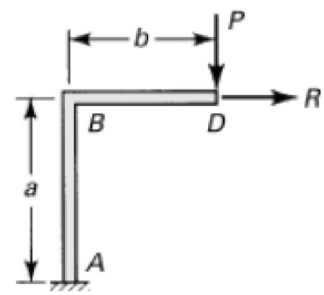
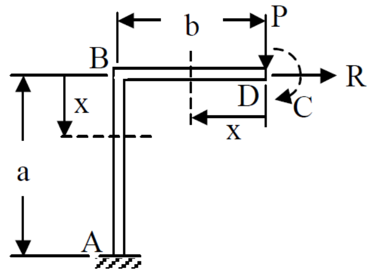
Homework 12 Solutions

1) A cantilever beam of constant AE and EI is loaded as shown. Determine the vertical and horizontal deflections and the angular rotation of the free end, considering the effects of normal force and bending moment. Employ Castigliano's theorem.





<u>Defections at point D</u>:
We write

$$M_{DB} = Px \qquad M_{BA} = Pb + Rx$$

Using Eq. (10.5),

$$U_{DB} = \int_0^b \left[\frac{(Px)^2}{2EI} + \frac{R^2}{2AE}\right]dx = \frac{P^2b^3}{6EI} + \frac{R^2 b}{2AE}$$

$$U_{BA} = \int_0^a \left[\frac{(Pb+Rx)^2}{2EI} + \frac{P^2}{2AE}\right]dx$$

$$= \frac{1}{2EI}\left[p^2ab^2 + PRa^2b + \tfrac{1}{3}R^2a^3\right] + \frac{P^2a}{2AE}$$

Total strain energy $U = U_{BD} + U_{BA}$. Vertical deflection at D is thus

$$\delta_v = \frac{\partial U}{\partial P} = \frac{1}{EI}\left[\frac{Ra^2b}{2} + P(ab^2 + \frac{b^3}{3})\right] + \frac{Pa}{AE}$$

Horizontal deflection at D:

$$\delta_h = \frac{\partial U}{\partial R} = \frac{1}{EI}\left[\frac{\overset{P}{\cancel{R}}a^2b}{2} + \frac{Ra^3}{3}\right] + \frac{Rb}{AE}$$

<u>Angular rotation of D</u>:
Introduce a couple moment C at D as shown in the figure. Then,

$$M_{DB} = Px + C \qquad M_{BA} = Pb + Rx + C$$

and

$$U_{DB} = \int_0^b \left[\frac{(Px+C)^2}{2EI} + \frac{R^2}{2AE}\right]dx$$

$$= \frac{1}{2EI}\left[\frac{P^2b^3}{3} + C^2b + PCb^2\right] + \frac{R^2b}{2AE}$$

$$U_{BA} = \frac{1}{2EI}\int_0^a [Pb + Rx + C]^2\,dx + \frac{R^2b}{2AE}$$

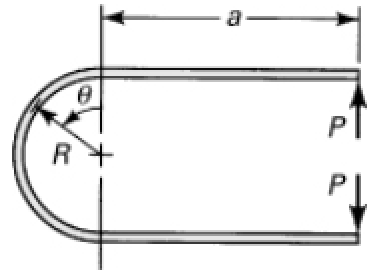
$$= \frac{1}{2EI}\left[P^2ab^2 + \frac{R^2a^3}{3} + Ca^2 + PRa^2b + 2PCab + CRa^2\right] + \frac{P^2a}{2AE}$$

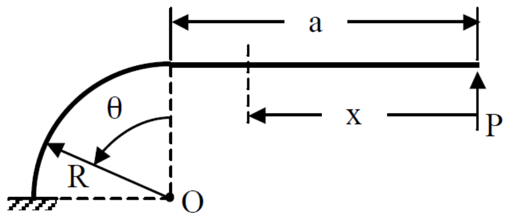
Hence,

$$\theta_D = \left.\frac{\partial U}{\partial C}\right|_{C=0} = \frac{1}{2EI}\left[Pb^2 + 2Pab + Ra^2\right]$$

Note that displacements of a simple (straight) cantilever beam may be readily found by setting b=0 in the foregoing results.

2) If a force P is applied to the steel spring (of uniform flexural rigidity) shown, determine the increase in the distance between the ends. Use Castigliano's theorem.





From symmetry,

$$M_1 = -Px \qquad 0 \le x \le a$$

$$M_2 = -Pa - PR\sin\theta \qquad 0 \le \theta \le \tfrac{\pi}{2}$$

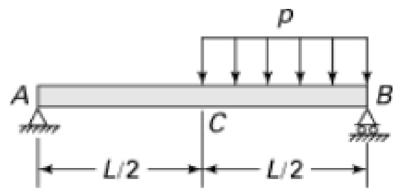
Hence, $\frac{\delta}{2} = \frac{1}{EI}\int_0^a M_1 \frac{\partial M_1}{\partial P} dx + \frac{1}{EI}\int_0^{\pi/2} M_2 \frac{\partial M_2}{\partial P} R d\theta$

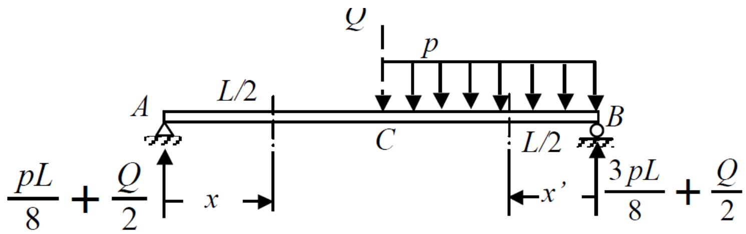
or

$$\delta = \frac{2}{EI}\left[\int_0^a Px^2 dx + \int_0^{\pi/2} P(a + R\sin\theta)^2 R d\theta\right]$$

$$= \frac{P}{6EI}(4a^3 + 6\pi Ra^2 + 24R^2 a + 3\pi R^3) \qquad \blacktriangleleft$$

3) A beam is loaded and supported as shown. Apply Castigliano's theorem to find the deflection at point C.





<u>Segment AC</u>

$$M_1 = \left(\frac{pL}{8}+\frac{Q}{2}\right)x \qquad \frac{\partial M_1}{\partial Q} = \frac{x}{2}$$

<u>Segment CB</u>

$$M_2 = \left(\frac{3pL}{8}+\frac{Q}{2}\right)x' - \frac{px'^2}{2} \qquad \frac{\partial M_2}{\partial Q} = \frac{x'}{2}$$

Let Q=0:

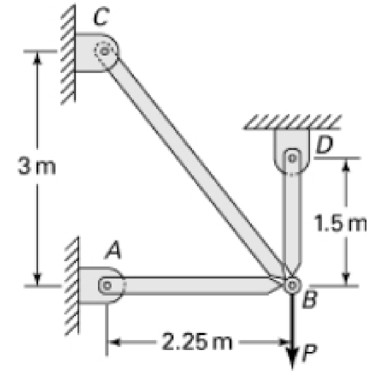
$$EIv_C = \int_0^{L/2} \frac{pLx}{8}\frac{x}{2}dx + \int_0^{L/2}\left(\frac{3pLx'}{8} - \frac{px'^2}{2}\right)\frac{x'}{2}dx'$$

$$= \cancel{p}\int_0^{L/2} \frac{pLx^2}{16}dx + \cancel{p}\int_0^{L/2}\left(\frac{3pLx'^2}{16} - \frac{px'^3}{4}\right)dx' = \frac{5pL^4}{768}$$

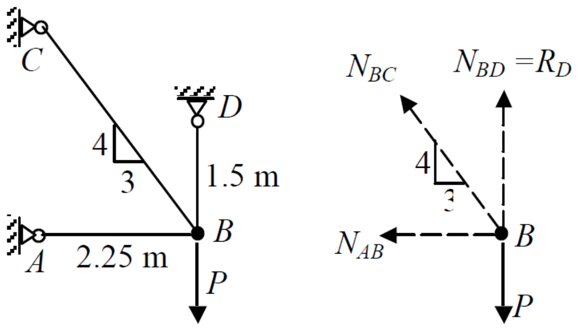
or $\quad v_C = \dfrac{5pL^4}{768EI} \downarrow$

◀

4) A load P is carried at joint B of a structure consisting of three bars of equal axial rigidity AE, as shown. Apply Castigliano's theorem to determine the force in each bar.



Consider R_D as redundant.



$$\frac{\partial N_{BD}}{\partial R_D} = 1$$

$$\sum F_y = 0:$$
$$N_{BC} = 1.25P - 1.25R_D$$

$$\sum F_x = 0: \quad N_{AB} = -0.75P + 0.75R_D$$

where $\quad \partial N_{BC}/\partial R_D = -1.25 \qquad \partial N_{AB}/\partial R_D = 0.75$

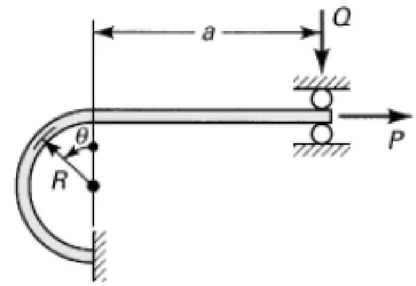
Thus,

$$AE\delta_D = \sum N_j L_j \frac{\partial N_j}{\partial R_D} = 0$$
$$= (1.25P - 1.25R_D)(7.5)(-1.25)$$
$$+(-0.75P + 0.75R_D)(4.5)(0.75) + R_D(3)1 = 0$$
$$= -14.25P + 17.25R_D = 0$$

or $\qquad R_D = 0.826P$

Then $\qquad N_{BD} = 0.826P \qquad N_{AB} = -0.131P \qquad N_{BC} = 0.22P$ ◀

5) A steel rod of constant flexural rigidity is shown. For force P applied at the simply supported end, derive a formula for roller reaction Q. Apply Castigliano's theorem.



We write

$$M_1 = Qx \qquad 0 \le x \le a$$

$$M_2 = Q(a + R\sin\theta) + PR(1-\cos\theta) \qquad 0 \le \theta \le \pi$$

Applying Eq. (10.6), with $\delta_Q = 0$:

$$\tfrac{\partial U}{\partial Q} = 0 = \tfrac{1}{EI}\int_0^a Qx^2 dx + \int_0^\pi [Q(a + R\sin\theta) + PR(1-\cos\theta)](a + R\sin\theta)Rd\theta$$

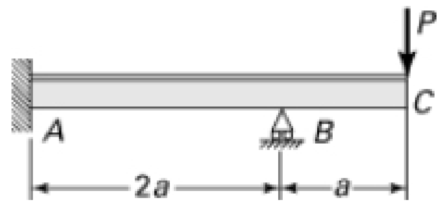
Integrating,

$$0 = \tfrac{QL^3}{3EI} + \tfrac{QR}{EI}(\pi a^2 + 4Ra + \tfrac{\pi}{2}R^2) + \tfrac{PR^2}{EI}(\pi a + 2R)$$

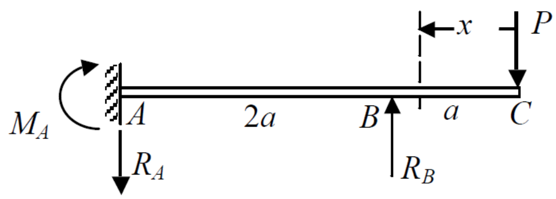
This may be written in the following form:

$$Q = \frac{-PR^3(\pi a + 2R)}{a^3[\frac{1}{3} + \frac{\pi R}{a} + 4(\frac{R}{a})^2 + \frac{\pi}{2}(\frac{R}{a})^3]} \quad \blacktriangleleft$$

6) A beam is supported and loaded as shown. Use Castigliano's theorem to determine the reactions.



Consider R_B as redundant.



Segment BC:

$$M_1 = -Px \qquad \partial M_1/\partial R_B = 0$$

Segment AB:

$$M_2 = -Px + R_B(x-a) \qquad \partial M_2/\partial R_B = x - a$$

Thus,

$$EIv_B = 0 = \int_0^a (-Px)(0)dx + \int_a^{3a} [-Px + R_B(x-a)](x-a)dx$$

$$= \int_a^{3a} [-Px^2 + R_B x^2 - 2R_B ax + Pax + R_B a^2]dx$$

$$= \left| -\frac{Px^3}{3} + \frac{R_B x^3}{3} - R_B ax^2 + \frac{Pax^2}{2} + R_B xa^2 \right|_a^{3a}$$

$$= -\frac{14}{3}P + \frac{3}{8}R_B$$

from which

$$R_B = \frac{7}{4}P \uparrow$$ ◀

Statics:

$$M_A = \frac{1}{2}Pa \circlearrowright \qquad R_A = \frac{3}{4}P \downarrow$$ ◀