,da^j ,$$

where

$$\sigma^{ij} = \epsilon_0\left(E_iE_j - \frac{1}{2}\delta_{ij}|\vec{E}|^2\right) + \frac{1}{\mu_0}\left(B_iB_j - \frac{1}{2}\delta_{ij}|\vec{B}|^2\right) \tag{2}$$

is the Maxwell stress tensor. This formula gives the rate of change of momentum of the field across the surface. We integrate only over that part of the cylinder surface facing against the direction of the the wave propagation, since we assume that that part of the surface already absorbs all of the wave. We obtain

$$\sigma^{ij} = -\left(\delta^{ix}\delta^{jx}\right)\epsilon_0 E_0^2 .$$

Using conservation of momentum at the surface tells us that the force on the cylinder is

$$\vec{F} = -\frac{d\vec{p}_{\text{field}}}{dt} = -\hat{x}\,\sigma^{xx}\,(2Rh) = \hat{x}\epsilon_0 2E_0^2 Rh ,$$

which agrees with our previous answer.

4. **Fields in a Hollow Cylinder:**

(a) The Poynting vector is

$$\vec{S} = \frac{1}{\mu_0}\vec{E}\times\vec{B} = \hat{z}\frac{1}{\mu_0}\frac{c_1c_2}{\rho^2} . \tag{3}$$

(b) The total flux of electromagnetic energy through a cross-sectional surface is (for one choice of what the "outside" of the surface is):

$$\int_S \vec{S}\cdot d\vec{A} = \int_S \vec{S}\cdot\hat{z}dA = \int_a^b d\rho\,\rho\int_0^{2\pi} d\varphi\,\frac{1}{\mu_0}\frac{c_1c_2}{\rho^2} = \frac{2\pi c_1c_2\ln(b/a)}{\mu_0} \tag{4}$$

(c) The electromagnetic energy per unit volume is given by

$$u = \frac{1}{2}\left(\epsilon_0|\vec{E}|^2 + \frac{1}{\mu_0}|\vec{B}|^2\right) = \frac{1}{2\rho^2}\left(\epsilon_0 c_1^2 + \frac{c_2^2}{\mu_0}\right) . \tag{5}$$

Therefore we have for the energy per unit length

$$\frac{d\mathcal{E}}{dz} = \int_S dA\, u = \int_a^b d\rho\, \rho \int_0^{2\pi} d\varphi\, \frac{1}{2\rho^2}\left(\epsilon_0 c_1^2 + \frac{c_2^2}{\mu_0}\right) \tag{6}$$

$$= \pi \ln\left(\frac{b}{a}\right)\left(\epsilon_0 c_1^2 + \frac{c_2^2}{\mu_0}\right) \tag{7}$$

Similarly, the electromagnetic momentum per unit volume is given by

$$\vec{g} = \epsilon_0 \vec{E} \times \vec{B} = \hat{z}\epsilon_0 \frac{c_1 c_2}{\rho^2}. \tag{8}$$

Thus the momentum per unit length is

$$\frac{d\vec{p}}{dz} = \int_S dA\, \vec{g} = \int_a^b d\rho\, \rho \int_0^{2\pi} d\varphi\, \hat{z}\epsilon_0 \frac{c_1 c_2}{\rho^2} = \hat{z}\epsilon_0 c_1 c_2 2\pi \ln(b/a)\,. \tag{9}$$

(d) A non-vanishing Poynting vector does not imply the energy of energy through radiation. The energy conservation law in its differential form reads

$$\vec{E} \cdot \vec{j} + \frac{\partial u}{\partial t} + \vec{\nabla} \cdot \vec{S} = 0\,. \tag{10}$$

Since only the divergence of $\vec{S}$ matters, a pure curl would not contribute. Let's consider now the integrated form of the energy conservation law. Energy loss or gain for some volume is present if the surface integral of the Poynting vector over an enclosing surface is non-vanishing. This integral can of course be zero even if $\vec{S}$ is not. Consider e.g. an electric monopole and a magnetic dipole at rest at the origin. There is a non-vanishing Poynting vector despite the absence of radiation.