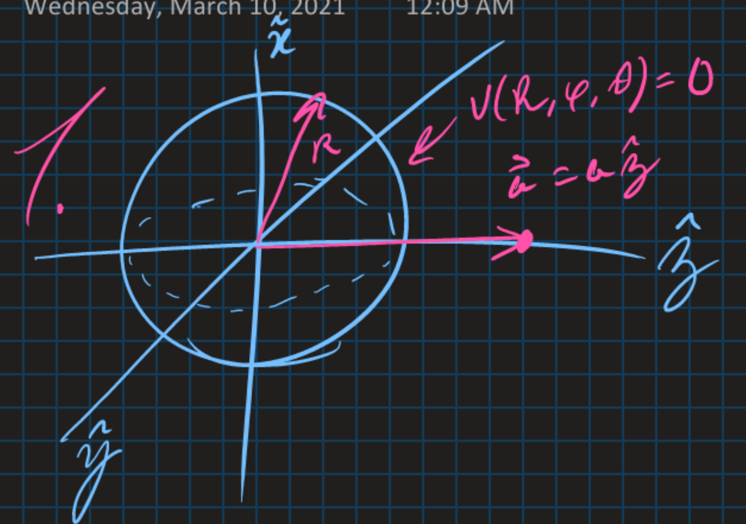
Homework 7

Wednesday, March 10, 2021 12:09 AM

1.



a) The potential is given by

$$\phi=\frac{1}{4\pi\varepsilon_0}\frac{q}{|\vec{r}-\vec{a}|}+\sum_{\ell=0}^{\infty}\left(A_\ell r^\ell+B_\ell r^{-\ell-1}\right)P_\ell(\cos\vartheta).$$

b) The boundary conditions are

$$\phi(r=R)=0,$$
$$\phi(r\to\infty)\to 0.$$

Hence $A_\ell=0$, and

$$0=\frac{1}{4\pi\varepsilon_0}\frac{q}{a-R}+\sum_{\ell=0}^{\infty}B_\ell R^{-\ell-1}P_\ell(\cos\vartheta).$$

This requires B_ℓ to be given by

$$B_\ell=-\frac{2\ell+1}{2}\frac{R^{\ell+1}}{4\pi\varepsilon_0}\frac{q}{a-R}\int_{-1}^{1}P_\ell(x)\,dx,$$
$$=\begin{cases}-\frac{R}{4\pi\varepsilon_0}\frac{q}{a-R}, & \ell=0\\ 0, & \ell\neq 0\end{cases}.$$

c) So,

$$\phi(\vec{r})=\frac{1}{4\pi\varepsilon_0}\frac{q}{|\vec{r}-\vec{a}|}-\frac{1}{4\pi\varepsilon_0}\frac{q}{a-R}\frac{R}{r},$$
$$=\frac{1}{4\pi\varepsilon_0}\frac{q}{|\vec{r}-\vec{a}|}-\frac{1}{4\pi\varepsilon_0}\frac{R}{a}\frac{q}{|\vec{r}-\vec{a}'|},$$

where $\vec{a}'=\frac{R^2}{a}\hat{z}$. This is equivalent to placing a charge of $q'=-\frac{R}{a}q$ at $\vec{a}'$.

d) The surface charge density is given by

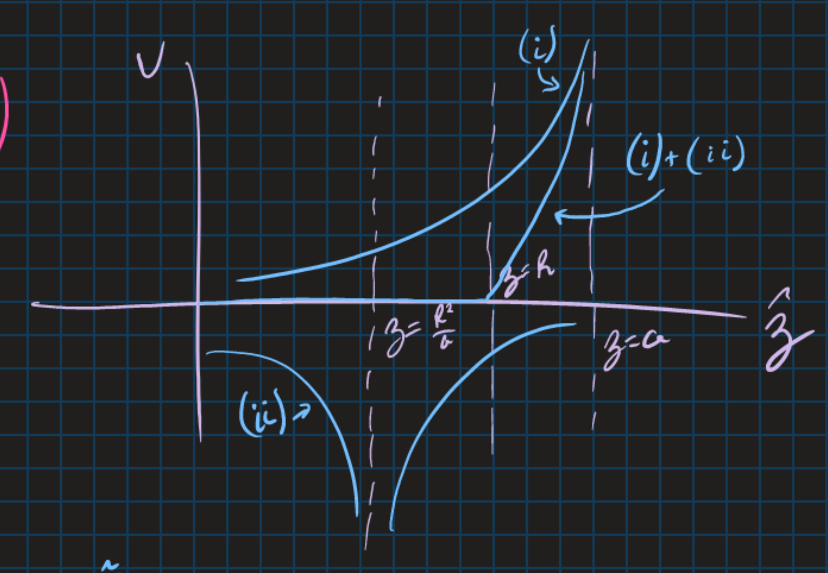
$$\sigma=\varepsilon_0\frac{\partial\phi}{\partial\hat{n}}\Big|_R=-\varepsilon_0\frac{\partial\phi}{\partial z}\Big|_{z=R}=-\frac{q}{4\pi R^2}\frac{R}{a}\frac{1-\frac{R^2}{a^2}}{\left(1+\frac{R^2}{a^2}-2\frac{R}{a}x\right)^{3/2}}.$$

As $a\to R$, $\sigma\to 0$.
As $a\to\infty$, $\sigma\to 0$.

e) Integrating this gives the total induced charge:

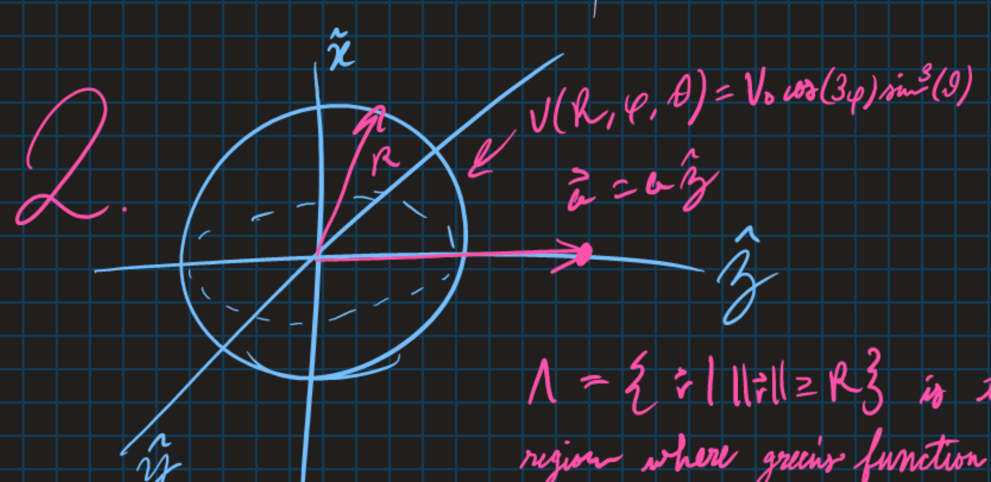
$$2\pi R^2\int_0^{\pi}\sigma(\cos\vartheta)\sin\vartheta\,d\vartheta=-\frac{R}{a}q.$$

f)



If the sphere is held at a fixed potential, I am not sure what the potential outside the sphere would look like.

2.



$\Lambda=\{\vec{r}\mid\|\vec{r}\|\geq R\}$ is the region where green's function is being considered.

So the unit normal vectors on $\partial\Lambda$ (the surface of Λ) are

$$\hat{n}_R=-\hat{r}\quad\text{and}\quad\hat{n}_\infty=\hat{r}.$$

For Dirichlet boundary conditions,

$$G_D(\vec{r},\vec{r}\,')=0\quad\text{when } \vec{r}\,'\in\partial\Lambda,$$

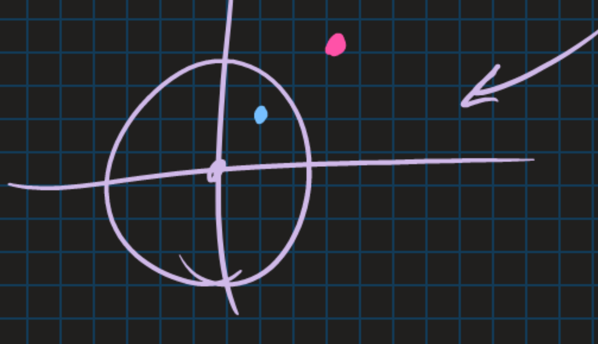
so,

$$G_D\big|_{r'=R}=0,\qquad\forall\vec{r}\in\Lambda$$
$$G_D\big|_{r'\to\infty}=0,\qquad\forall\vec{r}\in\Lambda.$$

Additionally,

$$\vec{\nabla}^2G_D(\vec{r},\vec{r}\,')=-4\pi\delta(\vec{r}-\vec{r}\,'),\qquad\forall\vec{r},\vec{r}\,'\in\Lambda.$$

To satisfy the $G=0$ conditions on the boundary, construct G with image charges:



same setting (basically) as the first problem.

$$\mathcal{G}(\vec{r},\vec{r}\,')=\frac{1}{|\vec{r}-\vec{r}\,'|}-\frac{R/r'}{|\vec{r}-(R^2/r'^2)\vec{r}\,'|}.$$

From this, the potential is given by

$$\phi=-\frac{1}{4\pi}\int dA'\,\frac{\partial\mathcal{G}(\vec{r},\vec{r}\,')}{\partial n'}\,\phi(\vec{r}\,').$$
$$=\frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi}V(\vartheta',\varphi')\frac{\partial}{\partial r'}\mathcal{G}(\vec{r},\vec{r}\,')R^2\sin\vartheta'\,d\varphi'\,d\vartheta'$$

Using the spherical expansion of $\partial\mathcal{G}/\partial r'$,

$$\frac{\partial\mathcal{G}}{\partial r'}=\frac{4\pi}{R^2}\sum_{\ell,m}\left(\frac{R}{r}\right)^{\ell+1}Y^*_{\ell m}(\vartheta',\varphi')Y_{\ell m}(\vartheta,\varphi),$$

gives

$$\phi=\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\left(\frac{R}{r}\right)^{\ell+1}\int d\Omega'\,V(\vartheta',\varphi')Y^*_{\ell m}(\vartheta',\varphi')\;Y_{\ell m}(\vartheta,\varphi).$$

In terms of $Y_{\ell m}$, $V(\vartheta,\varphi)$ is given by

$$V(\vartheta,\varphi)=-\sqrt{\frac{16}{35\pi}}\,V_0\left[Y_{3,3}(\vartheta,\varphi)-Y_{3,-3}(\vartheta,\varphi)\right].$$

Done the integral evaluates to

$$\int d\Omega'\,V(\vartheta',\varphi')Y^*_{\ell m}(\vartheta',\varphi')=-\sqrt{\frac{16}{35\pi}}\,V_0\left[\delta_{\ell,3}\delta_{m,3}-\delta_{\ell,3}\delta_{m,-3}\right].$$

Ainsi the potential is given by

$$\phi=-\sqrt{\frac{16}{35\pi}}\left(\frac{R}{r}\right)^4V_0\left[Y_{3,3}-Y_{3,-3}\right].$$
$$=\left(\frac{R}{r}\right)^4V(\vartheta,\varphi).$$