Problem Set 2

Brandon Henke[1]
[1] *Michigan State University - Department of Physics & Astronomy*
Last updated: September 14, 2022

I PAULI MATRICES

A

$$\langle +|S_x|+\rangle = \frac{\hbar}{2}\left(\langle +|+\rangle \langle -|+\rangle + \langle +|-\rangle \langle +|+\rangle\right), \tag{I.1}$$
$$= 0. \tag{I.2}$$
$$\langle -|S_x|+\rangle = \frac{\hbar}{2}\left(\langle -|+\rangle \langle -|+\rangle + \langle -|-\rangle \langle +|+\rangle\right), \tag{I.3}$$
$$= \frac{\hbar}{2}. \tag{I.4}$$
$$\langle +|S_x|-\rangle = \frac{\hbar}{2}\left(\langle +|+\rangle \langle -|-\rangle + \langle +|-\rangle \langle +|-\rangle\right), \tag{I.5}$$
$$= \frac{\hbar}{2}. \tag{I.6}$$
$$\langle -|S_x|-\rangle = \frac{\hbar}{2}\left(\langle -|+\rangle \langle -|-\rangle + \langle -|-\rangle \langle +|-\rangle\right), \tag{I.7}$$
$$= 0. \tag{I.8}$$
$$S_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{I.9}$$

$$\langle +|S_y|+\rangle = \frac{i\hbar}{2}\left(-\langle +|+\rangle \langle -|+\rangle + \langle +|-\rangle \langle +|+\rangle\right), \tag{I.10}$$
$$= 0. \tag{I.11}$$
$$\langle -|S_y|+\rangle = \frac{i\hbar}{2}\left(-\langle -|+\rangle \langle -|+\rangle + \langle -|-\rangle \langle +|+\rangle\right), \tag{I.12}$$
$$= \frac{i\hbar}{2}. \tag{I.13}$$
$$\langle +|S_y|-\rangle = \frac{i\hbar}{2}\left(-\langle +|+\rangle \langle -|-\rangle + \langle +|-\rangle \langle +|-\rangle\right), \tag{I.14}$$
$$= -\frac{i\hbar}{2}. \tag{I.15}$$
$$\langle -|S_y|-\rangle = \frac{i\hbar}{2}\left(-\langle -|+\rangle \langle -|-\rangle + \langle -|-\rangle \langle +|-\rangle\right), \tag{I.16}$$
$$= 0. \tag{I.17}$$
$$S_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}. \tag{I.18}$$

$$\langle +|S_z|+\rangle = \frac{\hbar}{2}\left(\langle +|+\rangle \langle +|+\rangle - \langle +|-\rangle \langle -|+\rangle\right), \tag{I.19}$$
$$= \frac{\hbar}{2}. \tag{I.20}$$
$$\langle -|S_z|+\rangle = \frac{\hbar}{2}\left(\langle -|+\rangle \langle +|+\rangle - \langle -|-\rangle \langle -|+\rangle\right), \tag{I.21}$$
$$= 0. \tag{I.22}$$
$$\langle +|S_z|-\rangle = \frac{\hbar}{2}\left(\langle +|+\rangle \langle +|-\rangle - \langle +|-\rangle \langle -|-\rangle\right), \tag{I.23}$$
$$= 0. \tag{I.24}$$
$$\langle -|S_z|-\rangle = \frac{\hbar}{2}\left(\langle -|+\rangle \langle +|-\rangle - \langle -|-\rangle \langle -|-\rangle\right), \tag{I.25}$$
$$= -\frac{\hbar}{2}. \tag{I.26}$$
$$S_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{I.27}$$

B

The eigenvalues for all of the following are ± 1.

$$|S_x, \pm\rangle = \frac{1}{\sqrt{2}}\left(|-\rangle \pm |+\rangle\right). \tag{I.28}$$
$$|S_y, \pm\rangle = \frac{1}{\sqrt{2}}\left(|-\rangle \pm i\,|+\rangle\right). \tag{I.29}$$
$$|S_z, \pm\rangle = |\pm\rangle. \tag{I.30}$$

C

$$S_x = \frac{\hbar}{2\sqrt{2}^2}\left(|-\rangle + |+\rangle\right)\left(\langle -| + \langle +|\right), \tag{I.31}$$
$$- \frac{\hbar}{2\sqrt{2}^2}\left(|-\rangle - |+\rangle\right)\left(\langle -| - \langle +|\right), \tag{I.32}$$
$$= \frac{\hbar}{4}\left(2\,|-\rangle\langle +| + 2\,|+\rangle\langle -|\right), \tag{I.33}$$
$$= \frac{\hbar}{2}\left(|-\rangle\langle +| + |+\rangle\langle -|\right). \tag{I.34}$$

$$S_y = \frac{\hbar}{2\sqrt{2}^2}\left(|-\rangle + i\,|+\rangle\right)\left(\langle -| + i\,\langle +|\right), \tag{I.35}$$
$$- \frac{\hbar}{2\sqrt{2}^2}\left(|-\rangle - i\,|+\rangle\right)\left(\langle -| - i\,\langle +|\right), \tag{I.36}$$
$$= \frac{\hbar}{4}\left(2i\,|-\rangle\langle +| - 2i\,|+\rangle\langle -|\right), \tag{I.37}$$
$$= \frac{i\hbar}{2}\left(|-\rangle\langle +| - |+\rangle\langle -|\right). \tag{I.38}$$

$$S_z = \frac{\hbar}{2}\,|+\rangle\langle +| - \frac{\hbar}{2}\,|-\rangle\langle -|, \tag{I.39}$$
$$= \frac{\hbar}{2}\left(|+\rangle\langle +| - |-\rangle\langle -|\right). \tag{I.40}$$

II SAKURAI 1.4

A

Since all Pauli matrices are traceless, $\mathrm{tr}\{X\} = 2a_0$. Additionally, $\mathrm{tr}\{\sigma_k X\} = 0 + \sum_{i=1}^{3} \mathrm{tr}\{\sigma_k \sigma_i\} = 2a_k$.

B

$$a_0 = \frac{1}{2}\left(X_{00} + X_{11}\right), \tag{II.1}$$
$$a_1 = \frac{1}{2}\left(X_{10} + X_{01}\right), \tag{II.2}$$
$$a_2 = -\frac{i}{2}\left(X_{10} - X_{01}\right), \tag{II.3}$$
$$a_3 = \frac{1}{2}\left(X_{00} - X_{11}\right). \tag{II.4}$$

III SAKURAI 1.10

First, the Pauli matrices anti-commutate:

$$\sigma_1\sigma_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \tag{III.1}$$
$$= \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \tag{III.2}$$
$$= -\sigma_2\sigma_1. \tag{III.3}$$

$$\sigma_2\sigma_3 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{III.4}$$
$$= \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \tag{III.5}$$
$$= -\sigma_3\sigma_2. \tag{III.6}$$

$$\sigma_3\sigma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{III.7}$$
$$= \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{III.8}$$
$$= -\sigma_1\sigma_3. \tag{III.9}$$

Here's a detail that will be used in a bit:

$$\sigma_1\sigma_2\sigma_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{III.10}$$
$$= i. \tag{III.11}$$

Since $\sigma_i^2 = 1$, and $\sigma_i\sigma_j = -\sigma_j\sigma_i$ (see equations III.1-III.9) for $i \neq j$:

$$\left\{\sigma_i, \sigma_j\right\} = 2\delta_{ij}. \tag{III.12}$$
$$\left[\sigma_i, \sigma_j\right] = 0. \quad (i = j) \tag{III.13}$$
$$\tag{III.14}$$

Since $-i = \sigma_3\sigma_2\sigma_1$ (eq. III.11),

$$\left[\sigma_i, \sigma_j\right] = 2\sigma_i\sigma_j, \tag{III.15}$$
$$= 2\sigma_i\sigma_j(-i^2)\sigma_k^2, \quad (i \neq j \neq k) \tag{III.16}$$
$$= 2i\left[\sigma_i\sigma_j\sigma_k\sigma_3\sigma_2\sigma_1\right]\sigma_k, \tag{III.17}$$
$$= 2i\epsilon_{ijk}\sigma_k. \tag{III.18}$$

Since $S_k = \frac{\hbar}{2}\sigma_k$, $\left[S_i, S_j\right] = \frac{\hbar^2}{4}\left[\sigma_i, \sigma_j\right]$, and the same for the anticommutator:

$$\therefore \left[S_i, S_j\right] = \hbar i\epsilon_{ijk}S_k. \tag{III.19}$$
$$\left\{S_i, S_j\right\} = \frac{\hbar^2}{2}\delta_{ij}. \tag{III.20}$$

IV SAKURAI 1.11

Consider the unit vector $n = n^k\sigma_k$. This can be written as

$$n = \sin\theta(\cos\phi\sigma_1 + \sin\phi\sigma_2) + \cos\theta\sigma_3. \tag{IV.1}$$

Let $s = \frac{\hbar}{2}n$. The expectation value for this spin operator with the spin aligned with n is $\hbar/2$:

$$1 = \sin\theta(\cos\phi\langle\sigma_1\rangle + \sin\phi\langle\sigma_2\rangle) + \cos\theta\langle\sigma_3\rangle \tag{IV.2}$$

The spin state is given by

$$|\psi\rangle = \alpha|+\rangle + \beta|-\rangle, \tag{IV.3}$$

where $|\alpha|^2 + |\beta|^2 = 1$. The expectation values for each of the Pauli matrices, in terms of α and β, are

$$\langle\sigma_1\rangle = \alpha\beta^* + \alpha^*\beta, \tag{IV.4}$$
$$\langle\sigma_2\rangle = i(\alpha\beta^* - \alpha^*\beta), \tag{IV.5}$$
$$\langle\sigma_3\rangle = \alpha\alpha^* + \beta\beta^*. \tag{IV.6}$$

These are equal to the respective components of the unit vector, n:

$$\sin\theta\cos\phi = \alpha\beta^* + \alpha^*\beta, \tag{IV.7}$$
$$\sin\theta\sin\phi = i(\alpha\beta^* - \alpha^*\beta), \tag{IV.8}$$
$$\cos\theta = \alpha\alpha^* + \beta\beta^*. \tag{IV.9}$$

Solving this for α and β gives (up to a global phase)

$$\alpha = \cos\frac{\theta}{2}, \tag{IV.10}$$
$$\beta = \sin\frac{\theta}{2}e^{i\phi}. \tag{IV.11}$$