Homework #6
ME EN 5210/6210 & CH EN 5203/6203 & ECE 5652/6652
Linear Systems & State-Space Control

Use this page as the cover page on your assignment, submitted as a single pdf.

Problem 1

Consider a system with state-space equations

$$\dot{\boldsymbol{x}} = A\boldsymbol{x} + B\boldsymbol{u}$$
$$\boldsymbol{y} = C\boldsymbol{x} + D\boldsymbol{u}$$

for state vector $\boldsymbol{x}$, and a change of coordinates defined by

$$\boldsymbol{z} = M\boldsymbol{x}$$

Write the state-space equations for the state vector $\boldsymbol{z}$, with the same inputs and outputs as the original system.

Problem 2

Find the companion-form equivalent equations of

$$\dot{\boldsymbol{x}} = \begin{bmatrix} -2 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & -2 & -2 \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & -1 & 0 \end{bmatrix} \boldsymbol{x}$$

Problem 3

For the same system from Problem 2, perform an equivalence transformation such that the new A matrix is in Jordan form. Provide the equivalent equations.

Problem 4

Discretize the following state-space equations for $T = 1$ and $T = \pi$.

$$\dot{\boldsymbol{x}} = \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 2 & 3 \end{bmatrix} \boldsymbol{x}$$

<u>Problem 5</u>

Solve for the analytic solution of $\boldsymbol{x}(t)$ for the unforced system

$$\dot{\boldsymbol{x}} = \begin{bmatrix} -3 & 1 & 0 & 0 \\ 0 & -3 & 1 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & -6 \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} 0 \\ 0 \\ 2 \\ -2 \end{bmatrix} u \qquad \text{with} \quad \boldsymbol{x}(0) = \begin{bmatrix} 10 \\ -8 \\ -4 \\ 5 \end{bmatrix}$$

(a) Solve for the analytic solution of $\boldsymbol{x}(t)$ for the unforced system (i.e., when $u(t) = 0$). Fully simplify your answer.

(b) What is the approximate amount of time it will take for this system to reach a steady-state value for any constant input?

(c) Use the lsim function in MATLAB to plot the zero-input response and the unit-step response. Choose a time duration that lets you see the states reach their steady-state values, but is not so long that the transients are hard to see. Include all of the states for a given input type on a single plot. Make sure your plots are clearly labeled, and include a legend. In addition to turning in your plots, turn in a printout of the .m file that you used to make them.

Problem 1

$$\dot{x} = Ax + Bu \qquad z = Mx$$
$$y = Cx + Du$$

$$z = Mx \Rightarrow x = M^{-1}z$$
$$\Downarrow \qquad\qquad \Downarrow$$
$$\dot{z} = M\dot{x} \qquad \dot{x} = M^{-1}\dot{z}$$

$$\dot{x} = Ax + Bu \Rightarrow M^{-1}\dot{z} = AM^{-1}\dot{z} + Bu$$
$$\dot{z} = MAM^{-1}\dot{z} + MBu$$

$$y = Cx + Du \Rightarrow y = CM^{-1}\dot{z} + Du$$

$$\left.\begin{aligned} \dot{z} &= MAM^{-1}z + MBu \\ y &= CM^{-1}z + Du \end{aligned}\right\}$$

4.4

Find the companion-form and modal-form equivalent equations of

$$\dot{\vec{x}}(t) = \begin{bmatrix} -2 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & -2 & -2 \end{bmatrix} \vec{x}(t) + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 1 & -1 & 0 \end{bmatrix} \vec{x}(t)$$

For the companion form, use the method of Section 3.4.

$$Q = [b \;\; Ab \;\; A^2b]$$

$$Ab = \begin{bmatrix} -2 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & -2 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \\ -2 \end{bmatrix}$$

$$A^2b = A(Ab) = \begin{bmatrix} -2 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & -2 & -2 \end{bmatrix} \begin{bmatrix} -2 \\ 2 \\ -2 \end{bmatrix} = \begin{bmatrix} 4 \\ -4 \\ 0 \end{bmatrix}$$

$$Q = \begin{bmatrix} 1 & -2 & 4 \\ 0 & 2 & -4 \\ 1 & -2 & 0 \end{bmatrix} \Rightarrow Q^{-1} = \begin{bmatrix} 1 & 1 & 0 \\ 0.5 & 0.5 & -0.5 \\ 0.25 & 0 & -0.25 \end{bmatrix}$$

$\tilde{A} = Q^{-1}AQ$ will be companion form

$$\tilde{A} = \begin{bmatrix} 0 & 0 & -4 \\ 1 & 0 & -6 \\ 0 & 1 & -4 \end{bmatrix}$$

The characteristic equation of both $\tilde{A}$ and A is

$$\lambda^3 + 4\lambda^2 + 6\lambda + 4 = 0$$

In the notation of Definition 4.1, $P = Q^{-1}$.

$$\tilde{B} = PB = Q^{-1}B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\tilde{C} = CP^{-1} = CQ = \begin{bmatrix} 1 & -4 & 8 \end{bmatrix}$$

4.4 cont.

$$\dot{\tilde{x}} = \tilde{A}\tilde{x} + \tilde{B}u$$
$$y = \tilde{C}\tilde{x}$$

$$\left.\begin{aligned}\dot{\tilde{x}} &= \begin{bmatrix} 0 & 0 & -4 \\ 1 & 0 & -6 \\ 0 & 1 & -4 \end{bmatrix}\tilde{x} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}u \\ y &= \begin{bmatrix} 1 & -4 & 8 \end{bmatrix}\tilde{x}\end{aligned}\right\}$$ companion-form equivalent

For the modal form, use the method of Example 4.4.3.

The eigenvalues are the roots of the characteristic equation:

$$\lambda_1 = -1+i, \quad \lambda_2 = -1-i, \quad \lambda_3 = -2$$

The Jordan form is

$$J = \begin{bmatrix} -1+i & 0 & 0 \\ 0 & -1-i & 0 \\ 0 & 0 & -2 \end{bmatrix} = Q^{-1}AQ$$

Q is formed from the associated eigenvectors:

$$Q = \begin{bmatrix} 0 & 0 & 0.7071 \\ -0.4082-0.4082i & -0.4082+0.4082i & 0 \\ 0.8165 & 0.8165 & -0.7071 \end{bmatrix}$$

$$Q^{-1} = \begin{bmatrix} 0.6124+0.6124i & 1.2247i & 0.6124+0.6124i \\ 0.6124-0.6124i & -1.2247i & 0.6124-0.6124i \\ 1.4142 & 0 & 0 \end{bmatrix}$$

Let $\bar{Q} = \begin{bmatrix} 0.5 & -0.5i & 0 \\ 0.5 & 0.5i & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $\bar{Q}^{-1} = \begin{bmatrix} 1 & 1 & 0 \\ i & -i & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$\tilde{A} = \bar{Q}^{-1} J \bar{Q} = \begin{bmatrix} -1 & 1 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\tilde{A} = \bar{Q}^{-1}(Q^{-1}AQ)\bar{Q} = (\bar{Q}^{-1}Q^{-1})A(Q\bar{Q}) = (Q\bar{Q})^{-1}A(Q\bar{Q})$$

In the notation of Definition 4.1, $P^{-1} = Q\bar{Q}$

$$P = (Q\bar{Q})^{-1}$$

4.4 cont.

$$P = \begin{bmatrix} 0 & 0 & 0.7071 \\ -0.4082 & -0.4082 & 0 \\ 0.8165 & 0 & -0.7071 \end{bmatrix}$$

$$P^{-1} = \begin{bmatrix} 1.2247 & 0 & 1.2247 \\ -1.2247 & -2.4495 & -1.2247 \\ 1.4142 & 0 & 0 \end{bmatrix}$$

$$\tilde{B} = PB = \begin{bmatrix} 0.7071 \\ -0.4082 \\ 0.1094 \end{bmatrix}$$

$$\tilde{C} = CP^{-1} = \begin{bmatrix} 2.4495 & 2.4495 & 2.4495 \end{bmatrix}$$

$$\left.\begin{aligned} \dot{\tilde{x}} &= \begin{bmatrix} -1 & 1 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \tilde{x} + \begin{bmatrix} 0.7071 \\ -0.4082 \\ 0.1094 \end{bmatrix} u \\ y &= \begin{bmatrix} 2.4495 & 2.4495 & 2.4495 \end{bmatrix} \tilde{x} \end{aligned}\right\} \text{modal-form equivalent} \leftarrow$$

Problem 3

Using the "eig" function in MATLAB, the eigenvalues of A are -2 and $-1 \pm i$.

We know $AV = VJ$, with $J = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -1+i & 0 \\ 0 & 0 & -1-i \end{bmatrix}$

where the columns of V are the sorted eigenvectors.

$\lambda_1 = -2$: $\begin{bmatrix} -2 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & -2 & -2 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = -2 \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

$-2a = -2a$

$a + c = -2b$

$-2b - 2c = -2c \Rightarrow -2b = 0 \Rightarrow b = 0$

$a + c = 0 \Rightarrow a = -c$

$\vec{V}_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

$\lambda_2 = -1 + i$: $\begin{bmatrix} -2 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & -2 & -2 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = (-1+i) \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

$-2a = (-1+i)a \Rightarrow a = 0$

$a + c = (-1+i)b \Rightarrow c = (-1+i)b$

$-2b - 2c = (-1+i)c = (-1+i)(-1+i)b = -2ib$

$c = -b + ib = (-1+i)b$ (same info as above)

$\vec{V}_2 = \begin{bmatrix} 0 \\ 1 \\ -1+i \end{bmatrix}$

$\lambda_3 = -1 + i$: $\begin{bmatrix} -2 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & -2 & -2 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = (-1-i) \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

Problem 3 cont.

$$-2a = (-1-i)\, a \Rightarrow a = 0$$

$$a + c = (-1-i)b \Rightarrow c = (-1-i)b$$

$$\vec{v}_3 = \begin{bmatrix} 0 \\ 1 \\ -1-i \end{bmatrix}$$

$$V = [\vec{v}_1 \ \vec{v}_2 \ \vec{v}_3]$$

$$\dot{\vec{x}} = A\vec{x} + Bu \quad , \quad A = VJV^{-1}$$

$$\dot{\vec{x}} = VJV^{-1}\vec{x} + Bu \Rightarrow V^{-1}\dot{\vec{x}} = JV^{-1}\vec{x} + V^{-1}B$$

$$\text{let } \vec{z} = V^{-1}\vec{x} \Rightarrow \dot{\vec{z}} = V^{-1}\dot{\vec{x}} \Rightarrow \dot{\vec{x}} = V\dot{\vec{z}}$$

$$\dot{\vec{z}} = J\vec{z} + V^{-1}Bu$$

$$y = C\vec{x} + Du = CV\vec{z} + Du$$

Using MATLAB : $V^{-1} = \begin{bmatrix} -1 & 0 & 0 \\ -0.5i & 0.5-0.5i & -0.5i \\ 0.5i & 0.5+0.5i & 0.5i \end{bmatrix}$

$$V^{-1}B = \begin{bmatrix} -1 \\ -i \\ i \end{bmatrix}$$

$$CV = [-1 \ -1 \ -1]$$

Problem 4

$$\dot{\vec{x}}(t) = \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix} \vec{x}(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t), \quad y = \begin{bmatrix} 2 & 3 \end{bmatrix} \vec{x}$$

$$\vec{x}[k+1] = A_d \vec{x}[k] + B_d u[k], \quad y[k] = C_d \vec{x}[k] + D_d u[k]$$

where

$$A_d = e^{AT}, \quad B_d = A^{-1}(A_d - I)B, \quad C_d = C, \quad D_d = D$$

We know $A = VJV^{-1}$ and $e^{AT} = e^{VJV^{-1}T} = Ve^{JT}V^{-1}$

Using MATLAB's "jordan" function:

$$J = \begin{bmatrix} -1-i & 0 \\ 0 & -1+i \end{bmatrix}, \quad V = \begin{bmatrix} -0.5+0.5i & -0.5-0.5i \\ 1 & 1 \end{bmatrix}$$

$$V^{-1} = \begin{bmatrix} -i & 0.5-0.5i \\ i & 0.5+0.5i \end{bmatrix}$$

$$A_d = Ve^{JT}V^{-1} = V \begin{bmatrix} e^{(-1-i)T} & 0 \\ 0 & e^{(-1+i)T} \end{bmatrix} V^{-1}$$

The remaining computations were done using MATLAB.

$$T = 1: \quad A_d = \begin{bmatrix} 0.508 & 0.310 \\ -0.619 & -0.111 \end{bmatrix}, \quad B_d = \begin{bmatrix} 1.05 \\ -0.182 \end{bmatrix}$$

$$T = \pi: \quad A_d = \begin{bmatrix} -0.0432 & 0 \\ 0 & -0.0432 \end{bmatrix}, \quad B_d = \begin{bmatrix} 1.56 \\ -1.04 \end{bmatrix}$$

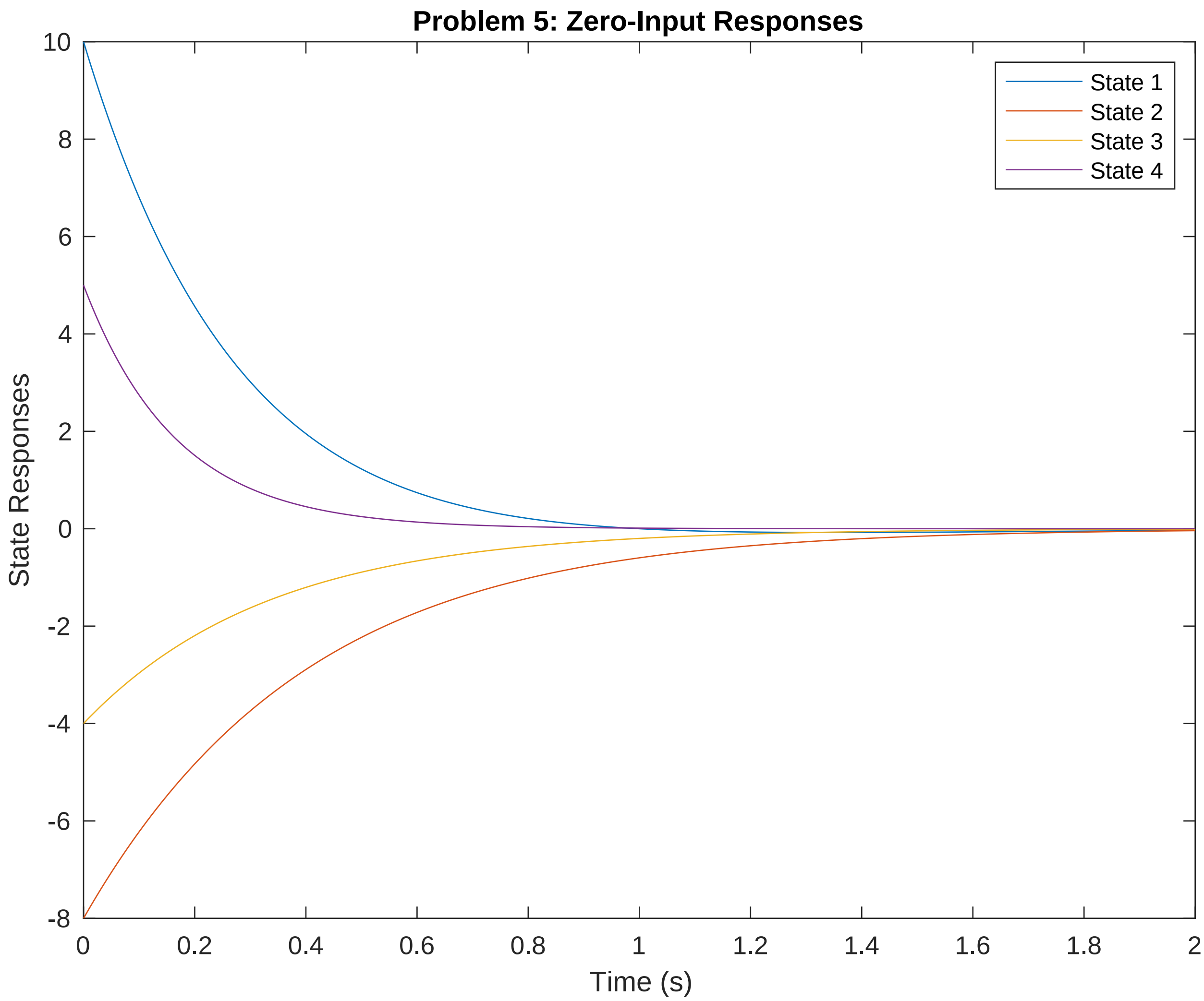
Problem 5

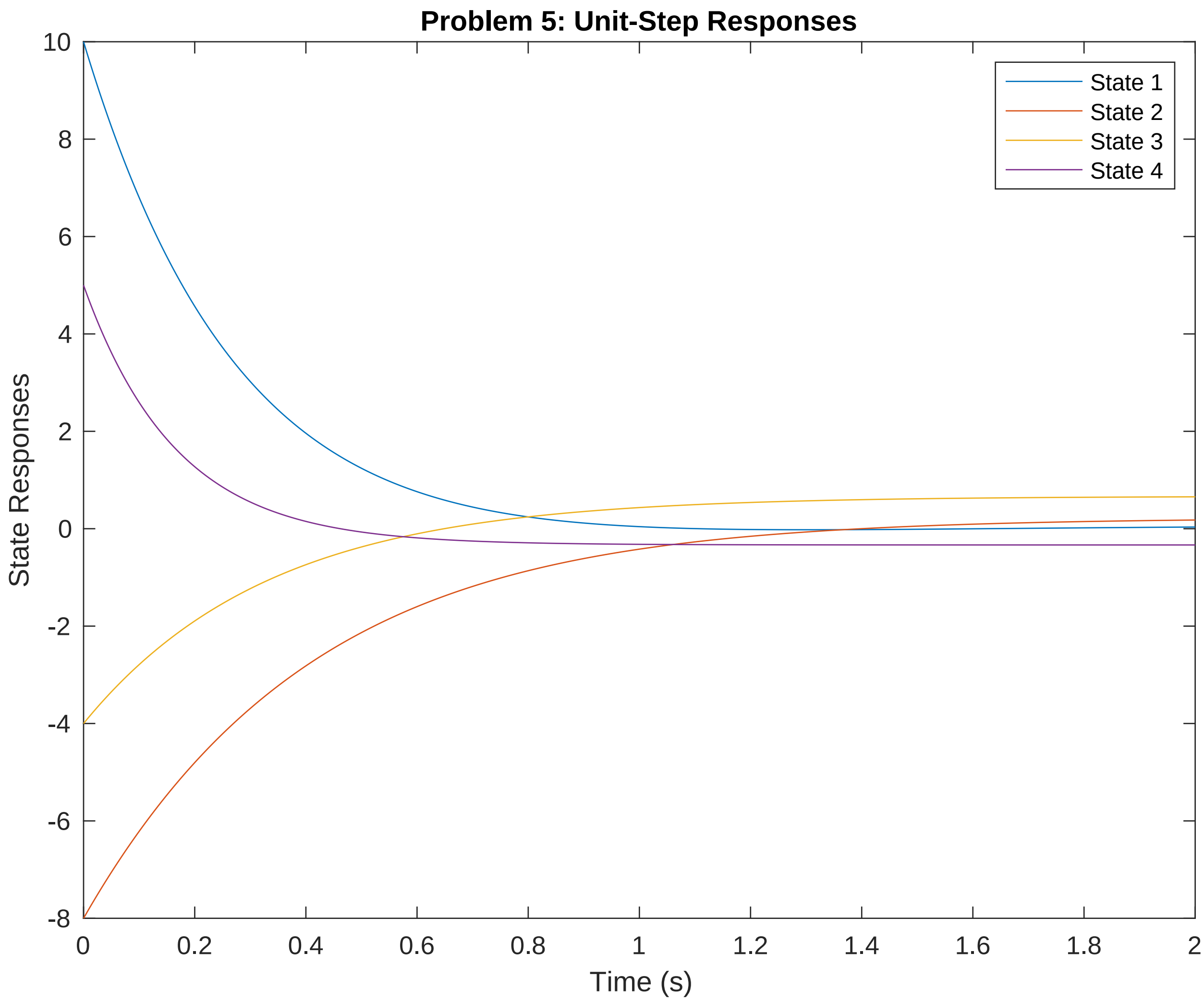
a) $\vec{x}(t) = e^{At}\vec{x}(0) = \begin{bmatrix} e^{-3t} & te^{-3t} & t^2e^{-3t}/2 & 0 \\ 0 & e^{-3t} & te^{-3t} & 0 \\ 0 & 0 & e^{-3t} & 0 \\ 0 & 0 & 0 & e^{-6t} \end{bmatrix} \begin{bmatrix} 10 \\ -8 \\ -4 \\ 5 \end{bmatrix}$

$$\vec{x}(t) = \begin{bmatrix} 10e^{-3t} - 8te^{-3t} - 2t^2e^{-3t} \\ -8e^{-3t} - 4te^{-3t} \\ -4e^{-3t} \\ 5e^{-6t} \end{bmatrix} = \begin{bmatrix} (10-8t-2t^2)e^{-3t} \\ (-8-4t)e^{-3t} \\ -4e^{-3t} \\ 5e^{-6t} \end{bmatrix}$$

b) There is a fast eigenvalue at $\lambda = -6$, and a slow (i.e., dominant) eigenvalue at $\lambda = -3$, so the time constant is $\tau \approx \frac{1}{3}$ second.

It with take approximately 5τ to reach steady-state (i.e., $5/3 = 1.67$ seconds).


Problem 5: Zero-Input Responses
State 1
State 2
State 3
State 4
State Responses
Time (s)
10
8
6
4
2
0
-2
-4
-6
-8
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2



Problem 5: Unit-Step Responses
State Responses
Time (s)
State 1
State 2
State 3
State 4
10
8
6
4
2
0
-2
-4
-6
-8
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2

```
% Homework 6, Problem 5
% Jake Abbott

A = [-3 1 0 0; 0 -3 1 0; 0 0 -3 0; 0 0 0 -6];
B = [0; 0; 2; -2];
C = eye(4); D = zeros(4,1); %Outputs are just the states
sys = ss(A,B,C,D);

X0 = [10; -8; -4; 5];

t = 0:0.001:2;
u_zi = zeros(size(t));
u_us = ones(size(t));

[Yzi,Tzi,Xzi] = lsim(sys,u_zi,t,X0); %Simulate zero-input response
[Yus,Tus,Xus] = lsim(sys,u_us,t,X0); %Simulate unit-step response

figure(1); clf; plot(Tzi,Yzi);
xlabel('Time (s)'); ylabel('State Responses');
title('Problem 5: Zero-Input Responses');
legend('State 1','State 2','State 3','State 4');

figure(2); clf; plot(Tus,Yus);
xlabel('Time (s)'); ylabel('State Responses');
legend('State 1','State 2','State 3','State 4');
title('Problem 5: Unit-Step Responses');
```