ME 5200/6200 and ECE 5615/6615 Classical Controls

Homework 04 Solutions

Do the following problems and show all your work for full credit. Note: not all problems will be graded, but you must complete all problems to get full credit.

Problem 1

Consider the two systems shown below, where each system has a zero.

(a) Use Matlab to plot the step responses for both systems. Label all axes and provide a hardcopy of your Matlab code that you used to generate the step response.

(b) Compare and contrast the step responses. Note that both systems have the same poles, but the zeros are different. How does this affect the response? (Hint – You may want to use partial fraction expansion to explain the output response when the input is a step. One of the systems is nonminimum phase – briefly describe what this behavior is. Refer to your text for additional details as well as the description in the notes).

$$G_1(s) = \frac{s+2}{s^2+3s+2};\ G_2(s) = \frac{-s+2}{s^2+3s+2}$$

(a)
See attached.

(b) if we look at each of the responses using L.T. expansion:

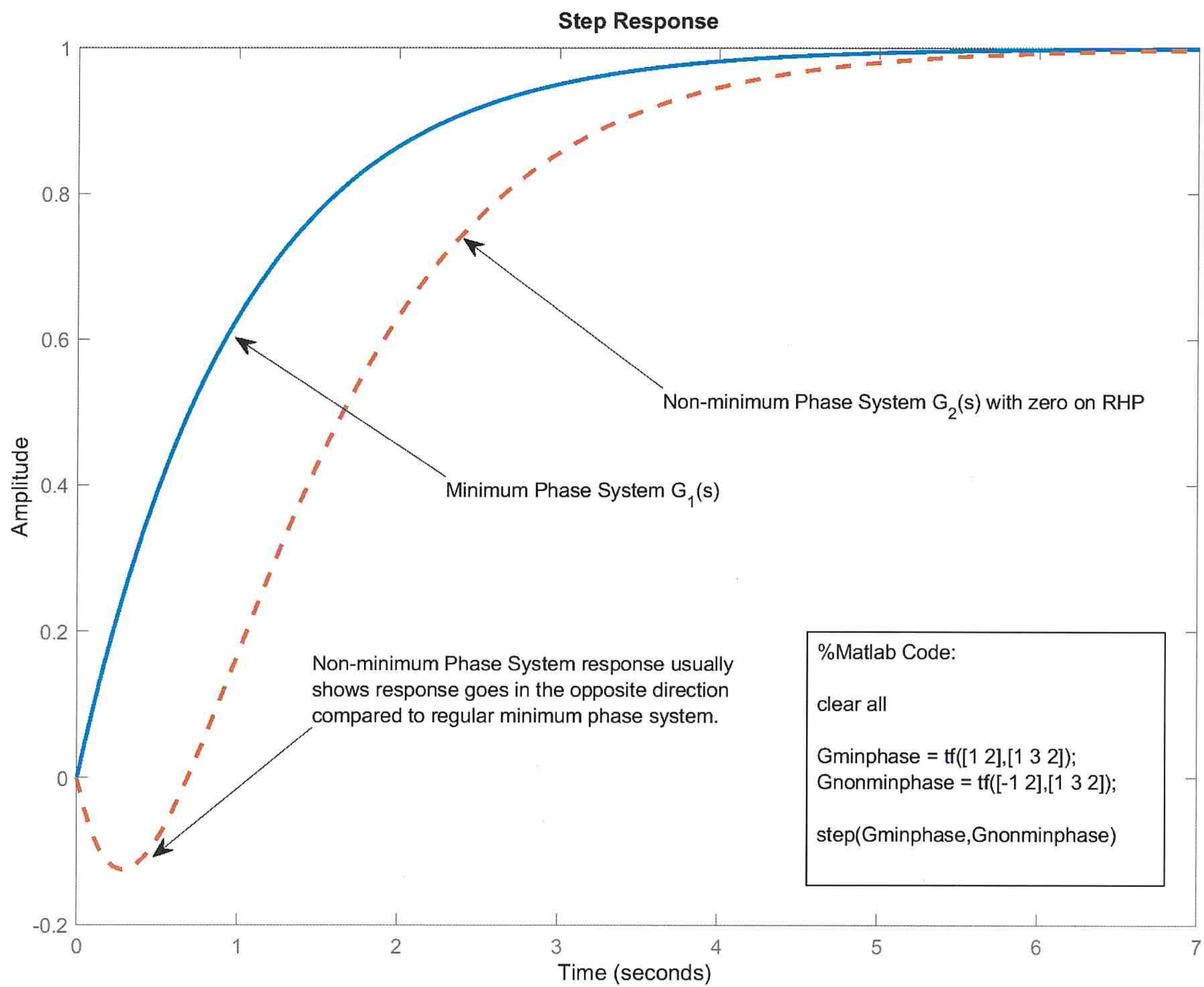
$$Y_1(s) = G_1(s)\,U(s) = \frac{s}{s^2+3s+2}U(s) + \frac{2}{s^2+3s+2}U(s)$$

versus:

$$Y_2(s) = G_2(s)\,U(s) = \underline{\frac{-s}{s^2+3s+2}}\,U(s) + \frac{2}{s^2+3s+2}$$

* zero in RHP causes the response to go in the negative direction at first!
non-min phase behavior!

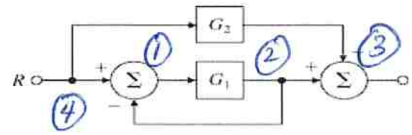
* This term goes negative at steady state and the transient goes in negative direction!


Step Response
Amplitude
Time (seconds)
1
0.8
0.6
0.4
0.2
0
-0.2
0
1
2
3
4
5
6
7
Non-minimum Phase System G2(s) with zero on RHP
Minimum Phase System G1(s)
Non-minimum Phase System response usually shows response goes in the opposite direction compared to regular minimum phase system.
%Matlab Code:
clear all
Gminphase = tf([1 2],[1 3 2]);
Gnonminphase = tf([-1 2],[1 3 2]);
step(Gminphase,Gnonminphase)

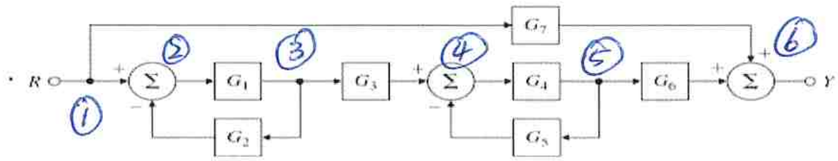
2

Problem 2

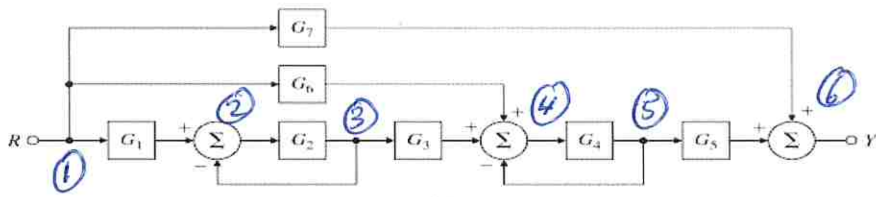
Find the transfer function from R to Y for each block diagram using the block diagram algebra reduction method. Your results for each should include detailed steps on how you manipulated the blocks.



(a)

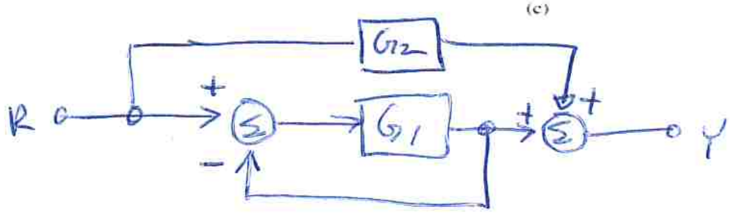


(b)

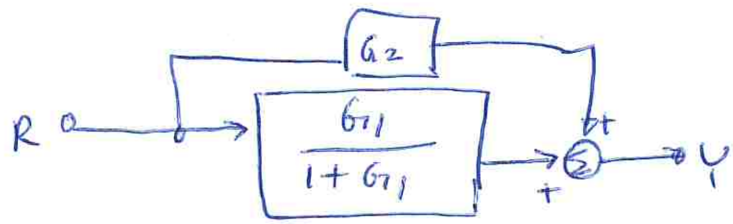


(c)

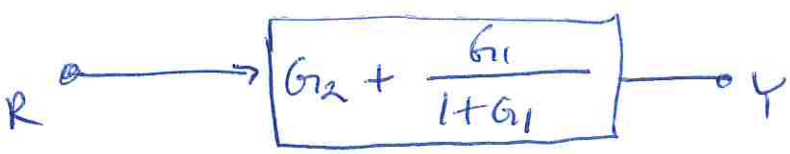
(a)



simplify inner feed back system



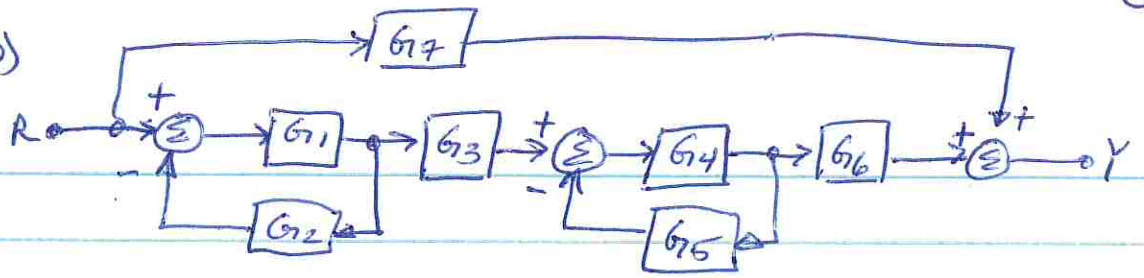
Now add



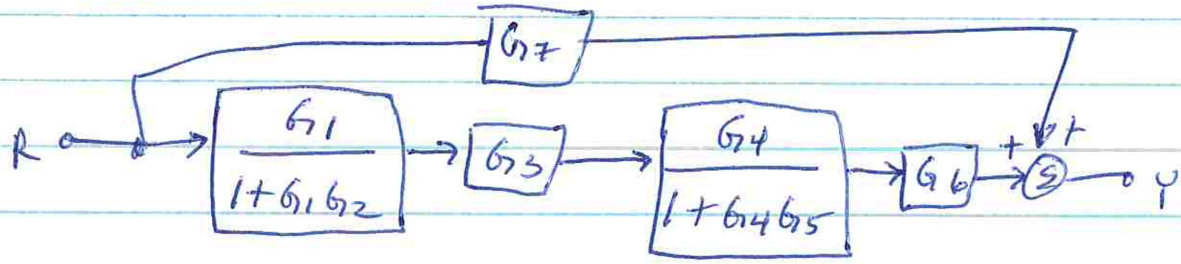
Simplify:

$$\frac{Y(s)}{R(s)} = \frac{G_2(1+G_1)+G_1}{1+G_1} \Rightarrow \boxed{\frac{Y(s)}{R(s)} = \frac{G_1+G_2+G_1G_2}{1+G_1}}$$

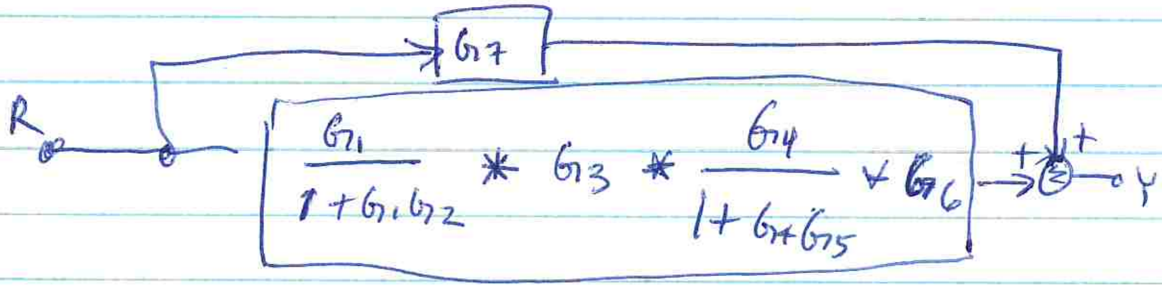
(b)



Simplify 2 inner feedback loops



cascade blocks



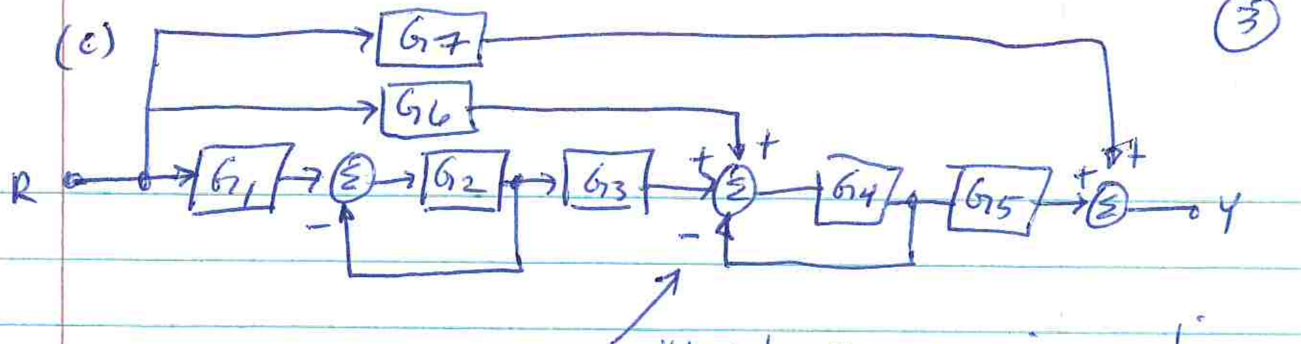
Add and simplify.

$$\frac{Y(s)}{R(s)} = G_7 + \frac{G_1 G_3 G_4 G_6}{(1+G_1G_2)(1+G_4G_5)}$$

$$= \frac{G_7(1+G_4G_5+G_1G_2+G_1G_2G_4G_5)+G_1G_3G_4G_6}{1+G_1G_2+G_4G_5+G_1G_2G_4G_5}$$

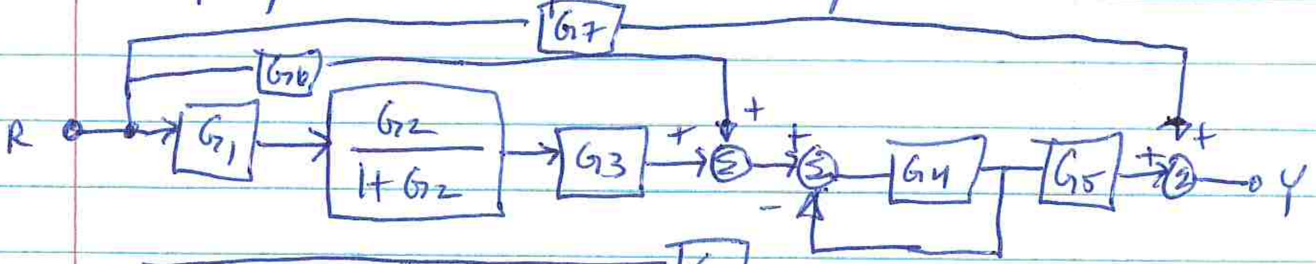
$$\boxed{\frac{Y}{R} = \frac{G_7+G_4G_5G_7+G_1G_2G_7+G_1G_2G_4G_5G_7+G_1G_2G_4G_6}{1+G_1G_2+G_4G_5+G_1G_2G_4G_5}}$$

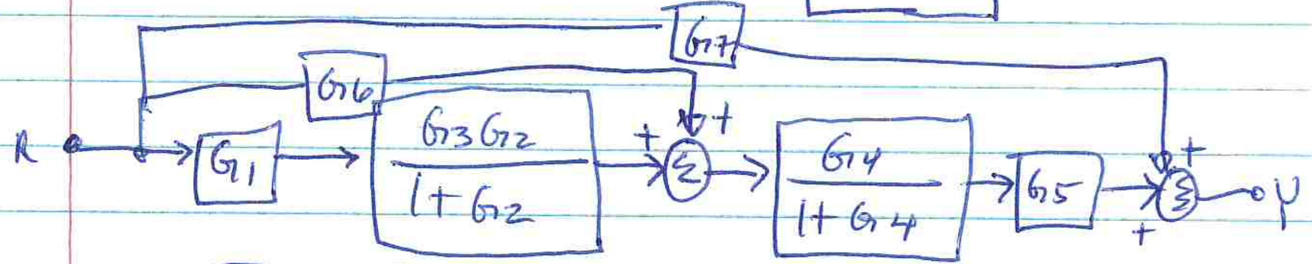
(c)

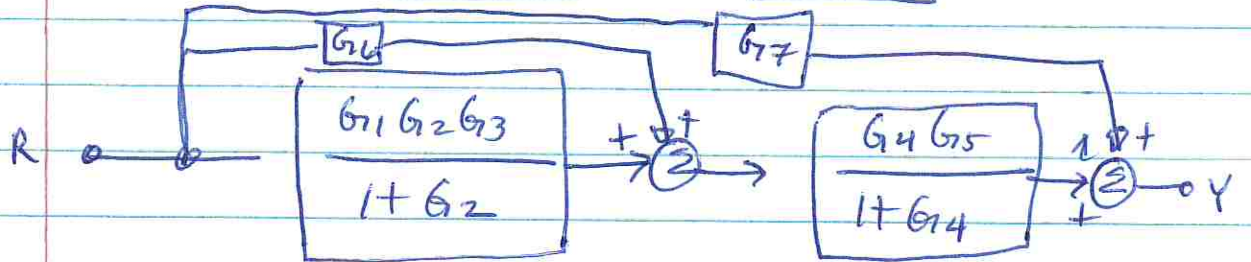


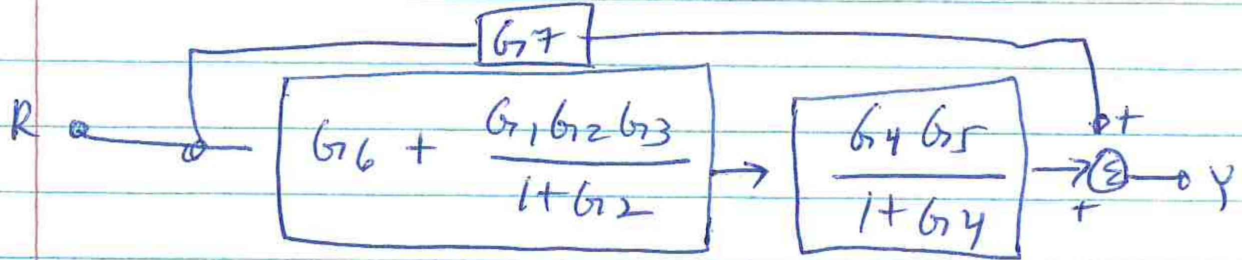
split into 2 summing junctions

Simplify 2 inner feedback loops:









$$\frac{Y}{R} = \left(\frac{G_6(1+G_2) + G_1 G_2 G_3}{1+G_2}\right)\left(\frac{G_4 G_5}{1+G_4}\right) + G_7$$

simplify

$$\frac{Y}{R} = \frac{G_4 G_5 G_6 (1+G_2) + G_1 G_2 G_3 G_4 G_5 + G_7 (1+G_2)(1+G_4)}{(1+G_2)(1+G_4)}$$

$$= \frac{G_4 G_5 G_6 + G_2 G_4 G_5 G_6 + G_1 G_2 G_3 G_4 G_5 + G_7 + G_7 G_4 + G_2 G_7 + G_7 G_4 G_2}{G_2 G_4 + G_2 + G_4 + 1}$$

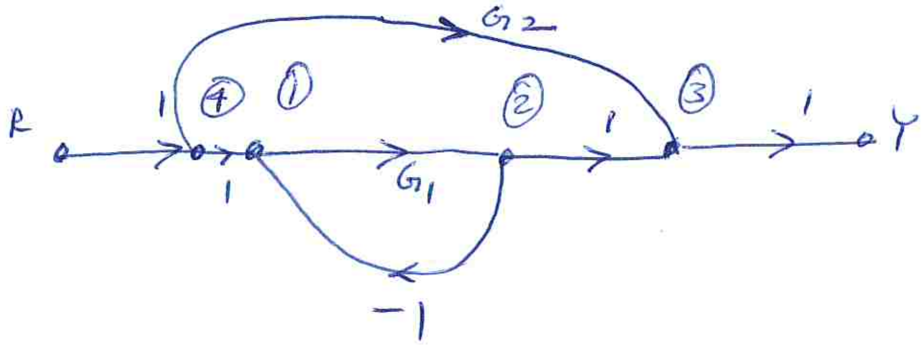
$$\Rightarrow \boxed{\frac{Y}{R} = \frac{G_4 G_5 G_6 + G_2 G_4 G_5 G_6 + G_1 G_2 G_3 G_4 G_5 + G_7 + G_7 G_4 + G_2 G_7 + G_1 G_2 G_4}{G_2 G_4 + G_2 + G_4 + 1}}$$

5

Problem 3

Find the transfer function from R to Y for Problem 2 above using Mason's rule. You need to show clearly the signal flow graph, labelling all nodes, branches, and listing all of the paths and loops. Your answers should match Problem 2.

(a)



$$F_1 = G_1 \qquad F_2 = G_2$$

$$L_1 = -G_1 \qquad \Delta = 1 + G_1$$

$$\Delta_1 = 1 \qquad \Delta_2 = 1 + G_1$$

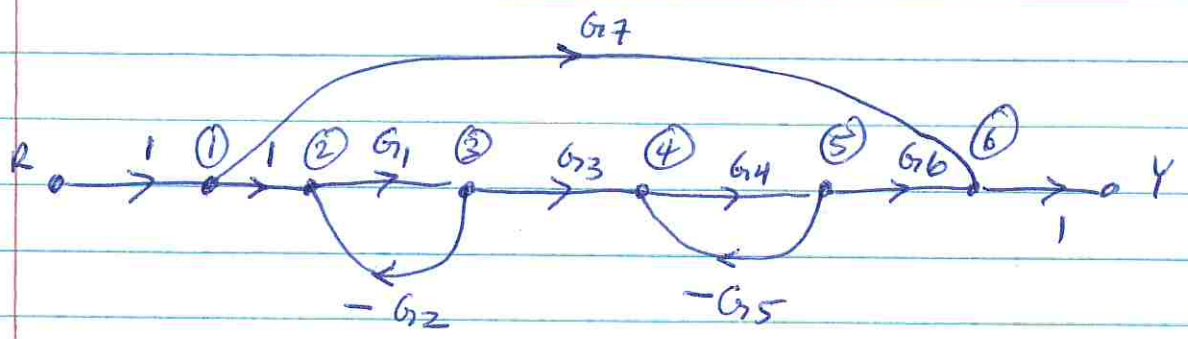
$$\frac{Y}{R} = \frac{1}{\Delta} \sum^{2} F_i \Delta_i = \frac{F_1 \Delta_1 + F_2 \Delta_2}{\Delta}$$

$$\frac{Y}{R} = \frac{G_1(1) + G_2(1+G_1)}{1+G_1}$$

$$\boxed{\frac{Y}{R} = \frac{G_1 + G_2 + G_1 G_2}{1+G_1}}$$

(6)

(b)



$F_1 = G_1 G_3 G_4 G_6 \qquad F_2 = G_7$

$L_1 = -G_1 G_2 \qquad L_2 = -G_4 G_5$

$\Delta = 1 + G_1 G_2 + G_4 G_5 + G_1 G_2 G_4 G_5$

$\Delta_1 = 1 \qquad \Delta_2 = 1 + G_1 G_2 + G_4 G_5 + G_1 G_2 G_4 G_5$

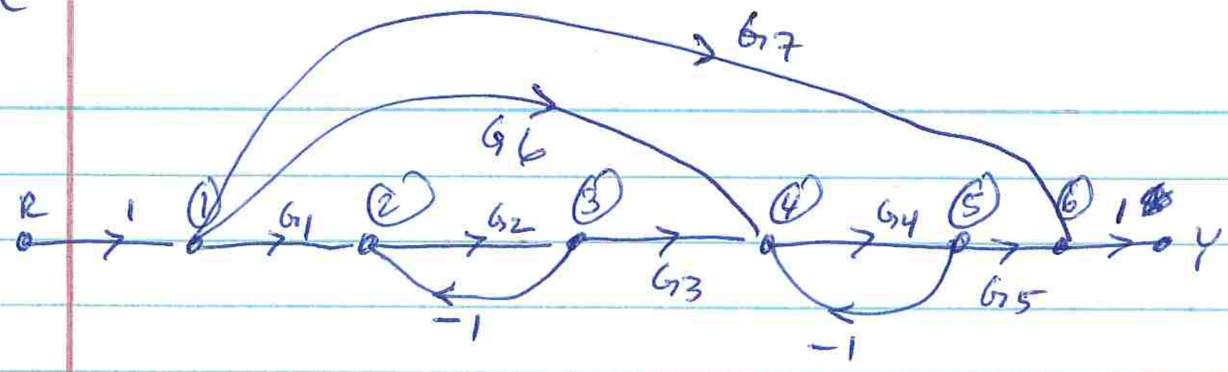
$$\frac{Y}{R} = \frac{1}{\Delta}\left(F_1 \Delta_1 + F_2 \Delta_2\right)$$

$$= \frac{G_1 G_3 G_4 G_6 (1) + G_7 (1 + G_1 G_2 + G_4 G_5 + G_1 G_2 G_4 G_5)}{1 + G_1 G_2 + G_4 G_5 + G_1 G_2 G_4 G_5}$$

$$\boxed{\frac{Y}{R} = \frac{G_1 G_3 G_4 G_6 + G_7 + G_1 G_2 G_7 + G_4 G_5 G_7 + G_1 G_2 G_4 G_5 G_7}{1 + G_1 G_2 + G_4 G_5 + G_1 G_2 G_4 G_5}}$$

(7)

(c)



$F_1 = G_1 G_2 G_3 G_4 G_5 \quad F_2 = G_6 G_4 G_5$

$F_3 = G_7 \quad L_1 = -G_2 \quad L_2 = -G_4$

$\Delta = 1 + G_2 + G_4 + G_2 G_4$

$\Delta_1 = 1 \quad \Delta_2 = 1 + G_2 \quad \Delta_3 = 1 + G_2 + G_4 + G_2 G_4$

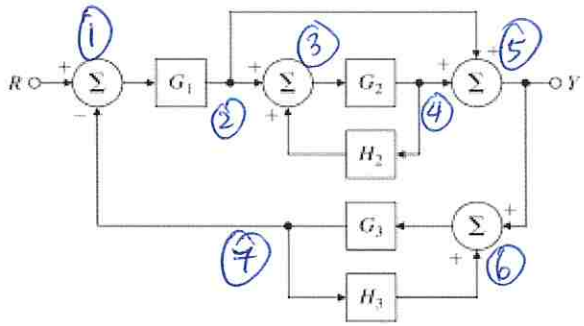
$$\frac{Y}{R} = \frac{1}{\Delta}\left(F_1 \Delta_1 + F_2 \Delta_2 + F_3 \Delta_3\right)$$

$$= \frac{G_1 G_2 G_3 G_4 G_5 (1) + G_4 G_5 G_6 (1 + G_2) + G_7 (1 + G_2 + G_4 + G_2 G_4)}{1 + G_2 + G_4 + G_2 G_4}$$

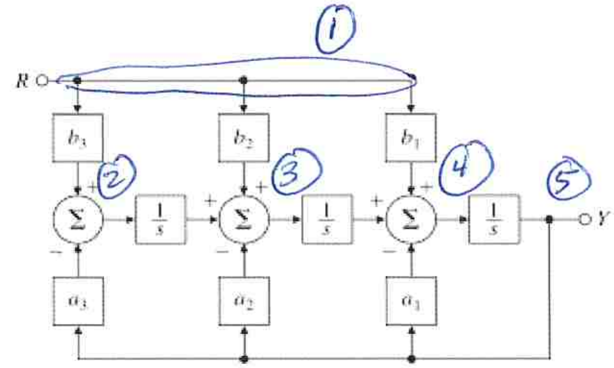
$$\boxed{\frac{Y}{R} = \frac{G_1 G_2 G_3 G_4 G_5 + G_4 G_5 G_6 + G_2 G_4 G_5 G_6 + G_7 + G_2 G_7 + G_4 G_7 + G_2 G_4 G_7}{1 + G_2 + G_4 + G_2 G_4}}$$

Problem 4

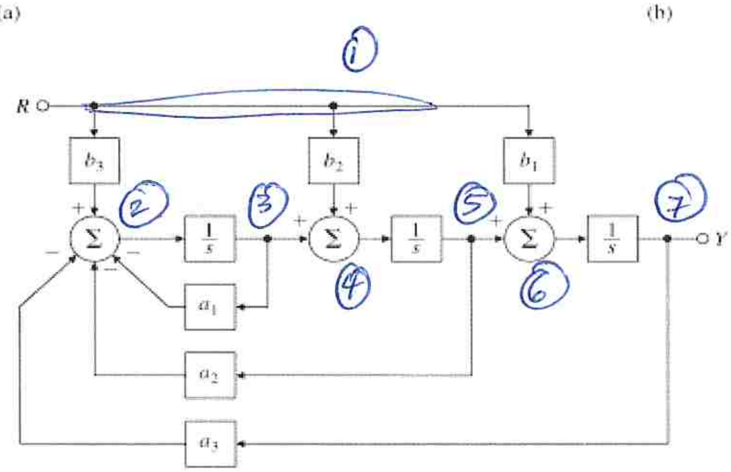
- Find the transfer function from R to Y for (a) using both block diagram algebra and Mason's rule. Show all your steps for full credit.
- Find the transfer function from R to Y using Mason's rule for (b)-(d). Show all your steps for full credit.



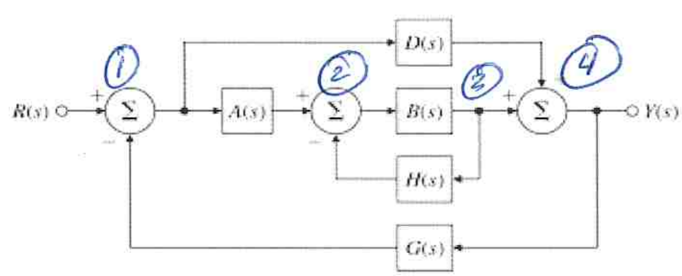
(a)



(b)

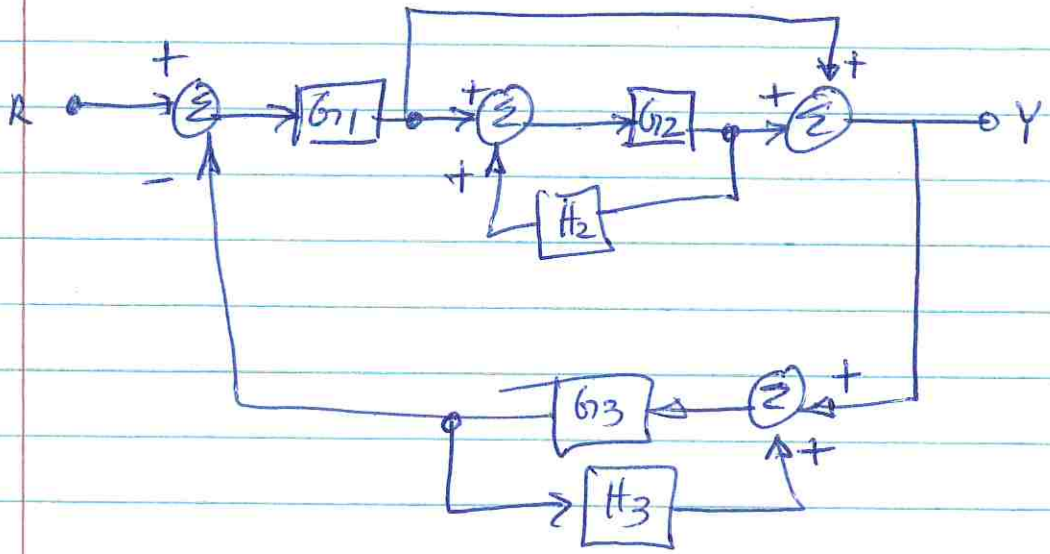


(c)

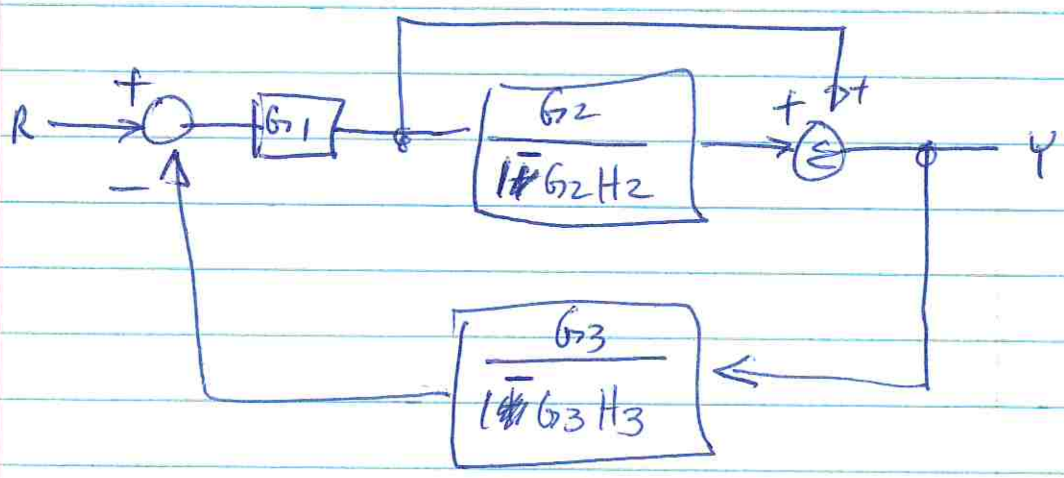


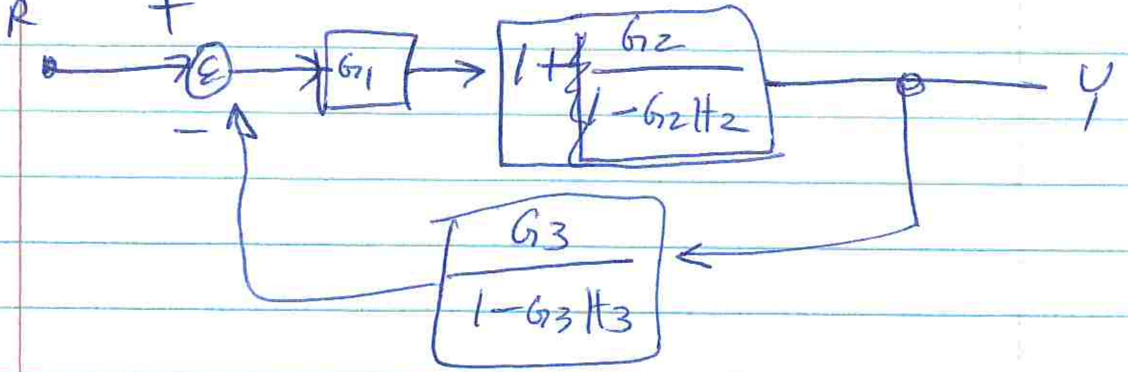
(d)

(a) Block algebra



Simplify ~~feed~~back blocks.



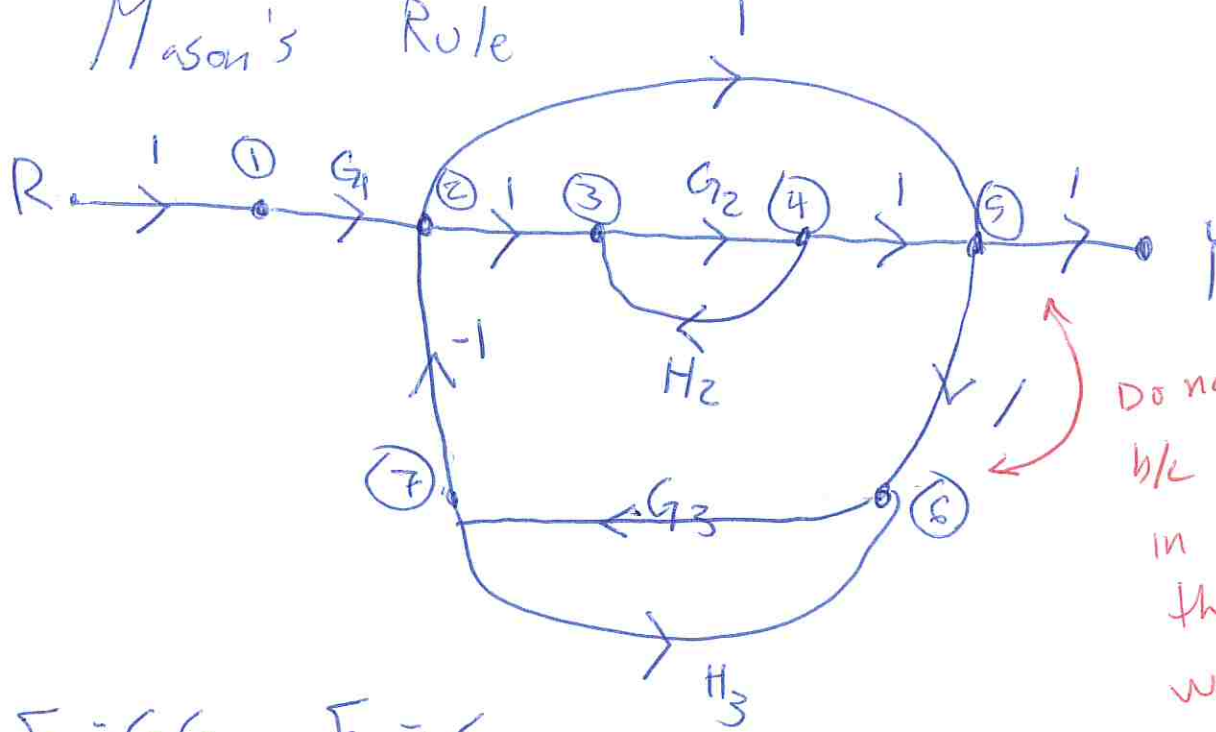


$$\frac{Y}{R} = \frac{G_1\left(\dfrac{1-G_2H_2+G_2}{1-G_2H_2}\right)}{1+G_1\left(\dfrac{1-G_2H_2+G_2}{1-G_2H_2}\right)\left(\dfrac{G_3}{1-G_3H_3}\right)}$$

$$= \frac{\left(G_1-G_1G_2H_2+G_1G_2\right)\left(1-G_3H_3\right)}{\left(1-G_2H_2\right)\left(1-G_3H_3\right)+\left(G_1-G_1G_2H_2+G_1G_2\right)G_3}$$

$$\boxed{\frac{Y}{R} = \frac{G_1-G_1G_2H_2+G_1G_2-G_1G_3H_3+G_1G_2G_3H_2H_3-G_1G_2G_3H_3}{1-G_3H_3-G_2H_2+G_2G_3H_2H_3+G_1G_3-G_1G_2H_2G_3+G_1G_2G_3}}$$

Mason's Rule



Do not touch b/c Y is in between them and we want $\frac{Y}{R}$ T.F.

$F_1 = G_1 G_2 \qquad F_2 = G_1$

$L_1 = G_2 H_2 \qquad L_2 = -G_1 G_3 \qquad L_3 = -G_1 G_2 G_3 \qquad L_4 = G_3 H_3$

$\Delta = 1 - G_2 H_2 + G_1 G_3 + G_1 G_2 G_3 - G_3 H_3 + G_2 H_2 G_3 H_3 - G_1 G_2 G_3 H_2$

$\Delta_1 = 1 - G_3 H_3$

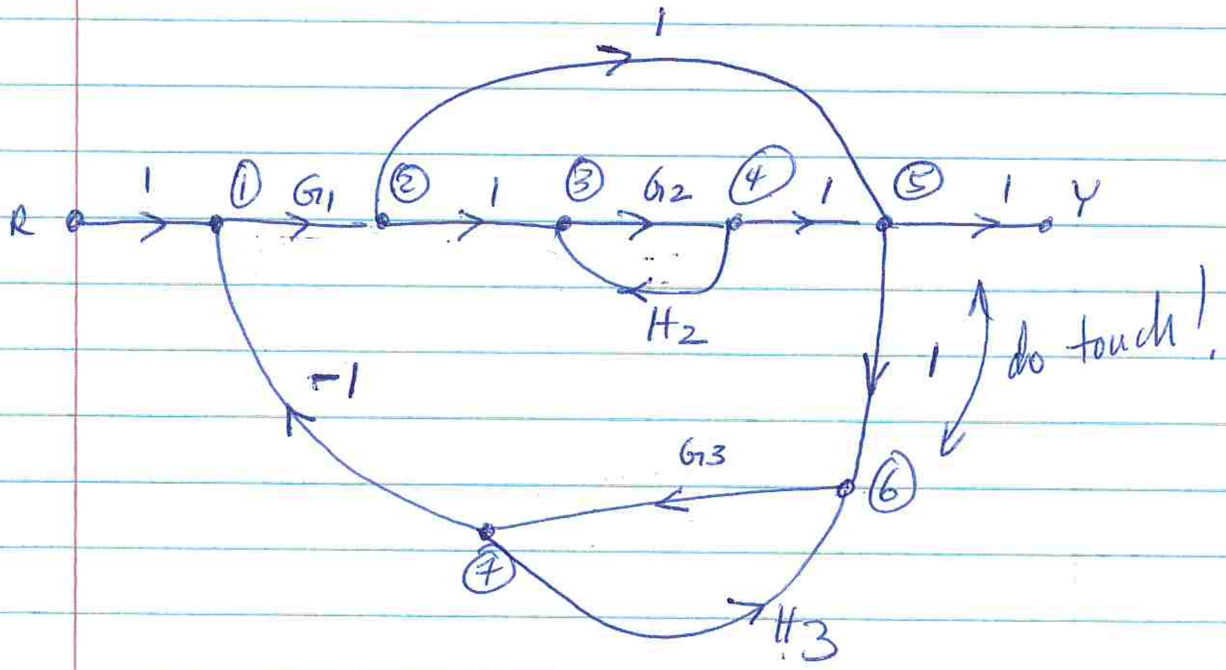
$\Delta_2 = 1 - G_2 H_2 - G_3 H_3 + G_2 G_3 H_2 H_3$

$$\frac{Y}{R} = \frac{1}{\Delta}\left(F_1 \Delta_1 + F_2 \Delta_2\right)$$

$$\frac{Y}{R} = \frac{G_1 G_2 (1 - G_3 H_3) + G_1 (1 - G_2 H_2 - G_3 H_3 + G_2 G_3 H_2 H_3)}{\Delta}$$

$$\frac{Y}{R} = \frac{G_1 G_2 - G_1 G_2 G_3 H_3 + G_1 - G_1 G_2 H_2 - G_1 G_3 H_3 + G_1 G_2 G_3 H_2 H_3}{1 - G_2 H_2 + G_1 G_3 + G_1 G_2 G_3 - G_3 H_3 + G_2 H_2 G_3 H_3 - G_1 G_2 G_3 H_2}$$

Mason's Rule:



$F_1 = G_1 G_2 \qquad F_2 = G_1$

$L_1 = G_2 H_2 \qquad L_2 = -G_1 G_3 \qquad L_3 = -G_1 G_2 G_3$

$L_4 = G_3 H_3$

$\Delta = 1 - G_2 H_2 + G_1 G_3 + G_1 G_2 G_3 - G_3 H_3 + G_2 H_2 G_3 H_3 + (-G_1 G_3) \times G_2 H_2$

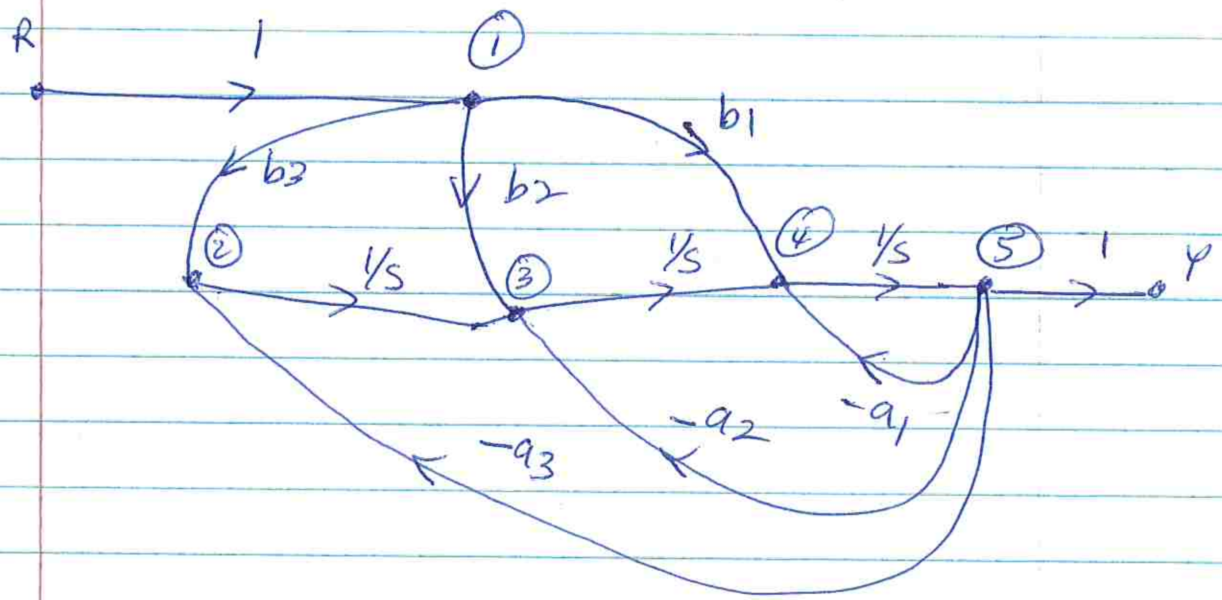
$\Delta_1 = 1 - G_3 H_3 \qquad \Delta_2 = 1 - G_2 H_2$

$\frac{Y}{R} = \frac{1}{\Delta}(F_1 \Delta_1 + F_2 \Delta_2)$

$$\frac{Y}{R} = \frac{G_1G_2(1 - G_3H_3) + G_1(1 - G_2H_2 - G_3H_3)}{1 - G_2H_2 + G_1G_3 + G_1G_2G_3 - G_3H_3}$$

$$\boxed{\frac{Y}{R} = \frac{G_1G_2 - G_1G_2G_3H_3 + G_1 - G_1G_2H_2}{1 - G_2H_2 + G_1G_3 + G_1G_2G_3 - G_3H_3 + G_2H_2G_3H_3 - G_1G_3G_2H_2}}$$

(b)



$F_1 = b_3\left(\frac{1}{s^3}\right) \quad F_2 = b_2\left(\frac{1}{s^2}\right) \quad F_3 = b_1\left(\frac{1}{s}\right)$

$L_1 = -a_1/s \quad L_2 = -a_2/s^2 \quad L_3 = \frac{-a_3}{s^3}$

$\Delta = 1 + \frac{a_1}{s} + \frac{a_2}{s^2} + \frac{a_3}{s^3}$

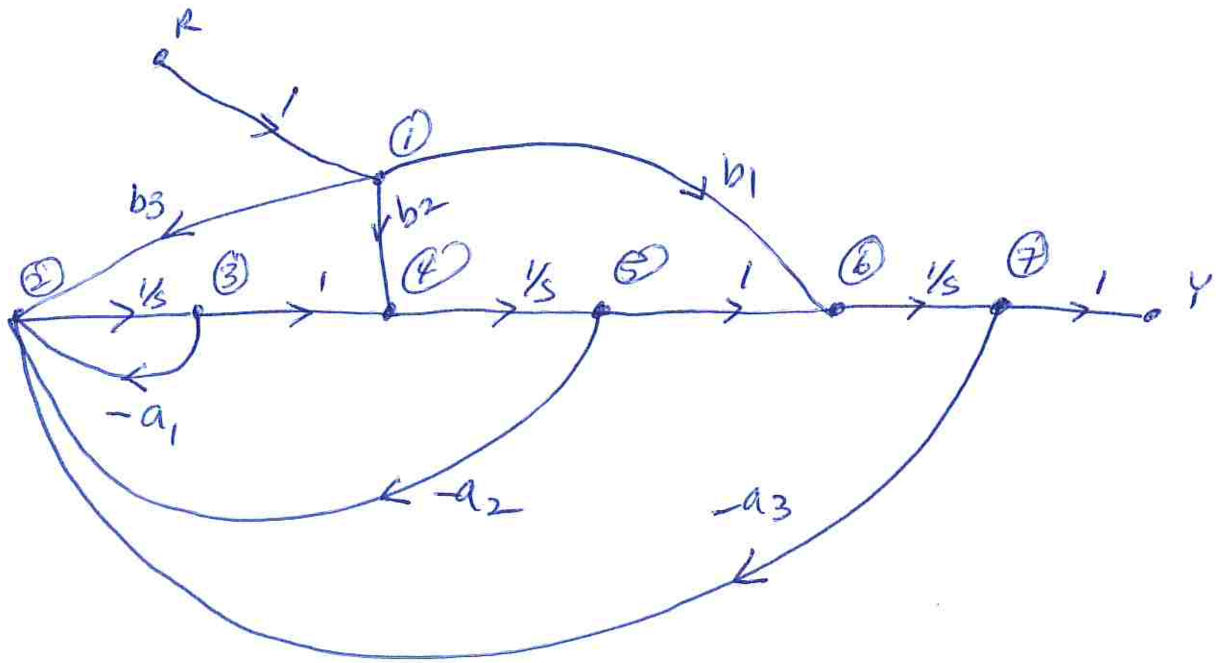
$\Delta_1 = 1 \quad \Delta_2 = 1 \quad \Delta_3 = 1$

$\frac{Y}{R} = \frac{1}{\Delta}\left(F_1\Delta_1 + F_2\Delta_2 + F_3\Delta_3\right)$

$\frac{Y}{R} = \frac{b_3/s^3 + b_2/s^2 + b_1/s}{s^3 + a_1 s^2 + a_2 s + a_3}$

$\Rightarrow \boxed{\frac{Y}{R} = \frac{b_1 s^2 + b_2 s + b_3}{s^3 + a_1 s^2 + a_2 s + a_3}}$

(c)



$$F_1 = \frac{b_3}{s^3} \qquad F_2 = \frac{b_2}{s^2} \qquad F_3 = \frac{b_1}{s}$$

$$L_1 = -\frac{a_3}{s^3} \qquad L_2 = -\frac{a_2}{s^2} \qquad L_3 = -\frac{a_1}{s}$$

$$\Delta = 1 + \frac{a_3}{s^3} + \frac{a_2}{s^2} + \frac{a_1}{s}$$

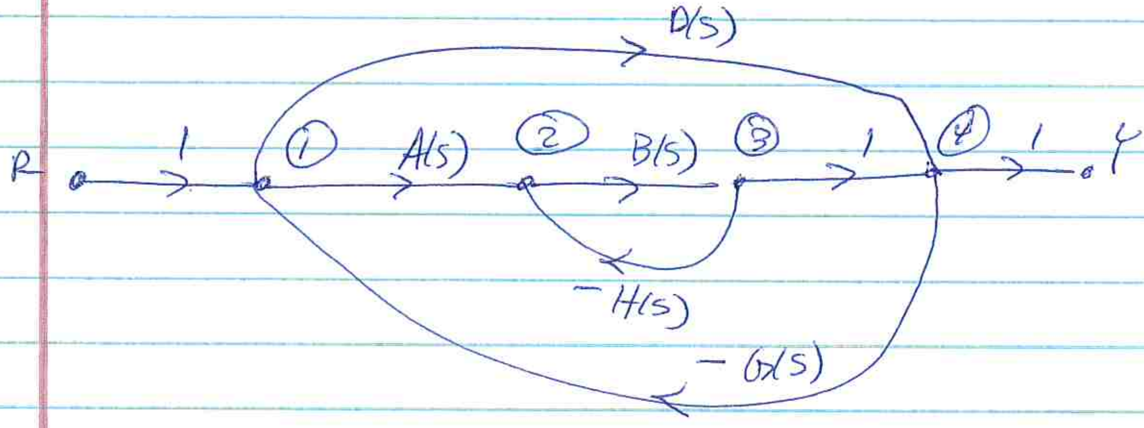
$$\Delta_1 = 1 \qquad \Delta_2 = 1 + \frac{a_1}{s} \qquad \Delta_2 = 1 + \frac{a_1}{s} + \frac{a_2}{s^2}$$

$$\frac{Y(s)}{R(s)} = \frac{F_1\Delta_1 + F_2\Delta_2 + F_3\Delta_3}{\Delta}$$

$$= \frac{(b_3/s^3)(1) + \left(\frac{b_2}{s^2}\right)\left(1 + \frac{a_1}{s}\right) + \frac{b_1}{s}\left(1 + \frac{a_1}{s} + \frac{a_2}{s^2}\right)}{1 + \frac{a_3}{s^3} + \frac{a_2}{s^2} + \frac{a_1}{s}}$$

$$\boxed{\frac{Y(s)}{R(s)} = \frac{b_1 s^2 + (a_1 b_1 + b_2)s + a_1 b_2 + a_2 b_1 + b_3}{s^3 + a_1 s^2 + a_2 s + a_3}}$$

(d)



$$F_1 = A(s)\,B(s) \qquad F_2 = D(s)$$

$$L_1 = -H(s)\,B(s) \qquad L_2 = -A(s)\,B(s)\,G(s)$$

$$L_3 = -G(s)\,D(s)$$

$$\Delta = 1 + H(s)B(s) + A(s)\,B(s)\,G(s) + G(s)\,D(s) + B(s)\,H(s)\,G(s)\,D(s)$$

$$\Delta_1 = 1 \qquad \Delta_2 = 1 + H(s)\,B(s)$$

$\cancel{\Delta_2}$

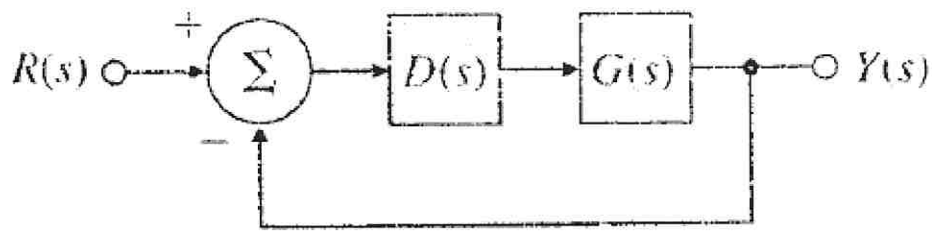
$$\frac{Y}{R} = \frac{1}{\Delta}\left(F_1\Delta_1 + F_2\Delta_2\right)$$

$$= \frac{A(s)\,B(s)\,(1) + D(s)\,(1) + D(s)H(s)B(s)}{1 + H(s)\,B(s) + A(s)\,B(s)\,G(s) + G(s)D(s) + B(s)H(s)G(s)D(s)}$$

$$\boxed{\frac{Y}{R} = \frac{A(s)\,B(s) + D(s) + D(s)H(s)B(s)}{1 + H(s)B(s) + A(s)B(s)G(s) + G(s)D(s) + B(s)H(s)D(s)G(s)}}$$

Problem 5

Suppose you have the following closed-loop system:



where

$$G(s) = \frac{1}{s(s+3)} \quad \text{and} \quad D(s) = \frac{K(s+z)}{s+p}.$$

Answer these questions (show all your work):

a) What is the order of the open-loop system?
b) What is the order of the controller?
c) What is the order of the closed-loop system?
d) Now suppose the controller is just a basic proportional controller with gain K (no zero or pole). Design a proportional controller (i.e., find the value of K) so that the step response of the closed-loop system is as **fast as possible without oscillation**. What value is K? Show all your work.
e) Prove that your design works by showing a step response of the closed-loop system using Matlab. Include of plot of output vs. time, and also your Matlab code (PDF of the m-file and/or Simulink model).

(a) Open-loop system is $G(s) = \frac{1}{s^2+3s}$

$\Rightarrow$ $\boxed{\text{2nd-order open-loop system}}$

(b) Controller $D(s) = \frac{K(s+z)}{s+p}$

$\Rightarrow$ $\boxed{\text{1st order system}}$

(c) $$\frac{Y(s)}{R(s)} = \frac{D(s)G(s)}{1+D(s)G(s)} = \frac{K(s+z)/(s+p)(s^2+3s)}{1+K\left(\frac{s+z}{s+p}\right)\left(\frac{1}{s^2+3s}\right)} = \frac{K(s+z)}{(s^2+3s)(s+p)+K(s+z)}$$

$\boxed{\text{3rd order!}}$

(d) Closed-loop w/ K controller is:

$$\frac{Y(s)}{R(s)} = \frac{K\,G(s)}{1 + K\,G(s)} = \frac{K}{s(s+3) + K}$$

$$\frac{Y(s)}{R(s)} = \frac{K}{s^2 + 3s + K} = T(s)$$

To get fastest response w/o oscillations, the system needs to be critically damped, so $\zeta = 1$!

From the denominator of $T(s)$, we have

$$s^2 + 3s + K = s^2 + 2\zeta\omega_n s + \omega_n^2$$

Thus:

$$3 = 2\zeta\omega_n \quad \text{where } \zeta = 1$$

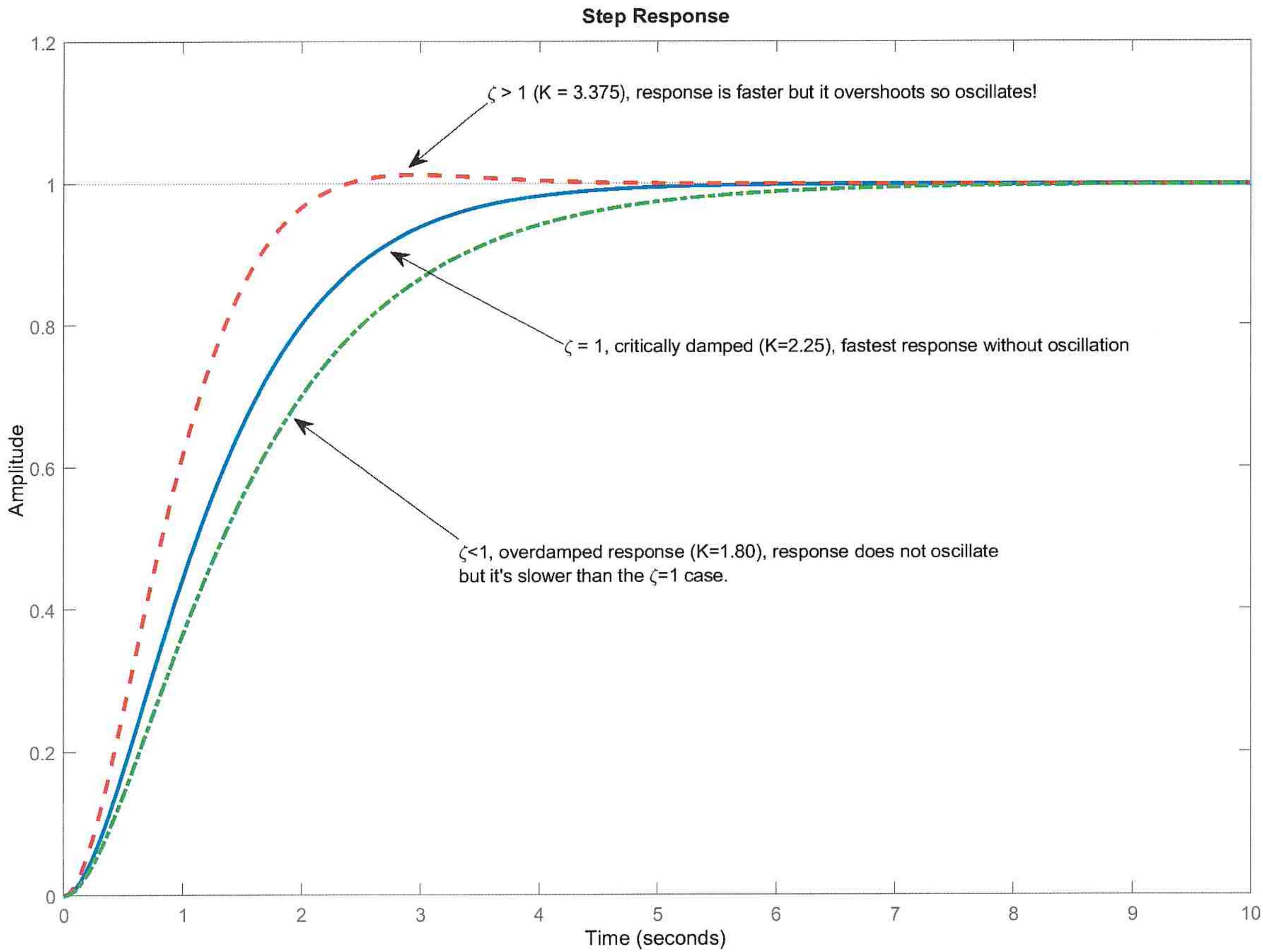
$$K = \omega_n^2$$

$$\Rightarrow \omega_n = \frac{3}{2\zeta} = \frac{3}{2}$$

$$\Rightarrow K = \omega_n^2 = \frac{9}{4}$$

we need $K = 9/4$ to get $\zeta = 1$!

Now let's test in Matlab $\rightarrow$ see plot.


Step Response
ζ > 1 (K = 3.375), response is faster but it overshoots so oscillates!
ζ = 1, critically damped (K=2.25), fastest response without oscillation
ζ<1, overdamped response (K=1.80), response does not oscillate but it's slower than the ζ=1 case.
Amplitude
Time (seconds)
0
0.2
0.4
0.6
0.8
1
1.2
0
1
2
3
4
5
6
7
8
9
10