ME EN 5700/6700 Homework #3 Fall 2015

1.) $u = -4U_{max}\left[\left(\frac{y}{h}\right)^2 - \frac{y}{h}\right]$, $v = 0$

a.) $e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$

$e_{i3} = e_{3j} =$

$e_{11} = \frac{1}{2}\left[\frac{\partial u}{\partial x} + \frac{\partial u}{\partial x}\right] = 0 \qquad e_{12} = \frac{1}{2}\left[\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right] = \frac{1}{2}\left[-4U_{max}\left(\frac{2y}{h^2} - \frac{1}{h}\right) + 0\right]$

$e_{21} = \frac{1}{2}\left[\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right] = \frac{1}{2}\left[0 + -4U_{max}\left(\frac{2y}{h^2} - \frac{1}{h}\right)\right] \qquad e_{22} = \frac{1}{2}\left[\frac{\partial v}{\partial y} + \frac{\partial v}{\partial y}\right] = 0$

$$\boxed{e_{21} = e_{12} = -2U_{max}\left(\frac{2y}{h^2} - \frac{1}{h}\right)}$$

b.) $\omega_i = \varepsilon_{ijk}\frac{\partial u_j}{\partial x_i} \qquad \omega_1 = \frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3} = \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} = 0$

$\omega_2 = \frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1} = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} = 0$

$\omega_3 = \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \left[0 - \left(-4U_{max}\left(\frac{2y}{h^2} - \frac{1}{h}\right)\right)\right]$

$$\boxed{= 4U_{max}\left[\left(\frac{2y}{h^2}\right) - \left(\frac{1}{h}\right)\right]}$$

c.) for incompressible flow, $\frac{\partial u_i}{\partial x_i} = 0 \Rightarrow \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$

since $u = f(y)$ (1), $v = 0$ (2), and 2D flow (3),

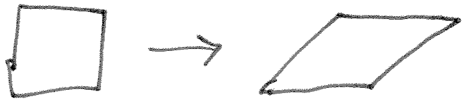
$\frac{\partial u_i}{\partial x_i} = \cancel{\frac{\partial u}{\partial x}}^{0\,(1)} + \cancel{\frac{\partial v}{\partial y}}^{0\,(2)} + \cancel{\frac{\partial w}{\partial z}}^{0\,(3)} \Rightarrow 0$ $\boxed{\text{Flow is incompressible}}$

d.) for $y/h = 0.5$ we have $e_{12} = e_{21} = -2U_{max}h\left[2\left(\frac{1}{2}\right) - 1\right] = 0$
and $\omega_3 = 4U_{max}h\left[2\left(\frac{1}{2}\right) - 1\right] = 0$, so we only have translation

A A'

1. cont.

e.) for $y/h = 0.25$ we will have non-zero strain-rate and vorticity



B $\rightarrow$ B'

2.) $u = 3x + y \quad v = 2x - 3y$

$$\Gamma = \int_A \vec{\omega} \cdot \partial \vec{A}, \quad (x-1)^2 + (y-6)^2 = 4$$

vorticity: $\vec{\omega} = \vec{\nabla} \times \vec{u}$

$$\omega_1 = \frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3} = \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} = 0 \; ; \quad \omega_2 = \frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1} = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} = 0$$

$$\omega_3 = \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 2 - 1 = 1 \qquad \vec{\omega} = 0\hat{i} + 0\hat{j} + 1\hat{k}$$

eqn of a circle: $(x-h)^2 + (y-k)^2 = r^2$, so $r = 2$

$$F = \int \vec{\omega}\, \partial A = \iint \vec{\omega} \cdot r \, \partial r \, \partial \theta \Rightarrow \int_0^{2\pi} \int_0^r 1 \cdot r \, \partial r \, \partial \theta$$

$$\int_0^{2\pi} 2 \, \partial \theta = \boxed{4\pi}$$

3.) $$\frac{D}{Dt} \int_{\forall(t)} F \, \partial V = \int_{\forall} \left[\frac{\partial F}{\partial t} + \frac{\partial (F u_j)}{\partial x_j} \right] \partial \forall \quad \text{if } f = F/\rho \quad F = f\rho$$

L.H.S. $\Rightarrow \dfrac{D}{Dt} \displaystyle\int_{\forall(t)} \rho f \, \partial \forall$ ✓

$$\text{RHS} \Rightarrow \int_{\forall} \left[\frac{\partial (\rho f)}{\partial t} + \frac{\partial (\rho f u_j)}{\partial x_j} \right] \partial \forall$$

$$\int_{\forall} \left[\rho \frac{\partial f}{\partial t} + f \frac{\partial \rho}{\partial t} + \rho f \frac{\partial u_j}{\partial x_j} + \rho u_j \frac{\partial f}{\partial x_j} + f u_j \frac{\partial \rho}{\partial x_j} \right] \partial \forall$$

$$\underbrace{\int_{\forall} \rho \left[\frac{\partial f}{\partial t} + u_j \frac{\partial f}{\partial x_j} \right] \partial \forall}_{\text{Use Material derivative Definition}} + \underbrace{\int_{\forall} f \left[\frac{\partial \rho}{\partial t} + \rho \frac{\partial u_j}{\partial x_j} + u_j \frac{\partial f}{\partial x_j} \right] \partial \forall}_{\text{by continuity} = 0}$$

LHS = RHS $\quad \dfrac{D}{Dt} \displaystyle\int_{\forall(t)} \rho f \, \partial \forall = \int_{\forall} \rho \frac{Df}{Dt} \, \partial \forall$

4.) $u(x,t)=u$, $0=v$, $0=w$, $u(0,t)=U$

$\rho=\rho_0(2-\cos(\omega t))$

Mass conservation: $\frac{D\rho}{Dt}+\rho\frac{\partial u_i}{\partial x_i}=\frac{\partial \rho}{\partial t}+\cancel{u_i\frac{\partial \rho}{\partial x_i}}^{\,0}+\rho\frac{\partial u_i}{\partial x_i}=0$

$$\frac{\partial \rho}{\partial t}+\rho\frac{\partial u_i}{\partial x_i}=\frac{\partial \rho}{\partial t}+\rho\frac{\partial u}{\partial x}+\cancel{\rho\frac{\partial v}{\partial y}}^{\,0}+\cancel{\rho\frac{\partial w}{\partial z}}^{\,0}=0 \Longrightarrow \frac{\partial \rho}{\partial t}+\rho\frac{\partial u}{\partial x}=0$$

$$\Longrightarrow \frac{\partial \rho}{\partial t}=\rho_0\omega\sin(\omega t)$$

$$\rho_0\omega\sin(\omega t)=-\rho\frac{\partial u}{\partial x} \Rightarrow \rho_0\omega\sin(\omega t)=-\rho_0(2-\cos(\omega t))\frac{\partial u}{\partial x}$$

$$\int\rho_0\omega\sin(\omega t)\,\partial x=\int\rho_0(\cos(\omega t)-2)\,\partial u$$

$$\int\frac{\rho_0\omega\sin(\omega t)}{\rho_0(\cos(\omega t)-2)}\,\partial x=\int\partial u$$

$$\frac{\omega\sin(\omega t)}{\cos(\omega t)-2}\cdot x+C_1=u(x,t)+C_2$$

$$\frac{\omega\sin(\omega t)}{\cos(\omega t)-2}\cdot x=u(x,t)+C_3$$

Use initial condition: $u(0,t)=U$

$$\frac{\omega\sin(\omega t)}{\cos(\omega t)-2}\cdot 0=u(0,t)+C_3$$

$$0=U+C_3$$

$$C_3=U$$

$$\boxed{u(x,t)=\frac{\omega\sin(\omega t)}{\cos(\omega t)-2}\cdot x+U}$$