Physics 851 **Problem Set 6** **Fall 2022**

NOTE: All problem numbers from Sakurai correspond to the 3^{rd} Edition.

1. **Time-dependence of Spin operators in Heisenberg picture**

   Consider the spin operators $\vec{S}(t)$ in the Heisenberg picture, evolving under a Hamiltonian given by $H = \omega S_z$ (*i.e.*, an electron sitting in a constant uniform magnetic field that points in the $\hat{z}$ direction).

   (a) Using the Heisenberg Equations of motion:
   $$\frac{dS_i(t)}{dt} = \frac{1}{i\hbar}\left[S_i(t), H\right] ,$$
   and the commutation relations for spins, obtain differential equations for the $S_i(t)$.

   (b) Solve the differential equations for $S_i(t)$. For $i = x, y$, they are coupled first-order differential equations. These can be uncoupled either by taking appropriate linear combinations of the equations or by taking another derivative to obtain uncoupled second-order equations.

   (c) If the expectation value at the initial time is given by $\langle\vec{S}\rangle(0) = \frac{\hbar}{2}\hat{n}$, with $\hat{n} = \sin\theta\,\hat{x} + \cos\theta\,\hat{z}$, what is $\langle\vec{S}\rangle(t)$? How does this compare to the classical solution for a magnetic moment sitting in a uniform magnetic field?

2. **Quantum Virial Theorem**

   Consider a Hamiltonian given by
   $$H = T + V(\vec{x}) = \frac{\vec{p}^2}{2m} + V(\vec{x})\,.$$

   (a) Treating the operators in the Heisenberg picture, show that they satisfy
   $$\frac{d}{dt}(\vec{x}\cdot\vec{p}) = \frac{1}{i\hbar}[\vec{x}\cdot\vec{p}, H] = \frac{\vec{p}^2}{m} - \vec{x}\cdot\vec{\nabla}V\,.$$

   (b) Now take the expectation value of the previous equation in an energy eigenstate $|E_i\rangle$ (satisfying $H|E_i\rangle = E_i|E_i\rangle$) to obtain the Quantum Virial theorem:
   $$2\langle T\rangle = \langle\vec{x}\cdot\vec{\nabla}V\rangle\,.$$
   [Hint: Evaluate $\langle E_i|[\vec{x}\cdot\vec{p}, H]E_i\rangle$. (It is crucial that the expectation value is in an energy eigenstate here.)]

   (c) If the potential energy function has a scaling behavior of $V(\lambda\vec{x}) = \lambda^s V(\vec{x})$, where s is a real constant, then the Virial theorem can be written
   $$\langle T\rangle = \frac{s}{2}\langle V\rangle\,.$$
   Verify this and give the scaling coefficient s for the Coulomb potential $V = -k/r$ and for the three-dimensional harmonic oscillator $V = \frac{k}{2}r^2$.

3. Sakurai 2.17 **Simple Harmonic Oscillator in momentum space**
   In addition to parts (a) and (b), do
   (c) Obtain the first two energy eigenfunctions in momentum-space, $\langle p'|0\rangle$ and $\langle p'|1\rangle$, and compare them to the same eigenfunctions in position-space, given by equations (2.151) and (2.152) in the text.

4. **Uncertainty relation for eigenstates of the simple harmonic oscillator**
   Using the expressions for x and p written in terms of annihilation and creation operators, (eq. (2.145) in the text), calculate $\langle(\Delta x)^2\rangle$ and $\langle(\Delta p)^2\rangle$, and show how your result compares against the Heisenberg uncertainty relation.

5. **Time dependence of simple harmonic oscillator state**
   Consider a state of the simple harmonic oscillator given at time $t = 0$ by

$$|\psi(0)\rangle \;=\; \frac{1}{\sqrt{2}}\Big(|0\rangle \,+\, e^{i\delta}|1\rangle\Big) \;,$$

   where δ is a real constant.

   (a) Calculate $\langle x\rangle(0)$ and $\langle p\rangle(0)$ in this state.
   (b) Calculate $|\psi(t)\rangle$, for $t > 0$.
   (c) Calculate $\langle x\rangle(t)$, expressing the results in terms of $\langle x\rangle(0)$ and $\langle p\rangle(0)$. How does this compare to the classical harmonic oscillator motion?