Problem Set 3

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I PERTURBATIVE HYDROGEN FINE STRUCTURE

A

$$\begin{aligned}
\Delta_l^{(1)} &= \langle nlm|H'_D|nlm\rangle\,, && \text{(I.1)}\\
&= \int \mathrm{d}^3r\,\frac{\pi\hbar^3\alpha}{2m^2c}\delta^{(3)}(\mathbf{r})\big|\Psi_{nl}^m(\mathbf{r})\big|^2, && \text{(I.2)}\\
&= \frac{\pi\hbar^3\alpha}{2m^2c}\big|\Psi_{nl}^m(\mathbf{0})\big|^2, && \text{(I.3)}\\
&= 0, \qquad \text{when } l\neq 0. && \text{(I.4)}
\end{aligned}$$

B

$$\begin{aligned}
\Delta_0^{(1)} &= \langle n00|H'_D|n00\rangle\,, && \text{(I.5)}\\
&= \frac{\pi\hbar^3\alpha}{2m^2c}\Big|\Psi_{n0}^0(\mathbf{0})\Big|^2, && \text{(I.6)}\\
&= \frac{\pi\hbar^3\alpha}{2m^2c}\Big|R_{n0}(0)\mathcal{Y}_0^0(\phi,\theta)\Big|^2, && \text{(I.7)}\\
&= \frac{\pi\hbar^3\alpha}{2m^2c}\frac{1}{4\pi}\left|\frac{2}{na_0^{3/2}}\right|^2, && \text{(I.8)}\\
&= \frac{\pi\hbar^3\alpha}{2m^2c}\frac{1}{4\pi}\frac{4}{n^2a_0^3}, && \text{(I.9)}\\
&= \frac{\alpha^4mc^2}{2n^3}. \qquad (a_0=\tfrac{\hbar}{mc\alpha}). && \text{(I.10)}
\end{aligned}$$

C

$$\begin{aligned}
E_{FS}^{(1)} &= E_K^{(1)} + \Delta_0^{(1)}, && \text{(I.11)}\\
&= E_n^{(0)}\alpha^2\left(\frac{1}{n(l+1/2)} - \frac{3}{4n^2}\right) + \frac{\alpha^4mc^2}{2n^3}, && \text{(I.12)}\\
E_n^{(0)} &= -\frac{1}{2}mc^2\frac{\alpha^2}{n^2}. && \text{(I.13)}\\
&= E_n^{(0)}\alpha^2\left(\frac{1}{n} - \frac{3}{4n^2}\right). && \text{(I.14)}\\
E_{n0,j=1/2}^{(1)} &= E_n^{(0)}\alpha^2\left(\frac{1}{n} - \frac{3}{4n^2}\right). \qquad \checkmark && \text{(I.15)}
\end{aligned}$$

II SAKURAI 5.6

The energy eigenstates of the infinite square well in two dimensions are:

$$\psi(\mathbf{r}) = \frac{2}{L}\sin\!\left(x\frac{n_x\pi}{L}\right)\sin\!\left(y\frac{n_y\pi}{L}\right). \tag{II.1}$$

The energies of these states are

$$E_{n_x,n_y} = \frac{\hbar^2}{2m}\left(\frac{(n_x+n_y)\pi}{L}\right)^2 \tag{II.2}$$

For a total $N=n_x+n_y$, there are $N+1$ states with the same energy. The ground state is $n_x=n_y=1$, and the first excited state is $n_x=2$ and $n_y=1$ or vice versa. Denote the energy states as $|n_x,n_y\rangle$.

$$\begin{aligned}
\langle 1,1|\hat V_1|1,1\rangle &= \lambda\,\langle 1,1|\hat x\hat y|1,1\rangle\,, && \text{(II.3)}\\
&= \lambda\frac{L^2}{4}. && \text{(II.4)}
\end{aligned}$$

$$\begin{aligned}
V &= \lambda\begin{pmatrix}\langle 1,2|xy|1,2\rangle & \langle 1,2|xy|2,1\rangle\\ \langle 2,1|xy|1,2\rangle & \langle 2,1|xy|2,1\rangle\end{pmatrix}, && \text{(II.5)}\\
&= \frac{\lambda L^2}{4\pi^4}\begin{pmatrix}\pi^4 & 1024/81\\ 1024/81 & \pi^4\end{pmatrix}, && \text{(II.6)}\\
\therefore\ \Delta_{1\pm}^{(1)} &= \frac{1}{81}(81\pi^4 \pm 1024). && \text{(II.7)}
\end{aligned}$$

The eigenvectors (zeroth-order energy eigenkets) are

$$|\Delta_{1\pm}^{(1)}\rangle = \frac{1}{\sqrt{2}}\big(|1,2\rangle \pm |2,1\rangle\big)\,. \tag{II.8}$$

III SAKURAI 5.19

$$H = \begin{pmatrix}E_2 + h\delta & 3ea_0\mathcal{E}\\ 3ea_0\mathcal{E} & E_2\end{pmatrix}. \tag{III.1}$$

The eigenvalues, E, are

$$E_\pm = -\frac{h\delta}{2} \pm \frac{1}{2}\sqrt{h^2\delta^2 + 36e^2\mathcal{E}^2a_0^2} + E_2. \tag{III.2}$$

Taking the limit where $\delta \ll 3ea_0\mathcal{E}$,

$$E_\pm \approx -\frac{h\delta}{2} \pm 3ea_0\mathcal{E} + E_2, \tag{III.3}$$

which is indeed linear in $\mathcal{E}$.

In the limit where $\delta \gg 3ea_0\mathcal{E}$,

$$E_\pm = -\frac{h\delta}{2} \pm \frac{1}{2}\left(h\delta + \frac{9e^2a_0^2\mathcal{E}^2}{2h\delta}\right) + E_2, \tag{III.4}$$

which is quadratic in $\mathcal{E}$.