Homework 3

Brandon Henke
PHY841
Andreas von Manteuffel

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3.1 Inhomogeneously Charged Sphere

a

The total charge is given by

$$
\begin{aligned}
Q &= \int \rho(\mathbf{r})\,\mathrm{d}^3 r\,, \\
&= \frac{4\pi\rho_0}{R^2}\int_0^R r^4\,\mathrm{d}r\,, \\
&= \frac{4\pi\rho_0 R^3}{5}.
\end{aligned}
$$

b

From Gauss' Law, the electric field of this charge distribution is given by

$$
\begin{aligned}
\mathbf{E}(\mathbf{r}) &= \frac{\hat{\mathbf{r}}}{4\pi\epsilon_0}\int_\Omega \rho(\mathbf{r})\,\mathrm{d}^3 r\,, \\
&= \frac{\hat{\mathbf{r}}}{4\pi\epsilon_0}\begin{cases} \frac{Qr^5}{R^5} & r \leq R \\ Q & r \geq R \end{cases}\,.
\end{aligned}
$$

c

See figure 3.1.1 for the graph of the magnitude of the electric field, as a function of the radius away from the center of the sphere. From the equation for the electric field, radii greater than or equal to the radius of the sphere, R, is simply the electric field of a point charge, $Q = 4\pi\rho_0 R^3/5$, located at the origin.

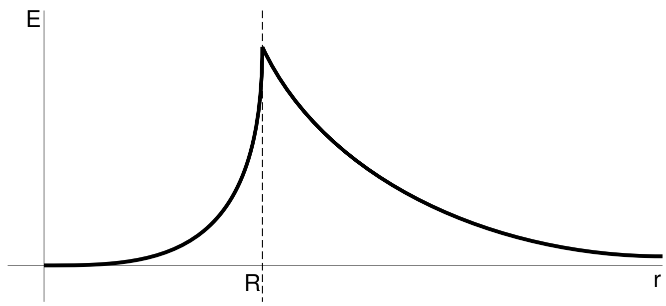

Figure 3.1.1: This figure shows a graph of the magnitude of the electric field, as a function of the radius away from the center of the charged sphere. For all radii less than or equal to R, the magnitude grows as r^5. For all radii greater than or equal to R, the magnitude decays as r^{-2}.

3.2 Spherical Cavity

a

Consider a sphere with a uniform charge density, ρ. Then the electric field inside the sphere is given by

$$
\begin{aligned}
\oint \mathbf{E}(\mathbf{r})\cdot \mathrm{d}^2\mathbf{r} &= \frac{1}{\epsilon_0}\int \rho \mathrm{d}^3 r, \\
E(\mathbf{r})4\pi r^2 &= \frac{4\pi r^3\rho}{3\epsilon_0}, \\
E(\mathbf{r}) &= \frac{\rho r}{3\epsilon_0}.
\end{aligned}
$$

Since the electric field is radial,

$$
\mathbf{E}(\mathbf{r}) = \frac{\rho\mathbf{r}}{3\epsilon_0}.
$$

If one were to super impose two of these spheres on top of each other, one centered at the origin with a radius of R and a uniform charge density of ρ, and another centered at $\mathbf{a}$ with a radius of $b < R - a$ and a charge density of $-\rho$, then this gives the exact same effect as the described in the problem. Thus, the electric field inside the cavity will be a superposition of the electric field inside the large sphere and the electric field inside the small sphere:

$$
\begin{aligned}
\mathbf{E}(\mathbf{r} = \mathbf{a} + \mathbf{r}') &= \frac{\rho\mathbf{r}}{3\epsilon_0} - \frac{\rho\mathbf{r} - \mathbf{a}}{3\epsilon_0}, \\
&= \frac{\rho\mathbf{a}}{3\epsilon_0},
\end{aligned}
$$

where $\mathbf{r}'$ is the position away from the center of the smaller sphere.

3.3 Charge Density of a Special Field

From Gauss's Law,

$$\begin{aligned}\rho(\mathbf{r}) &= \epsilon_0 \boldsymbol{\nabla} \cdot \mathbf{E}(\mathbf{r}),\\ &= \epsilon_0 \frac{1}{r^2}\frac{\partial(r^2 \mathbf{E}_r)}{\partial r},\\ &= -\frac{qr^2}{a^3\pi}e^{-2r/a}.\end{aligned}$$

b

The total charge of the system is given by

$$Q = 4\pi \int_0^\infty \rho(\mathbf{r}) r^2 \,\mathrm{d}r = -q.$$

3.4 Field of a Thin Disc

a

The volume charge density is given by

$$\rho(\mathbf{r}) = \rho(r, \phi, z) = \frac{q\Theta(r-R)}{\pi R^2}\delta(z).$$

Thus, the surface charge density is

$$\sigma(x, y) = \frac{q}{\pi R^2}\Theta\left(\sqrt{x^2+y^2} - R\right).$$

b

The electric field at an arbitrary point, $\mathbf{z} = z\hat{\mathbf{z}}$, above the center of the disk is found by evaluating the following integral:

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\int \rho(\mathbf{r}')\frac{\mathbf{z}-\mathbf{r}'}{|\mathbf{z}-\mathbf{r}'|^3}\,\mathrm{d}^3 r'\,.$$

However, there's a simplification that can be made. Due to the symmetry of the system, the electric field that is perpendicular to the $\hat{\mathbf{z}}$-axis will cancel out. Thus, the integral becomes

$$\mathbf{E} = \frac{z\hat{\mathbf{z}}}{4\pi\epsilon_0}\int \frac{\rho(\mathbf{r}')}{|\mathbf{z}-\mathbf{r}'|^3}\,\mathrm{d}^3 r'\,.$$

Inserting the charge density found in part a gives

$$\mathbf{E} = \frac{qz\hat{\mathbf{z}}}{2\pi\epsilon_0 R^2}\int_0^R \frac{r'}{(z^2+r'^2)^{3/2}}\,\mathrm{d}r'\,.$$

Evaluating the integral gives

$$\mathbf{E}(\mathbf{z}) = \frac{qz\hat{\mathbf{z}}}{2\pi\epsilon_0 R^2}\left[-\frac{1}{u^{1/2}}\right|_{z^2}^{z^2+R^2} = \frac{q\hat{\mathbf{z}}}{2\pi\epsilon_0 R^2}\left[1 - \frac{z}{\sqrt{z^2+R^2}}\right].$$

c

Using the binomial approximation, one can see that, for $z \gg R$, the electric field becomes

$$\mathbf{E}(\mathbf{z}) = \frac{qz\hat{\mathbf{z}}}{2\pi\epsilon_0 R^2}\left[1 - 1 + \frac{1}{2}\frac{R^2}{z^2}\right] = \frac{q\hat{\mathbf{z}}}{4\pi\epsilon_0 z^2},$$

and, for $z \ll R$, the electric field becomes

$$\mathbf{E}(\mathbf{z}) = \frac{q\hat{\mathbf{z}}}{2\pi\epsilon_0 R^2}\left[1 - \frac{z}{R} + \frac{1}{2}\frac{z^3}{R^3}\right].$$

It's easy to see that, for $z \gg R$, the electric field produced from a point charge is recovered. For $z \ll R$, if one makes a further approximation, namely $z/R \approx 0$, then one recovers the electric field produced from an infinite plane of charge with a surface charge density of $\sigma = q/\pi R^2$, which was found in part a.