Homework 10

Brandon Henke
Andreas von Manteuffel
PHY841

April 2, 2021

10.1 A Current Loop

From Jackson, the magnetic moment is given by

$$\mathbf{m} = \frac{I}{2} \oint \mathbf{x} \times \mathrm{d}\mathbf{l}\,.$$

The integral can be split into eight pieces, one for each side. Start with two sides that are parallel to the $\hat{\mathbf{z}}$-axis:

$$\begin{aligned}\mathbf{m}_1 &= -\frac{I}{2} \int_{-b}^{b} (x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}) \times \mathrm{d}z\,\hat{\mathbf{z}}, \\ &+ \frac{I}{2} \int_{-b}^{b} (-x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}) \times \mathrm{d}z\,\hat{\mathbf{z}}, \\ &= I\hat{\mathbf{y}} \int_{-b}^{b} x\,\mathrm{d}z\,, \\ &= 2Ib^2\hat{\mathbf{y}}. \qquad (x = b)\end{aligned}$$

Due to symmetry,

$$\mathbf{m} = 4Ib^2\hat{\mathbf{y}}.$$

10.2 Rotating Square Loop

On average, the charge disrobution is a charged hollow cylinder of radius a and height a. The outside of the cylinder has a surface charge density of $\sigma_1 = \lambda/2\pi a$. The top and botton have a surface charge density of $\sigma_2(r) = \lambda/2\pi r$. Thus, the charge density is given by

$$\begin{aligned}j(\mathbf{r}) &= \omega r \sigma_1 \delta(r-a)\Theta(a/2 - \left|z - a/2\right|) \\ &+ \omega r \sigma_2(r)(\delta(z) + \delta(z - a/2))\Theta(a - r).\end{aligned} \tag{10.2.1}$$

From these surface charge densities, one just needs to find the magnetic moment of the cylinder, rotating at an angular velocity of ω:

$$\mathbf{m} = \frac{1}{2} \iiint j(\mathbf{r})\mathbf{r} \times \hat{\boldsymbol{\phi}} r\,\mathrm{d}r\,\mathrm{d}\phi\,\mathrm{d}z\,.$$

Since $\mathbf{r}$ and $\hat{\boldsymbol{\phi}}$ are always perpendicular, and the symmetry of the situation causes the radial and azymuthal components to vanish, the magnetic moment becomes

$$\mathbf{m} = \pi\hat{\mathbf{z}} \iint j(\mathbf{r})\sqrt{r^2 + z^2}\cos(\theta) r\,\mathrm{d}r\,\mathrm{d}z\,,$$

where θ is the angle between the $\mathbf{r}$ and $\hat{\mathbf{z}}$. This simplifies to

$$\mathbf{m} = \pi\hat{\mathbf{z}} \iint j(\mathbf{r}) r z\,\mathrm{d}r\,\mathrm{d}z\,.$$

Plugging in 10.2.1 and integrating gives

$$\begin{aligned}\mathbf{m} &= \pi\omega\sigma_1\hat{\mathbf{z}} \int_0^a a^2 z\,\mathrm{d}z \\ &+ \pi\omega\hat{\mathbf{z}} \int_0^a r^2 a \sigma_2(r)\,\mathrm{d}r\,, \\ &= \omega\frac{\lambda}{4}\hat{\mathbf{z}}\left[a^4 + a^3\right].\end{aligned}$$

10.3 Non-relativistic Particle in a Magnetic Field

I didn't have time to finish this. :(