Hw 4

Wednesday, September 28, 2022 9:55

1). a. $[P, X^n] = PX^n - X^n P$

$= (XP - i\hbar)X^{n-1} - X^n P$

$= XPX^{n-1} - XX^{n-1}P - i\hbar X^{n-1}$

$= X[P, X^{n-1}] - i\hbar X^{n-1}$ ✓

$\vdots$

$= X^{n-1}[P, X] - (n-1)i\hbar X^{n-1}$

$= -i\hbar X^{n-1} - (n-1)i\hbar X^{n-1}$

$= -ni\hbar X^{n-1}$ ✓

$[X, P] = i\hbar = XP - PX$

$XP^n = (i\hbar + PX)P^{n-1}$

b.

c. $[P, F(X)] = [P, \sum_{n=0}^{\infty} \frac{1}{n!} F^{(n)}(0) X^n]$

$= \sum_{n=0}^{\infty} \frac{1}{n!} F^{(n)}(0) [P, X^n]$

$= \sum_{n=0}^{\infty} -\frac{1}{n!} F^{(n)}(0)\, n i\hbar X^{n-1}$

$= \sum_{n=0}^{\infty} -\frac{i\hbar}{(n-1)!} F^{(n)}(0) X^{n-1}$

$= -i\hbar \frac{dF}{dX}.$ ✓

d. $[X, P^n] = XP^n - P^n X$

$= (PX + i\hbar)P^{n-1} - P^n X$

$= PXP^{n-1} - PP^{n-1}X + i\hbar P^{n-1}$

$= P[X, P^{n-1}] + i\hbar P^{n-1}$ ✓

$\vdots$

$= P^{n-1}[X, P] + (n-1)i\hbar P^{n-1}$

$= i\hbar P^{n-1} + (n-1)i\hbar P^{n-1}$

$= ni\hbar P^{n-1}$ ✓

$[X, G(P)] = [X, \sum_{n=0}^{\infty} \frac{1}{n!} G^{(n)}(0) P^n]$

$= \sum_{n=0}^{\infty} \frac{1}{n!} G^{(n)}(0) [X, P^n]$

$= \sum_{n=0}^{\infty} \frac{1}{n!} G^{(n)}(0)\, n i\hbar P^{n-1}$

$= \sum_{n=0}^{\infty} \frac{i\hbar}{(n-1)!} G^{(n)}(0) P^{n-1}$

$$= \sum_{n=0} \frac{\cdots}{(n-1)!} G^{\cdots}(0)\, p^{n-\cdots}$$

$$= i\hbar \frac{dG}{dp}$$ ✓

2.) S 1.19

a. $H([A_1, A_2]|E\rangle) = E[A_1, A_2]|E\rangle$
$= E|[A_1, A_2]E\rangle$

$H|E\rangle = E|E\rangle$

$\therefore$ There are two different states with the same eigenvalue:

$$|[A_1, A_2]E\rangle,$$

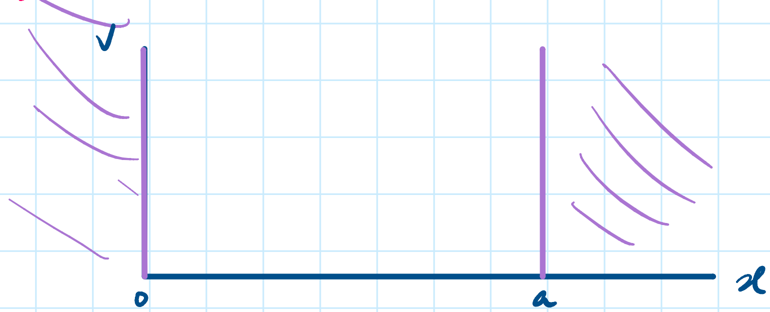
and

$$|E\rangle.$$

Hence the energy states are degenerate.

b. Yes, there are exceptions.

3) S 1.23 ✓



$\psi(x) = A \sin\left(\frac{n\pi x}{a}\right)$

$\int 1 = \psi^* \psi \, dx = A^2 \int_0^a \sin^2\left(\frac{n\pi x}{a}\right) dx$

$$A^{-2} = \frac{1}{2} \int_0^a \left(1 - \cos\left(\frac{2n\pi x}{a}\right)\right) dx$$
$$= \frac{1}{2}\left(a - \cancel{\frac{a}{2n\pi}\sin(2\pi n)}\right)$$
$$= \frac{a}{2}$$
$$A = \sqrt{\frac{2}{a}}$$

$$A = \sqrt{\tfrac{2}{a}}$$

$$\psi(x) = \sqrt{\tfrac{2}{a}} \sin\left(\frac{n\pi x}{a}\right)$$

$$\langle(\Delta x)^2\rangle = \langle x^2\rangle - \langle x\rangle^2 \qquad \langle(\Delta p)^2\rangle = \langle p^2\rangle - \langle p\rangle^2$$

$$\langle x^2\rangle = \int_0^a \langle\psi|x^2|x\rangle\langle x|\psi\rangle\, dx$$

$$= \int_0^a x^2 A^2 \sin^2\left(\frac{n\pi x}{a}\right) dx$$

$$= \frac{2}{a}\int_0^a x^2 \sin^2\left(\frac{n\pi}{a}x\right) dx$$

$$= a^2\left(\frac{1}{3} - \frac{1}{2n^2\pi^2}\right)$$

$$\langle x\rangle = \frac{2}{a}\int x \sin^2\left(\frac{n\pi}{a}x\right) dx$$
$$= \frac{2}{a}\frac{a^2}{4}$$
$$= \frac{a}{2}$$

$$\langle(\Delta x)^2\rangle = a^2\left(\frac{1}{3} - \frac{1}{2n^2\pi^2}\right) - \frac{a^2}{4}$$
$$= \frac{1}{12}a^2\left(1 - \frac{6}{n^2\pi^2}\right)$$

$$\langle p^2\rangle = \int \langle\psi|x\rangle\langle x|p^2|\psi\rangle\, dx$$

$$= \int \langle\psi|x\rangle(-\hbar^2\partial_x^2)\langle x|\psi\rangle\, dx$$
$$= \hbar^2 A^2 \int_0^a \sin^2\left(\frac{n\pi}{a}x\right)\left(\frac{n\pi}{a}\right)^2 dx$$
$$= \frac{\hbar^2}{a^2}(n\pi)^2$$

$$\langle p\rangle = i\hbar A^2 \int_0^a \sin\left(\frac{n\pi}{a}x\right)\cos\left(\frac{n\pi}{a}x\right) dx$$

$$= 0$$

$$\langle(\Delta p)^2\rangle = \left(\frac{n\pi\hbar}{a}\right)^2$$

$$\frac{1}{12}a^2\left(1 - \frac{6}{n^2\pi^2}\right)\left(\frac{n\pi\hbar}{a}\right)^2 = \frac{1}{12}\left(n^2\pi^2 - 6\right)\hbar^2 \geq \frac{\hbar^2}{4}$$

4.) a. From problem 1, $[x, g(p)] = i\hbar\,\partial_p\, g(p)$, 4.) a. From problem 1, $[x, \mathcal{G}(p)] = i\hbar \partial_p \mathcal{G}(p)$,

so, $[x_i, \mathcal{G}(p)] = i\hbar \partial_{p_i} \mathcal{G}(p)$.

$$[x_i, e^{-i\frac{p\cdot l}{\hbar}}] = i\hbar\left(\frac{-i l_i}{\hbar}\right) e^{-i\frac{p\cdot l}{\hbar}}$$
$$= l_i\, e^{-i\frac{p\cdot l}{\hbar}}.$$

b. $$\langle \psi | \hat{x} | \psi \rangle \to \left(\langle \psi | e^{i\frac{p\cdot l}{\hbar}}\right) \hat{x} \left(e^{-i\frac{p\cdot l}{\hbar}} | \psi \rangle\right)$$
$$= \int dx\, dy\, \langle \psi | y \rangle\langle y+l | \hat{x} | x+l \rangle\langle x | \psi \rangle$$
$$= \int dx\, dy\, (x+l)\, \delta(y-x)\, \langle \psi | y \rangle\langle x | \psi \rangle$$
$$= \int dy\, (y+l) \langle \psi | y \rangle\langle y | \psi \rangle$$
$$= \langle \hat{x} \rangle + l$$