Homework 5

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March 1, 2021

5.1 A Simple Poisson Equation

a

The Laplace operator, in cartesian coordinates, is given by

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}.$$

One can use a change of variables to find how the Laplace operator acts on a function that only depends on the distance from the origin, $f(r)$. The distance from the origin is $r = \sqrt{x^2 + y^2 + z^2}$. Taking the second derivative with respect to each of the cartesian coordinates is the same process:

$$\begin{aligned}
\frac{\partial^2}{\partial x^2} &= \frac{\partial}{\partial x}\left(\frac{\partial r}{\partial x}\frac{\partial}{\partial r}\right), \\
&= \frac{\partial^2 r}{\partial x^2}\frac{\partial}{\partial r} + \left(\frac{\partial r}{\partial x}\right)^2 \frac{\partial^2}{\partial r^2}, \\
&= \left(\frac{1}{r} - \frac{x^2}{r^3}\right)\frac{\partial}{\partial r} + \frac{x^2}{r^2}\frac{\partial^2}{\partial r^2}, \\
&= \frac{1}{r}\left(\left(1 - \frac{x^2}{r^2}\right)\frac{\partial}{\partial r} + \frac{x^2}{r}\frac{\partial^2}{\partial r^2}\right)
\end{aligned}$$

Since the processes is the same for each variable,

$$\begin{aligned}
\nabla^2 &= \frac{1}{r}\left(\left(3 - \frac{r^2}{r^2}\right)\frac{\partial}{\partial r} + \frac{r^2}{r}\frac{\partial^2}{\partial r^2}\right), \\
&= \frac{1}{r}\left(2\frac{\partial}{\partial r} + r\frac{\partial^2}{\partial r^2}\right), \\
\nabla^2 f(r) &= \frac{1}{r}\frac{\partial^2}{\partial r^2}\big(r f(r)\big).
\end{aligned}$$

b

The possion equation for this charge distribution is given by

$$\nabla^2 \phi(\mathbf{r}) = \frac{\rho_0}{\epsilon_0} e^{-\mu r}. \tag{5.1.1}$$

Since the charge distribution is radially symmetric, the potential must also be radially symmetric, since rotating the system in any way doesn't change the problem. From part a, 5.1.1 becomes

$$\frac{1}{r}\frac{\partial^2}{\partial r^2}\big(r\phi(r)\big) = \frac{\rho_0}{\epsilon_0} e^{-\mu r}.$$

One can easily find the solution to this through integration[1]:

$$\begin{aligned}
\frac{\partial}{\partial r}\big(r\phi(r)\big) &= \frac{\rho_0}{\epsilon_0}\int r e^{-\mu r}\,\mathrm{d}r\,, \\
&= c_0 - \frac{e^{-\mu r}(\mu r + 1)}{\mu^2}, \\
r\phi(r) &= \int c_0 - \frac{e^{-\mu r}(\mu r + 1)}{\mu^2}\,\mathrm{d}r\,, \\
&= c_1 + c_0 r + \frac{\rho_0 e^{-\mu r}\left(\frac{2}{\mu} + r\right)}{\mu^2 \epsilon_0}, \\
\phi(r) &= c_0 + \frac{c_1}{r} + \frac{\rho_0 e^{-\mu r}\left(\frac{2}{\mu} + r\right)}{\mu^2 \epsilon_0 r}.
\end{aligned}$$

c

Since there is no charge density at infinity, the potential can be set to zero. Hence $c_0 = 0$V. Something weird is going on. Using the integral form of Gauss' law gives an electric field and potential that do not blow up at $r = 0$m, yet solving Poisson's equation gave a solution that does blow up at $r = 0$m.

5.2 Potential for a Box

Being that there is no charge in the box, Gauss' law becomes

$$\nabla^2 \phi = 0. \tag{5.2.1}$$

The boundary conditions are

$$\begin{aligned}
\phi(0, y, z) = \phi(a, y, z) &= 0, && (5.2.2)\\
\phi(x, 0, z) = \phi(x, b, z) &= 0, && (5.2.3)\\
\phi(x, y, 0) &= 0, && (5.2.4)\\
\phi(x, y, c) &= c_0 x, && (5.2.5)
\end{aligned}$$

where c_0 is some constant.

a

Make the assumption that the solution to 5.2.1 is seperable:

$$\phi(x, y, z) = X(x)Y(y)Z(z).$$

Then 5.2.1 becomes

$$\frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} = 0.$$

[1] These integrals were done with the help of Mathematica.

For this to be the case, each term must be a constant, such that their sum vanishes:

$$
\begin{aligned}
\frac{X''}{X} &= -k_x^2, \\
\frac{Y''}{Y} &= -k_y^2, \\
\frac{Z''}{Z} &= k_z^2, \\
k_z^2 &= k_x^2 + k_y^2.
\end{aligned}
$$

These ODEs have the following solutions:

$$
\begin{aligned}
X(x) &= X_0 \sin(k_x x) + X_1 \cos(k_x x), \\
Y(y) &= Y_0 \sin(k_y y) + Y_1 \cos(k_y y), \\
Z(z) &= Z_0 \sinh(k_z z) + Z_1 \cosh(k_z z).
\end{aligned}
$$

b

Using 5.2.2-5.2.4,

$$
\begin{aligned}
X_1 = 0,\ k_x &= \frac{m\pi}{a}, \\
Y_1 = 0,\ k_y &= \frac{n\pi}{b}, \\
Z_1 &= 0.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
X(x) &= X_0 \sin\left(\frac{m\pi}{a} x\right), \\
Y(y) &= Y_0 \sin\left(\frac{n\pi}{b} y\right), \\
Z(z) &= Z_0 \sinh(k_{mn} \pi z),
\end{aligned}
$$

where

$$
k_{mn} = \sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}}.
$$

Using 5.2.5 gives

$$
V(x) = \sum_{m,n}^{\infty} A_{mn} \sin\left(\frac{m\pi}{a} x\right) \sin\left(\frac{n\pi}{b} y\right) \sinh(k_{mn} c\pi).
$$

In order to find the coefficients, A_{mn}, one can exploit the inner product

$$
\int_0^\tau \sin\left(\frac{m\pi}{\tau} s\right) \sin\left(\frac{n\pi}{\tau} s\right) \mathrm{d}s = \frac{\tau}{2} \delta_{mn}.
$$

Multiplying both sides by

$$
\sin\left(\frac{m\pi}{a} x\right) \sin\left(\frac{n\pi}{b} y\right),
$$

and using the identity

$$
\int_0^b \sin^2\left(\frac{n\pi}{b} \eta\right) \mathrm{d}\eta = \frac{b}{2},
$$

one can integrate over x and y to get[2]

$$
\begin{aligned}
A_{mn} &= \frac{b c_0}{2 \sinh(k_{mn} c\pi)} \int_0^a \xi \sin^2\left(\frac{m\pi}{a} \xi\right) \mathrm{d}\xi\,, \\
&= \frac{a^2 b c_0}{8 \sinh(k_{mn} c\pi)}.
\end{aligned}
$$

This gives the final solution to be

$$
\phi(x,y,z) = \sum_{m,n}^{\infty} A_{mn} \sin\left(\frac{m\pi}{a} x\right) \sin\left(\frac{n\pi}{b} y\right) \sinh(k_{mn} z\pi),
$$

where

$$
A_{mn} = \frac{a^2 b c_0}{8 \sinh(k_{mn} c\pi)} \quad \text{and} \quad k_{mn} = \sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}}.
$$

[2] This integral was done with the assistance of Mathematica.