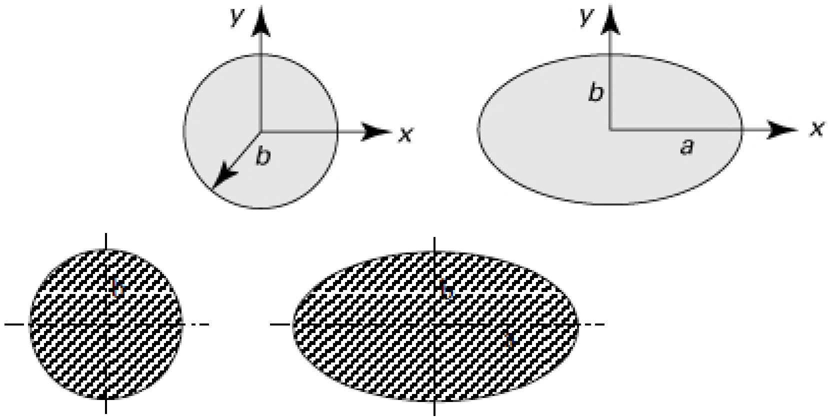
Homework 10 Solutions

1) Consider two bars, one having a circular section of radius b, the other an elliptic section with semiaxes a and b. Determine (a) for equal angles of twist, which bar experiences the larger shearing stress, and (b) for equal allowable shearing stresses, which bar resists a larger torque.



(a) For circular bar:

$$\theta_c = \frac{2T}{G\pi b^4} \qquad \tau_c = \frac{2T}{\pi b^3} = G\theta_c b$$

For elliptical bar:

$$\theta_e = \frac{T(a^2+b^2)}{\pi a^3 b^3 G}, \qquad T = \frac{\pi a^3 b^3 G}{a^2+b^2}\theta_e$$

$$\tau_e = \frac{2T}{\pi a b^2} = \frac{2\theta_e b a^3 G}{a^2+b^2}$$

We have

$$\frac{\tau_e}{\tau_c} = \frac{2\theta_e b a^2 G}{(a^2+b^2)\theta_c b G}$$

Setting $\theta_e = \theta_c$:

$$\frac{\tau_e}{\tau_c} = \frac{2a^2}{(a^2+b^2)}$$

Since $a > b$, $(a^2+b^2) < 2a^2$, and $\tau_e/\tau_c > 1$, or

$$\tau_e > \tau_c \qquad ◄$$

(b)

$$T_c = \frac{\theta_c \pi b^4 G}{2}; \qquad G\theta_c b = \tau_c$$

$$T_c = \frac{\tau_c}{Gb}\frac{\pi b^4 G}{2} = \frac{\tau_c \pi b^3}{2}$$

$$T_e = \frac{\theta_e \pi a^3 b^3 G}{a^2+b^2}, \qquad \tau_e = \frac{2\theta_e b a^2 G}{a^2+b^2}$$

Rearranging,

$$\theta_e = \frac{\tau_e(a^2+b^2)}{2ba^2 G}$$

Thus, $T_e = \frac{\tau_e \pi a b^2}{2}$

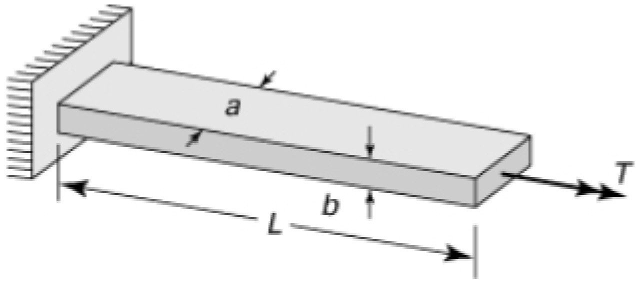
We obtain

$$\frac{T_c}{T_e} = \frac{(\tau_c \pi b^3/2)}{(\tau_e \pi a b^2/2)}$$

Setting $\tau_e = \tau_c$: $\qquad T_c/T_e = b/a$, or

$$T_e > T_c \qquad ◄$$

2) The torque T produces a rotation of 15° at the free end of the steel bar shown. Use $a = 24\ mm$, $b = 16\ mm$, $L = 400\ mm$, and $G = 80\ GPa$. What is the maximum shearing stress in the bar?



Referring to Table 6.2, we have

$$\phi = \theta L = \frac{TL}{\beta a b^2 G} \qquad \tau_{\max} = \frac{T}{\alpha a b^2}$$

b^3

Thus

$$\frac{\phi}{\tau} = \frac{L}{bG}\frac{\alpha}{\beta} \qquad (1)$$

For $a/b = 24/16 = 1.5$: $\alpha = 0.231$ and $\beta = 0.196$

Introducing given data into Eq. (1):

$$\frac{15\ \cancel{1.5}(\pi/180)}{\tau_{\max}} = \frac{0.4(0.231)}{(\cancel{0.024})(80\times10^9)(0.196)}$$

0.016

Solving,

$$\tau_{\max} = \cancel{106.6}\ MPa$$

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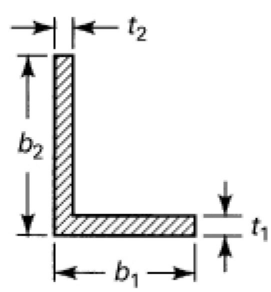
3) A steel bar ($G = 200\ GPa$) of cross section shown is subjected to a torque of $500\ N \cdot m$. Determine the maximum shearing stress and the angle of twist per unit length. The dimensions are $b_1 = 100$ mm, $b_2 = 125$ mm, $t_1 = 10$ mm, and $t_2 = 4$ mm.



Equation (6.20) yields

$$J_e = \sum \tfrac{1}{3} bt^3 = \tfrac{1}{3}(100)(10)^3 + \tfrac{1}{3}(115)(4)^3$$

$$= 3.5787(10^4)\ mm^4$$

Maximum shear stress occurs on the lower leg:

$$\tau_{\max} = \frac{Tt_1}{J_e} = \frac{500(0.1)}{3.5787(10^{-8})} = 139.7\ MPa$$

Note that answers can vary some depending on how geometry was divided up in calculation of J_e

Angle of twist per unit length is

$$\theta = \frac{T}{J_e G} = \frac{500}{3.5787(10^{-8})(200\times10^9)}$$

$$= 69.86(10^{-3})\ rad/m = 4.00^o \text{ per meter}$$

4) A torque T is applied to a thin-walled tube of a cross section in the form of a regular hexagon of constant wall thickness t and mean side length a. Derive relationships for the shearing stress τ and the angle of twist θ per unit length.

For a regular hexagon, we can write

$$A = \frac{3\sqrt{3}}{2}a^3 \qquad ds = 6a$$

Equation (6.22), substituting the value of A, results in the shear stress

$$\tau = \frac{T}{2At} = \frac{T\sqrt{3}}{9a^2t} \quad ◄$$

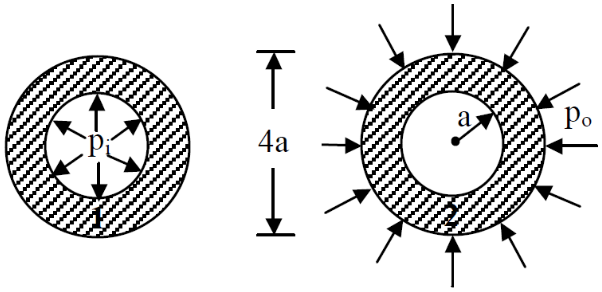
Angle of twist per unit length, using Eq. (6.23):

$$\theta = \frac{\tau(6a)}{2GA} = \frac{3Ta}{2A^2Gt}$$

or

$$\theta = \frac{2T}{9Ga^3t} \quad ◄$$

5) Two thick-walled, closed-ended cylinders of the same dimensions are subjected to internal and external pressure, respectively. The outer diameter of each is twice the inner diameter. What is the ratio of the pressures for the following cases? (a) The maximum tangential stress has the same absolute value in each cylinder. (b) The maximum tangential strain has the same absolute value in each cylinder. Assume axial strain is negligible and take $\nu = 1/3$.



(a) Equation (8.13) and (8.16) give at r=a:

$$\sigma_{\theta 1} = p_i \frac{b^2+a^2}{b^2-a^2}, \qquad \sigma_{\theta 2} = -2p_o \frac{b^2}{b^2-a^2}$$

Then, $|\sigma_{\theta 1}| = |\sigma_{\theta 2}|$; $\quad p_i \frac{b^2+a^2}{b^2-a^2} = 2p_o \frac{b^2}{b^2-a^2}$

or

$$\frac{p_i}{p_o} = \frac{2b^2}{a^2+b^2} = \frac{2(4a^2)}{a^2+4a^2} = 1.6$$ ◀

(b) By neglecting the strain ε_L in the longitudinal direction, (not justified without calculating to verify)

$$\varepsilon_{\theta 1} = \tfrac{1}{E}(\sigma_{\theta 1} + \nu p_i), \qquad \varepsilon_{\theta 2} = \tfrac{1}{E}(\sigma_{\theta 2} + \cancel{\nu} p_o)$$

Then,

$$|\varepsilon_{\theta 1}| = |\varepsilon_{\theta 2}|$$

(but strain is max on inside surface for p_o case, and $\sigma_r = 0$ there)

gives

$$\sigma_{\theta 1} + \nu p_i = \sigma_{\theta 2} - \cancel{\nu} p_o \qquad \text{(a)}$$

But

$$\frac{\sigma_{\theta 1}}{p_i} = \frac{a^2+b^2}{b^2-a^2} = \frac{a^2+4a^2}{4a^2-a^2} = 1.66$$

Hence, $\sigma_{\theta 1} = 1.66 p_i$ (b)

We also have

$$\frac{\sigma_{\theta 2}}{p_o} = \frac{2b^2}{b^2-a^2} = \frac{2(4a^2)}{4a^2-a^2} = 2.66$$

or

$$\sigma_{\theta 2} = 2.66 p_o \qquad \text{(c)}$$

Substituting Eqs. (b) and (c) into (a) and letting $\nu = 1/3$:

$$\frac{p_i}{p_o} = 1.33$$ ◀

6) A cylinder, subjected to internal pressure only, is constructed of aluminum having a tensile strength σ_{yp}. The internal radius of the cylinder is a, and the outer radius is $2a$. Based on the maximum energy of distortion and maximum shear stress theories of failure, predict the limiting values of internal pressure.

At r=a: from Eq. (8.18),

$$\sigma_{\theta,\max} = \frac{4a^2+a^2}{4a^2-a^2} p_i = \tfrac{5}{3} p_i$$

and from Eq. (8.12),

$$\sigma_{r,\max} = -p_i.$$

Energy of distortion theory:

$$p_i[(\tfrac{5}{3})^2 - (\tfrac{5}{3})(-1) + (-1)^2]^{\frac{1}{2}} = \sigma_{yp}$$

or $$p_i = 0.429\sigma_{yp} \quad \blacktriangleleft$$

Maximum shearing stress theory:

$$\tfrac{5}{3} p_i - (-p_i) = \sigma_{yp}$$

or $$p_i = 0.375\sigma_{yp} \quad \blacktriangleleft$$