Homework 01

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1.1

a

$$R(\phi) = \begin{pmatrix} cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{pmatrix}, \tag{1.1.1}$$

$$R\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}. \tag{1.1.2}$$

b

See equation (1.1.1).

c

Right circularly polarized light is given by

$$|R\rangle = \frac{1}{\sqrt{2}} \left(|x\rangle + i\,|y\rangle\right). \tag{1.1.3}$$

$$\phi(z,t) = \frac{1}{\sqrt{2}} e^{-i\omega t} \begin{pmatrix} e^{ik_x z} \\ e^{ik_y z} \end{pmatrix}. \tag{1.1.4}$$

Thus,

$$|\psi\rangle = \frac{1}{\sqrt{2}} \left(|x\rangle - ie^{i(k_x - k_y)Z}\,|y\rangle\right). \tag{1.1.5}$$

d

$$\rho_{|R\rangle} = |R\rangle\langle R|, \tag{1.1.6}$$

$$= \frac{1}{2}\left(|x\rangle + i\,|y\rangle\right)\left(\langle x| - i\,\langle y|\right), \tag{1.1.7}$$

$$= \frac{1}{2}\left(\hat{1} + \sigma_2\right). \tag{1.1.8}$$

e

$$\rho_\psi = \frac{1}{2}\,|x\rangle\langle x| + \frac{1}{2}\,|y\rangle\langle y|. \tag{1.1.9}$$

1.2

$$\langle x|R(\theta)|x\rangle = \langle x|(\cos\theta - i\sin\theta)|x\rangle\,, \tag{1.2.1}$$
$$= \cos\theta - i\sin\theta\,\langle x|\sigma_2|x\rangle\,, \tag{1.2.2}$$
$$= \cos\theta. \tag{1.2.3}$$

$$\langle x|R\left(\frac{\pi}{2}\right)|x\rangle = 0, \tag{1.2.4}$$
$$\langle x|R\,(\pi)|x\rangle = -1, \tag{1.2.5}$$
$$\langle x|R\,(2\pi)|x\rangle = 1. \tag{1.2.6}$$

1.3

$$\langle z,+|R_y(\theta)|z,+\rangle \tag{1.3.1}$$
$$= \langle z,+|\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)|z,+\rangle\,, \tag{1.3.2}$$
$$= \cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\,\langle z,+|\sigma_2|z,+\rangle\,, \tag{1.3.3}$$
$$= \cos\frac{\theta}{2}. \tag{1.3.4}$$

$$\langle z,+|R_y\left(\frac{\pi}{2}\right)|z,+\rangle = \frac{1}{\sqrt{2}}, \tag{1.3.5}$$
$$\langle z,+|R_y\,(\pi)|z,+\rangle = 0, \tag{1.3.6}$$
$$\langle z,+|R_y\,(2\pi)|z,+\rangle = -1. \tag{1.3.7}$$

1.4

Let U bet a unitary transformation. Then

$$UU^\dagger = U^\dagger U = \hat{1}.$$

Hence,

$$U^\dagger\hat{1}U = U^\dagger U\hat{1} = \mathbb{1}\mathbb{1} = \hat{1}.$$

1.5

$$U^\dagger = \frac{1}{\sqrt{2}}(\hat{1} + i\sigma_3). \tag{1.5.1}$$

a

$$U\sigma_1 U^\dagger = \frac{1}{2}(\hat{1} - i\sigma_3)\sigma_1(\hat{1} + i\sigma_3), \tag{1.5.2}$$
$$= \frac{1}{2}(\sigma_1 - i\sigma_3\sigma_1)(\hat{1} + i\sigma_3), \tag{1.5.3}$$
$$= \frac{1}{2}(\sigma_1 + i[\sigma_1, \sigma_3] - \sigma_1), \tag{1.5.4}$$
$$= \frac{1}{2}i(-2i\sigma_2), \tag{1.5.5}$$
$$= \sigma_2. \tag{1.5.6}$$

b

$$U\left|x,+\right\rangle = \frac{1}{2}(\hat{1} + i\sigma_3)(\left|\uparrow\right\rangle + \left|\downarrow\right\rangle), \tag{1.5.7}$$
$$= \frac{1}{2}(\left|\uparrow\right\rangle + \left|\downarrow\right\rangle - i\left|\uparrow\right\rangle + i\left|\downarrow\right\rangle), \tag{1.5.8}$$
$$= \frac{1}{\sqrt{2}}(\left|\uparrow\right\rangle + i\left|\downarrow\right\rangle). \tag{1.5.9}$$

This expression (1.5.9) is an eigenstate of σ_2.

1.6

$$U^\dagger = v_i^{(j)}. \tag{1.6.1}$$

a

$$UU^\dagger = v_j^{*(i)} v_i^{(j)}, \tag{1.6.2}$$
$$= \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}, \tag{1.6.3}$$
$$= \hat{1}. \tag{1.6.4}$$

b

$$U_{ij} v_i^{(n)} = v_j^{*(i)} v_i^{(n)}, \tag{1.6.5}$$
$$= \delta_{nj} \quad \text{(ortho.)} \tag{1.6.6}$$
$$K' = UKU^\dagger : \tag{1.6.7}$$
$$K'Uv^{(n)} = UKU^\dagger Uv^{(n)}, \tag{1.6.8}$$
$$= UKv^{(n)}, \tag{1.6.9}$$
$$= \lambda_n \left(Uv^{(n)}\right). \tag{1.6.10}$$

1.7

$$M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}. \tag{1.7.1}$$

a

$$\begin{aligned} &\begin{vmatrix} \lambda - 1 & 0 & 0 \\ 0 & \lambda & -1 \\ 0 & -1 & \lambda \end{vmatrix} \\ &= (\lambda - 1)(\lambda^2 - 1) = 0, && (1.7.2) \\ &\lambda = 1, \pm i. && (1.7.3) \end{aligned}$$

b

$$\begin{pmatrix} \lambda - 1 & 0 & 0 \\ 0 & \lambda & -1 \\ 0 & -1 & \lambda \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{1.7.4}$$

$$a = 0, \tag{1.7.5}$$

$$c = b\lambda. \tag{1.7.6}$$

$$v_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \tag{1.7.7}$$

$$v_i = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ i \end{pmatrix}, \tag{1.7.8}$$

$$v_{-i} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -i \end{pmatrix}. \tag{1.7.9}$$

1.8

$$K = \begin{pmatrix} A & C^* \\ C & B \end{pmatrix}. \tag{1.8.1}$$

a

$$(\lambda - A)(\lambda - B) - C^* C = 0, \tag{1.8.2}$$

$$\lambda_\pm = \frac{A + B \pm \sqrt{(A^2 - B^2) + 4C^* C}}{2}. \tag{1.8.3}$$

b

$$\begin{pmatrix}\lambda - A & -C^* \\ -C & \lambda - B\end{pmatrix}\begin{pmatrix}a \\ b\end{pmatrix} = \begin{pmatrix}0 \\ 0\end{pmatrix}, \tag{1.8.4}$$

$$a(\lambda - A) - bC^* = 0, \tag{1.8.5}$$

$$-aC + b(\lambda - B) = 0, \tag{1.8.6}$$

$$a = b\sqrt{\frac{C^*(\lambda - B)}{C(\lambda - A)}}, \tag{1.8.7}$$

$$v_\pm = \frac{1}{\sqrt{\frac{C^*(\lambda-B)}{C(\lambda-A} - 1}}\begin{pmatrix}\sqrt{\frac{C^*(\lambda-B)}{C(\lambda-A}}, \\ 1\end{pmatrix} \tag{1.8.8}$$

1.9

$$|\psi\rangle = \frac{1}{2}(\sqrt{3}\,|x\rangle + |y\rangle). \tag{1.9.1}$$

$$\left\langle (E - \langle E\rangle)^2\right\rangle^{1/2} = \sqrt{Np(1-p)}, \tag{1.9.2}$$

$$p = \left|\langle x|y\rangle\right|^2 = \frac{3}{4}, \tag{1.9.3}$$

$$\left\langle (E - \langle E\rangle)^2\right\rangle^{1/2} = \frac{\sqrt{3N}}{4}. \tag{1.9.4}$$

$$\langle E\rangle = Np, \tag{1.9.5}$$

$$= \frac{3N}{4}. \tag{1.9.6}$$

$$N = \frac{10\mathrm{J}}{h\nu} \approx 3.32 \cdot 10^{19}, \tag{1.9.7}$$

$$\frac{\left\langle (E - \langle E\rangle)^2\right\rangle^{1/2}}{\langle E\rangle} = \frac{1}{\sqrt{3N}}, \tag{1.9.8}$$

$$\approx 1.00 \cdot 10^{-10}\mathrm{J}^{-1/2}. \tag{1.9.9}$$