Intermediate Fluid Mechanics

Lecture 10: Forces Acting on a Fluid Particle

Marc Calaf

Department of Mechanical Engineering

University of Utah

ME 5700/6700

September 16, 2025

Chapter Overview

1. Chapter Objectives
2. Forces Acting on a Fluid
3. Stress at a Point

Lecture Objectives

In the previous lecture we learned how to write the differential form of the conservation of mass law, which is valid in the Eulerian framework.

In this lecture, we will prepare the way to write Newton's 2^{nd} law of motion in the Eulerian framework. Before we can do this, one must first study the different forces acting on a fluid particle.

Chapter Overview

Forces Acting on a Fluid

The forces acting on a fluid can be classified into three types:

1. body forces;
2. surface forces;
3. line forces.

(These are considered individually in the following slides).

Body Forces:

- Body forces are those that arise from *action at a distance* without physical contact.
- Body forces result when the material is placed in a force field such as a gravitational, magnetic, electrostatic, or electromagnetic field.
- Body forces are distributed throughout the mass of the fluid, and as such act over the entire volume of the fluid particle.
- For example, the gravitational force per unit mass is written as $\vec{g}$.
- Body forces can either be **conservative** or **non-conservative**.

Conservative body force:

- A **conservative force** is a force in which the work done on an object is independent of the path taken; it only depends on the original and final position.
- Conservative forces are referred as such because the resulting motion conserves the sum of kinetic and potential energy, if there are no dissipative processes.
- Thus, in a conservative force, the work done over a closed path is zero (the object returns to its starting position).
- Conservative forces have the advantage that can be derived from a scalar potential function ($\vec{F} = \vec{\nabla}\phi$)

Conservative body force: (continued ...)

A conservative body force may be written as the gradient of a potential function,

$$\vec{g} = -\vec{\nabla}\pi, \tag{1}$$

where π is a force potential.

Gravity is the most typical conservative body force that acts on a fluid. The force potential for gravity is

$$\pi = g\,z, \tag{2}$$

where g is the gravitational acceleration and z points vertically upward. Plugging π into the above equation,

$$\vec{g} = -\vec{\nabla}(g\,z) = -\left[\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right](g\,z) = -g\,\hat{k}, \tag{3}$$

which states that the gravitational force per unit mass is directed downward, in the $-z$ direction.

Surface Forces:

- **Surface forces** (SF) are those that are exerted on an area element by the surroundings through direct contact.
- **SF** are proportional to the extent of the area and are expressed per unit area.
- **SF** can be resolved into components normal and tangential to the surface.

$$\frac{dF_n}{dA} \equiv \tau_n \qquad \text{Normal stress} \tag{4}$$

$$\frac{dF_s}{dA} \equiv \tau_s \qquad \text{Shear stress} \tag{5}$$

+Note: The component of force tangential to the surface is a 2D vector in the surface.

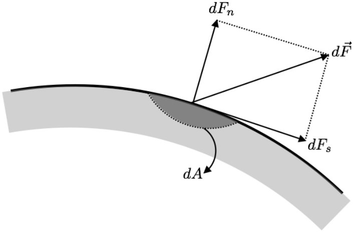

Figure: *(a) Differential force acting on a surface of fluid particle. dF_n is the component of $d\vec{F}$ normal to the surface, and dF_s is the tangential component.*

Surface Forces:

- **Line Forces**: Surface tension forces are called **line forces** because they act along a line and have a magnitude proportional to the extent of the line.
- Line forces appear at the interface between a liquid and gas, or at the interface between two immiscible liquids.
- Surface tension forces only appear in the boundary conditions and do not appear directly in the governing equations of motion.

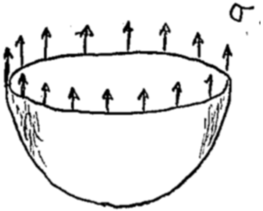

Figure: *(b) Illustration of a line force.*

Chapter Overview

Stress at a Point

The stress at a point can be completely specified by the nine components of the stress tensor τ,

$$\tau = \begin{pmatrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \tau_{32} \end{pmatrix} \tag{6}$$

On each face of an infinitesimal fluid cube, there is a normal stress and two shear (or tangential) stresses, denoted as,

$$\tau_{ij} \tag{7}$$

the i-indice indicates the direction of the unit normal to the surface on which the stress acts, and the j-indice indicates the direction in which the stress acts.

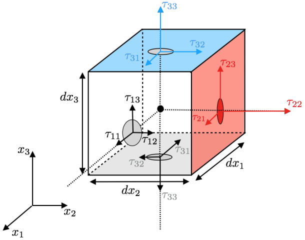

Figure: *Normal and shear stress on an infinitessimal cube with faces perpendicular to the coordinate axes.*

Forces acting on a surface with arbitrary orientation

- The components of τ_{ij} completely determine the state of the stress at that point in space.
- From this stress tensor, one can determine the stress on any arbitrary plane passing through the infinitessimal cube.

For example, what is the stress on face AC which is oriented at an angle θ_2 with respect to the horizontal.

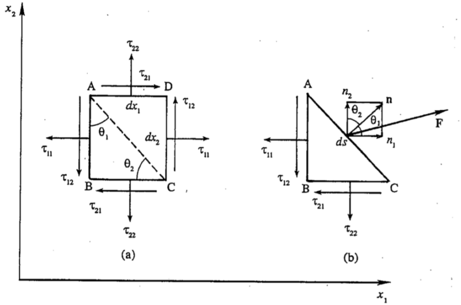

Figure: *Forces acting on a surface with arbitrary orientation.*

Forces acting on a surface with arbitrary orientation

The sum of forces in the x-direction is given by

$$F_1 = \tau_{21}\, dx_1 + \tau_{11}\, dx_2, \tag{8}$$

which dividing by ds to obtain the stress on face AC, one obtains that

$$f_1 = \frac{F_1}{ds} = \tau_{21}\frac{dx_1}{ds} + \tau_{11}\frac{dx_2}{ds}, \tag{9}$$

where $\vec{f} = \frac{\vec{F}}{ds}$ is the force per unit area. Now recognizing that

$$\frac{dx_1}{ds} = cos\theta_2 \qquad \text{and} \qquad \frac{dx_2}{ds} = cos\theta_1 \tag{10}$$

it can be rewritten that,

$$f_1 = \tau_{11}\, cos\theta_1 + \tau_{21}\, cos\theta_2. \tag{11}$$

Forces acting on a surface with arbitrary orientation (continued ...)

The unit vector $\hat{n} = n_1\,\hat{e}_1 + n_2\hat{e}_2$ represents the outward normal to face AC. From the figure, one can see that $n_1 = cos\theta_1$ and $n_2 = cos\theta_2$ since $|\hat{n}| = 1$. Therefore,

$$f_1 = \tau_{11}\,n_1 + \tau_{21}\,n_2. \tag{12}$$

One can perform a similar operation for f_2, which yields

$$f_2 = \tau_{12}\,n_1 + \tau_{22}\,n_2. \tag{13}$$

In general one then finds that

$$\boxed{f_i = \tau_{ji}\,n_i \qquad\qquad \text{or that} \qquad \vec{F} = \hat{n}\tau.} \tag{14}$$

Symmetry of the stress tensor

- The stress tensor has the special property of being symmetric.
- To show this, let's consider the torque on a fluid particle about an axis through the centroid that is parallel to x_3, as shown in the Figure.
- The net torque is equal to the sum of the moments generated by the shear stress acting on the four different faces.

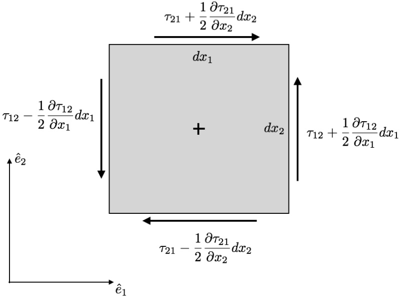

Figure: *Torque on a fluid particle about an axis through the centroid that is parallel to x_3.*

Symmetry of the stress tensor (continued ...)

- Let's start by writing the shear stress at each face based on a Taylor's series expansion of the shear stress at the centroid of the differential rectangle shown.

The net torque is,

$$T = \underbrace{\underbrace{\left[\tau_{12} + \frac{\partial \tau_{12}}{\partial x_1}\frac{dx_1}{2}\right] dx_2}_{\text{Force}} \ \underbrace{\left(\frac{dx_1}{2}\right)}_{\text{moment arm}}}_{\text{right face}} + \underbrace{\left[\tau_{12} - \frac{\partial \tau_{12}}{\partial x_1}\frac{dx_1}{2}\right] dx_2 \left(\frac{dx_1}{2}\right)}_{\text{left face}} \tag{15}$$

$$- \underbrace{\left[\tau_{21} - \frac{\partial \tau_{21}}{\partial x_2}\frac{dx_2}{2}\right] dx_1 \left(\frac{dx_2}{2}\right)}_{\text{bottom face}} - \underbrace{\left[\tau_{21} + \frac{\partial \tau_{21}}{\partial x_2}\frac{dx_2}{2}\right] dx_1 \left(\frac{dx_2}{2}\right)}_{\text{top face}}. \tag{16}$$

The derivative terms cancel leaving

$$T = \tau_{12}\, dx_2\, dx_1 - \tau_{21}\, dx_1\, dx_2. \tag{17}$$

Symmetry of the stress tensor (continued ...)

On the other hand, conservation of angular momentum requires that

$$T = I\,\dot{\omega}_3, \tag{18}$$

where I denotes the moment of inertia of the differential rectangle and $\dot{\omega}_3$ is the angular acceleration.

For the rectangular element,

$$I = dx_1\,dx_2\,(dx_1^2 + dx_2^2)\rho/12. \tag{19}$$

Note, the moment of inertia is defined as

$$I = \int_0^M r^2\,dm \tag{20}$$

where M represents the total mass of the object.

Symmetry of the stress tensor (continued ...)

Substituting in for T and I gives,

$$(\tau_{12} - \tau_{21})\, dx_1\, dx_2 = dx_1\, dx_2\, (dx_1^2 + dx_2^2)\rho/12\dot{\omega} \tag{21}$$

$$= (dx_1^2 + dx_2^2)\rho/12\dot{\omega}. \tag{22}$$

- Now, let the size of the rectangle shrink to zero (we are interested on the stress tensor at a point).
- As $dx_1 \to 0$ and $dx_2 \to 0$, the only way the equation above can be satisfied is if $\tau_{12} = \tau_{21}$.
- One can easily extend this analysis to three dimensions, wherein we should find that

  $$\boxed{\tau_{ij} = \tau_{ji}}, \tag{23}$$

  which says that **the stress tensor is symmetric**.