

Homework 02

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1.10

1.10.1

$$|45\rangle = \frac{1}{\sqrt{2}} (|R\rangle + i |L\rangle), \quad (1.10.1)$$

$$|135\rangle = \frac{1}{\sqrt{2}} (-|R\rangle + i |L\rangle). \quad (1.10.2)$$

1.10.2

$$\begin{aligned} \frac{e^{-i\pi/4}}{2} \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \\ = \begin{pmatrix} -\frac{i}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \end{pmatrix} \end{aligned} \quad (1.10.3)$$

The factor of $e^{-i\pi/4}$ is there to rotate the state into a nicer (but identical) form, since relative phase is all that matters.

1.10.3

For a unitary transformation, the adjoint is equal to the inverse.

$$\begin{aligned} \begin{pmatrix} -\frac{i}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} \end{pmatrix} \\ = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \end{aligned} \quad (1.10.4)$$

Since it has been shown that the adjoint is the transformation's inverse, the transformation is unitary.

1.11

1.11.1

$$P_x = |X\rangle\langle X| = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \quad (1.11.1)$$

1.11.2

The eigenvalues of P_x are

$$\lambda = 1, 0. \quad (1.11.2)$$

The eigenstates are

$$|\lambda\rangle = |X\rangle, |Y\rangle, \quad (1.11.3)$$

respectively.

1.11.3

In the RL basis, P_x is given by

$$P_x = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \frac{1}{2}(\mathbb{K} + \sigma_1). \quad (1.11.4)$$

From this, it's easy to see that the eigenvalues are, again,

$$\lambda = 1, 0. \quad (1.11.5)$$

1.12

1.12.1

$$\text{Tr}(U^\dagger AU) = U_{ij}^* A_{jk} U_{ki}, \quad (1.12.1)$$

$$= U_{ij}^* U_{ki} A_{jk}, \quad (1.12.2)$$

$$= \delta_{jk} A_{jk}, \quad (1.12.3)$$

$$= A_{jj} = \text{Tr}(A). \quad (1.12.4)$$

1.12.2

$$\text{Tr}(AB) = A_{ij} B_{ji}, \quad (1.12.5)$$

$$= B_{ji} A_{ij}, \quad (1.12.6)$$

$$= \text{Tr}(BA). \quad (1.12.7)$$

1.13

Let $\phi(z, t)$ be the operator

$$\phi(z, t) = e^{-i\omega t} \begin{pmatrix} e^{ik_x z} & 0 \\ 0 & e^{-ik_y z} \end{pmatrix}, \quad (1.13.1)$$

where z is the distance into the crystal. Then

$$\phi(z, t) |\psi\rangle = \frac{1}{\sqrt{2}} e^{-i\omega t} \begin{pmatrix} e^{ik_x z} \\ e^{ik_y z} \end{pmatrix}, \quad (1.13.2)$$

$$= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ e^{i(k_y - k_x)z} \end{pmatrix} \quad (1.13.3)$$

is the polarization of the photon at a distance z inside the crystal. If the photon comes out as righthand circularly polarized, then

$$e^{i(k_y - k_x)z} = i, \quad (1.13.4)$$

or

$$(k_y - k_x)z = \frac{\pi}{2}. \quad (1.13.5)$$

Thus

$$z = \frac{c}{4|n_y - n_x|\nu}. \quad (1.13.6)$$

1.14

Use geometric algebra!

$$R(\phi)\sigma_1 R^{-1}(\phi) = e^{-i\sigma_3\phi/2}\sigma_1 e^{i\sigma_3\phi/2}, \quad (1.14.1)$$

$$= \left(\cos(\phi/2) - i \sin(\phi/2)\sigma_3 \right) \sigma_1 \left(\cos(\phi/2) + i \sin(\phi/2)\sigma_3 \right), \quad (1.14.2)$$

$$= \left(\cos(\phi/2)\sigma_1 + \sin(\phi/2)\sigma_2\sigma_1\sigma_1 \right) \left(\cos(\phi/2) + \sin(\phi/2)\sigma_1\sigma_2 \right), \quad (1.14.3)$$

$$= \cos^2(\phi/2)\sigma_1 + 2\cos(\phi/2)\sin(\phi/2)\sigma_2 - \sin^2(\phi/2)\sigma_1, \quad (1.14.4)$$

$$= \sigma_1 \cos(\phi) + \sigma_2 \sin(\phi). \quad (1.14.5)$$

1.15

1.15.1

$$|x, +\rangle = \frac{1}{\sqrt{2}} (|\uparrow\rangle + |\downarrow\rangle), \quad (1.15.1)$$

$$|x, -\rangle = \frac{1}{\sqrt{2}} (|\uparrow\rangle - |\downarrow\rangle), \quad (1.15.2)$$

$$|y, +\rangle = \frac{1}{\sqrt{2}} (|\uparrow\rangle + i|\downarrow\rangle), \quad (1.15.3)$$

$$|y, -\rangle = \frac{1}{\sqrt{2}} (|\uparrow\rangle - i|\downarrow\rangle). \quad (1.15.4)$$

1.15.2

$$|z, +\rangle\langle z, +| = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad (1.15.5)$$

$$|z, -\rangle\langle z, -| = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad (1.15.6)$$

$$|x, +\rangle\langle x, +| = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad (1.15.7)$$

$$|x, -\rangle\langle x, -| = \frac{1}{2} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \quad (1.15.8)$$

$$|y, +\rangle\langle y, +| = \frac{1}{2} \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}, \quad (1.15.9)$$

$$|y, -\rangle\langle y, -| = \frac{1}{2} \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}. \quad (1.15.10)$$

1.15.3

$$\rho_{60/40} = 0.6 |z, +\rangle\langle z, +| + 0.4 |z, -\rangle\langle z, -| = \begin{pmatrix} 0.6 & 0 \\ 0 & 0.4 \end{pmatrix} \quad (1.15.11)$$

1.15.4

$$\langle y, + | S_z | y, + \rangle = \frac{1}{2} \text{Tr}(|y, +\rangle\langle y, + | S_z), \quad (1.15.12)$$

$$= 0. \quad (1.15.13)$$

1.17**1.17.1**

$$e^{-i\hat{H}t/\hbar} = e^{-i\left(\hat{1}\frac{m_\mu+m_\tau}{2} + \alpha\sigma_1 + \frac{m_\mu-m_\tau}{2}\sigma_3\right)t/\hbar}, \quad (1.17.1)$$

$$= e^{-i(m_\mu+m_\tau)t/(2\hbar)} e^{-i\left(\alpha\sigma_1 + \frac{m_\mu-m_\tau}{2}\sigma_3\right)t/\hbar}, \quad (1.17.2)$$

$$= e^{-i(m_\mu+m_\tau)t/(2\hbar)} e^{-i\sigma_n\omega t}, \quad (1.17.3)$$

$$= e^{-i(m_\mu+m_\tau)t/(2\hbar)} (\cos(\omega t) - i\sigma_n \sin(\omega t)). \quad (1.17.4)$$

1.17.2

$$P_{\mu \rightarrow \tau} = \left| \langle \tau | e^{-i\hat{H}t/\hbar} | \mu \rangle \right|^2, \quad (1.17.5)$$

$$= \sin^2(\omega t) \left| \langle \tau | \sigma_n | \mu \rangle \right|^2, \quad (1.17.6)$$

$$= \frac{\alpha^2}{\hbar^2 \omega^2} \sin^2(\omega t). \quad (1.17.7)$$

1.17.3

$$\tau_0 = \frac{\pi}{\omega}. \quad (1.17.8)$$

1.17.4

$$\tau = \gamma\tau_0 \approx \frac{\hbar kc}{m}\tau_0 \quad (1.17.9)$$

1.18

$$\text{Tr}(A_H(t)B_H(t)C_H(t)) = \text{Tr}\left(U(t)A_SU^\dagger(T)U(t)B_SU^\dagger(T)U(t)C_SU^\dagger(T)\right), \quad (1.18.1)$$

$$= \text{Tr}\left(U(t)A_SB_SC_SU^\dagger(T)\right), \quad (1.18.2)$$

$$= \text{Tr}(A_SB_SC_S). \quad (1.18.3)$$

2.1

Operating on the momentum eigenstates with the position operator doesn't make sense. If the momentum is clearly defined ($\hat{p}|q\rangle = q|q\rangle$), then the position isn't defined.

2.2

It's sufficient to show $\langle \hat{p}^2 \rangle \geq 0$:

$$\langle \hat{p}^2 \rangle = \hbar^2 \int dx \partial_x \psi^*(x) \partial_x \psi(x), \quad (2.2.1)$$

$$= \hbar^2 \int dx |\partial_x \psi(x)|^2 \geq 0. \quad (2.2.2)$$

2.3

The solutions for the problem when **shifted** by a to the right, in each region, are

$$\psi_I(x) = e^{kx}, \quad (2.3.1)$$

$$\psi_{II}(x) = A \cos(k_{II}x) + B \sin(k_{II}x), \quad (2.3.2)$$

$$\psi_{III}(x) = Ce^{-kx}, \quad (2.3.3)$$

where

$$k^2 = \frac{2m|E|}{\hbar^2} \quad (2.3.4)$$

and

$$k_{II}^2 = \frac{2m(V_0 - |E|)}{\hbar^2}. \quad (2.3.5)$$

From the boundary conditions,

$$A = 1, \quad (2.3.6)$$

$$B = \frac{k}{k_{II}}, \quad (2.3.7)$$

$$C = e^{2ka} (\cos(2k_{II}a) + B \sin(2k_{II}a)). \quad (2.3.8)$$

These are valid for $|E| \geq 0$, and

$$0 = \tan(2k_{II}a), \quad (2.3.9)$$

$$n\pi = 2k_{II}a, \quad (2.3.10)$$

$$|E| = V_0 - \frac{n^2\pi^2\hbar^2}{8a^2m}, \quad (2.3.11)$$

$$\therefore V_0 \geq \frac{n^2\pi^2\hbar^2}{8a^2m}. \quad (2.3.12)$$