Solutions Manual: Chapter 8

8th Edition

Feedback Control of Dynamic Systems

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Chapter 8

Digital Control

Problems and Solutions for Section 8.2: Dynamic Analysis of Discrete Systems

1. The z-transform of a discrete-time filter $h(k)$ at a $1Hz$ sample rate is

$$H(z) = \frac{1 + (1/2)z^{-1}}{[1 - (1/2)z^{-1}][1 + (1/3)z^{-1}]}.$$

(a) Let $u(k)$ and $y(k)$ be the discrete input and output of this filter. Find a difference equation relating $u(k)$ and $y(k)$.

(b) Find the natural frequency and damping coefficient of the filter's poles

(c) Is the filter stable?

Solution:

(a) Find a difference equation :

$$H(z) = \frac{Y(z)}{U(z)} = \frac{1 + (1/2)\, z^{-1}}{[1 - (1/2)z^{-1}]\,[1 + (1/3)z^{-1}]}$$

$$\Longrightarrow \quad Y(z) - \frac{1}{6}z^{-1}Y(z) - \frac{1}{6}z^{-2}y(z) = U(z) + \frac{1}{2}z^{-1}U(z)$$

$$\Longrightarrow \quad y(k) - \frac{1}{6}y(k-1) - \frac{1}{6}y(k-2) = u(k) + \frac{1}{2}u(k-1)$$

(b) Two poles at $z = 1/2$ and $z = -1/3$ in z-plane.

$$z = e^{sT} \Longrightarrow s = \frac{-0.693}{T} \text{ and } s = \frac{-1.10 + 3.14j}{T} \text{ in s-plane,}$$

where T is the sampling period. Since the sample rate is 1 Hz, $T = 1$ sec.

$$\begin{aligned}
\text{For } z &= \frac{1}{2},\ \omega_n = \frac{0.693}{T} = 0.693 \text{ rad/sec},\ \zeta = 1.0 \\
\text{For } z &= \frac{-1}{3},\ \omega_n = \frac{3.33}{T} = 3.33 \text{ rad/sec},\ \zeta = 0.330
\end{aligned}$$

(c) Yes, both poles are inside the unit circle.

2. Use the z-transform to solve the difference equation

$$y(k) - 3y(k-1) + 2y(k-2) = 2u(k-1) - 2u(k-2),$$

where

$$u(k) = \begin{cases} k, & k \geq 0, \\ 0, & k < 0, \end{cases}$$
$$y(k) = 0, \qquad k < 0.$$

Solution:

$$\begin{aligned}
\frac{Y(z)}{U(z)} &= \frac{2(z^{-1} - z^{-2})}{1 - 3z^{-1} - 2z^{-2}} = \frac{2}{z-2} \\
u(k) &= \begin{Bmatrix} k & k \geq 0 \\ 0 & k < 0 \end{Bmatrix}
\end{aligned}$$

$$\Longrightarrow U(z) = \frac{z}{(z-1)^2}$$

$$Y(z) = \frac{2}{z-2} \times \frac{z}{(z-1)^2} = \frac{2z}{z-2} - \frac{2z}{z-1} - \frac{2z}{(z-1)^2}$$

Taking the inverse z-transform from Table 8.1,

$$y(k) = 2(2^k - 1 - k) \quad (k \geq 0)$$

3. The one-sided z-transform is defined as

$$F(z) = \sum_0^\infty f(k) z^{-k}.$$

(a) Show that the one-sided transform of $f(k+1)$ is $\mathcal{Z}\{f(k+1)\} = zF(z) - zf(0)$.

(b) Use the one-sided transform to solve for the transforms of the Fibonacci numbers generated by the difference equation $u(k+2) = u(k+1) + u(k)$. Let $u(0) = u(1) = 1$. [*Hint:* You will need to find a general expression for the transform of $f(k+2)$ in terms of the transform of $f(k)$].

(c) Compute the pole locations of the transform of the Fibonacci numbers.

(d) Compute the inverse transform of the Fibonacci numbers.

(e) Show that, if $u(k)$ represents the kth Fibonacci number, then the ratio $u(k+1)/u(k)$ will approach $(1+\sqrt{5})/2$. This is the golden ratio valued so highly by the Greeks.

Solution:

(a)

$$\begin{aligned}\mathcal{Z}\{f()k+1)\} &= \sum_{k=0}^{\infty} f(k+1)z^{-1} = \sum_{j=1}^{\infty} f(j)z^{1(j-1)},\ k+1=j \\ &= z\sum_{0}^{\infty} f(j)z^{-1} - zf(0) \\ &= zF(z) - zf(0)\end{aligned}$$

(b)

$$u(k+2) - u(k+1) - u(k) = 0$$

We have :

$$\mathcal{Z}\{f(k+2)\} = z^2F(z) - z^2f(0) - zf(1)$$

Taking the z-transform,

$$z^2U(z) - z^2u(0) - zu(1) - [zU(z) - zu(0)] - U(z) = 0$$

$$\Longrightarrow (z^2 - z - 1)U(z) = (z^2 - z)u(0) + zu(1)$$

Since $u(0) = u(1) = 1$, we have :

$$U(z) = \frac{z^2}{z^2 - z - 1}$$

(c) The poles are at :

$$z = \frac{1 \pm \sqrt{5}}{2} = 1.618,\ -0.618 \triangleq \alpha_1, \alpha_2$$

(d) (i) By long division :

$$\begin{array}{r} 1 + z^{-1} + 2z^{-2} + 3z^{-3} + \cdots \\ 1 - z^{-1} - z^{-2}\overline{)\,1 \qquad\qquad\qquad\qquad\qquad} \\ \underline{1 - z^{-1} - z^{-2}} \qquad\qquad\quad \\ z^{-1} + z^{-2} \qquad\qquad\quad \\ \underline{z^{-1} - z^{-2} - z^{-3}} \qquad\quad \\ 2z^{-2} + z^{-3} \qquad\quad \\ \underline{2z^{-2} - 2z^{-3} - 2z^{-4}} \\ 3z^{-3} + 2z^{-4} \\ \cdots \end{array}$$

$$u(k) = 1,\ 1,\ 2,\ 3,\ 5, \cdots$$

(ii) By partial fraction expansion :

$$\begin{aligned}
U(z) &= \frac{1}{1 - z^{-1} - z^{-2}} = \frac{1}{(1-\alpha_1 z^{-1})(1-\alpha_2 z^{-1})} \\
&= \frac{\left(\dfrac{\alpha_1}{\alpha_1 - \alpha_2}\right)}{1-\alpha_1 z^{-1}} + \frac{\left(\dfrac{\alpha_2}{\alpha_2 - \alpha_1}\right)}{1-\alpha_2 z^{-1}} \\
u(k) &= \frac{\alpha_1}{\alpha_1 - \alpha_2}\alpha_1^k + \frac{\alpha_2}{\alpha_2 - \alpha_1}\alpha_2^k \\
&= \left(\frac{5+\sqrt{5}}{10}\right)\left(\frac{1+\sqrt{5}}{2}\right)^k + \left(\frac{5-\sqrt{5}}{10}\right)\left(\frac{1-\sqrt{5}}{2}\right)^k
\end{aligned}$$

(e) Since $|\alpha_2| < 1$, for large k the second term is $\cong 0$, and the ratio of $u(k+1)$ to $u(k)$ is $\alpha_1 = (1+\sqrt{5})/2$.

4. Prove the seven properties of the s-plane-to-z-plane mapping listed in Section 8.2.3.

Solution

(a) The stability boundary in s-plane is :

$$s = j\omega, \text{ for all } \omega \text{ between } [-\infty, \infty]$$

By $z = e^{sT}$, this boundary is mapped to :

$$\begin{aligned}
z &= e^{j\omega T} = \cos\omega T + j\sin\omega T \\
\Longrightarrow & \quad |z| = |\cos\omega T + j\sin\omega T|
\end{aligned}$$

Thus, the unit circle in z-plane represents the stability boundary.

(b) In the small vicinity around $s = 0$ in the s-plane,

$$s = -\sigma \pm j\omega_d$$

where $\sigma \ll \omega_s = \frac{2\pi}{T}$ and $\omega_d \ll \omega_s = \frac{2\pi}{T}$.
By $z = e^{sT}$, corresponding locations relative to 1 in the z-plane are :

$$\begin{aligned}
z - 1 &= e^{(-\sigma \pm j\omega_d)T} - 1 \\
&= e^{-\sigma T}(\cos\omega_d T \pm j\sin\omega_d T) - 1 \\
&\cong \left\{1 + \frac{(-\sigma T)}{1!} + \frac{(-\sigma T)^2}{2!} + ...\right\}(1 \pm j\omega_d T) - 1 \\
&= 1 - \sigma T \pm j\omega_d T \mp j\sigma T\omega_d T - 1 \\
&\cong -\sigma T \pm j\omega_d T
\end{aligned}$$

Thus, is approximately mapped to $z - 1 = -\sigma T \pm j\omega_d T$. The small vicinity around $z = +1$ in the z-plane is identical to the vicinity around $s = 0$ in the s-plane by a factor of $T = \frac{2\pi}{\omega_s}$.

(c) An arbitrary location in the s-plane is represented by :

$$\begin{aligned} s &= -\zeta\omega_n + j\omega_n\sqrt{1-\zeta^2} \\ &= x_s(\omega_n) + jy_s(\omega_n) \end{aligned}$$

where ω_n is in rad/sec and is nondimensional. Thus, the s-plane locations give response information in terms of frequency.

By $z = e^{sT} = e^{s\frac{2\pi}{\omega_s}}$, the corresponding location in the z-plane is :

$$\begin{aligned} z &= e^{-2\pi\frac{\omega_n}{\omega_s}} \cos\left(2\pi\sqrt{1-\zeta^2}\frac{\omega_n}{\omega_s}\right) + je^{-2\pi\frac{\omega_n}{\omega_s}} \sin\left(2\pi\sqrt{1-\zeta^2}\frac{\omega_n}{\omega_s}\right) \\ &= x_z\left(\frac{\omega_n}{\omega_s}\right) + jy_z\left(\frac{\omega_n}{\omega_s}\right) \end{aligned}$$

Since $\frac{\omega_n}{\omega_s}$ is nondimensional, the z-plane locations give response information normalized to the sample rate.

(d) Locations in the s-plane, $s = -\sigma \pm j\omega_d$, are mapped to z-plane locations :

$$z = e^{-\sigma T}(\cos\omega_d T \pm j\sin\omega_d T)$$

If z is on the negative real axis, we need :

$$\begin{aligned} \cos\omega_d T &< 0,\ \sin\omega_d T = 0 \\ &\Longrightarrow \omega_d T = 2\pi n + \pi,\ n = 0, 1, 2, ... \\ &\Longrightarrow \omega_d = (2n+1)\frac{\omega_s}{2},\ n = 0, 1, 2, ... \end{aligned}$$

Indeed, if $\omega_d = (2m+1)\frac{\omega_s}{2},\ n = 0, 1, 2, ...,$

$$z = -e^{-\sigma T} \Longrightarrow \text{negative real axis}$$

Thus, the negative real z-axis represents a horizontal line with a damped frequency :

$$\omega_d = (2n+1)\frac{\omega_s}{2},\ n = 0, 1, 2, ...$$

(e) An arbitrary vertical line in the left half of the s-plane is represented by :

$$s = -\sigma \pm j\omega_d,\ \sigma > 0,\ \text{for all } \omega_d \text{ between } [-\infty, \infty]$$

By $z = e^{sT}$, the vertical line is mapped to :

$$z = -e^{-\sigma T}e^{\pm j\omega_d T} = -e^{-\sigma T}(\cos\omega_d T \pm j\sin\omega_d T)$$

$$\Longrightarrow |z| = r = |-e^{-\sigma T}| = \text{constant} < 1$$

$$\angle z = 0 \longrightarrow 2\pi \text{ as } \omega_d = 2n\pi \longrightarrow (2n+1)\pi$$

Thus, vertical lines in the left half of the s-plane map into circles within the unit circle of the z-plane.

(f) An arbitrary horizontal line in the s-plane is represented by :

$$s = -\sigma \pm j\omega_d, \text{ for } \sigma \text{ between } [-\infty, \infty], \text{ at a given } \omega_d$$

By $z = e^{sT}$, the horizontal line is mapped to :

$$z = -e^{-\sigma T} e^{\pm j\omega_d T} = -e^{-\sigma T}(\cos\omega_d T \pm j\sin\omega_d T)$$

$$\Longrightarrow \angle z = \tan^{-1}\left(\frac{e^{-\sigma T}\sin\omega_d T}{e^{-\sigma T}\cos\omega_d T}\right) = \omega_d T = \text{ constant}$$

$$|z| = r = 0 \longrightarrow \infty \text{ as } \sigma = \infty \longrightarrow -\infty$$

Thus, a horizontal line in the s-plane maps into radial lines in the z-plane.

(g) Let s-plane locations s_1 and s_2 be :

$$\begin{aligned} s_1 &= -\sigma \pm j\omega_d \\ s_2 &= -\sigma \pm j\left(\omega_d + m\omega_s\right), \quad m = 1, 2, 3, ... \end{aligned}$$

where ω_d is between $\left[-\frac{2\pi}{\omega_s}, \ \frac{\omega_s}{2}\right]$, which is called the "primary strip".

By $z = e^{sT} = e^{s\frac{2\pi}{\omega_s}}$, these s-plane locations are mapped to z-plane locations :

$$\begin{aligned} z_1 &= e^{-\sigma\frac{2\pi}{\omega_s}}\left(\cos\omega_d\frac{2\pi}{\omega_s} + j\sin\omega_d\frac{2\pi}{\omega_s}\right) \\ z_2 &= e^{-\sigma\frac{2\pi}{\omega_s}}\left\{\cos\left(\omega_d + 2m\frac{\omega_s}{2}\right)\frac{2\pi}{\omega_s} + j\sin\left(\omega_d + 2m\frac{\omega_s}{2}\right)\frac{2\pi}{\omega_s}\right\} \\ &= e^{-\sigma\frac{2\pi}{\omega_s}}\left\{\cos\left(\omega_d\frac{2\pi}{\omega_s}\right)\cos 2m\pi - \sin\left(\omega_d\frac{2\pi}{\omega_s}\right)\sin 2m\pi + j\sin\left(\omega_d\frac{2\pi}{\omega_s}\right)\cos 2m\pi + ...\right\} \\ &= e^{-\sigma\frac{2\pi}{\omega_s}}\left\{\cos\left(\omega_d\frac{2\pi}{\omega_s}\right) + j\sin\left(\omega_d\frac{2\pi}{\omega_s}\right)\right\} \\ &= z_1 \end{aligned}$$

Thus, frequencies greater than $\frac{\omega_s}{2}$ appear in the z-plane on top of corresponding lower frequencies. Physically, this means that frequencies sampled faster than $\frac{\omega_s}{2}$ will appear in the samples to be at a much lower frequency. This is called "aliasing".

Problems and Solutions for Section 8.3: Design using Discrete Equivalents

5. A unity feedback system has an open-loop transfer function given by

$$G(s) = \frac{250}{s[(s/10) + 1]}.$$

The following lag compensator added in series with the plant yields a phase margin of 50°:

$$D_c(s) = \frac{s/1.25+1}{50s+1}.$$

Using the matched pole-zero approximation, determine an equivalent digital realization of this compensator for a sample rate of 20 Hz.

Solution:

(a) For the compensated closed-loop system, $\frac{D_c(s)G(s)}{1+D_c(s)G(s)}$, the crossover frequencey is approximatley 4 rad/sec, so the bandwidth is approximately 8 rad/sec. A very conservative sample rate would be faster than ω_{BW} by a factor of 20, yielding a sample rate, ω_s,

$$\omega_s = 20 \times 8 = 160 \text{ rad/sec} \cong 50 \text{ Hz}$$

So the problem statement's choice of 20 Hz is about 8 x ω_{BW},which is usually sufficient. For 20 Hz, the sample rate is $T = 0.05$ sec. Since

$$D_c(s) = \frac{1+s/1.25}{1+s/0.02} = 0.016\frac{s+1.25}{s+0.02},$$

an equivalent $D_c(z)$ is found for the matched pole-zero method by using the method summarized in Section 8.3.3. Step 1 maps the pole and zero, while Eq. (8.28) shows how to compute the gain. The result can also be calculated using Matlab's `c2d(sysD,T,'matched')` is,

$$D_c(s) = \frac{0.0165(z) - 0.0155}{z - 0.999}.$$

6. The following transfer function is a lead network designed to add about 60° of phase at $\omega_1 = 3$ rad/sec:

$$H(s) = \frac{s+1}{0.1s+1}.$$

(a) Assume a sampling period of $T = 0.25$ sec, and compute and plot in the z-plane the pole and zero locations of the digital implementations of $H(s)$ obtained using (1) Tustin's method and (2) pole-zero mapping. For each case, compute the amount of phase lead provided by the network at $z_1 = e^{j\omega_1 T}$

(b) Using a log-log scale for the frequency range $\omega = 0.1$ to $\omega = 100$ rad/sec, plot the magnitude Bode plots for each of the equivalent digital systems you found in part (a), and compare with $H(s)$. (*Hint:* Magnitude Bode plots are given by $|H(z)| = |H(e^{j\omega T})|$.)

Solution:

(a)

$$H(s) = \frac{s+1}{0.1s+1}, \ \angle H(j\omega)|_{\omega=3} = 54.87^{\circ}$$

From MATLAB, [mag,phasew1] = bode([1 1],[.1 1],3) yields phasew1 = 54.87.

(1) Tustin's method, analytically :

$$\begin{aligned} H(z) &= H(s)|_{s=\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}} = \frac{(2+T)+(T-2)z^{-1}}{(0.2+T)+(T-0.2)z^{-1}} \\ &= 5\frac{z-0.7778}{z+0.1111} \end{aligned}$$

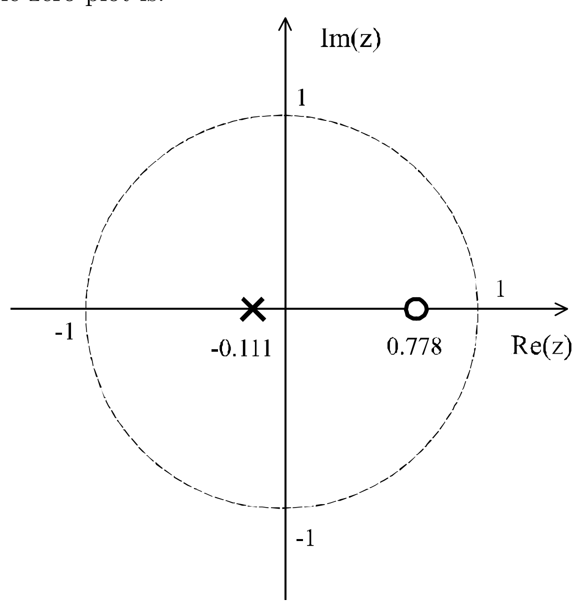
or, via MATLAB:

```
sysC = tf([1  1],[.1  1]);
sysDTust = c2d(sysC,T,'tustin')
```

Phase lead at $\omega_1 = 3$: $\angle H(e^{j\omega_1 T}) = 54.90^{\circ}$, which is most easily obtained by MATLAB

```
[mag,phasew1] = bode(sysDTust,3)
```

The pole-zero plot is:

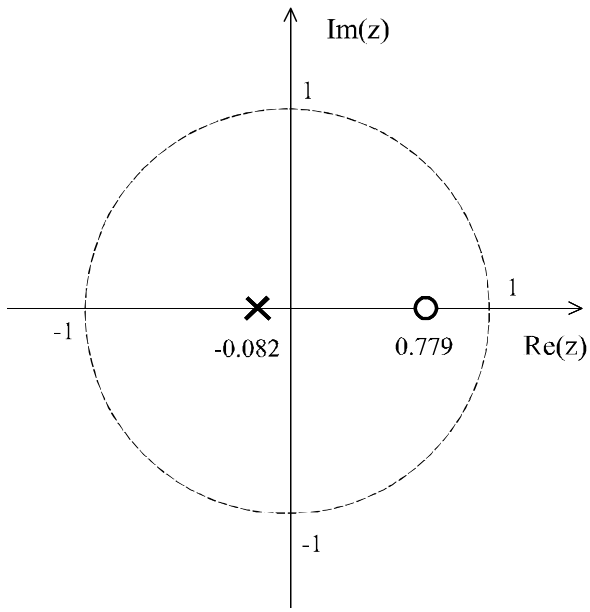


(2) Matched pole-zero method, analytically :

$$\begin{aligned} H(z) &= K\frac{z-e^{-1T}}{z-e^{-10T}} = 4.150\frac{z-0.7788}{z-0.0821} \\ K &= 4.150 \Longrightarrow |H(z)|_{z=1} = |H(s)|_{s=0} \end{aligned}$$

or, alternatively via MATLAB

sysDmpz = c2d(sysC,T,'matched')
will produce the same result.
Phase lead at $\omega_1 = 3$: $\angle H(e^{j\omega_1 T}) = 47.58°$ is obtained from [mag,phasew1] = bode(sysDmpz,3). The pole-zero plot is below. Note how similar the two pole-zero plots are.



(b) The Bode plots match fairly well until the frequency approaches the half sample frequency ($\cong$ 12 rad/sec), at which time the curves diverge.

7. The following transfer function is a lag network designed to introduce a gain attenuation of $10(-20\text{dB})$ at $\omega_1 = 3$ rad/sec:

$$H(s) = \frac{10s+1}{100s+1}.$$

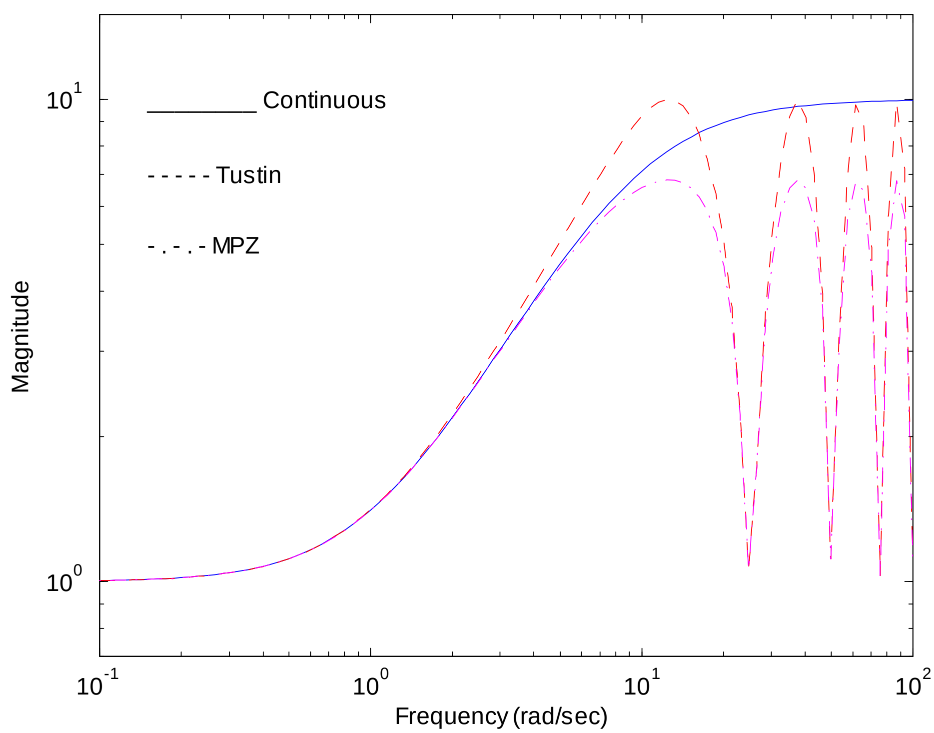
(a) Assume a sampling period of $T = 0.25\,\text{sec}$, and compute and plot in the z-plane the pole and zero locations of the digital implementations of $H(s)$ obtained using (1) Tustin's method and (2) pole-zero mapping. For each case, compute the amount of gain attenuation provided by the network at $z_1 = e^{j\omega_1 T}$.

(b) For each of the equivalent digital systems in part (a), plot the Bode magnitude curves over the frequency range $\omega = 0.01$ to 10 rad/sec.

Solution:

(a) First, we'll compute the attenuation of the continuous system,

$$H(s) = \frac{10s+1}{100s+1}, \; |H(j\omega)|_{\omega=3} = 0.1001 \;\; (-20 \text{ db})$$


Continuous
Tustin
MPZ
Magnitude
Frequency (rad/sec)
10^{1}
10^{0}
10^{-1}
10^{0}
10^{1}
10^{2}

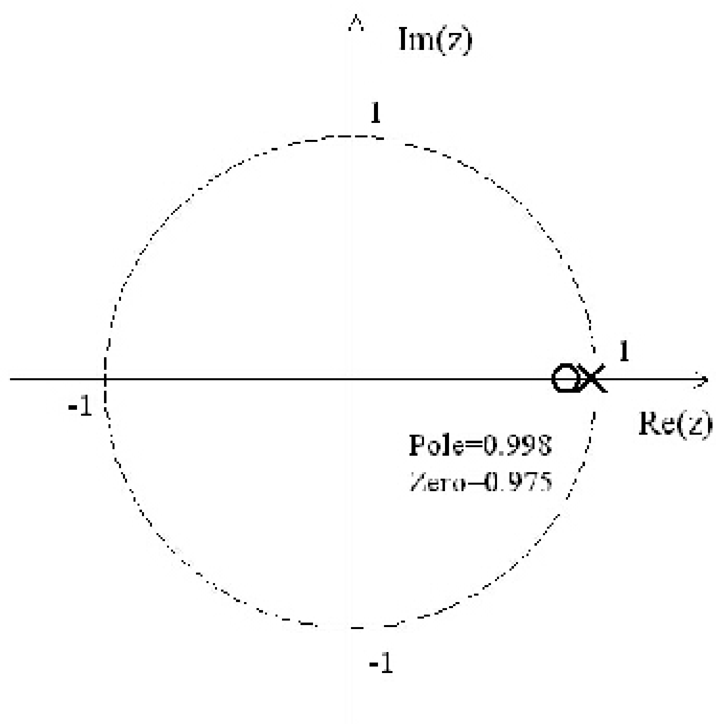
(1) Tustin's method :

$$\begin{aligned} H(z) &= H(s)|_{s=\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}} = \frac{(20+T)+(T-20)z^{-1}}{(200+T)+(T-200)z^{-1}} \\ &= 0.10112\frac{z-0.97531}{z+0.99750} \end{aligned}$$

or, use c2d as shown for problem 5.
Gain attenuation at $\omega_1 = 3$: $|H(e^{j\omega_1 T})| = 0.1000$ (-20 db), most easily computed from: [mag,phase]=bode(sysDTust,T,3).
(2) Matched pole-zero method :

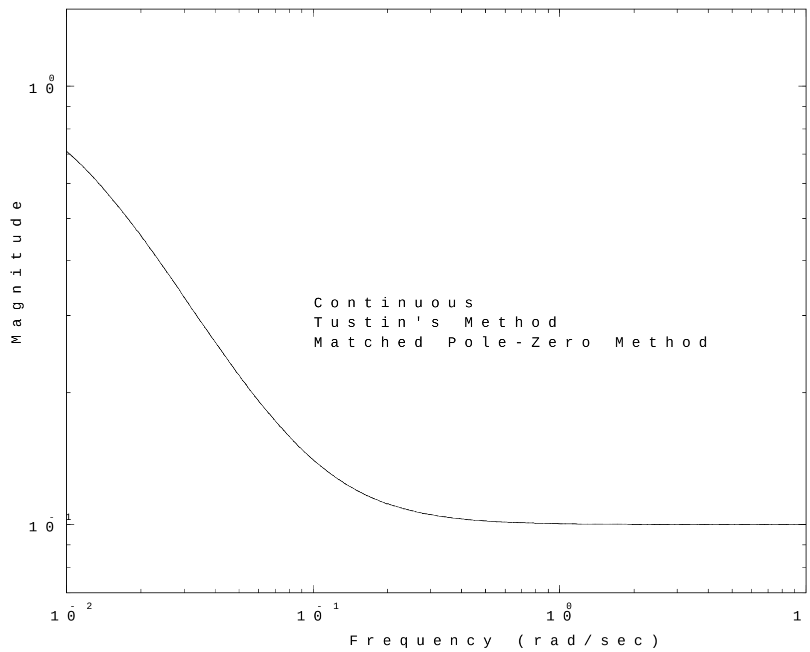
$$\begin{aligned} H(z) &= K\frac{z-e^{-0.1T}}{z-e^{-0.01T}} = 0.10113\frac{z-0.97531}{z-0.99750} \\ K &= 0.10113 \longleftarrow |H(z)|_{z=1} = |H(s)|_{s=0} \end{aligned}$$

Gain attenuation at $\omega_1 = 3$: $|H(e^{j\omega_1 T})| = 0.1001$ (-20 db), most easily computed from: [mag,phase]=bode(sysDmpz,T,3).



In this case, the sampling rate is so fast compared to the break frequencies that both methods give essentially the same equivalent, and both have a gain attenuation of a factor of 10 at $\omega_1 = 3$ rad/sec.

(b) All three are essentially the same and indistinguishable on the plot because the range of interest is below the half sample frequency (= 12 rad/sec).



Problems and Solutions for Section 8.5: Sample Rate Selection

8. For the system shown in Fig. 8.22, find values for K, T_D, and T_I so that the closed-loop poles satisfy $\zeta > 0.6$ and $\omega_n > 1$ rad/sec. Discretize the PID controller using:

(a) Tustin's method

(b) matched pole-zero method

Use MATLAB to simulate the step response of each of these digital implementations for sample times of $T = 1$, 0.1, and 0.01 sec.

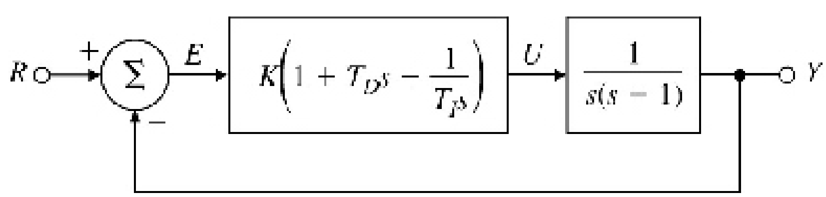


Fig. 8.22 Control system for Problem 8.8

Solution

(a) Continuous PID-controller design

Plant transfer function :

$$G(s) = \frac{1}{s(s+1)}$$

Continuous PID controller :

$$D_c(s) = K\left(1 + T_D s + \frac{1}{T_I s}\right)$$

The specification is that

$$\omega_n > 1 \text{ rad/sec}, \ \zeta > 0.6$$

There is no requirement that there be an integral term, so first let's look at a design without the integral term. To understand the difficulty, a sketch of the root locus with only proportional control (T_D =0) shows that $K = 1$ yields roots at $s = -0.5 \pm 0.86j$ which means that $\omega = 1$ rad/sec and $\zeta = 0.5$,thus the damping spec is not met. If we lower or raise the gain, there is not value that will meet both specs. It would certainly be useful to add a little derivative control in order to pull the locus to the left and allow for meeting the specs. One approach is to try some values of T_D and iterate with rlocus in MATLAB until a comfortable margin is reached on the two specs. Generally, it is also a good design feature to have some integral control in order to reduce steady state errors, so it would make sense to include the integral term. This term can also be designed iteratively by introducing a small amount (large T_I) and adjusting the other gains as needed to meet the specs. Clearly, this problem is underdetermined and there are many ways to meet the specs, a typical situation in control system design.

We somewhat arbitrarily select $T_I = 10.0$, which will provide a fairly low gain on the integral term. Iterating on K and T_D. We find that:

$$K = 1.18, \ T_D = 0.3, \ T_I = 10.0$$

yields

$$\omega_n = 1.02 \text{ rad/sec}, \ \zeta = 0.61$$

Re-arranging some, we have the continuous PID controller transfer function:

$$D_c(s) = \frac{KT_D(s+\alpha)(s+\beta)}{s}$$

where

$$
\begin{aligned}
\alpha &= \frac{1}{2T_D} + \frac{1}{2T_D}\sqrt{1 - 4\frac{T_D}{T_I}} \\
\beta &= \frac{1}{2T_D} - \frac{1}{2T_D}\sqrt{1 - 4\frac{T_D}{T_I}}
\end{aligned}
$$

(b) Discrete PID controller by Tustin's method can be obtained analytically as below or by using `c2d` in MATLAB :

$$
\begin{aligned}
D_d(z) &= D_c(s)|_{s=\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}} \\
&= \frac{K\left\{\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}} + T_D\left(\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}\right)^2 + \frac{1}{T_I}\right\}}{\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}} \\
&= \frac{\left(K + KT_D\frac{2}{T} + \frac{KT}{2T_I}\right) + \left(-2KT_D\frac{2}{T} + 2\frac{KT}{2T_I}\right)z^{-1} + \left(-K + KT_D\frac{2}{T} + \frac{KT}{2T_I}\right)z^{-1}}{1 - z^{-2}} \\
&= \left\{\begin{array}{cc}
\frac{3.3300 - 2.662z^{-1} - 0.305z^{-2}}{1-z^{-2}}, & T = 1 \\
\frac{16.042 - 28.414z^{-1} + 12.408z^{-2}}{1-z^{-2}} & T = 0.1 \\
\frac{143.980 - 284.322z^{-1} + 140.346z^{-2}}{1-z^{-2}} & T = 0.01
\end{array}\right\}
\end{aligned}
$$

(c) For the Matched Pole-zero approximation, note there is one more zero than pole, hence we need to add a pole at $z = -1$,

$$
D_d(z) = K_d\frac{(z - e^{-\alpha T})(z - e^{-\beta T})}{(z+1)(z-1)}
$$

There is no DC gain for this transfer function, so we can either match the K_v of $D(z)$ with that of $D(s)$ or match the gain at some other frequency. A good choice would be to match the gains at $s = j\omega_n$ for example. (ω_n is the closed-loop natural frequency.) Carrying this out,

$$
\begin{aligned}
D_c(s)|_{s=j\omega_n} &= KT_D\left\{\frac{1}{T_D} + \left(\omega_n - \frac{1}{T_I T_D \omega_n}\right)j\right\} \\
|D_c(s)|_{s=j\omega_n}| &= KT_D\sqrt{\left(\frac{1}{T_D}\right)^2 + \left(\omega_n - \frac{1}{T_I T_D \omega_n}\right)^2}
\end{aligned}
$$

$$
\begin{aligned}
D_d(z)|_{z=e^{j\omega_n T}} &= K_d\frac{A + Bj}{\{\cos(2\omega_n T) - 1\}^2 + \{\sin(2\omega_n T)\}^2} \\
|D_d(z)|_{z=e^{j\omega_n T}}| &= K_d\frac{\sqrt{A^2 + B^2}}{2 + 2\cos(2\omega_n T)}
\end{aligned}
$$

where

$$\begin{aligned} A &= \left\{\cos(2\omega_n T) - \left(e^{-\alpha T} + e^{-\beta T}\right)\cos(\omega_n T) + e^{-(\alpha+\beta)T}\right\}\{\cos(2\omega_n T) - 1\} \\ &\quad + \left\{\sin(2\omega_n T) - (e^{-\alpha T} + e^{-\beta T})\sin(\omega_n T)\right\}\sin(2\omega_n T) \\ B &= \left\{\cos(2\omega_n T) - \left(e^{-\alpha T} + e^{-\beta T}\right)\cos(\omega_n T) + e^{-(\alpha+\beta)T}\right\}\sin(2\omega_n T) \\ &\quad + \left\{\sin(2\omega_n T) - (e^{-\alpha T} + e^{-\beta T})\sin(\omega_n T)\right\}\{\cos(2\omega_n T) - 1\} \end{aligned}$$

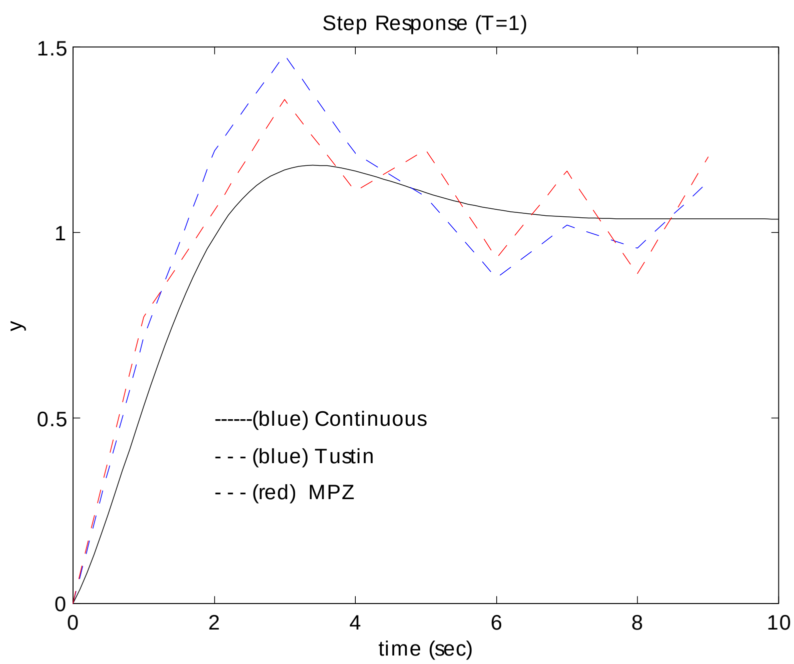
$$|D_c(s)|_{s=j\omega_n}| = |D_d(z)|_{z=e^{j\omega_n T}}|$$

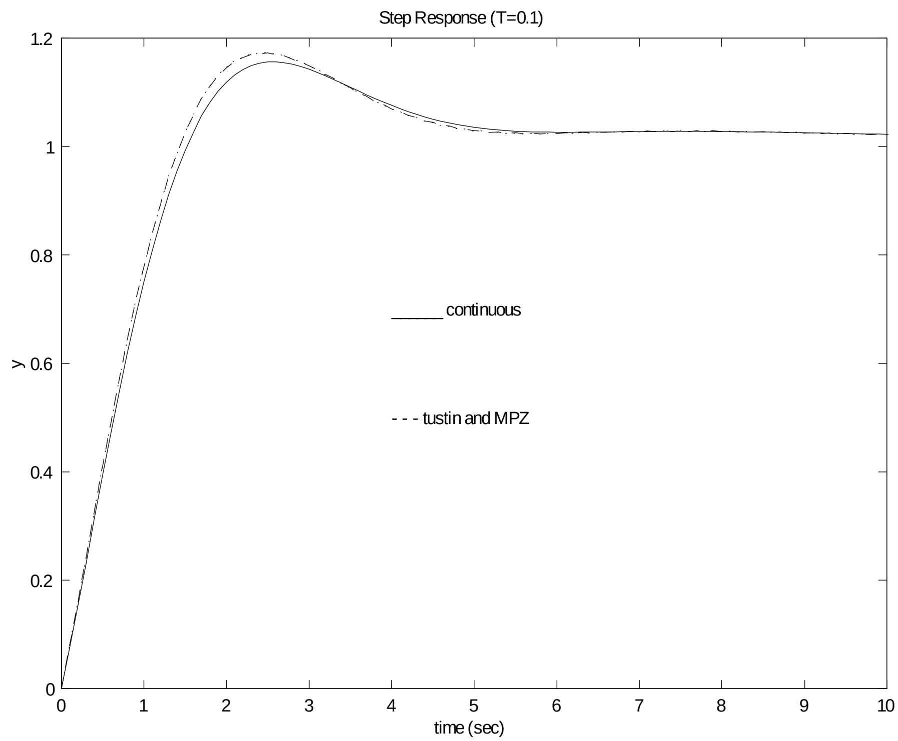
$$\Longrightarrow K_d = KT_D\sqrt{\left(\frac{1}{T_D}\right)^2 + \left(\omega_n - \frac{1}{T_I T_D \omega_n}\right)^2}\frac{2+2\cos(2\omega_n T)}{\sqrt{A^2+B^2}}$$

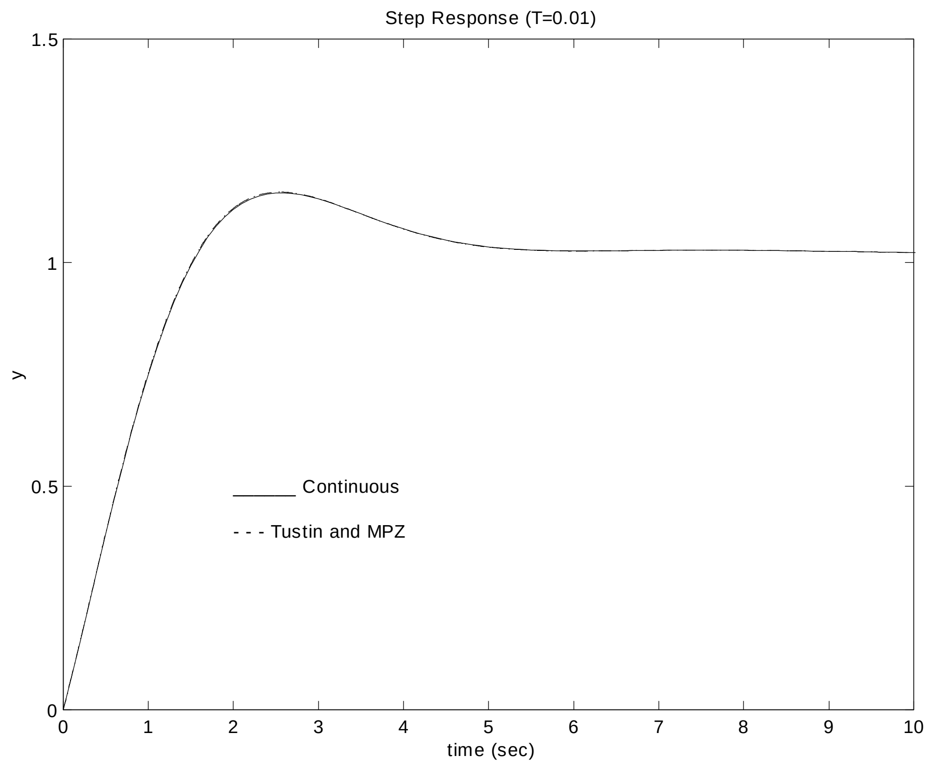
Thus,

$$\begin{aligned} D_d(z) &= K_d \frac{1 - \left(e^{-\alpha T} + e^{-\beta T}\right)z^{-1} + e^{-(\alpha+\beta)T}z^{-2}}{1 - z^{-2}} \\ &= \left\{ \begin{array}{cc} \frac{3.339 - 3.349z^{-1} - 0.263z^{-2}}{1-z^{-2}}, & T = 1 \\ \frac{16.092 - 28.518z^{-1} + 12.462z^{-2}}{1-z^{-2}} & T = 0.1 \\ \frac{143.985 - 284.333z^{-1} + 140.351z^{-2}}{1-z^{-2}} & T = 0.01 \end{array} \right\} \end{aligned}$$

Step responses $(T = 1,\ T = 0.1,\ T = 0.01)$:




Step Response (T=0.1)
1.2
1
0.8
y 0.6
0.4
0.2
0
0
1
2
3
4
5
6
7
8
9
10
time (sec)
______ continuous
- - - tustin and MPZ



Problems and Solutions for Section 8.6: Discrete Design

9. Consider the system configuration shown in Fig. 8.23, where

$$G(s) = \frac{40(s+2)}{(s+10)(s^2-1.4)}.$$

(a) Find the transfer function $G(z)$ for $T = 1$ assuming the system is preceded by a ZOH.

(b) Use MATLAB to draw the root locus of the discrete system with respect to K.

(c) What is the range of K for which the closed-loop discrete system is stable?

(d) Compare your results of part (c) to the case where an analog controller is used (that is, where the sampling switch is always closed). Which system has a larger allowable value of K?

(e) Use MATLAB to compute the step response of both the continuous and discrete systems with K chosen to yield a damping factor of $\zeta = 0.5$ for the continuous case.

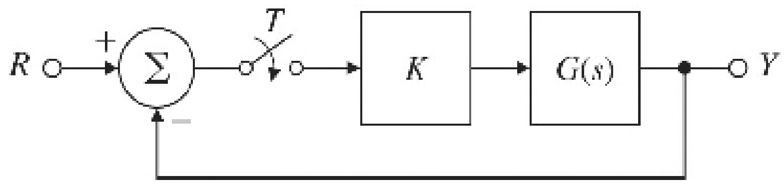


Fig. 8.23 Control system for Problem 8.9

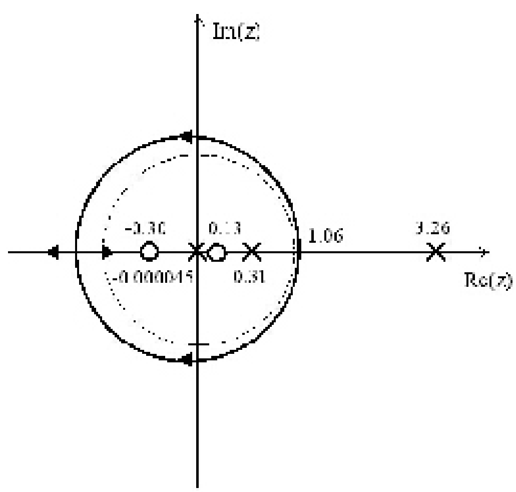
Solution

(a) Using partial fraction expansion along with Table 8.1,

$$
\begin{aligned}
G(z) &= \frac{z-1}{z}\mathcal{Z}\left\{\frac{G(s)}{s}\right\} = \frac{z-1}{z}\mathcal{Z}\left\{\frac{40(s+2)}{s(s+10)(s^2-1.4)}\right\} \\
&= \frac{z-1}{z}\mathcal{Z}\left\{40\left(-\frac{0.1429}{s}+\frac{0.0081}{s+10}+\frac{0.0331}{s+\sqrt{1.4}}+\frac{0.1017}{s-\sqrt{1.4}}\right)\right\} \\
&= \frac{z-1}{z}\mathcal{Z}\left\{40\left(-0.1429\frac{z}{z-1}+0.0081\frac{z}{z-e^{-10}}\right.\right. \\
&\quad \left.\left.+0.0331\frac{z}{z-e^{-\sqrt{1.4}}}+0.1017\frac{z}{z-e^{\sqrt{1.4}}}\right)\right\} \\
&= \frac{7.967z^{-1}+1.335z^{-2}-0.3245z^{-3}}{1-3.571z^{-1}+1.000z^{-2}-0.00004540z^{-3}}
\end{aligned}
$$

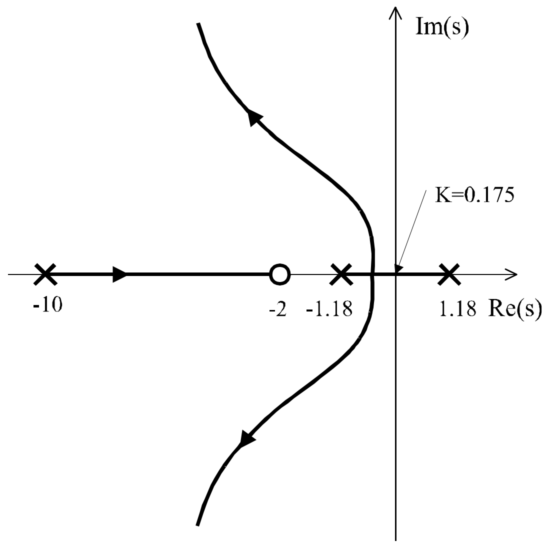
Alternately, we could compute the same result using c2d in MATLAB with $G(s)$.

(b) The z-plane root locus is shown.



(c) A portion of the locus is outside the unit circle for any value of K; therefore, the closed-loop system for the discrete case is unstable for all K.

(d) The s-plane root locus is shown. The closed-loop system is stable for $K > 0.175$. The analog case has a larger allowable K.



(e) Since $\zeta = 0.5$ must be achieved, an analytical approach would be to let a desired closed-loop pole be :

$$s_d = \sigma + \sqrt{3}\sigma j$$

Evaluate the continuous characteristic equation at s_d :

$$\left\{1 + \frac{40K(s+2)}{s(s+10)(s^2-1.4)}\right\}|_{s=\sigma+\sqrt{3}\sigma j} = 0$$

and find that a cubic results, i.e., there are three places on the locus where $\zeta = 0.5$.

$$\Longrightarrow (-8\sigma^3 - 20\sigma^2 + 40K\sigma - 1.4\sigma + 80K - 14)$$

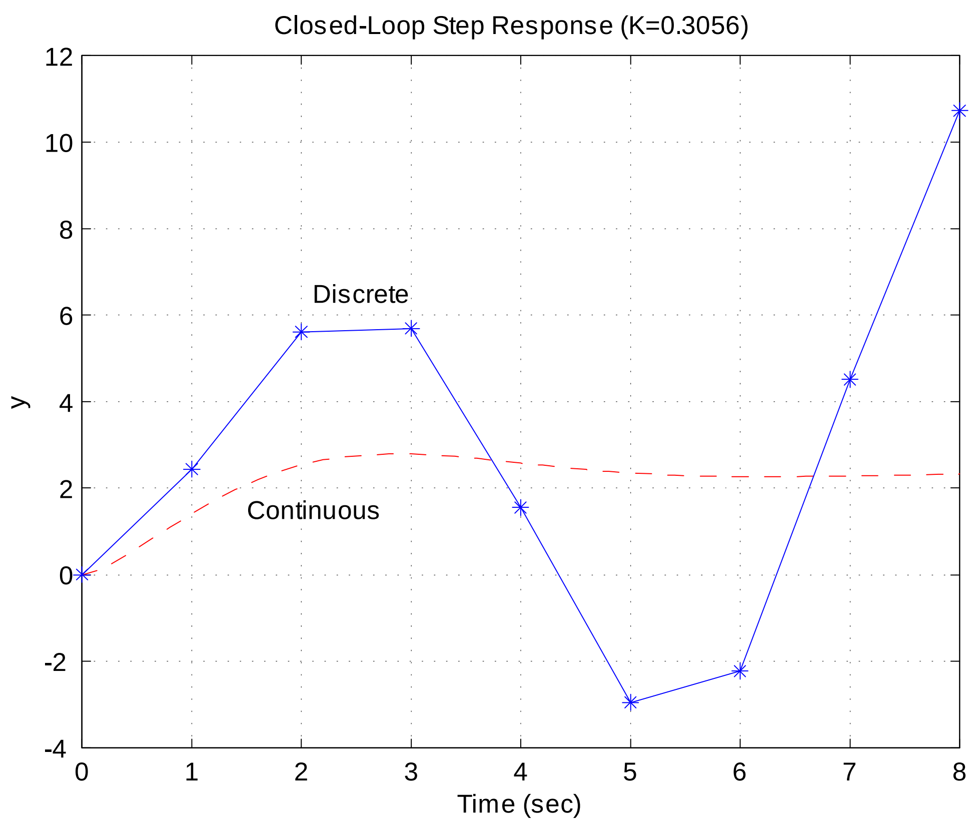
$$+(34.641\sigma^2 + 40\sqrt{3}K - 1.4\sqrt{3})j = 0$$

$$\Longrightarrow \quad \sigma = \begin{bmatrix} -3.7732 \\ -0.6857 \\ -0.5411 \end{bmatrix}, \quad K = \begin{bmatrix} 1.9216 \\ 0.3778 \\ 0.3056 \end{bmatrix}$$

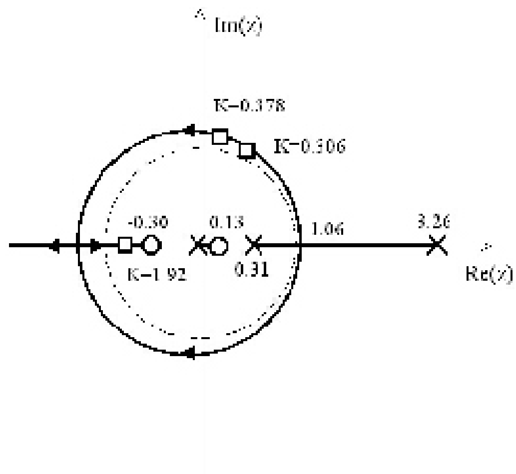
$$\Longrightarrow \quad s = \begin{bmatrix} -3.7732 - 6.5354j \\ -0.6857 - 1.1876j \\ -0.5411 - 0.9373j \end{bmatrix}$$

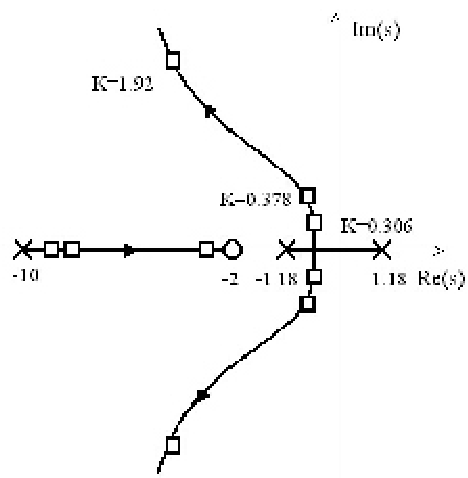
$$\Longrightarrow \quad \omega_n = \begin{bmatrix} 7.5456 \\ 1.3713 \\ 1.0823 \end{bmatrix}, \quad \zeta = 0.5$$

Any of these gains yield a damping factor of $\zeta = 0.5$ for the continuous case; however, we will use the lowest value of K.


Closed-Loop Step Response (K=0.3056)
Discrete
Continuous
y
Time (sec)
12
10
8
6
4
2
0
-2
-4
0
1
2
3
4
5
6
7
8

Alternatively, we could use rlocfind from MATLAB to determine K at the desired $\zeta = 0.5$.
Step responses for $K = 0.3056$:
As expected from the root loci, the discrete case is unstable for this case of quite slow sampling. The z-plane / s-plane root loci with closed-loop poles for 1.9216, 0.3778, 0.3056 marked are shown below:





10. *Single-axis Satellite Attitude Control:* Satellites often require attitude control for proper orientation of antennas and sensors with respect to Earth. Figure 2.7 shows a communication satellite with a three-axis attitude-control system. To gain insight into the three-axis problem we often consider one axis at a time. Figure 8.24 depicts this case where motion is only allowed about an axis perpendicular to the page. The equations of

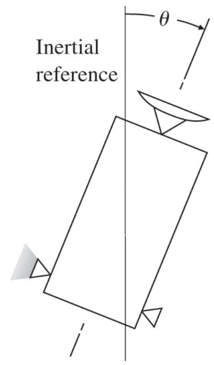


Figure 8.22: Fig. 8.24 Satellite control schematic for Problem 8.10

motion of the system are given by

$$I\ddot{\theta} = M_C + M_D,$$

where

I = moment of inertia of the satellite about its mass center,
$\mathrm{M_C}$ =control torque applied by the thrusters,
$\mathrm{M_D}$ =disturbance torques,
θ =angle of the satellite axis with respect to an inertial reference with no angular acceleration.

We normalize the equations of motion by defining

$$u = \frac{M_C}{I}, \quad w_d = \frac{M_D}{I},$$

and obtain

$$\ddot{\theta} = u + w_d.$$

Taking the Laplace transform yields

$$\theta(s) = \frac{1}{s^2}[u(s) + w_d(s)],$$

which with no disturbance becomes

$$\frac{\theta(s)}{u(s)} = \frac{1}{s^2} = G_1(s).$$

In the discrete case where u is applied through a ZOH, we can use the methods described in this chapter to obtain the discrete transfer function

$$G_1(z) = \frac{\theta(z)}{u(z)} = \frac{T^2}{2}\left[\frac{z+1}{(z-1)^2}\right].$$

(a) Sketch the root locus of this system by hand assuming proportional control.

(b) Draw the root locus using MATLAB to verify the hand sketch.

(c) Add a discrete velocity feedback to your controller so that the dominant poles correspond to $\zeta = 0.5$ and $\omega_n = 3\pi/(10T)$.

(d) What is the feedback gain if $T = 1\,\text{sec}$? If $T = 2\,\text{sec}$.

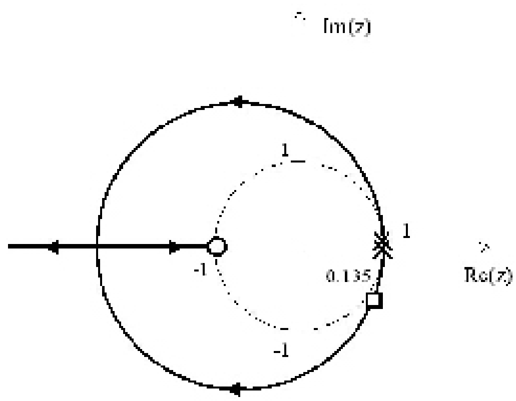
(e) Plot the closed-loop step response and the associated control time history for $T = 1\,\text{sec}$.

Solution

(a) The hand sketch will show that the loci branches depart vertically from $z = 1$; therefore, the system is marginally stable or unstable for any value of gain.

(b) The MATLAB version below confirms the situation.

$$G_1(z) = \frac{T^2}{2}\frac{(z+1)}{(z-1)^2} = K_0\frac{(z+1)}{(z-1)^2}$$



(c) To obtain the desired damping and frequency, Fig. 8.4 shows that a root locus branch should go through the desired poles at $z = 0.44 \pm 0.44j$. After some trial and error, you can find that this can be accomplished with the lead compensation :

$$D_d(z) = K\frac{(z-0.63)}{(z+0.44)}$$

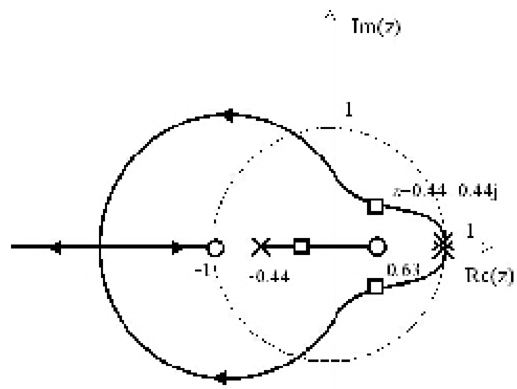
The specific value of K that yields the closed-loop poles are at :

$$z = 0.44 \pm 0.44j,\ -0.113$$

is $K = \frac{0.692}{K_0}$. The second-order pair give :

$$\begin{aligned} \omega_n &= \frac{0.917}{T} \text{ rad/sec} \\ \zeta &= 0.519 \end{aligned}$$

which is close enough to the desired specifications. The root locus for the compensated design is:



(d)

$$\begin{aligned} K &= \frac{0.692}{\frac{T^2}{2}} \\ &= \left\{ \begin{array}{ll} 1.383 & \text{for } T = 1 \text{ sec} \\ 0.3458 & \text{for } T = 2 \text{ sec} \end{array} \right\} \end{aligned}$$

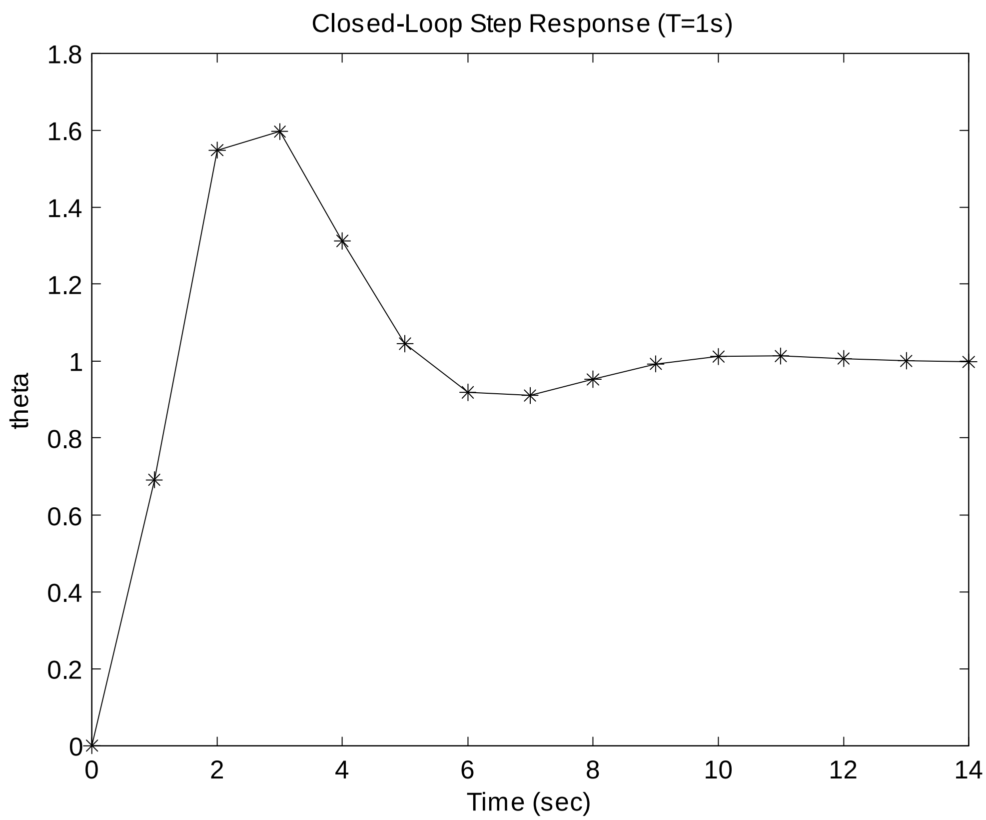
(e) Closed-loop step response :
and the closed-loop control time history.

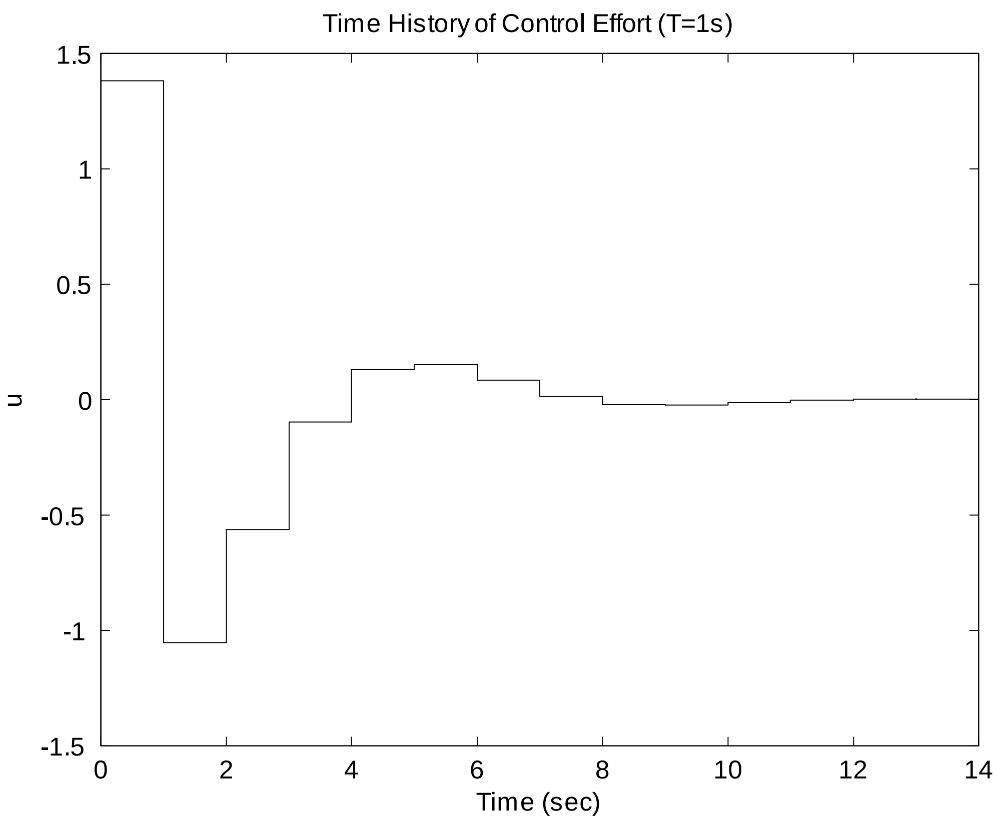
11. It is possible to suspend a mass of magnetic material by means of an electromagnet whose current is controlled by the position of the mass (Woodson and Melcher, 1968). The schematic of a possible setup is shown in Fig. 8.25, and a photo of a working system at Stanford University is shown in Fig. 9.1. The equations of motion are

$$m\ddot{x} = -mg + f(x, I),$$

where the force on the ball due to the electromagnet is given by $f(x, I)$. At equilibrium the magnet force balances the gravity force. Suppose we let I_0 represent the current at equilibrium. If we write $I = I_0 + i$, expand f about $x = 0$ and $I = I_0$, and neglect higher-order terms, we obtain the linearized equation

$$m\ddot{x} = k_1 x + k_2 i. \tag{1}$$


Closed-Loop Step Response (T=1s)
theta
Time (sec)
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
0
2
4
6
8
10
12
14


Time History of Control Effort (T=1s)
u
Time (sec)
1.5
1
0.5
0
-0.5
-1
-1.5
0
2
4
6
8
10
12
14

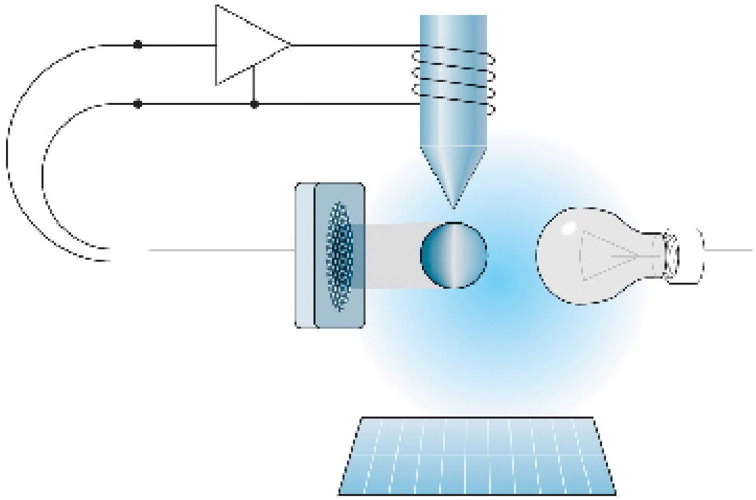


Figure 8.23: Fig. 8.25 Schematic of magnetic levitation device for Problem 8.11

Reasonable values for the constants in Eq. (1) are $m = 0.02$ kg, $k_1 = 20$ N/m, and $k_2 = 0.4$ N/A.

(a) Compute the transfer function from I to x, and draw the (continuous) root locus for the simple feedback $i = -Kx$.

(b) Assume the input is passed through a ZOH, and let the sampling period be 0.02 sec. Compute the transfer function of the equivalent discrete-time plant.

(c) Design a digital control for the magnetic levitation device so that the closed-loop system meets the following specifications: $t_r \leq 0.1$ sec, $t_s \leq 0.4$ sec, and overshoot $\leq 20\%$.

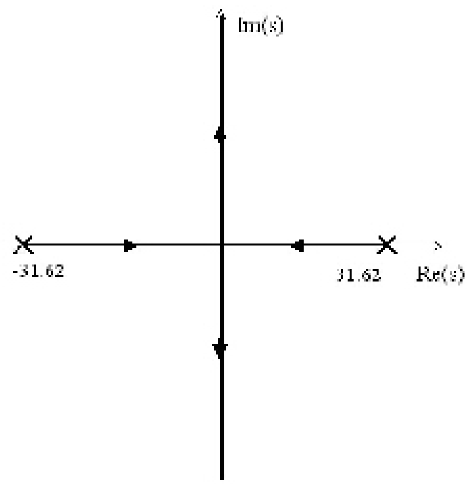
(d) Plot a root locus with respect to k_1 for your design, and discuss the possibility of using your closed-loop system to balance balls of various masses.

(e) Plot the step response of your design to an initial disturbance displacement on the ball, and show both x and the control current i. If the sensor can measure x only over a range of $\pm 1/4$cm and the amplifier can only provide a current of 1 A, what is the *maximum* displacement possible for control, neglecting the nonlinear terms in $f(x, I)$?

Solution:

(a)

$$\begin{aligned} G(s) &= \frac{X(s)}{I(s)} = \frac{k_2/m}{s^2 - k_1/m} \qquad (2) \\ &= \frac{20}{s^2 - 1000} \end{aligned}$$



(b) $T = 0.02$ sec,

$$\begin{aligned} G(z) &= (1 - z^{-1})\mathcal{Z}\left\{\frac{G(s)}{s}\right\} \\ &= 0.004135\frac{z+1}{(z-0.5313)(z-1.8822)} \end{aligned}$$

(c) The specifications imply that :

$$\begin{aligned} t_r &\leq 0.1 \text{ sec} \Longrightarrow \omega_n \geq \frac{1.8}{0.1} = 18 \text{ rad/sec} \\ t_s &\leq 0.4 \text{ sec} \Longrightarrow \sigma \geq \frac{4.6}{0.4} = 11.5 \text{ rad/sec} \\ &\Longrightarrow r = |z| \leq e^{-11.5\times 0.02} = 0.7945 \; (\longleftarrow z = e^{sT}) \\ M_p &\leq 20\% \Longrightarrow \zeta \geq 0.48 \end{aligned}$$

Thus, the closed-loop poles must be pulled into the unit circle near $r = 0.8$ and $\zeta = 0.5$. Using the template of Fig. 8.4, we experiment with lead compensation and select,

$$D(z) = 116\frac{z - 0.5313}{z - 0.093}$$

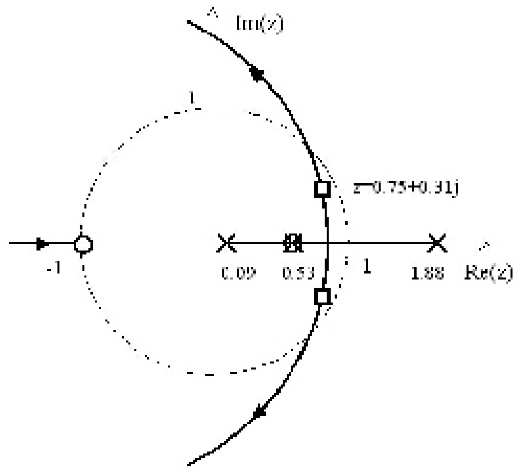
The closed-loop poles are :

$$z = 0.75 \pm 0.39j, \; 0.53$$

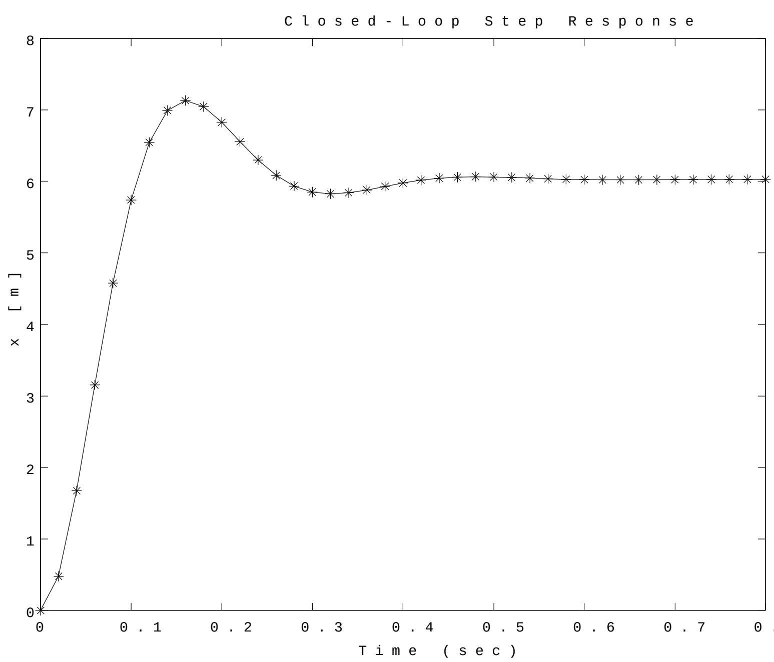
Performance :

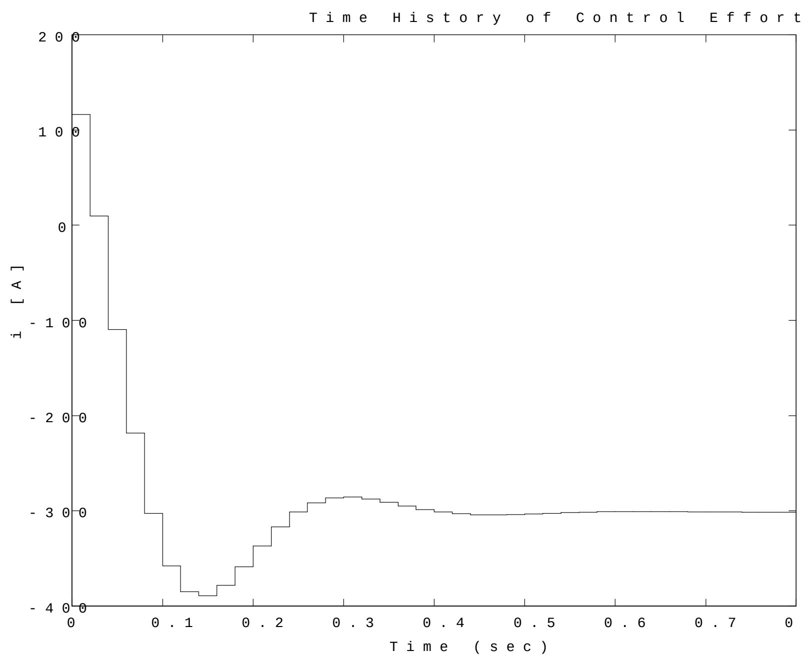
$$\begin{aligned} t_r &= 0.072 \\ t_s &= 0.397 \\ M_p &= 18.3\% \end{aligned}$$

which meet all the specifications. The root locus is below.

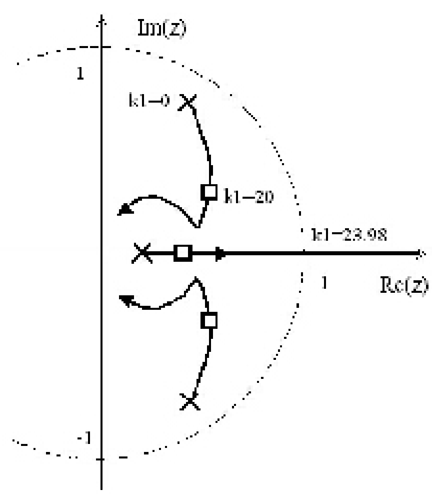


The step response shows M_p



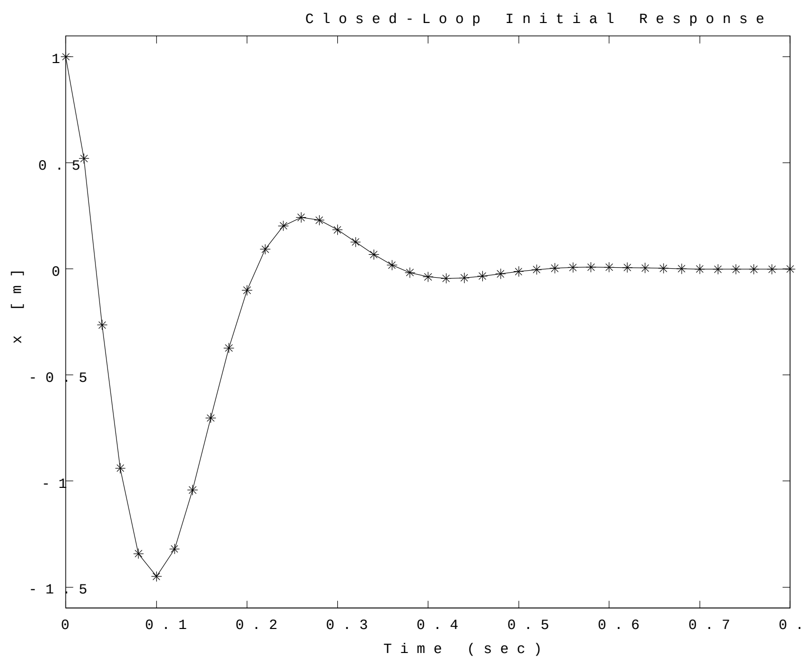


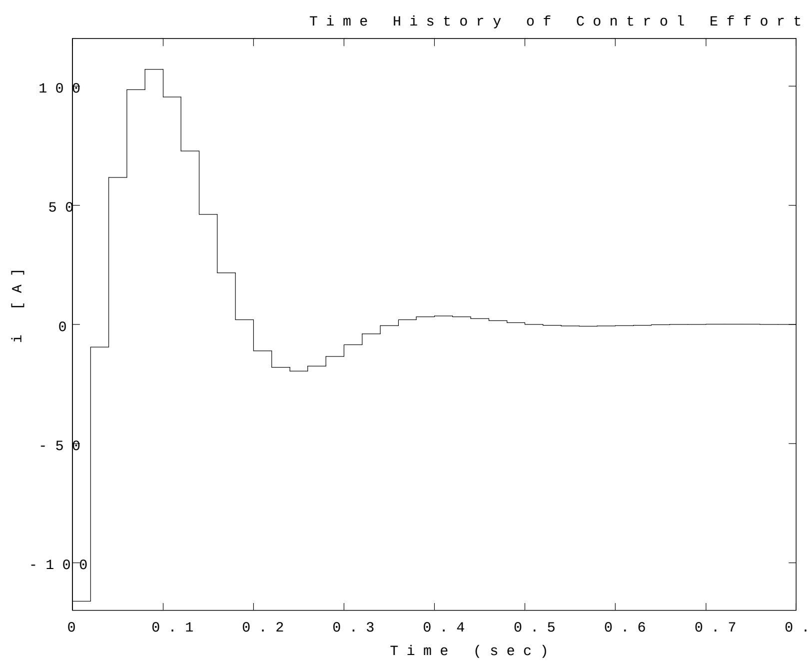
(d) As can be seen from Eq. (2), the loop gain and the open loop pole locations depend on the mass of the ball. Changing the mass will therefore affect the dynamic characteristics of the system and may render it unstable. A root locus of the closed-loop poles versus k_1 shows how the locus changes as a function of the mass:



The closed-loop system becomes unstable for $k_1 \geq 24$. Since a small increase in k_1 makes the system unstable and a decrease in m has the same effect on the system, it is difficult to balance balls of smaller masses.

(e) The response to an initial x displacement is shown :





The assumption here is that an allowable transient must stay in the range of the sensor and not require more than the limit of the current. From $I(z) = D(z)(0 - X(z))$, we have a difference equation :

$$i(k) - 0.093i(k-1) = -116\{x(k) - 0.5313x(k-1)\}$$

For $k = 0$, $i(0) = -116x(0)$. We see that if $x(0) = 1$ then $i(0) = -116$. Note that $i(0) = -D(\infty)x(0)$.
Thus, if i is to be kept below $1A$ then $x(0)$ must be kept below $1/116 = 0.00862$ m $= 0.862$ cm $= 0.339$ inch, which is greater than the sensor range. The current control can handle any displacement in the range of ± 0.25 inch.

12. Repeat Problem 5.26 in Chapter 5 by constructing discrete root loci and performing the designs directly in the z-plane. Assume that the output y is sampled, the input u is passed through a ZOH as it enters the plant, and the sample rate is 15 Hz.

Solution

(a) The most effective discrete design method is to start with some idea what the continuous design looks like, then adjust that as necessary with the discrete model of the plant and compensation. We refer to the solution for Probelm 5.27 for the starting point. It shows that the specs can be met with a lead compensation,

$$D_1(s) = K\frac{(s+1)}{(s+60)}$$

and a lag compensation,

$$D_2(s) = \frac{(s+0.4)}{(s+0.032)}.$$

Although it is stated in the solution to Problem 5.27 that a gain, $K = 240$ will satisfy the constraints, in fact, a gain of about $K = 270$ is actually required to meet the rise time constraint of $t_r \leq 0.4$ sec. So we will assume here that our reference continuous design is

$$D_1(s) = 270\frac{(s+1)}{(s+60)}\frac{(s+0.4)}{(s+0.032)}$$

It yields a rise time, $t_r \cong 0.38$, $M_p \cong 15\%$, and $K_v = (270)(\frac{1}{60})(\frac{0.4}{0.32}) = 56$. So all specs are met. For interest, use of the damp function shows that $\omega_n = 6.4$ rad/sec for the dominant roots, and those roots have a damping ratio, $\zeta \cong 0.7$. For the discrete case with $T = 15$ Hz, we should expect some degradation in performance, especially the damping, because the sample rate is approximately $15\times\omega_n$.

The discrete transfer function for the plant described by $G(s)$ and preceded by a ZOH is:

$$\begin{aligned} G(z) &= (1-z^{-1})\mathcal{Z}\left\{\frac{G(s)}{s}\right\} \\ &= \frac{z-1}{z}\mathcal{Z}\left\{\frac{10}{s(s+1)(s+10)}\right\} \end{aligned}$$

This is most easily determined via MATLAB,

```
sysC = tf([10],[1  11  10  0]);
T=1/15;
sysD = c2d(sysC,T,'zoh');
```

which produces:

$$G(z) = 0.00041424\frac{(z+3.136)(z+0.2211)}{(z-1)(z-0.9355)(z-0.5134}$$

The essential elements of the compensation are that the lead provides velocity feedback with a $T_D = 1$ and the lag provides some high frequency gain. The discrete equivalent of the proportional plus lead would be (see Eq. 8.42):

$$D_1(z) = K(1 + \frac{T_D}{T}(1-z^{-1})) = K\frac{(1+T_D/T)z - T_D/T}{z}$$

which for $T = 1/15 = 0.0667$ and $T_D = 1$ reduces to

$$D_1(z) = 270\frac{16z - 15z^{-1}}{z} = 4320\frac{z-0.9375}{z}.$$

The lag equivalent is best introduced by use of one of the approximation techniques, such as the matched pole-zero:

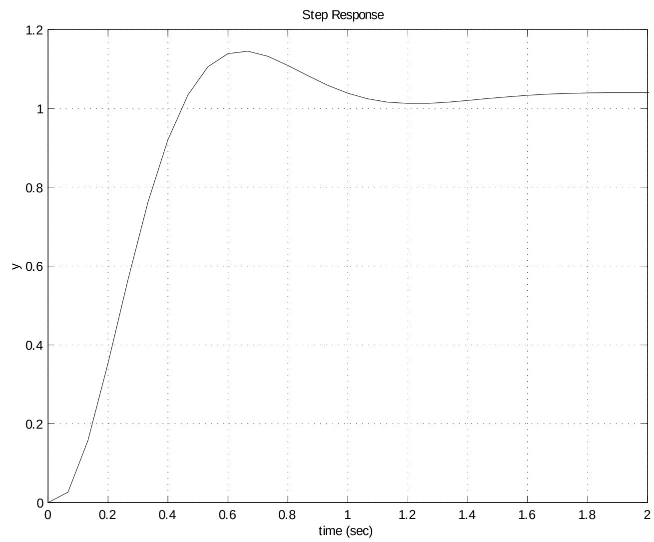
$$D_2(z) = \frac{z - e^{-0.4T}}{z - e^{-0.032T}} = \frac{z - .9737}{z - .9979}$$

as its whole function is to raise the gain at very low frequencies for error reduction. Examining the resulting discrete root locus and picking roots with `rlocfind` to yield the required damping shows that the gain, $K = 60$. While the use of `damp` indicates that the frequency and damping are acceptable, the time response shows an overshoot of about 20% and the rise time is slightly below spec. We therefore need to increase the lead (move the lead zero closer to $z = +1$) to decrease the overshoot and increase gain to speed up the rise time. Several iterations on these two quantities indicates that moving the lead zero from $z = 0.9375$ to $z = 0.96$ and increasing the gain from

$K = 60$ to $K = 65$ meets both specs. The velocity coefficient is found from and is also satisfied.

$$K_v = \frac{\lim}{z \to 1} \frac{(z-1)D(z)G(z)}{Tz}$$

The time response of the final design below shows that all specs are met.



13. Design a digital controller for the antenna servo system shown in Figs. 3.60 and 3.61 and described in Problem 3.35. The design should provide a step response with an overshoot of less than 10% and a rise time of less than 80 sec.

   (a) What should the sample rate be?

   (b) Use the matched pole-zero discrete equivalent method.

   (c) Use discrete design and the z-plane root locus.

Solution

   (a) The equation of motion is :

$$J\ddot{\theta} + B\theta = T_c$$

where

$$J = 600000 \text{ kg.m}^2, \; B = 20000 \text{ N.m.sec}$$

If we define :

$$a = \frac{B}{J} = \frac{1}{30}, \; U = \frac{T_c}{B}$$

after Laplace transform, we obtain :

$$G(s) = \frac{\theta(s)}{u(s)} = \frac{1}{s(30s+1)}$$

From the specifications,

$$\begin{aligned} M_p &< 10\% \Longrightarrow M_P \cong \left(1 - \frac{\zeta}{0.6}\right) 100 \Longrightarrow \zeta > 0.54 \\ t_r &< 80 \text{ sec} \Longrightarrow t_r \cong \frac{1.8}{\omega_n} < 80 \Longrightarrow \omega_n \cong \omega_{BW} > 0.0225 \end{aligned}$$

Note that $\omega_{pn} \cong 1/30 < \omega_n$.

If designing by emulation, a sample rate of 20 times the bandwidth is recommended. If using discrete design, the sample rate can be lowered somewhat to perhaps as slow as 10 times the bandwidth. However, to reject random disturbances, best results are obtained by sampling at 20 times the closed-loop bandwidth or faster. Thus, for both design methods, we choose :

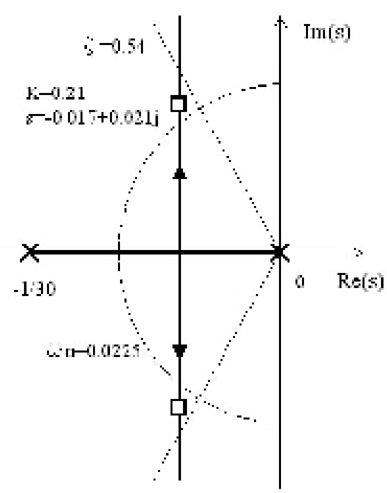
$$\begin{aligned} T &= 10 \text{ sec} \\ \omega_s &= 0.628 \text{ rad/sec, which is} > 20\omega_n = 0.45 \text{ rad/sec} \end{aligned}$$

(b) Continuous design :

Use a proportional controller :

$$u(s) = D_c(s)(\theta_r(s) - \theta(s)) = K(\theta_r(s) - \theta(s))$$

Root locus :



Choose $K = 0.210$.
The closed-loop pole location in s-plane :

$$s = -0.0167 \pm 0.0205j$$

The corresponding natural frequency and damping :

$$\omega_n = 0.0265, \ \zeta = 0.6299$$

Digitized the continuous controller with matched pole-zero method :

$$\begin{aligned} D_d(z) &= 0.0210 \\ T_c(z) &= Bu(z) = 420(\theta_r(z) - \theta(z)) \end{aligned}$$

Performance :

$$\begin{aligned} M_p &= 0.119 \\ t_r &= 67.3 \text{ sec} \end{aligned}$$

(c) With $u(k)$ applied through a ZOH, the transfer function for an equivalent discrete-time system is :

$$G(z) = K\frac{z+b}{(z-1)(z-e^{aT})}$$

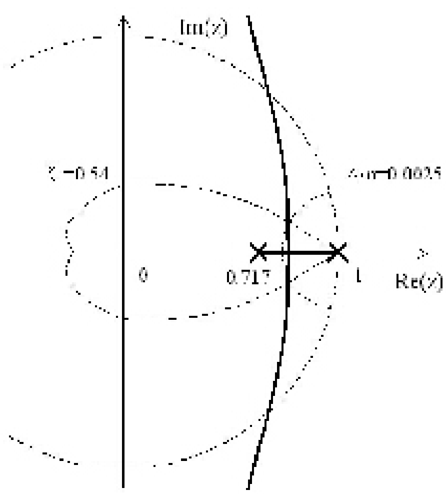
where

$$\begin{aligned} K &= \frac{aT - 1 + e^{-aT}}{a}, \ b = \frac{1 - e^{-aT} - aTe^{-aT}}{aT - 1 + e^{-aT}} \\ &\Longrightarrow \ G(z) = 1.4959\frac{z + 0.8949}{(z-1)(z-0.7165)} \end{aligned}$$

Use a proportional control of the form :

$$D_d(z) = K$$

Root locus :

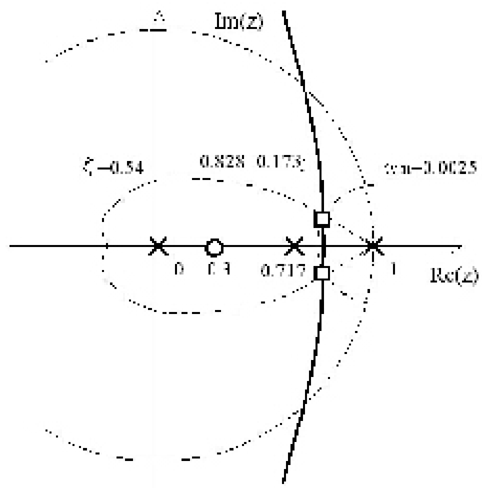


The specification can be achieved with the proportional control. However, we try to achieve the same closed-loop poles as the emulation design (part (b)) for comparison. These closed-loop pole locations are denoted by " + " in the root locus.

Use a PD control of the form :

$$D_d(z) = K\frac{z-\alpha}{z}$$

Root locus :



Choose $K = 0.0294,\ \alpha = 0.3$.

The resulting z-plane roots :

$$z = 0.8280 \pm 0.1725j,\ 0.0165$$

This corresponds to the s-plane roots :

$s = -0.0167 \pm 0.0205j$ (the design point of emulation design), -0.4104

which satisfy the specification :

$$\begin{aligned}\omega_n &= 0.0265,\ 0.4104\\ \zeta &= 0.6321,\ 1.000\end{aligned}$$

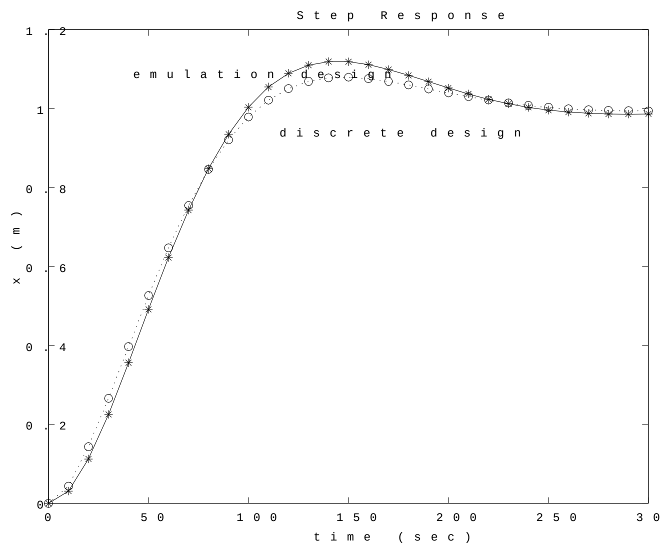
The control law :

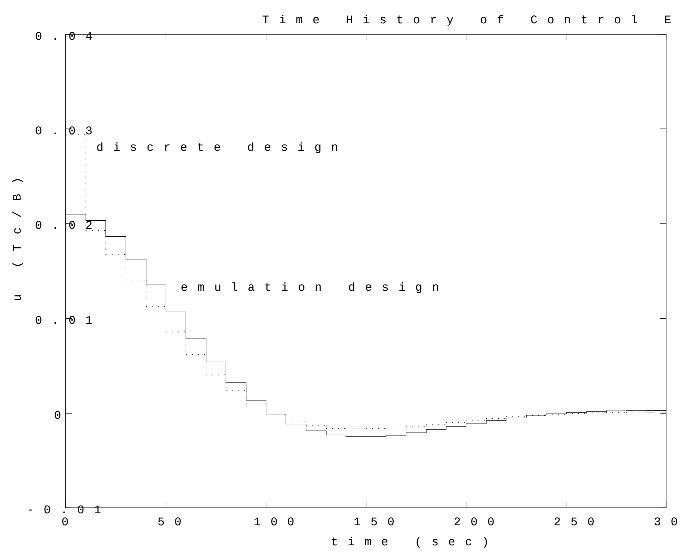
$$\begin{aligned}D_d(z) &= 0.0294\frac{z-0.3}{z}\\ T_c(z) &= Bu(z) = 588\frac{z-0.3}{z}\end{aligned}$$

Performance :

$$\begin{aligned} M_p &= 0.079 \\ t_r &= 71.3 \text{ sec} \end{aligned}$$

Step response :





14. The system

$$G(s) = \frac{1}{(s+0.1)(s+3)}$$

is to be controlled with a digital controller having a sampling period of $T = 0.1$ sec. Using a z-plane root locus, design compensation that will respond to a step with a rise time $t_r \leq 1$ sec and an overshoot $M_p \leq 5\%$. What can be done to reduce the steady-state error?

Solution

(a) Continuous plant :

$$G(s) = \frac{1}{(s+0.1)(s+3)}, \quad \text{Type 0 system}$$

Discrete model of $G(s)$ preceded by a ZOH ($T = 0.1$ sec) :

$$G(z) = 0.0045 \frac{z+0.9019}{(z-0.7408)(z-0.99)}$$

Specifications :

$$\begin{aligned} t_r &\leq 1 \text{ sec} \longrightarrow \omega_n \geq 1.8 \text{ rad/sec} \\ M_p &\leq 5\% \longrightarrow \zeta \geq 0.7 \end{aligned}$$

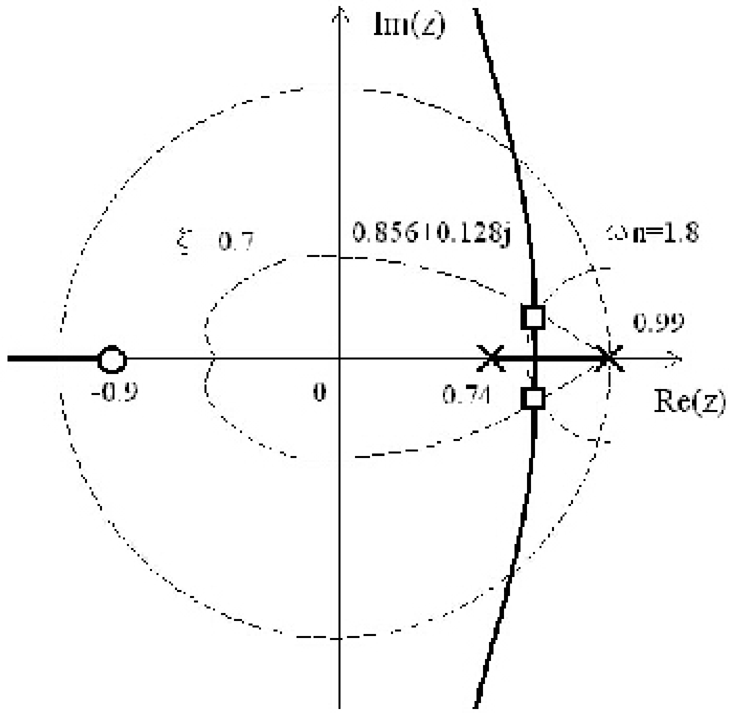
Discrete design : A simple proportional feedback, $D(z) = K = 4.0$, will bring the closed-loop poles to :

$$z = 0.8564 \pm 0.1278j$$

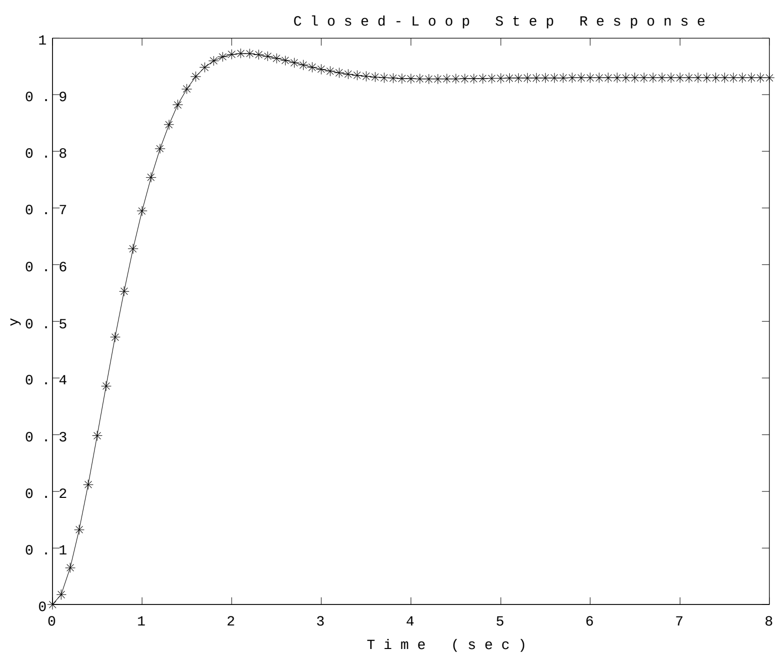
which are inside the specs region.

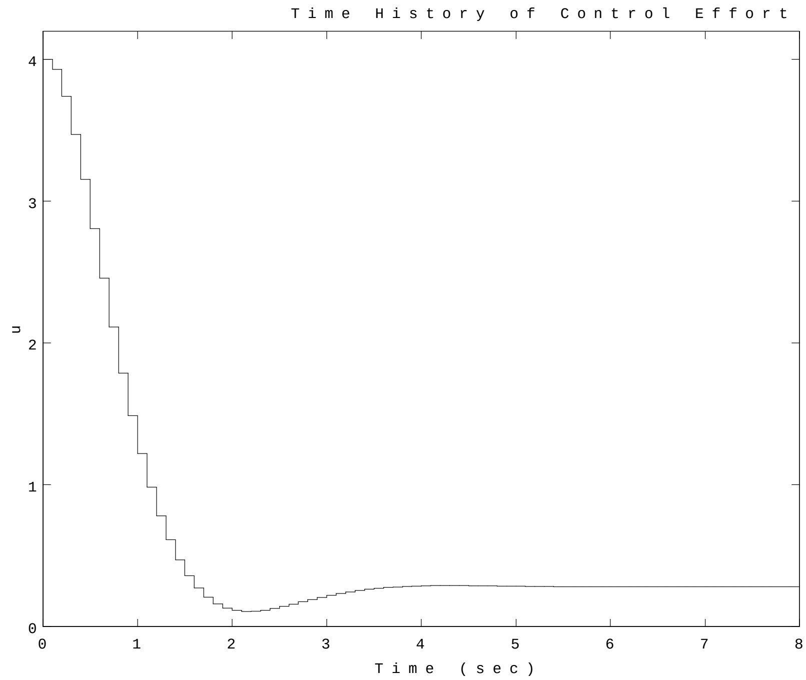
$$\omega_n = 2.07 \text{ rad/sec}, \ \zeta = 0.70$$

Root locus :



Step response :





The step response shows that :

$$t_r \cong 1.02 \text{ sec}$$
$$M_p \cong 4.7\%$$

However, since the system is type 0, steady-state error exists and is 7% in this case. An integral control of the form,

$$D_d(z) = \frac{K}{T_I}\frac{Tz}{z-1}$$

can be added to the proportional control to reduce the steady-state error, but this typically occurs at the cost of reduced stability.

15. The transfer function for pure derivative control is

$$D_d(z) = KT_D\frac{z-1}{Tz},$$

where the pole at $z = 0$ adds some destabilizing phase lag. Can this phase lag be removed by using derivative control of the form

$$D_d(z) = KT_D\frac{(z-1)}{T}?$$

Support your answer with the difference equation that would be required, and discuss the requirements to implement it.

Solution:

(a) No, we cannot use derivative control of the form :

$$D_d(z) = KT_D \frac{z-1}{T}$$

to remove the phase lag. The difference equation corresponding to

$$D_d(z) = K_p T_D \frac{z-1}{T} = \frac{U(z)}{E(z)}$$

is

$$u(k) = K_p T_D \frac{e(k+1) - e(k)}{T}$$

This is not a *causal* system since it needs the *future* error signal to compute the current control. In real time applications, it is not possible to implement a non-causal system.