Intermediate Fluid Mechanics

Lecture 22: Boundary Layer Flows

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October 28, 2025

Chapter Overview

1. Chapter Objectives
2. Introduction to Boundary Layer Theory
3. Laminar Boundary Layer Equations

Lecture Objectives

In this lecture we are starting a new chapter on Boundary layers.

- First, we will focus on the two-dimensional laminar boundary layer equations.
- Later, we will examine the turbulent boundary layer.

Chapter Overview

Introduction to Boundary Layer Theory

Whenever a fluid flows over a solid surface, there are two regions that must be considered:

1. In the immediate vicinity of the surface, a very thin layer of fluid exists, called the **boundary layer**, wherein viscous effects are extremely important.
2. Further away from the surface, viscous effects are insignificant and the flow may be considered inviscid.

Introduction to Boundary Layer Theory

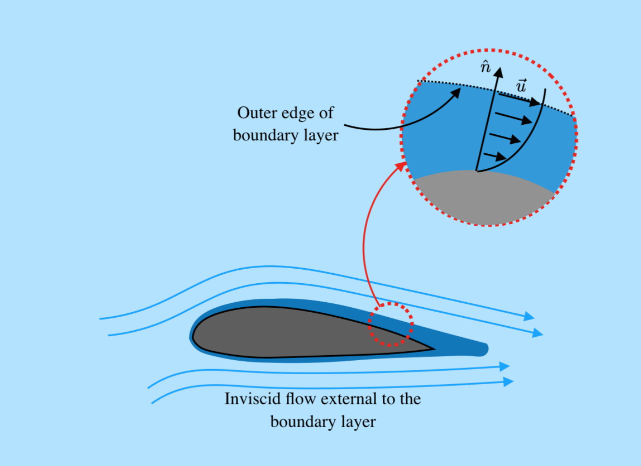

Figure: Sketch of a boundary layer around an airfoil.

Introduction to Boundary Layer Theory

- The concept of the boundary layer was first formulated by Ludwig Prandtl in 1905.
- In the boundary layer, very large velocity gradients exist because of the no-slip condition.
- If the surface is stationary, the velocity is zero at the surface and increases to the freestream velocity (U_∞) as you move normal to the surface.

$\Longrightarrow$ The region where the velocity goes from zero to U_∞ is the boundary layer.

Introduction to Boundary Layer Theory

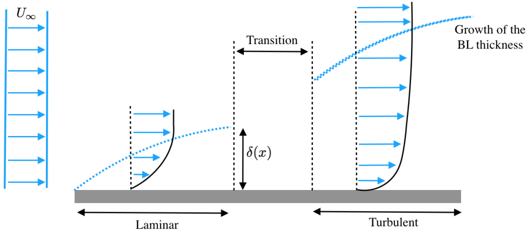

Figure: Sketch representing the evolution of a boundary layer over a flat plate.

- The thickness of the boundary layer is denoted by the symbol δ.
- There is a critical point when the boundary layer undergoes a transition from laminar flow to turbulent flow.
- The Reynolds number dictates this transition.

Chapter Overview

Laminar Boundary Layer Equations

To develop a set of equations for the BL, we will:

- Consider steady flow and assume the laminar flow is two-dimensional.
- Non-dimensionalize the NS and continuity equations using proper characteristic scales.

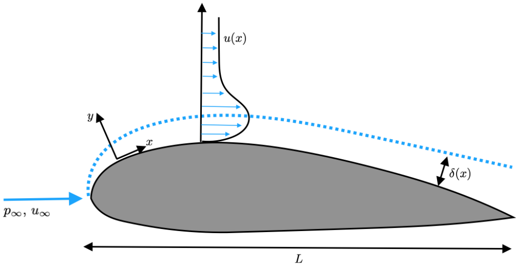

Figure: Boundary layer around a blade. The x-coordinate is tangential to the surface, and the y-coordinate is normal to the surface.

Laminar Boundary Layer Equations

For the non-dimensionalization, one should choose the following characteristic scales:

$$\tilde{u} = \frac{u}{U_\infty}, \quad \tilde{x} = \frac{x}{L}, \quad \tilde{y} = \frac{y}{\delta}, \quad \tilde{p} = \frac{p - p_\infty}{\rho U_\infty^2}. \tag{1}$$

- The key here is to recognize that the tangential and wall-normal coordinates scale differently due to the fact that the boundary layer is so thin.
- Intuitively, we would elect U_∞ as the characteristic scale for the tangential velocity.
- But what is the proper scale for the wall-normal velocity component, v?

Laminar Boundary Layer Equations

One can use the continuity equation to help us. If we assume an incompressible flow,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{2}$$

and let $\tilde{v} = v/V$, and substitute in the non-dimensional variables, one obtains that,

$$\frac{U_\infty}{L}\frac{\partial \tilde{u}}{\partial \tilde{x}} + \frac{V}{\delta}\frac{\partial \tilde{v}}{\partial \tilde{y}} = 0 \tag{3}$$

$$\frac{\partial \tilde{u}}{\partial \tilde{x}} + \frac{V\,L}{\delta\,U_\infty}\frac{\partial \tilde{v}}{\partial \tilde{y}} = 0. \tag{4}$$

Laminar Boundary Layer Equations

Since both,

$$\partial\tilde{u}/\partial\tilde{x} \qquad \text{and} \qquad \partial\tilde{v}/\partial\tilde{y} \sim \mathcal{O}(10^0) \tag{5}$$

the group of parameters $V\,L/\delta\,U_\infty$ must also be of order one; otherwise continuity can not be satisfied.

Therefore,

$$\frac{V\,L}{\delta\,U_\infty} \sim \mathcal{O}(1) \Rightarrow \boxed{V \sim U_\infty\delta/L}. \tag{6}$$

Since $\delta/L << 1$, then $v/U_\infty << 1$.

$\Longrightarrow$ This tells us that the wall-normal velocity is much smaller than the tangential velocity.

Laminar Boundary Layer Equations: x-Momentum

Assuming steady flow for the time being, the 2D x-momentum equation is

$$\text{x-momentum}: \quad u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial x^2} + \nu\frac{\partial^2 u}{\partial y^2}. \tag{7}$$

The non-dimensional wall-normal velocity is $\tilde{v} = \frac{v}{V} = \frac{v}{U_\infty \frac{\delta}{L}}$, which upon substitution in the x-momentum equation,

$$\frac{U_\infty^2}{L}\tilde{u}\frac{\partial \tilde{u}}{\partial \tilde{x}} + \frac{U_\infty^2 \delta}{\delta L}\tilde{v}\frac{\partial \tilde{u}}{\partial \tilde{y}} = -\frac{\rho U_\infty^2}{\rho L}\frac{\partial \tilde{p}}{\partial \tilde{x}} + \frac{U_\infty \nu}{L^2}\frac{\partial^2 \tilde{u}}{\partial \tilde{x}^2} + \frac{U_\infty \nu}{\delta^2}\frac{\partial^2 \tilde{u}}{\partial \tilde{y}^2} \tag{8}$$

Laminar Boundary Layer Equations: x-Momentum

Dividing the previous equation by U_∞^2/L,

$$\text{x-momentum}: \underbrace{\tilde{u}\frac{\partial \tilde{u}}{\partial \tilde{x}}}_{\mathcal{O}(1)} + \underbrace{\tilde{v}\frac{\partial \tilde{u}}{\partial \tilde{y}}}_{\mathcal{O}(1)} = \underbrace{-\frac{\partial \tilde{p}}{\partial \tilde{x}}}_{\mathcal{O}(1)} + \underbrace{\frac{\nu}{U_\infty L}}_{Re^{-1}} \underbrace{\frac{\partial^2 \tilde{u}}{\partial \tilde{x}^2}}_{\mathcal{O}(1)} + \underbrace{\frac{\nu}{U_\infty \delta}\left(\frac{L}{\delta}\right)}_{?} \underbrace{\frac{\partial^2 \tilde{u}}{\partial \tilde{y}^2}}_{\mathcal{O}(1)}. \tag{9}$$

- Viscous diffusion in the x-direction scales like Re^{-1}, where $Re = U_\infty L/\nu$.
- If Re is large enough, this term becomes negligible compared to advection and one can throw it out of the differential equation because it does not affect the dynamics of the flow.

Laminar Boundary Layer Equations: Viscous Diffusion in y

At this point we have already argued that we can neglect viscous diffusion in the x-direction (as long as the Re number is high enough).

So the working differential equation becomes,

$$\underbrace{\tilde{u}\frac{\partial \tilde{u}}{\partial \tilde{x}}}_{\mathcal{O}(1)} + \underbrace{\tilde{v}\frac{\partial \tilde{u}}{\partial \tilde{y}}}_{\mathcal{O}(1)} = \underbrace{-\frac{\partial \tilde{p}}{\partial \tilde{x}}}_{\mathcal{O}(1)} + \underbrace{\frac{\nu}{U_\infty \delta}\left(\frac{L}{\delta}\right)}_{?} \underbrace{\frac{\partial^2 \tilde{u}}{\partial \tilde{y}^2}}_{\mathcal{O}(1)} . \tag{10}$$

In order to move forward, let's consider the case of flow over a flat plate, wherein $\partial p/\partial x = 0$ since there is no imposed pressure gradient.

+<u>Note:</u> For flow over a curved surface, $\partial p/\partial x \neq 0$.

Laminar Boundary Layer Equations: Viscous Diffusion in y

For a flat plate then,

$$\underbrace{\tilde{u}\frac{\partial \tilde{u}}{\partial \tilde{x}}}_{\mathcal{O}(1)} + \underbrace{\tilde{v}\frac{\partial \tilde{u}}{\partial \tilde{y}}}_{\mathcal{O}(1)} = \left[\frac{\nu}{U_\infty \delta}\left(\frac{L}{\delta}\right)\right] \underbrace{\frac{\partial^2 \tilde{u}}{\partial \tilde{y}^2}}_{\mathcal{O}(1)} . \tag{11}$$

If the group of parameters inside the square brackets is not $\mathcal{O}(1)$, then there will be no force to drive inertia and we will not have a dynamical equation.

Laminar Boundary Layer Equations: Viscous Diffusion in y

Therefore, in order to satisfy the equality, we can see that,

$$\frac{\nu}{U_\infty \delta}\left(\frac{L}{\delta}\right) \sim \mathcal{O}(1). \tag{12}$$

Rearranging this equation leads to,

$$\boxed{\frac{\delta}{L} \sim Re^{-1/2}} \tag{13}$$

where $Re \equiv \frac{U_\infty L}{\nu}$.

Note: This tells us that the boundary layer grows like $\delta \sim \sqrt{\nu L / U_\infty}$. We observe that the growth will be slow because it is driven by the viscosity.

Laminar Boundary Layer Equations: y-Momentum

For steady flow, the y-momentum equation in 2D is,

$$\text{y-momentum}: \quad u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\frac{\partial^2 v}{\partial x^2} + \nu\frac{\partial^2 v}{\partial y^2}. \tag{14}$$

Substituting in for the non-dimensional variables,

$$\frac{U_\infty^2}{L}\left(\frac{\delta}{L}\right)\tilde{u}\frac{\partial \tilde{v}}{\partial \tilde{x}} + \frac{U_\infty^2}{\delta}\left(\frac{\delta}{L}\right)^2\tilde{v}\frac{\partial \tilde{v}}{\partial \tilde{y}} = -\frac{\rho U_\infty^2}{\rho\delta}\frac{\partial \tilde{p}}{\partial \tilde{y}} + \frac{U_\infty \nu\delta}{L^3}\frac{\partial^2 \tilde{u}}{\partial \tilde{x}^2} + \frac{U_\infty \nu\delta}{\delta^2 L}\frac{\partial^2 \tilde{v}}{\partial \tilde{y}^2}. \tag{15}$$

Divide by U_∞^2/δ,

$$\left(\frac{\delta^2}{L^2}\right)\tilde{u}\frac{\partial \tilde{v}}{\partial \tilde{x}} + \left(\frac{\delta^2}{L^2}\right)\tilde{v}\frac{\partial \tilde{v}}{\partial \tilde{y}} = -\frac{\partial \tilde{p}}{\partial \tilde{y}} + \frac{\nu}{U_\infty L}\left(\frac{\delta^2}{L^2}\right)\frac{\partial^2 \tilde{v}}{\partial \tilde{x}^2} + \frac{\nu}{U_\infty L}\frac{\partial^2 \tilde{v}}{\partial \tilde{y}^2}. \tag{16}$$

Laminar Boundary Layer Equations: y-Momentum

Next, one can use the fact that $\delta/L \sim Re^{-1/2}$,

$$\frac{1}{Re}\tilde{u}\frac{\partial \tilde{v}}{\partial \tilde{x}} + \frac{1}{Re}\tilde{v}\frac{\partial \tilde{v}}{\partial \tilde{y}} = -\frac{\partial \tilde{p}}{\partial \tilde{y}} + \frac{1}{Re^2}\frac{\partial^2 \tilde{v}}{\partial \tilde{x}^2} + \frac{1}{Re}\frac{\partial^2 \tilde{v}}{\partial \tilde{y}^2}. \tag{17}$$

Note:

- Relative to the non-dimensional x-momentum equation, all of the terms in the non-dimensional y-momentum equation (except the pressure gradient) are smaller by a factor Re^{-1} or Re^{-2}.
- This tells us that the y-momentum equation is insignificant compared to the x-momentum; *i.e.* fluid particles move further in the x-direction compared to the y-direction.
- This argument, of course, assumes the Reynolds number is large enough so that Re^{-1} is small relative to unity.

Laminar Boundary Layer Equations: y-Momentum

Because the non-dimensional pressure gradient is the only term in the y-momentum equation that is order one, one can write that

$$\frac{\partial \tilde{p}}{\partial \tilde{y}} = 0 \Rightarrow \tilde{p} = f(x). \qquad (18)$$

$\Longrightarrow$ This means that pressure is only a function of the $\tilde{x}-$ coordinate.

- This result is significant because it tells us that as long as the Reynolds number is high enough, the pressure does not vary across the boundary layer.
- Therefore, the pressure measured by a pressure tap at the surface is the same pressure as that at the edge of the boundary layer.

Laminar Boundary Layer Equations: y-Momentum

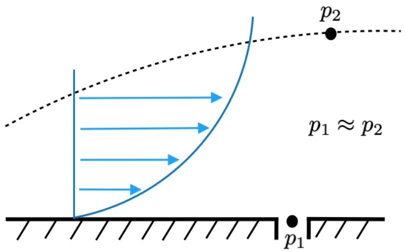

Figure: Boundary layer around a blade. The x-coordinate is tangential to the surface, and the y-coordinate is normal to the surface.

Laminar Boundary Layer Equations

In summary, the Laminar BL equations are,

$$\tilde{u}\frac{\tilde{u}}{\partial \tilde{x}} + \tilde{v}\frac{\tilde{v}}{\partial \tilde{y}} = -\frac{\partial \tilde{p}}{\partial \tilde{x}} + \frac{\partial^2 \tilde{u}}{\partial \tilde{y}^2} \qquad \text{x-momentum} \tag{19}$$

$$0 = -\frac{\partial \tilde{p}}{\partial \tilde{y}} \qquad \text{y-momentum} \tag{20}$$

$$0 = \frac{\partial \tilde{u}}{\partial \tilde{x}} + \frac{\partial \tilde{v}}{\partial \tilde{y}} \qquad \text{Continuity} \tag{21}$$

Note: The above equations are valid asymptotically as $Re \to \infty$, and as long as the flow remains laminar.