Time-Independent Perturbation Theory

Unperturbed states satisfy $H_0|n\rangle = \mathcal{E}_n|n\rangle$. Perturbation is H'.

Perturbative solution: $E_n = E_n^{(0)} + E_n^{(1)} + E_n^{(2)} + \dots,\qquad |\psi_n\rangle = |\psi_n^{(0)}\rangle + |\psi_n^{(1)}\rangle + \dots$

Non-degenerate case:

$$E_n^{(0)} = \mathcal{E}_n \qquad E_n^{(1)} = \langle n|H'|n\rangle \qquad E_n^{(2)} = \sum_{k\neq n} \frac{|\langle k|H'|n\rangle|^2}{\mathcal{E}_n - \mathcal{E}_k}$$

$$|\psi_n^{(0)}\rangle = |n\rangle \qquad |\psi_n^{(1)}\rangle = \sum_{k\neq n} |k\rangle \frac{\langle k|H'|n\rangle}{\mathcal{E}_n - \mathcal{E}_k}$$

Degenerate case: If the level n is k-fold degenerate, then $|\psi_n^{(0)}\rangle$ and $E_n^{(1)}$ are obtained from the eigenvectors and eigenvalues of the matrix

$$\begin{pmatrix} H'_{11} & H'_{12} & \cdots & H'_{1k} \\ H'_{21} & H'_{22} & \cdots & H'_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ H'_{k1} & H'_{k2} & \cdots & H'_{kk} \end{pmatrix},$$

where the subscripts label the k degenerate states.

Time-Dependent Perturbation Theory

Perturbation is $H'(t)$. State in interaction picture is $|\psi(t)\rangle_I = \sum_n c_n(t)|n\rangle$, with $|\psi(t_0)\rangle_I = |i\rangle$.

Perturbative solution: $c_n(t) = c_n^{(0)}(t) + c_n^{(1)}(t) + \dots$, with

$$c_n^{(0)}(t) = \delta_{ni} \qquad c_n^{(1)}(t) = -\frac{i}{\hbar}\int_{t_0}^{t} dt' e^{i\omega_{ni}t'} \langle n|H'(t')|i\rangle\,,$$

where $\omega_{ni} = (\mathcal{E}_n - \mathcal{E}_i)/\hbar$.

In Schrödinger picture: $|\psi(t)\rangle_S = e^{-iH_0t/\hbar}|\psi(t)\rangle_I$.

Simple Harmonic Oscillator

$$x = \sqrt{\frac{\hbar}{2m\omega}}\left(a + a^\dagger\right) \qquad p = i\sqrt{\frac{m\hbar\omega}{2}}\left(a^\dagger - a\right)$$

$$[a, a^\dagger] = 1 \qquad H = \hbar\omega\left(a^\dagger a + \frac{1}{2}\right)$$

$$a|0\rangle = 0 \qquad a|n\rangle = \sqrt{n}|n-1\rangle \qquad a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$$

Spherical Harmonics

$$Y_0^0 = \sqrt{\tfrac{1}{4\pi}}, \qquad Y_1^0 = \sqrt{\tfrac{3}{4\pi}}\cos\theta, \qquad Y_1^{\pm 1} = \mp\sqrt{\tfrac{3}{8\pi}}\sin\theta e^{\pm i\phi}$$