Intermediate Fluid Mechanics

Lecture 6: Vorticity

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Chapter Overview

Lecture Objectives

In the previous lecture, the strain rate of a fluid element was analyzed. This includes

- the normal strain rate, which tells how fast an individual line element is elongating or contracting.
- the shear-strain rate, which tells how fast the angle between two initially perpendicular line elements decreases.

In this lecture we will consider the rate of angular rotation of two line elements about an axis that is perpendicular to the plane containing the two line elements.

Chapter Overview

Vorticity

Vorticity ($\vec{\omega}$) is defined as twice the angular velocity of a fluid element.

The average rotation rate of the two lines $ab(AB)$ and $ac(AC)$ about the x_3-axis is:

$$\frac{1}{2}\left(\frac{d\beta}{dt} - \frac{d\alpha}{dt}\right) \tag{1}$$

.

Since the vorticity is defined as twice the average rotation rate, the component of $\vec{\omega}$ in the x_3-direction is

$$\omega_3 = \left(\frac{d\beta}{dt} - \frac{d\alpha}{dt}\right). \tag{2}$$

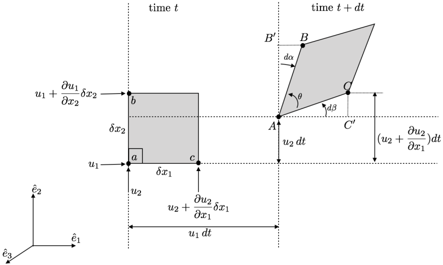

Figure: *Deformation of fluid element.*

Vorticity (continued ...)

Substituting the expressions for $d\beta$ and $d\alpha$, as obtained in lecture 4,

$$\omega_3 = \frac{1}{dt}\left[\left(\frac{\partial u_2}{\partial x_1}\delta x_1\right) dt \left(\frac{1}{\delta x_1}\right) - \left(\frac{\partial u_1}{\partial x_2}\delta x_2\right) dt \left(\frac{1}{\delta x_2}\right)\right. \tag{3}$$

$$= \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}. \tag{4}$$

Similarly for the other two directions,

$$\omega_1 = \frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}, \tag{5}$$

$$\omega_2 = \frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}. \tag{6}$$

Hence, Vorticity can be written compactly as,

$$\boxed{\vec{\omega} = \vec{\nabla} \times \vec{u}} \tag{7}$$

Vorticity (continued ...)

In index notation, Vorticity can alternatively be written as,

$$\omega_i = \varepsilon_{ijk} \frac{\partial u_k}{\partial x_j}. \tag{8}$$

Note:

- The vorticity of a fluid can be interpreted as the net swirl around a differential contour bounding a differential surface.
- A non-zero swirl means that as one moves around the closed path, the net sum of the velocity is non-zero.
- This will cause the fluid element to rotate or spin about the normal axis to the enclosed surface.

Chapter Overview

Irrotational Fluid

A flow is called irrotational if

$$\vec{\omega} = 0 \tag{9}$$

which also corresponds to:

$$\frac{\partial u_i}{\partial x_j} = \frac{\partial u_j}{\partial x_i} \text{ for } i \neq j. \tag{10}$$

Circulation

Circulation is defined as the line integral of the tangential component of velocity around a closed contour $\mathcal{L}$,

Using Stoke's theorem

$$\underbrace{\oint_{\mathcal{L}} \vec{u} \cdot d\vec{l}}_{I} = \underbrace{\int_{\mathcal{S}} \left(\vec{\nabla} \times \vec{u}\right) \cdot d\vec{\mathcal{S}}}_{II}, \qquad (11)$$

Circulation can alternatively be given in relation to the vorticity of the flow,

$$\Gamma = \int_{\mathcal{S}} \vec{\omega} \cdot d\vec{\mathcal{S}}, \qquad (12)$$

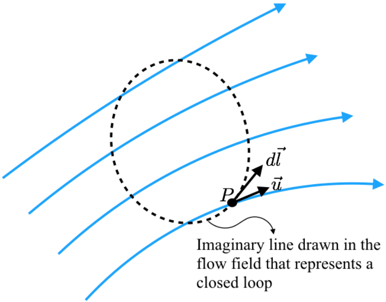

Figure: *Figure illustrating the concept of Circulation.*

where

- Term I: is the circulation around a closed contour $\mathcal{L}$.
- Term II: is the flux of vorticity through the surface $\mathcal{S}$ bounded by $\mathcal{L}$.

Chapter Overview

Relative Motion near a Point

The relative motion between two neighboring points in the flow can be written as the sum of two contributions:

- (i) motion due to local rotation,
- (ii) motion due to local deformation.

Lets now consider the velocity at point P, which is at a small distance $d\vec{x}$ away from point $\mathcal{O}$,

- The velocities at P and $\mathcal{O}$ are different.
- The relative velocity at time t between the two points is $d\vec{u}$ (or du_i).

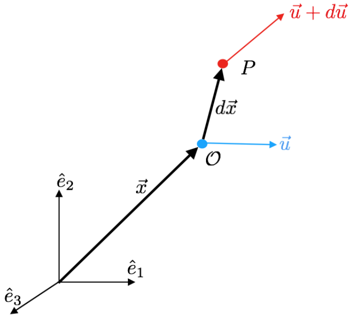

Figure: Representation of two fluid elements in relative movement from each other.

Relative Motion near a Point (continued ...)

Using Taylor's series expansion,

$$du_i = \frac{\partial u_i}{\partial x_j} dx_j, \tag{13}$$

or writing out the summation for $i = 1$,

$$du_1 = \frac{\partial u_1}{\partial x_1} dx_1 + \frac{\partial u_1}{\partial x_2} dx_2 + \frac{\partial u_1}{\partial x_3} dx_3. \tag{14}$$

The quantity $\frac{\partial u_i}{\partial x_j}$ is called the velocity gradient tensor, and it can be rewritten as,

$$\frac{\partial u_i}{\partial x_j} = \underbrace{\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)}_{\text{Strain-Rate tensor, } e_{ij}} + \underbrace{\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)}_{\text{Spin/Rotation tensor, } r_{ij}}, \tag{15}$$

Relative Motion near a Point (continued ...)

Note that, **the spin or rotation tensor** is an anti-symmetric tensor

$$r_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right), \tag{16}$$

Also Note that **the strain-rate tensor** (e_{ij}) is a symmetric tensor,

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \tag{17}$$

Relative Motion near a Point (continued ...)

The **spin or rotation tensor** is related to the vorticity based on their corresponding definitions through,

$$r_{ij} = -\frac{1}{2}\varepsilon_{ijk}\,\omega_k. \tag{18}$$

Since the vorticity represents twice the angular rotation rate of a fluid particle, the components of the spin tensor represent the actual angular rotation rate.

(That is why the factor $1/2$ is preferred in the definition of r_{ij}.)

Demonstration of the relationship between r_{ij} and ω_k

Let's substitute the definition of ω_k in the relation between r_{ij} and ω_k,

$$r_{ij} = -\frac{1}{2}\varepsilon_{ijk}\omega_k = -\frac{1}{2}\varepsilon_{ijk}\left(\varepsilon_{klm}\frac{\partial u_m}{\partial x_l}\right) \tag{19}$$

Now, one can use the epsilon-delta relation from last lecture,

$$\varepsilon_{ijk}\varepsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl} \tag{20}$$

and substitute into the expression for r_{ij},

$$r_{ij} = -\frac{1}{2}\left(\delta_{il}\delta_{jm}\frac{\partial u_m}{\partial x_l}\right) + \frac{1}{2}\left(\delta_{im}\delta_{jl}\frac{\partial u_m}{\partial x_l}\right). \tag{21}$$

Using the properties of the kronecker delta,

$$r_{ij} = -\frac{1}{2}\frac{\partial u_j}{\partial x_i} + \frac{1}{2}\frac{\partial u_i}{\partial x_j} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right) \tag{22}$$

which corresponds with the original definition given for r_{ij}.

Demonstration (continued ...)

Therefore, at this point one can rewrite equation 15 as

$$du_i = \underbrace{e_{ij}\, dx_j}_{\text{Term I}} - \underbrace{\varepsilon_{ijk}\, \omega_k\, dx_j}_{\text{Term II}}, \tag{23}$$

where

- Term I: represents the contribution due to stretching and deformation.
- Term II: represents the contribution due to rotation.

Demonstration (continued ...)

- This can be further illustrated for an infinitessimal sphere that is deformed and rotated into an ellipsoid.
- In general, the relative velocity in the $\{\hat{e}_1, \hat{e}_2, \hat{e}_3\}$ coordinate system is,

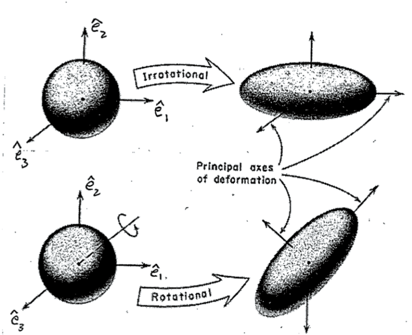

Figure: Rotation and deformation of a spheric fluid element.

$$\begin{bmatrix} du_1 \\ du_2 \\ du_3 \end{bmatrix} = \begin{bmatrix} e_{11} & e_{12} - \omega_3 & e_{13} + \omega_2 \\ e_{21} + \omega_3 & e_{22} & e_{23} - \omega_1 \\ e_{31} - \omega_2 & e_{32} + \omega_1 & e_{33} \end{bmatrix} \begin{bmatrix} dx_1 \\ dx_2 \\ dx_3 \end{bmatrix} \tag{24}$$