Solutions

1. **Cartesian Multipole Moments of a Charged Ring:**

   (a) The three-dimensional charge density of the ring in cylindrical coordinates $\{s, \phi, z\}$ is
   $\rho(\vec{r}) = \pm\lambda_0\delta(z)\delta(s-R)$, where the $+$ is for $0 \le \phi < \pi$ and the $-$ is for $\pi \le \phi < 2\pi$. The dipole moment is given by

   $$(p_x, p_y, p_z) \;=\; \int s\,ds\,dz\,d\phi\,\rho(\vec{r})\,(s\cos\phi, s\sin\phi, z)\;. \tag{1}$$

   Clearly, the integral of $z\delta(z)$ will give $p_z = 0$. In addition, we find $p_x = 0$ after integrating over ϕ. However, the value of $\sin\phi$ changes sign for the same range of ϕ as the charge density, so the contributions from the two half-circles add, and we get

   $$p_y \;=\; (\lambda_0 R^2)\,2\int_0^\pi d\phi\,\sin\phi \;=\; 4\lambda_0 R^2\;. \tag{2}$$

   So

   $$\vec{p} \;=\; (4\lambda_0 R^2)\,\hat{y}\;. \tag{3}$$

   (b) For the quadrupole tensor of the ring, we must calculate

   $$Q_{ij} = \int s\,ds\,dz\,d\phi\,\rho(\vec{r})\,(3r_i r_j - \delta_{ij}(s^2+z^2))$$

   Clearly, the piece of the integral proportional to δ_{ij} gives zero after integrating over ϕ. In addition, the components $Q_{xz} = Q_{zx} = Q_{yz} = Q_{zy} = 0$ because of the integral over $z\delta(z)$. Similarly, $Q_{zz} = 0$ because of the integral over $z^2\delta(z)$. Next, we find $Q_{xy} = Q_{yx} = 0$, due to the antisymmetry of $\cos\phi$ around $\pi/2$ and around $3\pi/2$. Finally, $Q_{yy} = Q_{xx} = 0$ due to canceling contributions from the change in sign of the charge density. In conclusion,

   $$Q_{ij} = 0\;.$$

   (c) The result for the dipole moment is independent of the choice of coordinates, since the total charge is zero. However, since the dipole moment is nonzero, the result for the quadrupole moment will be different if it is calculated around a point other than the center of the ring.

   (d) The torque on the dipole $\vec{p}_2$ with respect to its center is given by

   $$\vec{\tau}_2 \;=\; \vec{p}_2 \times \vec{E}_{\rm ring}(\vec{r}). \tag{4}$$

   We have for $\vec{r} = z\hat{z}$, $\vec{p} \;=\; (4\lambda R^2)\,\hat{y}$, and $p = |\vec{p}|$ for the electric field

   $$\vec{E}_{\rm ring} = \frac{1}{4\pi\epsilon_0}\cdot\frac{3(\vec{p}\cdot\hat{r}) - \vec{p}}{r^3} = -\frac{p}{4\pi\epsilon_0 r^3}\hat{y} \tag{5}$$

   with $p = |\vec{p}| = 4\lambda_0 R^2$. This gives

   $$\vec{\tau}_2 \;=\; -\frac{p}{4\pi\epsilon_0 z^3}\,(p_{2,x}\hat{z} - p_{2,z}\hat{x})\,. \tag{6}$$