Physics 852 **Problem Set 4** **Spring 2023**

1. This problem studies the Zeeman effect in Hydrogen for $n = 2$ states in the intermediate regime (i.e., $\epsilon_B \equiv \mu_B B \sim \Delta$, where $\mu_B \equiv \frac{|e|\hbar}{2m_e c}$ is the Bohr magneton, and Δ is the size of the fine structure splitting of the P-wave states.). This was sketched out in lecture; here you will fill in the details.

   (a) Verify that
   $$\langle 2, 1, \frac{1}{2}, m|S_z|2, 1, \frac{3}{2}, m\rangle = -\frac{\hbar}{3}\sqrt{\frac{9}{4} - m^2}$$

   (b) Construct and diagonalize the two 2×2 matrices that mix the $m = \pm\frac{1}{2}$ states of the $2P_{1/2}$ and $2P_{3/2}$ configurations. Write them in terms of ϵ_B and Δ.

   (c) Verify that the eigenvalues $E_m^{\pm}(\Delta, \epsilon_B)$ give the correct weak and strong field limits by Taylor expanding in the appropriate small parameter.

   (d) Analyze the eigenstates $|E_m^{\pm}\rangle$ in the strong field limit. You should find they are linear combinations of the $|ls; jm\rangle$ states, where the coefficients are just the CG coefficients that take you to the uncoupled basis $|lm_l s m_s\rangle$.

2. A harmonic oscillator potential is modified by adding a quartic potential to obtain
   $$\mathcal{H} = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 + \frac{\lambda}{3!}\hbar\omega\left(\frac{m\omega}{\hbar}\right)^2 x^4 .$$

   (a) Estimate the ground state energy using first order perturbation theory. (For this problem, I suggest calculating the first order correction using the wave function formalism, rather than the operator formalism.)

   (b) Estimate the ground state energy using the variational method with the trial function
   $$\psi(x) = \left(\frac{m\omega a}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega}{2\hbar}ax^2} ,$$
   where $a > 0$, by showing that the minimum occurs when a satisfies $a^3 - a - \lambda = 0$.

   (c) Show that the variational bound is given in terms of a as
   $$E_0 \leq \hbar\omega\left(\frac{1}{2a} + \frac{3\lambda}{8a^2}\right) .$$

   (d) Compare the perturbative results from part (a) with the variational results from part (c) for $\lambda = 0.1$ and for $\lambda = 1.0$. You will need to do some numerical work to accomplish this.

3. Find the lowest bound on the ground state of hydrogen that you can get using a gaussian trial wave function
   $$\psi(\vec{r}) \;=\; Ne^{-br^2} ,$$
   where N is determined by normalization and b is an adjustable parameter. Compare with the exact ground state energy of hydrogen.