Problem Set 2

Brandon Henke[1]

[1] *Michigan State University - Department of Physics & Astronomy*

Last updated: January 26, 2023

I SAKURAI 5.1

$$H_0 = \frac{\hat{p}^2}{2m} + \frac{1}{2}\omega^2 m^2 \hat{x}^2, \tag{I.1}$$

$$H = H_0 + b\hat{x}. \tag{I.2}$$

$$\Delta_0 = \langle 0^{(0)}|b\hat{x}|0^{(0)}\rangle + \sum_{k\neq 0} \frac{2b^2\left|\langle 1|x|0\rangle\right|^2}{\hbar\omega - (2k+1)\hbar\omega}, \tag{I.3}$$

$$= 0 + \sum_{k\neq 0} \frac{\hbar}{2m\omega} \frac{2b^2\left|\langle k|1\rangle\right|^2}{\hbar\omega - (2k+1)\hbar\omega}, \tag{I.4}$$

$$= \frac{\hbar}{2m\omega}\frac{2b^2}{-2\hbar\omega}, \tag{I.5}$$

$$= -\frac{b^2}{2m\omega^2} \tag{I.6}$$

$$V(x) \to \frac{1}{2}m\omega^2\left(x + \frac{b}{m\omega^2}\right)^2 - \frac{b^2}{2m\omega^2}. \tag{I.7}$$

$$x' = x + \frac{b}{m\omega^2}, \tag{I.8}$$

$$H = \frac{\hat{p}^2}{2m} + \frac{1}{2}\omega^2 m^2 \hat{x}'^2 - \frac{b^2}{2m\omega^2}. \tag{I.9}$$

This is a simple harmonic oscillator with an energy shift of $-\frac{b^2}{2m\omega^2}$, so the perturbation theory result gives the exact energy shift.

II SAKURAI 5.7

A

This is just two independent simple harmonic oscillators, so the energy is

$$E_{n_x,n_y} = \hbar\omega\left(n_x + n_y + 1\right). \tag{II.1}$$

Hence, the three lowest lying states ($|0,0\rangle$, $|1,0\rangle$, and $|0,1\rangle$) have energies of $E \in \{\hbar\omega, 2\hbar\omega, 2\hbar\omega\}$, respectively. Hence, the first excited state is doubly degenerate.

B

For the ground state, there is no degeneracy, so the energy shift is

$$\Delta_0^{(0)} = \delta m\omega^2 \langle 0,0|xy|0,0\rangle, \tag{II.2}$$

$$= 0. \qquad (x\,|0\rangle \propto |1\rangle) \tag{II.3}$$

For the first excited state:

$$V\,|l^{(0)}\rangle = \Delta_1^{(1)}\,|l^{(0)}\rangle = \sum_m V\,|m^{(0)}\rangle\,\langle m^{(0)}|l^{(0)}\rangle. \tag{II.4}$$

$$V = \delta m\omega^2 \begin{pmatrix} \langle 0,1|xy|0,1\rangle & \langle 1,0|xy|0,1\rangle \\ \langle 0,1|xy|1,0\rangle & \langle 1,0|xy|1,0\rangle \end{pmatrix}, \tag{II.5}$$

$$= \delta m\omega^2 \frac{\hbar}{2m\omega}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{II.6}$$

$$= \delta\omega\frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{II.7}$$

Solving the eigenvalue problem (eq II.4), gives

$$\Delta_1^{(1)} = \pm\delta\omega\hbar/2. \tag{II.8}$$

The eigenkets, which correspond to the zero-th order energy eigenkets, are

$$\Delta_1^{(1)} = \pm\delta\omega\hbar/2: \quad |l^{(0)}\rangle = \frac{1}{\sqrt{2}}(\,|0,1\rangle \pm |1,0\rangle). \tag{II.9}$$

C

The full Hamiltonian is

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 + \delta m\omega^2 xy, \tag{II.10}$$

$$= \frac{p^2}{2m} + \frac{1}{2}m\omega^2(x^2 + y^2 + 2xy\delta), \tag{II.11}$$

$$= \frac{p^2}{2m} + \frac{1}{2}m\omega^2(x'^2 + y'^2), \tag{II.12}$$

where $x' = \sqrt{1+\delta}\frac{x+y}{\sqrt{2}}$ and $y' = \sqrt{1-\delta}\frac{x-y}{\sqrt{2}}$.

From this, the energy of the energy eigenket $|n_{x'}, n_{y'}\rangle$ is

$$E_{n_{x'},n_{y'}} = \left(n_{x'} + \frac{1}{2}\right)\hbar\omega(1+\delta)^{1/2} + \left(n_{y'} + \frac{1}{2}\right)\hbar\omega(1-\delta)^{1/2}. \tag{II.13}$$

From this, Taylor expanding the energy of the energy eigenket $|0,0\rangle$ is $\approx \hbar\omega - \delta^2\hbar\omega/8$. The first term here, agrees with the perturbation theory result. The Taylor expanded energies of the energy eigenkets $|1,0\rangle$ and $|0,1\rangle$ are $\approx (2+\delta/2)\hbar\omega$ and $\approx (2-\delta/2)\hbar\omega$, respectively. These both agree with the perturbation theory results.

III SECOND ORDER GROUND STATE CORRECTION

The second order correction to the ground state of the perturbation $V = \delta m\omega^2 xy$ is

$$\Delta_0 = 0 + \sum_{k\neq 0} \delta^2 m^2\omega^4 \frac{\langle 0,0|xy|k\rangle\,\langle k|xy|0,0\rangle}{\hbar\omega - E_k^{(0)}}, \tag{III.1}$$

$$= \delta^2 m^2 \omega^4 \left(\frac{\hbar}{2m\omega}\right)^2 \frac{1}{\hbar\omega - 3\hbar\omega}, \tag{III.2}$$

$$= -\frac{\delta^2\hbar\omega}{8}. \tag{III.3}$$

This agrees with the second term in the taylor expansion of the energy for the exact ground state found in II-C.

IV SAKURAI 5.16

$$H_0 = \frac{\hat{p}^2}{2m} - \frac{e^2}{r}, \tag{IV.1}$$

$$H = H_0 + V(r), \tag{IV.2}$$

$$V(r) = \begin{cases} e^2\left(\frac{1}{r} + \frac{r^2}{2R^3} - \frac{3}{2R}\right) & r \le R \\ 0 & r \ge R \end{cases}. \tag{IV.3}$$

$$\Delta_{nl}^{(1)} = \langle nlm|V|nlm\rangle, \tag{IV.4}$$

$$= \int_0^R r^2\,\mathrm{d}r\, R_{nl}(r)V(r)R_{nl}(r). \tag{IV.5}$$

$$R_{10}(r) = 2\left(\frac{1}{a_0}\right)^{3/2} e^{-r/a_0}, \tag{IV.6}$$

$$R_{20}(r) = \frac{1}{\sqrt{3}}\left(\frac{1}{2a_0}\right)^{3/2}\left(1 - \frac{r}{2a_0}\right) e^{-r/2a_0}, \tag{IV.7}$$

$$R_{21}(r) = \frac{1}{\sqrt{3}}\left(\frac{1}{2a_0}\right)^{3/2}\left(\frac{r}{a_0}\right) e^{-r/2a_0}. \tag{IV.8}$$

Using Mathematica to evaluate the integrals gives

$$\Delta_{10}^{(1)} = \frac{2}{5}\frac{e^2}{a_0}\left(\frac{R}{a_0}\right)^2, \tag{IV.9}$$

$$\Delta_{20}^{(1)} = \frac{1}{20}\frac{e^2}{a_0}\left(\frac{R}{a_0}\right)^2, \tag{IV.10}$$

$$\Delta_{21}^{(1)} = \frac{1}{1120}\frac{e^2}{a_0}\left(\frac{R}{a_0}\right)^4. \tag{IV.11}$$

The wavefunction goes as r^l for $r \to 0$, so only states with $l = 0$ have a significant overlap with the proton.