Steady-State Error

Goal: Study Steady-state error for various inputs

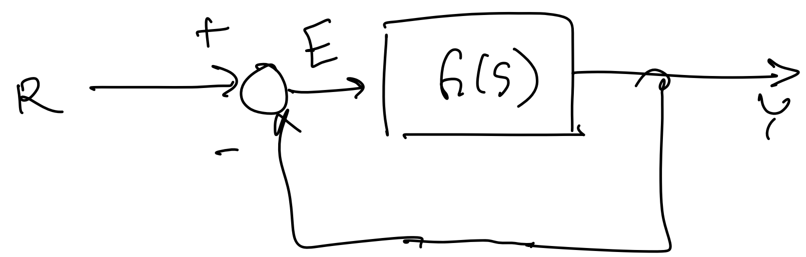
A. **What is error:**

error = desired behavior − Actual behavior

$$e(t) = y_d(t) - y(t)$$

B. Use the final value theorem

Consider the following (Note – unity feedback!)



$$E(s) = R(s) - Y(s) \quad \text{and} \quad \frac{Y(s)}{R(s)} = \frac{G(s)}{1+G(s)} = T(s)$$

$$\Rightarrow Y(s) = T(s)R(s)$$

From error: $E(s) = R(s) - Y(s)$

$$E(s) = R(s) - T(s)R(s)$$

$$\Rightarrow E(s) = R(s)[1 - T(s)] = R(s)\left[1 - \frac{G(s)}{1+G(s)}\right]$$

$$\Rightarrow E(s) = \left[\frac{1+G(s) - G(s)}{1+G(s)}\right]R(s)$$

$$\boxed{E(s) = \underbrace{\left[\frac{1}{1+G(s)}\right]R(s)}_{\text{for unity F.B. system}} = \underbrace{[1 - T(s)]R(s)}_{\text{for all C.L. systems}}}$$

Find s.s. error using F.V.T.:

$\Rightarrow$ Assume that poles of $sE(s)$ are in OLHP. $\Rightarrow$ poles of cloop system are in OLHP.

If the closed-loop system has unity feedback, then:

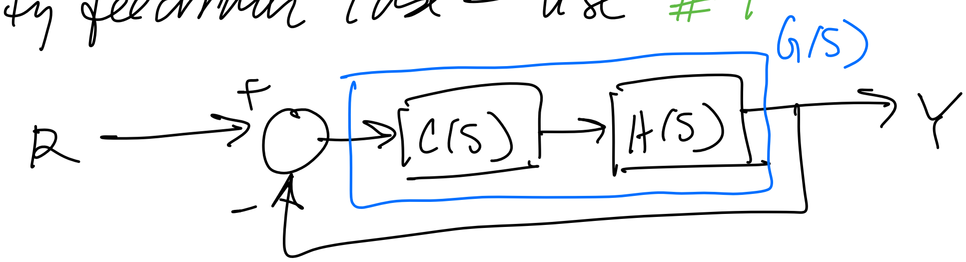
$$\boxed{e_{ss} = \lim_{s\to 0} sE(s) = \lim_{s\to 0} s\left[\frac{1}{1+G(s)}\right]R(s)} \quad \#1$$

Otherwise, we can use this expression

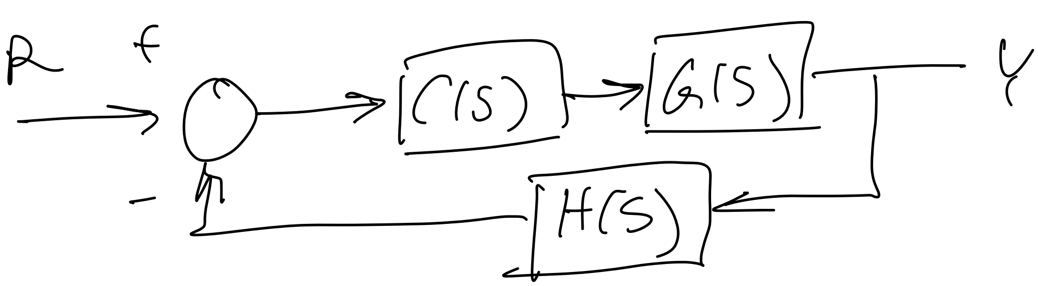
$$\boxed{e_{ss} = \lim_{s \to 0} s\left(1 - T(s)\right) R(s)} \quad \#2$$

Note these 2 expressions and when to use them!

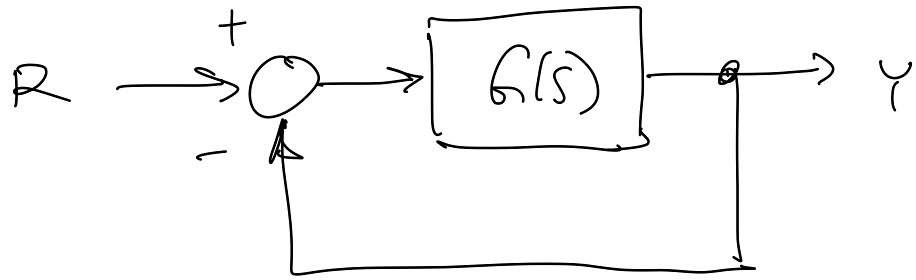
Unity feedback case – use #1



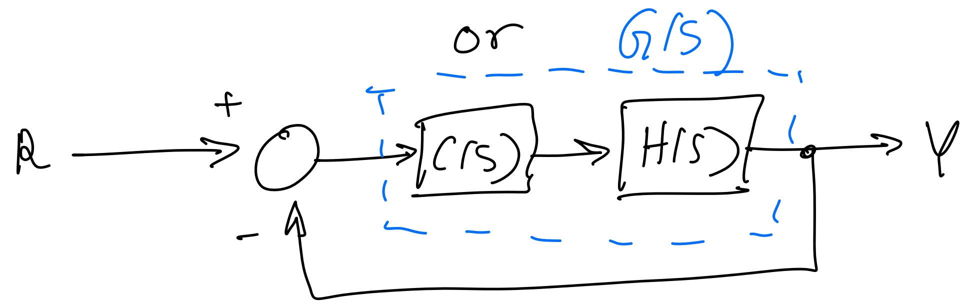
For non-unity or unity feedback – use #2



For unity feedback systems



or

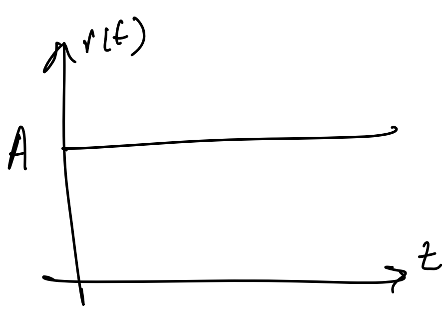


$$e_{ss} = \lim_{s \to 0} \frac{s\,R(s)}{1 + G(s)} = 0$$

We want: $e_{ss} = 0$ or $e_{ss} = \text{constant}$

we <u>do not</u> want $e_{ss} = \infty$

Suppose input is a step



$$r(t) = \begin{cases} A & t > 0 \\ 0 & t \le 0 \end{cases}$$

$$R(s) = \frac{A}{s}$$

$$e_{ss} = \lim_{s\to 0} \frac{s R(s)}{1+G(s)} = \lim_{s\to 0} \frac{\cancel{s}\left(A/\cancel{s}\right)}{1+G(s)}$$

$$= \lim_{s\to 0} \frac{A}{1+G(s)} = \frac{A}{1+\lim_{s\to 0} G(s)}$$

$$\boxed{e_{ss} = \frac{A}{1+\lim_{s\to 0} G(s)} = \frac{A}{1+K_p} \quad \text{where } K_p = \lim_{s\to 0} G(s)}$$

s.s. error expression for unity feedback system when reference is a step of mag. A.

we call $K_p = \lim_{s\to 0} G(s)$ = DC gain of $G(s)$

or K_p is called the position constant.

How do we get $e_{ss} = 0$?

$$e_{ss} = \frac{1}{1+K_p} = 0 \Rightarrow \underline{\underline{K_p = \infty}}$$

OK, then

$$K_p = \lim_{s \to 0} G(s) = \lim_{s \to 0} \left(\frac{(s+z_1)(s+z_2)\ldots}{s^n (s+p_1)(s+p_2)\ldots} \right)$$

for $n \geq 1 \Rightarrow K_p \to \infty$

we need at least 1 pole at origin!

We call n the <u>system type</u>

$n = 0$ type zero system

$e_{ss} = \text{constant}$

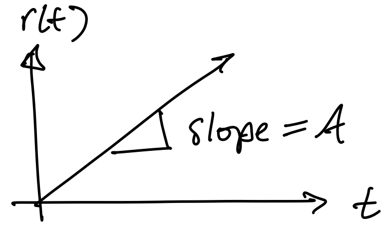
$n = 1$ type 1 system

$e_{ss} = 0$

$n = 2$ type 2 system

$e_{ss} = 0$

For ramp input:



constant velocity signal

$$R(s) = \frac{A}{s^2}$$

s.s. error:

$$e_{ss} = \lim_{s \to 0} \frac{s R(s)}{1 + G(s)} = \lim_{s \to 0} \frac{s\left(A/s^2\right)}{1 + G(s)}$$

$$e_{ss} = \lim_{s \to 0} \frac{A}{s + s G(s)} = \frac{A}{0 + \underbrace{\lim_{s \to 0} s G(s)}_{K_v}}$$

or we have:

$$\boxed{e_{ss} = \frac{A}{K_v} \quad \text{where } K_v = \lim_{s \to 0} s G(s)}$$

we call K_v the velocity constant.

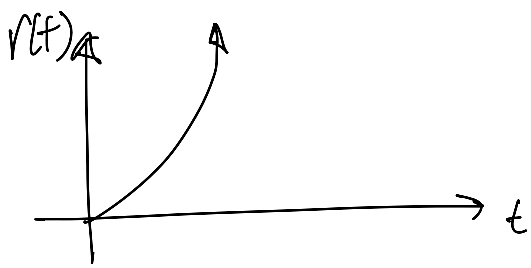
To get $e_{ss} = 0$ for a ramp input, we need $K_v \to \infty$

So,

$$K_v = \lim_{s \to 0} sG(s) = \lim_{s \to 0} s\left[\frac{(s+z_1)(s+z_2)\cdots}{s^n(s+p_1)(s+p_2)\cdots}\right]$$

$\Rightarrow$ we need $n \geq 2$ (type 2) to get $K_v = \infty \Rightarrow e_{ss} = 0$

For parabolic Input:



constant acceleration

$$R(s) = \frac{A}{s^3}$$

$$e_{ss} = \lim_{s \to 0} \frac{sR(s)}{1+G(s)} = \lim_{s \to 0} \frac{s(A/s^3)}{1+G(s)}$$

$$= \lim_{s \to 0}\left[\frac{A}{s^2+s^2G(s)}\right] = \frac{A}{\lim_{s \to 0} s^2G(s)}$$

$$\boxed{e_{ss} = \frac{A}{K_a} \quad \text{where } K_a = \lim_{s \to 0} s^2 G(s)}$$

we call K_a the acceleration constant

What system type do we need to get $e_{ss} = 0$ for a parabolic input?

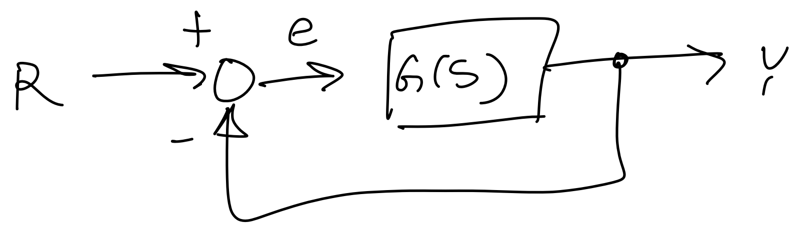
$$K_a = \lim_{s \to 0} s^2 G(s) = \lim_{s \to 0} s^2 \left[\frac{(s+z_1)(s+z_2)\cdots}{s^n (s+P_1)(s+P_2)\cdots} \right]$$

need $K_a = \infty$ to get $e_{ss} = 0$

so, we need $n \geq 3$ (type 3)!

Summary

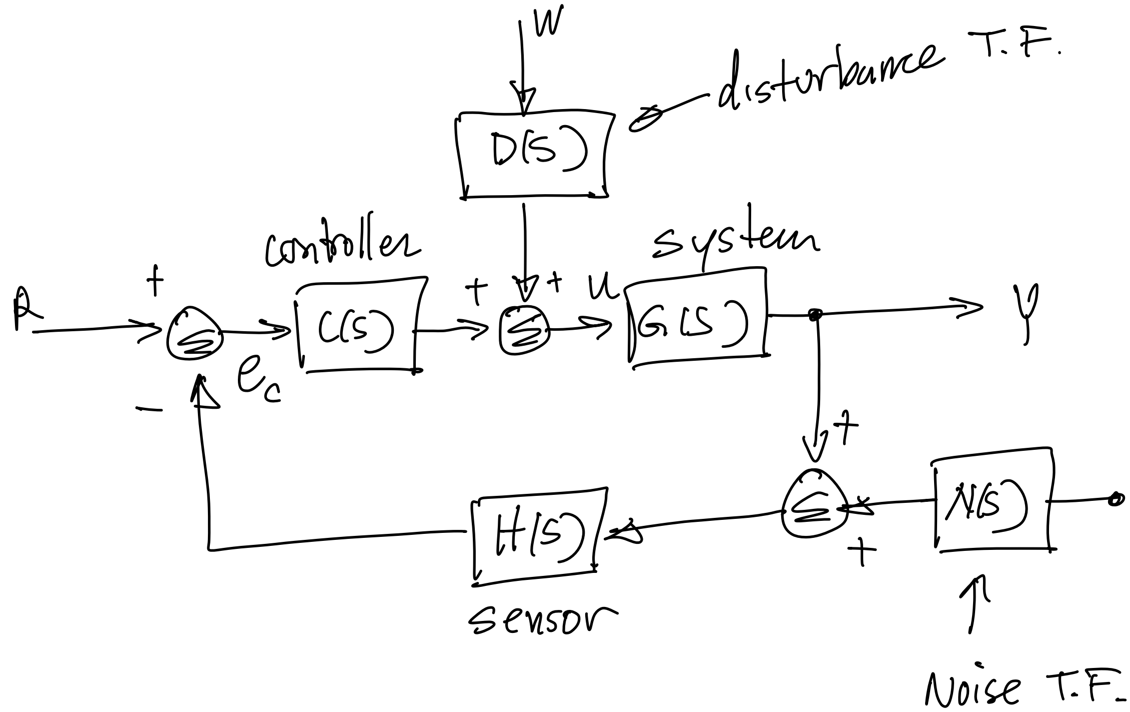
For unity feedback system:



$$e_{ss} = \lim_{s \to 0} \frac{R(s)}{1+G(s)}$$ where $G(s)$ is the "system"

system type	step Input	ramp input	parabolic input
0 $(n=0)$	$\frac{A}{1+K_p}$	∞	∞
I $(n=1)$	0	$\frac{A}{K_v}$	∞
II $(n=2)$	0	0	$\frac{A}{K_a}$
III $(n=3)$	0	0	0

Error due to disturbance and sensor noise



where:

W = disturbance

V = noise

D(S) and N(S) are T.F. that modify W and V before they affect the closed-loop system

Let's find the error due to r, w, and v:

$e = r - y$ (tracking error)

where $y = y_r + y_w + y_v$

From the block diagram:

$$\frac{Y_r(s)}{R(s)} = \frac{C(s)G(s)}{1 + C(s)G(s)H(s)} = T_R(s)$$

$$\frac{Y_w(s)}{W(s)} = \frac{D(s)G(s)}{1 + C(s)G(s)H(s)} = T_W(s)$$

$$\frac{Y_v(s)}{V(s)} = \frac{N(s)H(s)C(s)G(s)}{1 + C(s)G(s)H(s)} = T_V(s)$$

There fore:

$$Y(s) = Y_r(s) + Y_w(s) + Y_v(s)$$

$$Y(s) = \frac{C(s)G(s)}{1 + C(s)G(s)H(s)} R(s) + \frac{D(s)G(s)}{1 + C(s)G(s)H(s)} W(s) + \frac{N(s)H(s)C(s)G(s)}{1 + C(s)G(s)H(s)} V(s)$$

So,

$$E(s) = R(s) - Y(s)$$

$$E(s) = R(s) - T_R(s)R(s) - T_w(s)W(s) - T_v(s)V(s)$$

$$E(s) = [1 - T_R(s)]R(s) - T_W(s)W(s) - T_V(s)V(s)$$

To find e_{ss}:

$$e_{ss} = \lim_{s \to 0} s\,E(s)$$

$$e_{ss} = \lim_{s \to 0} \left\{ s\,[1 - T_R(s)]R(s) - T_W(s)W(s) - T_V(s)V(s) \right\}$$

So, we need to look at limits to determine final error!

Note, they all have same closed-loop poles!!