Homework 02

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PHY851
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September 20, 2021

1.10

1.10.1

$$|45\rangle = \frac{1}{\sqrt{2}}\left(|R\rangle + i\,|L\rangle\right), \tag{1.10.1}$$

$$|135\rangle = \frac{1}{\sqrt{2}}\left(-|R\rangle + i\,|L\rangle\right). \tag{1.10.2}$$

1.10.2

$$\begin{aligned}\frac{e^{-i\pi/4}}{2}\begin{pmatrix}\frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}}\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{pmatrix}\\ = \begin{pmatrix}-\frac{i}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}}\end{pmatrix}\end{aligned} \tag{1.10.3}$$

The factor of $e^{-i\pi/4}$ is there to rotate the state into a nicer (but identical) form, since relative phase is all that matters.

1.10.3

For a unitary transformation, the adjoint is equal to the inverse.

$$\begin{aligned}\begin{pmatrix}-\frac{i}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}}\end{pmatrix}\begin{pmatrix}\frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ -\frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}}\end{pmatrix}\\ = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}.\end{aligned} \tag{1.10.4}$$

Since it has been shown that the adjoint is the transformation's inverse, the transformation is unitary.

1.11

1.11.1

$$P_x = |X\rangle\langle X| = \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}. \tag{1.11.1}$$

1.11.2

The eigenvalues of P_x are

$$\lambda = 1, 0. \tag{1.11.2}$$

The eigenstates are

$$|\lambda\rangle = |X\rangle\,, |Y\rangle\,, \tag{1.11.3}$$

respectively.

1.11.3

In the RL basis, P_x is given by

$$P_x = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \frac{1}{2}(\mathbb{1} + \sigma_1). \tag{1.11.4}$$

From this, it's easy to see that the eigenvalues are, again,

$$\lambda = 1, 0. \tag{1.11.5}$$

1.12

1.12.1

$$\begin{aligned} \mathrm{Tr}\left(U^\dagger A U\right) &= U^*_{ij} A_{jk} U_{ki}, && (1.12.1)\\ &= U^*_{ij} U_{ki} A_{jk}, && (1.12.2)\\ &= \delta_{jk} A_{jk}, && (1.12.3)\\ &= A_{jj} = \mathrm{Tr}(A). && (1.12.4) \end{aligned}$$

1.12.2

$$\begin{aligned} \mathrm{Tr}(AB) &= A_{ij} B_{ji}, && (1.12.5)\\ &= B_{ji} A_{ij}, && (1.12.6)\\ &= \mathrm{Tr}(BA). && (1.12.7) \end{aligned}$$

1.13

Let $\phi(z,t)$ be the operator

$$\phi(z,t) = e^{-i\omega t}\begin{pmatrix} e^{ik_x z} & 0 \\ 0 & e^{-ik_y z} \end{pmatrix}, \tag{1.13.1}$$

where z is the distance into the crystal. Then

$$\begin{aligned} \phi(z,t)\,|\psi\rangle &= \frac{1}{\sqrt{2}} e^{-i\omega t}\begin{pmatrix} e^{ik_x z} \\ e^{ik_y z} \end{pmatrix}, && (1.13.2)\\ &= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ e^{i(k_y - k_x)z} \end{pmatrix} && (1.13.3) \end{aligned}$$

is the polarization of the photon at a distance z inside the crystal. If the photon comes out as righthand circularly polarized, then

$$e^{i(k_y-k_x)z} = i, \tag{1.13.4}$$

or

$$(k_y - k_x)z = \frac{\pi}{2}. \tag{1.13.5}$$

Thus

$$z = \frac{c}{4\left|n_y - n_x\right|\nu}. \tag{1.13.6}$$

1.14

Use geometric algebra!

$$\begin{aligned}
R(\phi)\sigma_1 R^{-1}(\phi) &= e^{-i\sigma_3\phi/2}\sigma_1 e^{i\sigma_3\phi/2}, && (1.14.1)\\
&= \left(\cos(\phi/2) - i\sin(\phi/2)\sigma_3\right)\sigma_1\left(\cos(\phi/2) + i\sin(\phi/2)\sigma_3\right), && (1.14.2)\\
&= \left(\cos(\phi/2)\sigma_1 + \sin(\phi/2)\sigma_2\sigma_1\sigma_1\right)\left(\cos(\phi/2) + \sin(\phi/2)\sigma_1\sigma_2\right), && (1.14.3)\\
&= \cos^2(\phi/2)\sigma_1 + 2\cos(\phi/2)\sin(\phi/2)\sigma_2 - \sin^2(\phi/2)\sigma_1, && (1.14.4)\\
&= \sigma_1\cos(\phi) + \sigma_2\sin(\phi). && (1.14.5)
\end{aligned}$$

1.15

1.15.1

$$\begin{aligned}
\left|x,+\right\rangle &= \frac{1}{\sqrt{2}}\left(\left|\uparrow\right\rangle + \left|\downarrow\right\rangle\right), && (1.15.1)\\
\left|x,-\right\rangle &= \frac{1}{\sqrt{2}}\left(\left|\uparrow\right\rangle - \left|\downarrow\right\rangle\right), && (1.15.2)\\
\left|y,+\right\rangle &= \frac{1}{\sqrt{2}}\left(\left|\uparrow\right\rangle + i\left|\downarrow\right\rangle\right), && (1.15.3)\\
\left|y,-\right\rangle &= \frac{1}{\sqrt{2}}\left(\left|\uparrow\right\rangle - i\left|\downarrow\right\rangle\right). && (1.15.4)
\end{aligned}$$

1.15.2

$$|z,+\rangle\langle z,+| = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \tag{1.15.5}$$

$$|z,-\rangle\langle z,-| = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \tag{1.15.6}$$

$$|x,+\rangle\langle x,+| = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \tag{1.15.7}$$

$$|x,-\rangle\langle x,-| = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \tag{1.15.8}$$

$$|y,+\rangle\langle y,+| = \frac{1}{2}\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}, \tag{1.15.9}$$

$$|y,-\rangle\langle y,-| = \frac{1}{2}\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}. \tag{1.15.10}$$

1.15.3

$$\rho_{60/40} = 0.6\,|z,+\rangle\langle z,+| + 0.4\,|z,-\rangle\langle z,-| = \begin{pmatrix} 0.6 & 0 \\ 0 & 0.4 \end{pmatrix} \tag{1.15.11}$$

1.15.4

$$\begin{aligned} \langle y,+|S_z|y,+\rangle &= \frac{1}{2}\,\mathrm{Tr}\big(|y,+\rangle\langle y,+|\,S_z\big), && (1.15.12)\\ &= 0. && (1.15.13) \end{aligned}$$

1.17

1.17.1

$$\begin{aligned} e^{-i\hat{H}t/\hbar} &= e^{-i\left(\hat{1}\frac{m_\mu+m_\tau}{2}+\alpha\sigma_1+\frac{m_\mu-m_\tau}{2}\sigma_3\right)t/\hbar}, && (1.17.1)\\ &= e^{-i(m_\mu+m_\tau)t/(2\hbar)}e^{-i\left(\alpha\sigma_1+\frac{m_\mu-m_\tau}{2}\sigma_3\right)t/\hbar}, && (1.17.2)\\ &= e^{-i(m_\mu+m_\tau)t/(2\hbar)}e^{-i\sigma_n\omega t}, && (1.17.3)\\ &= e^{-i(m_\mu+m_\tau)t/(2\hbar)}\left(\cos(\omega t) - i\sigma_n\sin(\omega t)\right). && (1.17.4) \end{aligned}$$

1.17.2

$$\begin{aligned} P_{\mu\to\tau} &= \left|\langle\tau|\,e^{-i\hat{H}t/\hbar}\,|\mu\rangle\right|^2, && (1.17.5)\\ &= \sin^2(\omega t)\big|\langle\tau|\,\sigma_n\,|\mu\rangle\big|^2, && (1.17.6)\\ &= \frac{\alpha^2}{\hbar^2\omega^2}\sin^2(\omega t). && (1.17.7) \end{aligned}$$

1.17.3

$$\tau_0 = \frac{\pi}{\omega}. \tag{1.17.8}$$

1.17.4

$$\tau = \gamma\tau_0 \approx \frac{\hbar kc}{m}\tau_0 \tag{1.17.9}$$

1.18

$$\begin{aligned}
\mathrm{Tr}\big(A_H(t)B_H(t)C_H(t)\big) &= \mathrm{Tr}\Big(U(t)A_SU^\dagger(T)U(t)B_SU^\dagger(T)U(t)C_SU^\dagger(T)\Big), &(1.18.1)\\
&= \mathrm{Tr}\Big(U(t)A_SB_SC_SU^\dagger(T)\Big), &(1.18.2)\\
&= \mathrm{Tr}(A_SB_SC_S). &(1.18.3)
\end{aligned}$$

2.1

Operating on the momentum eigenstates with the position operator doesn't make sense. If the momentum is clearly defined ($\hat{p}\,|q\rangle = q\,|q\rangle$), then the position isn't defined.

2.2

It's sufficient to show $\langle \hat{p}^2 \rangle \geq 0$:

$$\begin{aligned}
\langle \hat{p}^2 \rangle =& \hbar^2 \int \mathrm{d}x\, \partial_x\psi^*(x)\partial_x\psi(x), &(2.2.1)\\
=& \hbar^2 \int \mathrm{d}x\, \big|\partial_x\psi(x)\big|^2 \geq 0. &(2.2.2)
\end{aligned}$$

2.3

The solutions for the problem when **shifted** by a to the right, in each region, are

$$\begin{aligned}
\psi_I(x) &= e^{kx}, &(2.3.1)\\
\psi_{II}(x) &= A\cos(k_{II}x) + B\sin(k_{II}x), &(2.3.2)\\
\psi_{III}(x) &= Ce^{-kx}, &(2.3.3)
\end{aligned}$$

where

$$k^2 = \frac{2m|E|}{\hbar^2} \tag{2.3.4}$$

and

$$k_{II}^2 = \frac{2m(V_0 - |E|)}{\hbar^2}. \tag{2.3.5}$$

From the boundary conditions,

$$\begin{aligned}
A &= 1, &(2.3.6)\\
B &= \frac{k}{k_{II}}, &(2.3.7)\\
C &= e^{2ka}\left(\cos(2k_{II}a) + B\sin(2k_{II}a)\right). &(2.3.8)
\end{aligned}$$

These are valid for $|E| \geq 0$, and

$$0 = \tan(2k_{II}a), \tag{2.3.9}$$

$$n\pi = 2k_{II}a, \tag{2.3.10}$$

$$|E| = V_0 - \frac{n^2\pi^2\hbar^2}{8a^2m}, \tag{2.3.11}$$

$$\therefore V_0 \geq \frac{n^2\pi^2\hbar^2}{8a^2m}. \tag{2.3.12}$$