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Physics 472 **Exam 2** **Spring 2019**

Non-Degenerate Perturbation Theory

If a level n state with energy $E_n^{(0)}$ and eigenfunction $\psi_n^{(0)}$ is non-degenerate, then the perturbed energy and eigenfunction are given by

$$\begin{aligned} E_n &= E_n^{(0)} + \langle\, \psi_n^{(0)}|H'|\psi_n^{(0)}\rangle + \sum_{k\neq n} \frac{|\langle\, \psi_k^{(0)}|H'|\psi_n^{(0)}\rangle|^2}{E_n^{(0)} - E_k^{(0)}} + \dots \\ \psi_n &= \psi_n^{(0)} + \sum_{k\neq n} \frac{\psi_k^{(0)}\langle\, \psi_k^{(0)}|H'|\psi_n^{(0)}\rangle}{E_n^{(0)} - E_k^{(0)}} + \dots \end{aligned}$$

where H' is the perturbation.

Degenerate Perturbation Theory

If an energy level is n-fold degenerate with its n orthogonal eigenfunctions labelled as $\psi_1, \psi_2, \cdots, \psi_n$ and $H'_{jk} = \langle\psi_j|H'|\psi_k\rangle$, where H' is a perturbation, then the first order energy shifts to this level are determined by

$$\begin{vmatrix} H'_{11}-E & H'_{12} & H'_{13} & \cdots & H'_{1n} \\ H'_{21} & H'_{22}-E & H'_{23} & \cdots & H'_{2n} \\ H'_{31} & H'_{32} & H'_{33}-E & \cdots & H'_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ H'_{n1} & H'_{n2} & H'_{n3} & \cdots & H'_{nn}-E \end{vmatrix} = 0$$

Variational Method

If H is a Hamiltonian with ground state energy E_0, then for any normalized state $|\psi\rangle$,

$$E_0 \leq \langle\psi|H|\psi\rangle\,.$$

Symmetries

For hydrogen eigenstates $|n\ell m\rangle$, rotational invariance implies that $(m'-m)\langle n\ell' m'|z|n\ell m\rangle = 0$, $\ell' = \ell \pm 1$ and parity invariance implies $\langle n\ell' m'|z|n\ell m\rangle = (-1)^{\ell'+\ell+1}\langle n\ell' m'|z|n\ell m\rangle$.

Zeeman Effect

The perturbative Hamiltonian for the Zeeman effect gives, in first order

$$\langle H'_Z\rangle = \frac{\mu_B}{\hbar} B\langle \ell\, \tfrac{1}{2}\, j\, m_j|J_z + S_z|\ell\, \tfrac{1}{2}\, j\, m_j\rangle = \mu_B B g_L m_j\,,$$

where B is the magnitude of the external magnetic field.

Miscellaneous

$$\text{Harmonic Oscillator} \quad \left(\frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2\right)\psi_n = E_n\psi_n \qquad E_n = \hbar\omega(n+\tfrac{1}{2})\,,$$

$$\text{Hydrogen} \quad \left(\frac{\vec{p}^{\,2}}{2m} - \frac{\hbar c\alpha}{r}\right)\psi_{n\ell m} = E_n\psi_{n\ell m} \qquad E_n = -\frac{mc^2\alpha^2}{2n^2} = -\frac{13.6\text{eV}}{n^2}\,.$$

1. The ground state wave function and energy for the hydrogen atom are

$$\psi_0(\vec{r}) = \left(\frac{1}{\pi a^3}\right)^{\frac{1}{2}} e^{-r/a}, \quad E_0 = -\tfrac{1}{2}mc^2\alpha^2 = -13.6\,\mathrm{eV}\,,$$

where $a = \hbar/mc\alpha$ is the Bohr radius. The atom is perturbed by a potential linear in r resulting in the Hamiltonian

$$H = \frac{\vec{p}^2}{2m} - \frac{\hbar c\alpha}{r} + \frac{2}{3}mc^2\alpha^2\lambda\frac{r}{a}\,,$$

where $\lambda > 0$ and the term proportional to λ is the perturbation H'.

(a) (3 pt) Calculate the ground state energy using first order perturbation theory. (*I.e.,* calculate $E^{(0)} + E^{(1)}$ for the ground state.)

(b) (3 pt) To obtain an improved estimate of the ground state energy, use the trial wave function

$$\psi(\vec{r}) = \left(\frac{Z^3}{\pi a^3}\right)^{\frac{1}{2}} e^{-Zr/a}$$

and calculate $\langle\psi|H|\psi\rangle$. Note: With this trial wave function we have

$$\langle\psi|\,\vec{p}^2\,|\psi\rangle = m^2c^2\alpha^2Z^2, \quad \langle\psi|\,\frac{1}{r}\,|\psi\rangle = \frac{mc\alpha Z}{\hbar}, \quad \langle\psi|\,r\,|\psi\rangle = \frac{3a}{2Z}\,.$$

(c) (2 pt) Derive the equation satisfied by Z which determines the minimum of $\langle\psi|H|\psi\rangle$. (You do not have to solve this equation.)

2. A hydrogen atom is placed in a static electric field $\vec{\mathcal{E}}$ of magnitude $2 \times 10^4\,\mathrm{V/m}$. The atom experiences a perturbation given by the Stark interaction $H' = e\vec{r}\cdot\vec{\mathcal{E}}$.

   (a) (1 pt) Is it possible for this perturbation to shift the energy of the ground ($n = 1$) state? If so, in what order of perturbation theory does this first occur?

   (b) (1 pt) Is it possible for this perturbation to shift the energy of the first excited ($n = 2$) state? If so, in what order of perturbation theory does this first occur?

   (c) (2 pt) List the $n = 2$ states (ignoring spin) and state which matrix elements of H' between these states can be nonzero.

   (d) (2 pt) Calculate the leading order energy shifts to the $n = 2$ levels. You needn't evaluate any non-vanishing matrix elements that occur, but clearly define your notation.

   (e) (1 pt) Given that the Bohr radius $a = 0.53 \times 10^{-10}\,\mathrm{m}$, estimate the size of a typical Stark energy shift in electron volts for the given electric field strength $\mathcal{E} = 2{\times}10^4\,\mathrm{V/m}$.

3. Give a brief but reasoned answer to each of the following:

   (a) (2 pt) When describing the weak field Zeeman effect in hydrogen, the expectation value of the magnetic interaction is expressed in terms of the Landé g-factor g_L as

   $$E_{\text{Zee}} = \frac{\mu_B B}{\hbar} \langle\, n\ell\tfrac{1}{2} j m_j |\, (J_z + S_z)\, |\, n\ell\tfrac{1}{2} j m_j \rangle = \mu_B B g_L m_j \,,$$

   where $\mu_B = e\hbar/2m_e$ is the Bohr magneton and the eigenvalue of J_z is $\hbar m_j$. What is the value of g_L for an electron in the state $\psi_{3\,2j}^{\,m_j}(\vec{r}) = R_{3\,2}(r) Y_2^2(\theta,\phi)\chi^{1/2}$?

   (b) (1 pt) Given that $v/c \approx \alpha = 1/137$ in the hydrogen atom and that the relativistic kinetic energy correction is $H'_{KE} = -(\vec{p}^2)^2/8m^3c^2 = -(\vec{p}^2/2m)(\frac{\vec{p}^2}{4m^2c^2})$, estimate the size of the fine-structure corrections in eV. (An order of magnitude estimate is all that is wanted here.)

   (c) (2 pt) The state vectors $|njm_j\ell\tfrac{1}{2}\rangle$ are eigenvectors of the operator $\vec{L}\cdot\vec{S}$. What is the eigenvalue for an electron in the $4D_{\frac{3}{2}}$ state of hydrogen? (Note: here we use the notation $n\ell_j$ for neutral hydrogen, since $s = \frac{1}{2}$ is always assumed.)