

1. (a) (3 pt) The first order correction to the ground state is

$$E^{(1)} = \frac{2}{3a} mc^2 \alpha^2 \lambda \langle \psi_0 | r | \psi_0 \rangle = mc^2 \alpha^2 \lambda,$$

so

$$\boxed{E^{(0)} + E^{(1)} = mc^2 \alpha^2 \left(-\frac{1}{2} + \lambda\right)}.$$

- (b) (3 pt) Using the trial wave function, we obtain

$$\begin{aligned} \langle \psi | H | \psi \rangle &= \frac{1}{2m} \langle \psi | \vec{p}^2 | \psi \rangle - \hbar c \alpha \langle \psi | \frac{1}{r} | \psi \rangle + \frac{2}{3a} mc^2 \alpha^2 \lambda \langle \psi | r | \psi \rangle \\ &= \boxed{mc^2 \alpha^2 \left( \frac{Z^2}{2} - Z + \frac{\lambda}{Z} \right)}. \end{aligned}$$

where the expectation value is taken using the trial wave function.

- (c) (2 pt) To minimize  $E(Z) = \langle \psi | H | \psi \rangle$ , we just set  $dE(Z)/dZ = 0$  to obtain

$$Z - 1 - \frac{\lambda}{Z^2} = 0,$$

or

$$\boxed{Z^3 - Z^2 - \lambda = 0}$$

(Comment: For  $\lambda = 4$ , first order perturbation theory gives a shift in the energy of  $\Delta E_{\text{pert}} = 4 mc^2 \alpha^2$ , while the variational principle gives  $Z = 2$  and an energy shift of  $\Delta E_{\text{var}} = 2.5 mc^2 \alpha^2$ . These differ by 60%, so perturbation theory does not work well for this large value of  $\lambda$ , as expected. For  $\lambda = 0.121$ , first order perturbation theory gives  $\Delta E_{\text{pert}} = 0.121 mc^2 \alpha^2$ , while the variational principle gives  $Z = 1.1$  and  $\Delta E_{\text{var}} = 0.115 mc^2 \alpha^2$ . These differ by about 5%, showing that perturbation theory works better for this smaller value of  $\lambda$ .)

2. Use the notation  $|n\ell m_\ell\rangle$  for the eigenstates of hydrogen. (We can ignore spin here.)

- (a) (1 pt) The ground state ( $n = 1$ ) energy does not shift at first order, because  $\langle 100 | z | 100 \rangle = 0$ . It does shift at second order, because  $\langle n10 | z | 100 \rangle \neq 0$ .
- (b) (1 pt) For the first excited state ( $n = 2$ ), there are four degenerate states (ignoring spin). Using degenerate perturbation theory, there is a shift in the  $n = 2$  energy at first order, as shown below.
- (c) (2 pt) The  $n = 2$  states are  $|200\rangle$ ,  $|210\rangle$ ,  $|211\rangle$ , and  $|21-1\rangle$ . The only non-zero matrix elements of  $\mathcal{H}'$  between these states are  $\langle 210 | \mathcal{H}' | 200 \rangle = \langle 200 | \mathcal{H}' | 210 \rangle$ .

- (d) (2 pt) Let  $\langle 210|\mathcal{H}'|200\rangle = \langle 200|\mathcal{H}'|210\rangle = A$ . Ordering the state vectors as  $|\psi_1\rangle = |200\rangle$ ,  $|\psi_2\rangle = |210\rangle$ ,  $|\psi_3\rangle = |211\rangle$ ,  $|\psi_4\rangle = |21-1\rangle$ , we obtain the  $4 \times 4$  perturbation matrix:

$$H' = \begin{pmatrix} 0 & A & 0 & 0 \\ A & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then  $\det(H' - E_2^{(1)}I) = 0$  gives  $(E_2^{(1)})^4 - A^2(E_2^{(1)})^2 = 0$ , leading to the four solutions:

$$E_2^{(1)} = 0, 0, A, -A.$$

- (e) (1 pt) Using dimensional analysis, we have  $\langle r \rangle \sim a$ , so

$$\langle \mathcal{H}' \rangle \sim ea\mathcal{E} \approx e(0.5 \times 10^{-10} \text{ m})(2 \times 10^4 \text{ V/m}) = \boxed{10^{-6} \text{ eV}}.$$

This is the typical size of the Stark energy shift for the given electric field.

3. (a) (2 pt) The state  $\psi_{32j}^{m_j}(\vec{r}) = R_{32}(r)Y_2^2(\theta, \phi)\chi^{1/2}$  is an eigenstate of  $J_z$  with eigenvalue  $m_j\hbar = (2 + \frac{1}{2})\hbar = \frac{5}{2}\hbar$ . It is also an eigenstate of  $S_z$  with eigenvalue  $\frac{1}{2}\hbar$ . Thus we get

$$g_L = 1 + \frac{\langle S_z \rangle}{m_j\hbar} = \frac{6}{5}.$$

- (b) (1 pt) Using  $p^2/m^2c^2 = (v/c)^2 \approx \alpha^2$ , we get  $E_{FS}^{(1)} \approx E^{(0)}\alpha^2 \approx 10 \times (10^{-2})^2 \text{ eV} \approx \boxed{10^{-3} \text{ eV}}$ . Actually, keeping factors of 2 and  $n$ , give  $10^{-4} \text{ eV}$  as a better estimate, but either is fine.

- (c) (2 pt) The  $4D_{\frac{3}{2}}$  state of hydrogen has  $j = \frac{3}{2}$  and  $\ell = 2$ . Using  $\vec{L} \cdot \vec{S} = (\vec{J}^2 - \vec{L}^2 - \vec{S}^2)/2$ , we find that the eigenvector of  $\vec{L} \cdot \vec{S}$  for this state is

$$\frac{\hbar^2}{2}(j(j+1) - \ell(\ell+1) - \frac{3}{4}) = \boxed{-\frac{3}{2}\hbar^2}.$$