Physics 851 **Problem Set 2** **Fall 2022**

1. Denoting the ket vectors for the states of a spin-1/2 system by $|S_z;\pm\rangle \equiv |\pm\rangle$, the spin operators can be written

$$
\begin{aligned}
S_x &= \frac{\hbar}{2}\Big[\Big(|+\rangle\langle-|\Big) + \Big(|-\rangle\langle+|\Big)\Big] && (1)\\
S_y &= \frac{\hbar}{2}\Big[-i\Big(|+\rangle\langle-|\Big) + i\Big(|-\rangle\langle+|\Big)\Big] && (2)\\
S_z &= \frac{\hbar}{2}\Big[\Big(|+\rangle\langle+|\Big) - \Big(|-\rangle\langle-|\Big)\Big]. && (3)
\end{aligned}
$$

(a) The matrix representation of an operator A in the $|\pm\rangle$ basis can be written

$$\mathbf{A} = \begin{pmatrix} \langle+|A|+\rangle & \langle+|A|-\rangle \\ \langle-|A|+\rangle & \langle-|A|-\rangle \end{pmatrix}.$$

Verify that the representations of the spin matrices in this basis are $\mathbf{S}_i = (\hbar/2)\sigma_i$ for $i = x, y, z$, where

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{4}$$

are the Pauli matrices.

(b) Obtain the eigenvalues and <u>normalized</u> eigenvectors of S_x, S_y and S_z. (This last one should be trivial.) Express the eigenvectors in the form

$$|e\rangle = a_+|+\rangle + a_-|-\rangle$$

for some scalars a_+ and a_-.

(c) Given an operator A with eigenvalues $\lambda_{1,2}$ and normalized eigenvectors $|e_{1,2}\rangle$, we can express the operator as

$$A = \lambda_1\Big(|e_1\rangle\langle e_1|\Big) + \lambda_2\Big(|e_2\rangle\langle e_2|\Big).$$

Use the results of part (b) to express S_x, S_y, and S_z in this manner and verify that you recover equations (1)-(3). (Again this should be trivial for S_z.)

NOTE: All problem numbers from Sakurai correspond to the 3^{rd} Edition.

2. Sakurai 1.4

   [Hint: First verify that $\mathrm{tr}(\sigma_i) = 0$ and $\mathrm{tr}(\sigma_i \sigma_j) = \delta_{ij}$.
   Note that in this and all other problems from Sakurai, the σ_i are the same as in Equation (4) above, except we replaced (x, y, z) by $(1, 2, 3)$.]

3. Sakurai 1.10

   [Since the evaluation of the commutator and anti-commutator are similar, and there are many combinations to evaluate, only work out $\{S_x, S_x\}$ and $\{S_x, S_y\}$ among the anti-commutators. Use the explicit operator forms given in the text (equations (1)-(3) above), <u>not</u> the 2×2 matrix expressions that you found in problem 1(a).]

4. Sakurai 1.11

   [Hint: Express the unit vector $\hat{n}$ in terms of the angles α and β, and use the half-angle formulas: $\sin\beta = 2\sin(\beta/2)\cos(\beta/2)$ and $\cos\beta = 1 - 2\sin^2(\beta/2)$, <u>before</u> solving the eigenvector equation. This will make the resulting expressions easier to handle.]