Homework 2
(due Thurs, Jan 25)

1) The state of stress at a point relative to an x, y, z coordinate system is given by

$$\begin{bmatrix} 12 & 4 & 2 \\ 4 & -8 & -1 \\ 2 & -1 & 6 \end{bmatrix} \text{MPa}$$

Calculate the maximum shearing stress at the point.

Referring to Appendix B:

$$\sigma_1 = 13.212\ MPa \qquad \sigma_2 = 5.684\ MPa \qquad \sigma_3 = -8.896\ MPa$$

and

$$l_1 = 0.9556 \qquad m_1 = 0.1688 \qquad n_1 = 0.2416$$

Thus,

$$\tau_{max} = \tfrac{1}{2}(\sigma_1 - \sigma_3) = 11.054\ MPa \qquad \blacktriangleleft$$

2) At a point in a loaded member, the stresses relative to an x, y, z coordinate system are given by

$$\begin{bmatrix} 60 & 20 & 10 \\ 20 & -40 & -5 \\ 10 & -5 & 30 \end{bmatrix} \text{MPa}$$

Calculate the magnitude and direction of maximum principal stress. Give the direction as the components of a unit vector.

Referring to Appendix B:

$$\sigma_1 = 66.016\ MPa \qquad \sigma_2 = 28.418\ MPa \qquad \sigma_3 = -44.479\ MPa \qquad \blacktriangleleft$$

and

$$l_1 = 0.9556 \qquad m_1 = 0.1688 \qquad n_1 = 0.2416$$

3) The stresses (in MPa) with respect to an x, y, z coordinate system are described by

$$\sigma_x = x^2 + y, \qquad \sigma_z = -x + 6y + z$$
$$\sigma_y = y^2 - 5, \qquad \tau_{xy} = \tau_{xz} = \tau_{yz} = 0$$

At point (3, 1, 5), determine (a) the stress components with respect to x', y', z' if

$$l_1 = 1, \qquad m_2 = \frac{1}{2}, \qquad n_2 = \frac{\sqrt{3}}{2}, \qquad n_3 = \frac{1}{2}, \qquad m_3 = -\frac{\sqrt{3}}{2}$$

and (b) the stress components with respect to x'', y'', z'' if

$$l_1 = \frac{2}{\sqrt{5}}, \qquad m_1 = -\frac{1}{\sqrt{5}}, \qquad n_3 = 1$$

(c) Also show that the three invariants of the stress tensor, defined by Eq. (1.34), are, indeed, invariant under the transformations described for (a) and (b).

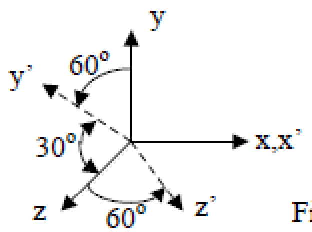


Figure (a)

(a) At point (3,1,5) with respect to xyz axis, we have $[\tau_{ij}]$:

$$\begin{bmatrix} 10 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 8 \end{bmatrix} MPa \qquad \text{(a)}$$

Then, Eqs. (1.34) result in

$$I_1 = 14\ MPa \qquad I_2 = 8\ (MPa)^2 \qquad I_3 = -320\ (MPa)^3 \qquad \blacktriangleleft$$

Direction cosines of x' y' z', referring to Fig. (a) are

$$l_1 = 1 \quad m_1 = 0 \quad n_1 = 0$$
$$l_2 = 0 \quad m_2 = 1/2 \quad n_2 = \sqrt{3}/2$$
$$l_3 = 0 \quad m_3 = -\sqrt{3}/2 \quad n_3 = 1/2$$

Now Eqs. (1.28) and (a) give $[\tau_{i'j'}]$:

$$\begin{bmatrix} 10 & 0 & 0 \\ 0 & 5 & 3\sqrt{3} \\ 0 & 3\sqrt{3} & -1 \end{bmatrix} MPa \qquad \blacktriangleleft$$

Thus, Eqs. (1.34) yield

$$I_1' = 14\ MPa \qquad I_2' = 8\ (MPa)^2 \qquad I_3' = -320\ (MPa)^3 \qquad \blacktriangleleft$$

as before.

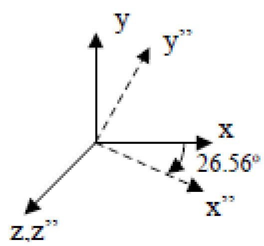


Figure (b)

(b) Direction cosines are (Fig. b):

$$\begin{array}{lll} l_1 = 2/\sqrt{5} & m_1 = -1/\sqrt{5} & n_1 = 0 \\ l_2 = 1/\sqrt{5} & m_2 = 2/\sqrt{5} & n_2 = 0 \\ l_3 = 0 & m_3 = 0 & n_3 = 1 \end{array}$$

With these and Eq. (a), Eqs. (1.28) yield $[\tau_{i'j'}]$:

$$\begin{bmatrix} 7.2 & 5.6 & 0 \\ 5.6 & -1.2 & 0 \\ 0 & 0 & 8 \end{bmatrix} MPa \qquad \blacktriangleleft$$

Thus, Eqs. (1.34) result in

$$I_1'' = 14\ MPa \qquad I_2'' = 8\ (MPa)^2 \qquad I_3'' = -320\ (MPa)^3 \qquad \blacktriangleleft$$

The I's are thus invariants.

4) At a point in a loaded body, the stresses relative to an x, y, z coordinate system are

$$\begin{bmatrix} 40 & 40 & 30 \\ 40 & 20 & 0 \\ 30 & 0 & 20 \end{bmatrix} \text{MPa}$$

Determine the normal stress σ and the shearing stress τ on a plane whose outward normal is oriented at angles of 40°, 75°, and 54° with the x, y, and z axes, respectively.

Direction cosines are

$$l = \cos 40^o = 0.766 \qquad m = \cos 75^o = 0.259 \qquad n = \cos 54^o = 0.588$$

Equation (1.40):

$$\begin{aligned} \sigma &= 40(0.766)^2 + 20(0.259)^2 + 20(0.588)^2 \\ &\quad + 2[40(0.766)(0.259) + 0 + 30(0.766)(0.588)] \\ &= 23.47 + 1.34 + 6.91 + 42.9 \\ &= 74.62 \ MPa \end{aligned} \qquad ◀$$

Equation (1.41) gives

$$\begin{aligned} \tau &= \{[40(0.766) + 40(0.259) - 30(0.588)]^2 \\ &\quad + [40(0.766) + 20(0.259) + 0]^2 + [30(0.766) + 0 + 20(0.588)]^2 - 74.62^2\}^{\frac{1}{2}} \\ &= [3436.3 + 1282.9 + 1206.7 - 5568.1]^{\frac{1}{2}} \\ &= 18.93 \ MPa \end{aligned} \qquad ◀$$

5) A displacement field in a body is given by

$$u = c(x^2 + 10)$$
$$v = 2cyz$$
$$w = c(-xy + z^2)$$

where $c = 10^{-4}$. Determine the state of strain on an element positioned at $(0, 2, 1)$.

Equations (2.4), for the given displacement field, yield $[\varepsilon_{ij}]$:

$$\begin{bmatrix} 2x & 0 & -y/2 \\ 0 & 2z & (2y-x)/2 \\ -y/2 & (2y-x)/2 & 2z \end{bmatrix} c$$

At point (0,2,1), we have $[\varepsilon_{ij}]$:

$$\begin{bmatrix} 0 & 0 & -100 \\ 0 & 200 & 200 \\ -100 & 200 & 200 \end{bmatrix} \mu$$

6) The plane displacement field and shear strain in a member have the form

$$u = a_0x^2y^2 + a_1xy^2 + a_2x^2y$$
$$v = b_0x^2y + b_1xy$$
$$\gamma_{xy} = c_0x^2y + c_1xy + c_2x^2 + c_3y^2$$

Determine the expressions for c_i (in terms of a_i and b_i) that must be satisfied for the given displacements and strain to be compatible.

First two of Eqs. (2.4) give

$$\varepsilon_x = 2a_0xy^2 + a_1y^2 + 2a_2xy$$
$$\varepsilon_y = b_0x^2 + b_1x$$

Equation (2.11): x

$$(4a_0 + 2a_1) + (2b_0) = 2c_0x + c_1$$

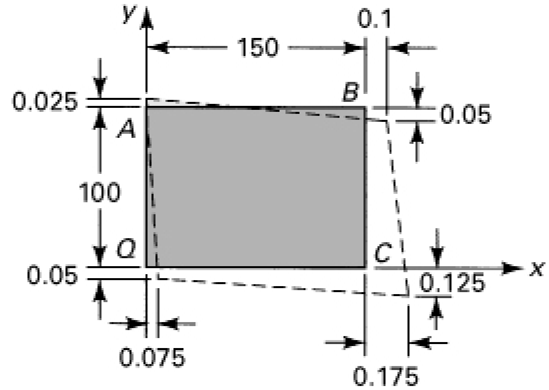
or

$$2(2a_0 - c_0)x + 2(a_1 + b_0) - c_1 = 0$$

This is satisfied if $x \neq 0$:

$$2a_0 - c_0 = 0, \qquad c_0 = 2a_0$$
$$2(a_1 + b_0) - c_1 = 0, \qquad c_1 = 2(a_1 + b_0)$$

7) A 100 mm by 150 mm rectangular plate $QABC$ is deformed into the shape shown by the dashed lines. All dimensions shown in the figure are in millimeters. Determine at point Q (a) the strain components ε_x, ε_y, γ_{xy}, and (b) the principal strains and the direction of the principal axes.



(a) Equations (2.4) give

$$\varepsilon_x = \frac{\partial u}{\partial x} = \frac{0.175-0.075}{150} = 667\ \mu$$

$$\varepsilon_y = \frac{\partial v}{\partial y} = \frac{0.025-(-0.05)}{100} = 750\ \mu$$

and

$$\gamma_{xy} = \frac{0-0.075}{100} + \frac{[-0.125-(-0.05)]}{150} = -1250\ \mu$$

(b) Equation (2.16) is therefore

$$\varepsilon_{1,2} = \frac{667+750}{2} \pm [(\frac{667-750}{2})^2 + 625^2]^{\frac{1}{2}}$$

or

$$\varepsilon_1 = 1335\ \mu \qquad \varepsilon_2 = 82\ \mu$$

When $\theta_p = \frac{1}{2}\tan^{-1}\frac{-1250}{667-750} = 43.1^o$

and stresses are substituted into Eq. (2.14a), we obtain $\varepsilon_{x'} = 82\ \mu$. Thus,

$$\theta_p'' = 43.1^o$$

8) At a point in a stressed body, the tensorial strains, related to the coordinate set xyz, are given by

$$\begin{bmatrix} 200 & 300 & 200 \\ 300 & -100 & 500 \\ 200 & 500 & -400 \end{bmatrix} \mu$$

Determine (a) the strain invariants; (b) the normal strain in the x' direction, which is directed at an angle $\theta = 30°$ from the x axis (in the x-y plane); (c) the principal strains ε_1, ε_2, and ε_3; and (d) the maximum shear strain.

(a) Applying Eqs. (2.21),

$$J_1 = 200 - 100 - 400 = -300\ \mu$$

$$J_2 = (-2 - 8 + 4 - 9 - 4 - 25)(10^4) = -44(10^4)\ (\mu)^2 \quad ◀$$

and

$$J_3 = \begin{vmatrix} 200 & 300 & 200 \\ 300 & -100 & 500 \\ 200 & 500 & -400 \end{vmatrix} = 58(10^6)\ (\mu)^3 \quad ◀$$

(b) Table of direction cosines:

	x	y	z
x'	$\sqrt{3}/2$	$1/2$	0
y'	$-1/2$	$\sqrt{3}/2$	0
z'	0	0	1

Thus, using Eqs. (2.18a),

$$\varepsilon_{x'} = \varepsilon_x l_1^2 + \varepsilon_y m_1^2 + \gamma_{xy} l_1 m_1$$

$$= 200(\tfrac{\sqrt{3}}{2})^2 - 100(\tfrac{1}{2})^2 + 600(\tfrac{\sqrt{3}}{2})(\tfrac{1}{2}) = 385\ \mu \quad ◀$$

(c) Use Table B.1 (with $\sigma \to \varepsilon$ and $\tau \to \gamma/2$):

$$\varepsilon_1 = 598\ \mu \qquad \varepsilon_2 = -126\ \mu \qquad \varepsilon_3 = -772\ \mu \quad ◀$$

(d)

$$\gamma_{\max} = 598 + 772 = 1370\ \mu \quad ◀$$