~~(4)~~ P 1

a.) Simplify x- and y- Momentum equations

$$x:\ \underbrace{\cancel{\rho\frac{\partial u}{\partial t}}}_{\text{Steady}} + \underbrace{\cancel{\rho u\frac{\partial u}{\partial x}}}_{\text{infinite}} + \underbrace{\cancel{\rho v\frac{\partial u}{\partial y}}}_{v=0} + \underbrace{\cancel{\rho w\frac{\partial u}{\partial z}}}_{2D} = -\frac{\partial P}{\partial x} + \underbrace{\cancel{\mu\frac{\partial^2 u}{\partial x^2}}}_{\text{infinite}} + \mu\frac{\partial^2 u}{\partial y^2} + \underbrace{\cancel{\mu\frac{\partial^2 u}{\partial z^2}}}_{2D}$$

$$\boxed{0 = -\frac{\partial P}{\partial x} + \mu\frac{\partial^2 u}{\partial y^2} = K + \mu\frac{\partial^2 u}{\partial y^2}}$$

$$y:\ \underbrace{\cancel{\rho\frac{\partial v}{\partial t}}}_{\text{Steady}} + \underbrace{\cancel{\rho u\frac{\partial v}{\partial x}}}_{\text{infinite}} + \underbrace{\cancel{\rho v\frac{\partial v}{\partial y}}}_{v=0} + \underbrace{\cancel{\rho w\frac{\partial v}{\partial y}}}_{2D} = -\frac{\partial P}{\partial y} + \underbrace{\cancel{\mu\frac{\partial^2 v}{\partial x^2}}}_{\text{infinite}} + \underbrace{\cancel{\mu\frac{\partial^2 v}{\partial y^2}}}_{v=0} + \underbrace{\cancel{\mu\frac{\partial^2 v}{\partial z^2}}}_{2D}$$

$$\boxed{\frac{\partial P}{\partial y} = 0}$$

b.) Solve for u Velocity:

B.C's $u(y=b) = U$
$u(y=0) = 0$

$$\frac{\partial^2 u}{\partial y^2} = \frac{-K}{\mu}$$

$$\int \frac{\partial^2 u}{\partial y^2}\,\partial y = \int \frac{-K}{\mu}\,\partial y$$

$$\int \frac{\partial u}{\partial y}\,\partial y = \int \frac{-K}{\mu}y + C_1\,\partial y$$

$$u = \frac{-K}{2\mu}y^2 + C_1 y + C_2$$

Apply BC's

$u(y=0) = 0 \ \therefore\ C_2 = 0$

$u(y=b) = U \Rightarrow C_1 = \left(\frac{U}{b} + \frac{Kb}{2\mu}\right)y$

$$\boxed{u = \frac{-K}{2\mu}y^2 + \left(\frac{U}{b} + \frac{Kb}{2\mu}\right)y}$$

c.) See Matlab Code

6.) cont...

d.) Stress Tensor $\sigma_{ij} = 2\mu e_{ij}$

$$= 2\mu\left[\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\right] = \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

$$\sigma_{12} = \mu\left(\frac{\partial u_1}{\partial x_2} + \overset{v=0}{\cancel{\frac{\partial u_2}{\partial x_1}}}\right) = \mu\left(\frac{-2k}{2\cancel{\mu}}y + \frac{U}{b} + \frac{kb}{2\cancel{\mu}}\right)$$

$$\sigma_{12} = -ky + \frac{1}{2}kb + \frac{U\mu}{b}$$

$$\sigma_{21} = \mu\left(\overset{0}{\cancel{\frac{\partial u_2}{\partial x_1}}} + \frac{\partial u_1}{\partial x_2}\right) = \sigma_{12}$$

$\sigma_{22} = 0$

$\sigma_{11} = 0$

$$\boxed{\sigma_{ij} = \begin{bmatrix} 0 & -ky + \frac{1}{2}kb + \frac{U\mu}{b} \\ -ky + \frac{1}{2}kb + \frac{U\mu}{b} & 0 \end{bmatrix}}$$

e.) $\sigma_{12} = -ky + \frac{1}{2}kb + \frac{U\mu}{b}$

@ lower wall

$$\left|\tau_{21}\right|_{(y=0)} = \left|\frac{kb}{2} + \frac{U\mu}{b}\right|$$

@ upper wall

$$\left|\tau_{12}\right|_{(y=b)} = -kb + \frac{1}{2}kb + \frac{U\mu}{b}$$

$$\boxed{= -\frac{kb}{2} + \frac{U\mu}{b}}$$

P2

[redacted]

a.) $u_1 = V_0\left(1 - \frac{a}{hL}x_1\right)^{-1}$

$\underline{x \text{ mom}}$

$$u_1 \frac{\partial u_1}{\partial x_1} = \mu \frac{\partial^2 u_1}{\partial x_1^2}$$

$\underline{y \text{ mom}}$

$$u_1 \frac{\partial u_2}{\partial x_1} + u_2 \frac{\partial u_2}{\partial x_2} = \mu\left(\frac{\partial^2 u_2}{\partial x_1^2}\right)$$

$$\frac{\partial u_2}{\partial x_2} = -\frac{\partial u_1}{\partial x_1}$$

$$= -\frac{\partial}{\partial x_1}\left(V_0\left(1 - \frac{a}{hL}x_1\right)^{-1}\right)$$

(check) ?

$$u_2 = \int \frac{V_0 a}{hL}\left(1 - \frac{a}{hL}x_1\right)^{-2} \partial x_2 = \frac{V_0 a}{hL}\left(1 - \frac{a}{hL}x_1\right)^{-2} x_2 + C$$

$\underline{\text{BC's}}$ $x_1 = 0$ & $x_2 = h$, $u_2 = 0$ $\quad \vec{u}\cdot\hat{n} = 0$

$$0 = \frac{V_0 a}{\cancel{h}L}(1-0)^{-2}\cancel{x_2} + c \qquad C = \frac{-V_0 a}{L}$$

$$\boxed{u_2 = \frac{V_0 a}{hL}\left(1 - \frac{a}{hL}x_1\right)^{-2} x_2 - \frac{V_0 a}{L}}$$

3.) a.) X momentum

$$\rho \cancel{\frac{\partial u}{\partial t}}^{0} + \cancel{u\frac{\partial u}{\partial x}}^{0} + \cancel{v\frac{\partial u}{\partial y}}^{0} = \rho g_x - \cancel{\frac{\partial P}{\partial x}}^{0} + \mu\left(\cancel{\frac{\partial^2 u}{\partial x^2}}^{0} + \frac{\partial^2 u}{\partial y^2}\right)$$

$$0 = \rho g_x + \mu\frac{\partial^2 u}{\partial y^2} \rightarrow \underline{\rho g \sin\alpha + \mu\frac{\partial^2 u}{\partial y^2} = 0}$$

$$g_x = g\sin\alpha$$

Y momentum

$$0 = \rho g_y - \frac{\partial P}{\partial y} + \mu\left(\cancel{\frac{\partial^2 v}{\partial x^2}}^{0} + \cancel{\frac{\partial^2 v}{\partial y^2}}^{0}\right) \rightarrow \underline{\rho g\cos\alpha - \frac{\partial P}{\partial y} = 0}$$

$$g_y = g\cos\alpha$$

b.) $\frac{\partial P}{\partial y} = -\rho g\cos\alpha \quad P = -\rho g\cos\alpha\, y + C$

$P(y=h) = P_{atm} \rightarrow C = P_{atm} + \rho g h\cos\alpha$

$$\boxed{P = +\rho g\cos\alpha(h-y) + P_{atm}}$$

c.) X momentum

$$\mu\frac{\partial^2 u}{\partial y^2} = -\rho g\sin\alpha \quad (+)$$

$$\frac{\partial^2 u}{\partial y^2} = -\frac{\rho g}{\mu}\sin\alpha$$

integrate Twice

Integration Const

$$u(y) = -\frac{\rho g}{2\mu}\sin\alpha\, y^2 + Ay + B$$

BC's

$y=0,\ u(y=0)=0 \quad B=0$

$y=h,\ \left.\frac{\partial u}{\partial y}\right|_{y=h} = 0 \qquad -\frac{\rho g}{\mu}\sin\alpha\, h + A = 0$

$$A = \frac{\rho g h}{\mu}\sin\alpha$$

$$\boxed{u(y) = -\frac{\rho g}{2\mu}\sin\alpha\, y^2 + \frac{\rho g h}{\mu}\sin\alpha\, y}$$

3.) continued

e.) $\left|\tau_{12}\right| = \left|\mu \frac{\partial u}{\partial y}\right|_{y=0}$ $u(y) \rightarrow$ part c.)

$$\boxed{\tau = \rho g h \sin\alpha}$$

4.) X momentum

a.) $$\rho \cancel{\frac{\partial u}{\partial t}} + \cancel{u\frac{\partial u}{\partial x}} + \cancel{v\frac{\partial u}{\partial y}} = \cancel{\rho g_x} - \frac{\partial P}{\partial x} + \mu\left(\cancel{\frac{\partial^2 u}{\partial x^2}} + \frac{\partial^2 u}{\partial y^2}\right)$$

$$\frac{\partial P}{\partial x} = \mu \frac{\partial^2 u}{\partial y^2}$$ Integrate twice

$$u(y) = \frac{1}{\mu}\frac{\partial P}{\partial x}\frac{y^2}{2} + \underbrace{Ay + B}_{\text{Int. Const}}$$

BC's $u(y=0) = 0 \Rightarrow B = 0$

$u(y=h) = 0 \Rightarrow A = \frac{-h}{2\mu}\frac{\partial P}{\partial x}$

$$u(y) = \frac{1}{2\mu}\frac{\partial P}{\partial x}(y^2 - hy)$$

Y Momentum

$$\cancel{\rho\frac{\partial v}{\partial t}} + \cancel{u\frac{\partial v}{\partial x}} + \cancel{v\frac{\partial v}{\partial y}} = \cancel{\rho g_y} - \frac{\partial P}{\partial y} + \cancel{\mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)}$$

$$\frac{\partial P}{\partial y} = 0$$

b.) Conservation of energy

$$\cancel{\rho c_p u \frac{\partial T}{\partial x}} + \cancel{\rho c_p v \frac{\partial T}{\partial y}} = \cancel{k\frac{\partial^2 T}{\partial x^2}} + k\frac{\partial^2 T}{\partial y^2} + \Phi$$

$$k\frac{\partial^2 T}{\partial y^2} = -\Phi$$

Viscous Dissipation

$$\Phi = \mu\left(\cancel{\left(\frac{\partial v}{\partial x}\right)^2} + \left(\frac{\partial u}{\partial y}\right)^2\right) = \mu\left(\frac{\partial u}{\partial y}\right)^2 = \cancel{\mu}\left[\frac{\partial}{\partial y}\left(\frac{1}{2\cancel{\mu}}\frac{\partial P}{\partial x}(y^2 - hy)\right)\right]^2$$

$$\Phi = \left[\frac{1}{2}\frac{\partial P}{\partial x}(2y - h)\right]^2$$

$$k\frac{\partial^2 T}{\partial y^2} = -\left[\frac{1}{2}\frac{\partial P}{\partial x}(2y-h)\right]^2$$

#4 b.) Cont...

$$\frac{\partial^2 T}{\partial y^2} = \frac{-1}{k}\left[\left(\frac{\partial p}{\partial x}\right)^2 \frac{(2y-h)^2}{4}\right] = \frac{-1}{4k}\left(\frac{\partial p}{\partial x}\right)^2 (4y^2-4yh+h^2)$$

integrate Twice

Int. const.

$$T = \frac{-1}{4k\mu}\left(\frac{\partial p}{\partial x}\right)^2\left[\frac{y^4}{3} - \frac{2y^3h}{3} + \frac{h^2y^2}{2}\right] + Ay + B$$

BC's

$T = T_0$ @ $y=0$ $\quad B = T_0$

$T = T_1$ @ $y=h$ $\quad A = \left(\frac{T_1 - T_0}{h}\right) + \frac{1}{4k\mu}\left(\frac{\partial p}{\partial x}\right)^2\left(\frac{h^3}{6}\right)$

$$\boxed{T = \frac{-1}{4k}\left(\frac{\partial p}{\partial x}\right)^2\left[\frac{y^4}{3} - \frac{2y^3h}{3} + \frac{h^2y^2}{2}\right] + \left(\frac{T_1-T_0}{h}\right)y + \frac{1}{4k}\left(\frac{\partial p}{\partial x}\right)^2\left(\frac{yh^3}{6}\right) + T_0}$$

c) $\tilde{y} = y/h \quad \tilde{T} = \frac{T-T_0}{\Delta T} \quad \chi = \frac{h^4}{144k\Delta T\mu}\left(\frac{\partial p}{\partial x}\right)^2$

multiply by 36

$$\frac{T-T_0}{\Delta T} = \frac{1}{144k}\left(\frac{\partial p}{\partial x}\right)^2 h^4\left[-12\tilde{y}^4 + 24\tilde{y}^3 - 18\tilde{y}^2 + 6\tilde{y}\right] + \frac{\Delta T\tilde{y}}{\Delta T}$$

$$\boxed{\frac{T-T_0}{\Delta T} = \chi\left[-12\tilde{y}^4 + 24\tilde{y}^3 - 18\tilde{y}^2 + 6\tilde{y}\right] + \tilde{y}}$$

$$T - T_0 = \frac{1}{4k\mu}\left(\frac{\partial p}{\partial x}\right)^2\left[-\frac{y^4}{3} + \frac{2y^3h}{3} - \frac{h^2y^2}{2} + \frac{yh^3}{6}\right]$$

$$= \frac{1}{144k\mu}\left(\frac{\partial p}{\partial x}\right)^2\left[-\frac{36y^4}{3} + \frac{72y^3h}{3} - \frac{36h^2y^2}{2} + 6yh^3\right]$$

$$\frac{T-T_0}{\Delta T} = \frac{1h^4}{\Delta T144k\mu}\left(\frac{\partial p}{\partial x}\right)^2\left[-12\tilde{y}^4 + 24\tilde{y}^3\cancel{h} - 18\cancel{h^2}\tilde{y}^2 + 6\tilde{y}\cancel{h^3}\right] + \tilde{y}$$