# Energy Methods

- Recall previous discussion of strain energy
- Work done by force acting through deformation

$$U = \int \underline{P} \cdot d\underline{s}$$

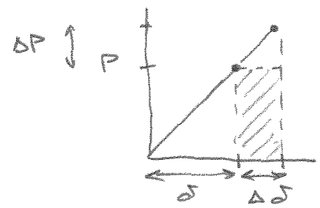
  - If response is linear and direction of force is same as deformation,

$$U = \int_0^\delta k s\, ds = \frac{1}{2} k \delta^2 = \frac{1}{2} P \delta$$

- Reciprocity Theorem
  - Consider two stages, or sets, of loading on simple cantilever

1) $P$, causing $\delta$

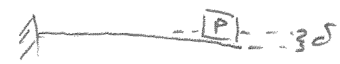
$$U_1 = \frac{1}{2} P \delta$$

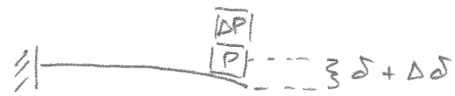



2) $\Delta P$, causing $\Delta\delta$ more

$$U_2 = \frac{1}{2} \Delta P \Delta \delta$$

$$U_{1,2} = P \Delta \delta$$

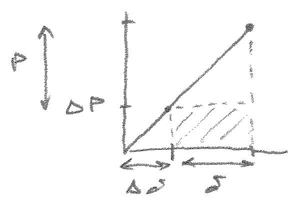



- Now change the order

1) $\Delta P$, causing $\Delta\delta$

$$U_1' = \frac{1}{2} \Delta P \Delta \delta$$




2) $P$, causing $\delta$ more

$$U_2' = \frac{1}{2} P \delta$$

$$U_{2,1} = \Delta P \delta$$

- Since $U = U'$, $U_1 + U_2 + U_{1,2} = U_1' + U_2' + U_{2,1}$

$$\Rightarrow U_{1,2} = U_{2,1} \quad \Rightarrow \quad P \Delta \delta = \Delta P \delta$$

- Can generalize to include any number of forces

$$\Rightarrow \sum P_k' \delta_k'' = \sum P_j'' \delta_j'$$

$\Rightarrow$ work done by one set of forces through displacement due to second set = work done by 2nd set on displacements due to 1st set

- Castigliano's Theorem
  - Consider again $P$ and $\Delta P$
  - $$\Delta U_{1-2} = U_2 + U_{1,2} = \frac{1}{2}\Delta P \Delta\delta + \underbrace{P\Delta\delta}_{\Delta P\,\delta}$$

$$\Rightarrow \quad \frac{\Delta U}{\Delta P} = \frac{1}{2}\Delta\delta + \delta$$

$$\cancelto{\frac{\partial U}{\partial P}}{\lim_{\Delta P\to 0}\frac{\Delta U}{\Delta P}} = \frac{1}{2}\cancelto{0}{\Delta\delta} + \delta$$

$$\Rightarrow \quad \frac{\partial U}{\partial P} = \delta$$

- Again, we generalize here to allow any number of forces be applied as part of set 1. Then

$$\frac{\partial U}{\partial P_i} = \delta_i$$

and we can find the displacement $\delta_i$ associated with the action of any force $P_i$ by taking $\frac{\partial U}{\partial P_i}$.

- Can similarly show,

$$\frac{\partial U}{\partial C_i} = \theta_i$$ , where $C_i$ is any applied couple moment and $\theta_i$ is the slope at the point of application

- Express $U$ as fnc of load. In general,

$$U = \int \frac{N^2}{2AE}\,dx + \int \frac{M^2}{2EI}\,dx + \int \frac{\alpha V^2}{2AG}\,dx + \int \frac{T^2}{2JG}\,dx$$

where $N \equiv$ axial load, $M \equiv$ bending moment, $V \equiv$ shearing force, $T \equiv$ torque, and $\alpha = \frac{A}{I^2}\int \frac{Q^2}{b^2}\,dA \equiv$ form factor for shear

$$= \int_L \int_A \frac{\sigma^2}{2E}\, dA\, dx = \int_L \frac{\sigma^2}{2E} \int_A dA\, dx = \int_L \frac{P^2}{A^2}\frac{1}{2E} A\, dx$$

## Strain Energy Expressions for Common Applications

$$\cancel{\left[\int_V \frac{P^2}{2EA^2}\, dV = \int_0^L \frac{P^2}{2EA^2} \int_A dA\, dx = \int_0^L \frac{P^2 A}{2EA^2}\, dx\right]}$$

- Axial Loading
  - $$U = \int_V U_0\, dV = \int_V \frac{\sigma^2}{2E}\, dV = \int_V \frac{P^2}{2EA^2}\, dV = \int_0^L \frac{P^2}{2EA^2} A\, dx = \underline{\int_0^L \frac{P^2}{2EA}\, dx}$$
  - For prismatic bar, $U = \dfrac{P^2 L}{2EA}$

- Torsion
  - $$U = \int_V U_0\, dV = \int_V \frac{\tau_{xy}^2}{2G}\, dV = \int_V \frac{1}{2G}\left(\frac{T\rho}{J}\right)^2 dV = \int_L \frac{1}{2G}\frac{T^2}{J^2} \overbrace{\int_A \rho^2\, dA}^{J}\, dx$$
    $$= \underline{\int_0^L \frac{T^2}{2GJ}\, dx}$$

- Bending
  - $$U = \int_V \frac{\sigma^2}{2E}\, dV = \int_V \frac{1}{2E}\left(\frac{My}{I}\right)^2 dA\, dx = \int_L \frac{M^2}{2EI^2} \overbrace{\int_A y^2\, dA}^{I}\, dx = \underline{\int_0^L \frac{M^2}{2EI}\, dx}$$

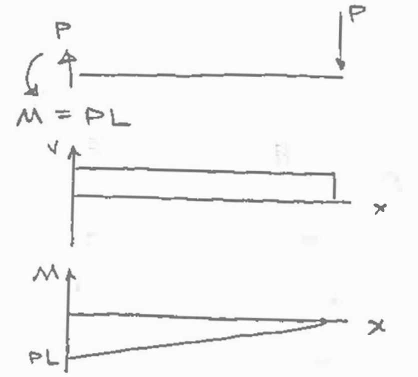
Example

Find deflection of cantilever beam using Castigliano's Method. Compare to traditional solution.


- $\delta_i = \dfrac{\partial U}{\partial P_i}$

$$\Rightarrow \delta_P = \frac{\partial U}{\partial P} = \frac{\partial}{\partial P}\int \frac{M^2}{2EI}dx = \frac{1}{2EI}\int \frac{\partial}{\partial P}(M^2)\,dx = \frac{1}{2EI}\int \frac{\partial (M^2)}{\partial M}\frac{\partial M}{\partial P}dx$$

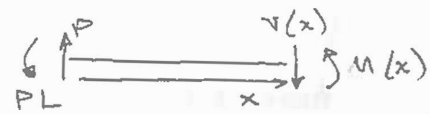
$$= \frac{1}{EI}\int M\frac{\partial M}{\partial P}dx \qquad \text{(ignoring shear, } V\text{)}$$




- Need $M(x)$
  - $\Sigma M_x = 0 \Rightarrow M(x) + PL - Px$

$$\Rightarrow M(x) = Px - PL$$

$$\Rightarrow \frac{\partial M}{\partial P} = x - L$$




$$\Rightarrow \delta_P = \frac{1}{EI}\int_0^L (Px - PL)(x - L)\,dx$$

$$= \frac{1}{EI}\int_0^L (Px^2 - 2PLx + PL^2)\,dx$$

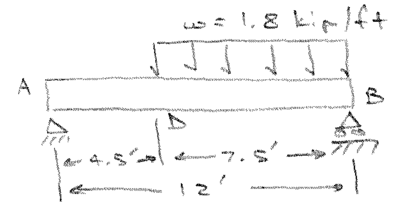
$$= \frac{1}{EI}\left[P\frac{x^3}{3} - \cancel{2}PL\frac{x^2}{\cancel{2}} + PL^2x\right]_0^L$$

$$= \frac{1}{EI}\left[\frac{PL^3}{3} - PL^3 + PL^3\right] = \underline{\frac{PL^3}{3EI}}$$

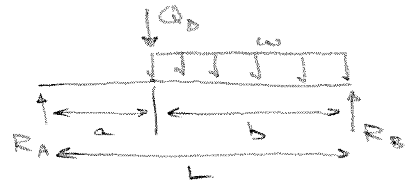
* Table D.4 gives same answer
* Note that we neglected the shear term, so neither sol'n is exact, but we will see that shear term is small

Example (S.P. 11.6, Beer 7, p. 813)

Given the beam and loading shown, find the deflection @ D using Castigliano's Method.




- Sol'n




- No load @ D → apply dummy $Q_D$

$$y_D = \frac{\partial U}{\partial Q_D} = \int_0^L \frac{M}{EI}\frac{\partial M}{\partial Q_D}\,dx$$

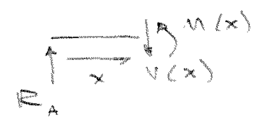
* Need to do this separately for AD and BD since $M(x)$ changes @ D

- Reactions

- $\Sigma M_B = 0 = Q_D b + \frac{wb^2}{2} - R_A L \Rightarrow R_A = Q_D \frac{b}{L} + \frac{wb^2}{2L}$

- $\Sigma M_A = 0 = R_B L - Q_D a - wb\left(a + \frac{b}{2}\right) \Rightarrow R_B = Q_D \frac{a}{L} + \frac{wb}{L}\left(a + \frac{b}{2}\right)$

- Section AD




- $\Sigma M_{cut} = 0 = M(x) - R_A x \Rightarrow M(x) = R_A x$

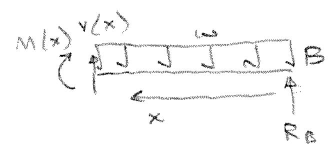
$$\Rightarrow M(x) = Q_D \frac{b}{L}x + \frac{wb^2}{2L}x$$

$$\frac{\partial M}{\partial Q_D} = \frac{b}{L}x$$

$$\Rightarrow \frac{1}{EI}\int M\frac{\partial M}{\partial Q_D}dx = \frac{1}{EI}\int_0^a\left(\cancel{Q_D}^{\,0}\frac{b}{L}x + \frac{wb^2}{2L}x\right)\left(\frac{b}{L}x\right)dx = \frac{1}{EI}\int_0^a \frac{wb^3}{2L^2}x^2\,dx$$

$$= \frac{wb^3}{2EIL^2}\left.\frac{x^3}{3}\right|_0^a = \frac{wa^3b^3}{6EIL^2}$$

- Section BD




- $\Sigma M_{cut} = 0 = R_B x - w\frac{x^2}{2} - M(x)$

$$\Rightarrow M(x) = R_B x - \frac{wx^2}{2} = Q_D \frac{a}{L}x + \frac{wb}{L}\left(a + \frac{b}{2}\right)x - \frac{wx^2}{2}$$

$$\frac{\partial M}{\partial Q_D} = \frac{a}{L}x$$

$$\Rightarrow \frac{1}{EI}\int M\frac{\partial M}{\partial Q_D}dx = \frac{1}{EI}\int_0^b\left[\frac{wb}{L}\left(a+\frac{b}{2}\right)x - \frac{wx^2}{2}\right]\frac{a}{L}x\,dx$$

$$= \frac{1}{EI}\left[\frac{wab}{L^2}\left(a+\frac{b}{2}\right)\frac{x^3}{3} - \frac{wa}{2L}\frac{x^4}{4}\right]_0^b = \frac{1}{EI}\left[\frac{\overset{8}{wab^4}}{\underset{24}{3L^2}}\left(a+\frac{b}{2}\right) - \frac{\overset{3}{wab^4}}{\underset{24}{8L}}\right]$$

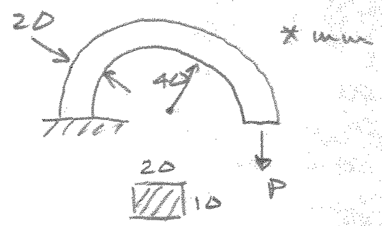
$$= \frac{wab^3}{24EIL}\left[\frac{8ab}{L} + \frac{4b^2}{L} - 3b\frac{L}{L}\right] = \frac{wab^3}{24EIL^2}\left(5ab + b^2\right)$$

$$\Rightarrow y_D = y_{AD} + y_{BD} = \frac{wab^3}{24EIL^2}\underbrace{\left(4a^2 + 5ab + b^2\right)}_{(4a+b)(a+b)} = \frac{wab^3}{24EIL}(4a+b) = \underline{0.262\text{ in}}$$

* Superposition w/ strain energy

**Example**: Castigliano - deflection of half ring (U+F E10.4)

Given: A load of $P = 5\,kN$ is applied to curved steel bar. $E = 200\,GPa$, $G = 80\,GPa$




Find: vertical deflection of free end (considering all stresses)

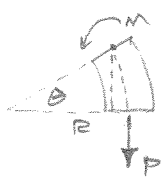
Sol'n:

- FBD of section shows contributions from M, V, and N


$$\Rightarrow \delta_P = \frac{1}{AE}\int N\frac{\partial N}{\partial P}dx + \frac{1}{EI}\int M\frac{\partial M}{\partial P}dx + \frac{1}{AG}\int \alpha V\frac{\partial V}{\partial P}dx$$

- Find expressions for N, M, + V
  - $\sum F_V = 0 = V - P\sin\theta \Rightarrow V = P\sin\theta \Rightarrow \frac{\partial V}{\partial P} = \sin\theta$
  - $\sum F_N = 0 = N - P\cos\theta \Rightarrow N = P\cos\theta \Rightarrow \frac{\partial N}{\partial P} = \cos\theta$
  - $\sum M = 0 = M - P(R - R\cos\theta) \Rightarrow M = PR(1-\cos\theta)$

$$\Rightarrow \frac{\partial M}{\partial P} = R(1-\cos\theta)$$




- Form factor for shear $\alpha = 6/5$ for rectangular section
- Need to integrate over length, but circular $\Rightarrow dx = Rd\theta$

$$\Rightarrow \delta_P = \frac{1}{AE}\int_0^{\pi} P\cos^2\theta\, R\,d\theta + \frac{1}{EI}\int_0^{\pi} PR^2(1-\cos\theta)^2 R\,d\theta + \frac{6}{5AG}\int_0^{\pi} P\sin^2\theta\, R\,d\theta$$

$$= \frac{PR}{AE}\int_0^{\pi}\cos^2\theta\,d\theta + \frac{PR^3}{EI}\int_0^{\pi}(1-\cos\theta)^2 d\theta + \frac{6PR}{5AG}\int_0^{\pi}\sin^2\theta\,d\theta$$

* Recall $\cos^2\theta = \frac{1+\cos 2\theta}{2}$ and $\sin^2\theta = \frac{1-\cos 2\theta}{2}$

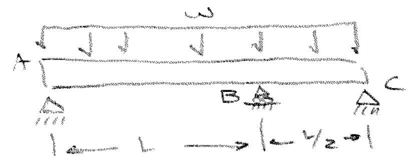
$$\Rightarrow \delta_P = \frac{\pi PR}{2AE} + \frac{3\pi PR^3}{2EI} + \frac{3\pi PR}{5AG}$$

$$= (\underbrace{0.01}_{\text{axial}} + \underbrace{2.21}_{\text{bending}} + \underbrace{0.03}_{\text{shear}}) \times 10^{-3}\,m = \underline{2.25\,mm}$$

* axial + shear contribute little! $\Rightarrow$ common to omit these when $R/c > 4$

Example (S.P. 11.7, Beer 7, p. 815)

Determine reactions @ supports

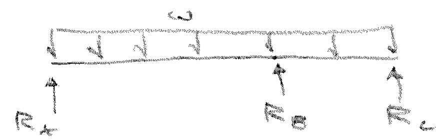



· Sol'n

· Indeterminate: Remove one of the supports to make determinant; choose A to remove


· Solve, then require $y_A = 0$

$$\Rightarrow \quad y_A = \frac{1}{EI}\int M \frac{\partial M}{\partial R_A}\,dx$$




· Find reactions in terms of $R_A$ and $\omega$

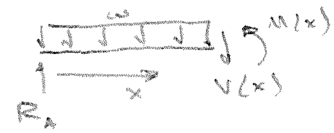
· $\Sigma M_c = 0 = \omega \frac{3}{2}L\left(\frac{3L}{4}\right) - R_A \frac{3L}{2} - R_B \frac{L}{2}$

$$\Rightarrow R_B = \frac{9}{4}\omega L - 3R_A$$

· $\Sigma M_B = 0 = \omega \frac{3L}{2}\underbrace{\left(\frac{3L}{4} - \frac{2L}{4\,2}\right)}^{L/4} + R_C \frac{L}{2} - R_A L$

$$\Rightarrow R_C = 2\left(R_A - \frac{3}{8}\omega L\right) = 2R_A - \frac{3}{4}\omega L$$

· Segment AB

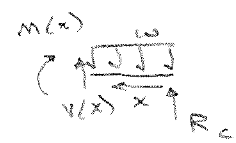



· $\Sigma M_{cut} = 0 = M(x) + \frac{\omega x^2}{2} - R_A x$

$$\Rightarrow M(x) = R_A x - \frac{\omega x^2}{2} \quad ; \quad \frac{\partial M}{\partial R_A} = x$$

$$\Rightarrow \frac{1}{EI}\int_0^L \left(R_A x - \frac{\omega x^2}{2}\right) x\,dx = \frac{1}{EI}\left(R_A \frac{x^3}{3} - \omega \frac{x^4}{8}\right)\Big|_0^L = \frac{1}{EI}\left(R_A \frac{L^3}{3} - \omega \frac{L^4}{8}\right)$$

· Segment BC




· $\Sigma M_{cut} = 0 = -M(x) - \omega \frac{x^2}{2} + R_C x$

$$\Rightarrow M(x) = R_C x - \omega \frac{x^2}{2} = 2R_A x - \frac{3}{4}\omega L x - \omega \frac{x^2}{2}$$

$$\frac{\partial M}{\partial R_A} = 2x$$

$$\Rightarrow \frac{1}{EI}\int_0^{L/2} \left(\cancel{2}^{4} R_A x - \frac{3}{\cancel{4}_{2}}\omega L x - \omega \frac{x^2}{\cancel{2}}\right) \cancel{2} x\,dx = \frac{1}{EI}\left[4R_A \frac{x^3}{3} - \frac{3}{2}\omega L \frac{x^3}{3} - \omega \frac{x^4}{4}\right]_0^{L/2}$$

$$= \frac{1}{EI}\left(\frac{4}{3}R_A \frac{L^3}{8\,2} - \frac{\cancel{3}}{\cancel{2}\cancel{8}}\frac{\omega L^4}{16\,\cancel{8}}_{64} - \underbrace{\frac{\omega}{4}\frac{L^4}{16}}_{64}\right) = \frac{1}{EI}\left(R_A \frac{L^3}{6} - \frac{5}{64}\omega L^4\right)$$

· Now, $y_A = 0 = \frac{1}{EI}\left(R_A \frac{L^3}{3} - \omega \frac{L^4}{8}\right) + \frac{1}{EI}\left(R_A \frac{L^3}{6} - \frac{5}{64}\omega L^4\right)$

$$= \frac{R_A L^3}{2} - \frac{13}{64}\omega L^4 \Rightarrow R_A \frac{\cancel{L^3}}{\cancel{2}} = \frac{13}{\cancel{64}_{32}}\omega L^{\cancel{4}} \Rightarrow \underline{R_A = \frac{13}{32}\omega L}$$

$$\Rightarrow R_B = \frac{9^{72}}{4_{32}}\omega L - 3\left(\frac{13}{32}\omega L\right) = \underline{\frac{33}{32}\omega L} \quad ; \quad \underline{R_C = \frac{\omega L}{16}}$$