Intermediate Fluid Mechanics

Lecture 19: Dimensional Analysis III

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Chapter Overview

Lecture Objectives

- In this lecture we will continue to expand in the realm of dimensional analysis.

- For this, we will consider once again the NS equation in non-dimensional form considering additional terms.

Chapter Overview

The Non-dimensional form of the NS equations

Let's consider the non-dimensional Navier-Stokes equation with the **body force**.

- If we also choose the characteristic pressure to be the dynamic pressure, *i.e.* $\tilde{p} = (p - p_\infty)/\rho U^2$,
- The characteristic time to be the ratio of the length and velocity scales, *i.e.* $T = L/U$,

$\implies$ the non-dimensional NS equationis given by:

$$\frac{\partial \tilde{u}_i}{\partial \tilde{t}} + \tilde{u}_j \frac{\partial \tilde{u}_i}{\partial \tilde{x}_j} = -\frac{\partial \tilde{p}}{\partial \tilde{x}_i} - \frac{1}{F_r}\delta_{i3} + \frac{1}{R_e}\frac{\partial^2 \tilde{u}_i}{\partial \tilde{x}_j^2}. \tag{1}$$

Note:

When gravitational effects are dynamically important, one must consider the Froude number. $\implies$ Applications involving surface waves, for example, must consider the Froude number.

Chapter Overview

Example of Tidal Estuary: Matching the Froude number

Let's imagine that one is interested in developing a physical model of a tidal estuary, with tidal period of about 12h.

(*A tidal estuary is the confluence of seawater and freshwater as can occur in river outlets to the ocean*)

Let's say we are interested in studying some of the details of the fluid flow in a tidal estuary and would like to build a scale model to facilitate these studies.

The question is:

$\longrightarrow$ What should the tidal period be in the scaled model? – Assume that the geometry in the physical model is scaled by a factor 1:500.

Example of Tidal Estuary (continued ....)

If we want the physical model to behave in a manner dynamically similar to the actual flow, then one has to **match the relevant non-dimensional parameters.**

$\implies$ In this flow, the Froude number is more important than the Reynolds number. Hence,

$$Fr_m = Fr_a$$
$$\left(\frac{U^2}{gL}\right)_m = \left(\frac{U^2}{gL}\right)_a \tag{2}$$

where $\square_a$ stands for *actual*, and $\square_m$ for *model*.

- Since we can not change the gravitational constant, it implies that $g_m = g_a$.
- Additionally, we also know that $L_a = 500\, L_m$,

Example of Tidal Estuary (continued ....)

This means that,

$$\frac{U_m}{U_a} = \left(\frac{L_m}{L_a}\right)^{1/2} = \left(\frac{L_m}{500\,L_m}\right)^{1/2} = 0.045. \tag{3}$$

$\implies$ So the flow in the model must be slower than the actual flow by a factor of 0.045.

Now, to determine the appropriate tidal period in the model, the non-dimensional time $\tilde{t}$ in the model must behave similarly to the real (actual) non-dimensional time,

$$\tilde{t}_m = \tilde{t}_a \tag{4}$$

$$\frac{t_m}{T_m} = \frac{t_a}{T_a} \tag{5}$$

where T is a characteristic time scale.

Example of Tidal Estuary (continued ....)

- In this case the characteristic time should be

$$T = \frac{L}{U}, \tag{6}$$

(*The tidal period is based on the time it takes a fluid particle with a velocity U to move a distance L.*)

- Here, U represents the velocity of the tidal current

- L represents the distance traveled by a fluid particle as it is swept toward the shore and back out again to the ocean over the course of the tidal period.

Example of Tidal Estuary (continued ....)

As a result, the dimensional time associated with the tidal period in the model system is

$$t_m = \left(\frac{T_m}{T_a}\right)t_a = \left(\frac{L_m/U_m}{L_a/U_a}\right)t_a \tag{7}$$

$$t_m = \left(\frac{L_m}{L_a}\right) \underbrace{\left(\frac{U_a}{U_m}\right)}_{\substack{\text{From similarity} \\ \text{on } Fr \text{ numbers}}} t_a \tag{8}$$

$$t_m = \left(\frac{L_m}{L_a}\right)\left(\frac{L_a}{L_m}\right)^{1/2} t_a \tag{9}$$

$$t_m = \left(\frac{L_m}{L_a}\right)^{1/2} t_a = \left(\frac{1}{500}\right)^{1/2}(12hr) = 0.54\, hr. \tag{10}$$

$\implies$ The scaled-down model system will have a tidal period of half hour, compared to the 12 hours in the actual real system.

Chapter Overview

Non-dimensional Navier-Stokes equations: Compressible flows

Let's now consider the compressible form of the Navier-Stokes equations, allowing for non-constant viscosity, as would be the case in a compressible flow.

In that case, the NS equations are rewritten in dimensional form as,

$$\rho\frac{\partial u_i}{\partial t} + \rho u_j\frac{\partial u_i}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_j}\Big[\mu\Big(\frac{\partial u_i}{\partial x_j}\Big) + \frac{1}{3}\mu\frac{\partial u_m}{\partial x_m}\Big]. \tag{11}$$

To non-dimensionalize this equation, we choose the following characteristic scales,

$$\tilde{u}_i = \frac{u_i}{U}, \quad \tilde{x}_i = \frac{x_i}{L}, \quad \tilde{p} = \frac{p}{p_0}, \quad \tilde{\rho} = \frac{\rho}{\rho_0}, \quad \tilde{\mu} = \frac{\mu}{\mu_0}, \quad \tilde{t} = \frac{t\,U}{L} \tag{12}$$

Compressible flows (continued ...)

Performing the non-dimensionalization in the same manner as before we find,

$$\tilde{\rho}\frac{\partial \tilde{u}_i}{\partial \tilde{t}} + \tilde{\rho}\tilde{u}_j\frac{\partial \tilde{u}_i}{\partial \tilde{x}_j} = -\underbrace{\frac{p_0}{\rho_0\, U^2}}_{*}\frac{\partial \tilde{p}}{\partial \tilde{x}_i} + \frac{1}{Re}\frac{\partial}{\partial \tilde{x}_j}\Big[\tilde{\mu}\Big(\frac{\partial \tilde{u}_i}{\partial \tilde{x}_j}\Big) + \frac{1}{3}\tilde{\mu}\frac{\partial \tilde{u}_m}{\partial \tilde{x}_m}\Big]. \qquad (13)$$

Next, we will rewrite term (*) in terms of a Mach number, which is defined as

$$M = \frac{U}{C} \qquad \text{where } C \text{ is the speed of sound.} \qquad (14)$$

Compressible flows (continued ...)

Taking into consideration that in an ideal gas,

$$C = \sqrt{\frac{\gamma\, p}{\rho}} \qquad \text{where } \gamma = \frac{C_p}{C_v} \text{ is the ratio of specific heats,} \qquad (15)$$

one can then represent the non-dimensional parameter associated with the pressure gradient as

$$\frac{p_0}{\rho_0 U^2} = \frac{C^2}{\gamma U^2} = \frac{1}{\gamma\, M^2}. \qquad (16)$$

Therefore, upon substitution,

$$\tilde{\rho}\frac{\partial \tilde{u}_i}{\partial \tilde{t}} + \tilde{\rho}\tilde{u}_j\frac{\partial \tilde{u}_i}{\partial \tilde{x}_j} = -\frac{1}{\gamma\, M^2}\frac{\partial \tilde{p}}{\partial \tilde{x}_i} + \frac{1}{Re}\frac{\partial}{\partial \tilde{x}_j}\Big[\tilde{\mu}\Big(\frac{\partial \tilde{u}_i}{\partial \tilde{x}_j}\Big) + \frac{1}{3}\tilde{\mu}\frac{\partial \tilde{u}_m}{\partial \tilde{x}_m}\Big]. \qquad (17)$$

Model Testing: Determining the Drag of an Object

Let's consider the situation where one has just designed the body of a new automobile.

$\longrightarrow$ One would like to know how much drag this vehicle will experience because more drag translates into higher operational expenses.

Ideally, one would like to estimate the drag prior to actually building a prototype, so that one can make design modifications accordingly.

$\longrightarrow$ Therefore, one could conduct experiments in a wind tunnel using a $1/5$ scale model of the vehicle.

(*For this study one ensures geometric similarity by scaling all lengths proportionally*).

Drag of an Object (continued ....)

To achieve dynamic similarity, one needs to consider several elements:

1)- If we want to be able to relate the measured drag on the scale model in the wind tunnel to the actual drag expected on the prototype, then one must match the drag coefficients,

$$C_{D\,\text{model}} = C_{D\,\text{prototype}}, \tag{18}$$

where $C_D = \frac{F_D}{1/2\rho U^2 A}$.

2)- From the non-dimensional form of the momentum equations, we know that the relevant non-dimensional parameter is the **Reynolds number.**

Therefore, to achieve dynamic similarity we must also match the Reynolds number,

$$Re_m = Re_p \tag{19}$$

$$\frac{U_m \, L_m}{\nu_m} = \frac{U_p \, L_p}{\nu_p} \tag{20}$$

- Since the prototype is designed to operate in air, under atmospheric conditions and assuming that the air in the wind tunnel is neither heated nor compressed, then $\nu_p = \nu_m$.

- We are also given that $L_p/L_m = 5$.

Drag of an Object (continued ....)

Let's say that we want to estimate the drag on the prototype when it is traveling at $30 m/s$ (67 mph).

$\implies$ Then $U_p = 30 m/s$ and based on the above relation, we would need to operate our wind tunnel at,

$$U_m = \left(\frac{L_p}{L_m}\right) U_p = 150 \, m/s \, (335 \, mph) \tag{21}$$

3)- This U_m is quite fast. Furthermore, the speed of sound of dry air is $\sim 330 m/s$, so the Mach number of the model is

$$M_m = \frac{150 \, m/s}{330 \, m/s} = 0.45, \tag{22}$$

whereas the Mach number of the actual flow is

$$M_m = \frac{30 \, m/s}{330 \, m/s} = 0.09 \tag{23}$$

Drag of an Object (continued ....)

$\implies$ Clearly, we will have a problem attaining dynamic similarity under this scenario because the dynamics of the flow in the model case are different than that in the actual value due to the mismatch in M.

Specifically, the effect of the term $\partial \tilde{p}/\partial \tilde{x}_i$ will be different in both cases; therefore, the resultant solution (*i.e.* velocity field) cannot be expected to be the same.

Question: How can we overcome this troublesome situation?