Homework 05

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3.5

$$\mathbf{B} = B\hat{\mathbf{z}}, \tag{3.5.1}$$
$$\mathbf{E} = 0, \quad \text{and} \tag{3.5.2}$$
$$\mathbf{A} = Bx\hat{\mathbf{y}}. \tag{3.5.3}$$

For $\mathbf{E} = 0$ one can assume $A_0 = \Phi = 0$. Boosting at v_y in the $\hat{\mathbf{y}}$ direction gives

$$A_0' = \gamma\left(A_0 - \boldsymbol{\beta} \cdot \mathbf{A}\right), \tag{3.5.4}$$
$$= -\gamma v_y Bx. \tag{3.5.5}$$

For nonrelativistic velocities, $\gamma \approx 1$, so,

$$A_0' = -v_y Bx. \tag{3.5.6}$$

Thus

$$\mathbf{E}' = -\boldsymbol{\nabla} A_0' = v_y B\hat{\mathbf{x}}. \tag{3.5.7}$$

4.1

The angular momentum operator for the $\hat{\mathbf{z}}$ axis is given by

$$\hat{L}_3 = \hat{r}_1\hat{p}_2 - \hat{r}_2\hat{p}_1. \tag{4.1.1}$$

From this, the commutator with $\hat{r}^2$ is

$$\left[\hat{r}^2, \hat{L}_3\right] = \hat{r}_1\hat{r}^2\hat{p}_2 - \hat{r}_2\hat{r}^2\hat{p}_1 - \hat{r}_1\hat{p}_2\hat{r}^2 + \hat{r}_2\hat{p}_1\hat{r}^2, \tag{4.1.2}$$
$$= \hat{r}_1\left[\hat{r}^2, \hat{p}_2\right] - \hat{r}_2\left[\hat{r}^2, \hat{p}_1\right], \tag{4.1.3}$$
$$= i\hbar\hat{r}_1\hat{r}_2 - i\hbar\hat{r}_1\hat{r}_2, \tag{4.1.4}$$
$$= 0. \tag{4.1.5}$$

Hence, the two operators commute.

4.2

Let $i = \sigma_{123} = \sigma_1\sigma_2\sigma_3$, and

$$\alpha = \sum_{n=1}^{3} \alpha_n \sigma_n, \tag{4.2.1}$$
$$\beta = \sum_{n=1}^{3} \beta_n \sigma_n, \quad \text{and} \tag{4.2.2}$$
$$\gamma = \sum_{n=1}^{3} \gamma_n \sigma_n. \tag{4.2.3}$$

Then

$$\begin{aligned}
e^{i\alpha/2}e^{i\beta/2} &= \left(\cos\left(\frac{|\alpha|}{2}\right) + i\sin\left(\frac{|\alpha|}{2}\right)\frac{\alpha}{|\alpha|}\right)\left(\cos\left(\frac{|\beta|}{2}\right) + i\sin\left(\frac{|\beta|}{2}\right)\frac{\beta}{|\beta|}\right), && (4.2.4)\\
&= \cos\left(\frac{|\alpha|}{2}\right)\cos\left(\frac{|\beta|}{2}\right) - \frac{\alpha\beta}{|\alpha||\beta|}\sin\left(\frac{\alpha}{2}\right)\sin\left(\frac{|\beta|}{2}\right) && (4.2.5)\\
&+ i\left(\frac{\alpha}{|\alpha|}\sin\left(\frac{|\alpha|}{2}\right)\cos\left(\frac{|\beta|}{2}\right) + \frac{\beta}{|\beta|}\sin\left(\frac{|\alpha|}{2}\right)\cos\left(\frac{|\beta|}{2}\right)\right). && (4.2.6)
\end{aligned}$$

By definition of the geometric product, $\alpha\beta = \alpha\cdot\beta + i(\alpha\times\beta)$. Thus

$$\begin{aligned}
e^{i\alpha/2}e^{i\beta/2} = {}&\cos\left(\frac{|\alpha|}{2}\right)\cos\left(\frac{|\beta|}{2}\right) - \frac{\alpha\cdot\beta}{|\alpha||\beta|}\sin\left(\frac{\alpha}{2}\right)\sin\left(\frac{|\beta|}{2}\right)\\
&+ i\left[\frac{\alpha}{|\alpha|}\sin\left(\frac{|\alpha|}{2}\right)\cos\left(\frac{|\beta|}{2}\right) + \frac{\beta}{|\beta|}\sin\left(\frac{|\alpha|}{2}\right)\cos\left(\frac{|\beta|}{2}\right)\right.\\
&\left.+ \frac{\alpha\times\beta}{|\alpha||\beta|}\sin\left(\frac{\alpha}{2}\right)\sin\left(\frac{|\beta|}{2}\right)\right] && (4.2.7)
\end{aligned}$$

If one expands $e^{i\gamma/2}$ in the same way and equates the scalar and bivector parts, then one gets the desired relationships:

$$\begin{aligned}
\cos\left(\frac{|\gamma|}{2}\right) &= \cos\left(\frac{|\alpha|}{2}\right)\cos\left(\frac{|\beta|}{2}\right) - \sin\left(\frac{|\alpha|}{2}\right)\sin\left(\frac{|\beta|}{2}\right). && (4.2.8)\\
\frac{\gamma}{|\gamma|}\sin\left(\frac{|\gamma|}{2}\right) &= \frac{\alpha}{|\alpha|}\sin\left(\frac{|\alpha|}{2}\right)\cos\left(\frac{|\beta|}{2}\right) + \frac{\beta}{|\beta|}\sin\left(\frac{|\alpha|}{2}\right)\cos\left(\frac{|\beta|}{2}\right)\\
&+ \frac{\alpha\times\beta}{|\alpha||\beta|}\sin\left(\frac{\alpha}{2}\right)\sin\left(\frac{|\beta|}{2}\right) && (4.2.9)
\end{aligned}$$

4.3

4.3.1

$$[S_3, S_1] = S_3S_1 - S_1S_3, \tag{4.3.1}$$

$$= \frac{\hbar^2}{\sqrt{2}}\left[\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}\right], \tag{4.3.2}$$

$$= i\hbar\epsilon_{312}S_2. \tag{4.3.3}$$

4.3.2

$$\sum_{n=1}^{3} S_n^2 = \hbar^2\left[\frac{1}{2}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right], \tag{4.3.4}$$

$$= 2\hbar^2\hat{1}. \tag{4.3.5}$$

4.4

This is just a rotation of the $\hat{x}$ operator about the $\hat{\mathbf{z}}$ axis:

$$\hat{x}(\phi) = e^{i\hat{L}_3\phi/2}\hat{x}e^{-i\hat{L}_3\phi/2}, \tag{4.4.1}$$

$$= \hat{x}\cos\phi + \hat{y}\sin\phi. \tag{4.4.2}$$

4.5

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 1 & 5 & 6 & 4 \\ 3 & 1 & 2 & 6 & 4 & 5 \\ 4 & 6 & 5 & 1 & 3 & 2 \\ 5 & 4 & 6 & 2 & 1 & 3 \\ 6 & 5 & 4 & 3 & 2 & 1 \end{pmatrix} \tag{4.5.1}$$