Homework 01

Brandon Henke
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Scott Pratt

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1.1

a

$$R(\phi) = \begin{pmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{pmatrix}, \quad (1.1.1)$$

$$R\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}. \quad (1.1.2)$$

b

See equation (1.1.1).

c

Right circularly polarized light is given by

$$|R\rangle = \frac{1}{\sqrt{2}} \left(|x\rangle + i\,|y\rangle \right). \quad (1.1.3)$$

$$\phi(z,t) = \frac{1}{\sqrt{2}} e^{-i\omega t} \begin{pmatrix} e^{ik_x z} \\ e^{ik_y z} \end{pmatrix}. \quad (1.1.4)$$

Thus,

$$|\psi\rangle = \frac{1}{\sqrt{2}} \left(|x\rangle - i e^{i(k_x - k_y)Z} |y\rangle \right). \quad (1.1.5)$$

d

$$\rho_{|R\rangle} = |R\rangle\langle R|, \quad (1.1.6)$$

$$= \frac{1}{2} \left(|x\rangle + i\,|y\rangle \right) \left(\langle x| - i\,\langle y| \right), \quad (1.1.7)$$

$$= \frac{1}{2} \left(\mathbb{1} + \sigma_2 \right). \quad (1.1.8)$$

e

$$\rho_\psi = \frac{1}{2} |x\rangle\langle x| + \frac{1}{2} |y\rangle\langle y|. \quad (1.1.9)$$

1.2

$$\langle x|R(\theta)|x\rangle = \langle x|(\cos\theta - i\sin\theta)|x\rangle, \quad (1.2.1)$$

$$= \cos\theta - i\sin\theta\, \langle x|\sigma_2|x\rangle, \quad (1.2.2)$$

$$= \cos\theta. \quad (1.2.3)$$

$$\langle x|R\left(\frac{\pi}{2}\right)|x\rangle = 0, \quad (1.2.4)$$

$$\langle x|R(\pi)|x\rangle = -1, \quad (1.2.5)$$

$$\langle x|R(2\pi)|x\rangle = 1. \quad (1.2.6)$$

1.3

$$\langle z,+|R_y(\theta)|z,+\rangle \quad (1.3.1)$$

$$= \langle z,+|\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)|z,+\rangle, \quad (1.3.2)$$

$$= \cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\, \langle z,+|\sigma_2|z,+\rangle, \quad (1.3.3)$$

$$= \cos\frac{\theta}{2}. \quad (1.3.4)$$

$$\langle z,+|R_y\left(\frac{\pi}{2}\right)|z,+\rangle = \frac{1}{\sqrt{2}}, \quad (1.3.5)$$

$$\langle z,+|R_y(\pi)|z,+\rangle = 0, \quad (1.3.6)$$

$$\langle z,+|R_y(2\pi)|z,+\rangle = -1. \quad (1.3.7)$$

1.4

Let U bet a unitary transformation. Then

$$UU^\dagger = U^\dagger U = \mathbb{1}.$$

Hence,

$$U^\dagger \mathbb{1} U = U^\dagger U \mathbb{1} = \mathbb{1}\mathbb{1} = \mathbb{1}.$$

1.5

$$U^\dagger = \frac{1}{\sqrt{2}}(\mathbb{1} + i\sigma_3). \qquad (1.5.1)$$

a

$$U\sigma_1 U^\dagger = \frac{1}{2}(\mathbb{1} - i\sigma_3)\sigma_1(\mathbb{1} + i\sigma_3), \qquad (1.5.2)$$

$$= \frac{1}{2}(\sigma_1 - i\sigma_3\sigma_1)(\mathbb{1} + i\sigma_3), \qquad (1.5.3)$$

$$= \frac{1}{2}(\sigma_1 + i[\sigma_1,\sigma_3] - \sigma_1), \qquad (1.5.4)$$

$$= \frac{1}{2}i(-2i\sigma_2), \qquad (1.5.5)$$

$$= \sigma_2. \qquad (1.5.6)$$

b

$$U\,|x,+\rangle = \frac{1}{2}(\mathbb{1} + i\sigma_3)(|\uparrow\rangle + |\downarrow\rangle), \qquad (1.5.7)$$

$$= \frac{1}{2}(|\uparrow\rangle + |\downarrow\rangle - i\,|\uparrow\rangle + i\,|\downarrow\rangle), \qquad (1.5.8)$$

$$= \frac{1}{\sqrt{2}}(|\uparrow\rangle + i\,|\downarrow\rangle). \qquad (1.5.9)$$

This expression (1.5.9) is an eigenstate of σ_2.

1.6

$$U^\dagger = v_i^{(j)}. \qquad (1.6.1)$$

a

$$UU^\dagger = v_j^{*(i)}v_i^{(j)}, \qquad (1.6.2)$$

$$= \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}, \qquad (1.6.3)$$

$$= \mathbb{1}. \qquad (1.6.4)$$

b

$$U_{ij}v_i^{(n)} = v_j^{*(i)}v_i^{(n)}, \qquad (1.6.5)$$

$$= \delta_{nj} \quad \text{(ortho.)} \qquad (1.6.6)$$

$$K' = UKU^\dagger : \qquad (1.6.7)$$

$$K'Uv^{(n)} = UKU^\dagger Uv^{(n)}, \qquad (1.6.8)$$

$$= UKv^{(n)}, \qquad (1.6.9)$$

$$= \lambda_n\left(Uv^{(n)}\right). \qquad (1.6.10)$$

1.7

$$M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}. \qquad (1.7.1)$$

a

$$\begin{vmatrix} \lambda - 1 & 0 & 0 \\ 0 & \lambda & -1 \\ 0 & -1 & \lambda \end{vmatrix}$$

$$= (\lambda - 1)(\lambda^2 - 1) = 0, \qquad (1.7.2)$$

$$\lambda = 1, \pm i. \qquad (1.7.3)$$

b

$$\begin{pmatrix} \lambda - 1 & 0 & 0 \\ 0 & \lambda & -1 \\ 0 & -1 & \lambda \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{1.7.4}$$

$$a = 0, \tag{1.7.5}$$

$$c = b\lambda. \tag{1.7.6}$$

$$v_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \tag{1.7.7}$$

$$v_i = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ i \end{pmatrix}, \tag{1.7.8}$$

$$v_{-i} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -i \end{pmatrix}. \tag{1.7.9}$$

1.8

$$K = \begin{pmatrix} A & C^* \\ C & B \end{pmatrix}. \tag{1.8.1}$$

a

$$(\lambda - A)(\lambda - B) - C^*C = 0, \tag{1.8.2}$$

$$\lambda_\pm = \frac{A + B \pm \sqrt{(A^2 - B^2) + 4C^*C}}{2}. \tag{1.8.3}$$

b

$$\begin{pmatrix} \lambda - A & -C^* \\ -C & \lambda - B \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{1.8.4}$$

$$a(\lambda - A) - bC^* = 0, \tag{1.8.5}$$

$$-aC + b(\lambda - B) = 0, \tag{1.8.6}$$

$$a = b\sqrt{\frac{C^*(\lambda - B)}{C(\lambda - A)}}, \tag{1.8.7}$$

$$v_\pm = \frac{1}{\sqrt{\frac{C^*(\lambda-B)}{C(\lambda-A)} - 1}} \begin{pmatrix} \sqrt{\frac{C^*(\lambda-B)}{C(\lambda-A)}}, \\ 1 \end{pmatrix} \tag{1.8.8}$$

1.9

$$|\psi\rangle = \frac{1}{2}(\sqrt{3}\,|x\rangle + |y\rangle). \tag{1.9.1}$$

$$\langle (E - \langle E\rangle)^2 \rangle^{1/2} = \sqrt{Np(1-p)}, \tag{1.9.2}$$

$$p = \left| \langle x|y \rangle \right|^2 = \frac{3}{4}, \tag{1.9.3}$$

$$\langle (E - \langle E\rangle)^2 \rangle^{1/2} = \frac{\sqrt{3N}}{4}. \tag{1.9.4}$$

$$\langle E \rangle = Np, \tag{1.9.5}$$

$$= \frac{3N}{4}. \tag{1.9.6}$$

$$N = \frac{10\mathrm{J}}{h\nu} \approx 3.32 \cdot 10^{19}, \tag{1.9.7}$$

$$\frac{\langle (E - \langle E\rangle)^2 \rangle^{1/2}}{\langle E \rangle} = \frac{1}{\sqrt{3N}}, \tag{1.9.8}$$

$$\approx 1.00 \cdot 10^{-10}\mathrm{J}^{-1/2}. \tag{1.9.9}$$