Problem Set 1

Friday, September 2, 2022 10:45

Physics 851 **Problem Set 1** **Fall 2022**

The problems in this homework are a review of topics that are typically found in Modern Physics, Introductory Quantum Mechanics, and Math classes taken by undergraduate physics majors.

1. **The Bohr atom.** Consider the electron in the Hydrogen atom. Start from Bohr's assumption that the orbital angular momentum is quantized in integer units of some constant $\tilde{h}$ that is of the same dimensions and order-of-magnitude as Planck's constant,

$$mvr = n\tilde{h}$$

$m\ddot{r} = f$, $m\ddot{r} = -\frac{n\tilde{h}}{r^2}\dot{r}$, $f = -\frac{e^2}{r^2}k$, $n\tilde{h}\dot{r} = e^2k$, $\boxed{\dot{r}_n = \frac{e^2k}{n\tilde{h}}}$

$r_n = \frac{n\tilde{h}}{m\dot{r}_n} = \boxed{\frac{(n\tilde{h})^2}{e^2km}}$

$E_n = \frac{1}{2}m\dot{r}_n^2 + \frac{ke^2}{r_n} = \frac{1}{2}\frac{me^4k^2}{n^2\tilde{h}^2} - \frac{k^2e^4m}{n^2\tilde{h}^2} = \frac{-mk^2e^4}{2(n\tilde{h})^2}$

   (a) Using the classical equations for circular motion, obtain expressions for the quantized radii r_n, velocity v_n, and energy E_n. Also obtain an expression for the frequency ν of a photon emitted when the electron makes a transition from orbit n to n'. At this point, you should NOT assume that $\tilde{h} = \hbar$.

$E = h\nu$

$\nu = -\frac{1}{h}(E_{n'} - E_n) = \frac{(n')^2 - n^2}{(nn')^2}\frac{mk^2e^4}{2h\tilde{h}^2}$

   (b) We wish to show that $\tilde{h} = \hbar$ by invoking Bohr's famous Correspondence Principle, which says that the results of classical physics should apply for large quantum numbers (i.e., large n). Assume $n \gg 1$ and $n' = n - 1$, and equate your expression for ν in the previous part to the classical frequency of the electron in it's orbit, $\nu = \frac{v}{2\pi r}$. You should find $\tilde{h} = \frac{h}{2\pi} = \hbar$.

$n \gg 1$, $n' = n-1$

$\frac{\dot{r}}{2\pi r} = \frac{e^2k}{2\pi n\tilde{h}}\left(\frac{e^2km}{(n\tilde{h})^2}\right) = \frac{e^4k^2m}{2\pi(n\tilde{h})^3} = \frac{n^2-(n')^2}{(nn')^2}\frac{mk^2e^4}{2h\tilde{h}^2} \Rightarrow \frac{1}{2\pi(n\tilde{h})^3} \approx \frac{2}{2n^3}\frac{1}{h\tilde{h}^2}$

$\therefore \tilde{h} = \frac{h}{2\pi}$

2. **Simple Wave function problem.** A wave function of a particle in one dimension is given by the following function

$$\psi(x) = \begin{cases} 0 & \text{if } -\infty < x < 0 \\ Ae^{-(x/\lambda)} & \text{if } 0 \le x < \infty \end{cases}$$

where A is a real constant.

   (a) Find the value of the constant A that normalizes the wavefunction.

   (b) What is the probability for the particle to lie in the region $0 \le x \le \lambda$.

$1 = A^2\int_0^\infty e^{-2x/\lambda}dx = A^2\left[-\frac{\lambda}{2}e^{-2x/\lambda}\right]_0^\infty = A^2\left[0 + \frac{\lambda}{2}\right]$

$\therefore \boxed{A = \sqrt{\frac{2}{\lambda}}}$

$\frac{2}{\lambda}\int_0^\lambda e^{-2x/\lambda}dx = \left[-e^{-2x/\lambda}\right]_0^\lambda = -e^{-2} + 1$

$\boxed{P([0,\lambda]) = 1 - \frac{1}{e^2}}$

3. **Wave mechanics problem.** Consider a non-relativistic particle of mass m in one-dimension, confined to a potential that vanishes for $0 < x < L$, and is infinite everywhere else.

   (a) Starting directly from the Time-Independent Schrödinger Equation obtain the general form of the eigenfunctions. Apply the appropriate boundary conditions to obtain the energy eigenvalues and eigenfunctions. Finally, normalize the eigenfunctions.

   (b) Suppose the particle is placed in a state with $\psi(x) = Nx(L-x)$, where N is a normalization constant. If the energy is measured, what is the probability for obtaining the ground state energy (the lowest energy eigenvalue). (Feel free to use Mathematica, Wolfram Alpha, or any other method to obtain the necessary integral.)

$1 = \int_0^L \psi^*\psi\,dx = N^2\int_0^L x^2(L-x)^2dx$

$N = \sqrt{\frac{30}{L^5}}$

$B\int_0^L \psi^* \sin\left(\frac{\pi}{L}x\right)dx = \boxed{b.\ \frac{4\sqrt{30L}}{\pi^3}}$

$-\frac{\hbar}{2m}\nabla^2\psi = E\psi \rightarrow \psi = A\cos(\alpha x) + B\sin(\alpha x)$

$\alpha = \sqrt{\frac{2Em}{\hbar}} \rightarrow$ a. $\boxed{\psi_n(x) = \sqrt{\frac{2}{L}}\sin(\alpha x)}$

$\psi(0) = \psi(L) = 0 \rightarrow A = 0$ & $\alpha L = n\pi \rightarrow E_n = \left(\frac{n\pi}{L}\right)^2\left(\frac{\hbar}{2m}\right)$

$1 = \int_0^L B^2\sin^2(\alpha x)dx = \frac{B^2}{2}\int_0^L(1 - \cos(2\alpha x))dx = \frac{B^2}{2}\left[L - \cancel{\frac{\sin(2\alpha L)}{2\alpha}}\right] \rightarrow B = \sqrt{\frac{2}{L}}$

4. **Eigenvalues and vectors of a 3×3 Matrix.** Find the eigenvalues and eigenvectors of the following matrix:

$$A = \begin{pmatrix} 0 & a & 0 \\ a & 0 & a \\ 0 & a & 0 \end{pmatrix}.$$

$\boxed{\text{eigenvalues: } 0,\ \pm a\sqrt{2} \quad \text{eigenvectors: } \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\-1\end{pmatrix},\ \frac{1}{2}\begin{pmatrix}1\\ \pm\sqrt{2}\\ 1\end{pmatrix}}$

$\begin{vmatrix} \lambda & -a & 0 \\ -a & \lambda & -a \\ 0 & -a & \lambda \end{vmatrix} = \lambda(\lambda^2 - a^2) + \lambda(\lambda^2) + \lambda(\lambda^2 - a^2) = \lambda(3\lambda^2 - 2a^2)$

$av^2 - \lambda v^1 = 0$

$a(v^1 + v^3) - \lambda v^2 = 0$

$av^2 - \lambda v^3 = 0$

$\frac{1}{\sqrt{2 + \frac{2}{3}}} = \sqrt{\frac{3}{8}}$

$$\frac{1}{\sqrt{2+\frac{2}{3}}} = \sqrt{\frac{\quad}{8}}$$

$$= \lambda(3\lambda^2 - 2a^2)$$

$$a v^2 - \lambda v^1 = 0$$
$$a(v^1 + v^3) - \lambda v^2 = 0$$
$$a v^2 - \lambda v^3 = 0$$

$$\lambda = 0, \pm a\sqrt{2}$$

$$v^2 = \frac{\lambda}{a} v^1$$
$$v^1 + v^3 = \frac{\lambda}{a} v^2$$
$$v^2 = \frac{\lambda}{a} v^3$$

$\lambda = 0$:
$$v^2 = 0$$
$$v^1 = -v^3$$

$\lambda = \pm a\sqrt{2}$:
$$v^2 = \pm\sqrt{2}\,v^1$$
$$= \pm\sqrt{2}\,v^3$$