Problem Set 3

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Last updated: September 22, 2022

I SAKURAI 1.5

It's a rotation, of course it's invariant:

$$\det(ABC) = \det(A)\det(B)\det(C), \tag{I.1}$$

$$\therefore \det\left(e^{in\phi/2}a^k\sigma_k e^{-in\phi/2}\right) \tag{I.2}$$

$$= \det\left(e^{in^k\sigma_k\phi/2}\right)\det\left(a^k\sigma_k\right)\det\left(e^{-in^k\sigma_k\phi/2}\right), \tag{I.3}$$

$$= e^{\mathrm{tr}\left(in^k\sigma_k\phi/2\right)}\cdot\det\left(a^k\sigma_k\right)\cdot e^{\mathrm{tr}\left(-in^k\sigma_k\phi/2\right)}, \tag{I.4}$$

$$= \det\left(a^k\sigma_k\right). \tag{I.5}$$

$$a' = a^k e^{i\sigma_3\phi/2}\sigma_k e^{-i\sigma_3\phi/2}, \tag{I.6}$$

$$= a^k\left(\cos\frac{\phi}{2} + i\sigma_3\sin\frac{\phi}{2}\right)\left(\sigma_k\cos\frac{\phi}{2} - i\sigma_k\sigma_3\sin\frac{\phi}{2}\right), \tag{I.7}$$

$$= a^k\left(\sigma_k\cos^2\frac{\phi}{2} + i[\sigma_3,\sigma_k]\sin\frac{\phi}{2}\cos\frac{\phi}{2} \right. \tag{I.8}$$

$$\left. + \sigma_3\sigma_k\sigma_3\sin^2\frac{\phi}{2}\right). \tag{I.9}$$

If $k = 3$, $[\sigma_3, \sigma_k] = 0$, and $\sigma_3^3 = \sigma_3$. Thus,

$$\begin{aligned} a' = a^1 &\left(\sigma_1\cos^2\frac{\phi}{2} + i[\sigma_3,\sigma_1]\sin\frac{\phi}{2}\cos\frac{\phi}{2} + \sigma_3\sigma_1\sigma_3\sin^2\frac{\phi}{2}\right) \\ &+ a^2\left(\sigma_2\cos^2\frac{\phi}{2} + i[\sigma_3,\sigma_2]\sin\frac{\phi}{2}\cos\frac{\phi}{2} + \sigma_3\sigma_2\sigma_3\sin^2\frac{\phi}{2}\right) \\ &+ a^3\sigma_3, \end{aligned} \tag{I.10}$$

$$= a^1\left(\sigma_1\cos^2\frac{\phi}{2} - \sigma_2\sin\phi - \sigma_1\sin^2\frac{\phi}{2}\right) + a^2\left(\sigma_2\cos^2\frac{\phi}{2} + \sigma_1\sin\phi - \sigma_2\sin^2\frac{\phi}{2}\right) + a^3\sigma_3, \tag{I.11}$$

$$= a^1\left(\sigma_1\cos\phi - \sigma_2\sin\phi\right) + a^2\left(\sigma_2\cos\phi + \sigma_1\sin\phi\right) + a^3\sigma_3, \tag{I.12}$$

$$= (a^1\cos\phi + a^2\sin\phi)\sigma_1 + (-a^1\sin\phi + a^2\cos\phi)\sigma_2 + a^3\sigma_3. \tag{I.13}$$

$$a'^1 = a^1\cos\phi + a^2\sin\phi. \tag{I.14}$$

$$a'^2 = -a^1\sin\phi + a^2\cos\phi. \tag{I.15}$$

$$a'^3 = a^3. \tag{I.16}$$

This is a rotation of the vector $a = a^k\sigma_k$ through an angle of ϕ in the xy plane ($\sigma_1\sigma_2$ plane).

II SAKURAI 1.14

A

$$\left|\langle S_x;+|\psi\rangle\right|^2 = \left|\frac{1}{\sqrt{2}}\left(\langle+| + \langle-|\right)\left(\cos\frac{\gamma}{2}|+\rangle \pm \sin\frac{\gamma}{2}|-\rangle\right)\right|^2, \tag{II.1}$$

$$= \frac{1}{2}\left(\cos\frac{\gamma}{2} \pm \sin\frac{\gamma}{2}\right)^2, \tag{II.2}$$

$$= \frac{1}{2}\left(1 \pm \sin\gamma\right). \tag{II.3}$$

B

$$\left\langle(S_x - \langle S_x\rangle)^2\right\rangle = \left\langle S_x^2\right\rangle - \langle S_x\rangle^2. \tag{II.4}$$

$$\left\langle S_x^2\right\rangle = \frac{\hbar^2}{4}\left(\cos^2\frac{\gamma}{2} + \sin^2\frac{\gamma}{2}\right), \tag{II.5}$$

$$= \frac{\hbar^2}{4}. \tag{II.6}$$

$$\langle S_x\rangle^2 = \frac{\hbar^2}{4}4\sin^2\frac{\gamma}{2}\cos^2\frac{\gamma}{2}, \tag{II.7}$$

$$= \frac{\hbar^2}{4}\sin^2\gamma. \tag{II.8}$$

$$\left\langle S_x^2\right\rangle - \langle S_x\rangle^2 = \frac{\hbar^2}{4}\cos^2\gamma. \tag{II.9}$$

III SAKURAI 1.15

This is the same as polarization of photons, but with twice the angle: the intensity of the final $s_z = -\hbar/2$ beam is $I(\beta) = \sin^2(\beta)/4$.

The intensity of the final $s_z = -\hbar/2$ beam is maximized when the second device is measuring anywhere in the xy-plan: $\beta = \pi/2$.

IV SAKURAI 1.21

A

$$\left\langle S_x^2 \right\rangle - \langle S_x \rangle^2 = \frac{\hbar^2}{4} - 0, \tag{IV.1}$$

$$= \frac{\hbar^2}{4}. \left\langle S_y^2 \right\rangle - \left\langle S_y \right\rangle^2 \quad = \frac{\hbar^2}{4}. \tag{IV.2}$$

$$\frac{1}{4}\left|\left\langle [S_x, S_y] \right\rangle\right|^2 = \frac{\hbar^4}{16}. \tag{IV.3}$$

$$\left\langle (\Delta S_x)^2 \right\rangle \left\langle (\Delta S_y)^2 \right\rangle \geq \frac{1}{4}\left|\left\langle [S_x, S_y] \right\rangle\right|^2, \tag{IV.4}$$

$$\frac{\hbar^4}{16} \geq \frac{\hbar^4}{16}. \quad \checkmark \tag{IV.5}$$

B

$$\left\langle S_x^2 \right\rangle - \langle S_x \rangle^2 = \frac{\hbar^2}{4} - \frac{\hbar^2}{4}, \tag{IV.6}$$

$$= 0. \tag{IV.7}$$

$$\frac{1}{4}\left|\left\langle [S_x, S_y] \right\rangle\right|^2 = 0. \tag{IV.8}$$

$$\left\langle (\Delta S_x)^2 \right\rangle \left\langle (\Delta S_y)^2 \right\rangle \geq \frac{1}{4}\left|\left\langle [S_x, S_y] \right\rangle\right|^2, \tag{IV.9}$$

$$0 \geq 0. \quad \checkmark \tag{IV.10}$$

V SAKURAI 1.25

A

The operator, B, exhibits a degenerate spectrum (repeated eigenvalues):

$$\det(\lambda - B) = 0 = (\lambda - b)(\lambda^2 - b^2), \tag{V.1}$$

$$\lambda \in \{b, b, -b\}. \tag{V.2}$$

B

$$AB = \begin{pmatrix} a & 0 & 0 \\ 0 & -a & 0 \\ 0 & 0 & -a \end{pmatrix} \begin{pmatrix} b & 0 & 0 \\ 0 & 0 & -ib \\ 0 & ib & 0 \end{pmatrix}, \tag{V.3}$$

$$= \begin{pmatrix} ab & 0 & 0 \\ 0 & 0 & iab \\ 0 & -iab & 0 \end{pmatrix}, \tag{V.4}$$

$$= \begin{pmatrix} b & 0 & 0 \\ 0 & 0 & -ib \\ 0 & ib & 0 \end{pmatrix} \begin{pmatrix} a & 0 & 0 \\ 0 & -a & 0 \\ 0 & 0 & -a \end{pmatrix}, \tag{V.5}$$

$$= BA. \tag{V.6}$$

C

The eigenvalues of AB are $\lambda_{ab} \in \{ab, ab, -ab\}$.

$$\begin{pmatrix} ab & 0 & 0 \\ 0 & 0 & iab \\ 0 & -iab & 0 \end{pmatrix} \begin{pmatrix} v^1 \\ v^2 \\ v^3 \end{pmatrix} = \lambda_{ab} \begin{pmatrix} v^1 \\ v^2 \\ v^3 \end{pmatrix}, \tag{V.7}$$

$$abv^1 = \lambda_{ab} v^1, \tag{V.8}$$

$$iabv^3 = \lambda_{ab} v^2, \tag{V.9}$$

$$-iabv^2 = \lambda_{ab} v^3. \tag{V.10}$$

$$\rightarrow v \in \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ i \\ 1 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ -i \\ 1 \end{pmatrix} \right\}. \tag{V.11}$$

The respective eigenvalues are given above.