Solutions

1. **Sphere at fixed potential:**

(a) Since the potential on the surface is azimuthally symmetric, we can assume that the solution for the potential will also be azimuthally symmetric. Thus, we have

$$\phi(r,\theta,\varphi) \;=\; \sum_l \left(A_l r^l + B_l r^{-(l+1)}\right) P_l(\cos\theta)\ . \tag{1}$$

As $r\to\infty$, we must have $\phi\to 0$, so $A_l = 0$ for all l. To find the B_l, we use

$$\phi_0 \sin^2\theta \;=\; \sum_l B_l R^{-(l+1)} P_l(\cos\theta)\ . \tag{2}$$

We could take the integral over $\cos\theta$ of both sides of this equation with the Legendre Polynomials, $P_l(\cos\theta)$; however, it is easier to note that

$$\begin{aligned}\phi_0 \sin^2\theta \;&=\; \phi_0\left[1-\cos^2\theta\right] \;=\; \phi_0\left[\frac{2}{3}-\frac{2}{3}\left(\frac{3}{2}\cos^2\theta-\frac{1}{2}\right)\right]\\ &=\; \frac{2\phi_0}{3}P_0(\cos\theta) - \frac{2\phi_0}{3}P_2(\cos\theta)\ ,\end{aligned} \tag{3}$$

where $P_0(\cos\theta) = 1$ and $P_2(\cos\theta) = (3\cos^2\theta - 1)/2$. Thus, we obtain $B_l = 0$ for $l \neq 0, 2$, and

$$B_0 \;=\; \frac{2\phi_0 R}{3} \tag{4}$$

$$B_2 \;=\; -\frac{2\phi_0 R^3}{3}\ , \tag{5}$$

and the electrostatic potential outside the sphere is

$$\phi(r,\theta,\varphi) = \frac{2\phi_0}{3}\left(\frac{R}{r} - \frac{R^3}{r^3}P_2(\cos\theta)\right) \tag{6}$$

(b) The electric field is obtained from

$$\begin{aligned}\vec{E} = -\vec{\nabla}\phi \;&=\; -\frac{\partial\phi}{\partial r}\,\hat{r} - \frac{1}{r}\frac{\partial\phi}{\partial\theta}\,\hat{\theta} - \frac{1}{r\sin\theta}\frac{\partial\phi}{\partial\varphi}\,\hat{\varphi}\\ &= \frac{2\phi_0}{3}\left(\frac{R}{r^2} - \frac{3R^3}{r^4}P_2(\cos\theta)\right)\hat{r} - \phi_0\sin(2\theta)\frac{R^3}{r^4}\,\hat{\theta}\ .\end{aligned} \tag{7}$$

(c) Our solution is a special case of a spherical multipole expansion with $m = 0$ due to the azimuthal symmetry:

$$\phi = \frac{1}{4\pi\epsilon_0}\sum_{l=0}^{\infty}\frac{4\pi}{2l+1}\frac{q_{l0}}{r^{l+1}}Y_{l0}(\theta,\varphi) \tag{8}$$

$$= \frac{1}{4\pi\epsilon_0}\sum_{l=0}^{\infty}\sqrt{\frac{4\pi}{2l+1}}\frac{q_{l0}}{r^{l+1}}P_l(\cos\theta) \tag{9}$$

(where we used $Y_{l0}(\theta,\varphi) = P_l(\cos\theta)\sqrt{2l+1}/\sqrt{4\pi}$). We see that the potential is the sum of a monopole and a quadrupole potential, while moments for the dipole and multipoles higher than the quadrupole vanish. We have

$$q_{00} = \sqrt{4\pi}\epsilon_0 2\phi_0 R/3 \qquad \Rightarrow Q_{\rm tot} = q_{00}\sqrt{4\pi} = (4\pi\epsilon_0)\, 2\phi_0 R/3, \tag{10}$$

$$q_{20} = -\sqrt{4\pi}\epsilon_0\sqrt{5}\, 2\phi_0 R^3/3, \qquad \Rightarrow Q_{zz} = q_{20}\, 2\sqrt{4\pi}/\sqrt{5} = -(4\pi\epsilon_0)\, 4\phi_0 R^3/3 \tag{11}$$

$$q_{lm} = 0 \text{ else} \qquad \Rightarrow Q_{xx} = Q_{yy} = -Q_{zz}/2, \tag{12}$$

$$Q_{ij} = 0 \text{ for } i \neq j \tag{13}$$

In these equation, we have also translated the spherical results to Cartesian multipoles in order to make a connection to the previously discussed Cartesian expansion. In the Cartesian formulation, we have for the monopole and the quadrupole contributions

$$E_k = \frac{1}{4\pi\epsilon_0}\left(\frac{Q_{\rm tot}\hat{r}_k}{r^2} + \frac{Q_{ij}(\delta_{ik}\hat{r}_j + \delta_{jk}\hat{r}_i - 5(\hat{r}_i\hat{r}_j)\hat{r}_k}{2!\, r^4})\right) \tag{14}$$

$$= \frac{1}{4\pi\epsilon_0}\left(\frac{Q_{\rm tot}\hat{r}_k}{r^2} + \frac{2Q_{kk}\hat{r}_k - 5\hat{r}_k Q_{xx}(1-3\hat{r}_z^2)}{2r^4})\right) \tag{15}$$

and therefore

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\left(\frac{Q_{\rm tot}\hat{r}}{r^2} + \frac{Q_{xx}(\hat{r} - 3\hat{r}_z\hat{z} + 5\hat{r}P_2(\cos\theta))}{r^4}\right) \tag{16}$$

$$= \frac{1}{4\pi\epsilon_0}\left(\frac{Q_{\rm tot}\hat{r}}{r^2} + \frac{3Q_{xx}(\hat{r}P_2(\cos\theta) + \sin\theta\cos\theta\,\hat{\theta})}{r^4}\right) \tag{17}$$

$$= \frac{2\phi_0}{3}\left(\frac{R}{r^2} - \frac{3R^3}{r^4}P_2(\cos\theta)\right)\hat{r} - \phi_0\sin(2\theta)\frac{R^3}{r^4}\,\hat{\theta}\ . \tag{18}$$

in agreement with the result above. In the derivation $\hat{z} = \cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}$ was used.

2. **Spherical multipole moments of discrete charges:**

(a) Let us consider two different charge distributions:

i. A charge $-q$ at $(-a/2, a/2, 0)$, a charge $+q$ at $(a/2, a/2, 0)$, a charge $+q$ at $(-a/2, -a/2, 0)$, a charge $-q$ at $(a/2, -a/2, 0)$.

ii. A charge $+q$ at $(-a/2, a/2, 0)$, a charge $+q$ at $(a/2, a/2, 0)$, a charge $-q$ at $(-a/2, -a/2, 0)$, a charge $-q$ at $(a/2, -a/2, 0)$.

The problem asked only for the multipole expansion of charge distribution (i), but it is interesting to compare it to the case (ii). The spherical multipole moments are given by

$$q_{lm} = \int d^3r'\, Y^*_{lm}(\varphi,\theta)\, r'^l \rho(\vec{r}') \tag{19}$$

$$= \int d^3r' \sqrt{\frac{(2l+1)(l-m)!}{4\pi(l+m)!}} P_l^m(\cos\theta) e^{-im\varphi} r'^l \rho(\vec{r}') \tag{20}$$

For the first set of charges we have

$$\begin{aligned}\rho^{(i)}(\vec{r}) = q[&-\delta(x-(-a/2))\,\delta(y-a/2)\delta(z)\,+\delta(x-a/2)\delta(y-a/2)\delta(z)\\ &+\delta(x-(-a/2))\delta(y-(-a/2))\,\delta(z)\,-\delta(x-a/2)\delta(y-(-a/2))\delta(z)]\end{aligned} \tag{21}$$

and for the second set of charges

$$\begin{aligned}\rho^{(ii)}(\vec{r}) = q[&+\delta(x-(-a/2))\,\delta(y-a/2)\delta(z)\,+\delta(x-a/2)\delta(y-a/2)\delta(z)\\ &-\delta(x-(-a/2))\delta(y-(-a/2))\,\delta(z)\,-\delta(x-a/2)\delta(y-(-a/2))\delta(z)]\end{aligned} \tag{22}$$

We therefore have

$$\begin{aligned}q^{(i)}_{lm} &= q\sqrt{\frac{(2l+1)(l-m)!}{4\pi(l+m)!}}P_l^m(0)(a/\sqrt{2})^l\\ &\quad\times(-e^{-im3\pi/4}+e^{-im\pi/4}+e^{im3\pi/4}-e^{im\pi/4}) \qquad (23)\\ &= \begin{cases} q\sqrt{\frac{(2l+1)(l-m)!}{4\pi(l+m)!}}P_l^m(0)(a/\sqrt{2})^l 4i^{m+1}\sin\left(\frac{m\pi}{4}\right) & \text{if } l \text{ and } m \text{ even}\\ 0 & \text{if } l \text{ or } m \text{ odd}\end{cases} \qquad (24)\end{aligned}$$

and

$$\begin{aligned}q^{(ii)}_{lm} &= q\sqrt{\frac{(2l+1)(l-m)!}{4\pi(l+m)!}}P_l^m(0)(a/\sqrt{2})^l\\ &\quad\times(e^{-im3\pi/4}+e^{-im\pi/4}-e^{im3\pi/4}-e^{im\pi/4}) \qquad (25)\\ &= \begin{cases} 0 & \text{if } l \text{ or } m \text{ even}\\ q\sqrt{\frac{(2l+1)(l-m)!}{4\pi(l+m)!}}P_l^m(0)(a/\sqrt{2})^l(-4)i\sin\left(\frac{m\pi}{4}\right) & \text{if } l \text{ and } m \text{ odd}\end{cases} \qquad (26)\end{aligned}$$

where the integration (over Cartesian coordinates) collapsed due to the delta functions. Here we used that $P_l^m(0) = 0$ if $l+m$ odd. For the first couple of l we obtain

$$q^{(i)}_{0,0} = 0, \tag{27}$$

$$q^{(i)}_{1,m} = (0,0,0), \tag{28}$$

$$q^{(i)}_{2,m} = \left(iqa^2\sqrt{\frac{15}{8\pi}}, 0, 0, 0, -iqa^2\sqrt{\frac{15}{8\pi}}\right), \tag{29}$$

$$q^{(i)}_{3,m} = (0,0,0,0,0,0,0), \tag{30}$$

$$q^{(i)}_{4,m} = \left(0, 0, -iqa^4\sqrt{\frac{45}{128\pi}}, 0, 0, 0, iqa^4\sqrt{\frac{45}{128\pi}}, 0, 0\right) \tag{31}$$

and

$$q_{0,0}^{(ii)} = 0, \tag{32}$$

$$q_{1,m}^{(ii)} = \left(iqa\sqrt{\frac{3}{2\pi}}, 0, iqa\sqrt{\frac{3}{2\pi}} \right), \tag{33}$$

$$q_{2,m}^{(ii)} = (0,0,0,0,0), \tag{34}$$

$$q_{3,m}^{(ii)} = \left(iqa^3\sqrt{\frac{35}{64\pi}}, 0, -iqa^3\sqrt{\frac{21}{64\pi}}, 0, -iqa^3\sqrt{\frac{21}{64\pi}}, 0, iqa^3\sqrt{\frac{35}{64\pi}} \right). \tag{35}$$

(b) The leading term of charges (i) is the quadrupole moment. In Cartesian coordinates we obtain from the relations discussed in the lecture and the results for $q_{2,m}$ above

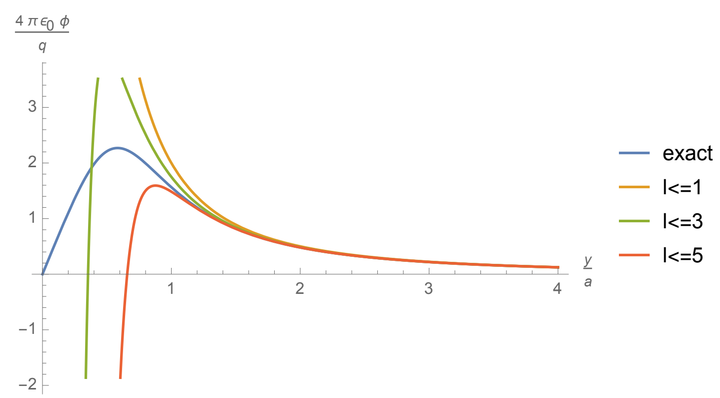
$$Q_{12} = Q_{21} = 3a^2 q, \quad Q_{ij} = 0 \text{ else}. \tag{36}$$

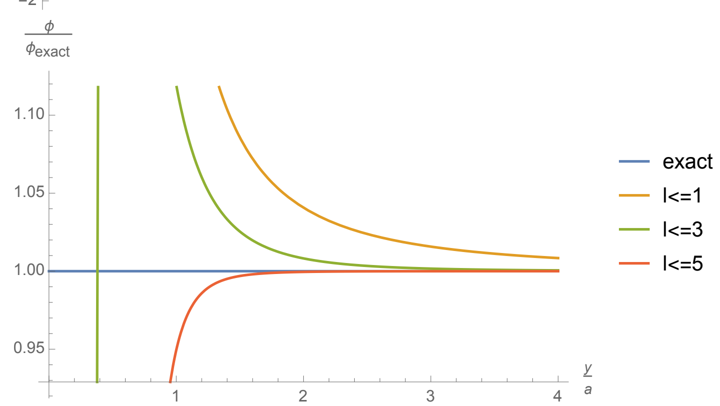
This reproduces the usual Cartesian quadrupole moments.

(c) We have $p_x = p_z = 0$ and $p_y = 2aq$ and thus

$$\phi^{(ii)} = \frac{1}{4\pi\epsilon_0}\frac{2aqy}{r^3} + \ldots \tag{37}$$

We want to plot the exact potential (obtained from the Coulomb potentials of the point charges) and the multipole expansion up to some maximal value of l. For higher l it is convenient to use a computer algebra system which has the spherical harmonics built in. In this way we obtain for $x = z = 0$ along the positive y axis the following graphs of approximations up to different l:





It is clearly visible that adding more terms improves the approximation at larger distances. One can also see clear hints that the series does not converge inside the charge distribution.

3. **Spherical multipole moments of a charged ring:**

   (a) The multipole moments are

$$q_{lm} = \int s\,ds\,d\varphi\,dz\,\rho(\vec{r})Y^*_{lm}(\theta,\varphi)\sqrt{s^2+z^2}^{\,l} \tag{38}$$

$$= R^{l+1}\lambda_0\left(\int_0^\pi d\varphi\,Y^*_{lm}(\pi/2,\varphi) - \int_\pi^{2\pi} d\varphi\,Y^*_{lm}(\pi/2,\varphi)\right) \tag{39}$$

$$= 2R^{l+1}\lambda_0\int_0^\pi d\varphi\,Y^*_{lm}(\pi/2,\varphi)\,. \tag{40}$$

   The charge is clearly zero and for the dipole moments we obtain

$$q_{1m} = 2R^2\lambda_0(i\sqrt{3/(2\pi)}, 0, i\sqrt{3/(2\pi)})\,. \tag{41}$$

   (b) If m is even, the integration of $e^{-im\varphi}$ over $[0,\pi]$ leads to a zero multipole moment. If m is odd and l is even $P_l^m(0) = 0$ and we have a zero multipole moment. In other words, only if m and l is odd, there can be multipole moments. There is no reason that all of the higher moments beyond the dipole should vanish and indeed a quick explicit check with a computer algebra system reveals that e.g. $l = 3$ has non-zero contribution.

   (c) For the alternative charge density we find

$$q_{1m} = R^2\lambda_0(i\sqrt{3/(8\pi)}, 0, i\sqrt{3/(8\pi)})\,. \tag{42}$$