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Problems

1. **Simple antenna:** The simplest linear antenna can be approximated by a thin wire of length $l$ carrying an alternating current. The wire is aligned with the $z$ axis and centered at the origin. Determine the power of the radiation averaged over the period of the oscillating current, assuming that the wavelength of the radiation is much larger than the size of the antenna.
   (a) (20 pts) Solve the problem for a current $I(t) = I_0 \cos(\omega t)$.
   (b) (5 pts) Consider the problem for a current $I(t) = I_0 \cos(\pi z/l)\cos(\omega t)$. Solve it using a similar kind of approximation as above. Is this a concise description of the half-wave dipole antenna, where $l = \lambda/2$ ?
   (c) (5 pts) Consider receivers placed at the following positions (all far away from the antenna): $\vec{a} = (1,0,0)\,\mathrm{m}$, $\vec{b} = (0,0,1/2)\,\mathrm{m}$, $\vec{c} = (1,1,1)/\sqrt{3}\,\mathrm{m}$. Determine which of these positions allow to receive the largest signal.

$P = \frac{2}{3}\frac{|\ddot{p}|^2}{c^3}$

$\dot{\vec{p}} = \hat{z}\frac{d}{dt}\int_{-l/2}^{l/2} I(z,t)\,dz$

$= \hat{z}\frac{d}{dt} I_0 \int_{-l/2}^{l/2} \cos(\omega t)\,dz$

$= \hat{z}\frac{d}{dt} I_0 l \cos(\omega t) = -\hat{z} I_0 l \omega \sin(\omega t)$

$P = \frac{2}{3}\frac{1}{c^3} I_0^2 l^2 \omega^2 \sin^2(\omega t)$

$\langle P\rangle = \frac{1}{\tau}\int_0^\tau P\,dt = \frac{2}{3}\frac{I_0^2 l^2 \omega^2}{c^3}\frac{1}{\tau}\int_0^\tau \frac{1}{2}(1-\cos(2\omega t))\,dt$

$\boxed{a) = \frac{I_0^2 l^2 \omega^2}{3c^3}}$

If $I(t) = I_0 \cos(\pi z/l)\cos(\omega t)$

$\dot{\vec{p}} = \hat{z}\frac{d}{dt}\int_{-l/2}^{l/2} I_0 \cos(\pi z/l)\cos(\omega t)\,dz$

$= -\hat{z}\,\omega \sin(\omega t) I_0 \frac{2l}{\pi}$

$\langle P\rangle = \frac{2}{3}\frac{1}{c^3}\omega^2 I_0^2 \frac{2^2 l^2}{\pi^2}\frac{1}{2}$

$\boxed{b) = \frac{16\pi^2\omega^2 I_0^2}{3c^3\lambda^2}}$

c) Since $\hat{c}$ is perpendicular to $\ddot{\vec{p}}$, where the others are not, $\hat{c}$ receives the best signal. ($\vec{b}$ receives none, $\hat{c}$ and $\hat{a}$ are equidistant to the source but $\hat{c}$ isn't perpendicular to $\ddot{\vec{p}}$)

2. **Non-relativistic particle in $\vec{E}$ field:** A non-relativistic particle with mass $m$, charge $q > 0$ and initial velocity $\vec{v}_0$ crosses the space between the plates of a capacitor where a constant voltage $V$ is applied. The distance between the plates is equal to $b$, and the initial angle between $\vec{v}_0$ and the electric field in the capacitor is $\theta < \pi/2$.
   (a) (20 pts) Find the energy loss by dipole radiation during the time of flight of the particle inside the capacitor. (Neglect the influence of radiation on the trajectory.)
   (b) (5 pts) Assume the particle has the mass of an electron and an initial kinetic energy of 4 eV, the voltage is $V = 10$ kV, the width of the capacitor is $b = 1$ cm, and $\theta = 0$. Determine if the approximation of non-relativistic kinematics and if the approximation that the trajectory is unperturbed by radiation are justified. (For the latter determination, compare the total energy loss to the initial energy.)

$\frac{d\mathcal{E}}{dt} = -P = \frac{2}{3}\frac{q^2}{c^3}|\dot{v}|^2 \leftarrow \dot{\vec{v}} = \frac{\vec{F}}{m} = \frac{q\vec{E}}{m}$

$= \frac{2}{3}\frac{q^4}{m^2c^3}\vec{E}\cdot\vec{E}$

$= \frac{2}{3}\frac{q^4}{m^2c^3}\frac{V^2}{b^2}$

3. **Non-relativistic charge in $\vec{B}$ field:** A non-relativistic proton is moving in the plane perpendicular to a uniform static magnetic field $\vec{B}$. Its energy at the initial time $t = 0$ is $\mathcal{E}(0) = \mathcal{E}_0$.
   (a) (20 pts) Find an expression for the energy as a function of time as it decreases due to the emission of dipole radiation.

$\frac{d\mathcal{E}}{dt} = -P$

! PLEASE SEE NEXT PAGE FOR PROBLEM 4 !

$\mathcal{E}_0 = \frac{1}{2}mv^2$

$P(t) = \frac{2}{3}\frac{q^2}{c^3}|\dot{v}|^2$

$\dot{\vec{v}} = \frac{\vec{F}}{m} = \frac{q}{m}\vec{v}\times\vec{B} \leftarrow$ these two are perpendicular

$|\dot{\vec{v}}| = \frac{q}{m}|\vec{v}||\vec{B}|$

$\frac{d\mathcal{E}}{dt} = -\frac{2}{3}\frac{q^2}{c^3}\frac{q^2}{m^2}|\vec{v}|^2|\vec{B}|^2$

$= -\frac{q^4}{3c^3}\frac{2}{m^2}v^2B^2 \leftarrow \mathcal{E} = \frac{mv^2}{2}$

$= -\frac{4q^4}{3c^3m^3}\mathcal{E}B^2$

$= -\alpha\mathcal{E}$

$\boxed{\mathcal{E}(t) = \mathcal{E}_0 e^{-\alpha t}}$

4. **Precessing magnetic dipole:** Consider a magnetic dipole $\vec{m}$ pointing in the $xy$-plane in the presence of a magnetic field $\vec{B} = B\,\hat{z}$. Assume that the ratio of the charge distribution to the mass distribution is constant for the magnetic dipole, and the total charge and mass of the magnetic dipole are $q > 0$ and $m$, respectively.
   (a) (5 pts) What is the Larmor frequency $\omega_L$ for precession of $\vec{m}$ around the $\vec{B}$ field?
   (b) (10 pts) Find the angular distribution of the radiated power in spherical coordinates when the magnetic moment is at $\vec{m} = \mu\left[\hat{x}\cos(\omega_L t) - \hat{y}\sin(\omega_L t)\right]$.
   (c) (5 pts) Find the distribution of the radiated power averaged over one period of precession.
   (d) (5 pts) Find the total time-averaged power emitted by the precessing magnetic dipole.

$\vec{\tau} = \vec{m}\times\vec{B} = \frac{q}{2m}\vec{L}\times\vec{B} = \frac{d\vec{L}}{dt}$

$= \frac{q}{2m}\begin{pmatrix} 0 & -L_3 & L_2 \\ L_3 & 0 & -L_1 \\ -L_2 & L_1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ B \end{pmatrix} = \frac{d}{dt}\begin{pmatrix} L_1 \\ L_2 \\ L_3 \end{pmatrix}$

$= \underbrace{\frac{qB}{2m}}_{\alpha}\begin{pmatrix} L_2 \\ -L_1 \\ 0 \end{pmatrix} = \frac{d}{dt}\begin{pmatrix} L_1 \\ L_2 \\ L_3 \end{pmatrix}$

$\frac{dL_3}{dt} = 0 \rightarrow L_3 = C_3$

$\left.\begin{aligned}\frac{dL_1}{dt} &= \alpha L_2 \\ \frac{dL_2}{dt} &= -\alpha L_1\end{aligned}\right\} \rightarrow \frac{d^2}{dt^2}L_2 = -\alpha\frac{dL_1}{dt} = -\alpha^2 L_2$

$\frac{d^2L_1}{dt^2} = -\alpha^2 L_1$

$\boxed{\omega_L = \alpha = \frac{qB}{2m}}$

$\frac{dP}{d\Omega} = \frac{\mu_0}{4\pi c}\frac{|\ddot{m}|^2}{4\pi}\sin^2[\alpha(t)]$ ← angle between $\hat{r}$ and $\ddot{m}$

$\ddot{m} = -\omega_L^2 m$

$\hat{\ddot{m}}\cdot\hat{r} = \cos(\alpha(t))$

$= -\sin\theta\cos(\omega_L t + \varphi)$

$\frac{dP}{d\Omega} = \frac{\mu_0}{4\pi c}\frac{\omega_L^4\mu^2}{4\pi}\left(1 - \sin^2\theta\cos^2(\omega_L t + \varphi)\right)$

$\langle\cos^2(\omega_L t + \varphi)\rangle = \frac{1}{2}$

$\left\langle\frac{dP}{d\Omega}\right\rangle = \frac{\mu_0\omega_L^4\mu^2}{16\pi^2 c}\left(1 - \frac{1}{2}\sin^2\theta\right)$

$\langle P\rangle = \int\left\langle\frac{dP}{d\Omega}\right\rangle d\Omega$

$= \frac{\mu_0\omega_L^4\mu^2}{16\pi^2 c}\iint\left(1 - \frac{1}{2}\sin^2\theta\right)\sin\theta\,d\varphi\,d\theta$

$= \frac{\mu_0\omega_L^4\mu^4}{6\pi c}$