Homework 01

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1.1

a

$$R(\phi) = \begin{pmatrix} cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{pmatrix}, \tag{1.1.1}$$

$$R\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}. \tag{1.1.2}$$

b

See equation (1.1.1).

c

Right circularly polarized light is given by

$$|R\rangle = \frac{1}{\sqrt{2}} \left(|x\rangle + i\,|y\rangle \right). \tag{1.1.3}$$

$$\phi(z,t) = \frac{1}{\sqrt{2}} e^{-i\omega t} \begin{pmatrix} e^{ik_x z} \\ e^{ik_y z} \end{pmatrix}. \tag{1.1.4}$$

Thus,

$$|\psi\rangle = \frac{1}{\sqrt{2}} \left(|x\rangle - ie^{i(k_x - k_y)Z}\,|y\rangle \right). \tag{1.1.5}$$

d

$$\begin{aligned} \rho_{|R\rangle} &= |R\rangle\langle R|, & (1.1.6) \\ &= \frac{1}{2} \left(|x\rangle + i\,|y\rangle \right) \left(\langle x| - i\,\langle y| \right), & (1.1.7) \\ &= \frac{1}{2} \left(\not{1} + \sigma_2 \right). & (1.1.8) \end{aligned}$$

e

$$\rho_\psi = \frac{1}{2}\,|x\rangle\langle x| + \frac{1}{2}\,|y\rangle\langle y|. \tag{1.1.9}$$

1.2

$$\begin{aligned} \langle x|R(\theta)|x\rangle &= \langle x|(\cos\theta - i\sin\theta)|x\rangle, & (1.2.1) \\ &= \cos\theta - i\sin\theta\,\langle x|\sigma_2|x\rangle, & (1.2.2) \\ &= \cos\theta. & (1.2.3) \end{aligned}$$

$$\begin{aligned} \langle x|R\left(\frac{\pi}{2}\right)|x\rangle &= 0, & (1.2.4) \\ \langle x|R(\pi)|x\rangle &= -1, & (1.2.5) \\ \langle x|R(2\pi)|x\rangle &= 1. & (1.2.6) \end{aligned}$$

1.3

$$\begin{aligned} &\langle z,+|R_y(\theta)|z,+\rangle & (1.3.1) \\ &= \langle z,+|\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)|z,+\rangle, & (1.3.2) \\ &= \cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\,\langle z,+|\sigma_2|z,+\rangle, & (1.3.3) \\ &= \cos\frac{\theta}{2}. & (1.3.4) \end{aligned}$$

$$\begin{aligned} \langle z,+|R_y\left(\frac{\pi}{2}\right)|z,+\rangle &= \frac{1}{\sqrt{2}}, & (1.3.5) \\ \langle z,+|R_y(\pi)|z,+\rangle &= 0, & (1.3.6) \\ \langle z,+|R_y(2\pi)|z,+\rangle &= -1. & (1.3.7) \end{aligned}$$

1.4

Let U bet a unitary transformation. Then

$$UU^\dagger = U^\dagger U = \mathbb{1}.$$

Hence,

$$U^\dagger \mathbb{1} U = U^\dagger U \mathbb{1} = \mathbb{1}\mathbb{1} = \mathbb{1}.$$

1.5

$$U^\dagger = \frac{1}{\sqrt{2}}(\mathbb{1} + i\sigma_3). \tag{1.5.1}$$

a

$$\begin{aligned}
U\sigma_1 U^\dagger &= \frac{1}{2}(\mathbb{1} - i\sigma_3)\sigma_1(\mathbb{1} + i\sigma_3), && (1.5.2)\\
&= \frac{1}{2}(\sigma_1 - i\sigma_3\sigma_1)(\mathbb{1} + i\sigma_3), && (1.5.3)\\
&= \frac{1}{2}(\sigma_1 + i[\sigma_1, \sigma_3] - \sigma_1), && (1.5.4)\\
&= \frac{1}{2}i(-2i\sigma_2), && (1.5.5)\\
&= \sigma_2. && (1.5.6)
\end{aligned}$$

b

$$\begin{aligned}
U\left|x,+\right\rangle &= \frac{1}{2}(\mathbb{1} + i\sigma_3)(\left|\uparrow\right\rangle + \left|\downarrow\right\rangle), && (1.5.7)\\
&= \frac{1}{2}(\left|\uparrow\right\rangle + \left|\downarrow\right\rangle - i\left|\uparrow\right\rangle + i\left|\downarrow\right\rangle), && (1.5.8)\\
&= \frac{1}{\sqrt{2}}(\left|\uparrow\right\rangle + i\left|\downarrow\right\rangle). && (1.5.9)
\end{aligned}$$

This expression (1.5.9) is an eigenstate of σ_2.

1.6

$$U^\dagger = v_i^{(j)}. \tag{1.6.1}$$

a

$$\begin{aligned}
UU^\dagger &= v_j^{*(i)} v_i^{(j)}, && (1.6.2)\\
&= \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}, && (1.6.3)\\
&= \mathbb{1}. && (1.6.4)
\end{aligned}$$

b

$$\begin{aligned}
U_{ij} v_i^{(n)} &= v_j^{*(i)} v_i^{(n)}, && (1.6.5)\\
&= \delta_{nj} \quad \text{(ortho.)} && (1.6.6)\\
K' = UKU^\dagger : & && (1.6.7)\\
K' U v^{(n)} &= UKU^\dagger U v^{(n)}, && (1.6.8)\\
&= UKv^{(n)}, && (1.6.9)\\
&= \lambda_n \left(Uv^{(n)}\right). && (1.6.10)
\end{aligned}$$

1.7

$$M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}. \tag{1.7.1}$$

a

$$\begin{vmatrix} \lambda - 1 & 0 & 0 \\ 0 & \lambda & -1 \\ 0 & -1 & \lambda \end{vmatrix} = (\lambda - 1)(\lambda^2 - 1) = 0, \tag{1.7.2}$$

$$\lambda = 1, \pm i. \tag{1.7.3}$$

b

$$\begin{pmatrix} \lambda-1 & 0 & 0 \\ 0 & \lambda & -1 \\ 0 & -1 & \lambda \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{1.7.4}$$

$$a = 0, \tag{1.7.5}$$

$$c = b\lambda. \tag{1.7.6}$$

$$v_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \tag{1.7.7}$$

$$v_i = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ i \end{pmatrix}, \tag{1.7.8}$$

$$v_{-i} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -i \end{pmatrix}. \tag{1.7.9}$$

1.8

$$K = \begin{pmatrix} A & C^* \\ C & B \end{pmatrix}. \tag{1.8.1}$$

a

$$(\lambda - A)(\lambda - B) - C^*C = 0, \tag{1.8.2}$$

$$\lambda_\pm = \frac{A + B \pm \sqrt{(A^2 - B^2) + 4C^*C}}{2}. \tag{1.8.3}$$

b

$$\begin{pmatrix} \lambda - A & -C^* \\ -C & \lambda - B \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{1.8.4}$$

$$a(\lambda - A) - bC^* = 0, \tag{1.8.5}$$

$$-aC + b(\lambda - B) = 0, \tag{1.8.6}$$

$$a = b\sqrt{\frac{C^*(\lambda - B)}{C(\lambda - A}}, \tag{1.8.7}$$

$$v_\pm = \frac{1}{\sqrt{\frac{C^*(\lambda-B)}{C(\lambda-A} - 1}} \begin{pmatrix} \sqrt{\frac{C^*(\lambda-B)}{C(\lambda-A}}, \\ 1 \end{pmatrix} \tag{1.8.8}$$

1.9

$$|\psi\rangle = \frac{1}{2}(\sqrt{3}\,|x\rangle + |y\rangle). \tag{1.9.1}$$

$$\left\langle (E - \langle E\rangle)^2 \right\rangle^{1/2} = \sqrt{Np(1-p)}, \tag{1.9.2}$$

$$p = \left|\langle x|y\rangle\right|^2 = \frac{3}{4}, \tag{1.9.3}$$

$$\left\langle (E - \langle E\rangle)^2 \right\rangle^{1/2} = \frac{\sqrt{3N}}{4}. \tag{1.9.4}$$

$$\langle E\rangle = Np, \tag{1.9.5}$$

$$= \frac{3N}{4}. \tag{1.9.6}$$

$$N = \frac{10\text{J}}{h\nu} \approx 3.32 \cdot 10^{19}, \tag{1.9.7}$$

$$\frac{\left\langle (E - \langle E\rangle)^2 \right\rangle^{1/2}}{\langle E\rangle} = \frac{1}{\sqrt{3N}}, \tag{1.9.8}$$

$$\approx 1.00 \cdot 10^{-10}\text{J}^{-1/2}. \tag{1.9.9}$$