The Equations of Motion

Differential Form (for a fixed volume element)

These form a closed set when two thermodynamic relations are specified

The Continuity equation

$$\frac{D\rho}{Dt} = -\rho\nabla.\mathbf{V}$$

The Navier Stokes' equations

$$\rho\frac{Du}{Dt} = \rho\,\mathrm{f}_x - \frac{\partial p}{\partial x} + \frac{\partial}{\partial x}\left(2\mu(\frac{\partial u}{\partial x} - \frac{1}{3}\nabla.\mathbf{V})\right) + \frac{\partial}{\partial y}\left(\mu(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y})\right) + \frac{\partial}{\partial z}\left(\mu(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x})\right)$$

$$\rho\frac{Dv}{Dt} = \rho\,\mathrm{f}_y - \frac{\partial p}{\partial y} + \frac{\partial}{\partial x}\left(\mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\right) + \frac{\partial}{\partial y}\left(2\mu\left(\frac{\partial v}{\partial y} - \frac{1}{3}\nabla.\mathbf{V}\right)\right) + \frac{\partial}{\partial z}\left(\mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)\right)$$

$$\rho\frac{Dw}{Dt} = \rho\,\mathrm{f}_z - \frac{\partial p}{\partial z} + \frac{\partial}{\partial x}\left(\mu(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x})\right) + \frac{\partial}{\partial y}\left(\mu(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y})\right) + \frac{\partial}{\partial z}\left(2\mu(\frac{\partial w}{\partial z} - \frac{1}{3}\nabla.\mathbf{V})\right)$$

The Viscous Flow Energy Equation

$$\rho\frac{D\left(e + \frac{1}{2}V^2\right)}{Dt} = \rho\mathbf{f}.\mathbf{V} - \nabla(p\mathbf{V}) + \nabla.(k\nabla T) + \frac{\partial}{\partial x}\left[2\mu u\left(\frac{\partial u}{\partial x} - \frac{1}{3}\nabla.\mathbf{V}\right) + \mu v\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) + \mu w\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)\right]$$

$$+\frac{\partial}{\partial y}\left[\mu u\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) + 2\mu v\left(\frac{\partial v}{\partial y} - \frac{1}{3}\nabla.\mathbf{V}\right) + \mu w\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)\right] + \frac{\partial}{\partial z}\left[\mu u\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) + \mu v\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) + 2\mu w\left(\frac{\partial w}{\partial z} - \frac{1}{3}\nabla.\mathbf{V}\right)\right]$$

Getting to Ideal Flow

1st new Assumption: Constant density flow

RESTRICTS US TO LOW MACH NO ($M<0.3$) FLOWS OF A HOMOGENEOUS FLUID WITH SMALL TEMPERATURE VARIATIONS

Constant Density Flow

The Continuity equation

$$\cancel{\frac{D\rho}{Dt}} = \cancel{-\rho}\nabla.\mathbf{V}$$

$\Rightarrow \vec{\nabla}\cdot\vec{V} = 0$

constant viscosity (μ = const)

The Navier Stokes' equations

$$\rho\frac{Du}{Dt} = \rho\,\mathrm{f}_x - \frac{\partial p}{\partial x} + \frac{\partial}{\partial x}\left(2\mu(\frac{\partial u}{\partial x} - \frac{1}{3}\nabla.\mathbf{V})\right) + \frac{\partial}{\partial y}\left(\mu(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y})\right) + \frac{\partial}{\partial z}\left(\mu(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x})\right) \rightarrow \mu\nabla^2 u$$

$$\rho\frac{Dv}{Dt} = \rho\,\mathrm{f}_y - \frac{\partial p}{\partial y} + \frac{\partial}{\partial x}\left(\mu(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y})\right) + \frac{\partial}{\partial y}\left(2\mu(\frac{\partial v}{\partial y} - \frac{1}{3}\nabla.\mathbf{V})\right) + \frac{\partial}{\partial z}\left(\mu(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y})\right) \rightarrow \mu\nabla^2 v$$

$$\rho\frac{Dw}{Dt} = \rho\,\mathrm{f}_z - \frac{\partial p}{\partial z} + \frac{\partial}{\partial x}\left(\mu(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x})\right) + \frac{\partial}{\partial y}\left(\mu(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y})\right) + \frac{\partial}{\partial z}\left(2\mu(\frac{\partial w}{\partial z} - \frac{1}{3}\nabla.\mathbf{V})\right) \rightarrow \mu\nabla^2 w$$

$$\rho\frac{D\vec{V}}{Dt} = \rho\vec{f} - \nabla P + \mu\nabla^2(\vec{V})$$

The Viscous Flow Energy Equation uncouples

Constant Density Flow

The Viscous Flow Energy Equation

$$\rho\frac{D(e+\frac{1}{2}V^2)}{Dt}=\rho\mathbf{f}.\mathbf{V}-\nabla(p\mathbf{V})+\nabla.(k\nabla T)+\frac{\partial}{\partial x}\left[2\mu u(\frac{\partial u}{\partial x}-\cancel{\frac{1}{3}\nabla.\mathbf{V}})+\mu v(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y})+\mu w(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x})\right]$$
$$+\frac{\partial}{\partial y}\left[\mu u(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y})+2\mu v(\frac{\partial v}{\partial y}-\cancel{\frac{1}{3}\nabla.\mathbf{V}})+\mu w(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y})\right]+\frac{\partial}{\partial z}\left[\mu u(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x})+\mu v(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y})+2\mu w(\frac{\partial w}{\partial z}-\cancel{\frac{1}{3}\nabla.\mathbf{V}})\right]$$

Subtract **V.**(Momentum), use continuity and take viscosity as constant…

$$\rho\frac{D(e)}{Dt}=\nabla.(k\nabla T)+2\mu\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2+\left(\frac{\partial w}{\partial z}\right)^2+\frac{1}{2}\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)^2+\frac{1}{2}\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)^2+\frac{1}{2}\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)^2\right]$$

This equation of thermal energy controls how thermal energy is conducted into the fluid and generated by viscous action are redistributed by the flow. This redistribution however has no impact back on the velocity or pressure fields of the flow, which are determined entirely by the continuity and momentum eqns.

Getting to Ideal Flow

1st new Assumption: Constant density flow

Restricts us to:
Low Mach number (<0.3 say), nearly isothermal flows of homogeneous fluids.

Implies:

The Continuity equation $\vec{\nabla} \cdot \vec{V} = 0$

The Navier Stokes' equations $\frac{D\vec{V}}{Dt} = \vec{f} - \nabla p / \rho + \underline{\nu \nabla^2 \vec{V}}$

Energy Equation decouples

2nd new Assumption: ??

Constant Density Inviscid Flow?

- $\nu = 0$ ONLY GOOD FOR FLOWS WITH NO VISCOUS EFFECTS.
- $\dfrac{D\vec{V}}{Dt} = \vec{f} - \nabla p/\rho$

PARTIAL SOLUTION: $\cancel{\dfrac{\partial \vec{V}}{\partial t}}^{\,0} + \vec{V}\cdot\vec{\nabla}\vec{V} = -g\vec{k} - \nabla p/\rho$

MUST ASSUME STEADY FLOW: $\nabla\left(V^2/2\right) - \vec{V}\times(\underbrace{\vec{\nabla}\times\vec{V}}_{\Omega}) = -\nabla(gz) - \nabla(p/\rho)$

MULTIPLY BY (-1) & REARRANGE: $\vec{V}\times\vec{\Omega} = \nabla\left(p/\rho + V^2/2 + gz\right)$

SO $p/\rho + V^2/2 + gz$ IS CONSTANT ALONG STREAMLINES & VORTEX LINES.

BERNOULLI'S EQUATION FOR STEADY, ROTATIONAL FLOW !!

Constant Density Irrotational Flow?

$\nabla \times \vec{V} = 0$ CAN BE SOLVED BY WRITING $\vec{V} = \nabla \phi$

- RESTRICTED TO REGIONS THAT HAVE NOT EXPERIENCED VISCOUS FORCE TORQUES

$$\frac{D\vec{V}}{Dt} = \vec{f} - \nabla p/\rho + \nu \nabla^2 \vec{V} \quad \text{But} \quad \nabla^2 \vec{V} = \nabla(\cancel{\nabla \cdot \vec{V}})^{0} - \cancel{\nabla \times \vec{V}}^{0}$$

$$\overset{\nabla\phi}{\frac{\partial \vec{V}}{\partial t}} + \nabla\left(V^2/2\right) + \vec{V} \times \cancel{\vec{\nabla} \times \vec{V}}^{0} = -\nabla(gz) - \nabla(p/\rho)$$

$$\left(\frac{\partial \nabla\phi}{\partial t} = \nabla \frac{\partial \phi}{\partial t}\right) \Rightarrow \nabla \frac{\partial \phi}{\partial t} + \nabla\left(V^2/2\right) = -\nabla(gz) - \nabla(p/\rho)$$

$$\nabla\left(\frac{\partial \phi}{\partial t} + p/\rho + V^2/2 + gz\right) = 0$$

$$\frac{\partial \phi}{\partial t} + p/\rho + V^2/2 + gz = C(t)$$

UNSTEADY BERNOULLI'S EQUATION.

VALID EVERYWHERE.

Governing Equations of Ideal Flow

- Constant density irrotational flow $\rightarrow \vec{V} = \nabla \phi$ $\left(\vec{\nabla} \cdot \vec{V} = \vec{\nabla} \cdot \vec{\nabla} \phi = \nabla^2 \phi \right)$

Continuity $\nabla . \mathbf{V} = 0$ or $\nabla^2 \phi = 0$

Laplace's equation

Momentum $$\frac{\partial \phi}{\partial t} + \frac{V^2}{2} + gz + \frac{p}{\rho} = C(t)$$

Condition of irrotationality $\nabla \times \mathbf{V} = 0$

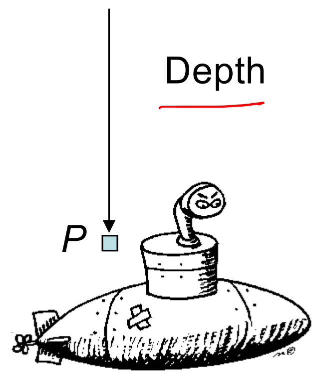
- Decoupled equations
- Solve them separately
- Any solution to these equations are actually solutions to the N-S equations

Understanding Ideal Flow

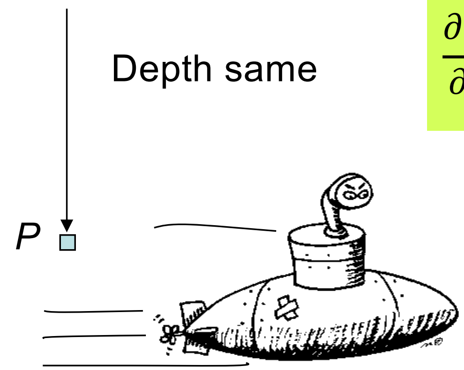
1. Gravity
   – When can you ignore it?
2. Boundary Conditions
3. Principle of Superposition
4. Velocity Potential
   – What is it? … connection with circulation.
   – How does it behave?
     • In a flow?
     • At infinity?
5. Forces on a General Body in Arbitrary Motion

1. Gravity – When can you ignore it?

$$\frac{\partial \phi}{\partial t} + \frac{V^2}{2} + gz + \frac{p}{\rho} = C(t)$$



Stationary submarine in stagnant ocean



Moving submarine in ocean

$$\frac{P_h}{\rho} + gz = C_h$$

$$\frac{\partial \phi}{\partial t} + V^2/2 + gz + \frac{P}{\rho} = C(t)$$

Subtracting
$$\frac{\partial \phi}{\partial t} + V^2/2 + \frac{P - P_h}{\rho} = C(t) - C_h$$

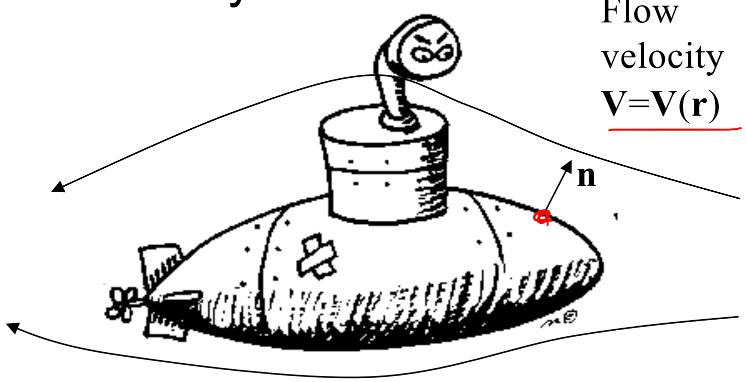
$$\frac{\partial \phi}{\partial t} + V^2/2 + \frac{P}{\rho} = C_1(t),$$ where P is now measured relative to hydrostatic pressure

VALID AS LONG AS ρ = Const (VEHICLE DOES NOT DISTURB THE ~~surf~~ FREE SURFACE)

2. Boundary Conditions?

NO FLOW THROUGH SURFACE

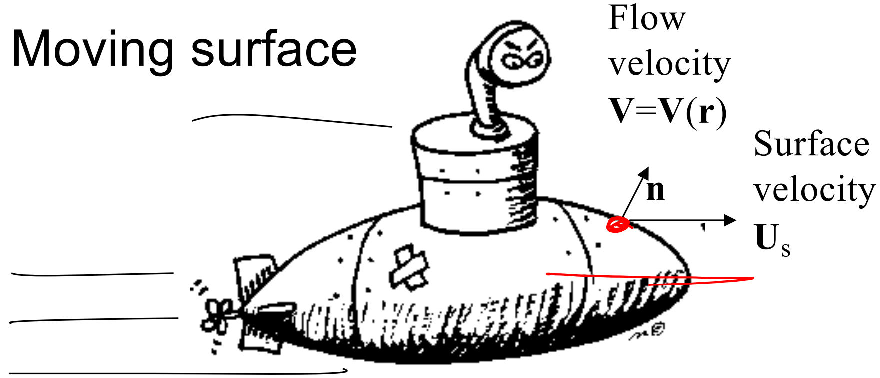
Stationary surface



$$\vec{V}\cdot n = 0 = \vec{\nabla}\phi\cdot\vec{n} = 0$$

$$\frac{\partial \phi}{\partial n} = 0$$

Moving surface



$$\left(\vec{V} - \vec{U}_s\right)\cdot\vec{n} = 0$$

$$\vec{V}\cdot\vec{n} = \vec{U}_s\cdot\vec{n}$$

$$\frac{\partial \phi}{\partial n} = \vec{U}_s\cdot\vec{n}$$

Initial Conditions

- Initial conditions are not needed because the instantaneous state of an ideal flow is entirely determined by the instantaneous state of its boundary conditions

See proof, Karamcheti section 9.9

3. The Principle of Superposition

- Laplace's equation is linear, i.e. different solutions to Laplace's equation can be added together to make new solutions

$$\nabla^2 \phi_1 = 0 \qquad \nabla^2 \phi_2 = 0$$

$$\nabla^2(\phi_1 + \phi_2) = \nabla^2 \phi_1 + \nabla^2 \phi_2 = 0$$

i.e $\phi_1 + \phi_2$ is a new solution.

WE SOLVE COMPLEX PROBLEMS BY ADDING SOLUTIONS TO SIMPLE PROBLEMS

4a. The Potential – What is it?

$$V = \nabla\phi$$

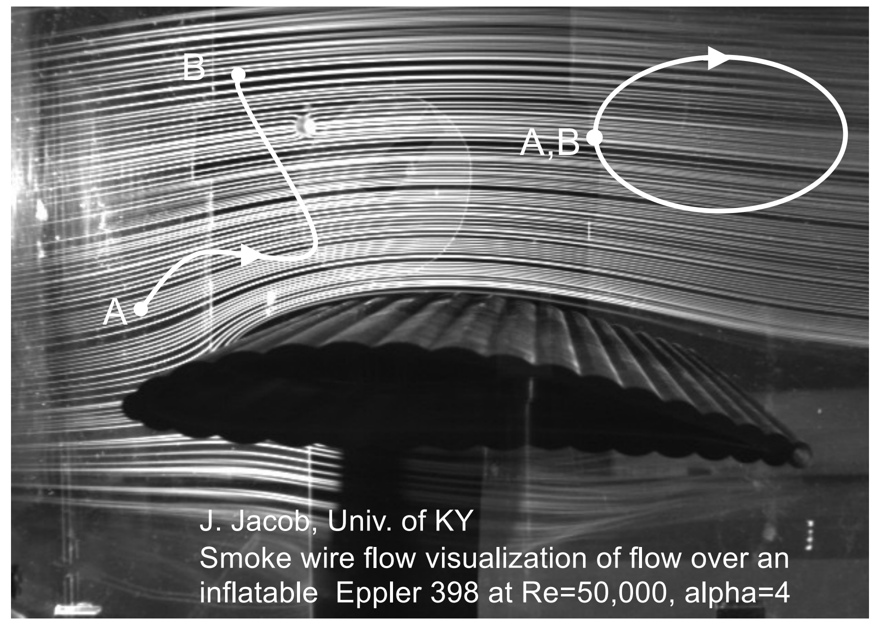
- Explicit equation for the potential

$$\nabla\phi\cdot d\vec{s} = \nabla\phi\cdot\vec{e}_s\, ds = \frac{\partial\phi}{\partial s}\cdot ds = d\phi$$

$$\oint_A^B \vec{V}\cdot d\vec{s} = \oint_A^B \nabla\phi\cdot d\vec{s} = \oint_A^B d\phi = \phi(B) - \phi(A)$$

$$\Gamma = \oint_A^B \vec{V}\cdot d\vec{s} = \phi(B) - \phi(A)$$

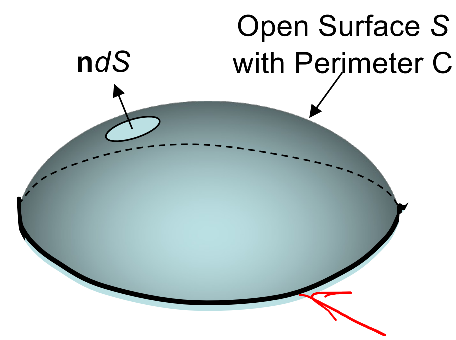
SO POTENTIAL CAN BE MULTI-VALUED FUNCTION



J. Jacob, Univ. of KY
Smoke wire flow visualization of flow over an inflatable Eppler 398 at Re=50,000, alpha=4

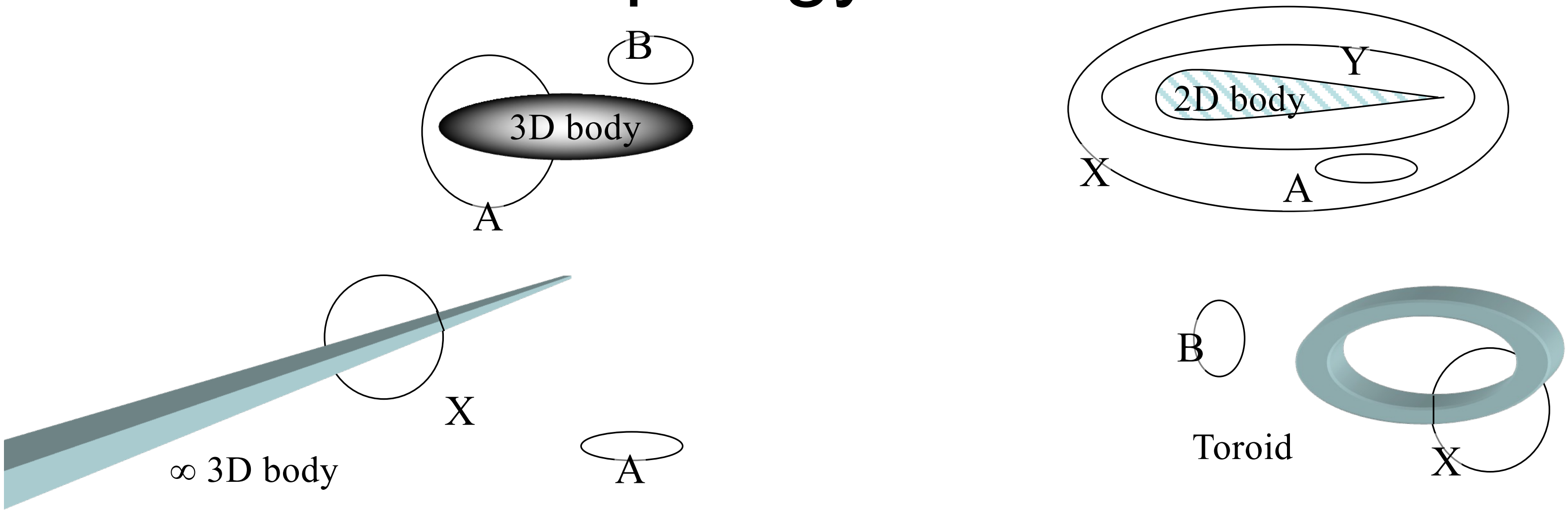
Can the circulation be non-zero?

- Stokes' theorem $\int_S \boldsymbol{\Omega}.\mathbf{n}dS \cancel{=} \oint_C \mathbf{V}.d\mathbf{s} = \Gamma_c$



- However $\boldsymbol{\Omega}=\nabla\times\mathbf{V}=0$, so the circulation is zero whenever Stokes' theorem can be applied

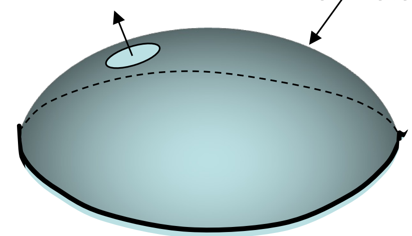
Topology of Potential Flow



- Loops A and B are *reducible* – they may be contracted to a point without passing out of the fluid domain. Hence X and Y are *irreducible*.
- For the 3D body loop A is *reconcilable* with loop B – A may be made coincident with B by moving and deforming them without passing out of the fluid domain. Hence for the 2D body X and Y are *reconcilable* but A and X are *irreconcilable*.
- A space which contains no irreducible loops is termed *simply connected.*
- A space containing irreducible loops is called *doubly* or *multiply connected* depending on the number of irreducible irreconcilable loops that may be drawn

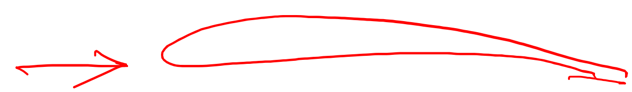
Can the circulation be non-zero?

- Stokes' theorem $$\int_S \mathbf{\Omega} \,.\, \mathbf{n} dS = \oint_C \mathbf{V} . d\mathbf{s} = \Gamma_c$$



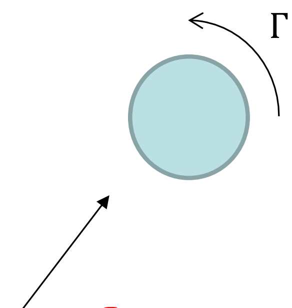
- However $\mathbf{\Omega} = \nabla \times \mathbf{V} = 0$, so the circulation is zero whenever Stokes' theorem can be applied

- CIRCULATION MAY BE NON-ZERO ON ANY IRREDUCIBLE LOOP
- CIRCULATION IS SAME ON RECONCILABLE LOOPS



Kauffman Stadium (Kansas City Royals)

$$\boldsymbol{V} = \nabla\phi$$



$\phi = x^2 + 10$

$\nabla\phi = 2x$

$\phi = x^2 + 106$

$\nabla\phi = 2x$

4b. Behavior Of The Velocity Potential And Related Quantities Within A Flow

Within a flow -

$\nabla^2 \phi = 0$

1. the velocity potential may not reach a maximum or minimum,
2. the velocity components may not reach a maximum or minimum, $\nabla^2 V = 0 \Rightarrow \nabla^2 u = 0$, $\nabla^2 v = 0$
3. the velocity magnitude may not reach a maximum,
4. the pressure may not reach a minimum.
5. the velocity potential may only be determined up to an additive constant.
6. the flow through a multiply connected region is only uniquely determined if the circulation(s) is specified.

See proofs, Karamcheti section 9.18

4c. Behavior Of The Velocity Potential And Related Quantities At Infinity

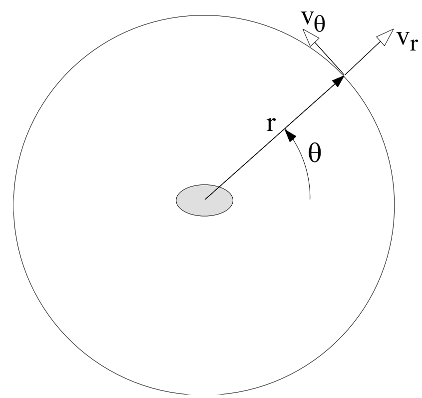
As r$\rightarrow\infty$

$v_r = \partial\phi/\partial r \sim$

$1/r^3$	for a 3D rigid body
$1/r^2$	for a 3D dilating body
$1/r^2$	for a 2D rigid body
$1/r$	for a 2D dilating body

$v_\theta = \frac{1}{r}\partial\phi/\partial\theta \sim$

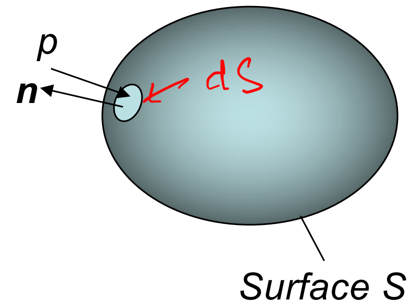
$1/r^3$	for any 3D body (also v_ϕ)
$1/r^2$	for a 2D body with no circulation
$1/r$	for a 2D body with circulation



See proofs, Karamcheti section 9.16, 9.17

5. Forces on a General Body in Arbitrary Motion

Review



1. Relationship between the velocity and velocity potential?

   $\vec{V} = \nabla \phi$

2. An integral giving the force due to pressure on a body with surface S

   $-\oint_S p \vec{n} \, dS$

3. The rate of change of a scalar ϕ seen by an observer moving through a flowfield $\phi = \phi(x,y,z,t)$ with velocity $\mathbf{U_b}$

   $\left.\frac{\partial \phi}{\partial t}\right|_b = \frac{\partial \phi}{\partial t} + \vec{U}_b \cdot \nabla \phi$ $\nabla\phi \to \vec{V}$

4. How fast the disturbance velocity produced by a 3D rigid body decays with distance from the body *r*

   $\propto \frac{1}{r^3}$

Force On A Rigid Body In Motion Through An Otherwise Undisturbed Fluid

Force on body $-\oint_S p\vec{n}\,dS$

Using Bernoulli's eqn.

$$\rho\frac{\partial\phi}{\partial t} + \frac{\rho V^2}{2} + p = \text{const} = P_\infty$$

And,

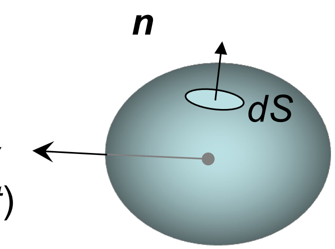
$$\left.\frac{\partial\phi}{\partial t}\right|_B = \frac{\partial\phi}{\partial t} + \vec{U}_b\cdot\nabla\phi \;\rightarrow\; \vec{V}$$

So

$$p = P_\infty - \rho\left.\frac{\partial\phi}{\partial t}\right|_B + \rho\vec{U}_b\cdot\vec{V} - \rho\frac{V^2}{2}$$

So force

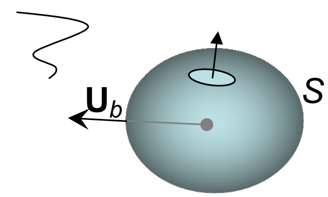
$$\vec{F} = -\cancelto{0}{\oint_S P_\infty\vec{n}\,dS} + \rho\oint_S\left.\frac{\partial\phi}{\partial t}\right|_B\vec{n}\,dS + \rho\oint_S\left[\frac{V^2}{2} - \vec{U}_b\cdot\vec{V}\right]\vec{n}\,dS$$



Body velocity
$\mathbf{U}_b = \mathbf{U}_b(t)$

Flow velocity due to disturbance by body
$\mathbf{V} = \mathbf{V}(\mathbf{r},t)$, $\phi = \phi(\mathbf{r},t)$
Goes to 0 at ∞

Simplifying…



$$\mathbf{F} = \rho \oint_S \left.\frac{\partial\varphi}{\partial t}\right|_s \mathbf{n}dS + \rho \oint_S \frac{1}{2}V^2\mathbf{n} - (\mathbf{U}_b.\mathbf{V})\mathbf{n}dS$$

Now $\quad \mathbf{U}_b\times(\mathbf{n}\times\mathbf{V}) = (\mathbf{U}_b.\mathbf{V})\mathbf{n} - \cancel{(\mathbf{U}_b.\mathbf{n})}\mathbf{V}$ $\quad (\mathbf{V}.\mathbf{n})\mathbf{V}$

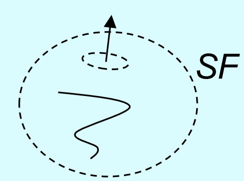
So $\quad (\mathbf{U}_b.\mathbf{V})\mathbf{n} = \mathbf{U}_b\times(\mathbf{n}\times\mathbf{V}) + (\mathbf{V}.\mathbf{n})\mathbf{V}$

$$\mathbf{F} = \rho \oint_S \left.\frac{\partial\varphi}{\partial t}\right|_s \mathbf{n}dS - \rho\mathbf{U}_b\times \oint_S \mathbf{n}\times\mathbf{V}dS \;+\; \oint_S \frac{1}{2}V^2\mathbf{n} - (\mathbf{V}.\mathbf{n})\mathbf{V}dS$$

This last term is zero. Showing this takes 2 steps. First we show that this is zero for a surface containing fluid (rather than a body):

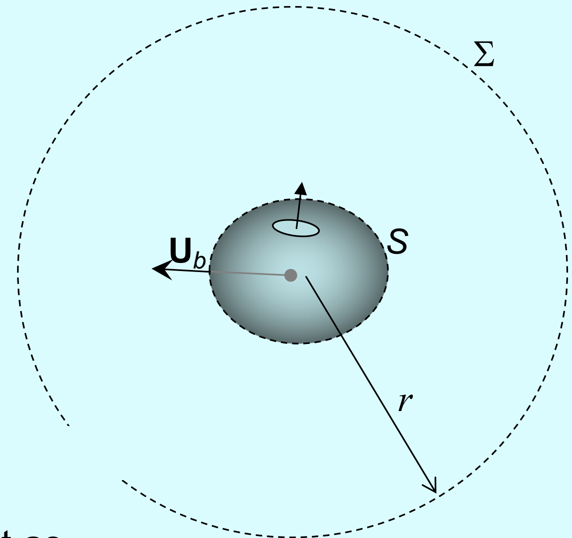
$$\oint_{SF} \frac{1}{2}V^2\mathbf{n}dS \xrightarrow[\text{Grad theorem}]{} \frac{1}{2}\int_{VF} \nabla V^2 d\tau \xrightarrow[\nabla\times\mathbf{V}=0]{} \int_{VF} \mathbf{V}.\nabla\mathbf{V}d\tau$$

$$\oint_{SF} (\mathbf{V}.\mathbf{n})\mathbf{V}dS \xrightarrow[\text{Split into components}]{} \xrightarrow[\text{Div theorem, } \nabla.\mathbf{V}=0]{} \int_{VF} \mathbf{V}.\nabla\mathbf{V}d\tau$$



So $$\oint_{SF} \frac{1}{2}V^2\mathbf{n} - (\mathbf{V}.\mathbf{n})\mathbf{V}dS = 0$$

Then we consider a surface containing only fluid *SF* consisting of the body surface *S* and a spherical surface Σ surrounding the body of large radius r.



$$\oint_{S+\Sigma} \frac{1}{2}V^2\mathbf{n} - (\mathbf{V}.\mathbf{n})\mathbf{V}dS = 0$$

$$\oint_{S} \frac{1}{2}V^2\mathbf{n} - (\mathbf{V}.\mathbf{n})\mathbf{V}dS = -\oint_{\Sigma} \frac{1}{2}V^2\mathbf{n} - (\mathbf{V}.\mathbf{n})\mathbf{V}dS$$

$$= -\oint_{\Sigma} \frac{1}{2}(v_r^2 + v_\theta^2 + v_\varphi^2)\mathbf{e}_r - (v_r)(v_r\mathbf{e}_r + v_\theta\mathbf{e}_\theta + v_\varphi\mathbf{e}_\varphi)dS$$

This should be true however large we make r, but as

$r \to \infty \quad v_r, v_\theta, v_\phi \sim 1/r^3$ and $dS \sim r^2$ So, the integral over Σ is zero and thus

and thus the integral over *S* must be zero.

Thus, the total force on the body is just $$\boxed{\mathbf{F} = \rho \oint_S \left.\frac{\partial \varphi}{\partial t}\right|_s \mathbf{n}dS - \rho \mathbf{U}_b \times \oint_S \mathbf{n}\times\mathbf{V}dS}$$

Interpretation of $\oint_S \mathbf{n}\times\mathbf{V}\,dS$

$\vec{A} = (x^2+y^2)\hat{i} + z^2\hat{j} + y^3\hat{k}$

$\hat{i}\cdot\vec{A} = (x^2+y^2)$

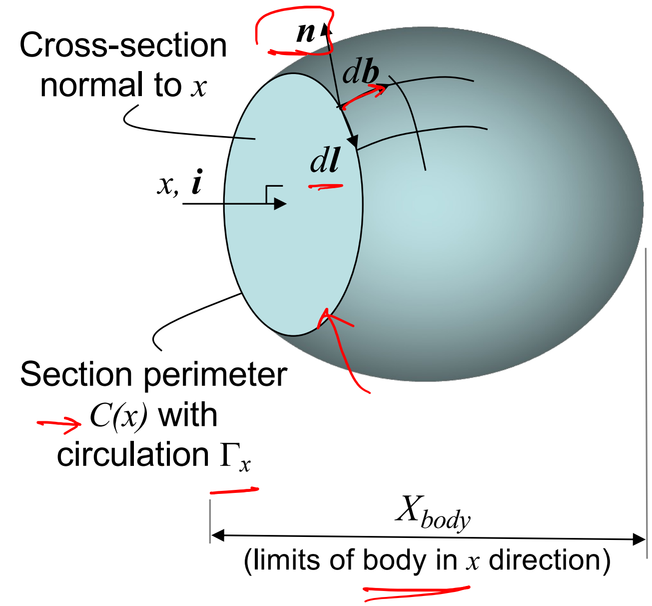
$$\mathbf{i}.\oint_S \mathbf{n}\times\mathbf{V}\,dS = \oint \hat{i}\cdot\vec{n}\times\vec{V}\,dS = -\oint \hat{i}\cdot\vec{V}\times\vec{n}\,dS$$

$$= -\oint \hat{i}\cdot\vec{V}\times(d\vec{l}\times d\vec{b})$$

$$= -\oint \hat{i}\cdot\left[(\vec{V}\cdot d\vec{b})\,d\vec{l} - (\vec{V}\cdot d\vec{l})\,d\vec{b}\right]$$

$$= -\oint (\vec{V}\cdot d\vec{b})\,\cancelto{0}{\hat{i}\cdot d\vec{l}} - (\vec{V}\cdot d\vec{l})\,\cancelto{dx}{\hat{i}\cdot d\vec{b}}$$

$$= \int_{X_{body}}\oint_{C(x)} (\vec{V}\cdot d\vec{l})\,dx = \int_{X_{Body}} \Gamma_x\,dx$$



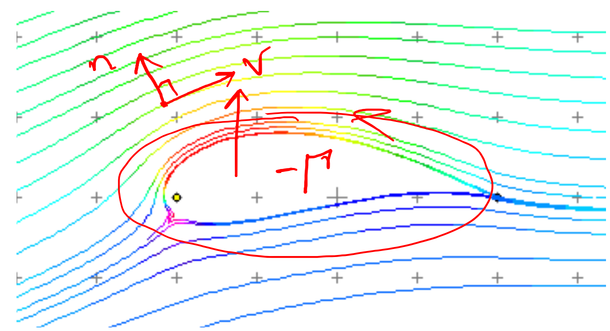
Conclusions

$$\mathbf{F} = \rho \oint_S \frac{\partial \varphi}{\partial t}\bigg|_s \mathbf{n}\,dS - \rho \mathbf{U}_b \times \left(\mathbf{i} \int_{X\ body} \Gamma_x dx + \mathbf{j} \int_{Y\ body} \Gamma_y dy + \mathbf{k} \int_{Z\ body} \Gamma_z dz \right)$$

A body in motion through an otherwise undisturbed ideal fluid experiences:

1. If flow is steady ($\partial/\partial t = 0$) and acyclic ($\Gamma = 0$), NO net force
   THIS IS D'ALEMBERT'S PARADOX.
2. A FORCE ~~TO~~ ~~BE~~ PERPENDICULAR TO ITS MOTION DUE TO CIRCULATION (LIFT)
3. A FORCE IN ANY DIRECTION DUE TO UNSTEADINESS (i.e acceleration)

2D Steady Ideal Flow



- Governing equations
  - Continuity $\nabla\cdot V = 0 \Rightarrow \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$; $\nabla^2\phi = 0$
  - Bernoulli $p + \frac{1}{2}\rho V^2 = \text{const}$
  - Irrotationality $\nabla \times V = 0$ (or) $\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} = 0$; $\nabla\phi = \vec{V}$

$\Rightarrow \nabla\times\nabla\times V = 0 = \nabla(\cancel{\vec{\nabla}\cdot\vec{V}}) - \nabla^2 V = 0 \Rightarrow \nabla^2 V = 0$

- Other results that matter
  - Circulation must be specified around closed bodies
  - Boundary condition **V.n**=0 is same as condition on any streamline
  - Kutta-Joukowski theorem (force on a body in undisturbed fluid)

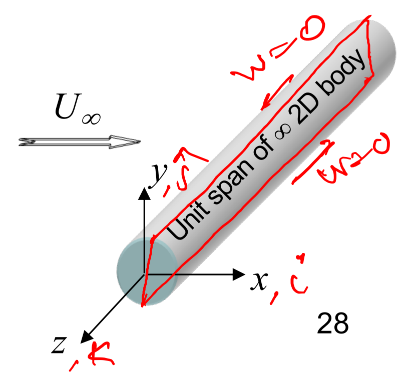
$$\mathbf{F} = -\rho\mathbf{U}_b \times \left(\mathbf{i}\cancel{\int_{X\ body}\Gamma_x dx} + \mathbf{j}\cancel{\int_{Y\ body}\Gamma_y dy} + \mathbf{k}\int_{Z\ body}\Gamma_z dz\right) = -\rho\vec{U}_b \times \hat{k}\,\Gamma\cdot Z$$

FORCE PER UNIT SPAN

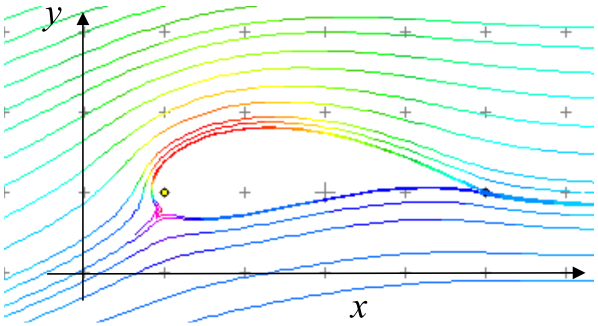
$$F/Z = -\rho\vec{U}_b \times \hat{k}\,\Gamma = -\rho(-\hat{i}\,U_\infty)\times\hat{k}\,\Gamma$$

$$= -\rho U_\infty \Gamma\,\hat{j}$$

Lift force per unit span $\Rightarrow l = -\rho U_\infty \Gamma$ (K-J theorem)



Relations for flow in the x, y plane



Independent Variables	x	y		
Dependent Variables	u	v	ϕ	ψ
Governing Equations	$\nabla^2 u = 0$	$\nabla^2 v = 0$	$\nabla^2 \phi = 0$	$\nabla^2 \psi = 0$?
Relationships	$\frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y}$	$\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}$	$u = \frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}$?	$v = \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}$?

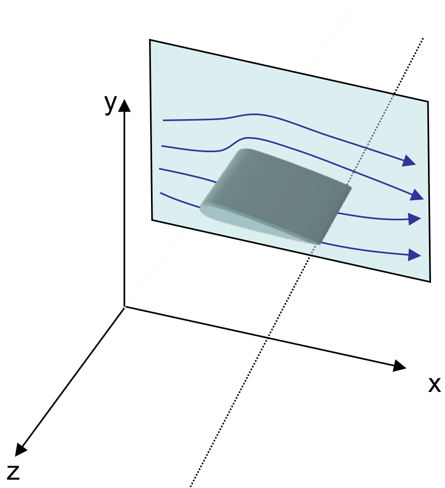
$$\rightarrow \phi(B) - \phi(A) = \int_A^B \vec{V} \cdot d\vec{s} \quad ; \quad \psi(B) - \psi(A) = \int_A^B \vec{V} \cdot \vec{n}\, ds \ ?$$

Streamfunction ψ

$\alpha\vec{V} = \nabla\psi_1 \times \nabla\psi_2$

$\alpha = 1$

$\boxed{V = \nabla\psi_1 \times K}$



Take $\psi_2 = z \quad \Rightarrow \quad \nabla\psi_2 = \mathbf{k}$

$$\Rightarrow \quad \cancel{\alpha}\mathbf{V} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial\psi_1}{\partial x} & \dfrac{\partial\psi_1}{\partial y} & \dfrac{\partial\psi_1}{\partial z} \\ 0 & 0 & 1 \end{vmatrix}$$

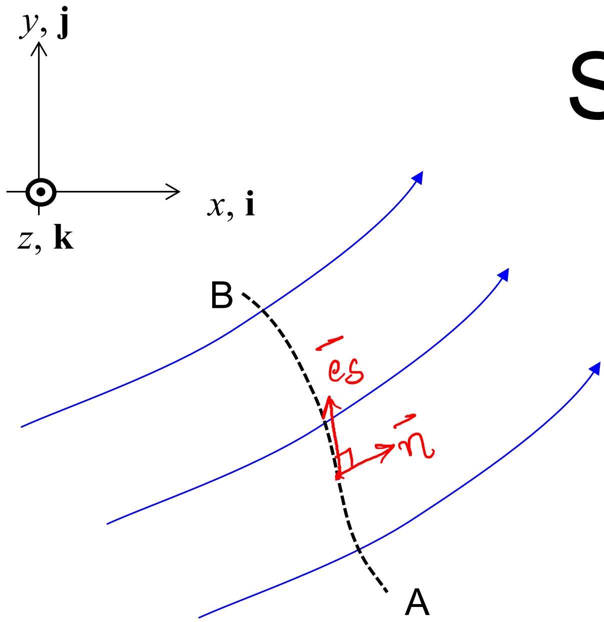
$$\Rightarrow \cancel{\alpha}u = \frac{\partial\psi_1}{\partial y}, \quad \cancel{\alpha}v = -\frac{\partial\psi_1}{\partial x}$$

$$\vec{\nabla}\times\vec{V} = 0 = \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} = 0 \Rightarrow \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial x^2} = 0 \Rightarrow \nabla^2\psi = 0$$

$$\vec{\nabla}\cdot V = 0$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \Rightarrow \frac{\partial^2\psi}{\partial x\partial y} - \frac{\partial^2\psi}{\partial x\partial y} = 0$$

Streamfunction ψ



$$\vec{V} = \vec{\nabla\psi} \times \vec{k}$$

$$Q_{AB} = \int_A^B \vec{V}\cdot\vec{n}\,ds = \int_A^B \left(\vec{\nabla}\psi \times \vec{k}\right)\cdot\vec{n}\,ds$$

A	B	C
B	C	A
C	A	B

$$= \int_A^B \left(\vec{k}\times\vec{n}\right)\cdot\nabla\psi\,ds$$

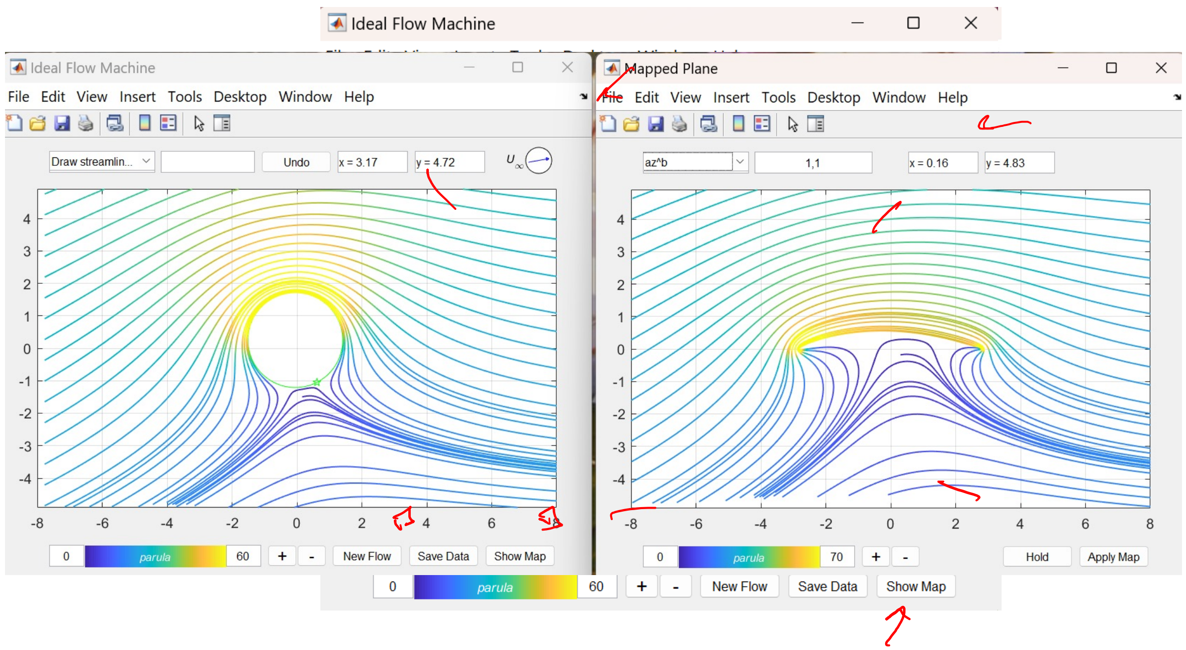
$$= \int_A^B \vec{e}_s\cdot\nabla\psi\,ds = \int_A^B \frac{\partial\psi}{\partial s}\,ds$$

$$Q_{AB} = \int_A^B d\psi = \psi(B) - \psi(A)$$

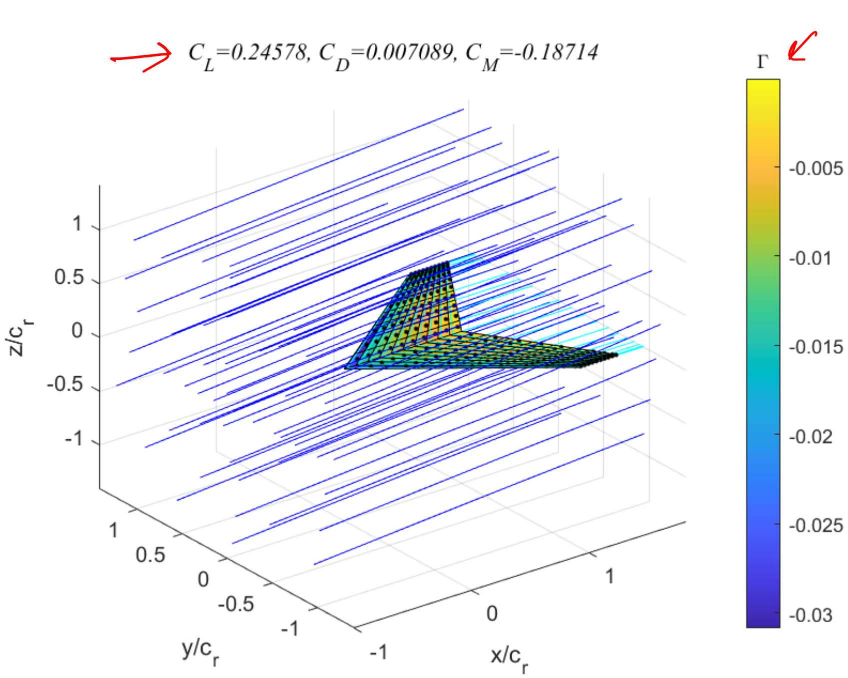
$$\vec{V} = \nabla\psi_1 \times \nabla\psi_2$$

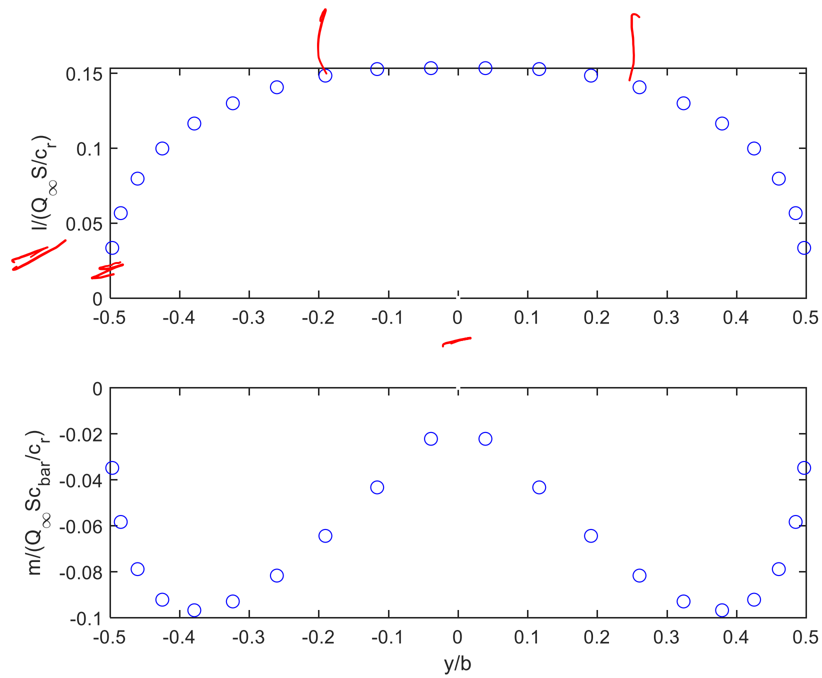
$$\nabla\phi = \nabla\psi_1 \times \nabla\psi_2$$

2D IDEAL FLOW MACHINE



3D PANEL METHOD CODE

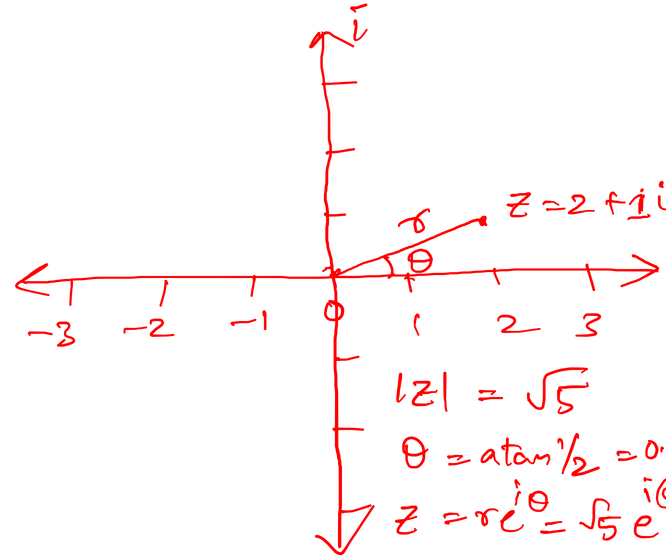




Complex Numbers

$3-6i$, $2\cdot 3$, $3i$, $-76-4.35i$

- A complex number z can *ALWAYS* be separated into its real x and imaginary iy parts and that separation is *UNIQUE*.
- Magnitude $=|z|=\sqrt{(x^2+y^2)}$
- Argument (or angle) $=arg(z)=\arctan(y/x)$ $=\theta$
- A single complex variable can be used to represent two real independent variables with the same dimensions. E.g. Coordinates, velocities, streamfunction/potential
- Any complex variable may be expressed in polar form using the complex exponential. E.g. Coordinates $z=x+iy=r\cos\theta+ir\sin\theta=re^{i\theta}$.
- Multiplying two complex numbers/variables together multiplies their magnitudes and <u>adds</u> their angles.
- Multipication of any complex number by $e^{i\theta}$ is equivalent to rotation about the origin through angle θ
- Raising a complex number to a power <u>multiplies</u> its argument by that power.



$|z| = \sqrt{5}$

$\theta = \text{atan}\,1/2 = 0.5\,\text{rad}$

$z = re^{i\theta} = \sqrt{5}\,e^{i(0.5)}$

COMPLEX POSITION $\Rightarrow z = x+iy$

COMPLEX VELOCITY $\Rightarrow W(z) = u-iv$

COMPLEX POTENTIAL $\Rightarrow \phi$

$F(z) = \phi + i\psi$

$z_1 = 2 + i$

$z_2 = 0 + i$

$z_1 + z_2 = 2 + 2i$

$z_1 - z_2 = 2$

$$z_1 \cdot z_2 = (2+i)(0+i)$$

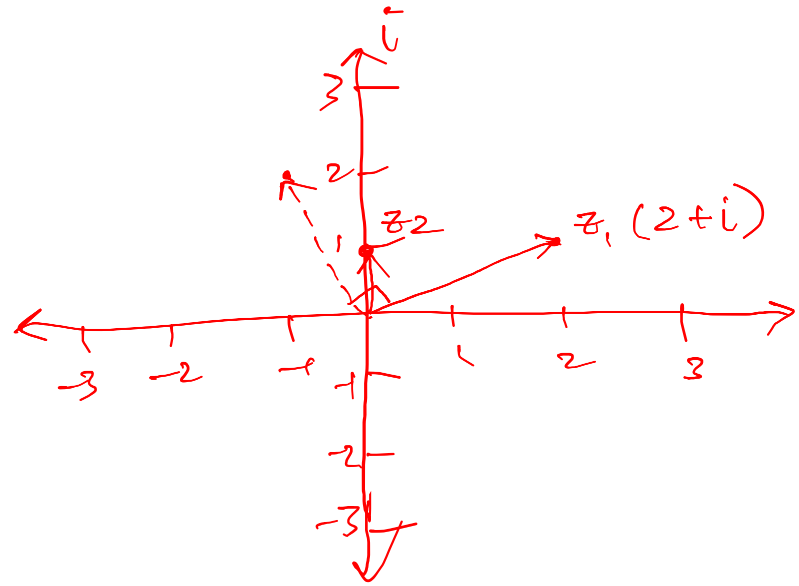
$$= 2\cdot 0 + 2i + i(0) + i^2$$

$$= 2i + i^2$$

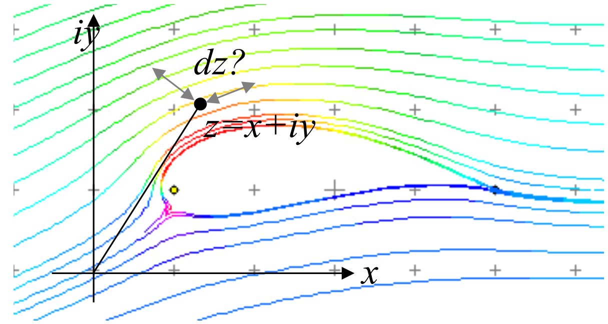
$$z_1 \cdot z_2 = -1 + 2i$$

$$z_1 \cdot z_2 = \sqrt{5}\, e^{i0.5} \cdot e^{i\pi/2}$$

$$= \sqrt{5}\, e^{i(0.5 + \pi/2)}$$



Complex Functions I



• A complex function [e.g. $F(z)$, $W(z)$, $\sin(z)$, $\zeta(z)$] produces a complex number at every position z. That number can always be split into real and imaginary parts, say $\zeta(z)=\xi(x,y)+i\eta(x,y)$ where ξ and η are real functions.

zeta xi

eg $F(z)$ $\quad \xi \equiv \phi \quad \eta \equiv \psi$

$W(z)$ $\quad \xi \equiv u \quad \eta \equiv -v$

$\zeta(z) = \sin(z)$. $\dfrac{d\zeta}{dz} = \cos(z)$

$$\frac{d\zeta}{dz} = \frac{d\zeta}{dx} = \frac{d\zeta}{d(iy)} = \frac{d\zeta}{i\,dy} = -i\frac{d\zeta}{dy}$$

Complex Functions II

- The derivatives contain no reference to the direction of z so

$$\zeta = \xi(x,y) + i\eta(x,y)$$

$$\frac{d\zeta}{dz} = \frac{\partial\zeta}{\partial x} = \frac{\partial\zeta}{\partial(iy)} = -i\frac{\partial\zeta}{\partial y} \longrightarrow \frac{\partial\zeta}{\partial x} = -i\frac{\partial\zeta}{\partial y}$$

$$\frac{\partial\xi}{\partial x} + i\frac{\partial\eta}{\partial x} = -i\frac{\partial\xi}{\partial y} - i.i\frac{\partial\eta}{\partial y}$$

$$\boxed{\frac{\partial\xi}{\partial x} = \frac{\partial\eta}{\partial y}, \quad \frac{\partial\xi}{\partial y} = -\frac{\partial\eta}{\partial x}}$$

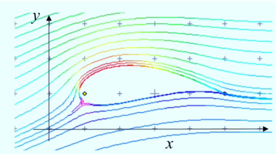
- This result is called the CAUCHY RIEMANN CONDITIONS

Differentiate $\frac{\partial^2\xi}{\partial x^2} = \frac{\partial^2\eta}{\partial x\partial y}$ $\qquad \frac{\partial^2\xi}{\partial y^2} = -\frac{\partial^2\eta}{\partial y\partial x}$

Adding $\frac{\partial^2\xi}{\partial x^2} + \frac{\partial^2\xi}{\partial y^2} = 0$, $\nabla^2\xi = 0$; likewise $\nabla^2\eta = 0$

SO, ANY $\underbrace{\text{DIFFERENTIABLE}}_{\text{"ANALYTIC"}}$ FUNCTION OF z IS A SOLUTION TO LAPLACE'S EQUATION

Relations for flow in the x, y plane

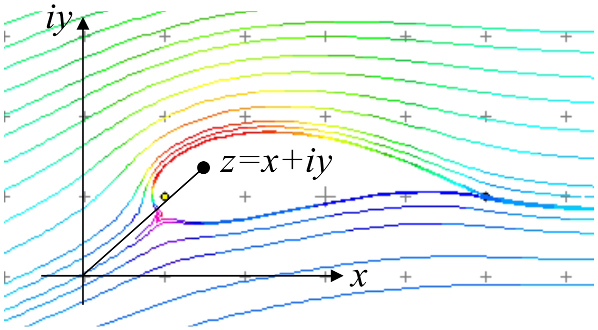


Independent Variables	x	y		
Dependent Variables	u	v	ϕ	ψ
Governing Equations	$\nabla^2 u = 0$	$\nabla^2 v = 0$	$\nabla^2\phi = 0$	$\nabla^2\psi = 0$?
Relationships	$\frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y}$	$\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}$	$u = \frac{\partial\phi}{\partial x} = \frac{\partial\psi}{\partial y}$?	$v = \frac{\partial\phi}{\partial y} = -\frac{\partial\psi}{\partial x}$?

$\rightarrow \phi(B) - \phi(A) = \int_A^B \vec{V}\cdot d\vec{s}$; $\psi(B) - \psi(A) = \int_A^B$?

2D Ideal Flow in the Complex Plane

Summary



- Position: $z=x+iy$
- Complex velocity: $W(z)=u-iv$
- Complex potential: $F(z)=\phi+i\psi$
- As analytic functions $W(z)$ and $F(z)$ automatically satisfy the governing equations and the relations between dependent variables.

$W = \frac{dF}{dz}$ $F = \int W\,dz$

$V_r - iV_\theta = W(z)\cdot e^{i\theta}$ $\Gamma + iq = \oint W\,dz$

$$W = dF/dz$$
$$F = \int W dz$$

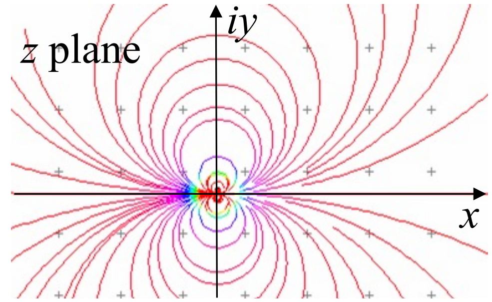
$$F = \phi + i\psi$$

$$\frac{dF}{dz} = \frac{dF}{dx} = \frac{\partial \phi}{\partial x} + i\frac{\partial \psi}{\partial x}$$
$$= u - iv$$
$$= W$$

$$\Gamma + iQ = \oint W dz$$
$$v_r - iv_\theta = (u-iv)e^{i\theta} = We^{i\theta}$$

$$\oint W dz = \oint (u-iv)(dx + i dy)$$
$$= \oint (u dx + v dy) + i \oint (u dy - v dx)$$
$$= \oint \left(\frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy \right) + i \oint \left(\frac{\partial \psi}{\partial y} dy + \frac{\partial \psi}{\partial x} dx \right)$$
$$= \oint d\phi + i \oint d\psi$$
$$= \Gamma + iq$$

Elementary Ideal Flows



$W = u - iv = Az^n$

ALL FUNCTIONS OF FORM $W(z) = A z^n$

INTEGER

COMPLEX CONSTANT

$a + ib$; $c e^{i\beta}$

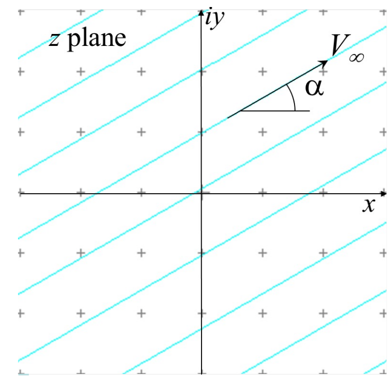
$$\frac{dW}{dz} = A n z^{n-1}$$

if $n \geq 0$ $W(z)$ IS ANALYTIC EVERYWHERE

if $n < 0$ $W(z)$ IS ANALYTIC EVERYWHERE EXCEPT THE ORIGIN

$\Longrightarrow$ $a+ib$ or $ce^{i\beta}$

$$\boxed{W(z)=Az^n}$$



1. Uniform Flow ($n=0$)

$$W(z) = u - iv = A$$

$$|W| = \sqrt{u^2+v^2} = C \equiv V_\infty$$ FREESTREAM VELOCITY

$$\arg(W) = \tan^{-1}\left(-\frac{v}{u}\right) = -\text{(ve) ANGLE OF THE FLOW} \equiv -\alpha$$

UNIFORM FLOW OF VELOCITY V_∞, AT AN ANGLE α TO THE X-AXIS

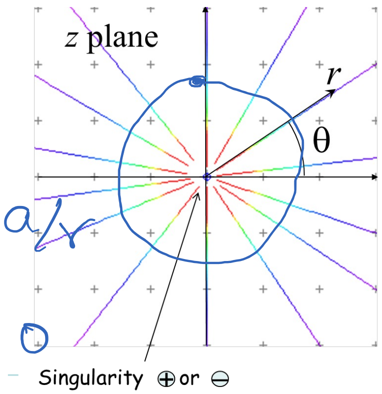
$$W(z) = V_\infty e^{-i\alpha} \qquad F(z) = V_\infty e^{-i\alpha} z$$

2. Source or Sink Flow

(n=-1, A=real=a)

$a+ib$ or $ce^{i\beta}$

$$\boxed{W(z)=Az^n}$$



$$W(z) = \frac{a}{z} = \frac{a}{re^{i\theta}} \quad \text{so} \quad \underline{V_r - iV_\theta} = We^{i\theta} = \frac{a}{r} \Rightarrow \begin{aligned} V_r &= a/r \\ V_\theta &= 0 \end{aligned}$$

$$F(z) \equiv \int W dz = a \log_e z = a \log_e re^{i\theta} = a \log_e r + ai\theta \quad \text{so,} \quad \begin{aligned} \phi &= a \log_e r \\ \psi &= a\theta \end{aligned}$$

$(F(z) = \phi + i\psi)$

$\Gamma = 0$, $q = 2\pi a$ ← <u>STRENGTH</u>

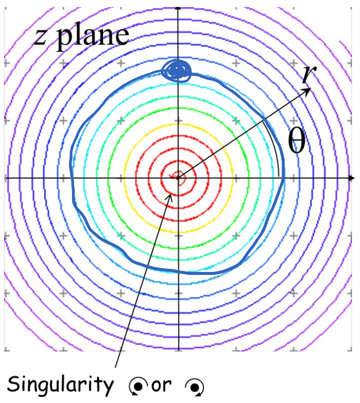
→ SOURCE OF STRENGTH q at ORIGIN

$$W(z) = \frac{q}{2\pi}\frac{1}{z} \qquad F(z) = \frac{q}{2\pi}\log(z)$$

3. Point Vortex Flow
(n=-1, A=imag.=ib)

$a+ib$ or $ce^{i\beta}$

$$W(z)=Az^n$$



$$W(z) = \frac{ib}{z} = \frac{ib}{re^{i\theta}}\ ; \quad V_r - iV_\theta = \frac{ib}{r} \Rightarrow \begin{array}{l} V_r = 0 \\ V_\theta = -\frac{b}{r} \end{array}$$

$$F(z) = ib\log_e z = -b\theta + ib\log_e r \qquad \begin{array}{l} \phi = -b\theta \\ \psi = b\log_e r \end{array} \qquad \{F = \phi + i\psi\}$$

$$\Gamma = \Delta\phi = -2\pi b\ , \quad q = 0$$

VORTEX OF STRENGTH Γ ~~AT~~ ~~ORIGIN~~ z_1

$$W(z) = \frac{-i\Gamma}{2\pi(z - z_1)} \qquad F(z) = \frac{-i\Gamma}{2\pi}\log(z - z_1)$$

4. Doublet (n=-2)

$a+ib$ or $ce^{i\beta}$

$$\boxed{W(z)=Az^n}$$

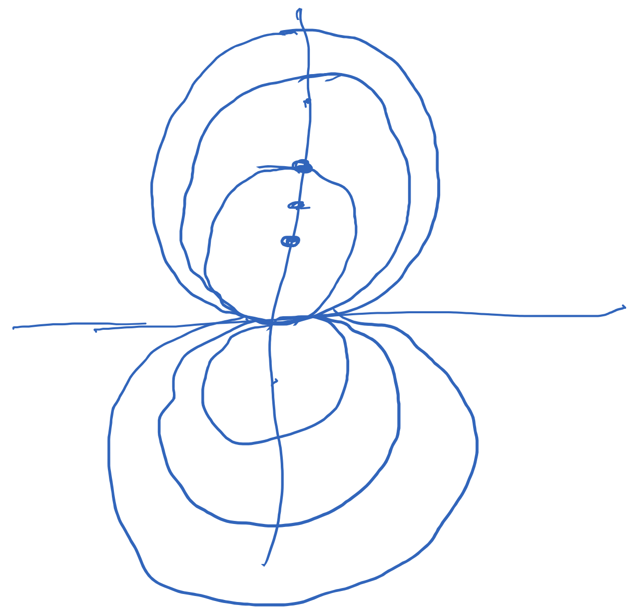
$z = x + iy$

$\bar{z} = x - iy$

$$W(z) = \frac{A}{z^2} \; ; \; F(z) = \frac{-A}{z} = \frac{-ce^{i\beta}}{z} = -\frac{c}{z} \quad \text{Rotated by } \beta$$

$$\text{For } \beta = 0, \quad \psi = \mathrm{Im}\left(\frac{-c}{z}\right) = \mathrm{Im}\left(\frac{-c\bar{z}}{z\bar{z}}\right) = \frac{cy}{x^2+y^2} = \text{CONST ON STREAMLINES}$$

$$\text{SO} \quad x^2+y^2 = c_1 y \quad \text{or} \quad x^2 + (y - c_1/2)^2 = (c_1/2)^2$$



DOUBLET STRENGTH $\mu = 2\pi c$

SO A DOUBLET OF STRENGTH μ AT z_1
WITH ANGLE β

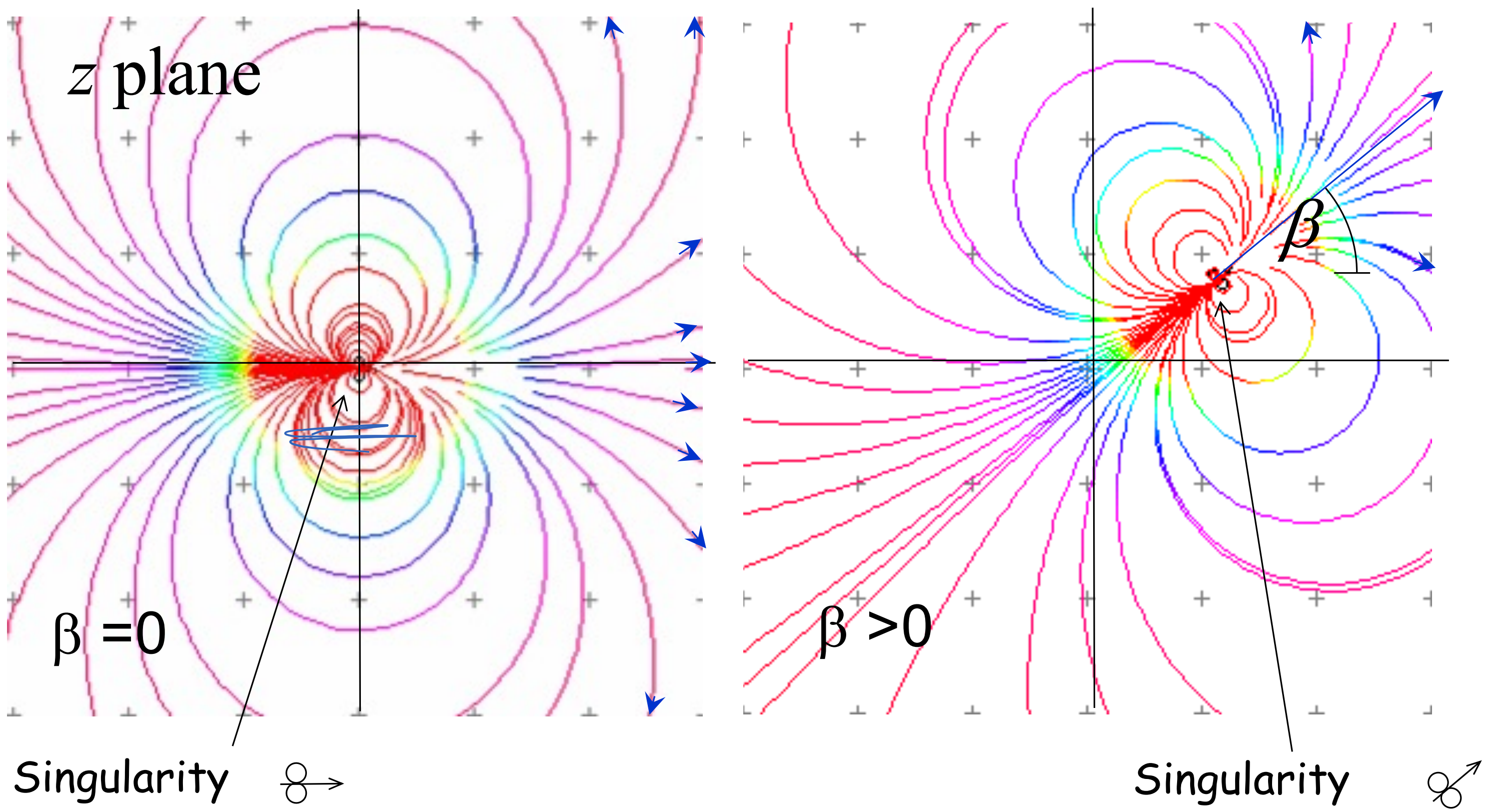
$$W(z) = \frac{\mu}{2\pi}\frac{e^{i\beta}}{(z-z_1)^2} \; ; \; F(z) = -\frac{\mu}{2\pi}\frac{e^{i\beta}}{(z-z_1)}$$

4. Doublet (n=-2)

$a+ib$ or $ce^{i\beta}$

$$\boxed{W(z)=Az^n}$$

Ideal flow Machine



Summary / Crib

1. Uniform flow of V_∞ at angle α to the x axis $$W(z) = V_\infty e^{-i\alpha} \qquad F(z) = V_\infty z e^{-i\alpha}$$
2. Source at z_1 flow producing volume flowrate q $$W(z) = \frac{q}{2\pi(z - z_1)} \qquad F(z) = \frac{q}{2\pi}\log_e(z - z_1)$$
3. Vortex at z_1 producing circulation Γ $$W(z) = -\frac{i\Gamma}{2\pi(z - z_1)} \qquad F(z) = -\frac{i\Gamma}{2\pi}\log_e(z - z_1)$$
4. Doublet at z_1 strength μ aligned at angle β to the x axis $$W(z) = \frac{\mu e^{i\beta}}{2\pi(z - z_1)^2} \qquad F(z) = -\frac{\mu e^{i\beta}}{2\pi(z - z_1)}$$
5. Flow of velocity V_∞ at angle α past a circular cylinder of radius a at z_1 with circulation Γ