HW5

Wednesday, October 5, 2022 19:03

1. $\int dx\, |x\rangle\langle x|\psi\rangle$

a. $\langle p\rangle = \int dx\, \langle\psi|p|x\rangle\langle x|\psi\rangle = \int_0^\infty dx\, \psi(x)\, i\hbar\, \psi'(x) = 0$ (Mathematica)

b. $\langle p|\psi\rangle = \int dx\, \langle p|x\rangle\langle x|\psi\rangle$

$$= \frac{2}{a^{3/2}}\int_0^\infty dx\, \frac{1}{\sqrt{2\pi\hbar}}\, e^{-ipx/\hbar}\, x\, e^{-x/a}$$

$$= \frac{\sqrt{2a\hbar^3/\pi}}{(iap+\hbar)^2}$$ (Thanks mathematica)

c. $P\left(-\frac{\hbar}{a}\le p\le\frac{\hbar}{a}\right) = \int_{-\frac{\hbar}{a}}^{\frac{\hbar}{a}} dp\, \tilde\psi^*(p)\,\tilde\psi(p) = \frac{1}{2} + \frac{1}{\pi}$ (Mathematica)

d. $\langle p\rangle = \int_{-\infty}^{\infty} dp\, p\,\tilde\psi^*(p)\,\tilde\psi(p)$

$= 0$ (Mathematica)

2. $A|a_i\rangle = a_i|a_i\rangle$

$f(A)|a_i\rangle = f(a_i)|a_i\rangle$

$\sum_{ji} |a_i\rangle\langle a_i|f(A)|a_j\rangle\langle a_j|$

$= \sum_i f(a_i)\, |a_i\rangle\langle a_i|$

$f(A)|a_i\rangle = f(a_i)|a_i\rangle$

$M|a_i\rangle = \sum_j |b_j\rangle\langle b_j|a_i\rangle$

$\langle b_j|M f(A) M^{-1}|b_i\rangle =$

$\langle b_i|f(A)|b_j\rangle = \sum_k\sum_l \langle b_i|a_l\rangle\langle a_l|f(A)|a_k\rangle\langle a_k|b_j\rangle$

$$\boxed{\text{a.} \quad = \sum_k f(a_k)\, \langle b_i|a_k\rangle\langle a_k|b_j\rangle}$$

$\langle \vec p_i|f(\hat r)|\vec p_j\rangle = \int dx^3\, f(r)\, \langle \vec p_i|\vec x\rangle\langle \vec x|\vec p_j\rangle$

$$= \int dx^3\, f(r)\frac{1}{(2\pi\hbar)^3}\, e^{-i\vec p_i\cdot\vec x\,\hbar^{-1}}\, e^{i\vec p_j\cdot\vec x\,\hbar^{-1}}$$

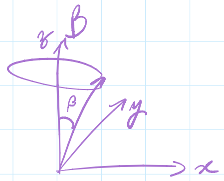
$$= \int dx^3\, f(r)\frac{1}{(2\pi\hbar)^3}\, e^{-i(\vec p_i-\vec p_j)\cdot\vec x\,\hbar^{-1}}$$

$$= \frac{1}{(2\pi)^2\hbar^3}\int_0^\infty\int_0^\pi r^2\, d\theta\, dr\, f(r)\, e^{-i\Delta p_{ij} r\cos\theta\,\hbar^{-1}}\sin\theta \qquad \Delta p_{ij} = |\vec p_i - \vec p_j|$$

$$= \frac{1}{(2\pi)^2\hbar^3}\int_0^\infty dr\, r^2 f(r)\left(e^{i\Delta p_{ij} r\,\hbar^{-1}} - e^{-i\Delta p_{ij} r\,\hbar^{-1}}\right)$$

$$\boxed{\text{b.} \quad = \frac{1}{2\pi^2\hbar^3}\int_0^\infty r^2 f(r)\sin\left(\frac{\Delta p_{ij}\, r}{\hbar}\right)}$$

3.



$H = -M\cdot B$

$= \omega S_z, \quad \omega \equiv -\frac{q_e B}{mc}$

$U(t) = e^{-I\omega S_z t/\hbar}$

$\langle +_x|U(t)|\psi\rangle$

$\frac{1}{\sqrt2}\left(\langle+| + \langle-|\right) e^{-I\omega S_z t/2\hbar}\left(\cos\left(\frac{\beta}{2}\right)|+\rangle + \sin\left(\frac{\beta}{2}\right)|-\rangle\right)$

$\langle +_x | U(t) | \psi \rangle$

$\frac{1}{\sqrt{2}}\left(\langle +| + \langle -|\right) e^{-I\omega S_z t/2\hbar}\left(\cos\left(\frac{\beta}{2}\right)|+\rangle + \sin\left(\frac{\beta}{2}\right)|-\rangle\right)$

$= \frac{1}{\sqrt{2}}\left(e^{-I\omega t/2}\cos\frac{\beta}{2} + e^{I\omega t/2}\sin\frac{\beta}{2}\right)$

$P = |\langle +_x | U(t) | \psi \rangle|^2$

$= \frac{1}{2}\left(e^{I\omega t/2}\cos\frac{\beta}{2} + e^{-I\omega t/2}\sin\frac{\beta}{2}\right)\left(e^{-I\omega t/2}\cos\frac{\beta}{2} + e^{I\omega t/2}\sin\frac{\beta}{2}\right)$

$= \frac{1}{2}\left(\cos^2\frac{\beta}{2} + \sin^2\frac{\beta}{2} + \cos\frac{\beta}{2}\sin\frac{\beta}{2}\left(e^{I\omega t} + e^{-I\omega t}\right)\right)$

$= \frac{1}{2} + \cos(\omega t)\cos\frac{\beta}{2}\sin\frac{\beta}{2}$

$\boxed{a.\ P = \frac{1}{2}\left(1 + \sin\beta\cos(\omega t)\right)}$

$\langle S_x \rangle = \langle \psi | U^\dagger S_x U | \psi \rangle$

$= \frac{\hbar}{2}\langle \psi | \left(\cos\omega t\,\sigma_1 + \sin\omega t\,\sigma_2\right) | \psi \rangle$

$= \frac{\hbar}{2}\left(\cos\omega t\,\langle \sigma_1 \rangle + \sin\omega t\,\cancel{\langle \sigma_2 \rangle}\right)$

$\boxed{b.\ \langle S_x \rangle = \frac{\hbar}{2}\cos\omega t\,\sin\beta}$

c. $\beta = 0 \rightarrow P = \frac{1}{2},\ \langle S_x \rangle = 0$

$\beta = \frac{\pi}{2} \rightarrow P = \frac{1}{2} + \cos\omega t,\ \langle S_x \rangle = \frac{\hbar}{2}\cos\omega t$

These make sense.

4. $|\nu_e\rangle = \cos\vartheta|\nu_1\rangle - \sin\vartheta|\nu_2\rangle$

$|\nu_\mu\rangle = \sin\vartheta|\nu_1\rangle + \cos\vartheta|\nu_2\rangle$

$H|\nu_1\rangle = E_1|\nu_1\rangle$
$H|\nu_2\rangle = E_2|\nu_2\rangle \rightarrow U = e^{-iHt/\hbar}$

$P(\nu_e \to \nu_e) = \langle \nu_e | U^\dagger | \nu_e \rangle\langle \nu_e | U | \nu_e \rangle$

$= \langle \nu_e | e^{iHt/\hbar} | \nu_e \rangle\langle \nu_e | e^{-iHt/\hbar} | \nu_e \rangle$

$= \langle \nu_e | \left(e^{iE_1t/\hbar}\cos\vartheta|\nu_1\rangle - e^{iE_2t/\hbar}\sin\vartheta|\nu_2\rangle\right)\left(\langle \nu_1 | e^{-iE_1t/\hbar}\cos\vartheta - \langle \nu_2 | e^{-iE_2t/\hbar}\sin\vartheta\right)|\nu_e\rangle$

$= \langle \nu_e | \left(\cos^2\vartheta|\nu_1\rangle\langle \nu_1| + \sin^2\vartheta|\nu_2\rangle\langle \nu_2| - \cos\vartheta\sin\vartheta\left(e^{\frac{i(E_1-E_2)t}{\hbar}}|\nu_1\rangle\langle \nu_2| + e^{\frac{-i(E_1-E_2)t}{\hbar}}|\nu_2\rangle\langle \nu_1|\right)\right)|\nu_e\rangle$

$= \cos^2\vartheta\langle \nu_e|\nu_1\rangle\langle \nu_1|\nu_e\rangle + \sin^2\vartheta\langle \nu_e|\nu_2\rangle\langle \nu_2|\nu_e\rangle$
$- 2\cos\left(\frac{E_1-E_2}{\hbar}t\right)\langle \nu_e|\nu_1\rangle\langle \nu_2|\nu_e\rangle\cos\vartheta\sin\vartheta$

$= \cos^4\vartheta + \sin^4\vartheta$
$+ 2\cos\left(\frac{E_1-E_2}{\hbar}t\right)\cos^2\vartheta\sin^2\vartheta$

$= \cos^4\vartheta + \sin^4\vartheta + 2\left(1 - 2\sin^2\left(\frac{E_1-E_2}{2\hbar}t\right)\right)\cos^2\vartheta\sin^2\vartheta$

$= 1 - \sin^2\left(\frac{E_1-E_2}{2\hbar}t\right)\sin^2 2\vartheta$

$E_1 - E_2 = E\left(1 + \frac{m_1^2c^4}{2E}\right) - E\left(1 + \frac{m_2^2c^4}{2E}\right)$

$= \frac{\Delta m^2c^4}{2E}$

$= 1 - \sin^2\left(\frac{\Delta m^2c^4}{4E\hbar c}L\right)\sin^2 2\vartheta$ ✓

$$= 1 - \sin^2\left(\frac{\Delta m^2 c^4}{4E\hbar c} L\right)\sin^2 2\vartheta \quad \checkmark$$

$$\tau = (70 - 34)\,\frac{km}{MeV} \qquad \alpha = \tfrac{1}{2}(0.78 - 0.4)$$

$$\frac{4\hbar c\pi}{\Delta m^2 c^4} = 36\,\frac{km}{MeV} \qquad \sin^2 2\vartheta = \tfrac{1}{2}(0.38)$$

$$\boxed{\begin{aligned} \Delta m^2 c^4 &= \frac{\hbar c \pi}{9}\,\frac{MeV}{km} & \vartheta &= \tfrac{1}{2}\arcsin\left(\sqrt{\tfrac{0.38}{2}}\right) \\ &= 6.87\times 10^{-17}\,eV^2 & &= 0.2255\ldots \end{aligned}}$$