Intermediate Fluid Mechanics

Lecture 25: Boundary Layer Flows IV

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ME 5700/6700
November 10, 2025

Chapter Overview

1. Chapter Objectives
2. Von Karman Momentum Integral Method
3. Application of VK's Momentum Integral Method to the BL over flat plate

Lecture Objectives

In the previous lectures we examined the differential equation governing fluid motion in the boundary layer. The solution to this differential equation tells us the velocity everywhere within the boundary layer.

$\longrightarrow$ However, exact solutions to the boundary layer equations are possible only in simple cases, such as that over a flat plate.

In this lecture,

- We will discuss the Von Karman Momentum Integral Analysis, which is an approximate method that allows one to find approximate values of the wall shear stress and boundary layer thickness with relative ease, regardless of the boundary layer complexity.

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Von Karman Momentum Integral Method

For two-dimensional, incompressible, steady flows, the boundary layer equation is

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = U\frac{dU}{dx} + \nu\,\frac{\partial^2 u}{\partial y^2} \tag{1}$$

Note that the pressure gradient term has been rewritten through the equivalent inertia at the edge of the boundary layer, where U denotes the horizontal velocity at the edge of the boundary layer.

The continuity equation is,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \tag{2}$$

Next, we multiply the continuity equation by the momentum deficit $(u - U)$ and add it to equation 1, such that

$$(u-U)\frac{\partial u}{\partial x} + (u-U)\frac{\partial v}{\partial y} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - U\frac{dU}{dx} = +\nu\,\frac{\partial^2 u}{\partial y^2}$$

$$U\frac{\partial u}{\partial x} - u\frac{\partial u}{\partial x} + U\frac{\partial v}{\partial y} - u\frac{\partial v}{\partial y} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - U\frac{dU}{dx} = +\nu\,\frac{\partial^2 u}{\partial y^2} \qquad (3)$$

Note: Realize that this is possible because the continuity equation is $= 0$, hence one can always add and subtract zero to a given equation.

Von Karman Momentum Integral Method

Using the product rule it is possible to obtain that

$$(a)\;\; U\frac{\partial u}{\partial x} = \frac{\partial (u\,U)}{\partial x} - u\,\underbrace{\frac{dU}{dx}}_{\substack{\text{Assume}\\ U=U(x)}} \tag{4}$$

$$(b)\;\; u\frac{\partial u}{\partial x} = \frac{\partial (u^2)}{\partial x} - u\frac{\partial u}{\partial x} \tag{5}$$

$$(c)\;\; v\frac{\partial u}{\partial y} = \frac{\partial (u\,v)}{\partial y} - u\frac{\partial v}{\partial y} \tag{6}$$

$$(d)\;\; U\frac{\partial v}{\partial y} = \frac{\partial (U\,v)}{\partial y} - v\frac{\partial U}{\partial y} \underbrace{=}_{\substack{\text{Since}\\ U=U(x)}} \frac{\partial (U\,v)}{\partial y}. \tag{7}$$

Von Karman Momentum Integral Method

These relationships can be used in equation 3 to obtain,

$$\frac{dU}{dx}(U-u)+\frac{\partial}{\partial x}(u\,U-u^2)+\frac{\partial}{\partial y}(U\,v-u\,v)=-\nu\,\frac{\partial^2 u}{\partial y^2}. \tag{8}$$

Next, each term can be multiplied by dy and integrated between $y=0$ and $y\to\infty$ such that,

$$\frac{dU}{dx}\underbrace{\int_{y=0}^{\infty}(U-u)dy}_{I}+\frac{\partial}{\partial x}\underbrace{\int_{y=0}^{\infty}(u\,U-u^2)\,dy}_{II}+\underbrace{\int_{y=0}^{\infty}\frac{\partial}{\partial y}(U\,v-u\,v)\,dy}_{III}= \tag{9}$$

$$\underbrace{-\int_{y=0}^{\infty}\nu\,\frac{\partial^2 u}{\partial y^2}\,dy}_{IV}. \tag{10}$$

This can be further rearranged by re-introducing the definitions of displacement thickness and momentum thickness,

$$\delta^*=\frac{1}{U}\int_{y=0}^{\infty}(U-u)\,dy \qquad\qquad \theta=\frac{1}{U^2}\int_{y=0}^{\infty}(u\,U-u^2)\,dy. \tag{11}$$

Von Karman Momentum Integral Method

Specifically, replacing integral I with δ^* and integral II with θ, results in

$$\frac{dU}{dx}(U\delta^*) + \frac{\partial}{\partial x}(U^2\theta) + (U\,v - u\,v)|_{y=0}^{\infty} = -\nu\,\frac{\partial u}{\partial y}|_{y=0}^{\infty}. \tag{12}$$

where,

$$(U\,v - u\,v)\Big|_{y=0}^{\infty} = (U-u)v\Big|_{y\to\infty} - (U-u)v\Big|_{y=0} \tag{13}$$

and

$$\frac{\partial u}{\partial y}\Big|_{y=0}^{\infty} = \frac{\partial u}{\partial y}\Big|_{y\to\infty} - \frac{\partial u}{\partial y}\Big|_{y=0}. \tag{14}$$

Von Karman Momentum Integral Method

For a Newtonian fluid, the definition of the wall shear stress is

$$\tau_w = \mu \frac{\partial u}{\partial y}\Big|_{y=0} \quad \Longrightarrow \quad \frac{\tau_w}{\rho} = \nu \frac{\partial u}{\partial y}\Big|_{y=0}. \tag{15}$$

Also, we realize that $v|_{y=0} = v_w$; hence only if there is suction or blowing at the wall, $v_w \neq 0$; further $v|_{y\to\infty} = 0$ and $du/dy|_{y\to\infty} = 0$.

$\Longrightarrow$ Therefore, the previous integral equation may be rewritten as

$$(U\delta^*)\frac{dU}{dx} + \frac{d}{dx}(U^2\theta) - Uv_w = \frac{\tau_w}{\rho}. \tag{16}$$

Von Karman Momentum Integral Method

In the case that there is no injection of fluid at the wall, then

$$\boxed{(U\delta^*)\frac{dU}{dx} + \frac{d}{dx}(U^2\theta) = \frac{\tau_w}{\rho}.} \tag{17}$$

This is the so-called **Von Karman Momentum Integral Equation**.

Note:

- For a given geometry, we would like to determine δ^*, θ, and τ_w, which are unknowns in Von Karman's integral equation.
- No assumptions have been made on whether the flow inside the boundary layer was laminar or turbulent. Therefore, VK's integral equation is valid both for laminar and turbulent boundary layers.

Von Karman Momentum Integral Method

- The procedure for applying the integral approach is to assume a reasonable velocity distribution, satisfying as many conditions as possible.
- From thereon, VK's integral equation can predict the boundary layer thickness and other parameters.
- This approximate method is only useful in situations where an exact solution does not exist.

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VK's Momentum Integral Method to the BL over flat plate

In the specific case of a BL over a flat plate (i.e. no external pressure graddeints) $\implies U\frac{dU}{dx} = 0$,

hence, VK's integral equation reduces to:

$$\frac{d}{dx}\int_0^\delta u(U-u)dy = \frac{\tau_0}{\rho} \tag{18}$$

Assuming a cubic velocity profile within the BL,

$$\frac{u}{U} = a + b\frac{y}{\delta} + c\left(\frac{y}{\delta}\right)^2 + d\left(\frac{y}{\delta}\right)^3 \tag{19}$$

Together with the corresponding boundary conditions for the velocity profile,

$$u = 0, \qquad \text{and} \qquad \frac{\partial^2 u}{\partial y^2} = 0 \qquad \text{at} \quad y = 0,$$

$$u = U, \qquad \text{and} \qquad \frac{\partial u}{\partial y} = 0 \qquad \text{at} \quad y = \delta, \tag{20}$$

(21)

VK's Momentum Integral Method to the BL over flat plate

Results in a velocity profile that looks like,

$$\frac{u}{U} = \frac{3}{2}\frac{y}{\delta} - \frac{1}{2}\left(\frac{y}{\delta}\right)^3 \tag{22}$$

Note: The boundary condition $\frac{\partial^2 u}{\partial y^2} = 0$ at $y = 0$, is necessary for the velocity field to satisfy,

$$\nu \frac{\partial^2 u}{\partial y^2}\bigg|_0 = U\frac{dU}{dx} = 0 \tag{23}$$

VK's Momentum Integral Method to the BL over flat plate

Substituting the velocity profile defined above,

$$\frac{u}{U} = \frac{3}{2}\frac{y}{\delta} - \frac{1}{2}\left(\frac{y}{\delta}\right)^3 \tag{24}$$

into the VK's integral equation, leads to the left and right hand side of the equation to be

$$\frac{d}{dx}\int_0^\delta u(U-u)dy = \frac{39}{280}U^2\delta \tag{25}$$

$$\frac{\tau_0}{\rho} = \nu\frac{\partial u}{\partial y}\Big|_0 = \frac{3}{2}\frac{U\nu}{\delta} \tag{26}$$

which means that VK's integral equation reduces to an ODE for δ,

$$\frac{39}{280}U^2\frac{d\delta}{dx} = \frac{3}{2}\frac{U\nu}{\delta} \tag{27}$$

VK's Momentum Integral Method to the BL over flat plate

Upon resolution, one finds,

$$\delta = 4.64\sqrt{\frac{\nu x}{U}} \tag{28}$$

which is remarkably close to the Blasius solution obtained earlier.

Similarly, one can also obtain a solution for the friction factor,

$$C_f = \frac{\tau_0}{(1/2)\rho U^2} = \frac{(3/2)U\nu/\delta}{(1/2)U^2} = \frac{0.646}{\sqrt{Re_x}} \tag{29}$$

which is also very close to the Blasius solution.

VK's Momentum Integral Method to the BL over flat plate

- Pohlhausen found that a 4th-degree polynomial was necessary to exhibit sensitivity of the velocity profile to the pressure gradient.
- Adding another therm to the hypothesized velocity profile such as $e\left(\frac{y}{\delta}\right)^4$ requires an additional boundary condition, $\frac{\partial^2 u}{\partial y^2} = 0$ at $y = \delta$.

$\Longrightarrow$ This equation is solved in detail in the course notes.