Intermediate Fluid Mechanics

Lecture 16: Thermal Energy of a Fluid System

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Chapter Overview

Lecture Objectives

- In the previous lecture, we considered an equation describing the rate of change of mechanical energy (kinetic energy) of a fluid.
- In this lecture, we will consider an equation describing the rate of change of the thermal energy in a fluid.

Chapter Overview

First Law of Thermodynamics

The first law of thermodynamics states:

the rate of change of stored energy in a system	=	the rate of heat added to the system	+	the rate of work done on the system by its surroundings

Mathematically, one has that

$$\frac{dE_T}{dt}\Big|_{sys} = \frac{dQ}{dt} + \frac{dW}{dt}, \tag{1}$$

where,

- E_T indicates the total energy,
- Q heat,
- W, the work done on the system.

First Law of Thermodynamics (continued ...)

In the case of a fluid, when one talks about a '*system*', we refer to a collection of fluid particles.

$\Longrightarrow$ Hence, it is preferable to express the total energy of the system as the density times the energy per unit mass,

$$\frac{dE_T}{dt}\Big|_{sys} = \frac{d}{dt}\int_{\forall(t)} \rho\, e_T\, d\forall \tag{2}$$

Comments about the total energy

We will write the total energy as the sum of the **internal energy** plus the **kinetic energy** of the fluid, (representing the *stored* energy of the system)

$$e_T = e + \frac{1}{2} u_i u_i \tag{3}$$

where e is the internal energy per unit mass.

Note

Since the **potential energy** is not added on the left-hand side, then it needs to be included as a body-force on the right-hand side.

The **internal energy** of a system of fluid particles results from microscopic motions such as random translational motion, molecular vibrations, molecular rotation, and any other microscopic energy modes.

Work Rate

From the previous lecture, we had found that the total work rate done on the system is comprised of two parts:

- One due to **body forces**
- One due to **surface forces**

$$\frac{dW}{dt} = \frac{dW_b}{dt} + \frac{dW_s}{dt}. \tag{4}$$

Work Rate (continued ...)

From the previous lecture we also know that the work rate of the volume forces is written as,

$$\frac{dW_b}{dt} = \int_{\forall} \rho\, u_i\, g_i\, d\forall, \tag{5}$$

and that for the surface forces as,

$$\frac{dW_s}{dt} = \int_{\forall} u_i \frac{\partial \tau_{ij}}{\partial x_j}\, d\forall + \int_{\forall} \tau_{ij} \frac{\partial u_i}{\partial x_j}\, d\forall \tag{6}$$

$$\frac{dW_s}{dt} = \int_{volume} \left[u_i \frac{\partial \tau_{ij}}{\partial x_j} - \rho \frac{\partial u_i}{\partial x_i} + \phi - \frac{2}{3}\mu \left(\frac{\partial u_i}{\partial x_i} \right)^2 \right] d\forall. \tag{7}$$

Comments about the total energy (continued ....)

Let's now consider a case where:

- The surface forces are zero or negligible.
- The internal energy of the system is uniform everywhere

Then:

A rising fluid particle ($\vec{u} \cdot \vec{g} = u_i g_i < 0$) $\Longrightarrow$ must undergo a $\downarrow$ of $\frac{1}{2} u_i u_i$ by

$$\frac{d}{dt} \int_{\forall(t)} \rho \, e_T \, d\forall = \int_{\forall} \rho \, u_i \, g_i \, d\forall \tag{8}$$

Heat Transfer

The heat flux is the sum of all molecular modes of energy transfer, namely:

- Conduction is the most common type of molecular energy transfer.
- Other modes are radiation and diffusion of different chemical species.

Here, we adopt a sign convention of positive heat flux $\vec{q}$ from the inside of the system to the outside.

$\implies$ As a result, the net rate of heat gain by the system is,

$$\frac{dQ}{dt} = -\int_A q_i\, n_i\, dA + \int_{\forall} S\, d\forall, \tag{9}$$

where A indicates the area of integration and S is used to represent a source or generation of heat inside the volume of the system.

Heat Transfer (continued ...)

Using Gauss theorem to rewrite the surface integral as a volume integral and neglecting the source term,

$$\frac{dQ}{dt} = -\int_{\forall} \frac{\partial q_i}{\partial x_i}\, d\forall \qquad (10)$$

If all the contributions to the heat flux are neglected except for conduction, *i.e.* neglecting radiation and chemical reaction;

$\Longrightarrow$ Then we may use Fourier's law of conduction to write the heat fluxes,

$$q_i = -k\,\frac{\partial T}{\partial x_i}, \qquad (11)$$

where k is the **thermal conductivity**, which is a material property.

First Law of Thermodynamics in Eulerian framework

One can rewrite the 1st law of Thermodynamics (eq. 1) as

$$\frac{d}{dt}\int_{\forall} \rho\, e_T \, d\forall = \frac{dW_b}{dt} + \frac{dW_s}{dt} + \frac{dQ}{dt}. \tag{12}$$

Which upon substitution leads to

$$\int_{\forall} \rho \, \frac{De_T}{Dt} \, d\forall = \int_{\forall} \rho \, u_i \, g_i \, d\forall + \int_{\forall} u_i \frac{\partial \tau_{ij}}{\partial x_j} d\forall + \int_{\forall} \tau_{ij} \frac{\partial u_i}{\partial x_j} d\forall - \int_{\forall} \frac{\partial q_i}{\partial x_i} d\forall. \tag{13}$$

Because this equality must hold regardless of the volume $\forall$,

$$\boxed{\rho \, \frac{De_T}{Dt} = \rho \, u_i \, g_i + u_i \frac{\partial \tau_{ij}}{\partial x_j} + \tau_{ij} \frac{\partial u_i}{\partial x_j} - \frac{\partial q_i}{\partial x_i}} \tag{14}$$

Note: This is the 1st Law of Thermodynamics in differential form, which has both mechanical and thermal energy terms in it.

First Law of Thermodynamics in Eulerian framework

Recalling now from the previous lecture that the equation for kinetic energy is,

$$\rho \frac{D}{Dt}\left(\frac{1}{2}\, u_i u_i\right) = \rho\, g_i\, u_i + \frac{\partial}{\partial x_j}(u_i\, \tau_{ij}) + \rho \frac{\partial u_i}{\partial x_i} - \phi, \tag{15}$$

one can subtract this from equation 14,

$\Longrightarrow$ an **equation for the time rate of change of the internal energy**,

$$\boxed{\rho \frac{De}{Dt} = -\rho \frac{\partial u_i}{\partial x_i} + \phi - \frac{\partial q_i}{\partial x_i}} \tag{16}$$

Chapter Overview

Equation for temperature, T

$\Longrightarrow$ In practice, it is useful to recast e and q_i in terms of the temperature, T.

To do this, one can start by introducing the thermodynamic definition of enthalpy,

$$h = e + \frac{p}{\rho}. \tag{17}$$

Rearranging this gives that $e = h - \frac{p}{\rho}$, and the differential of e is then,

$$de = dh - \frac{dp}{\rho} - \frac{p}{\rho} d\rho. \tag{18}$$

Equation for temperature, T (continued ...)

We know that there is an **equation of state** for h of the form $h = h(p, T)$.

$\longrightarrow$ Performing a Taylor series expansion gives,

$$dh = \underbrace{\left.\frac{\partial h}{\partial T}\right|_p}_{\substack{\text{Specific heat,}\\ C_p}} dT + \left.\frac{\partial h}{\partial p}\right|_T dp. \tag{19}$$

The second term on the right hand side can be rewritten using entropy relations (not shown here) such that,

$$dh = C_p\, dT + \frac{1}{\rho^2}\left(\rho - T\left.\frac{\partial \rho}{\partial T}\right|_p\right) dp \tag{20}$$

Equation for temperature, T (continued ...)

By defining the coefficient of thermal expansion β as,

$$\beta = -\frac{1}{\rho}\frac{\partial \rho}{\partial T}|_p. \tag{21}$$

The differential equation for Enthalpy can be rewritten as:

$$dh = C_p\, dT + \frac{1}{\rho}(1 + T\,\beta)\, dp. \tag{22}$$

Equation for temperature, T (continued ...)

Substituting the Enthalpy equation

$$dh = C_p \, dT + \frac{1}{\rho}(1 + T\,\beta)\, dp. \tag{23}$$

into the internal energy equation

$$de = dh - \frac{dp}{\rho} - \frac{p}{\rho} d\rho. \tag{24}$$

and taking the time derivative gives,

$$\frac{De}{Dt} = \frac{dh + \frac{dp}{\rho} - \frac{p}{\rho} d\rho}{dt} = \frac{dh}{dt} - \frac{1}{\rho}\frac{dp}{dt} - \frac{p}{\rho}\frac{d\rho}{dt} \tag{25}$$

Equation for temperature, T (continued ...)

Which can be further operated to,

$$\frac{De}{Dt} = \left[C_p \frac{dT}{dt} + \cancel{\frac{1}{\rho}\frac{dp}{dt}} + \frac{T\beta}{\rho}\frac{dp}{dt}\right] - \cancel{\frac{1}{\rho}\frac{dp}{dt}} - \underbrace{\frac{p}{\rho}\frac{d\rho}{dt}}_{\substack{\text{Use Continuity eq.} \\ \frac{1}{\rho}\frac{d\rho}{dt} = -\frac{\partial u_i}{\partial x_i}}} \tag{26}$$

$$\frac{De}{Dt} = C_p \frac{dT}{dt} + \frac{T\beta}{\rho}\frac{dp}{dt} + p\left(\frac{\partial u_i}{\partial x_i}\right). \tag{27}$$

This last equation, can be substituted into the previous equation for the internal energy (eq. 16), obtaining an **equation for Temperature**

$$\boxed{\rho\left[C_p \frac{DT}{Dt} + \frac{T\beta}{\rho}\frac{Dp}{Dt} + p\left(\frac{\partial u_i}{\partial x_i}\right)\right] = -\rho\frac{\partial u_i}{\partial x_i} + \phi - \frac{\partial q_i}{\partial x_i}} \tag{28}$$

Chapter Overview

Review of Governing equations

For a steady, incompressible, two dimensional flow, the governing equations (continuity, momentum, and energy) are:

- Continuity: $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0.$
- x-momentum: $\rho\, u \frac{\partial u}{\partial x} + \rho\, v \frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \mu \frac{\partial^2 u}{\partial x^2} + \mu \frac{\partial^2 u}{\partial y^2}.$
- y-momentum: $\rho\, u \frac{\partial v}{\partial x} + \rho\, v \frac{\partial v}{\partial y} = -\frac{\partial p}{\partial y} + \mu \frac{\partial^2 v}{\partial x^2} + \mu \frac{\partial^2 v}{\partial y^2}.$
- Energy: $\rho\, C_p\, u \frac{\partial T}{\partial x} + \rho\, C_p\, v \frac{\partial T}{\partial y} = k \frac{\partial^2 T}{\partial x^2} + k \frac{\partial^2 T}{\partial y^2} + \mu\phi.$

where the viscous dissipation for an incompressible flow is

$$\phi = 2\mu\, e_{ij}\, e_{ij} = 2\mu\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right] + \mu\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)^2. \tag{29}$$

Chapter Overview

Example: Couette flow with heated walls

For the Couette flow shown in the Figure, in the absence of a pressure gradient, we found that:

- From continuity: $v = 0$.
- From the x-momentum equation: $u = (U/h)y$.

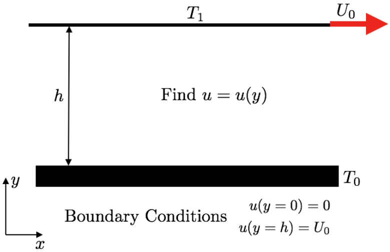

Figure: Couette flow with heated walls.

$\Longrightarrow$ At this time, we are interested to find the temperature distribution for the case when the upper and lower walls are held at two different temperatures.

Example: Couette flow with heated walls (continued ...)

For this purpose, one needs to consider the **energy equation** for this specific study case,

$$\rho\, C_p\, u \underbrace{\cancel{\frac{\partial T}{\partial x}}}_{\text{Fully developed}} + \rho\, C_p \underbrace{\cancel{v}}_{v=0} \frac{\partial T}{\partial y} = k \underbrace{\cancel{\frac{\partial^2 T}{\partial x^2}}}_{\text{Fully developed}} + k \frac{\partial^2 T}{\partial y^2} + \mu\phi. \tag{30}$$

For this specific case the **dissipation term** also gets simplified,

$$\phi = 2\mu\left[\cancel{\left(\frac{\partial u}{\partial x}\right)^2} + \cancel{\left(\frac{\partial v}{\partial y}\right)^2}\right] + \mu\left(\cancel{\frac{\partial v}{\partial x}} - \frac{\partial u}{\partial y}\right)^2. \tag{31}$$

hence,

$$\phi = \mu\left(\frac{\partial u}{\partial y}\right)^2 = \mu\left(\frac{U}{h}\right)^2. \tag{32}$$

Example: Couette flow with heated walls (continued ...)

Thus, in this example case, the energy equation reduces to

$$k\frac{d^2T}{dy^2} = -\mu\left(\frac{U}{h}\right)^2. \tag{33}$$

Upon integrating twice, one finds that,

$$T = \frac{-\mu}{2k}\frac{U^2}{h^2}\,y^2 + A\,y + B \tag{34}$$

where A and B are constants of integration for which once can find the corresponding values using the boundary conditions.

Example: Couette flow with heated walls (continued ...)

By leveraging the BCs, it is found that:

$$T(y=0) = T_0: \quad T_0 = B \tag{35}$$

$$T(y=h) = T_1: \quad T_1 = \frac{-\mu}{2k}\frac{U^2}{h^2}\,h^2 + A\,h + T_0 \quad \Rightarrow \quad A = \frac{T_1 - T_0}{h} + \frac{\mu\,U^2}{2\,k\,h}. \tag{36}$$

Therefore, the final solution is,

$$T(y) = \frac{-\mu\,U^2}{2\,k\,h^2}\,y^2 + \left(\frac{T_1 - T_0}{h} + \frac{\mu\,U^2}{2\,k\,h}\right)y + T_0. \tag{37}$$

Example: Couette flow with heated walls (continued ...)

Let's non-dimensionalize this solution:

$\Longrightarrow$ To do this, one needs to consider a **characteristic length** and a **characteristic temperature**.

(a) **Length Scale**: An obvious choice is h, so that

$$\tilde{y} = \frac{y}{h}. \tag{38}$$

As a result, equation 37 becomes,

$$T(\tilde{y}) = \frac{-\mu\, U^2}{2\, k}\, \tilde{y}^2 + \left(T_1 - T_0 + \frac{\mu\, U^2}{2\, k} \right) \tilde{y} + T_0. \tag{39}$$

Example: Couette flow with heated walls (continued ...)

(b) **Temperature Scale**: Since we have two temperature scales in the flow (T_0 and T_1), it is best to use, $\Delta T = T_1 - T_0$.

One can then rewrite the previous equation as,

$$\frac{T(\tilde{y}) - T_0}{\Delta T} = \frac{-\mu\, U^2}{2\, k\, \Delta T}\left(\tilde{y}^2 - \tilde{y}\right) + \tilde{y}. \tag{40}$$

Defining $\tilde{T} = \frac{T(\tilde{y}) - T_0}{T_1 - T_0}$, the non-dimensional solution can then be rewritten as,

$$\tilde{T}(\tilde{y}) = \frac{-\mu\, U^2}{2\, k\, \Delta T}\left(\tilde{y}^2 - \tilde{y}\right) + \tilde{y}. \tag{41}$$

Example: Couette flow with heated walls (continued ...)

$\Longrightarrow$ Note that a group of physical parameters appears in the non-dimensional solution

$$\frac{-\mu\, U^2}{2\, k\, \Delta T} \tag{42}$$

Checking the corresponding dimensions of this parameter group,

$$\left[\frac{\mu\, U^2}{k\, \Delta T}\right] = \frac{\frac{kg}{m\, s}\, \frac{m^2}{s^2}}{\frac{J}{m\, s\, K}\, K} = \boxed{\text{Void}} \quad \Rightarrow \quad \text{`Non-dimensional group of variables'.} \tag{43}$$

The value of this non-dimensional parameter will affect the shape of the non-dimensional temperature profile as shown in the following Figure.

Example: Couette flow with heated walls (continued ...)

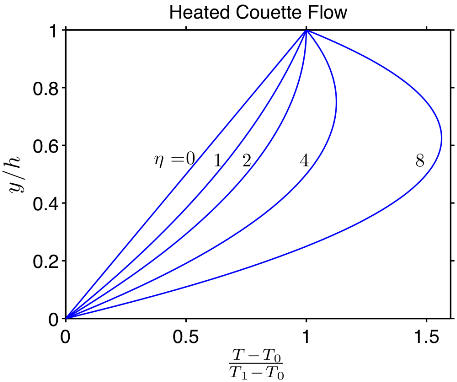

Figure: Distribution of the non-dimensionalized temperature as a function of non-dimensionalized distance. Each line represents a different value of $\eta = \frac{\mu U^2}{k \Delta T}$.

Example: Special case when $T_1 = T_0$

For the unheated case, when the two walls are at the same temperature, the solution in equation 39 reduces to,

$$T(\tilde{y}) - T_0 = \frac{-\mu\, U^2}{2\, k}\left(\tilde{y}^2 - \tilde{y}\right). \tag{44}$$

This solution provides a parabolic profile with a maximum value at the center of the gap $\tilde{y} = 0.5$. The maximum temperature is,

$$T_{max} = T_0 + \frac{\mu\, U^2}{8\, k}. \tag{45}$$

Note:

- The fluid inside the gap gets hot due to viscous effects.
- Knowing the functional form of the viscous dissipation allows us to calculate this rise in temperature.

Example:

Let's calculate the expected temperature rise in the oil of a journal bearing using the Couette flow result.

- Viscosity of oil at moderate temperature (say 30°C) is $\mu = 0.4\, kg/m\, s$;
- The thermal conductivity is $k = 0.14\, J/m\, s\,^{\circ}C$.
- $\Longrightarrow$ At a speed of $U = 5m/s,\ T_{max} - T_0 = 9^{\circ}C$.
- $\Longrightarrow$ At a speed of $U = 10m/s,\ T_{max} - T_0 = 36^{\circ}C$.

This large temperature rise will have an effect on the material properties of the oil. Therefore, in order to get a more accurate solution it may be necessary to allow for a temperature-dependent viscosity.

Example: Power (work rate) required to drive the top plate

The equation for kinetic energy is,

$$\rho \frac{D}{Dt}\left(\frac{1}{2}\, u_i u_i\right) = \rho\, g_i\, u_i + \frac{\partial}{\partial x_j}(u_i\, \tau_{ij}) + p e_{ii} - \phi, \tag{46}$$

Knowing that:

- $\rho \frac{D}{Dt}\left(\frac{1}{2}\, u_i u_i\right) = 0$ given that the flow is steady,
- It is fully developed in x and $v = 0$.
- The flow is incompressible, hence $e_{ii} = 0$,
- Neglecting body-forces.

Then the above equation reduces to,

$$\frac{\partial}{\partial y}(u\, \tau_{xy}) = \phi. \tag{47}$$

Example: Power (work rate) required to drive the top plate (continued ...)

To get the work rate required to drive the fluid between the two plates, one must integrate this equation over the dotted region:

$$\int_{x=0}^{L} \int_{y=0}^{y=h} \frac{\partial}{\partial y}(u\, \tau_{xy})\, dx\, dy =$$

$$= \int_{x=0}^{L} \int_{y=0}^{y=h} \phi\, dx\, dy. \quad (48)$$

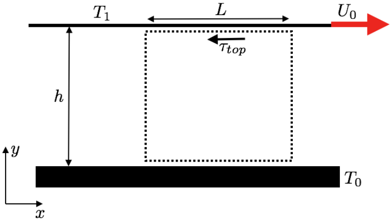

Figure: Sketch of the region over which the work rate done by the viscous forces is computed.

Example: Power (work rate) required to drive the top plate (continued ...)

Nothing depends on x, hence the x-integrals only provide a factor L on both sides, which cancels. Therefore, one only needs to consider the y-integrals,

$$u\,\tau_{xy}\Big|_{y=0}^{y=h} = \int_0^h \phi\, dy \tag{49}$$

$$\underbrace{U\tau_{top}}_{\substack{\text{Work rate done}\\ \text{moving top plate}}} = \mu\frac{U^2}{h}. \tag{50}$$

- The power that must go into driving the top plate is entirely consumed by viscous dissipation, which only serves to heat up the fluid.
- One can calculate the power required to drive the top plate by multiplying τ_{top} with the velocity of the top plate.