Physics 852 **Problem Set 9** **Spring 2023**

NOTE: All problem numbers from Sakurai correspond to the 3rd Edition.

1. Sakurai 6.3

2. Consider a spherically-symmetric potential $V(r) = V_0 e^{-r^2/(2a^2)}$.

   (a) Obtain the scattering amplitude, $f(\mathbf{k}', \mathbf{k})$, and the differential cross section, $d\sigma/d\Omega$, for the scattering of a particle off this potential, in the first-order Born approximation.

   (b) Find the total cross section in this approximation.

3. This problem is a crude illustration of renormalization and scale-dependent ("running") couplings.

   Consider a so-called "separable potential" $V = |v\rangle\langle v| \;\Rightarrow\; \langle \mathbf{k}'|V|\mathbf{k}\rangle = v(\mathbf{k}')v^*(\mathbf{k})$.

   (a) For $E = \frac{\hbar^2 k^2}{2m}$ and elastic scattering ($|\mathbf{k}'| = |\mathbf{k}|$), solve for the T-matrix, $T(\mathbf{k}', \mathbf{k})$, exactly.

   [Hint: This can be solved to all orders by iterating the right hand side of the T-matrix equation and making use of the separable nature of the potential. Your final result should be written in terms of an unevaluated momentum integral over $d^3\mathbf{q}$.]

   (b) Consider the special case where $v(\mathbf{k}) = \sqrt{g}$, where g is a constant. What is the form of the potential in coordinate representation, $\langle \mathbf{r}'|V|\mathbf{r}\rangle$?

   (c) Due to the fact that $v(\mathbf{k})$ doesn't die off at high $\mathbf{k}$, the solution of the T-matrix equation is ill-defined, because of ultra-violet (UV) divergences that arise in the intermediate state momentum integrals (these are called loop integrals in quantum field theory). For instance, if we put a UV cutoff of $q < \Lambda$ on the momentum integrals, with Λ much larger than k, then the 2nd-order perturbative contribution to the T-matrix goes as

   $$T^{(2)}(\mathbf{k}', \mathbf{k}) = \frac{2m}{\hbar^2} g^2 \int^{\Lambda} d^3\mathbf{q} \frac{1}{k^2 - q^2 + i\epsilon} \sim \Lambda$$

   which diverges when we take $\Lambda \to \infty$. One way to deal with this problem would be to leave a finite UV cutoff in all momentum integrals, but then the calculated T-matrix (which is related to observables like scattering amplitudes) would depend on this arbitrary value of Λ. The solution is to let the coupling constant g depend on Λ such that observables like scattering amplitudes come out Λ-independent for momentum scales $\ll \Lambda$. This is the (dramatically oversimplified) essence of how similar UV divergences are eliminated in quantum field theory, and how it leads one to the inescapable consequence that coupling "constants" such as the fine structure $\alpha = e/\hbar c$ in electromagnetism and the analogous quantities in quantum chromodynamics (the strong force) are not actually constant, but depend on the momentum/energy scale of the particular process under consideration. For instance, $\alpha \sim 1/137$ at low energies, but for processes involving momentum/energy transfers of the order of the Z-boson mass $M_Z = 90$ GeV, the relevant coupling is $\alpha \sim 1/127$.

Suppose the T-matrix is measured at zero energy to be $T(0,0) = a$. By leaving the UV cutoff Λ in all momentum integrals, solve for g as an explicit function of a and Λ.

(d) Using your expression for the "running coupling" $g(\Lambda)$, show that $\frac{d}{d\Lambda}T(\mathbf{k}', \mathbf{k}) \approx 0$ provided that $k \ll \Lambda$.
[Hint: The algebra is easier if you show $\frac{d}{d\Lambda}\frac{1}{T(\mathbf{k}',\mathbf{k})} = 0$.]