PHY-841: Classical Electrodynamics / Midterm Exam 1 / Feb 18, 2021

Please read all of the following before starting the exam:

- Please access the exam problems on Gradescope only after we agree to start *in the Zoom meeting*
- You then have 2 hours to solve the problems.
- Please upload your solutions to Gradescope and indicate for each problem where the relevant pages are.
- Please stay connected to Zoom during the entire duration of your exam and switch on your camera.
- To solve the problems, you may use a simple calculator, but no computer algebra systems, external notes, books, etc.
- All problems are in S.I. units unless stated otherwise. Please give your answers in terms of the given variables and units.
- A complete answer usually includes a derivation of the result (unless stated otherwise). Show all work as neatly and logically as possible to maximize your credit. State clearly which equations were used. Circle or otherwise indicate your final answers.
- Please ask if the problem description is unclear.
- *Good luck !*

$$
\begin{aligned}
\vec{a}\times(\vec{b}\times\vec{c}) &= \vec{b}(\vec{a}\cdot\vec{c})-\vec{c}(\vec{a}\cdot\vec{b}),\\
\vec{a}\cdot(\vec{b}\times\vec{c}) &= \vec{b}\cdot(\vec{c}\times\vec{a})=\vec{c}\cdot(\vec{a}\times\vec{b}),\\
(\vec{a}\times\vec{b})\cdot(\vec{c}\times\vec{d}) &= (\vec{a}\cdot\vec{c})(\vec{b}\cdot\vec{d})-(\vec{a}\cdot\vec{d})(\vec{b}\cdot\vec{c}),\\
\vec{\nabla}\times(\vec{\nabla}\psi) &= 0,\\
\vec{\nabla}\cdot(\vec{\nabla}\times\vec{a}) &= 0,\\
\vec{\nabla}\times(\vec{\nabla}\times\vec{a}) &= \vec{\nabla}(\vec{\nabla}\cdot\vec{a})-\nabla^2\vec{a},\\
\vec{\nabla}\cdot(\psi\vec{a}) &= \vec{a}\cdot\vec{\nabla}\psi+\psi\vec{\nabla}\cdot\vec{a},\\
\vec{\nabla}\times(\psi\vec{a}) &= \vec{\nabla}\psi\times\vec{a}+\psi\vec{\nabla}\times\vec{a},\\
\vec{\nabla}(\vec{a}\cdot\vec{b}) &= (\vec{a}\cdot\vec{\nabla})\vec{b}+(\vec{b}\cdot\vec{\nabla})\vec{a}+\vec{a}\times(\vec{\nabla}\times\vec{b})+\vec{b}\times(\vec{\nabla}\times\vec{a}),\\
\vec{\nabla}\cdot(\vec{a}\times\vec{b}) &= \vec{b}\cdot(\vec{\nabla}\times\vec{a})-\vec{a}\cdot(\vec{\nabla}\times\vec{b}),\\
\vec{\nabla}\times(\vec{a}\times\vec{b}) &= \vec{a}(\vec{\nabla}\times\vec{b})-\vec{b}(\vec{\nabla}\times\vec{a})+(\vec{b}\cdot\vec{\nabla})\vec{a}-(\vec{a}\cdot\vec{\nabla})\vec{b},\\
\vec{\nabla}\cdot\vec{r} &= 3,\\
\vec{\nabla}\times\vec{r} &= 0,\\
\vec{\nabla}\cdot\hat{r} &= 2/r,\\
\vec{\nabla}\times\hat{r} &= 0,\\
\vec{\nabla}r &= \hat{r},\\
\vec{\nabla}\frac{1}{r} &= -\frac{\hat{r}}{r^2},\\
\vec{\nabla}\cdot(\hat{r}f(r)) &= \frac{2}{r}f+\frac{df}{dr},\\
(\vec{a}\cdot\vec{\nabla})\hat{r} &= \frac{1}{r}[\vec{a}-\hat{r}(\vec{a}\cdot\hat{r})]=\frac{\vec{a}_\perp}{r},\\
\vec{\nabla}^2\left(\frac{1}{r}\right) &= -4\pi\delta(\vec{r}),\\
\int_V d^3r\ \vec{\nabla}\cdot\vec{A} &= \int_S d\vec{S}\cdot\vec{A},\\
\int_V d^3r\ \vec{\nabla}\psi &= \int_S \psi\ d\vec{S},\\
\int_V d^3r\ \vec{\nabla}\times\vec{A} &= \int_S d\vec{S}\times\vec{A},\\
\int_V d^3r\ (\phi\nabla^2\psi+\vec{\nabla}\phi\cdot\vec{\nabla}\psi) &= \int_S \phi\ d\vec{S}\cdot\vec{\nabla}\psi,\\
\int_V d^3r\ (\phi\nabla^2\psi-\psi\nabla^2\phi) &= \int_S (\phi\vec{\nabla}\psi-\psi\vec{\nabla}\phi)\cdot d\vec{S},\\
\int_S (\vec{\nabla}\times\vec{A})\cdot d\vec{S} &= \oint d\vec{\ell}\cdot\vec{A},\\
\int_S d\vec{S}\times\vec{\nabla}\psi &= \oint_C d\vec{\ell}\ \psi.
\end{aligned}
$$

$$
\begin{aligned}
\mathscr{L} &= -\frac{1}{4\mu_0} F_{\mu\nu}F^{\mu\nu} - J_\mu A^\mu \\
L &= \frac{1}{\gamma}(-mc^2 - qA_\mu u^\mu) \\
\frac{dp^\mu}{d\tau} &= qF^{\mu\nu}u_\nu \\
\partial_\mu F^{\mu\nu} &= \mu_0 J^\nu \\
\partial_\mu \tilde{F}^{\mu\nu} &= 0 \\
\partial_\mu J^\mu &= 0 \\
(g_{\mu\nu}) &= \mathrm{diag}(1,-1,-1,-1), \\
\vec{\beta} &= \vec{v}/c, \\
\gamma &= 1/\sqrt{1-\beta^2}, \\
x'^\mu &= \Lambda^\mu{}_\nu x^\nu \\
(\Lambda^\mu{}_\nu) &= \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},
\end{aligned}
$$

$$
\begin{aligned}
x^\mu &= (ct, x, y, z), \\
\partial^\mu &= ((1/c)\partial/\partial t, -\vec{\nabla}) \\
k^\mu &= (\omega/c, \vec{k}), \\
u^\mu &= (\gamma c, \gamma\vec{v}), \\
p^\mu &= (E/c, \vec{p}), \\
A^\mu &= (\phi/c, \vec{A}), \\
J^\mu &= (c\rho, \vec{j}), \\
F^{\mu\nu} &= \partial^\mu A^\nu - \partial^\nu A^\mu, \\
\tilde{F}^{\mu\nu} &= \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}
\end{aligned}
$$

$$
\begin{aligned}
\vec{\nabla}\cdot\vec{E} &= \frac{1}{\epsilon_0}\rho, & \vec{\nabla}\cdot\vec{B} &= 0, \\
\vec{\nabla}\times\vec{B} - \frac{1}{c^2}\frac{\partial\vec{E}}{\partial t} &= \mu_0\vec{j}, & \vec{\nabla}\times\vec{E} + \frac{\partial\vec{B}}{\partial t} &= 0,
\end{aligned}
$$

$$
\begin{aligned}
\frac{d\vec{p}}{dt} &= q(\vec{E} + \vec{v}\times\vec{B}) \\
\vec{E} &= -\vec{\nabla}\phi - \frac{\partial\vec{A}}{\partial t} \\
\vec{B} &= \vec{\nabla}\times\vec{A} \\
p_i &= \int d^3\, r'\rho(\vec{r}')\, r'_i \\
Q_{ij} &= \int d^3r'\,\rho(\vec{r}')\,(3r'_ir'_j - \delta_{ij}r'^2) \\
\phi &= \frac{1}{4\pi\epsilon_0}\left(\frac{Q_{\mathrm{tot}}}{r} + \frac{p_i\hat{r}_i}{r^2} + \frac{1}{2!}Q_{ij}\frac{\hat{r}_i\hat{r}_j}{r^3} + \ldots\right) \\
U &= Q_{\mathrm{tot}}\phi(\vec{r}) - p_iE_i(\vec{r}) - \frac{1}{6}Q_{ij}\partial_iE_j + \ldots \\
\vec{E} &= \frac{1}{4\pi\epsilon_0}\left(Q_{\mathrm{tot}}\frac{\hat{r}}{r^2} + \frac{3(\vec{p}\cdot\vec{r})\hat{r} - \vec{p}}{r^3} + \ldots\right)
\end{aligned}
$$

Alice's birthday

Alice leaves Earth with her spaceship traveling at constant speed $v = (12/13)c$. Bob says goodbye to her from the control room on Earth and realizes that Alice could have celebrated her birthday during the launch if it had been scheduled 24h later. He decides to send her a message using radio waves, such that it reaches Alice exactly when it is her birthday *according to the clock in her spaceship*. [Consider just a single spatial direction x.]

1. (5 pts) Draw a Minkowski (ct, x) diagram in the Earth frame and denote the worldlines of Alice and of the radio message. In the Earth frame, denote the time t_S when Bob sends the message and the time t_R when Alice receives it.

2. (20 pts) According to Bob's clock on Earth, derive the time t_S after launch at which he should send the message.

Particle decay

A particle decays into two particles. The energies and three-momenta of the two final state particles are

$$E_1 = 52 \text{ GeV}, \qquad \vec{p}_1 = (36, 27, -27) \text{ GeV}/c, \quad \text{and} \tag{1}$$

$$E_2 = 156\text{GeV}, \qquad \vec{p}_2 = (68, 51, 131) \text{ GeV}/c\,. \tag{2}$$

respectively. [Feel free to set $c = 1$ here.]

3. (15 pts) Derive the mass M in GeV/c^2 of the decaying particle.

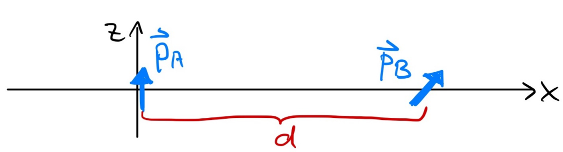
Three-vectors

We derived the relativistic theory of electromagnetic fields and point charges.

4. (10 pts) Which of the following quantities transform like the three-vector $\vec{x}$ if observed from a uniformly moving coordinate system: $\vec{v}$, $\vec{A}$, $\vec{j}$, $\vec{E}$, $\vec{B}$?

Flipping a dipole

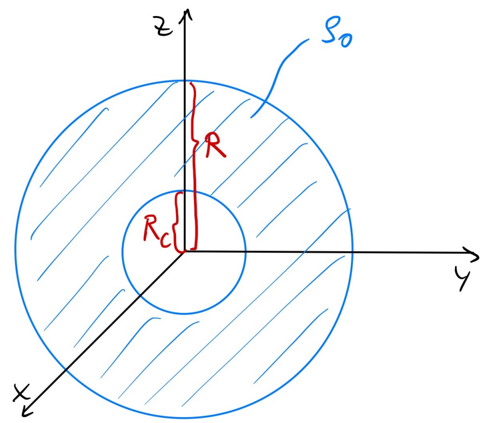
One electric dipole with moment $\vec{p}_A = p_A\hat{z}$ is located at the origin, another one with moment $\vec{p}_B = (p_{B,x}, p_{B,y}, p_{B,z})$ is at $\vec{r} = (d, 0, 0)$.



5. (10 pts) Derive, how much work W is done *on the dipole B* when flipping its orientation.

Charged sphere with cavity

Consider a solid sphere of radius R with a spherical cavity of radius R_c at its center. The sphere is centered at the origin and charged with a constant charge density ρ_0, except inside the cavity, where the charge density is zero.

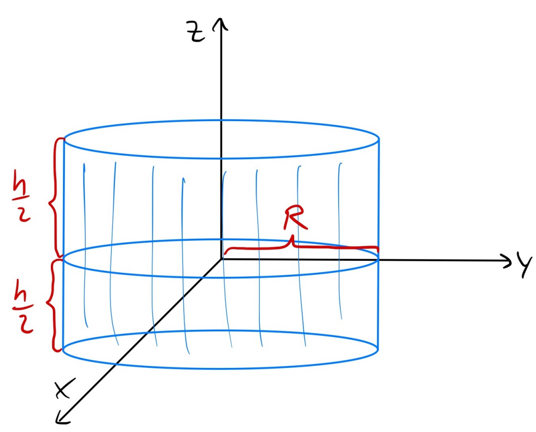


6. (15 pts) *Derive* the electric field $\vec{E}(\vec{r})$ at a point $\vec{r}$ inside the cavity ($|\vec{r}| < R_c$).

Charged cylinder

Consider a *non-uniformly* charged cylinder of radius R and height h, centered at the origin. Using cylindrical coordinates s, φ, z, the charge density is given by

$$\rho(s,\varphi,z) = \begin{cases} \rho_0 \left(\frac{s}{R}\right)^3 & \text{if } s = \sqrt{x^2+y^2} \leq R \;\text{ and } -h/2 \leq z \leq h/2 \;\text{ (inside the cylinder)} \\ 0 & \text{otherwise} \end{cases} \tag{3}$$



7. (25 pts) Derive the total charge, the dipole moment and the quadrupole moment of the cylinder. Give all components and try to use symmetry arguments to avoid explicit calculations whenever possible.