ME 5200/6200 Classical Controls

Homework 03 Solutions

Do the following problems and show all your work for full credit. Note: not all problems will be graded, but you must complete all problems to get full credit.

Problem 1

Consider the following first-order system:

$$G(s) = \frac{4}{s + 102}$$

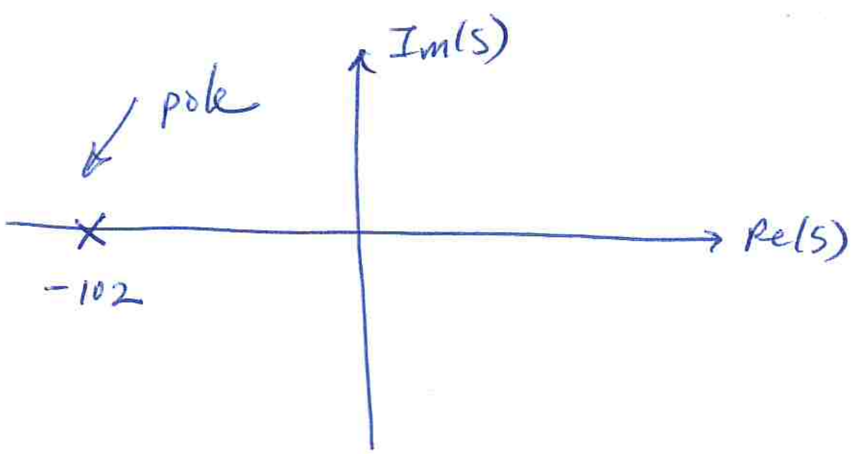
(a) The DC gain of G(s)
(b) Plot the location of the poles and zeros (if any) in the s-plane. Label your axes and clearly indicate the location of the poles and zeros.
(c) The final value of the output for a unit step input (hint: use the final-value theorem)
(d) The time constant
(e) Use Matlab to make a plot of the step response
(~~d~~ f) From the Matlab step response plot, estimate the time constant. Explain any differences in your results with part (d).

(a) DC gain is magnitude of $|G(s)|$ when $s=0$

$$\text{DCGain} = \left|G(s)\right|_{s=0} = \left|\frac{4}{0+102}\right| = \left|\frac{4}{102}\right| \simeq \underline{\underline{0.039}}$$

(b) zeros : None

poles : $s + 102 = 0 \Rightarrow s = -102$



(c) Final value : use final value thereom

$$\frac{Y(s)}{U(s)} = G(s) \Rightarrow Y(s) = G(s)\,U(s)$$

$$Y(s) = \left(\frac{4}{s+102}\right)\left(\frac{1}{s}\right)$$

$$y_{ss} = \lim_{s \to 0} s\,Y(s) = \lim_{s \to 0} \cancel{s}\left(\frac{4}{s+102}\right)\left(\frac{1}{\cancel{s}}\right)$$

$$\Rightarrow y_{ss} = \frac{4}{102} \cong \underline{\underline{0.039}}$$

(d) From the transfer fuction, we see that

$$a = 102 \Rightarrow \tau = 1/a = 1/102$$

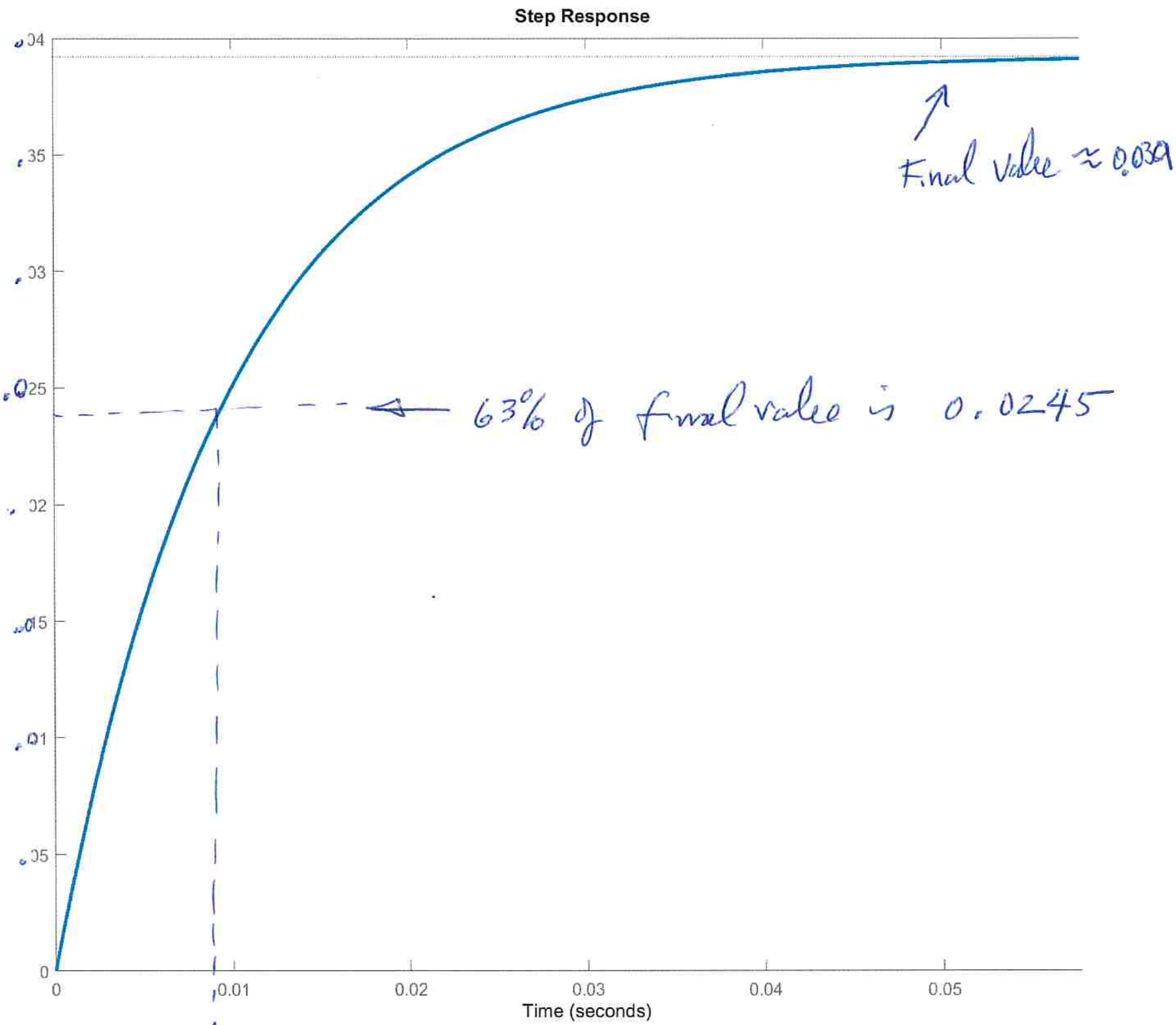
$$\underline{\underline{\tau = 9.8 \times 10^{-3} \text{ seconds}}}$$

(e) see plot.

(f) see plot

(e) step response from Matlab

(f) see below on finding τ



$\tau \cong 0.0098$

Problem 2

Consider the following second-order system:

$$G(s) = \frac{2}{s^2 + s + 2}$$

Determine:

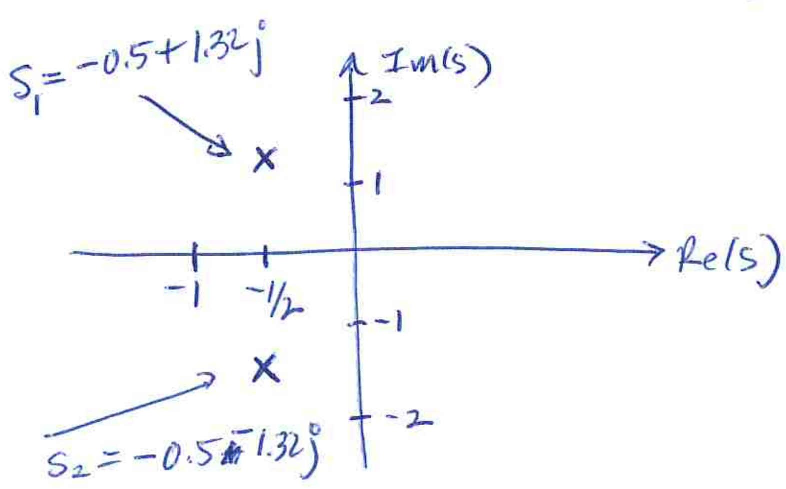
(a) The DC gain of G(s)
(b) Plot the location of the poles and zeros (if any) in the s-plane. Label your axes and clearly indicate the location of the poles and zeros.
(c) The final value of the output for a unit step input (hint: use the final-value theorem)
(d) The damping ratio and natural frequency of the system
(e) The rise-time, time-to-peak, settling time, and percent overshoot. Find these values using the equations provided in class.
(f) Use Matlab to make a plot of the step response
(g) From the Matlab step response plot, estimate the rise-time, time-to-peak, settling time, and percent overshoot and compare to your answers in part (d). Explain any differences in your results.

(a) $\text{DC Gain} = \left|G(s)\right|_{s=0} = \left|\frac{2}{0^2+0+2}\right| \Rightarrow \underline{\text{DC Gain} = 1}$

(b) poles: $s^2 + s + 2 = 0$ use quadratic formula

$$s_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-1 \pm \sqrt{1^2 - 4(1)(2)}}{2}$$

$$s_{1,2} = \frac{-1 \pm j\sqrt{7}}{2} = -\frac{1}{2} \pm \frac{\sqrt{7}}{2} j = \underline{-0.5 \pm 1.32j}$$



(c) Final value:

$$Y(s) = G(s)U(s) \quad \text{where } U(s) = 1/s \text{ (unit step)}$$

$$y_{ss} = \lim_{s \to 0} s\,Y(s) = \lim_{s \to 0} \cancel{s}\left(\frac{2}{s^2+s+2}\right)\left(\frac{1}{\cancel{s}}\right)$$

$$\Rightarrow \underline{\underline{y_{ss} = 1}}$$

(d) From the transfer function, the denominator term: $s^2 + 2\zeta\omega_n s + \omega_n^2$

So: $2\zeta\omega_n = 1$

$$\omega_n^2 = 2 \Rightarrow \underline{\underline{\omega_n = \sqrt{2}}}$$

and
$$\zeta = \frac{1}{2\sqrt{2}} \Rightarrow \zeta = \frac{\sqrt{2}}{4} \simeq \underline{\underline{0.355}}$$

(e)

rise time: $t_r \cong \dfrac{1.8}{\omega_n} = \dfrac{1.8}{\sqrt{2}}$

$$\Rightarrow \underline{\underline{t_r \simeq 1.27\,s}}$$

time-to-peak: $t_p = \dfrac{\pi}{\omega_n\sqrt{1-\zeta^2}} = \dfrac{\pi}{\sqrt{2}\,(1-.707^2)}$

$$\Rightarrow \underline{\underline{t_p = 2.539\,s}}$$

settling time: $t_s \cong \frac{4}{\zeta \omega_n} = \frac{4}{(0.707)\sqrt{2}}$

$$\Rightarrow \boxed{t_s \simeq 8.0\,s}$$

percent overshoot:

$$\%OS = 100\, e^{-\pi\zeta/\sqrt{1-\zeta^2}} = 100\, e^{-\pi(.707)/\sqrt{1-0.707^2}}$$

$$\%OS \approx \overset{30.5\%}{\cancel{4.3}\%}$$

(f) See plot

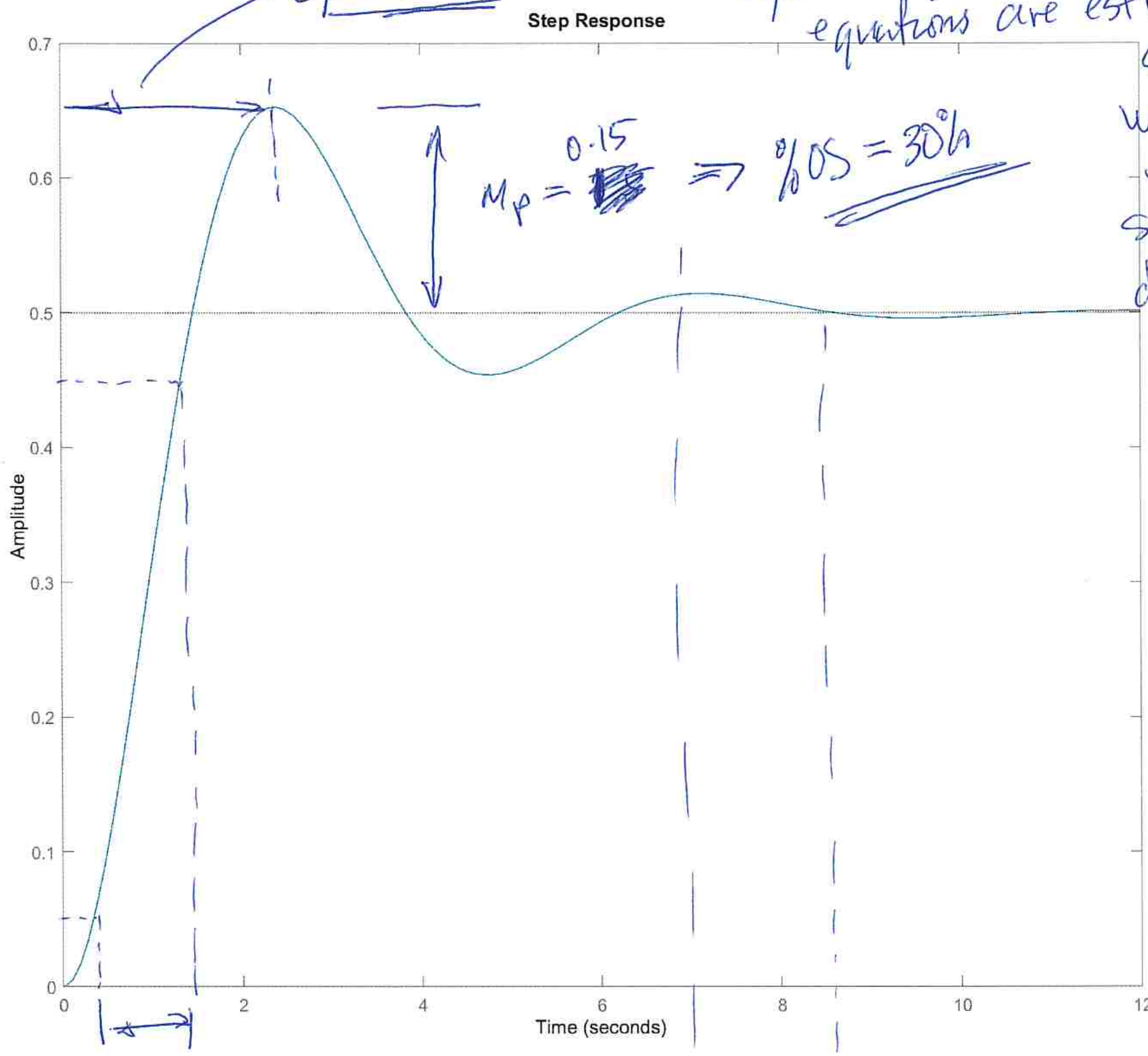
(g) see plot

(e) Step response from Matlab

(f) see below

Note that values from plot are off w/ the values from equations. This is because equations are estimates, so we would expect some discrepancies!

$t_p \simeq 2.25\,s$



$M_p = 0.15 \Rightarrow \%OS = 30\%$

$t_r \simeq 1\,s$

$t_s \simeq 7 - 8.2\,s$

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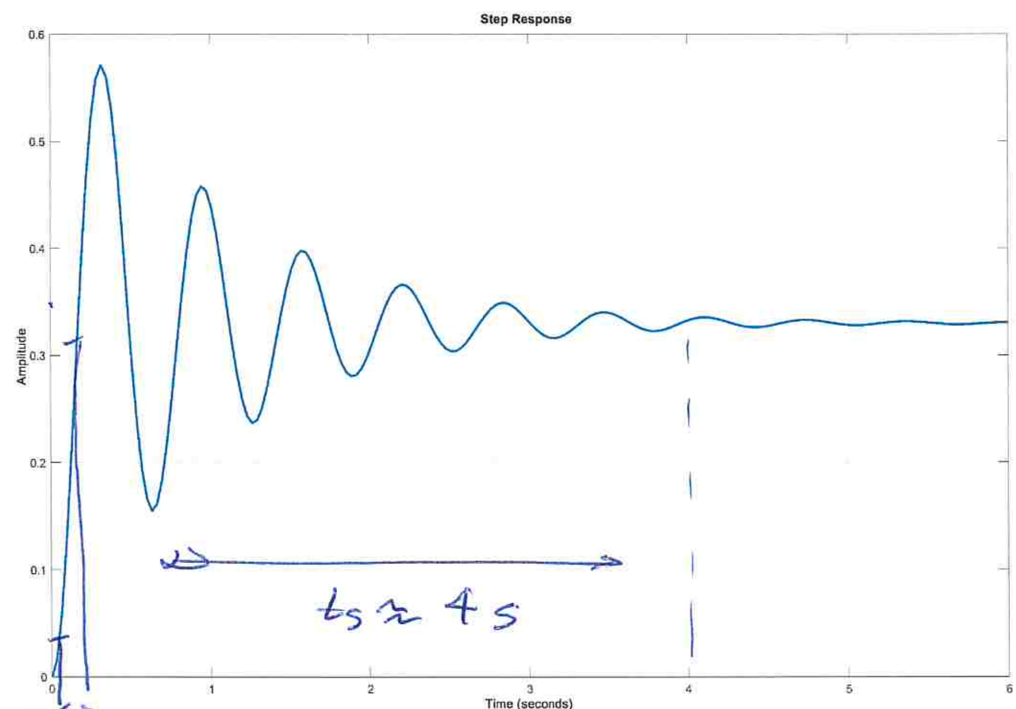
Problem 3

Consider the following step response for a generic second-order system. Estimate the transfer function G(s) for the second-order system. Hint, you need to find the constants $\bar{K}$, ξ, and ω_n, then write out your estimated transfer function with the estimated parameters you found from the step response.



$\bar{K} \approx 0.33$

t_r

$$G(s) = \bar{K}\left(\frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}\right) \qquad \text{DC Gain} = \bar{K}$$

From plot, we know that $\underline{\underline{\bar{K} = 0.33}}$

To find ξ, ω_n, we use:

$$t_s \approx \frac{4}{\xi\omega_n} \quad \text{and} \quad t_r \approx \frac{1.8}{\omega_n}$$

From plot: $t_r \approx 0.2\,s$ and $t_s \cong 4\,s$

$$\Rightarrow \frac{1.8}{\omega_n} = 0.2 \Rightarrow \omega_n = 9 \text{ rad/s} \quad \text{and} \quad \xi = \frac{4}{t_s\omega_n} = \frac{4}{4(9)} \cong 0.11$$

Thus, the T.F. $G(s)$ is:

$$G(s) = \bar{K}\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = 0.33\frac{81}{s^2 + 2(.11)(9)s + 81}$$

$$\Rightarrow G(s) = \frac{26.7}{s^2 + 1.98s + 81}$$

This answer is very close to the actual $G(s) = \frac{33}{s^2 + 2s + 100}$ that was used to generate the step response shown!

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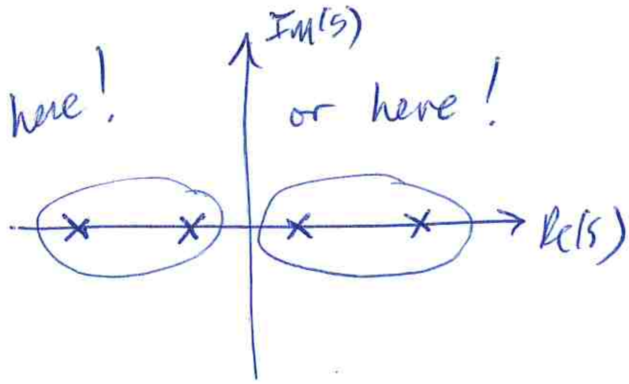
Problem 4

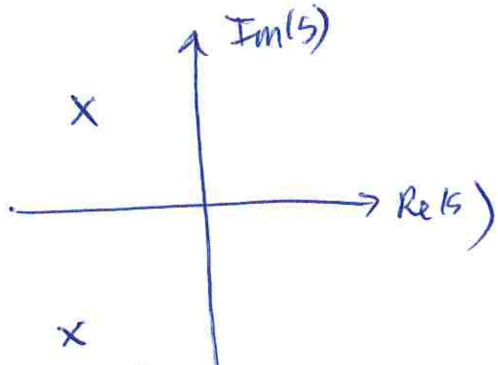
(a) If the step response of a second-order system does not oscillate, sketch the location of the poles in the s-plane.
(b) If the step response of a second-order system oscillates and decays, sketch the location of the poles in the s-plane.
(c) If the step response of a second-order system oscillates but the oscillations never decays, sketch the location of the poles in the s-plane.
(d) If the step response of a second-order system oscillates but the oscillations grows without bound as time increases, sketch the location of the poles in the s-plane.

(a)



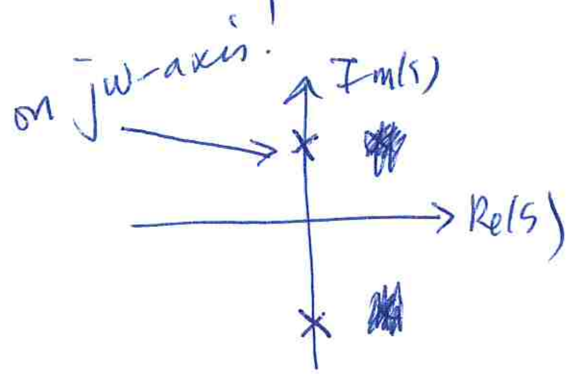
poles are on the real axis; can also be on the right-hand side, too!

(b)



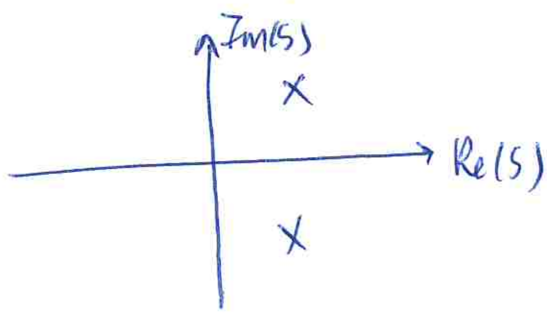
complex poles on the left-hand side of jω-axis.

(c)



complex poles on the ~~right-hand side of~~ jω-axis!

(d)



complex poles on the right-hand side of jω-axis!