

# Intermediate Fluid Mechanics

## Lecture 11: Newton's Second Law

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ME 5700/6700

September 16, 2025

# Chapter Overview

- ① Chapter Objectives
- ② Newton's Second Law
- ③ Cauchy's Equation of Motion

# Lecture Objectives

In this lecture we combine elements of the last two lectures to write Newton's second law of motion describing the dynamics of the fluid in an Eulerian framework.

# Chapter Overview

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② Newton's Second Law

③ Cauchy's Equation of Motion

# Newton's Second Law

From previous courses on dynamics, recall that Newton's second law of motion states that the inertia (mass times acceleration) of a system is equal to the net force acting on that system.

$$\vec{F} = m \vec{a}. \quad (1)$$

## Note:

Note that this equation is a vectorial equation with three components in a 3-dimensional vectorial space.

## Newton's Second Law (continued ...)

Since, in fluid mechanics one doesn't work with discrete masses, one can rewrite the left hand side of Newton's 2nd law in terms of the time rate of change of the momentum,

$$\frac{D(m\vec{u})}{Dt} = \vec{F}. \quad (2)$$

Note that, this can be rewritten this way because of Conservation of Mass ( $Dm/Dt = 0$ ),

$$\frac{D(m\vec{u})}{Dt} = m\frac{D\vec{u}}{Dt} + \vec{u}\frac{Dm}{Dt} = m\frac{D\vec{u}}{Dt}, \quad (3)$$

⇒ Now, by expressing mass as the integral of the density over some volume,

$$\boxed{\frac{D}{Dt} \int_{V(t)} \rho \vec{u} dV = \vec{F}.} \quad (4)$$

## Newton's Second Law (continued ...)

$$\boxed{\frac{D}{Dt} \int_{\mathcal{V}(t)} \rho \vec{u} d\mathcal{V} = \vec{F}.} \quad (5)$$

On the right hand side, the forces ( $\vec{F}$ ) on a fluid particle can be due to:

- (i) body forces, namely gravity (in classic fluids),
- (ii) surface forces (i.e stresses).

# Surface Forces

To determine the Surface Forces, it is convenient to consider a differential fluid element, and Taylor's series expansion. With these one can write the stresses on each face based on the stress tensor at the center of the differential element,

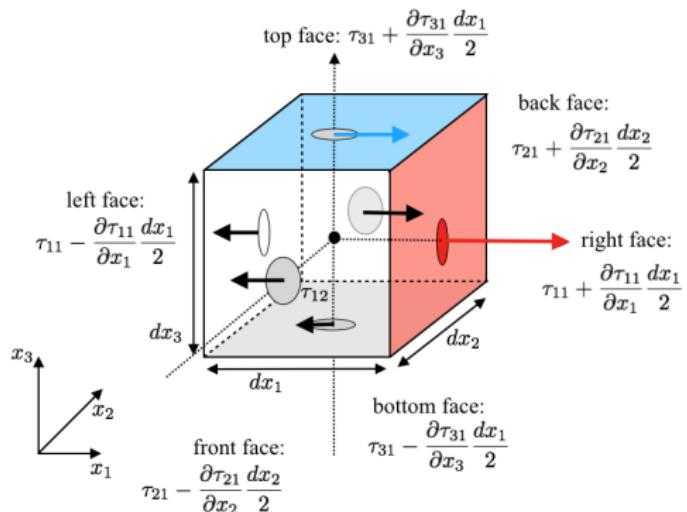


Figure: Schematic diagram illustrating the forces on each face of the differential fluid particle.

# Direction of the Surface Forces (Stresses) on the Fluid Element

- On each face of a fluid element there will be 3 stresses (i.e. 1 normal, and 2 of shear).
- Given that the surface normals  $\hat{n}$  are pointing in opposite directions for a given cartesian-axis, the corresponding stresses must have different sign.

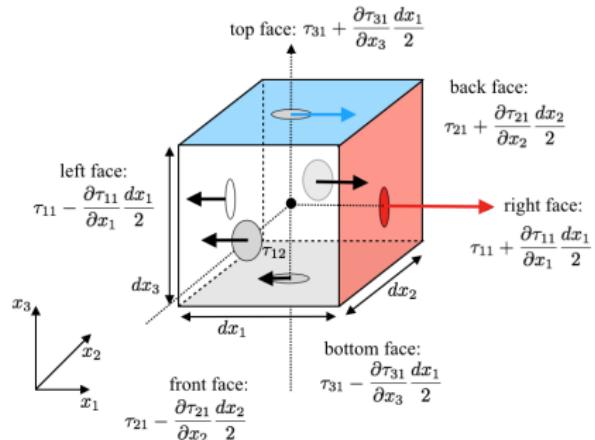


Figure: Schematic diagram illustrating the forces on each face of the differential fluid particle.

# Sum of stresses in the $x_1$ -direction

Let's now sum all of the stresses in the  $x_1$ -direction,

$$\underbrace{dF_{s_{(1)}}}_{\text{surface force in } x_1 \text{ direction}} = \left[ \underbrace{\left( \tau_{11} + \frac{\partial \tau_{11}}{\partial x_1} \frac{dx_1}{2} \right)}_{\text{right face}} - \underbrace{\left( \tau_{11} - \frac{\partial \tau_{11}}{\partial x_1} \frac{dx_1}{2} \right)}_{\text{left face}} \right] \underbrace{dx_2 dx_3}_{\text{area surface}} \quad (6)$$

$$+ \left[ \underbrace{\left( \tau_{21} + \frac{\partial \tau_{21}}{\partial x_2} \frac{dx_2}{2} \right)}_{\text{right face}} - \underbrace{\left( \tau_{21} - \frac{\partial \tau_{21}}{\partial x_2} \frac{dx_2}{2} \right)}_{\text{left face}} \right] \underbrace{dx_1 dx_3}_{\text{area surface}} \quad (7)$$

$$+ \left[ \underbrace{\left( \tau_{31} + \frac{\partial \tau_{31}}{\partial x_3} \frac{dx_3}{2} \right)}_{\text{right face}} - \underbrace{\left( \tau_{31} - \frac{\partial \tau_{31}}{\partial x_3} \frac{dx_3}{2} \right)}_{\text{left face}} \right] \underbrace{dx_1 dx_2}_{\text{area surface}} . \quad (8)$$

The summation in the above expression simplifies to

$$dF_{s_1} = \left( \frac{\tau_{11}}{\partial x_1} + \frac{\tau_{21}}{\partial x_2} + \frac{\tau_{31}}{\partial x_3} \right) dx_1 dx_2 dx_3, \quad (9)$$

## Sum of stresses in the $x_1$ -direction (continued ...)

$$dF_{s_1} = \left( \frac{\tau_{11}}{\partial x_1} + \frac{\tau_{21}}{\partial x_2} + \frac{\tau_{31}}{\partial x_3} \right) dx_1 dx_2 dx_3, \quad (10)$$

In index notation this can be written as,

$$dF_{s_1} = \frac{\partial \tau_{j1}}{\partial x_j} dV = \frac{\partial \tau_{1j}}{\partial x_j} dV. \quad (11)$$

This can be further generalized to the other directions such that,

$$dF_{s_i} = \frac{\partial \tau_{ij}}{\partial x_j} dV \quad (12)$$

# Body forces on a differential element

The **Body Forces** act over the entire volume, and it is fair to assume that the force is constant over the differential volume:

$$d\vec{F}_b = \rho \vec{g} dV \quad \text{or,} \quad dF_{bi} = \rho g_i dV. \quad (13)$$

For the specific case of gravity, assuming  $\hat{k}$  points vertically upward,

$$\vec{g} = -g \hat{k}, \quad (14)$$

with  $g$  being the gravitational constant.

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# Cauchy's Equation of Motion

Substituting equations 11 & 12 into equation 4, and integrating the differential forces over the control volume  $\mathcal{V}$ , one finds that,

$$\int_{\mathcal{V}} \rho \frac{Du_i}{Dt} d\mathcal{V} = \int_{\mathcal{V}} (dF_{s_i} + dF_{B_i}) = \int_{\mathcal{V}} \left( \frac{\partial \tau_{ij}}{\partial x_j} + \rho g_i \right) d\mathcal{V}. \quad (15)$$

But this should hold for any arbitrary volume; therefore,

$$\implies \rho \frac{Du_i}{Dt} = \frac{\partial \tau_{ij}}{\partial x_j} + \rho g_i. \quad (16)$$

# Cauchy's Equation of Motion (continued ...)

At this point, using the definition of the material derivative, one obtains

$$\underbrace{\rho \frac{\partial u_i}{\partial t}}_{I} + \underbrace{\rho u_j \frac{\partial u_i}{\partial x_j}}_{II} = \underbrace{\frac{\partial \tau_{ij}}{\partial x_j}}_{III} + \underbrace{\rho g_i}_{IV}. \quad (17)$$

- Term I: represents the local rate of change of momentum at a fixed point in space.
- Term II: advection of momentum accounts for the fact that the momentum of a fluid particle might be changing as it passes through a fixed point in space (apparent acceleration).
- Term III: Stress gradient causes a net surface force on the fluid particle occupying the fixed point in space at time  $t$ .
- Term IV: body force acting on the fluid particle occupying the fixed point in space at time  $t$ .

# Cauchy's Equation of Motion (continued ...)

## Note:

The ultimate goal in fluid mechanics is to know the velocity everywhere in the domain at all times. Therefore, one needs to solve Cauchy's equations everywhere for  $\vec{u}(\vec{x}, t)$ . How is this done?

- We know that we need boundary conditions and possibly an initial condition if the flow is unsteady.
- We also notice that there are actually 9 unknowns in equation 15: 3-velocity components and 6-stress components. However, we only have 4 equations: 3-momentum equations, and 1 conservation of mass equations. Therefore, we need a relationship between  $\tau_{ij}$  and  $u_i$ .  $\implies$  One needs a **Constitutive law**.

# Boundary Conditions

Consider the situation illustrated in Figure 3, where one has medium 1 (fluid) and medium 2 (solid or liquid immiscible with fluid 1) separated by an interface. Here,

$$d\vec{A}_1 = dA_1 \hat{n} \quad \text{and} \quad d\vec{A}_2 = dA_2 \hat{n}, \quad (18)$$

To conserve mass across the interface, as  $l \rightarrow 0$ , one must satisfy (assuming the interface to be stationary) that,

$$\rho_1 \vec{u} \cdot \hat{n} = \rho_2 \vec{u} \cdot \hat{n} \quad (19)$$

at each point along the interface. If medium 2 is solid, then  $\vec{u}_2 = 0$  and we have that

$$\vec{u}_1 \cdot \hat{n} = 0. \quad (20)$$

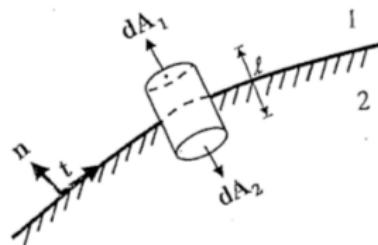


Figure: Illustration of the boundary in a fluid element.

# Boundary Conditions (continued ...)

- Also, as one allows the volume to go to zero,  $\Rightarrow$  the contribution from the body force will go to zero (since body forces act over the volume).
- Therefore, we are left with the balance of stresses, assuming surface tension maybe neglected, *i.e.*

$$n_i \tau_{ij} \quad \text{is continuous across interface} \quad (21)$$

The last boundary condition we will consider is the no-slip boundary condition, which states that the velocity component of the fluid tangential to a stationary solid is zero,

$$\vec{u}_1 \cdot \hat{t} = 0 \quad (\text{along the surface}). \quad (22)$$

# Boundary Conditions (continued ...)

## Comments on the no-slip condition:

Violations of the no-slip condition occur in several cases:

- (i) superfluid helium at or below 2.17 K has an essentially zero viscosity.
- (ii) when water-based fluids flow over superhydrophobic (strongly water repellent) coated surfaces, the water appears to slip over the surface.
- (iii) slip may be exhibited in highly rarefied gases, where in the mean distance between intermolecular collisions become of the same order of magnitude of the length scales of interest in the flow. In this case, the continuum hypothesis becomes suspect.