Physics 851 **Problem Set 5** **Fall 2022**

NOTE: All problem numbers from Sakurai correspond to the 3^{rd} Edition.

1. A particle in one-dimension is described by the state $|\psi\rangle = \int_{-\infty}^{\infty} dx\, |x\rangle\, \psi(x)$, where the (normalized) position-space wave function is

$$\psi(x) \;=\; \begin{cases} 0 & x < 0 \\ \frac{2}{a^{3/2}} x e^{-x/a} & 0 \le x < \infty\,. \end{cases}$$

   (a) Using the position-space wave function, obtain the expectation value of momentum, $\langle p \rangle$, in this state.

   (b) The state can also be expanded in terms of momentum eigenstates, $|\psi\rangle = \int_{-\infty}^{\infty} dp\, |p\rangle\, \widetilde{\psi}(p)$. Obtain the momentum-space wave function, $\widetilde{\psi}(p)$.

   (c) If the momentum is measured in this state, what is the probability to obtain a value satisfying $-\frac{\hbar}{a} \le p \le \frac{\hbar}{a}$?

   (d) Using the momentum-space wave function, calculate $\langle p \rangle$, and verify that you obtain the same result as in part (a).

2. Sakurai 1.29

   [Hints: For part (a), use the completeness relation, inserting a complete set of $|a'\rangle$ states. For part (b), do the same, using position eigenstates $|\vec{x}'\rangle$. Do the three-dimensional integral in spherical polar coordinates (r', θ', ϕ'). With a judicious choice for the $\hat{z}$ direction, you should be able to do the θ' and ϕ' integrals for any $F(r)$. (Note that here I am using the Sakurai notation, where primed quantities indicate numbers, not operators.)]

3. Sakurai 2.3

   [Hint: You can use the answer to Sakurai 1.11 to obtain the state vector at $t = 0$.]

4. Sakurai 2.4. You can read the text of Section 2.1.6 for general information on neutrino oscillations, but I will lead you through the derivation here.

   In the two neutrino approximation, neutrinos are produced and detected in their gauge eigenstates, either as an electron neutrino, $|\nu_e\rangle$, or a muon neutrino, $|\nu_\mu\rangle$. These are related to the Energy/Mass eigenstates, $|\nu_1\rangle$ and $|\nu_2\rangle$, by the relation

$$\begin{aligned} |\nu_e\rangle &= \cos\theta|\nu_1\rangle - \sin\theta|\nu_2\rangle \\ |\nu_\mu\rangle &= \sin\theta|\nu_1\rangle + \cos\theta|\nu_2\rangle\,, \end{aligned}$$

   where θ is the neutrino mixing angle.

   (a) Let the energy eigenvalues be E_1 and E_2. If a reactor produces an electron neutrino at time $t = 0$, so that $|\nu(0)\rangle = |\nu_e\rangle$, what is the state, $|\nu(t)\rangle$ of the neutrino at a time $t > 0$ later?

   (b) Derive the probability that the neutrino is detected as an electron neutrino at the time t. [You should obtain the result $P(\nu_e \to \nu_e) = 1 - \sin^2(2\theta)\sin^2\left(\frac{(E_1-E_2)t}{2\hbar}\right)$.]

   (c) The neutrino masses are very small, so the neutrinos are very relativistic. For fixed momentum p, the energies can be approximated by

$$E_i = (p^2c^2 + m_i^2c^4)^{1/2} \approx pc\left(1 + \frac{m_i^2c^2}{2p^2}\right) \approx E\left(1 + \frac{m_i^2c^4}{2E^2}\right)\,,$$

   where in the last expression, we let $E = pc$, which is the energy for both eigenstates up to the tiny mass corrections. Using this, obtain equation (2.65) in the textbook, for the probability that the neutrino is detected as an electron neutrino after traveling a distance $L \approx ct$.

   (d) Using your result from part (c), along with the data in Figure 2.2 from the text, estimate the values of the mixing angle θ and of $\Delta m^2c^4 = (m_1^2 - m_2^2)c^4$ (in units of eV^2).