Chapter 7

State-Space Design

Problems and Solutions for Section 7.3: Block diagrams and State Space

1. Write the dynamic equations describing the circuit in Fig.7.83. Write the equations as a second-order differential equation in $y(t)$. Assuming a zero input, solve the differential equation for $y(t)$ using Laplace-transform methods for the parameter values and initial conditions shown in the figure. Verify your answer using the initial command in MATLAB.

   Solution:

$$i = C\frac{dy}{dt} \qquad (1)$$
$$v = L\frac{di}{dt} \qquad (2)$$
$$u(t) - L\frac{di}{dt} - Ri(t) - y(t) = 0$$
$$\frac{di}{dt} = \frac{u}{L} - \frac{R}{L}i - \frac{1}{C}y \qquad (3)$$

   Substituting the given values for L, R, and C we have for equation (3):

$$\frac{di}{dt} = u - 2\frac{dy}{dt} - y$$
$$\ddot{y} + 2\dot{y} + y = u$$

   Characteristic equation:

$$\begin{aligned} s^2 + 2s + 1 &= 0 \\ (s+1)^2 &= 0 \end{aligned}$$

   So:

$$y(t) = A_1 e^{-t} + A_2 t e^{-t}$$

Figure 7.83: Circuit for Problem 7.1

Solving for the coefficients:

$$\begin{aligned} y(t) &= A_1 e^{-t} + A_2 t e^{-t} \\ y(t_0) &= A_1 e^{-t_0} + A_2 t_0 e^{t_0} = 1 \\ \dot{y}(t) &= -A_1 e^{-t} + A_2 e^{-t} - A_2 t e^{-t} \\ \dot{y}(t_0) &= -A_1 e^{-t_0} + A_2 e^{-t_0} - A_2 t e^{-t_0} = 0 \end{aligned}$$

$$\Rightarrow A_2 = e^{t_0} \quad \text{and} \quad A_1 = (1 - t_0)e^{t_0}$$

$$y(t) = (1 - t_0)e^{t_0 - t} + t e^{t_0 - t}.$$

To verify the solution using MATLAB, re-write the differential equation in state space form,

$$\begin{aligned} \begin{bmatrix} \dot{y} \\ \ddot{y} \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} y \\ \dot{y} \end{bmatrix} + \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix} u = \mathbf{A}\mathbf{x} + \mathbf{B}u \\ y &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} y \\ \dot{y} \end{bmatrix} = \mathbf{C}\mathbf{x}\ , \end{aligned}$$

where $\mathbf{x} = [y\ \dot{y}]^T$. $L = 1$. Then the following MATLAB statements,

```
A=[0,1;-1,-2];
B=[0;1];
C=[1,0];
D=[0];
sys=ss(A,B,C,D);
xo=[1;0];
```

```
[y,t,x]=initial(sys,xo);
plot(t,y);
grid on;
xlabel('Time (sec)');
ylabel('y(t)');
title('Initial condition response');
```

generate the initial condition response shown below that agrees with the analytical solution above.

Problem 7.1: Initial condition response.

2. A schematic for the satellite and scientific probe for the Gravity Probe-B (GP-B) experiment that was launched on April 30, 2004 is sketched in Fig. 7.84. Assume that the mass of the spacecraft plus helium tank, m_1, is 2000 kg and the mass of the probe, m_2, is 1000 kg. A rotor will float inside the probe and will be forced to follow the probe with a capacitive forcing mechanism. The spring constant of the coupling, k, is 3.2×10^6. The viscous damping b is 4.6×10^3.

Figure 7.84: Schematic diagram of the GP-B satellite and probe.

(a) Write the equations of motion for the system consisting of masses m_1 and m_2 using the inertial position variables, y_1 and y_2.

(b) The actual disturbance u is a micrometeorite, and the resulting motion is very small Therefore, rewrite your equations with the scaled variables $z_1 = 10^6 y_1$, $z_2 = 10^6 y_2$, and $v = 1000u$.

(c) Put the equations in state-variable form using the state $\mathbf{x} = [z_1 \quad \dot{z}_1 \quad z_2 \quad \dot{z}_2]^T$, the output $y = z_2$, and the input an impulse, $u = 10^{-3}\delta(t)$ N·sec on mass m_1.

(d) Using the numerical values, enter the equations of motion into MATLAB in the form

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}v \tag{1}$$
$$y = \mathbf{C}\mathbf{x} + Dv \tag{2}$$

and define the MATLAB system: sysGPB = ss(A,B,C,D). Plot the response of y caused by the impulse with the MATLAB command impulse(sysGPB). This is the signal the rotor must follow.

(e) Use the MATLAB commands p = eig(A) to find the poles (or roots) of the system and z = tzero(A,B,C, D) to find the zeros of the system.

Solution:

(a) The rotor is not part of the problem and can be ignored in writing the equations of motion

$$\begin{aligned} m_1\ddot{y}_1 &= u - k\,(y_1 - y_2) - b\,(\dot{y}_1 - \dot{y}_2) \\ m_2\ddot{y}_2 &= -k\,(y_2 - y_1) - b\,(\dot{y}_2 - \dot{y}_1) \end{aligned}$$

(b) Let's put in the values for the parameters as well as scale the variables as requested.

$$\begin{aligned} 2000\left(10^{-6}\ddot{z}_1\right) &= \frac{1}{1000}v - 10^{-6}(3.2\times 10^6)\,(z_1 - z_2) - 10^{-6}(4.6\times 10^3)\,(\dot{z}_1 - \dot{z}_2) \\ 1000\left(10^{-6}\ddot{z}_2\right) &= -10^{-6}(3.2\times 10^6)\,(z_2 - z_1) - 10^{-6}(4.6\times 10^3)\,(\dot{z}_2 - \dot{z}_1) \end{aligned}$$

which becomes

$$\begin{aligned}\ddot{z}_1 &= -(1.6\times 10^3)(z_1 - z_2) - (2.3)(\dot{z}_1 - \dot{z}_2) + \frac{1}{2}v \\ \ddot{z}_2 &= -(3.2\times 10^3)(z_2 - z_1) - (4.6)(\dot{z}_2 - \dot{z}_1)\end{aligned}$$

(c) The state-variable form for $\mathbf{x} = [z_1 \quad \dot{z}_1 \quad z_2 \quad \dot{z}_2]^T$ is

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -(1.6\times 10^3)(x_1 - x_3) - (2.3)(x_2 - x_4) + \frac{1}{2}v \\ \dot{x}_3 &= x_4 \\ \dot{x}_4 &= -(3.2\times 10^3)(x_3 - x_1) - (4.6)(x_4 - x_2)\end{aligned}$$

or, in matrix form

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\\ \dot{x}_4\end{bmatrix} = \begin{bmatrix}0 & 1 & 0 & 0\\ -1.6\times 10^3 & -2.3 & 1.6\times 10^3 & 2.3\\ 0 & 0 & 0 & 1\\ 3.2\times 10^3 & 4.6 & -3.2\times 10^3 & -4.6\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix} + \begin{bmatrix}0\\ \frac{1}{2}\\ 0\\ 0\end{bmatrix} v$$

and the output equation is

$$y = [0 \quad 0 \quad 1 \quad 0]\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix} + 0$$

(d) The system matrices

$$\mathbf{A} = \begin{bmatrix}0 & 1 & 0 & 0\\ -1.6\times 10^3 & -2.3 & 1.6\times 10^3 & 2.3\\ 0 & 0 & 0 & 1\\ 3.2\times 10^3 & 4.6 & -3.2\times 10^3 & -4.6\end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix}0\\ \frac{1}{2}\\ 0\\ 0\end{bmatrix} \quad \text{and} \quad \mathbf{C} = [0 \quad 0 \quad 1 \quad 0] \quad \text{and} \quad D = 0$$

plus the MATLAB statements:

```
sysGPB = ss(A,B,C,D);
t=0:0.001:1;
y=impulse(sysGPB,t);    % u = 10^-3 implies that v = 1
plot(t,y)
```

produce the next plot.

Problem 7.2: Impulse response.

The micro meteorite hits the first mass and imparts a velocity of 0.33μm/sec to the two mass system. It also excites the resonant mode of relative motion between the masses that dies out in less than a second.

Problems and Solutions for Section 7.4: Analysis of the State Equations

3. Give the state description matrices in control-canonical form for the following transfer functions:

   (a) $G(s) = \dfrac{1}{7s+1}$.

   (b) $G(s) = \dfrac{6(s/3+1)}{(s/10+1)}$.

   (c) $G(s) = \dfrac{7s+1}{s^2+3s+2}$.

   (d) $G(s) = \dfrac{s+7}{s(s^2+2s+2)}$.

(e) $G(s) = \dfrac{(s+7)(s^2+s+25)}{s^2(s+2)(s^2+s+36)}$.

Solution:

(a) $A = -1/7,\ B = 1,\ C = 1/7,\ D = 0.$

(b) $G(s) = \frac{20s+60}{s+10} = 20 - \frac{140}{s+10}; A = -10,\ B = 1,\ C = -140,\ D = 20$.
Hint: Do a partial fraction expansion to find the D term first.

(c)

$$\mathbf{A}_c = \begin{bmatrix} -3 & -2 \\ 1 & 0 \end{bmatrix},\ \mathbf{B_c} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \tag{3}$$

$$\mathbf{C}_c = \begin{bmatrix} 7 & 1 \end{bmatrix},\ D_c = [0]. \tag{4}$$

(d)

$$\mathbf{A}_c = \begin{bmatrix} -2 & -2 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},\ \mathbf{B}_c = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},\ \mathbf{C_c} = \begin{bmatrix} 0 & 1 & 7 \end{bmatrix},\ D_c = [0].$$

(e)

$$\mathbf{A}_c = \begin{bmatrix} -3 & -38 & -72 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix},\ \mathbf{B}_c = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},\ \mathbf{C}_c = \begin{bmatrix} 0 & 1 & 8 & 32 & 175 \end{bmatrix},\ D_c = [0].$$

4. Use the MATLAB function tf2ss to obtain the state matrices called for Problem7.3.

Solution:

In all cases, simply form *num* and *den* given below and then use the MATLAB command [A,B,C,D] = tf2ss(num,den).

(a) $num = \begin{bmatrix} 0 & 0.5 \end{bmatrix}$, $den = \begin{bmatrix} 1 & 0.5 \end{bmatrix}$.

(b) $num = \begin{bmatrix} 20 & 60 \end{bmatrix}$, $den = \begin{bmatrix} 1 & 10 \end{bmatrix}$.

(c) $num = \begin{bmatrix} 0 & 8 & 1 \end{bmatrix}$, $den = \begin{bmatrix} 1 & 3 & 2 \end{bmatrix}$.

(d) $num = \begin{bmatrix} 0 & 0 & 1 & 7 \end{bmatrix}$, $den = \begin{bmatrix} 1 & 2 & 2 & 0 \end{bmatrix}$.

(e) num=conv([1 10],[1 1 25]), den=conv([1 2 0 0],[1 1 36]); $num = \begin{bmatrix} 0 & 0 & 1 & 11 & 35 & 250 \end{bmatrix}$, $den = \begin{bmatrix} 1 & 3 & 38 & 72 & 0 & 0 \end{bmatrix}$.

Note that the answers are the same as for Problem 7.2.

Hint: The MATLAB function conv will save time when forming the numerator and denominator for part (e).

5. Give the state description matrices in normal-mode form for the transfer functions of Problem7.3. Make sure that all entries in the state matrices are real valued by keeping any pairs of complex conjugate poles together, and realize them as a separate subblock in control canonicalform.

Solution:

(a) $A = -0.5,\ B = 1,\ C = 0.5,\ D = 0.$

(b) $A = -10,\ B = 1,\ C = -140,\ D = 20.$

(c)

$$\frac{8s+1}{s^2+3s+2} = \frac{8s+1}{(s+1)(s+2)} = \frac{-7}{s+1} + \frac{15}{s+2},$$

The computation can also be done using the residue command in MATLAB: [r,p,k]=residue(num,den).

Block diagram for Problem 7.5 (c).

$$\mathbf{A} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix},\ \mathbf{B} = \begin{bmatrix} 1 \\ 1 \end{bmatrix},\ \mathbf{C} = \begin{bmatrix} -7 & 15 \end{bmatrix},\ D = [0].$$

(d)

$$\frac{s+7}{s(s^2+2s+2)} = \frac{7/2}{s} - \frac{7/2s+6}{s^2+2s+2},$$

The computation can also be done using the residue command in MATLAB: [r,p,k]=residue(num,den).

Block diagram for Problem 7.5 (d).

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -2 & -2 \\ 0 & 1 & 0 \end{bmatrix}, \ \mathbf{B} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \ \mathbf{C} = \begin{bmatrix} \frac{7}{2} & -\frac{7}{2} & -2 \end{bmatrix}, \ D = [0].$$

(e) The hard part is getting the expansion,

$$\frac{(s+10)(s^2+s+25)}{s^2(s+2)(s^2+s+36)} = \frac{-1.346s+3.472}{s^2} + \frac{1.421}{s+2} + \frac{-0.0746s+0.2952}{s^2+s+36}$$

You can use the MATLAB function residue to obtain this: [r,p,k]=residue(num,den). From the figure, we have,

Block diagram for Problem 7.5 (e).

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & -1 & -36 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \ \mathbf{B} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \ \mathbf{C} = \begin{bmatrix} -1.346 & 3.472 & 1.421 & -0.0746 & 0.2952 \end{bmatrix}, \ D = [0].$$

6. A certain system with state $\mathbf{x}$ is described by the state matrices,

$$\mathbf{A} = \begin{bmatrix} -2 & 1 \\ -2 & 0 \end{bmatrix}, \ \mathbf{B} = \begin{bmatrix} 1 \\ 3 \end{bmatrix},$$
$$\mathbf{C} = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad D = 0.$$

Find the transformation $\mathbf{T}$ so that if $\mathbf{x} = \mathbf{T}\mathbf{z}$, the state matrices describing the dynamics of $\mathbf{z}$ are in control canonical form. Compute the new matrices $\bar{\mathbf{A}}$, $\bar{\mathbf{B}}$, $\bar{\mathbf{C}}$, and$\bar{D}$.

Solution:

Following the procedure outlined in the chapter, we have,

$$\mathcal{C} = \begin{bmatrix} \mathbf{B} & \mathbf{AB} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 3 & -2 \end{bmatrix}.$$

$$\begin{aligned} t_2 &= \begin{bmatrix} 0 & 1 \end{bmatrix} \mathcal{C}^{-1} = \frac{1}{5} \begin{bmatrix} 3 & -1 \end{bmatrix}, \\ t_1 &= t_2 \mathbf{A} = \frac{1}{5} \begin{bmatrix} -4 & 3 \end{bmatrix}. \end{aligned}$$

Thus,

$$\begin{aligned} \mathbf{T}^{-1} &= \begin{bmatrix} -4/5 & 3/5 \\ 3/5 & -1/5 \end{bmatrix} \Longrightarrow \mathbf{T} = \begin{bmatrix} 1 & 3 \\ 3 & 4 \end{bmatrix}, \\ \bar{\mathbf{A}} &= \mathbf{T}^{-1}\mathbf{A}\mathbf{T} = \begin{bmatrix} -2 & -2 \\ 1 & 0 \end{bmatrix}, \ \bar{\mathbf{B}} = \mathbf{T}^{-1}\mathbf{B} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \\ \bar{\mathbf{C}} = \mathbf{C}\mathbf{T} &= \begin{bmatrix} 1 & 3 \end{bmatrix}, \ \bar{D} = D = 0. \end{aligned}$$

7. Show that the transfer function is not changed by a linear transformation of state.

Solution:

Assume the original system is,

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}u, \\ y &= \mathbf{C}\mathbf{x} + Du, \\ G(s) &= \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + D. \end{aligned}$$

Assume a change of state from $\mathbf{x}$ to $\mathbf{z}$ using the nonsingular transformation $\mathbf{T}$,

$$\mathbf{x} = \mathbf{T}\mathbf{z}.$$

The new system matrices are,

$$\bar{\mathbf{A}} = \mathbf{T}^{-1}\mathbf{A}\mathbf{T}, \ \bar{\mathbf{B}} = \mathbf{T}^{-1}\mathbf{B}, \ \bar{\mathbf{C}} = \mathbf{C}\mathbf{T}, \ \bar{D} = D.$$

The transfer function is,

$$\begin{aligned} G_z(s) &= \bar{\mathbf{C}}(s\mathbf{I} - \bar{\mathbf{A}})^{-1}\bar{\mathbf{B}} + \bar{D} \\ &= \mathbf{C}\mathbf{T}(s\mathbf{I} - \mathbf{T}^{-1}\mathbf{A}\mathbf{T})^{-1}\mathbf{T}^{-1}\mathbf{B} + D. \end{aligned}$$

If we factor $\mathbf{T}$ on the left and $\mathbf{T}^{-1}$ on the right of the $(s\mathbf{I} - \mathbf{T}^{-1}\mathbf{A}\mathbf{T})^{-1}$ term, we obtain,

$$\begin{aligned} G_z(s) &= \mathbf{C}\mathbf{T}(s\mathbf{T}\mathbf{T}^{-1} - \mathbf{T}^{-1}\mathbf{A}\mathbf{T})^{-1}\mathbf{T}^{-1}\mathbf{B} + D \\ &= \mathbf{C}\mathbf{T}\mathbf{T}^{-1}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{T}\mathbf{T}^{-1}\mathbf{B} + D = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + D = G(s). \end{aligned}$$

8. Use block-diagram reduction or Mason's rule to find the transfer function for the system in observer canonical form depicted by Fig.7.31.

Solution:

(a) We will show the process for the general third-order case shown below. Combine the feedforward terms to produce the top figure on the next page. Then, reduce the last loop to get the figure on the bottom.

Problem 7.8: Observer canonical form.

Observer canonical form: feedforward terms combined.

Observer canonical form: one loop reduced.

(a) Using Mason's rule:

Forward path gains:

$$p_1 = \frac{b_3}{s^3}, \quad p_2 = \frac{b_2}{s^2}, \quad p_3 = \frac{b_1}{s}$$

Loop path gains:

$$\ell_1 = -\frac{a_3}{s^3}, \quad \ell_2 = -\frac{a_2}{s^2}, \quad \ell_3 = -\frac{a_1}{s}$$

$$\frac{Y}{R} = \frac{p_1 + p_2 + p_3}{1 - \ell_1 - \ell_2 - \ell_3} = \frac{\frac{b_3}{s^3} + \frac{b_2}{s^2} + \frac{b_1}{s}}{1 + \frac{a_3}{s^3} + \frac{a_2}{s^2} + \frac{a_1}{s}} = \frac{b_1 s^2 + b_2 s + b_3}{s^3 + a_1 s^2 + a_2 s + a_3}.$$

9. Suppose we are given a system with state matrices $\mathbf{A}, \mathbf{B}, \mathbf{C}$ ($D = 0$ in this case). Find the transformation $\mathbf{T}$ so that, under Eqs.(7.21) and (7.22), the new state description matrices will be in observer canonicalform.

Solution:

Express the transformation matrix in terms of its column vectors,

$$\mathbf{T} = [t_1 \;\; t_2 \;\; t_3]$$

Then if $\mathbf{A}$ is in observer canonical form,

$$\begin{aligned} \mathbf{T}\mathbf{A}_o &= [* \;\; t_1 \;\; t_2] = \mathbf{A}\mathbf{T} = [\mathbf{A}t_1 \;\; \mathbf{A}t_2 \;\; \mathbf{A}t_3] \\ \mathbf{C}_o &= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} = \mathbf{C}\mathbf{T} = [\mathbf{C}t_1 \;\; \mathbf{C}t_2 \;\; \mathbf{C}t_3]. \end{aligned}$$

From these,

$$\begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \mathbf{CA}^2 \end{bmatrix} t_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \Longrightarrow t_3 = \mathcal{O}^{-1} \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T, \; t_2 = \mathbf{A}t_3, \; t_1 = \mathbf{A}t_2.$$

10. Use the transformation matrix in Eq.(7.38) to explicitly multiply out the equations at the end of Example7.9.

Solution:

$$\mathbf{A}_m = \mathbf{T}^{-1}\mathbf{A}_c\mathbf{T} = \begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} -7 & -12 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 4 & -3 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} -4 & 0 \\ 0 & -3 \end{bmatrix}.$$

$$\begin{aligned} \mathbf{B}_m &= \mathbf{T}^{-1}\mathbf{B}_c = \begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \\ \mathbf{C}_m &= \mathbf{C}_c\mathbf{T} = \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} 4 & -3 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 \end{bmatrix}. \end{aligned}$$

11. Find the state transformation that takes the observer canonical form of Eq.(7.32) to the modal canonicalform.

Solution:

We wish to find the transformation $\mathbf{T}$ such that,

$$\mathbf{A}_m = \mathbf{T}^{-1}\mathbf{A}_o\mathbf{T} = diag(\lambda_1, \lambda_2, \ldots, \lambda_n).$$

$$\begin{aligned} \mathbf{B}_m &= \mathbf{T}^{-1}\mathbf{B}_o, \\ \mathbf{C}_m &= \mathbf{C}_o\mathbf{T}. \end{aligned}$$

The columns of $\mathbf{T}$ are the eigenvectors of $\mathbf{A}_o$. The eigenvectors of $\mathbf{A}_o$ are all of the form (which can be proved by induction):

$$\begin{aligned} t_i &= \begin{bmatrix} 1 \\ \lambda_i + a_1 \\ \lambda_i^2 + a_1\lambda_i + a_2 \\ \lambda_i^3 + a_1\lambda_i^2 + a_2\lambda_i + a_3 \\ \vdots \\ \lambda_i^{n-1} + a_1\lambda_i^{n-2} + a_2\lambda_i^{n-3} + \ldots + a_{n-1} \end{bmatrix}, \ i = 1, 2, \ldots, n. \\ \mathbf{T} &= \begin{bmatrix} t_1 & t_2 & \ldots & t_n \end{bmatrix}. \end{aligned}$$

For the cases where there are *repeated* eigenvalues, and a full set of linearly *independent* eigenvectors do not exist, then the generalized eigenvectors need to be computed to transform the system to Jordan form (see Strang, 1988).

12. a) Find the transformation $\mathbf{T}$ that will keep the description of the airplane system of Example7.10 in modal canonical form but will convert each element of the input matrix $\mathbf{B}_m$ to unity.

b) Use MATLAB to verify that your transformation does thejob.

Solution:

(a) We would like to find a transformation matrix $\mathbf{T}_1$such that, $\mathbf{B}_m = \mathbf{T}_1^{-1}\mathbf{B} = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}^T$.

where t_i are the eigenvectors, n_i are scalars and $\mathsf{N}=\mathsf{diag}(\mathsf{n}_1,\cdots,\mathsf{n}_5)$. Now,

$$\mathbf{G} = \mathbf{T}_1\mathbf{B}_m = \mathbf{TNB}_m = \mathbf{Tn}, \Rightarrow \mathbf{n} = \mathbf{T}^{-1}\mathbf{B},$$

and $\mathbf{n} = [n_1\ n_2 \cdots n_5]^T$. Thus, to find the transformation $\mathbf{T}_1$, we compute $\mathbf{n}$ and multiply to get $\mathbf{T}_1 = \mathbf{TN} = \mathbf{T}\mathsf{diag}(n_1\ n_2 \cdots n_5)$.

(b) In MATLAB,

$$T = \begin{bmatrix} 0.0307 & -0.0336 & 0.0005 & 0.0000 \\ -0.0068 & -0.0024 & -0.0000 & -0.0019 \\ 0.0001 & 0.0011 & -0.0078 & 0.0005 \\ 0.0002 & -0.0001 & 0.0040 & 0.0316 \end{bmatrix},$$

$$\mathsf{n} = \mathsf{T}\backslash\mathsf{B},$$

$$\mathsf{N} = \mathsf{diag}(\mathsf{n}),$$

$$\mathsf{T}1 = \mathsf{T} * \mathsf{N},$$

$$T1 = \begin{bmatrix} 0.2388 & 0.7627 & -0.0015 & 0.000 \\ -0.0529 & 0.0535 & 0.000 & -0.0006 \\ 0.0006 & -0.0258 & 0.0258 & 0.0002 \\ 0.0012 & 0.0019 & -0.0128 & 0.0097 \end{bmatrix},$$

$$B_m = T1\backslash B = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}^T,$$

$$A_m = T1\backslash A * T1 = \begin{bmatrix} -2.5000 & -16.9423 & -0.0000 & -0.0000 \\ 1.9921 & -2.5000 & -0.0000 & -0.0000 \\ -0.0000 & -0.0000 & -0.0150 & -0.0234 \\ 0.0000 & -0.0000 & 2.5526 & -0.0150 \end{bmatrix}.$$

13. a) Find the state transformation that will keep the description of the airplane system of Example7.10 in modal canonical form but will cause the poles to be displayed in $\mathbf{A}_m$ in order of increasing magnitude.
b) Use MATLAB to verify your result in part(a), and give the complete new set of state matrices as $\bar{\mathbf{A}}$, $\bar{\mathbf{B}}$, $\bar{\mathbf{C}}$, and $\bar{D}$.

Solution:

(a) To change the order of the eigenvalues (poles), p_i, in A_m, all we need to do is re-order the eigenvectors in T. In this case, Thus, $T_2 = [t_3\ t_4\ t_1\ t_2]$.

(b) Our solution uses the MATLAB's sort command to re-order eigenvectors. Note that this approach is independent of the size of the system matrix $\mathbf{A}_m$. $\mathbf{T}$ is the same as in Example 7.10. In MATLAB,

```
p = eig(A);

[f, indices] = sort(abs(p));

T2 = T(:,indices);

n = T2\B;

T3 = T2*diag(n);
```

Am2 = T3\A*T3;

$$Am2 = \begin{bmatrix} -0.015 & -0.0234 & -0.0000 & 0.0000 \\ 2.5526 & -0.0150 & -0.0000 & -0.0000 \\ -0.0000 & -0.0000 & -2.5000 & -16.9423 \\ -0.0000 & -0.0000 & 1.9921 & -2.50000 \end{bmatrix},$$

Bm2 = T3\B;

Bm2=$\begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}^{\mathsf{T}}$

Cm2 = C3*T3;

$$Cm2 = \begin{bmatrix} 9.1885 & 2.7014 & -7.7879 & -4.1020 \end{bmatrix},$$

Dm2 = 0;

14. Find the characteristic equation for the modal-form matrix $\mathbf{A}_m$ of Eq.(7.14a) using Eq.(7.55).

Solution:

$$\det(s\mathbf{I} - \mathbf{A}) = \det \begin{bmatrix} s+4 & 0 \\ 0 & s+3 \end{bmatrix} = (s+4)(s+3)$$

Since $\mathbf{A}_m$ was already in modal form, your solution is easily checked by inspection.

15. Given the system,

$$\dot{\mathbf{x}} = \begin{bmatrix} -5 & 1 \\ -2 & -1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u,$$

with zero initial conditions, find the steady-state value of $\mathbf{x}$ for a step input u.

Solution:

We are given $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$. Steady-state means that $\dot{\mathbf{x}} = 0$ and a step input (or unit step) means $u = 1(t)$. Thus, assuming that the system is stable and $\mathbf{A}$ is invertible (which you can check), we have,

$$\mathbf{0} = \mathbf{A}\mathbf{x}_{ss} + \mathbf{B} \Longrightarrow \mathbf{x}_{ss} = -\mathbf{A}^{-1}\mathbf{B} = -\begin{bmatrix} -5 & 1 \\ -2 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/7 \\ 5/7 \end{bmatrix}.$$

This can be verified in MATLAB with step(A,B,C,D,1) where C=eye(size(A)) and D=[0;0] as shown in the ensuing plot.

Figure 7.85: A block diagram for Problem 7.16

Solution for Problem 7.15.

16. Consider the system shown in Fig.7.85:

    a) Find the transfer function from U to Y.
    b) Write state equations for the system using the state variables indicated.

    Solution:

    (a) The system is equivalent to the block diagram shown. Following the block diagram back to

known state variables,

$$y = x_1 \left[1 + 2s + \frac{1}{s} \right],$$

and,

$$x_1 = \frac{1}{2s(s+4)} \left[u - \frac{1}{s} \left(2s + \frac{1}{s} \right) x_1 \right],$$

resulting in,

$$\frac{Y(s)}{U(s)} = \frac{2s^3 + s^2 + s}{2s^4 + 8s^3 + 2s^2 + 1}.$$

Block diagram for solution of Problem 7.16 (a).

Another possible solution is in terms of Mason's rule.

(b)

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & \frac{1}{2} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u \tag{5}$$

$$y = \begin{bmatrix} 1 & 1 & 0 & 1 \end{bmatrix} \mathbf{x} \tag{6}$$

With practice, you should be able to see quickly that is simply the input to block , which is the sum of and . But this is nothing more than the third row of the matrix equation above. Your results can be checked for consistency using MATLAB's command ss2tf.

17. Using the indicated state variables, write the state equations for each of the systems shown in Fig.7.86. Find the transfer function for each system using both block-diagram manipulation and matrix algebra [as in Eq.(7.45)].

Solution:

(a) Performing a partial fraction expansion on $\frac{s+2}{s+3}$, Fig. 7.86(a) can be redrawn as shown below.

(a)

(b)

Figure 7.86: Block diagrams for Problem 7.17.

Redrawn block diagram for solution to Problem 7.17(a).

By inspection of the block diagram, the state equations are,

$$\begin{aligned}\dot{\mathbf{x}} &= \begin{bmatrix} -5 & 1 & 1 \\ 0 & -3 & -2 \\ 0 & 0 & -4 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} u, \\ y &= \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}\mathbf{x}.\end{aligned}$$

Computing , the following transfer function is obtained:

$$\frac{Y(s)}{U(s)} = \frac{s^2 + 10s + 20}{s^3 + 12s^2 + 47s + 60}.$$

We can use the MATLAB ss2tf command to verify this result.

(b) Using the second block diagram given in Fig. 7.86(b), we can write,

$$\begin{aligned}\dot{\mathbf{x}} &= \begin{bmatrix} -3 & 1 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & -1 & -2 & 1 \\ 0 & 0 & 0 & -10 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} u, \\ y &= \begin{bmatrix} 1 & 0 & 1 & 0 \end{bmatrix}\mathbf{x}.\end{aligned}$$

Computing (using MATLAB's ss2tf), we obtain the following transfer function,

$$\frac{Y(s)}{U(s)} = \frac{s^3 + 14s^2 + 37s + 44}{s^4 + 15s^3 + 60s^2 + 112s + 120}.$$

18. For each of the listed transfer functions, write the state equations in both control and observer canonical form. In each case draw a block diagram and give the appropriate expressions for **A**, **B** , and **C** .
a) (control of an inverted pendulum by a force on the cart)
b)

Solution:

(a)

$$\frac{Y(s)}{U(s)} = G(s) = \frac{s^2 - 2}{s^2(s^2 - 1)}.$$

This transfer function can be realized in controller canonical form as shown below. From the figure, we have,

$$\begin{aligned}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} u, \\ y &= \begin{bmatrix} 0 & 1 & 0 & -2 \end{bmatrix}\mathbf{x}.\end{aligned}$$

Controller canonical form for the transfer function of Problem 7.18(a).

The block diagram for observer canonical form is shown below. From the figure, we have,

$$\begin{aligned}
\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ -2 \end{bmatrix} u, \\
y &= \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \mathbf{x}.
\end{aligned}$$

Observer canonical form for the transfer function of Problem 7.18(a).

(b)

$$\frac{Y(s)}{U(s)} = G(s) = \frac{3s+4}{s^2+2s+2}.$$

This transfer function can be realized in Controller canonical form as shown below. From the figure, we have,

$$\begin{aligned}
\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} -2 & -2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u, \\
y &= \begin{bmatrix} 3 & 4 \end{bmatrix} \mathbf{x}.
\end{aligned}$$

Controller canonical form for the transfer function of Problem 7.18(b).

Figure 7.87: Observer canonical form for the transfer function of Problem 7.18(b).

The block diagram for observer canonical form is shown below. From the figure, we have:

$$\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} -2 & 1 \\ -2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 3 \\ 4 \end{bmatrix} u, \\ y &= \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x}. \end{aligned}$$

19. Consider the transfer function,

$$G(s) = \frac{s+7}{s^2+5s+6},$$

a) By rewriting Eq.(7.283) in the form,

$$G(s) = \frac{1}{s+3}\left(\frac{s+7}{s+2}\right),$$

find a **series realization** of as a cascade of two first-order systems.
b) Using a partial-fraction expansion of $G(s)$, find a **parallel realization** of $G(s)$.
c) Realize in control canonical form.

Solution:

(a) The series realization shown below is given by:

$$G(s) = \left(\frac{1}{s+3}\right)\left(\frac{s+7}{s+2}\right) = \hat{g}_2(s)\hat{g}_1(s).$$

$$\begin{aligned} \text{For } \hat{g}_1(s),\ \dot{x}_1 &= -2x_1 + u_1,\ \ y_1 = -x_1 + u_1. \\ \text{For } \hat{g}_2(s),\ \dot{x}_2 &= -3x_2 + u_2,\ \ y_2 = x_2. \end{aligned}$$

The series interconnections result in $u = u_1, y = y_2, u_2 = y_1$. Therefore,

$$\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} -2 & 0 \\ -1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u, \\ y &= \begin{bmatrix} 0 & 1 \end{bmatrix} \mathbf{x}. \end{aligned}$$

Figure 7.88: Series connection of for Problem 7.19(a).

Figure 7.89: Parallel connection of for Problem 7.19(b).

The MATLAB command series can also be used.

(b) The parallel realization, shown below, is given by:

$$G(s) = \frac{-4}{s+3} + \frac{5}{s+2} = \hat{g}_1(s) + \hat{g}_2(s).$$

$$\begin{array}{lrcl} \text{For } \hat{g}_1(s), & \dot{x}_1 & = & -3x_1 + u_1, \quad y_1 = 2x_1. \\ \text{For } \hat{g}_2(s), & \dot{x}_2 & = & -2x_2 + u_2, \quad y_2 = -x_2. \end{array}$$

The interconnections are

$$u = u_{1,} y = y_2, u_2 = y_1. \tag{7}$$

Therefore,

$$\begin{array}{rcl} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} & = & \begin{bmatrix} -3 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u, \\ y & = & \begin{bmatrix} 2 & -1 \end{bmatrix} \mathbf{x}. \end{array}$$

The MATLAB command parallel can also be used.

Figure 7.90: Controller canonical form of for Problem 7.19(c).

(c) Control canonical form, shown in Fig. 7.17, is realized by simply picking off the appropriate coefficients of the original (strictly proper) transfer function. If the original transfer function is not strictly proper, then it should be reduced to a feedthrough term (D) plus a strictly proper transfer function term.

We have,

$$\mathbf{A} = \begin{bmatrix} -5 & -6 \\ 1 & 0 \end{bmatrix}, \ \mathbf{B} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \tag{8}$$

$$\mathbf{C} = \begin{bmatrix} 7 & 1 \end{bmatrix}, \ D = 0. \tag{9}$$

20. Show that the impulse response of the system ($\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, D) is given by

$$h(t) = \mathbf{C}e^{\mathbf{A}t}\mathbf{B} + D\delta(t),$$

where is the matrix exponential defined by

$$e^{\mathbf{A}t} = \left(\mathbf{I} + \mathbf{A}t + \frac{\mathbf{A}^2t^2}{2!} + \cdots\right) = \sum_{k=1}^{\infty} \frac{\mathbf{A}^k t^k}{k!}.$$

Solution: See web Appendix W.7.13.1,"Solution of State Equations."

Problems and Solutions for Section 7.5: Control-Law Design for Full-State Feedback

21. Consider the plant described by,

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ 7 & -4 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} u,$$
$$y = \begin{bmatrix} 1 & 3 \end{bmatrix}\mathbf{x}.$$

a) Draw a block diagram for the plant with one integrator for each state variable.

b) Find the transfer function using matrix algebra.
c) Find the closed-loop characteristic equation if the feedback is
(1) $u = -\begin{bmatrix} K_1 & K_2 \end{bmatrix}\mathbf{x}$; (2) $u = -Ky$.

Solution:

Problem 7.21 (a): State realization showing integrators explicitly.

(a) See figure.
(b) Using the formula $G(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}$, we obtain,

$$G(s) = \frac{Y(s)}{U(s)} = \frac{7s + 27}{s^2 + 4s - 7}.$$

The MATLAB command ss2tf can also be used.
(c)
(i) State feedback, $u = -\begin{bmatrix} K_1 & K_2 \end{bmatrix}\mathbf{x}$.

$$\begin{aligned} \det(\lambda\mathbf{I} - \mathbf{A} + \mathbf{BK}) &= \det\begin{bmatrix} \lambda + K_1 & -1 + K_2 \\ -7 + 2K_2 & \lambda + 4 + 2K_2 \end{bmatrix} \\ &= \lambda^2 + \lambda(4 + 2K_2 + K_1) + (6K_1 + 7K_2 - 7) = 0. \end{aligned}$$

(ii) Output feedback,

$$u = -Ky = -K\begin{bmatrix} 1 & 3 \end{bmatrix}\mathbf{x} = -\begin{bmatrix} K & 3K \end{bmatrix}\mathbf{x}.$$

This yields the following closed-loop characteristic equation:

$$\lambda^2 + \lambda(7K + 4) + (27K - 7) = 0.$$

Hints: If you have already solved the case for state feedback, simply plug $K_1 = K$ and $K_2 = 3K$ into the characteristic equation for state feedback and find the characteristic equation for output feedback. The output vector $\mathbf{C}$ fixes the ratio among the state variables. Secondly, although there were products of K_1 and K_2 when we were forming the determinant, they should all cancel in your final answer. The reason for this is that the characteristic equation is *linear* in $\mathbf{K}$.

22. For the system,

$$\begin{aligned}\dot{\mathbf{x}} &= \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \\ y &= \begin{bmatrix} 1 & 0 \end{bmatrix}\mathbf{x},\end{aligned}$$

design a state feedback controller that satisfies the following specifications:

(1) Closed-loop poles have a damping coefficient $\zeta = 0.707$.

(2) Step-response peak time is under 3.14sec.

Verify your design with MATLAB.

Solution:

For a second-order system, the specification on rise time can be translated into a value of ω_n by the equation $\omega_d = \omega_n\sqrt{1-\zeta^2}$. Then determine from . This yields . Using full state feedback, we would like the a characteristic equation to be,

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = s^2 + 2s + 2 = 0.$$

Using state feedback $u = -\mathbf{K}\mathbf{x}$, we get,

$$\dot{\mathbf{x}} = (\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{x} = \begin{bmatrix} 0 & 1 \\ -6-k_1 & -5-k_2 \end{bmatrix}\mathbf{x}.$$

Hence the closed-loop characteristic equation is,

$$s^2 + (5+k_2)s + (6+k_1) = 0.$$

Comparing coefficients, $k_1 = -4$ and $k_{2\,=\,-3}$. The MATLAB command place can also be used. The reference step can be simulated in MATLAB with , and the MATLAB command step, as shown below.

Step response for Problem 7.22.

23. a) Design a state feedback controller for the following system so that the closed-loop step response has an overshoot of less than 25% and a 1% settling time under 0.115sec.:

$$\begin{aligned} \dot{\mathbf{x}} &= \begin{bmatrix} 0 & 1 \\ 0 & -10 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \\ y &= \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x}. \end{aligned}$$

b) Use the `step` command in MATLAB to verify that your design meets the specifications. If it does not, modify your feedback gains accordingly.

Solution:

(a) For the overshoot specification,

$$M_p = e^{\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}} < 25\% \Longrightarrow \zeta \cong 0.4.$$

For the settling time specification, we use,

$$e^{-\zeta\omega_n t_s} = 0.01 \Longrightarrow \omega_n = \frac{4.6}{\zeta t_s}.$$

(b) This can be implemented in MATLAB with the following code:

```
A = [0,1;0,-10];
B = [0;1];
C = [1,0];
D = 0;
zeta = 0.404; % Tweak values slightly so that specs are met.
ts = 0.114;
wn = 4.6/(ts*zeta);
p = roots([1, 2*zeta*wn, wn^2]);
k = place(A,B,p);
sysCL=ss(A-B*k,B,C,D)
step(sysCL);
```

The step response is shown next.

Step response for Problem 7.23.

24. Consider the system,

$$\begin{aligned}\dot{\mathbf{x}} &= \begin{bmatrix} -1 & -2 & -2 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} u, \\ y &= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}\mathbf{x}.\end{aligned}$$

a) Design a state feedback controller for the system so that the closed-loop step response has an overshoot of less than 5% and a 1% settling time under 4.6sec.
b) Use the step command in MATLAB to verify that your design meets the specifications. If it does not, modify your feedback gains accordingly.

Solution:

(a) There are many different approaches to designing the control law. We will attack the problem using a symmetric root locus. We assume the output is x_1. Although the system is third-order, we can still use the second-order order rules of thumb in order to get an estimate of where we would like the closed loop poles.

$$\begin{aligned}\sigma &= \frac{4.6}{t_s} \Longrightarrow \zeta\omega_n = 1, \\ M_p &\leq 5\% \Longrightarrow \zeta > 0.7.\end{aligned}$$

The open-loop poles are at $s = -1.45, -0.77 \pm j1.47$ and the open-loop zeros are at -1.37 and 0.37. The symmetric root locus is shown on the next page and was generated using the following MATLAB code:

```
%%%
```

Symmetric root locus for Problem 7.24.

Figure 7.91: System for Problem 7.25

Closed-loop step response for Problem 7.24.

25. Consider the system in Fig.7.87.
a) Write a set of equations that describes this system in the control canonical form as $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$ and $y = \mathbf{C}\mathbf{x}$.
b) Design a control law of the form,

$$u = -\begin{bmatrix} K_1 & K_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix},$$

which will place the closed-loop poles at $s = -2 \pm j2$.

Solution:

(a) Let's write this system in the control canonical form,

$$\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} 0 & -4 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u, \\ y &= \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x}. \end{aligned}$$

(b) If $u = -\begin{bmatrix} K_1 & K_2 \end{bmatrix} \mathbf{x}$, the poles of the closed-loop system satisfy $\det(s\mathbf{I} - \mathbf{A} + \mathbf{BK}) = 0$. Thus,

$$\det \begin{bmatrix} s + K_1 & -1 + K_2 \\ 4 & s \end{bmatrix} = 0 \Longrightarrow s^2 + K_1 s + 4 + K_2 = 0.$$

The closed-loop characteristic equation is,

$$(s + 2 - 2j)(s + 2 + 2j) = s^2 + 4s + 8 = 0.$$

Comparing coefficients, we have $K_1 = 4$ and $K_2 = 4$. The MATLAB command place can also be used to verify this result.

26. *Output Controllability*: In many situations a control engineer may be interested in controlling the output y rather than the state $\mathbf{x}$. A system is said to be **output controllable** if at any time you are able to transfer the output from zero to any desired output in a finite time using an appropriate control signal u^*. Derive necessary and sufficient conditions for a continuous system ($\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$) to be output controllable. Are output and state controllability related? If so, how?

Solution:

Because we are considering *linear* systems, if you can take the state from some initial state to some final condition in a finite time with a finite input, then you can also take it to the same state in infinitesimal time using impulsive inputs. To express this mathematically, let be,

$$u(t) = g_1 \delta(t) + g_2 \delta^{(1)}(t) + \cdots + g_n \delta^{(n-1)}(t),$$

where represents a delta function, represents the first derivative of a delta function (a unit doublet), etc., and the are scalars to be determined. Let,

$$u^* = [g_1 \;\; g_2 \cdots g_n]^T,$$

then

$$\mathbf{x}(0+) - \mathbf{x}(0-) = \mathcal{C}\, u^*.$$

Hence, we have found a control signal that will drive the state to arbitrary values given the non-singularity of the controllability matrix, .

In fact, the invertibility of is a necessary and sufficient condition for state controllability. For output controllability,

$$\begin{aligned} \mathbf{C}\mathbf{x}(0+) - \mathbf{C}\mathbf{x}(0-) &= \mathbf{C}\mathcal{C}\, u^*, \\ y(0+) - y(0-) &= \mathbf{C}\mathcal{C}\, u^*. \end{aligned}$$

Assuming (without loss of generality) that , we have,

$$y(0+) = [\mathbf{CB} \;\; \mathbf{CAB} \cdots \mathbf{CA}^{n-1}\mathbf{B}]u^*.$$

Therefore, a system is output controllable if and only if,

$$[\mathbf{CA} \;\; \mathbf{CAB} \cdots \mathbf{CA}^{n-1}\mathbf{B}] \text{ is full rank .}$$

This is always true (for a single-input single-output system) unless the transfer function is zero. Of course, state controllability implies output controllability, but output controllability does *not* imply state controllability.

27. Consider the system,

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 4 & 0 & 0 \\ -1 & -4 & 0 & 0 \\ 5 & 7 & 1 & 15 \\ 0 & 0 & 3 & -3 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} u.$$

a) Find the eigenvalues of this system. (*Hint:* Note the block-triangular structure.)
b) Find the controllable and uncontrollable modes of this system.
c) For each of the uncontrollable modes, find a vector such that,

$$\mathbf{v}^T\mathbf{B} = 0, \quad \mathbf{v}^T\mathbf{A} = \lambda\mathbf{v}^T.$$

d) Show that there are an infinite number of feedback gains $\mathbf{K}$ that will relocate the modes of the system to -5, -3, -2, and -2.
e) Find the unique matrix $\mathbf{K}$ that achieves these pole locations and prevents initial conditions on the uncontrollable part of the system from ever affecting the controllable part.

Solution:

(a) Because the system is block lower triangular, we can determine the eigenvalues of the system by taking the union of the eigenvalues of each of the blocks along the main (block)diagonal.

$$\begin{aligned} s^2 + 4s + 4 &= 0 \Longrightarrow -2, -2. \\ s^2 + 2s - 48 &= 0 \Longrightarrow 6, -8 \end{aligned}$$

Thus, the eigenvalues of the system are , and . (Easily checked with MATLAB's eig command).

(b) To find the controllable or uncontrollable modes of the system, we follow method learned in Problem 7.28. Specifically, we find an orthogonal similarity transformation which transforms (A,B , C) to $(\bar{\mathbf{A}}, \bar{\mathbf{B}}, \bar{\mathbf{C}})$, where $\bar{\mathbf{A}}$ is an upper-Hessenberg matrix. (See Problem 7.30 for details). Observe that this system is almost in the desired form already! Simply by interchanging the state variables x_2 and x_4, we can transform the system into the proper form.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_4 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 4 & 0 & 0 \\ -1 & -4 & 0 & 0 \\ 0 & 0 & -3 & 3 \\ 5 & 7 & 15 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_4 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u.$$

Now the controllable and uncontrollable modes can be determined by inspection. The uncontrollable modes correspond to the eigenvalues of the block, so and are both uncontrollable modes. Similarly, the controllable modes from the block are and .

MATLAB function ctrbf will give a similar result, although the order of the state variables may be switched.

(c) Notice that we need the left eigenvectors of that is orthogonal to . The only left eigenvector of that is orthogonal to is

(d) Because the modes at and are uncontrollable, we expect that state feedback will not have any affect on these modes. Writing an expression for the feedback we have,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 4 & 0 & 0 \\ -1 & -4 & 0 & 0 \\ 5-k_1 & 7-k_2 & 1-k_3 & 15-k_4 \\ 0 & 0 & 3 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}.$$

Notice that the system matrix is still block diagonal. The characteristic equation of the block gives,

$$\det(s\mathbf{I} - \mathbf{A}_{22}) = s^2 + (2+k_3)s + (3k_3 + 3k_4 - 48) = 0.$$

Picking $k_3 = 6$ and $k_4 = 15$ will place the controllable roots at -3 and -5. Since and are arbitrary, there are an infinite number of feedback gains that will relocate the modes of the system to the desired locations.

(e) To completely decouple the controllable and uncontrollable portions of the system, we make the A_{21} block identically zero by setting $k_1 = 5$ and $k_2 = 7$.

28. Two pendulums, coupled by a spring, are to be controlled by two equal and opposite forces u, which are applied to the pendulum bobs as shown in Fig.7.88. The equations of motion are

$$ml^2\ddot{\theta}_1 = -ka^2(\theta_1 - \theta_2) - mgl\theta_1 - lu,$$
$$ml^2\ddot{\theta}_2 = -ka^2(\theta_2 - \theta_1) - mgl\theta_2 + lu.$$

a) Show that the system is uncontrollable. Can you associate a physical meaning with the controllable and uncontrollable modes?
b) Is there any way that the system can be made controllable?

Solution:

(a) Using the state vector $\mathbf{x} = \begin{bmatrix} \theta_1 & \dot{\theta}_1 & \theta_2 & \dot{\theta}_2 \end{bmatrix}^T$

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\left(\frac{ka^2}{ml^2} + \frac{g}{l}\right) & 0 & \frac{ka^2}{ml^2} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{ka^2}{ml^2} & 0 & -\left(\frac{ka^2}{ml^2} + \frac{g}{l}\right) & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ -\frac{1}{ml} \\ 0 \\ \frac{1}{ml} \end{bmatrix} u.$$

Figure 7.92: Coupled pendulums for Problem 7.28

The controllability matrix is determined as,

$$\begin{aligned}
\mathcal{C} &= \begin{bmatrix} \mathbf{B} & \mathbf{AB} & \mathbf{A}^2\mathbf{B} & \mathbf{A}^3\mathbf{B} \end{bmatrix} \\
&= \begin{bmatrix} 0 & -\frac{1}{ml} & 0 & \frac{1}{ml}\left(\frac{ka^2}{ml^2}+\frac{g}{l}\right)+\frac{ka^2}{m^3l^3} \\ -\frac{1}{ml} & 0 & \frac{1}{ml}\left(\frac{ka^2}{ml^2}+\frac{g}{l}\right)+\frac{ka^2}{m^3l^3} & 0 \\ 0 & \frac{1}{ml} & 0 & -\frac{ka^2}{m^3l^3}-\frac{1}{ml}\left(\frac{ka^2}{ml^2}+\frac{g}{l}\right) \\ \frac{1}{ml} & 0 & -\frac{ka^2}{m^3l^3}-\frac{1}{ml}\left(\frac{ka^2}{ml^2}+\frac{g}{l}\right) & 0 \end{bmatrix}
\end{aligned}$$

Then $(\mathbf{A},\mathbf{B})$ is uncontrollable since $\det(\mathcal{C}) = 0$. If we re-write the state equations in terms of the state vector,

$$\mathbf{z} = \begin{bmatrix} \alpha & \dot{\alpha} & \beta & \dot{\beta} \end{bmatrix}^T,$$

where, $\alpha = \theta_1 + \theta_2$ and, $\beta = \theta_1 - \theta_2$, then the resulting equations of motion are,

$$\begin{aligned}
ml^2\ddot{\alpha} &= -mgl\alpha \\
ml^2\ddot{\beta} &= -2ka^2\beta - mgl\beta - 2lu.
\end{aligned}$$

Clearly, α, the "pendulum mode" (or the symmetric mode, i.e., the pendulums swinging together), is uncontrollable and, β, the "spring mode" (i.e., the unsymmetric mode) is controllable.

(b) Yes, make the forces unequal, i.e., let $u_1 \neq u_2$ or eliminate one of the forces, i.e., let $u_1 = 0$ or let $u_2 = 0$.

29. The state-space model for a certain application has been given to us with the following state

description matrices:

$$\mathbf{A} = \begin{bmatrix} 0.174 & 0 & 0 & 0 & 0 \\ 0.157 & 0.645 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} -0.207 \\ -0.005 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{C} = [\ 1 \quad 0 \quad 0 \quad 0 \quad 0\ ].$$

a) Draw a block diagram of the realization with an integrator for each state variable.
b) A student has computed $\det \mathcal{C} = 2.3 \times 10^{-7}$and claims that the system is uncontrollable. Is the student right or wrong? Why?
c) Is the realization observable?

Solution:

(a) The block diagram is shown below.

Block diagram for Problem 7.29.

(b) The system is controllable because a control signal (command) reaches all the state variables of the system through the integrators in Fig. 7.26. The determinant of the controllability matrix is small, , due to poor scaling of system variables. For example, if the control signal is scaled by 100, then $\det(\mathcal{C}) = -2.3 \times 10^3$.

(c) The realization is unobservable. You can check $\det(\mathcal{O})$ or just observe from the block diagram that there is no path from the state variables x_2, x_3, x_4, or x_5 to the output .

30. *Staircase Algorithm (Van Dooren et al., 1978)*: Any realization (**A, B, C**) can be transformed by an **orthogonal similarity transformation** to ($\bar{\mathbf{A}}$,$\mathbf{B}$,$\bar{\mathbf{C}}$), where is an **upper Hessenberg matrix** (having one nonzero diagonal above the main diagonal):

$$\bar{\mathbf{A}} = \mathbf{T}^T\mathbf{A}\mathbf{T} = \begin{bmatrix} * & \alpha_1 & \mathbf{0} & 0 \\ * & * & \ddots & 0 \\ * & * & \ddots & \alpha_{n-1} \\ * & * & \cdots & * \end{bmatrix}, \quad \bar{\mathbf{B}} = \mathbf{T}^T\mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ g_1 \end{bmatrix},$$

where $g_1 \neq 0$, and,

$$\bar{\mathbf{C}} = \mathbf{CT} = [\bar{c}_1 \bar{c}_2 \cdots \bar{c}_n], \quad \mathbf{T}^{-1} = \mathbf{T}^T.$$

Orthogonal transformations correspond to a **rotation** of the vectors (represented by the matrix columns) being transformed with no change in length.
a) Prove that if $\alpha_i = 0$ and $\alpha_{i+1}, ...\alpha_{n-1} \neq 0$, for some i, then the controllable and uncontrollable modes of the system can be identified after this transformation has been done.
b) How would you use this technique to identify the observable and unobservable modes of (**A**,**B**, **C**)?
c) What advantage does this approach for determining the controllable and uncontrollable modes have over transforming the system to any other form?
d) How can we use this approach to determine a basis for the controllable and uncontrollable subspaces, as in Problem7.44?
This algorithm can be used to design a numerically stable algorithm for pole placement [see Minimis and Paige (1982)]. The name of the algorithm comes from the multi-input version in which the are the blocks that make resemble a staircase.

Solution:

(a) If $\alpha_i = 0$,

$$\bar{\mathbf{A}} = \mathbf{T}^T\mathbf{AT} = \begin{bmatrix} * & \alpha_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & \alpha_2 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & \alpha_{i+1} & 0 & 0 & 0 \\ * & * & * & * & * & \ddots & 0 & 0 \\ * & * & * & * & * & * & * & \alpha_{n-1} \\ * & * & * & * & * & * & * & * \end{bmatrix} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{0} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix}, \quad \bar{\mathbf{B}} = \mathbf{T}^T\mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ g_1 \end{bmatrix},$$

This suggests naturally splitting up the state vector into two parts $\mathbf{x} = \begin{bmatrix} \mathbf{x}_1^T & \mathbf{x}_2^T \end{bmatrix}^T$ where $\mathbf{x}_1$ and $\mathbf{x}_2$ are vectors of the appropriate size (depending upon which $\alpha_i = 0$). Then recognize that the equations are,

$$\begin{aligned} \dot{x}_1 &= \mathbf{A}_{11}x_1, \\ \dot{x}_2 &= \mathbf{A}_{21}x_1 + \mathbf{A}_{22}x_2 + g_1 u. \end{aligned}$$

Notice that the control signal u and the state x_2 do *not* effect the state x_1. Thus, all of the modes associated with the $\mathbf{A}_{11}$ block are uncontrollable. All of the states in x_2 are controllable. This is easily checked by forming the controllability matrix associated with the pair $\left(\mathbf{A}_{22}, \begin{bmatrix} 0 \\ g_1 \end{bmatrix}\right)$. Hence, the system has been split into its controllable and uncontrollable parts.

(b) Use duality, i.e., transform $\left[\mathbf{A}^T, \mathbf{C}^T\right]$ into the Hessenberg form.

(c) Because $\mathbf{T}^{-1} = \mathbf{T}^T$, three advantages are recognized:

(i) Better numerical accuracy.

(ii) The controllable-uncontrollable decomposition is immediate.

(iii) Repeated roots are handled.

(d) Simply split and extract the controllable and uncontrollable subspaces,

$$\mathbf{T} = [\underbrace{\mathbf{T}_1}_{i} \ \underbrace{\mathbf{T}_2}_{n-i}], \tag{10}$$

$$\mathbf{T}_1 = \mathcal{N}(\mathcal{C}^T), \quad \mathbf{T}_2 = \mathcal{R}(\mathcal{C}). \tag{11}$$

See the MATLAB ctrbf (and obsvf) functions.

Problems and Solutions for Section 7.6: Selection of Pole Locations for Good Design

31. The normalized equations of motion for an inverted pendulum at angle on a cart are,

$$\ddot{\theta} = \theta + u, \quad \ddot{x} = -\beta\theta - u,$$

where is the cart position, and the control input is a force acting on the cart.
a) With the state defined as $\mathbf{x} = \begin{bmatrix} \theta & \dot{\theta} & x & \dot{x} \end{bmatrix}^T$ find the feedback gain $\mathbf{K}$ that places the closed-loop poles at $s = -1, -1, -1 \pm j1$.
For parts (b) through (d), assume that $\beta = 0.5$.
b) Use the symmetric root locus to select poles with a bandwidth as close as possible to those of part(a), and find the control law that will place the closed-loop poles at the points you selected.
c) Compare the responses of the closed-loop systems in parts(a) and (b) to an initial condition of $\theta = 10°$. You may wish to use the initial command in MATLAB.
d) Compute $\mathbf{N_x}$ and N_u and for zero steady-state error to a constant command input on the cart position, and compare the step responses of each of the two closed-loop systems.

Solution:

(a) The state space equations of motion are,

$$\begin{bmatrix} \dot{\theta} \\ \ddot{\theta} \\ \dot{x} \\ \ddot{x} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -\beta & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \theta \\ \dot{\theta} \\ x \\ \dot{x} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} u.$$

We require the closed-loop characteristic equation to be,

$$\alpha_c(s) = (s+1)^2(s^2+2s+2) = s^4 + 4s^3 + 7s^2 + 6s + 2.$$

From the above state equations,

$$\det(s\mathbf{I} - \mathbf{A} + \mathbf{BK}) = s^4 + (k_2 - k_4)s^3 + (k_1 - k_3 - 1)s^2 + k_4(1-\beta)s + k_3(1-\beta) \equiv \alpha_c(s)$$

Comparing coefficients yields:

$$\begin{aligned} k_1 &= \frac{10-8\beta}{1-\beta}, \ k_2 = \frac{10-4\beta}{1-\beta}, \ k_3 = \frac{2}{1-\beta}, \ k_4 = \frac{6}{1-\beta}, \\ \mathbf{K} &= \begin{bmatrix} 12 & 16 & 4 & 12 \end{bmatrix}. \end{aligned}$$

(b) The symmetric root locus is shown below, where we have chosen $\mathbf{C} = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}$. The following MATLAB commands can be used to generate the symmetric root locus,

```
% Symmetric root locus

a=[A, 0*A;-C'*C, -A'];

b=[B;0*B];

c=[0*C, B'];

d=0;

rlocus(a,b,c,d);
```

The chosen pole locations, shown on the symmetric root locus, result in a feedback gain of (using MATLAB's place command),

$$\mathbf{K} = \begin{bmatrix} 13.5 & 18.36 & 3.9 & 13.98 \end{bmatrix}.$$

(c) The initial condition response to for both control designs in (a) and (b) is shown on the next page.

(d) To compute and for zero steady-state error to a constant command input on cart position, , we solve the equations,

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C}_2 & D \end{bmatrix} \begin{bmatrix} \mathbf{N_x} \\ N_u \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix}.$$

This yields $\mathbf{N_x} = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}^T$ and $N_u = 0$. The step responses for each of the closed-loop systems (using the MATLAB step command) are shown next.

Symmetric root locus for Problem for Problem 7.31.

Initial condition response with for Problem 7.31.

Step response using for Problem 7.31.

32. An asymptotically stable type I system with input and output is described by the closed-loop system matrices $(\mathbf{A}, \mathbf{B}, \mathbf{C}, D = 0)$. Suppose the input is given by the ramp $r = at$, for $t > 0$. Show that the velocity error coefficient, K_v, for this system is given by

$$K_v = \left[\mathbf{C}\mathbf{A}^{-2}\mathbf{B}\right]^{-1}.$$

Solution: The closed-loop transfer function is

$$T(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}.$$

Asymptotic stability implies that is invertible. We can write that

$$(s\mathbf{I} - \mathbf{A})(-\mathbf{A}^{-1} - s\mathbf{A}^{-2} - s^2\mathbf{A}^{-3} - \cdots) = \mathbf{I},$$

which leads to the series

$$(s\mathbf{I} - \mathbf{A})^{-1} = -\mathbf{A}^{-1} - s\mathbf{A}^{-2} - s^2\mathbf{A}^{-3} - \cdots,$$

Figure 7.93: Nyquist plot for an optimal regulator

Therefore

$$T(s) = -\mathbf{CA}^{-1}\mathbf{B} - s\mathbf{CA}^{-2}\mathbf{B} - s^2\mathbf{CA}^{-3}\mathbf{B} - \cdots.$$

Now,

$$R(s) = \frac{a}{s^2},$$

The tracking error is given by

$$E(s) = [1 - T(s)]R(s).$$

Since the system is Type I, its DC gain is unity, so that

$$\text{DC gain } = T(0) = -\mathbf{CA}^{-1}\mathbf{B} = 1.$$

$$E(s) = [1 + \mathbf{CA}^{-1}\mathbf{B} + s\mathbf{CA}^{-2}\mathbf{B} + s^2\mathbf{CA}^{-3}\mathbf{B} + \cdots]\frac{a}{s^2},$$

Using the Final Value Theorem,

$$\begin{aligned} e_{\text{ ramp }}(\infty) &= \lim_{s\to 0} sE(s) = \lim_{s\to 0} s\left[1 + \mathbf{CA}^{-1}\mathbf{B} + s\mathbf{CA}^{-2}\mathbf{B} + s^2\mathbf{CA}^{-3}\mathbf{B} + \cdots\right]\frac{a}{s^2}, \\ &= \lim_{s\to 0} s\left[1 - 1 + s\mathbf{CA}^{-2}\mathbf{B} + s^2\mathbf{CA}^{-3}\mathbf{B} + \cdots\right]\frac{a}{s^2}, \\ &= \mathbf{CA}^{-2}\mathbf{B}a. \end{aligned}$$

is defined as the ratio of the input rate to the steady-state error,

$$K_v = \frac{a}{e_{\text{ ramp }}(\infty)} = \left[\mathbf{CA}^{-2}\mathbf{B}\right]^{-1}.$$

33. Prove that the Nyquist plot for LQR design avoids a circle of radius one centered at the -1 point as shown in Fig. 7.89. Show that this implies that GMthe "upward" gain margin is GM and there is a "downward" GM, and the phase margin is at least PM. Hence the LQR gain matrix, $\mathbf{K}$, can be multiplied by a large scalar or reduced by half with guaranteed closed-loop system stability.

Solution:

It has been proved (Anderson and Moore, 1990) that the Nyquist plot for LQR design avoids a circle of radius one centered at the point as shown in Fig. 7.89. This leads to extraordinary phase and gain margin properties as shown below. First note that the state-feedback system can be re-drawn in the usual feedback configuration as shown on the next page. Using Eq. (7.72) and factoring , we have

$$\begin{aligned}\alpha_c(s) &= \det[s\mathbf{I} - (\mathbf{A} - \mathbf{BK})] \\ &= \det\{(s\mathbf{I} - \mathbf{A})[\mathbf{I} + (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{BK}]\} \\ &= \det(s\mathbf{I} - \mathbf{A})\det([\mathbf{I} + (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{BK}]) \\ &= Den(s)[1 + \mathbf{K}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}].\end{aligned} \tag{12a}$$

Now[1], using the above equation and Eq. (7.95) we can write,

$$\begin{aligned}\alpha_c(s)\alpha_c(-s) &= Den(s)Den(-s)[1 + \mathbf{K}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}][1 + \mathbf{K}(-s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}], \\ &= 1 + \rho G_0(s)G_0(-s).\end{aligned} \tag{13a}$$

Setting we obtain,

$$\begin{aligned}\alpha_c(j\omega)\alpha_c(-j\omega) &= |Den(j\omega)|^2|[1 + \mathbf{K}(j\omega\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}]|^2 \\ &= |Den(j\omega)|^2|[1 + \rho|G(j\omega)|^2|.\end{aligned} \tag{14a}$$

But since we can conclude that the return difference must satisfy,

$$|1 + \mathbf{K}(j\omega\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}| \geq 1. \tag{15}$$

Let us re-write the loop gain as sum of its real and imaginary parts,

$$L(j\omega) = \mathbf{K}(j\omega\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} = \mathrm{Re}\,[L(j\omega)) + j\,\mathrm{Im}(L(j\omega)]\,. \tag{16}$$

Finally, Eq. (14) implies that,

$$([\mathrm{Re}(L(j\omega)] + 1)^2 + [\mathrm{Im}(L(j\omega)]^2 \geq 1, \tag{17}$$

which means that the Nyquist plot must indeed avoid a circle centered at -1 with unit radius. The Nyquist plot approaches the origin for large frequencies, and we find that the "upward" gain margin GM= ∞ . The only other point on the negative real axis, in the proximity of the same circle, that the Nyquist plot can possibly cross is close to -2. This implies a "downward" gain margin of GM= $\frac{1}{2}$. Hence the LQR gain matrix can be multiplied by a large scalar or reduced by half with guaranteed closed-loop system stability. As far as the determination of the phase margin PM is concerned, the closest possible approach point of the Nyquist plot to the -1 point is shown in Fig. 7.89. From this figure we conclude that the PM is at least $60°$. These margins are remarkable and it is not realistic to assume that they can be achieved in practice because of the presence of modeling errors and lack of sensors!

[1]We have used the following result from matrix theory: if $\mathbf{A}$ is nm matrix and $\mathbf{B}$ is mn then See Appendix WB.

Optimal regulator in a feedback configuration.

Problems and Solutions for Section 7.7: Estimator Design

34. Consider the system

$$\mathbf{A} = \begin{bmatrix} -2 & 1 \\ 1 & 0 \end{bmatrix}, \ \mathbf{B} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \ \mathbf{C} = [1 \ \ 2],$$

and assume that you are using feedback of the form , where is a reference input signal.
a) Show that (**A,**C) is observable.
b) Show that there exists a **K** such that **(A-BK, C),** is unobservable.

c) Compute a **K** of the form $\mathbf{K} = [1, K_2]$ that will make the system unobservable as in part (b); that is, find so that the closed-loop system is not observable.
d) Compare the open-loop transfer function with the transfer function of the closed-loop system of part (c). What is the unobservability due to?

Solution:

(a)

$$\mathcal{O} = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix},$$

is nonsingular. Therefore, (**A,**C) is observable.

(b) Let,

$$\mathcal{O}_{unobs} = \begin{bmatrix} \mathbf{C} \\ \mathbf{C}(\mathbf{A} - \mathbf{BK}) \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -K_1 & 1 - K_2 \end{bmatrix}.$$

So if

$$\det(\mathcal{O}_{unobs}) = 1 - K_2 + 2K_1, \tag{18}$$

then $(\mathbf{A} - \mathbf{BK}, \mathbf{C})$ is unobservable.

(c)

$$K_1 = 1, 1 - K_2 + 2 = 0, K_2 = 3. \tag{19}$$

The result can be verified using MATLAB's place command.

(d)

$$\begin{aligned} G_{ol}(s) &= \mathbf{C}(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B} = \frac{s+2}{s^2+2s-1} = \frac{s+2}{(s-0.414)(s+2.414)}. \\ G_{cl}(s) &= \mathbf{C}(s\mathbf{I}-\mathbf{A}+\mathbf{BK})^{-1}\mathbf{B} = \frac{s+2}{s^2+3s+2} = \frac{s+2}{(s+2)(s+1)} = \frac{1}{(s+1)}. \end{aligned}$$

The computations can be carried out using MATLAB's ss2tf command. So the unobservability is due to a **cancellation** of one of the closed-loop poles with the zero of the system. In other words, this closed-loop mode is unobservable from the output.

35. Consider a system with the transfer function,

$$G(s) = \frac{9}{s^2-9}.$$

a) Find ($\mathbf{A}_o$, $\mathbf{B}_o$, $\mathbf{C}_o$) for this system in observer canonical form.
b) Is ($\mathbf{A}_o$, $\mathbf{B}_o$) controllable?
c) Compute $\mathbf{K}$ so that the closed-loop poles are assigned to $s = -3 \pm j3$.
d) Is the closed-loop system of part(c) observable?
e) Design a full-order estimator with estimator-error poles at $s = -12 \pm j12$.
f) Suppose the system is modified to have a zero:

$$G_1(s) = \frac{9(s+1)}{s^2-9}.$$

Prove that if $u = -\mathbf{Kx} + r$ there is a feedback gain $\mathbf{K}$ that makes the closed-loop system unobservable. [Again assume an observer canonical realization for $G_1(s)$.]

Solution:

(a) For a transfer function,

$$G(s) = \frac{b_1 s + b_2}{s^2 + a_1 s + a_2},$$

the observer canonical form becomes,

$$\mathbf{A}_o = \begin{bmatrix} -a_1 & 1 \\ -a_2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 9 & 0 \end{bmatrix}, \ \mathbf{B}_o = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 9 \end{bmatrix}, \ \mathbf{C}_o = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

(b) To check whether is controllable we form the controllability matrix,

$$\mathcal{C} = \begin{bmatrix} \mathbf{B}_o & \mathbf{A}_o\mathbf{B}_o \end{bmatrix} = \begin{bmatrix} 0 & 9 \\ 9 & 0 \end{bmatrix} \Rightarrow \det(\mathcal{C}) = -81 \neq 0.$$

Thus, the system is controllable.

(c) $\mathbf{K} = [3, 2/3]$. The result can be verified using MATLAB's place command.

(d) The system is in observer canonical form. Hence, it is guaranteed to be observable. To check,

$$\mathcal{O} = \begin{bmatrix} \mathbf{C}_o \\ \mathbf{C}_o\mathbf{A}_o \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Longrightarrow \det(\mathcal{O}) = 1 \neq 0 \neq 0.$$

(e) Solving for $\det(sI - \mathbf{A}_o + \mathbf{L}\mathbf{C}_o) = (s+12)^2$ for $\mathbf{L}$ yields $\mathbf{L} = [24, 297]^T$. The result can be verified using MATLAB's place command.

(f) The realization for,

$$G(s) = \frac{9(s+1)}{s^2 - 9},$$

in observer canonical form yields,

$$\mathbf{A}_o = \begin{bmatrix} 0 & 1 \\ 9 & 0 \end{bmatrix}, \ \mathbf{B}_o = \begin{bmatrix} 9 \\ 9 \end{bmatrix}, \ \mathbf{C}_o = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

Thus, for the system with feedback,

$$\mathcal{O} = \begin{bmatrix} \mathbf{C}_o \\ \mathbf{C}_o(\mathbf{A}_o - \mathbf{B}_o\mathbf{K}) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -9k_1 & 1 - 9k_2 \end{bmatrix} \Longrightarrow \tag{20}$$

$$det(\mathcal{O}) = 1 - 9k_2. \tag{21}$$

Thus,

$$\{(k_1, k_2) \ \ |k_2 = 1/9\},$$

is the set of all k_1 and k_2 that make the system unobservable. So we have shown that there exists a feedback gain $\mathbf{K}$ which makes the closed-loop system unobservable. Note that for k_2=1/9

$$\det(s\mathbf{I} - \mathbf{A}_o + \mathbf{B}_o\mathbf{K}) = \det \begin{bmatrix} s + 9k_1 & -1 + 9k_2 \\ -9 + 9k_1 & s + 9k_2 \end{bmatrix} = (s + 9k_1)(s+1).$$

Thus, the reason why the system becomes unobservable is that the pole at is canceled by a zero.

36. Explain how the controllability, observability, and stability properties of a linear system are related.

Solution:

controllability

$$\det(\mathcal{C}) \neq 0 \tag{22}$$

observability

$$\det(\mathcal{O}) \neq 0 \tag{23}$$

stability eigenvalues.

$$\mathrm{Re}(eig(\mathbf{A})) < 0 \tag{24}$$

So, in general, there is no connection between these three properties. However, for a minimal realization (controllable, observable), internal and external stabilities are the same.

Note that the mathematical relations given above are idealizations. In practice, it is important to consider the singular values of the controllability or observability matrices and the proximity of the eigenvalues to the $j\omega$ axis. For example, if two of the eigenvectors of the controllability matrix are nearly parallel, then the system is nearly uncontrollable and large actuator signals may be required to get the system to a particular (final) state in state space.

Figure 7.94: Electric circuit for Problem 7.37

37. Consider the electric circuit shown in Fig.7.90.
a) Write the internal (state) equations for the circuit. The input $u(t)$ is a current, and the output y is a voltage. Let $x_1 = i_L$ and $x_2 = v_c$.
b) What condition(s) on R, L and C will guarantee that the system is controllable?
c) What condition(s) on , R, L and C will guarantee that the system is observable?

Solution:

(a) Apply Kirchhoff's voltage and current laws, with $x_1 = i_L$ and $x_2 = v_c$, we obtain,

$$\begin{aligned} L\dot{x}_1 + Rx_1 &= x_2 + RC\dot{x}_2, \\ \dot{x}_2 &= u - x_1, \\ y &= (u - x_1)R \end{aligned}$$

Thus,

$$\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} -2R/L & 1/L \\ -1/C & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} R/L \\ 1/C \end{bmatrix} u, \\ y &= \begin{bmatrix} -R & 0 \end{bmatrix} \mathbf{x} + Ru. \end{aligned}$$

(b) The condition for the system to be uncontrollable is .

$$\begin{aligned} \mathcal{C} &= \begin{bmatrix} \mathbf{B} & \mathbf{AB} \end{bmatrix} = \begin{bmatrix} R/L & -2R^2/L^2 + 1/LC \\ 1/C & -R/LC \end{bmatrix}. \\ \det(\mathcal{C}) &= R^2/L^2C - 1/LC^2. \end{aligned}$$

Thus, the system is controllable if $R^2 \neq L/C$.

(c) The condition for the system to be unobservable is,

$$\begin{aligned} \mathcal{O} &= \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \end{bmatrix} = \begin{bmatrix} -R & 0 \\ 2R^2/L & -R/L \end{bmatrix}. \\ \det(\mathcal{O}) &= R^2/L. \end{aligned}$$

Figure 7.95: Block diagram for Problem 7.38

Since $\det(\mathcal{O}) \neq 0$ for any (R, L, C)) except for $R = 0$ or $L = \infty$, the system is observable.

38. The block diagram of a feedback system is shown in Fig.7.91. The system state is,

$$\mathbf{x} = \begin{bmatrix} \mathbf{x}_p \\ \mathbf{x}_f \end{bmatrix},$$

and the dimensions of the matrices are as follows:

$$\begin{aligned} \mathbf{A} &= n \times n, & \mathbf{L} &= n \times 1, \\ \mathbf{B} &= n \times 1, & \mathbf{x} &= 2n \times 1, \\ \mathbf{C} &= 1 \times n, & r &= 1 \times 1, \\ \mathbf{K} &= 1 \times n, & y &= 1 \times 1, \end{aligned}$$

a) Write state equations for the system.
b) Let $\mathbf{x} = \mathbf{T}\mathbf{z}$ where

$$\mathbf{T} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{I} & -\mathbf{I} \end{bmatrix}.$$

Show that the system is not controllable.
c) Find the transfer function of the system from r to y.

Solution:

(a) We have,

$$\begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\mathbf{x}}_f \end{bmatrix} = \begin{bmatrix} \mathbf{A} & -\mathbf{BK} \\ \mathbf{LC} & \mathbf{A} - \mathbf{LC} - \mathbf{BK} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{x}_f \end{bmatrix} + \begin{bmatrix} \mathbf{B} \\ \mathbf{B} \end{bmatrix} r.$$

(b) In order to apply our transformation of coordinates, we need $\mathbf{T}^{-1}$,

$$\mathbf{T} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{I} & -\mathbf{I} \end{bmatrix} \Longrightarrow \mathbf{T}^{-1} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{I} & -\mathbf{I} \end{bmatrix}.$$

Thus,

$$\begin{aligned} \mathbf{A}_{cl} &= \mathbf{T}^{-1} \begin{bmatrix} \mathbf{A} & -\mathbf{BK} \\ \mathbf{LC} & \mathbf{A} - \mathbf{LC} - \mathbf{BK} \end{bmatrix} \mathbf{T} = \begin{bmatrix} \mathbf{A} - \mathbf{BK} & -\mathbf{BK} \\ \mathbf{0} & \mathbf{A} - \mathbf{LC} \end{bmatrix}, \\ \mathbf{B}_{cl} &= \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{I} & -\mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{B} \\ \mathbf{B} \end{bmatrix} = \begin{bmatrix} \mathbf{B} \\ \mathbf{0} \end{bmatrix}, \\ \mathbf{C}_{cl} &= \begin{bmatrix} \mathbf{C} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{I} & -\mathbf{I} \end{bmatrix} = \begin{bmatrix} \mathbf{C} & \mathbf{0} \end{bmatrix}. \end{aligned}$$

Figure 7.96: Electric circuit for Problem 7.39

In the new coordinate system, we have,

$$\begin{aligned}\dot{\mathbf{z}} &= \begin{bmatrix} \mathbf{A}-\mathbf{BK} & -\mathbf{BK} \\ \mathbf{0} & \mathbf{A}-\mathbf{LC} \end{bmatrix}\mathbf{z} + \begin{bmatrix} \mathbf{B} \\ \mathbf{0} \end{bmatrix} r, \\ y &= \begin{bmatrix} \mathbf{C} & \mathbf{0} \end{bmatrix}\mathbf{z}.\end{aligned}$$

Observe that the system is now decomposed into controllable and uncontrollable parts. Hence, we have shown that it is an uncontrollable system.

(c) The transfer function is,

$$T(s) = \mathbf{C}[s\mathbf{I} - (\mathbf{A} - \mathbf{BK})]^{-1}\mathbf{B}.$$

39. This problem is intended to give you more insight into controllability and observability. Consider the circuit in Fig.7.92, with an input voltage source $u(t)$ and an output current $y(t)$.
a) Using the capacitor voltage and inductor current as state variables, write state and output equations for the system.
b) Find the conditions relating R_1, R_2, C and L that render the system uncontrollable. Find a similar set of conditions that result in an unobservable system.
c) Interpret the conditions found in part(b) physically in terms of the time constants of the system.
d) Find the transfer function of the system. Show that there is a pole-zero cancellation for the conditions derived in part(b) (that is, when the system is uncontrollable or unobservable).

Solution:

(a) From Figure 7.92,

$$\begin{aligned}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} -\frac{1}{R_1C} & 0 \\ 0 & \frac{R_2}{L} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \frac{1}{R_1C} \\ \frac{1}{L} \end{bmatrix} u, \\ y &= \begin{bmatrix} -\frac{1}{R_1} & 1 \end{bmatrix}\mathbf{x} + \frac{1}{R_1}u.\end{aligned}$$

(b) First, form the controllability matrix,

$$\begin{aligned}
\mathcal{C} &= \begin{bmatrix} \mathbf{B} & \mathbf{AB} \end{bmatrix} = \begin{bmatrix} \frac{1}{R_1C} & -\frac{1}{(R_1C)^2} \\ \frac{1}{L} & -\frac{R_2}{L^2} \end{bmatrix}, \\
\det(\mathcal{C}) &= -R_2/R_1CL^2 + 1/L(R_1C)^2.
\end{aligned}$$

For uncontrollability,

$$\det(\mathcal{C}) = 0 \tag{25}$$

implies

$$R_1R_2C = L. \tag{26}$$

Next, form the observability matrix,

$$\begin{aligned}
\mathcal{O} &= \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \end{bmatrix} = \begin{bmatrix} -\frac{1}{R_1} & 1 \\ \frac{1}{R_1^2C} & -\frac{R_2}{L} \end{bmatrix}, \\
\det(\mathcal{O}) &= R_2/R_1L - 1/R_1^2C.
\end{aligned}$$

For unobservability,

$$\det(\mathcal{O}) = 0 \tag{27}$$

implies, again that,

$$R_1R_2C = L. \tag{28}$$

(c) When the system is unobservable/uncontrollable, we have so that:

$$\begin{aligned}
\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} -\frac{1}{R_1C} & 0 \\ 0 & -\frac{1}{R_1C} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \frac{1}{R_1C} \\ \frac{1}{L} \end{bmatrix} u, \\
y &= \begin{bmatrix} -\frac{1}{R_1} & 1 \end{bmatrix} \mathbf{x} + [1/R_1]u.
\end{aligned}$$

The two modes of the system have the same time constant and hence cannot be changed independently using only one control u, i.e., it is not controllable. The system output is a linear combination of modes having the same frequency. It is thus impossible to determine the composition of the combination, i.e., it is not observable.

(d)

$$\begin{aligned}
G(s) &= \frac{s^2 + \frac{R_1+R_2}{L}s + \frac{1}{LC}}{R_1(s+1/R_1C)(s+R_2/L)} \\
&= \frac{(s+a)(s+b)}{R_1(s+1/R_1C)(s+R_2/L)},
\end{aligned}$$

where,

$$a, b = \frac{1}{2}\left[\frac{R_1+R_2}{L} \pm \sqrt{\frac{(R_1+R_2)^2}{L^2} - \frac{4}{LC}}\right].$$

Substituting $1/C = R_1R_2/L$, we find $a = \frac{R_2}{L}$, and $b = \frac{R_1}{L}$. Thus,

$$\begin{aligned} G(s) &= \frac{(s+R_2/L)(s+R_1/L)}{R_1(s+1/R_1C)(s+R_2/L)} \\ &= \frac{s+R_1/L}{R_1(s+1/R_1C)}. \end{aligned}$$

Observe the pole-zero *cancellation* in the transfer function.

40. The linearized equations of motion for a satellite are,

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}, \\ \mathbf{y} &= \mathbf{C}\mathbf{x}, \end{aligned}$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 3\omega^2 & 0 & 0 & 2\omega \\ 0 & 0 & 0 & 1 \\ 0 & -2\omega & 0 & 0 \end{bmatrix}, \ \mathbf{B} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, \ \mathbf{C} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$
$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \ \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}.$$

The inputs u_1 and u_2 are the radial and tangential thrusts, the state variables x_1 and x_2 are the radial and angular deviations from the reference (circular) orbit, and the outputs y_1 and y_2 are the radial and angular measurements, respectively.
a) Show that the system is controllable using both control inputs.
b) Show that the system is controllable using only a single input. Which one is it?
c) Show that the system is observable using both measurements.
d) Show that the system is observable using only one measurement. Which one is it?

Solution:

(a) Checking the controllability matrix:

$$\mathcal{C} = \begin{bmatrix} \mathbf{B} & \mathbf{AB} & \mathbf{A}^2\mathbf{B} & \mathbf{A}^3\mathbf{B} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 & \\ 1 & 0 & 0 & 2\omega & \cdots \\ 0 & 0 & 0 & 1 & \\ 0 & 1 & -2\omega & 0 & \end{bmatrix}.$$

Considering only the first four columns of the controllability matrix, the rank is already 4 and hence it is controllable. Incidentally, you could also show this part of the problem by first doing part (b) and then recognizing that if the system is controllable from a single actuator, it will surely be controllable from the same actuator plus any other additional actuators you care to add. This could be useful in multivariable system design. For example, when actuators are expensive, one design criterion could be to minimize the number of actuators while maintaining controllability of the system.

(b) Consider only the first (radial) thruster u_1 (i.e., $u_2 = 0$). Then $\mathbf{B}_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}^T$.

So we have,

$$\mathcal{C}_1 = \begin{bmatrix}\mathbf{B}_1 & \mathbf{AB}_1 & \mathbf{A}^2\mathbf{B}_1 & \mathbf{A}^3\mathbf{B}_1\end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & -\omega^2 \\ 1 & 0 & -\omega^2 & 0 \\ 0 & 0 & -2\omega & 0 \\ 0 & -2\omega & 0 & 2\omega^3 \end{bmatrix}.$$

The rank is 3, hence the satellite is uncontrollable using only the radial thruster, u_1. Now consider the second (tangential) thruster, u_2 (i.e., $u_1 = 0$). Then, $\mathbf{B}_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}^T$.So we have,

$$\mathcal{C}_2 = \begin{bmatrix}\mathbf{B}_2 & \mathbf{AB}_2 & \mathbf{A}^2\mathbf{B}_2 & \mathbf{A}^3\mathbf{B}_2\end{bmatrix} = \begin{bmatrix} 0 & 0 & 2\omega & 0 \\ 0 & 2\omega & 0 & -2\omega^3 \\ 0 & 1 & 0 & -4\omega^2 \\ 1 & 0 & -4\omega^2 & 0 \end{bmatrix}.$$

The rank is 4, hence the satellite is controllable using only the tangential thruster,u_2. The tangential thruster can be used to control the angular velocity of the satellite, and hence radial deviations can be controlled.

(c) Checking the observability matrix:

$$\mathcal{O} = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \mathbf{CA}^2 \\ \mathbf{CA}^3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ & & \vdots & \end{bmatrix}.$$

Since the rank of the first four rows is already 4, the system is observable.

(d) Using only the first measurement, y_1, we have,

$$\mathcal{O}_1 = \begin{bmatrix} \mathbf{C}_1 \\ \mathbf{C}_1\mathbf{A} \\ \mathbf{C}_1\mathbf{A}^2 \\ \mathbf{C}_1\mathbf{A}^3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 3\omega^2 & 0 & 0 & 2\omega \\ 0 & -\omega^2 & 0 & 0 \end{bmatrix}.$$

This has a rank of 3, hence the system's state is unobservable using only a radial measurement. Now considering the tangential measurement, y_2,

$$\mathcal{O}_2 = \begin{bmatrix} \mathbf{C}_2 \\ \mathbf{C}_2\mathbf{A} \\ \mathbf{C}_2\mathbf{A}^2 \\ \mathbf{C}_2\mathbf{A}^3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -2\omega & 0 & 0 \\ -6\omega^3 & 0 & 0 & -4\omega^2 \end{bmatrix}.$$

Since this matrix has a rank of 4, the system is observable using only the tangential measurement.

41. Consider the system in Fig.7.93.
a) Write the state-variable equations for the system, using $\begin{bmatrix} \theta_1 & \theta_2 & \dot{\theta}_1 & \dot{\theta}_2 \end{bmatrix}$ as the state vector and F as the single input.
b) Show that all the state variables are observable using measurements of θ_1 alone.

Figure 7.97: Coupled pendulums for Problem 7.41

c) Show that the characteristic polynomial for the system is the product of the polynomials for two oscillators. Do so by first writing a new set of system equations involving the state variables

$$\begin{bmatrix} y_1 \\ y_2 \\ \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} \theta_1 + \theta_2 \\ \theta_1 - \theta_2 \\ \dot{\theta}_1 + \dot{\theta}_2 \\ \dot{\theta}_1 - \dot{\theta}_2 \end{bmatrix}.$$

Hint: If $\mathbf{A}$ and $\mathbf{D}$ are invertible matrices, then,

$$\begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{D} \end{bmatrix}^{-1} = \begin{bmatrix} \mathbf{A}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{D}^{-1} \end{bmatrix}.$$

d) Deduce the fact that the spring mode is controllable with F but the pendulum mode is not.

Solution:

The equations of motion for the system given in Fig. 7.93

$$\begin{aligned} ml^2\ddot{\theta}_1 &= -kd^2(\theta_1 - \theta_2) - mgl\theta_1 + lu, \\ ml^2\ddot{\theta}_2 &= -kd^2(\theta_2 - \theta_1) - mgl\theta_2 - lu, \end{aligned}$$

where u is the force from the gas jet, l is the pendulum length, m is the pendulum mass, d and is as given in the figure. Letting $\omega^2 = g/l$, $K = kd^2/ml^2$, and $F = u/ml$ we obtain:

$$\begin{aligned} \ddot{\theta}_1 &= -\omega^2\theta_1 - K(\theta_1 - \theta_2) + F, \\ \ddot{\theta}_2 &= -\omega^2\theta_2 - K(\theta_2 - \theta_1) - F. \end{aligned}$$

(a) Using the state vector $\mathbf{x} = \begin{bmatrix} \theta_1 & \theta_2 & \dot{\theta}_1 & \dot{\theta}_2 \end{bmatrix}^T$

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -(\omega^2 + K) & K & 0 & 0 \\ K & -(\omega^2 + K) & 0 & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} F.$$

(b) Considering only the measurement of θ_1, then:

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \mathbf{x}.$$

Observability:

$$\mathcal{O} = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \mathbf{CA}^2 \\ \mathbf{CA}^3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -(\omega^2+K) & K & 0 & 0 \\ 0 & 0 & -(\omega^2+K) & K \end{bmatrix}.$$

Since $K \neq 0$, $\det(\mathcal{O}) \neq 0$. Hence, the state is observable with θ_1.

(c) Define a state vector $\mathbf{x}$, such that:

$$\mathbf{x} = \begin{bmatrix} y_1 & \dot{y}_1 & y_2 & \dot{y}_2 \end{bmatrix}^T = \begin{bmatrix} \theta_1+\theta_2 & \dot{\theta}_1+\dot{\theta}_2 & \theta_1-\theta_2 & \dot{\theta}_1-\dot{\theta}_2 \end{bmatrix}^T \tag{29}$$

Note that the order of the state variables is chosen such that the resulting plant $\mathbf{A}$ matrix is block diagonal. With this state vector,

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\omega^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -(\omega^2+2K) & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 2 \end{bmatrix} F.$$

The characteristic equation of the system is,

$$\det(s\mathbf{I}-\mathbf{A}) = (s^2+\omega^2)(s^2+\omega^2+2K).$$

Thus, it is the product of two oscillators with frequencies ω and $\sqrt{\omega^2+2K}$.

(d) From the state equations in part (c), note that they are block diagonal. Thus there is no coupling between the *spring mode* (θ_1-θ_2) and the *pendulum mode* ($\theta_1-\theta_2$). Because the gas jets, via F are only connected to the spring mode, we conclude that spring mode is controllable while the pendulum mode is not.

42. A certain fifth-order system is found to have a characteristic equation with roots at 0, -1, and $-1 \pm j1$. A decomposition into controllable and uncontrollable parts discloses that the controllable part has a characteristic equation with roots 0, and $-1 \pm j1$. A decomposition into observable and nonobservable parts discloses that the observable modes are at 0, -1 and -2.
a) Where are the zeros of $b(s) = \mathbf{C}adj(s\mathbf{I}-\mathbf{A})\mathbf{B}$ for this system?
b) What are the poles of the reduced-order transfer function that includes only controllable and observable modes?

Solution:

(a) $b(s) = \mathbf{C}adj(s\mathbf{I}-\mathbf{A})\mathbf{B}$

controllable modes: 0, $-1 \pm j1$.

observable modes: 0, -1, and -2 .

Hence, mode 0 is the only mode that is both controllable and observable. Therefore, $b(s)$ as zeros at -1,-2, and $-1 \pm j1$.

(b) Reduced transfer function has only one pole which is at the origin, i.e., 0.

Figure 7.98: Block diagrams for Problem 7.43: (a) series; (b) parallel; (c) feedback

43. Consider the systems shown in Fig.7.94, employing series, parallel, and feedback configurations.

a) Suppose we have controllable-observable realizations for each subsystem:

$$\begin{aligned} \dot{\mathbf{x}}_i &= \mathbf{A}_i\mathbf{x}_i + \mathbf{B}_i u_i, \\ y_i &= \mathbf{C}_i\mathbf{x}_i, \quad \text{where} \quad i = 1, 2. \end{aligned}$$

Give a set of state equations for the combined systems in Fig.7.94.
b) For each case, determine what condition(s) on the roots of the polynomials N_i and D_i is necessary for each system to be controllable and observable. Give a brief reason for your answer in terms of pole-zero cancellations.

Solution:

(a) Series connection,

$$\begin{aligned} \begin{bmatrix} \dot{\mathbf{x}}_1 \\ \dot{\mathbf{x}}_2 \end{bmatrix} &= \begin{bmatrix} \mathbf{A}_1 & \mathbf{0} \\ \mathbf{B}_2\mathbf{C}_1 & \mathbf{A}_2 \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} + \begin{bmatrix} \mathbf{B}_1 \\ \mathbf{0} \end{bmatrix} u, \\ y &= \begin{bmatrix} \mathbf{0} & \mathbf{C}_2 \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix}. \end{aligned}$$

See the MATLAB series command.

(b) Parallel connection,

$$\begin{aligned} \begin{bmatrix} \dot{\mathbf{x}}_1 \\ \dot{\mathbf{x}}_2 \end{bmatrix} &= \begin{bmatrix} \mathbf{A}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_2 \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} + \begin{bmatrix} \mathbf{B}_1 \\ \mathbf{B}_2 \end{bmatrix} u, \\ y &= \begin{bmatrix} \mathbf{C}_1 & \mathbf{C}_2 \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix}. \end{aligned}$$

See the MATLAB parallel command.

(c) Feedback connection,

$$\begin{aligned} \begin{bmatrix} \dot{\mathbf{x}}_1 \\ \dot{\mathbf{x}}_2 \end{bmatrix} &= \begin{bmatrix} \mathbf{A}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_2 \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} + \begin{bmatrix} \mathbf{B}_1(r - y_2) \\ \mathbf{B}_2 y_1 \end{bmatrix}, \\ &= \begin{bmatrix} \mathbf{A}_1 & -\mathbf{A}_1\mathbf{C}_2 \\ \mathbf{B}_2\mathbf{C}_1 & \mathbf{A}_2 \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} + \begin{bmatrix} \mathbf{B}_1 \\ \mathbf{0} \end{bmatrix} r, \\ y &= \begin{bmatrix} \mathbf{C}_1 & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix}. \end{aligned}$$

See the MATLAB feedback command.

(d) Since each sub-system is in controllable-observable realization, each transfer function $G_1(s)$ and $G_2(s)$ is minimal (i.e., no cancellations).

(e) Series connection:

$$\frac{Y(s)}{U(s)} = G_2(s)G_1(s) = \frac{N_2(s)}{D_2(s)}.\frac{N_1(s)}{D_1(s)}$$

For observability, $N_2(s)$ and $D_1(s)$ must be coprime (i.e., no common factors). Otherwise, a mode of $D_1(s)$ masked from the output. For controllability, $N_1(s)$ and $D_2(s)$ must be coprime. Otherwise, a mode of $D_2(s)$ is masked from the output.

(f) Parallel connection:

$$\frac{Y(s)}{U(s)} = G_1(s) + G_2(s) = \frac{N_1(s)}{D_1(s)} + \frac{N_2(s)}{D_2(s)} = \frac{N_1(s)D_2(s) + D_1(s)N_2(s)}{D_1(s)D_2(s)}.$$

For observability (controllability), $D_1(s)$ and $D_2(s)$ must be coprime. Otherwise, the two modes appear as a single mode from the output (input).

(g) Feedback connection:

$$\frac{Y(s)}{R(s)} = \frac{G_1(s)}{1 + G_1(s)G_2(s)} = \frac{N_1(s)D_2(s)}{D_1(s)D_2(s) + N_1(s)N_2(s)}.$$

For observability, $N_1(s)$ and $D_2(s)$ must be coprime (i.e., no common factors). For controllability, $N_2(s)$ and $D_1(s)$ must be coprime.

44. Consider the system $\ddot{y} + 3y + 2\dot{y} = u + \dot{u}$.
a) Find the state matrices $\mathbf{A}_c, \mathbf{B}_c, \mathbf{C}_c$ in control canonical form that correspond to the given differential equation.
b) Sketch the eigenvectors of $\mathbf{A}_c$ in the (x_1, x_2) plane, and draw vectors that correspond to the completely observable and the completely unobservable state variables.
c) Express and in terms of the observability matrix $\mathcal{O}$.
d) Give the state matrices in observer canonical form and repeat parts (b) and (c) in terms of controllability instead of observability.

Solution:

(a) The Laplace transform of the differential equation gives the transfer function,

$$G(s) = \frac{Y(s)}{U(s)} = \frac{s+1}{s^2 + 3s + 2}.$$

Hence in controller canonical form,

$$\mathbf{A}_c = \begin{bmatrix} -3 & -2 \\ 1 & 0 \end{bmatrix}, \ \mathbf{B}_c = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \ \mathbf{C}_c = \begin{bmatrix} 1 & 1 \end{bmatrix}.$$

(b) First, we find the eigenvectors of $\mathbf{A}_c$ or the modal directions of the system,

$$\det(s\mathbf{I} - \mathbf{A}_c) = 0 \Longrightarrow s = -1, -2.$$

$$
\begin{aligned}
(\mathbf{A}_c + \mathbf{I})\mathbf{v}_1 &= 0 \Longrightarrow \mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}. \\
(\mathbf{A}_c + 2\mathbf{I})\mathbf{v}_2 &= 0 \Longrightarrow \mathbf{v}_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}.
\end{aligned}
$$

Using partial-fraction expansion of $G(s)$, we can determine which modes are unobservable and which are observable. The mode $s = -1$, this mode has a pole-zero cancellation in $G(s)$, so is the unobservable mode shape. The mode $s = -2$, does appear in the minimal transfer function, so is the observable mode shape. Therefore, the completely unobservable direction is equal to $\mathbf{v}_1$. The completely observable direction is $\mathbf{v}_2^{\perp}$, where $\mathbf{v}_2^{\perp}$ is the projection of $\mathbf{v}_2$ on the orthogonal direction to $\mathbf{v}_1$. These vectors are drawn in the figure below. Note from the figure that is, in fact, the same as $\mathbf{C}$. Also, the observable mode, *is observable* since the projection of $\mathbf{v}_2$ onto $\mathbf{x}_o$ is not zero, i.e.,

$$\mathbf{x}_o^T \mathbf{v}_2 = \mathbf{C}_c \mathbf{v}_2 \neq 0.$$

Observable and unobservable state directions for Problem 7.44(b).

(c) Observability measures the ability to reconstruct the realization state variables given an output and its derivatives. Consider the determination of the state initial condition, $\mathbf{x}(0)$, given the initial output measurement and its derivatives, $\mathbf{Y}(0)$, where,

$$\mathbf{Y}(\mathbf{0}) = \begin{bmatrix} y(0) \\ \dot{y}(0) \\ \vdots \\ y^{(n-1)}(0) \end{bmatrix} = \begin{bmatrix} \mathbf{C}_c \\ \mathbf{C}_c\mathbf{A}_c \\ \vdots \\ \mathbf{C}_c\mathbf{A}_c^{(n-1)} \end{bmatrix} \mathbf{x}(\mathbf{0}) = \mathcal{O}\mathbf{x}(\mathbf{0}).$$

So the determination of $\mathbf{x}(0)$ is equivalent to the solution of $\mathbf{Y}(0) = \mathcal{O}\mathbf{x}(0)$. From linear algebra, the unobservable state variables are in the null space of the observability matrix, and the observable state variables are in the left-range space of :

$$\mathbf{x}_{\bar{o}} \in \mathcal{N}(\mathcal{O}), \qquad \mathbf{x}_o \in \mathcal{R}(\mathcal{O}^T).$$

So that,

$$\mathbf{x}_{\bar{o}} = \mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \ \mathbf{x}_o = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

(d) In observer canonical form,

$$\mathbf{A}_o = \begin{bmatrix} -3 & 1 \\ -2 & 0 \end{bmatrix}, \ \mathbf{B}_o = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \ \mathbf{C}_o = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

The eigenvectors of $\mathbf{A}_o$ are,

$$\det(s\mathbf{I} - \mathbf{A}_o) = 0 \Longrightarrow s = -1, -2.$$

$$\begin{aligned} (\mathbf{A}_o + \mathbf{I})\mathbf{v}_1 &= 0 \Longrightarrow \mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}. \\ (\mathbf{A}_o + 2\mathbf{I})\mathbf{v}_2 &= 0 \Longrightarrow \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \end{aligned}$$

The mode $s = -1$, has a pole-zero cancellation from $G(s)$, so $\mathbf{v}_1$ is the uncontrollable mode shape. The mode $s = -2$, appears in $G(s)$ so $\mathbf{v}_2$ is the controllable mode shape. These vectors are drawn in the figure below.

Controllable and uncontrollable state directions for Problem 7.44(d).

Controllability measures the ability to drive the states to arbitrary values. Consider the use of $u(t)$ to move the state vector, $\mathbf{x}(0-)$, to an arbitrary value, say $\mathbf{x}(0+)$, where,

$$u(t) = g_1\delta(t) + g_2\dot{\delta}(t) + \cdots + g_n\delta^{(n-1)}(t).$$

So that,

$$\mathbf{x}(0+) - \mathbf{x}(0-) = \mathcal{C}u^*,$$

Figure 7.99: Diagram of a station-keeping satellite in orbit for Problem 7.45.

where $u^* = \begin{bmatrix} g_1 & g_2 & \dots & g_n \end{bmatrix}^T$. Hence, a controllable state is one in which some vector u^* exists such that $\mathbf{x}_c = \mathcal{C}u^*$. From linear algebra,

$$\mathbf{x}_{\overline{c}} \in \mathcal{N}(\mathcal{C}^T), \qquad \mathbf{x}_c \in \mathcal{R}\{\mathcal{C}\}.$$

So that,

$$\mathbf{x}_{\overline{c}} = \mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \ \mathbf{x}_c = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

45. The equations of motion for a station-keeping satellite (such as a weather satellite) are

$$\ddot{x} - 2\omega\dot{y} - 3\omega^2 x = 0, \quad \ddot{y} + 2\omega\dot{x} = u,$$

where,

$$\begin{aligned} x &= \text{radial perturbation}, \\ y &= \text{longitudinal position perturbation}, \\ u &= \ \text{engine thrust in the y} - \text{direction}, \end{aligned}$$

as depicted in Fig.7.94. If the orbit is synchronous with the earth's rotation, then $\omega = 2\pi/(3600\times 24)$rad/sec.

a) Is the state $\begin{bmatrix} x & \dot{x} & y & \dot{y} \end{bmatrix}^T$ observable?

b) Choose $\mathbf{x} = \begin{bmatrix} x & \dot{x} & y & \dot{y} \end{bmatrix}^T$ as the state vector and y as the measurement, and design a full-order observer with poles placed at $s = -2\omega, -3\omega$ and $s = -3\omega \pm 3\omega j$.

Solution:

(a) There is not enough information to answer this question. Recall, as mentioned in the chapter, that both observability and controllability are properties of realizations. Thus if you are only

given differential equations or transfer functions you will *not* be able to conclude anything about the observability or controllability of the system. This problem was designed to heighten the readers awareness of this issue.

(b) Choosing $x_1 = x, x_2 = \dot{x}, x_3 = y, x_4 = \dot{y}$ and z as the output of the system (so that it doesn't conflict with the variable y), we have the following in state space equations.

$$\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 3\omega^2 & 0 & 0 & 2\omega \\ 0 & 0 & 0 & 1 \\ 0 & -2\omega & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u, \\ z &= \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix} \mathbf{x}. \end{aligned}$$

Now that we have a realization for the system, we can check the observability to verify that we can arbitrarily place the estimator poles. The observability matrix is,

$$\mathcal{O} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -2\omega & 0 & 0 \\ -6\omega^3 & 0 & 0 & -4\omega^2 \end{bmatrix}.$$

Since $\mathcal{O}$ is full rank, the realization can now be declared observable. The desired and actual estimator characteristic equations are,

$$\begin{aligned} \alpha_{e,desired}(s) &= (s+2\omega)(s+3\omega)(s+3\omega-j3\omega)(s+3\omega+j3\omega) \\ &= s^4 + 11\omega s^3 + 54\omega^2 s^2 + 126\omega^3 s + 108\omega^4 \\ \alpha_e(s) &= \det(s\mathbf{I} - \mathbf{A} + \mathbf{LC}) = s^4 + l_3 s^3 + (l_4 + \omega^2)s^2 + (-2\omega l_2 + \omega^2 l_3)s + (-3\omega^2 l_4 - 6\omega^3 l_1). \end{aligned}$$

Equating coefficients gives,

$$l_1 = -44.5\omega,\ l_2 = -57.5\omega^2,\ l_3 = 11\omega,\ l_4 = 53\omega^2.$$

46. The linearized equations of motion of the simple pendulum in Fig.7.96 are

$$\ddot{\theta} + \omega^2\theta = u.$$

a) Write the equations of motion in state-space form.
b) Design an estimator (observer) that reconstructs the state of the pendulum given measurements of $\dot{\theta}$. Assume $\omega = 5$rad/sec, and pick the estimator roots to be at $-10 \pm 10j$.
c) Write the transfer function of the estimator between the measured value of $\dot{\theta}$ and the estimated value of θ.
d) Design a controller (that is, determine the state feedback gain $\mathbf{K}$) so that the roots of the closed-loop characteristic equation are at $s = -4 \pm 4j$.

Solution:

(a) Defining $x_1 = \theta$ and $x_2 = \dot{\theta}$, and anticipating that the measured variable in part (b) is $\dot{\theta}$, we have,

$$\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ -\omega^2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \\ y &= \begin{bmatrix} 0 & 1 \end{bmatrix} \mathbf{x}. \end{aligned}$$

Figure 7.100: Pendulum diagram for Problem 7.46

(b) From,

$$\begin{aligned}
\det(s\mathbf{I} - \mathbf{A} + \mathbf{LC}) &= 0, \\
\det\left\{\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -\omega^2 & 0 \end{bmatrix} + \begin{bmatrix} l_1 \\ l_2 \end{bmatrix}\begin{bmatrix} 0 & 1 \end{bmatrix}\right\} &= s^2 + l_2 s + \omega^2(-l_1 + 1) = 0.
\end{aligned}$$

Using $\omega = 5$ and the specified roots for the estimator, we compute $l_1 = -7$ and $l_2 = 20$. This result can be verified using MATLAB's place command.

(c) To find the transfer function from the measured value of $\dot{\theta}$, to the estimated value of θ, i.e. $\hat{\theta}$, we use the estimator equations,

$$\begin{aligned}
\dot{\hat{\mathbf{x}}} &= \mathbf{A}\hat{\mathbf{x}} + \mathbf{B}u + \mathbf{L}(y - \mathbf{C}\hat{x}) \\
&= (\mathbf{A} - \mathbf{LC})\hat{\mathbf{x}} + \mathbf{B}u + \mathbf{L}y.
\end{aligned}$$

Since this is in state space form, we can now directly compute the transfer function from y to $\hat{\theta}$. It is simply,

$$\begin{aligned}
\frac{\hat{\Theta}(s)}{Y(s)} &= \begin{bmatrix} 1 & 0 \end{bmatrix}(s\mathbf{I} - \mathbf{A} + \mathbf{LC})^{-1}\mathbf{L} \\
&= \frac{-7(s - 20/7)}{s^2 + 20s + 200}.
\end{aligned}$$

(d) For controller gain $\mathbf{K} = [k_1, k_2]$, we require,

$$\det(s\mathbf{I} - \mathbf{A} + \mathbf{BK}) = 0 \Longrightarrow s^2 + k_2 s + \omega^2 + k_1 = 0.$$

Comparing this with the specified roots equation:

$$(s + 4 + j4)(s + 4 - j4) = s^2 + 8s + 32 = 0,$$

we obtain $k_1 = 7$, and $k_2 = 8$. This result can be verified using MATLAB's place command.

47. An error analysis of an inertial navigator leads to the set of normalized state equations

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u,$$

where

$$\begin{aligned}
x_1 &= \text{east} - \text{velocity error}, \\
x_2 &= \text{platform tilt about the north axis}, \\
x_3 &= \text{north} - \text{gyro drift}, \\
u &= \text{gyro drift rate of change}.
\end{aligned}$$

Design a reduced-order estimator with $y = x_1$ as the measurement, and place the observer error poles at -0.1 and -0.1. Be sure to provide all the relevant estimator equations.

Solution:

Partitioning the system matrices yields,

$$\begin{aligned}
\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} &= \begin{bmatrix} \mathbf{A}_{aa} & \mathbf{A}_{ab} \\ \mathbf{A}_{ba} & \mathbf{A}_{bb} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u, \\
&= \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u, \\
y &= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \mathbf{x}.
\end{aligned}$$

The characteristic equation of the reduced order estimator is then given by,

$$\det(s\mathbf{I} - \mathbf{A}_{bb} + \mathbf{L}\mathbf{A}_{ab}) = s^2 - l_1 s - l_2 = 0.$$

The desired characteristic equation for the reduced order estimator poles is

$$\alpha_e(s) = (s + 0.1)^2 = s^2 + 0.2s + 0.01.$$

Thus, $l_1 = -0.2$, and $l_2 = -0.01$. This result can be verified using MATLAB's acker command.The reduced order estimator equations are given by Eq. (7.157) - Eq. (7.158).

Problems and Solutions for Section 7.8: Compensator Design: Combined Control Law and Estimator

48. A certain process has the transfer function $G(s) = \frac{4}{s^2-4}$.
a) Find $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$ for this system in observer canonical form.
b) If $u = -\mathbf{K}\mathbf{x}$, compute $\mathbf{K}$ so that the closed-loop control poles are located at $s = -2 \pm 2j$.
c) Compute $\mathbf{L}$ so that the estimator-error poles are located at $s = -10 \pm 10j$.
d) Give the transfer function of the resulting controller (for example, using Eq. (7.177)).
e) What are the gain and phase margins of the controller and the given open-loop system?

Solution:

(a) From the transfer function, we can read off the elements that will give observer canonical form,

$$\begin{aligned}
\dot{\mathbf{x}} &= \mathbf{A}_o\mathbf{x} + \mathbf{B}_o u, \\
y &= \mathbf{C}_o\mathbf{x}, \\
\mathbf{A}_o &= \begin{bmatrix} 0 & 1 \\ 4 & 0 \end{bmatrix}, \ \mathbf{B}_o = \begin{bmatrix} 0 \\ 4 \end{bmatrix}, \ \mathbf{C}_o = \begin{bmatrix} 1 & 0 \end{bmatrix}.
\end{aligned}$$

(b) With $u = -\mathbf{K}\mathbf{x}$, we want to achieve the following closed-loop characteristic equation:

$$\alpha_c(s) = (s + 2 + 2j)(s + 2 - 2j) = s^2 + 4s + 8 = 0.$$

From $\det(s\mathbf{I} - \mathbf{A} + \mathbf{B}\mathbf{K}) = 0$, we obtain,

$$s^2 + 4k_2 s + 4k_1 - 4 = 0.$$

Comparing the coefficients yields $k_1 = 3$, and $k_2 = 1$. This result can be verified using MATLAB's place command.

(c) The estimator roots are determined by the equation $\alpha_e(s) = 0$. We want to find l_1 and l_2 such that,

$$\alpha_e(s) = (s + 10 + 10j)(s + 10 - 10j) = s^2 + 20s + 200.$$

$$\begin{aligned}
\alpha_e(s) &= \det(s\mathbf{I} - \mathbf{A} + \mathbf{LC}) \\
&= \det\left(\begin{bmatrix} s & -1 \\ -4 & s \end{bmatrix} + \begin{bmatrix} l_1 \\ l_2 \end{bmatrix}\begin{bmatrix} 1 & 0 \end{bmatrix}\right) \\
&= \det\begin{bmatrix} s + l_1 & -1 \\ -4 + l_2 & s \end{bmatrix} = s^2 + l_1 s + l_2 - 4.
\end{aligned}$$

Comparing the coefficients yields $l_1 = 10, l_2 = 204$. This result can be verified using MATLAB's place command.

(d) The transfer function of the resulting compensator is,

$$\begin{aligned}
D_c(s) &= \frac{U(s)}{Y(s)} = -\mathbf{K}(s\mathbf{I} - \mathbf{A} + \mathbf{BK} + \mathbf{LC})^{-1}\mathbf{L}, \\
&= -\begin{bmatrix} 3 & 1 \end{bmatrix}\begin{bmatrix} s + 20 & -1 \\ 212 & s + 4 \end{bmatrix}^{-1}\begin{bmatrix} 20 \\ 204 \end{bmatrix} = \frac{-264s - 692}{s^2 + 24s + 292}.
\end{aligned}$$

This result can be verified using MATLAB's ss2tf command.

(e) The next figure shows the Nyquist plot generated by MATLAB (using the nyquist command), note that there is both a positive and a negative gain margin. The Nyquist plot has a "upward" gain margin of 0.422 (i.e., the gain can be increased by $1/0.422 = 2.37$) and a "downward" gain margin of 5.46 (i.e., the gain can be decreased by $1/5.46 = 0.183$) before the number of encirclements of the -1 point changes.

Nyquist plot for Problem 7.48.

49. The linearized longitudinal motion of a helicopter near hover (Fig.7.97) can be modeled by the normalized third-order system,

$$\begin{bmatrix} \dot{q} \\ \dot{\theta} \\ \dot{u} \end{bmatrix} = \begin{bmatrix} -0.4 & 0 & -0.01 \\ 1 & 0 & 0 \\ -1.4 & 9.8 & -0.02 \end{bmatrix} \begin{bmatrix} q \\ \theta \\ u \end{bmatrix} + \begin{bmatrix} 6.3 \\ 0 \\ 9.8 \end{bmatrix} \delta,$$

where,

$q =$ pitch rate,
$\theta =$ pitch angle of fuselage,
$u =$ horizontal velocity (standard aircraft notation),
$\delta =$ rotor tilt angle (control variable).

Suppose our sensor measures the horizontal velocity as the output; that is, $y = u$.
a) Find the open-loop pole locations.
b) Is the system controllable?

Figure 7.101: Helicopter for Problem 7.49

c) Find the feedback gain that places the poles of the system at $s = -1 \pm 1j$ and $s = -2$.
d) Design a full-order estimator for the system, and place the estimator poles at -8 and $-4 \pm 4\sqrt{3}j$.
e) Design a reduced-order estimator with both poles at -4. What are the advantages and disadvantages of the reduced-order estimator compared with the full-order case?
f) Compute the compensator transfer function using the control gain and the full-order estimator designed in part (d), and plot its frequency response using MATLAB. Draw a Bode plot for the closed-loop design, and indicate the corresponding gain and phase margins.
g) Repeat part(f) with the reduced-order estimator.
h) Draw the symmetrical root locus (SRL) and select roots for a control law that will give a control bandwidth matching the design of part(c), and select roots for a full-order estimator that will result in an estimator error bandwidth comparable to the design of part(d). Draw the corresponding Bode plot and compare the pole placement and SRL designs with respect to bandwidth, stability margins, step response, and control effort for a unit-step rotor-angle input. Use MATLAB for the computations.

Solution:

Again, the equations of motion for the helicopter are,

$$\begin{bmatrix} \dot{q} \\ \dot{\theta} \\ \dot{u} \end{bmatrix} = \begin{bmatrix} -0.4 & 0 & -0.01 \\ 1 & 0 & 0 \\ -1.4 & 9.8 & -0.02 \end{bmatrix} \begin{bmatrix} q \\ \theta \\ u \end{bmatrix} + \begin{bmatrix} 6.3 \\ 0 \\ 9.8 \end{bmatrix} \delta.$$

(a) The open-loop poles are the eigenvalues of $\mathbf{A}$. Solving gives the open-loop poles as $s = -0.6565$ and $s = -0.1183 \pm j0.3678$. In MATLAB, use eig(A). We also note that the zeros of the plant are in the RHP at $0.25 \pm j2.5$ and can be computed using the MATLAB tzero command.

(b) To determine controllability, we want to look at the rank of the controllability matrix. For the helicopter,

$$\text{rank}\,\{\mathcal{C}\} = \text{rank}\,\begin{bmatrix} \mathbf{B} & \mathbf{AB} & \mathbf{A}^2\mathbf{B} \end{bmatrix} = 3.$$

Thus, the system is controllable. Alternatively, you can find the singular values of the matrix using the MATLAB svd command. This will give an indication of how large the actuator signals will need to be.

(c) When the order of the system gets larger than two, it is often convenient to let the computer do the necessary calculations. Using MATLAB's place command and the specified pole locations, we find the control gains,

$$\mathbf{K} = \begin{bmatrix} 0.4706 & 1.0 & 0.0627 \end{bmatrix}.$$

If we have to do this computation by hand, the approach would be the following. Form the desired characteristic equation and compare it with the equation,

$$\det(s\mathbf{I} - \mathbf{A} + \mathbf{BK}) = 0.$$

to obtain the values for $\mathbf{K}$.

(d) Using the duality principle, we find the estimator gains using MATLAB's place command as well. We find,

$$\mathbf{L} = \begin{bmatrix} 44.7097 & 18.8130 & 15.5800 \end{bmatrix}^T.$$

(e) The notation of this solution follows Equation 7.147 in the text. Reordering the system matrix, we have,

$$\begin{aligned}
\begin{bmatrix} \dot{u} \\ \dot{q} \\ \dot{\theta} \end{bmatrix} &= \begin{bmatrix} -0.02 & -1.4 & 9.8 \\ -0.01 & -0.4 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} u \\ q \\ \theta \end{bmatrix} + \begin{bmatrix} 9.8 \\ 0 \\ 6.3 \end{bmatrix} \delta \\
&= \begin{bmatrix} \mathbf{A}_{aa} & \mathbf{A}_{ab} \\ \mathbf{A}_{ba} & \mathbf{A}_{bb} \end{bmatrix} \begin{bmatrix} x_a \\ x_b \end{bmatrix} + \begin{bmatrix} G_a \\ G_b \end{bmatrix} \delta.
\end{aligned}$$

To design the reduced order estimator, we need to solve the characteristic equation,

$$\det(s\mathbf{I} - \mathbf{A}_{bb} + \mathbf{L}\mathbf{A}_{ab}) = 0.$$

So that the estimator gains, $\mathbf{L}$, place the poles at the desired locations. Using MATLAB's place command, we find,

$$\mathbf{L} = \begin{bmatrix} 1.2510 & 0.9542 \end{bmatrix}^T.$$

The advantages of the reduced order estimator are that the resulting estimator is simpler (in terms of the number of flops, Floating point operations, count necessary to implement the estimator on a real system). Another issue is that you are using the measurement of the state directly. This would be advantageous if the measured signal was relatively noise free. However, if the signal was noisy, then it would be better to use the full order estimator because it provides filtering of noisy measurement.

(f) The compensator for the controller in part (c) and estimator in part (d) is,

$$\begin{aligned}
D_c(s) &= -\mathbf{K}(s\mathbf{I} - \mathbf{A} + \mathbf{B}\mathbf{K} + \mathbf{L}\mathbf{C})^{-1}\mathbf{L} = \frac{40.8s^2 + 61.0s + 31.9}{s^3 + 19.58s^2 - 210.4s + 814.7} \\
&= \frac{40.8(s + 0.75 \pm j0.47)}{(s + 28.1)(s - 4.26 \pm j3.29)}.
\end{aligned}$$

The figures on the next page show the Bode plot of the compensator transfer function using the full-order estimator, and the Bode plot of the plant and compensator. The Gain and Phase margins for the system are -2.6 db and 22.0 deg respectively.

Problem 7.49: Bode plots of compensator using the full-order estimator alone.

Problem 7.49: Bode plot of plant and compensator combined.

(g) Compensator for the controller in part (c) and estimator in part (e) (i.e., the reduced-order estimator) is,

$$D_{cr}(s) = \mathbf{C}_r(s\mathbf{I} - \mathbf{A}_r)^{-1}\mathbf{B}_r + D_r.$$

$$\begin{aligned}
K_a &= 0.0627,\ \mathbf{K}_b = [0.4706, 1], \\
\mathbf{A}_r &= \mathbf{A}_{bb} - \mathbf{L}\mathbf{A}_{ab} - (\mathbf{B}_b - \mathbf{L}\mathbf{B}_a)\mathbf{K}_b = \begin{bmatrix} 4.16 & -6.30 \\ 6.74 & 0.00 \end{bmatrix}, \\
\mathbf{B}_r &= \mathbf{A}_r\mathbf{L} + \mathbf{A}_{ba} - \mathbf{L}\mathbf{A}_{aa} - (\mathbf{B}_b - \mathbf{L}\mathbf{B}_a)K_a = \begin{bmatrix} -0.423 \\ 9.034 \end{bmatrix}, \\
\mathbf{C}_r &= -\mathbf{K}_b = \begin{bmatrix} -0.4706 & -1 \end{bmatrix}, \\
D_r &= -K_a - \mathbf{K}_b\mathbf{L} = -1.61.
\end{aligned}$$

(h) Thus, the transfer function for the compensator using the reduced-order estimator is,

$$D_{cr}(s) = \mathbf{C}_r(s\mathbf{I} - \mathbf{A}_r)^{-1}\mathbf{B}_r + D_r = \frac{-1.61s^2 - 2.16s + 0.97}{s^2 - 4.16s + 42.44}.$$

The figure below shows the open-loop transfer function for the compensator designed using the reduced-order estimator. The Bode plot for the plant and the compensator is also shown on the next page.

Problem 7.49: Bode plot of the compensator transfer function using the reduced-order estimator.

Problem 7.49: Bode plot of the compensated system.

50. Suppose a DC drive motor with motor current is connected to the wheels of a cart in order to control the movement of an inverted pendulum mounted on the cart. The linearized and normalized equations of motion corresponding to this system can be put in the form

$$\ddot{\theta} = \theta + v + u,$$
$$\dot{v} = \theta - v - u,$$

where,

$$\theta = \text{angle of the pendulum},$$
$$v = \text{velocity of the cart}.$$

a) We wish to control θ by feedback to u of the form,

$$u = -K_1\theta - K_2\dot{\theta} - K_3 v.$$

Find the feedback gains so that the resulting closed-loop poles are located at $-1, -1 \pm j\sqrt{3}$.
b) Assume that θ and v are measured. Construct an estimator for θ and θdot of the form,

$$\dot{\hat{x}} = \mathbf{A}\hat{\mathbf{x}} + \mathbf{L}(y - \hat{y}),$$

where $\mathbf{x} = \begin{bmatrix} \theta & \dot{\theta} \end{bmatrix}^T$ and $y = \theta$. Treat both v and u as known. Select $\mathbf{L}$ so that the estimator poles are at -2 and -2.
c) Give the transfer function of the controller, and draw the Bode plot of the closed-loop system, indicating the corresponding gain and phase margins.
d) Using MATLAB, plot the response of the system to an initial condition on θ, and give a physical explanation for the initial motion of the cart.

Solution:

(a) Defining the state $\mathbf{x} = \begin{bmatrix} \theta & \dot{\theta} & v \end{bmatrix}^T$, the system is written as,

$$\begin{aligned} \begin{bmatrix} \dot{\theta} \\ \ddot{\theta} \\ \dot{v} \end{bmatrix} &= \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} \theta \\ \dot{\theta} \\ v \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} u, \\ \dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}u. \end{aligned}$$

Using $\det(s\mathbf{I} - \mathbf{A} + \mathbf{B}\mathbf{K}) = 0$ with $\mathbf{K} = \begin{bmatrix} k_1 & k_2 & k_3 \end{bmatrix}$ we find the characteristic equation,

$$s^3 + s^2(1 - k_3 + k_2) + s(k_1 - 1) + 2(k_3 - 1) = 0.$$

The desired characteristic equation is,

$$(s+1)((s+1)^2 + 3) = s^3 + 3s^2 + 6s + 4 = 0.$$

Comparing coefficients, $\mathbf{K} = \begin{bmatrix} 7 & 5 & 3 \end{bmatrix}$. This result can be verified using the MATLAB place command.

(b) The estimator equations (both explicitly and symbolically) for estimating $\hat{\mathbf{x}}$ are,

$$\begin{aligned} \begin{bmatrix} \hat{\dot{\theta}} \\ \hat{\ddot{\theta}} \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{\theta} \\ \hat{\dot{\theta}} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} v + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + \mathbf{L}(y - \hat{y}), \\ &= \mathbf{A}_e \widehat{\mathbf{x}} + \mathbf{B}_v v + \mathbf{B}_u u + \mathbf{L}(y - \hat{y}). \end{aligned}$$

where u and v are assumed to be known. The output equations for the plant and the estimator are,

$$\begin{aligned} y &= \mathbf{C}\mathbf{x} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \mathbf{x}, \\ \hat{y} &= \mathbf{C}_e \hat{\mathbf{x}} = \begin{bmatrix} 1 & 0 \end{bmatrix} \hat{\mathbf{x}}. \end{aligned}$$

With $\mathbf{L} = \begin{bmatrix} \ell_1 & \ell_2 \end{bmatrix}^T$, the estimator error characteristic equation becomes,

$$\det(s\mathbf{I} - \mathbf{A}_e + \mathbf{L}\mathbf{C}_e) = s^2 + sl_1 + l_2 - 1 = 0.$$

Equating coefficients with the desired characteristic equation,

$$(s+2)(s+2) = s^2 + 4s + 4,$$

we have $\mathbf{L} = \begin{bmatrix} 4 & 5 \end{bmatrix}^T$. This result can be verified using the MATLAB place command.

(c) Construct the feedback u in terms of both the measured signal v and the estimated state $\hat{\mathbf{x}}$. Using the feedback gains from (a), we have,

$$\begin{aligned} u &= -K_1\hat{\theta} - K_2\dot{\hat{\theta}} - K_3 v, \\ &= -\mathbf{K}_e\hat{\mathbf{x}} - K_3 v. \end{aligned}$$

Plugging this expression for u into the estimator equation we have,

$$\begin{aligned} \dot{\hat{\mathbf{x}}} &= (\mathbf{A}_e - \mathbf{B}_u\mathbf{K}_e - \mathbf{L}\mathbf{C}_e)\hat{\mathbf{x}} + (\mathbf{B}_v - \mathbf{B}_u K_3)v + \mathbf{L}y, \\ u &= -\mathbf{K}_e\hat{\mathbf{x}} - K_3 v. \end{aligned}$$

The transfer function from y to u can now be read directly from these two equations by setting all of the auxiliary inputs to zero, i.e., $v = 0$. Thus,

$$D_c(s) = -\mathbf{K}_e(s\mathbf{I} - \mathbf{A}_e + \mathbf{B}_u\mathbf{K}_e + \mathbf{L}\mathbf{C}_e)^{-1}\mathbf{L} = \frac{-(53s + 55)}{s^2 + 9s + 31}.$$

The Bode plots are shown next.

Open-loop Bode plot of compensator transfer function for Problem 7.50.

Bode plot of the compensator and plant together for Problem 7.50.

(d) One approach to simulating the system is to augment the plant and estimator equations into one matrix. Recognizing that $v = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \mathbf{x} = \mathbf{C}_v \mathbf{x}$, we can eliminate u and v.

$$\begin{aligned}
\dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}u = (\mathbf{A} - \mathbf{B}K_3\mathbf{C}_v)\mathbf{x} - \mathbf{B}\mathbf{K}_e\hat{\mathbf{x}} \\
\dot{\hat{\mathbf{x}}} &= \mathbf{A}_e\hat{\mathbf{x}} + \mathbf{B}_v v + \mathbf{B}_u u + \mathbf{L}(y - \hat{y}) \\
&= (\mathbf{B}_v\mathbf{C}_v - \mathbf{B}_u K_3\mathbf{C}_v + \mathbf{L}\mathbf{C})\mathbf{x} + (\mathbf{A}_e - \mathbf{L}\mathbf{C}_e - \mathbf{B}_u\mathbf{K}_e)\hat{\mathbf{x}}.
\end{aligned}$$

This is now easily implemented using the MATLAB command lsim. The figure on the next page shows the closed-loop system response due to an initial angle of $\theta = 0.1$ rad with respect to a vertical line. The initial motion of the cart is in the direction that the pendulum is leaning (due to the initial condition). Physically, if the cart moved away from the direction that the pendulum was leaning, then it would cause the angle to increase eventually toppling the pendulum.

Angle of pendulum and the velocity of the cart given an initial angle for Problem 7.50.

51. Consider the control of

$$G(s) = \frac{Y(s)}{U(s)} = \frac{10}{s(s+1)}.$$

a) Let $y = x_1$ and $\dot{x}_1 = x_2$, and write state equations for the system.
b) Find K_1 and K_2 so that $u = -K_1x_1 - K_2x_2$ yields closed-loop poles with a natural frequency $\omega_n = 3$ and a damping ratio $\zeta = 0.5$.
c) Design a state estimator for the system that yields estimator error poles with $\omega_{n1} = 15$ and $\zeta_1 = 0.5$.
d) What is the transfer function of the controller obtained by combining parts(a) through (c)?
e) Sketch the root locus of the resulting closed-loop system as plant gain (nominally 10) is varied.

Solution:

The state equations are,

$$\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 10 \end{bmatrix} u, \\ y &= \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x}. \end{aligned}$$

(b) K=place(A,B, roots([1 2*zeta*wn wn*wn])) = [0.9 0.2].

(c) L=place(A',C',roots([1 2*zeta*wn wn*wn]))' = $[14\ 211]^{\mathrm{T}}$.

(d) The transfer function for the controller is,

$$\begin{aligned} D_c(s) &= -\mathbf{K}(s\mathbf{I} - \mathbf{A} + \mathbf{BK} + \mathbf{LC})^{-1}\mathbf{L} \\ &= \frac{-(54.8s + 202.5)}{s^2 + 17s + 262}. \end{aligned}$$

(e) The figure below shows the root locus around a nominal gain of 10, which is indicated by asterisk.

Problem 7.51: Root locus of the closed-loop system as plant gain is varied.

52. Unstable equations of motion of the form,

$$\ddot{x} = x + u,$$

arise in situations where the motion of an upside-down pendulum (such as a rocket) must be controlled.
a) Let $u = -Kx$ (position feedback alone), and sketch the root locus with respect to the scalar gain K.
b) Consider a lead compensator of the form,

$$U(s) = K\frac{s+a}{s+10}X(s).$$

Select and so that the system will display a rise time of about 2 sec and no more than 25% overshoot. Sketch the root locus with respect to K.

c) Sketch the Bode plot (both magnitude and phase) of the uncompensated plant.
d) Sketch the Bode plot of the compensated design, and estimate the phase margin.
e) Design state feedback so that the closed-loop poles are at the same locations as those of the design in part(b).
f) Design an estimator for x and $\dot{x}$ using the measurement of $x = y$, and select the observer gain $\mathbf{L}$ so that the equation for $\tilde{x}$ has characteristic roots with a damping ratio $\zeta = 0.5$ and a natural frequency $\omega_n = 8$ rad/sec.
g) Draw a block diagram of your combined estimator and control law, and indicate where and appear. Draw a Bode plot for the closed-loop system, and compare the resulting bandwidth and stability margins with those obtained using the design of part(b).

Solution:

(a) The root locus using position feedback alone is shown below. Notice that no matter how large the gain is made, the closed-loop roots are never strictly in the LHP.

(b) First of all, we need to translate the specifications into values for ω_n and ζ. Although the closed system with a lead compensator is third-order, we assume the rules of thumb for a second-order system are valid and then validate our design after settling on values for a and K.

$$M_p < 25\% \Longrightarrow \zeta > 0.4, \omega_n = \frac{1.8}{t_r} = \frac{1.8}{2} = 0.9.$$

Try $\omega_n = 1$ and $\zeta = 0.4$ for the design. Because the form of the compensator is specified, we can calculate the closed-loop transfer function to be,

$$\frac{Y(s)}{R(s)} = T(s) = \frac{s + 10}{s^3 + 10s^2 + (K - 1)s + (Ka - 10)}.$$

Note that we have subtly introduced r as a reference input to the plant. The desired closed loop poles should be placed at (taking $\alpha = 10$),

$$(s + \alpha)(s^2 + 2\zeta\omega_n s + \omega_n^2) = (s + 10)(s^2 + 0.8s + 1) = s^3 + 10.8s^2 + 9s + 10.$$

Although the coefficient for the s^2 term doesn't match exactly, we just want to get a ballpark estimate for K and a. So comparing the other coefficients, we find $K = 10$ and $a = 2$. Using these values, the root locus for using the lead compensator is shown. To verify that our design is acceptable, we also check the step response of the system. This is shown on the last figure in this section.

Problem 7.52: Root locus with position feedback alone.

Root locus for Problem 7.52.

(c) The Bode plot of $\frac{1}{s^2-1}$ is shown below.

(d) The Bode plot of the compensated design is also shown on the next page. The phase margin is approximately 23°. The gain margin is 0.5.

Problem 7.52: Bode plots for the open-loop system.

Problem 7.52: Compensator and plant combined.

(e) Although the design in part (b) has three closed-loop poles (due to the lead compensator), full state feedback on a second-order system does not introduce an extra pole. Recognizing this, we keep the poles closest to the plant's open loop poles, $-0.433 \pm 0.953j$. The feedback gains can now be determined using MATLAB's place command,

$$\mathsf{K} = \mathsf{place}(A, B, [-0.433 + 0.953 * \mathsf{j}; -0.433 - 0.953 * \mathsf{j}]) = [2.09 \;\; 0.87].$$

(f) The estimator gains are just as easy to produce. With $\zeta = 0.5$ and $\omega_n = 8$, we have,

$$\begin{aligned} [A, B, C, D] &= \mathsf{tf2ss}([0 \;\; 0 \;\; 1], [1 \;\; 0 \;\; -1]) \\ \mathsf{pe} &= [1 \;\; 2 * \mathsf{zeta} * wn \;\; wn * wn] \\ \mathsf{L} &= \mathsf{place}(\mathsf{A}\prime, \mathsf{C}\prime, \mathsf{pe})\prime = [8 \;\; 65]^T. \end{aligned}$$

(g) The estimator equations are,

$$\begin{aligned} \dot{\hat{\mathbf{x}}} &= \mathbf{A}\hat{\mathbf{x}} + \mathbf{B}u + \mathbf{L}(y - \mathbf{C}\hat{\mathbf{x}}), \\ u &= -\mathbf{K}\hat{\mathbf{x}}. \end{aligned}$$

and are shown in block diagram form on top of the next page.

Block diagram of the combined estimator and control law in Problem 7.52.

The Bode plot of the controller and plant designed using pole placement techniques is shown below. The phase margin is approximately 22° and the gain margin now has a limitation both for increasing and decreasing the gain. The gain can be increased by a factor of $1/0.14 = 7.14 = 17$db and decreased by a factor of $1/1.96 = 0.51 = -5.8$db. So the lead compensator has roughly equivalent stability margins.

Problem 7.52: Bode plot of plant and compensator design with pole placement.

The step responses for both designs are shown on the next page using the MATLAB `step` command. They differ slightly because the DC gain of the compensator designed using pole placement hasn't been adjusted for unity gain. Also the specification for less than overshoot has not been met with the pole placement design. This can be attributed to an estimator roots which are too slow. Increasing the ω_n of the estimator to 10 rad/sec will meet the specification.

Problem 7.52: Closed-loop step responses.

53. A simplified model for the control of a flexible robotic arm is shown in Fig.7.98, where

$$k/M = 900 \text{ rad/sec}^2 ,$$
$$y = \text{output, the mass position,}$$
$$u = \text{input, the position of the end of the spring.}$$

a) Write the equations of motion in state-space form.
b) Design an estimator with roots as $s = -100 \pm 100j$.
c) Could both state variables of the system be estimated if only a measurement of $\dot{y}$ was available?
d) Design a full-state feedback controller with roots at $s = -20 \pm 20j$.
e) Would it be reasonable to design a control law for the system with roots at $s = -200 \pm 200j$? State your reasons.
f) Write equations for the compensator, including a command input for y. Draw a Bode plot for the closed-loop system, and give the gain and phase margins for the design.

Solution:

Figure 7.102: Simple robotic arm for Problem 7.53

(a) Defining $x_1 = y$ and $x_2 = \dot{y}$, we have,

$$\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ -k/M & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ k/M \end{bmatrix} u, \\ y &= \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x}. \end{aligned}$$

(b) Comparing coefficients of like powers of s,

$$\begin{aligned} (s + 100 + j100)(s + 100 - j100) &= s^2 + 200s + 20000 = 0 \\ &= \det(s\mathbf{I} - \mathbf{A} + \mathbf{LC}) = s^2 + l_1 s + l_2 + 900 = 0, \end{aligned}$$

yields $\mathbf{L} = \begin{bmatrix} 200 & 19100 \end{bmatrix}^T$. This result can be verified using the MATLAB place command.

(c) Let's check if x_1 is observable with $y = x_2$. $\det(\mathcal{O}) = k/M \neq 0$. So y is observable and (not surprisingly) both state variables can be estimated from $\dot{y}$.

(d) Comparing coefficients of like powers of s,

$$\begin{aligned} (s + 20 + j20)(s + 20 - j20) &= s^2 + 40s + 800 = 0 \\ &= \det(s\mathbf{I} - \mathbf{A} + \mathbf{BK}) = s^2 + 900k_2 s + 900(k_1 + 1) = 0, \end{aligned}$$

yields $\mathbf{K} = \begin{bmatrix} -0.111 & 0.044 \end{bmatrix}^T$. This result can be verified using MATLAB's place command.

(e) No.

(i) The bandwidth of the spring is about rad/sec and system roots at rad/sec means that large control levels will be required.

(ii) When using an estimated state feedback, you would like the estimates of the state to have converged "to some extent" before generating a control signal from the estimate. This is the

reason for the rule of thumb about picking the estimator roots 3 to 10 times faster than the control roots.

(f) We can express the compensator as,

$$\begin{aligned}\dot{\hat{\mathbf{x}}} &= (\mathbf{A} - \mathbf{BK} - \mathbf{LC})\hat{\mathbf{x}} + \mathbf{L}y, \\ u &= -\mathbf{K}\hat{\mathbf{x}}.\end{aligned}$$

Thus the loop gain is,

$$D_c(s)G(s) = -\frac{744000(s - 12.2)}{(s + 120 \pm 116j)(s \pm 30j)}.$$

Note that the compensator has a zero in the RHP (*non-minimum phase*). This result can be verified using MATLAB's ss2tf command. The root locus of the compensated system is shown below. From the Bode plot of the loop gain, we find that the gain margin is approximately 8.8 db (can be verified from the root locus as well) and the phase margin is approximately $-64°$. Note that the closed-loop system is stable despite the fact that the phase margin is negative. This is true because the closed-loop system is non-minimum phase.

Root locus for Problem 7.53.

Problem 7.53: Bode plot of plant and compensator for robot arm.

54. The linearized differential equations governing the fluid-flow dynamics for the two cascaded tanks in Fig.7.99 are

$$\delta\dot{h}_1 + \sigma\delta h_1 = \delta u,$$
$$\delta\dot{h}_2 + \sigma\delta h_2 = \sigma\delta h_1,$$

where,

$$\delta h_1 = \text{deviation of depth in tank 1 from the nominal level,}$$
$$\delta h_2 = \text{deviation of depth in tank 2 from the nominal level,}$$
$$\delta u = \text{deviation in fluid inflow rate to tank 1 (control).}$$

a) *Level Controller for Two Cascaded Tanks:* Using state feedback of the form,

$$\delta u = -K_1\delta h_1 - K_2\delta h_2,$$

choose values of K_1 and K_2 that will place the closed-loop eigenvalues at,

$$s = -2\sigma(1 \pm j).$$

Figure 7.103: Coupled tanks for Problem 7.54

b) *Level Estimator for two Cascaded Tanks:* Suppose that only the deviation in the level of tank2 is measured (that is, $y = \delta h_2$. Using this measurement, design an estimator that will give continuous, smooth estimates of the deviation in levels of tank1 and tank2, with estimator error poles at $-8\sigma(1 \pm j)$.
c) *Estimator/Controller for Two Cascaded Tanks:* Sketch a block diagram (showing individual integrators) of the closed-loop system obtained by combining the estimator of part(b) with the controller of part(a).
d) Using MATLAB, compute and plot the response at to an initial offset in δh_1. Assume $\sigma = 1$ for the plot.

Solution:

(a) Comparing coefficients of like powers of s,

$$\begin{aligned} \det \begin{bmatrix} s+\sigma+K_1 & K_2 \\ -\sigma & s+\sigma \end{bmatrix} &= s^2 + (2\sigma + K_1)s + \sigma^2 + \sigma(K_1 + K_2) = 0. \\ &= (s + 2\sigma + 2\sigma j)(s + 2\sigma - 2\sigma j) = s^2 + 4\sigma s + 8\sigma^2 = 0, \end{aligned}$$

gives $K_1 = 2\sigma$ and $K_2 = 5\sigma$.

(b) Comparing coefficients of like powers of s,

$$\det\begin{bmatrix} s+\sigma & l_1 \\ -\sigma & s+\sigma+l_2 \end{bmatrix} = s^2+(2\sigma+l_1)s+\sigma(l_1+l_2)+\sigma^2=0$$
$$= (s+8\sigma+8\sigma j)(s+8\sigma-8\sigma j) = s^2+16\sigma s+128\sigma^2=0.$$

gives $\ell_1 = 113\sigma$ and $\ell_2 = 14\sigma$.

(c) The figure below shows a block diagram of the system.

Block diagram for of closed-loop system for Problem 7.54 (shown for $\sigma = 1$.

(d) The response to an initial condition on $\delta h_1(0)$ is shown next using the MATLAB initial command.

Problem 7.54: Initial condition response for offset in $\delta h_1(0)$.

55. The lateral motions of a ship that is 100 m long, moving at a constant velocity of 10 m/sec, are described by

$$\begin{bmatrix} \dot{\beta} \\ \dot{r} \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} -0.0895 & -0.286 & 0 \\ -0.0439 & -0.272 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \beta \\ r \\ \psi \end{bmatrix} + \begin{bmatrix} 0.0145 \\ -0.0122 \\ 0 \end{bmatrix} \delta,$$

where

$$\begin{aligned} \beta &= \text{sideslip angle, deg,} \\ \psi &= \text{heading angle,} \\ \delta &= \text{ rudder angle, deg,} \\ r &= \text{yaw rate. See Fig. 7.100.} \end{aligned}$$

a) Determine the transfer function from δ to ψ and the characteristic roots of the uncontrolled-ship.

b) Using complete state feedback of the form,

$$\delta = -K_1\beta - K_2 r - K_3(\psi - \psi_d),$$

where ψ_d is the desired heading, determine values of K_1, K_2 and K_3 that will place the closed-loop roots at $s = -0.2, -0.2 \pm 0.2j$.
c) Design a state estimator based on the measurement of ψ (obtained from a gyrocompass, for example). Place the roots of the estimator error equation at $s = -0.8$ and $-0.8 \pm 0.8j$.
d) Give the state equations and transfer function for the compensator $D_c(s)$ in Fig.7.101, and plot its frequency response.
e) Draw the Bode plot for the closed-loop system, and compute the corresponding gain and phase margins.
f) Compute the feed-forward gains for a reference input, and plot the step response of the system to a change in heading of 5°.

Solution:

(a) With ψ as the measurement,

$$y = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \beta \\ r \\ \psi \end{bmatrix} = \mathbf{Cx}.$$

The transfer function from δ to ψ is (using MATLAB's ss2tf command),

$$\frac{\psi(s)}{\delta(s)} = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} = \frac{-0.0122(s + 0.142)}{s(s + 0.326)(s + 0.036)}.$$

The roots of the uncontrolled ship are the poles of the above transfer function: $s = 0, -0.326, -0.036$.
(b) Define $\mathbf{K} = \begin{bmatrix} K_1 & K_2 & K_3 \end{bmatrix}$ and let $\delta = -\mathbf{Kx}$. Then,

$$\det[s\mathbf{I} - \mathbf{A} + \mathbf{BK}] = \det \begin{bmatrix} s + 0.0895 + 0.0145K_1 & 0.286 + 0.0145K_2 & 0.0145K_3 \\ 0.0439 - 0.0122K_1 & s + 0.272 - 0.0122K_2 & -0.0122K_3 \\ 0 & -1 & s \end{bmatrix},$$

gives roots at $s = -0.2, -0.2 \pm 0.2j$, when,

$$K_1 = 0.276,\ K_2 = -19.22,\ K_3 = -9.26.$$

This result can be verified using MATLAB's place command.
(c) With $\mathbf{L} = \begin{bmatrix} \ell_1 & \ell_2 & \ell_3 \end{bmatrix}^T$,

$$\det[s\mathbf{I} - \mathbf{A} + \mathbf{LC}] = \det \begin{bmatrix} s + 0.0895 & 0.286 & l_1 \\ 0.0439 & s + 0.272 & l_2 \\ 0 & -1 & s + l_3 \end{bmatrix},$$

gives roots at $s = -0.8, -0.8 \pm 0.8j$, when

$$l_1 = -19.09,\ l_2 = 1.81,\ l_3 = 2.04.$$

Figure 7.104: View of ship from above for Problem 7.55

Figure 7.105: Ship control block diagram for Problem 7.55

Again, this result can be verified using MATLAB's place command.

(d) The compensator state equations are,

$$\begin{aligned}\dot{\hat{\mathbf{x}}} &= \mathbf{A}\hat{\mathbf{x}} + \mathbf{B}u + \mathbf{L}(y - \mathbf{C}\hat{\mathbf{x}}) = (\mathbf{A} - \mathbf{BK} - \mathbf{LC})\hat{\mathbf{x}} + \mathbf{L}y, \\ u &= -\mathbf{K}\hat{\mathbf{x}}.\end{aligned}$$

The compensator transfer function is given by (using the MATLAB ss2tf command),

$$D_c(s) = -\mathbf{K}(s\mathbf{I} - \mathbf{A} + \mathbf{BK} + \mathbf{LC})^{-1}\mathbf{L} = 58.96\frac{s^2 + 0.753s + 0.161}{s^3 + 2.64s^2 + 3.2s + 1.05}.$$

(e) The Bode plot of the closed-loop system is shown on the next page. The MATLAB command Bode or Margin can be used to create this figure. Note that when you find the Bode plot, the gain and phase margins only make sense if you consider the transfer function:

$$G(s) = -D_c(s)\frac{\psi(s)}{\delta(s)}.$$

Since the margins on a Bode plot assume negative feedback, the negative sign incorporated in $D_c(s)$ must be removed. The gain and phase margins are Gain Margin = 6.15 db, Phase Margin = 52.8°.

Problem 7.55: Bode plot of closed-loop system for ship control.

(f) Consider the determination of the feedforward gains $\mathbf{N_x}$ and N_u by,

$$\begin{bmatrix} \mathbf{N_x} \\ N_u \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & D \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

This gives $\mathbf{N_x} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T$, $N_u = 0$. Hence the control becomes,

$$\delta = \mathbf{K}(\mathbf{N_x}\psi_d - \hat{\mathbf{x}}) = \mathbf{K}N_{\mathbf{x}}\psi_d - \mathbf{K}\hat{\mathbf{x}}.$$

The complete closed-loop system step response is shown on the next page. The MATLAB command `step` can be used to create this figure.

Problem 7.55: 5° step response of closed-loop system for ship control.

Problems and Solutions for Section 7.9: Introduction of the Reference Input with the Estimator

56. As mentioned in footnote9 in Section7.9.2, a reasonable approach for selecting the feed-forward gain in Eq.(7.202) is to choose $\bar{N}$ such that when r and y are both unchanging, the DC gain from r to u is the negative of the DC gain from y to u.Derive a formula for $\bar{N}$ based on this selection rule. Show that if the plant is type 1, this choice is the same as that given by Eq.(7.202).

Solution:

The system equations with the feedforward gains included are,

$$\begin{aligned} \dot{\hat{\mathbf{x}}} &= (\mathbf{A} - \mathbf{BK} - \mathbf{LC})\hat{\mathbf{x}} + \mathbf{L}y + \mathbf{M}r, \\ u &= -\mathbf{K}\hat{\mathbf{x}} + \bar{N}r. \end{aligned}$$

To find the DC gain from y to u, we let,

$$\hat{\mathbf{x}} = \hat{\mathbf{x}}_0,\ r = 0,\ y = y_0,\ u = u_0.$$

Then,

$$\begin{aligned} \mathbf{0} &= (\mathbf{A} - \mathbf{BK} - \mathbf{LC})\hat{\mathbf{x}}_0 + \mathbf{L}y_0 \\ u_0 &= -\mathbf{K}\hat{\mathbf{x}}_0. \end{aligned}$$

So that the DC gain from y to u is given by,

$$u_0 = \mathbf{K}(\mathbf{A} - \mathbf{BK} - \mathbf{LC})^{-1}\mathbf{L}y_0.$$

Similarly, to find the DC gain from r to u, we let,

$$\hat{\mathbf{x}} = \hat{\mathbf{x}}_0,\ y = 0,\ r = r_0,\ u = u_{0.}$$

Then,

$$\begin{aligned} \mathbf{0} &= (\mathbf{A} - \mathbf{BK} - \mathbf{LC})\hat{\mathbf{x}}_0 + \mathbf{M}r_0, \\ u_0 &= -\mathbf{K}\hat{\mathbf{x}}_0 + \bar{N}r_0. \end{aligned}$$

So that the DC gain from r to u is given by,

$$u_0 = (\mathbf{K}(\mathbf{A} - \mathbf{BK} - \mathbf{LC})^{-1}\mathbf{M} + \bar{N})r_0.$$

From the footnote in the Servo design section, we set the DC gain from r to u equal to the negative of the DC gain from y to u,

$$-\mathbf{K}(\mathbf{A} - \mathbf{BK} - \mathbf{LC})^{-1}\mathbf{L} = \mathbf{K}(\mathbf{A} - \mathbf{BK} - \mathbf{LC})^{-1}\mathbf{M} + \bar{N}.$$

Therefore,

$$\bar{N} = -\mathbf{K}(\mathbf{A} - \mathbf{BK} - \mathbf{LC})^{-1}(\mathbf{L} + \mathbf{M}).$$

We can show, in general, if $\bar{N}$ is chosen as the footnote implies, then the system DC gain is unity for a Type I plant. Consider the general closed-loop system block diagram shown below. Assuming a Type I plant,

$$G(s) = \frac{1}{s}\bar{G}(s),$$

then the closed-loop DC gain is simply,

$$\begin{aligned} \frac{Y(0)}{R(0)} &= \lim_{s\to 0} \bar{N}D_r \frac{G(s)}{1 - \bar{G}(s)D_y} \\ &= \lim_{s\to 0} \bar{N}D_r \frac{G(0)}{1 - \bar{G}(0)D_y} \\ &= -\bar{N}\frac{D_r(0)}{D_y(0)}. \end{aligned}$$

So, if $\bar{N}\ D_r = -D_y$ then the DC gain of the system is unity. The selection approach of $\bar{N}$ mentioned in the servo design section is exactly the condition $\bar{N}\ D_r = -D_y$. Hence, since Eq. 7.202 is a direct result of setting the DC gain to unity, then the above expression for $\bar{N}$, that was derived from the footnote hint, is equivalent to Eq. 7.202.

General closed-loop system block diagram for Problem 7.56.

Problems and Solutions for Section 7.10: Integral Control and Robust Tracking

57. Assume that the linearized and time-scaled equation of motion for the ball-bearing levitation device is $\ddot{x} - x = u + w$. Here w is a constant bias due to the power amplifier. Introduce integral error control, and select three control gains $\mathbf{K} = \begin{bmatrix} K_1 & K_2 & K_3 \end{bmatrix}$ so that the closed-loop poles are at -1 and $-1j\pm$ and the steady-state error to and to a (step) position command will be zero. Let $y = x$ and the reference input $r = y_{ref}$ be a constant. Draw a block diagram of your design showing the locations of the feedback gains . Assume that both $\dot{x}$ and x can be measured. Plot the response of the closed-loop system to a step command input and the response to a step change in the bias input. Verify that the system is type1. Use MATLAB (Simulink) software to simulate the system responses.

Solution:

The equations of motion are given by,

$$\begin{aligned} \ddot{x} - x &= u + w, \\ \dot{w} &= 0. \end{aligned}$$

A realization of these equations is,

$$\begin{aligned} \begin{bmatrix} \dot{x} \\ \ddot{x} \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w, \\ y &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \end{bmatrix}. \end{aligned}$$

In order to incorporate integral control, we augment the state vector with an integral state, x_I, such that,

$$\dot{x}_I = y - r.$$

With the augmented state vector, $\mathbf{z} = \begin{bmatrix} x_I & x & \dot{x} \end{bmatrix}^T$, the augmented state matrices become,

$$\mathbf{A}_a = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \ \mathbf{B}_a = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \ \mathbf{C}_a = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}.$$

The design of the state feedback vector, $\mathbf{K}$, is now done using the above augmented state matrices. For closed-loop poles of $s = -1, -1 \pm j$,

$$\det(s\mathbf{I} - \mathbf{A}_a + \mathbf{B}_a\mathbf{K}) = 0,$$

when,

$$\mathbf{K} = [\ K_1 \quad K_2 \quad K_3\ ] = [\ 2 \quad 5 \quad 3\ ].$$

This result can be verified using the MATLAB place command.

The closed-loop system is given by,

$$\begin{aligned} \dot{\mathbf{z}} &= (\mathbf{A}_a - \mathbf{B}_a\mathbf{K})\mathbf{z} + \mathbf{B}_a w + \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} r, \\ y &= \mathbf{C}_a\mathbf{z}. \end{aligned}$$

To show that the system is Type I, show that $y = 0$ for any constant w in the steady-state, i.e., $\dot{\mathbf{z}} = 0$. For the closed-loop system we have,

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -K_1 & 1-K_2 & -K_3 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} w.$$

This immediately gives $z_2 = 0$ and $y = z_2$. Thus, in steady-state $y = 0$ for any constant w in the steady-state. The figure below shows a simulation of the closed-loop system to a commanded step r, at $t = 0$. At $t = 8$, a step in the constant bias is applied. This figure was generated using the MATLAB lsim command.

Problem 7.57: Response of closed-loop system to a unit step input at $t = 0$ and step disturbance at $t = 8$.

The simulation of the closed-loop system in Simulink is shown on the next page.

r
To Workspace
e
To Workspace2
y
To Workspace1
Scope2
Scope1
Out1
Gain5
Step
r
e
Integrator
-2
Gain1
u
Integrator2
x2
Integrator3
x1
Scope
-3
Gain2
-5
Gain3

Simulink simulation for Problem 7.57.

58. Consider a system with state matrices,

$$\mathbf{A} = \begin{bmatrix} -2 & 1 \\ 0 & -3 \end{bmatrix}, \ \mathbf{B} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \ \mathbf{C} = \begin{bmatrix} 1 & 3 \end{bmatrix}.$$

a) Use feedback of the form $u = -\mathbf{K}\mathbf{x}(t) + \bar{N}r(t)$, where $\bar{N}$ is a nonzero scalar, to move the poles to $-3 \pm 3j$.
b) Choose $\bar{N}$ so that if r is a constant, the system has zero steady-state error; that is $y(\infty) = r$.
c) Show that if $\mathbf{A}$ changes to $\mathbf{A} + \delta\mathbf{A}$, where $\delta\mathbf{A}$ is an arbitrary 2×2 matrix, then your choice of in part(b) will no longer make $y(\infty) = r$. Therefore, the system is not robust under changes to the system parameters in $\mathbf{A}$.
d) The system steady-state error performance can be made robust by augmenting the system with an integrator and using unity feedback; that is, by setting $\dot{x}_I = r - y$, where x_I is the state of the integrator. To see this, first use state feedback of the form $u = -\mathbf{K}\mathbf{x} - K_1 x_I$ so that the poles of the augmented system are at $-3, -2 \pm j\sqrt{3}$.
e) Show that the resulting system will yield $y(\infty) = r$ no matter how the matrices $\mathbf{A}$ and $\mathbf{B}$ are changed, as long as the closed-loop system remains stable.
f) For part (d), use MATLAB (Simulink) software to plot the time response of the system to a constant input. Draw Bode plots of the controller as well as the sensitivity function ($\mathcal{S}$) and the complementary sensitivity function ($\mathcal{T}$).

Solution:

(a) Using feedback of the form, $u = -\mathbf{K}\mathbf{x} + Nr$ we have,

$$\det(s\mathbf{I} - \mathbf{A} + \mathbf{B}\mathbf{K}) = (s + 2 + k_1)(s + 3 + k_2) + k_1(1 - k_1) = s^2 + 6s + 18,$$

when $\mathbf{K} = \begin{bmatrix} 5 & -4 \end{bmatrix}$. This result can be verified using the MATLAB place command.

(b) We can find the desired value for N by setting the DC gain from r to y equal to unity. The closed-loop system equations are,

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}(-\mathbf{K}\mathbf{x} + Nr) = (\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{x} + \mathbf{B}Nr, \\ y &= \mathbf{C}\mathbf{x}. \end{aligned}$$

Therefore, the transfer function is,

$$D_c(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A} + \mathbf{B}\mathbf{K})^{-1}\mathbf{B}N,$$

and the DC gain is simply,

$$D_c(0) = \mathbf{C}(-\mathbf{A} + \mathbf{B}\mathbf{K})^{-1}\mathbf{G}N = \frac{5}{9}N = 1.$$

Hence, we choose $N = \frac{9}{5}$.

(c) Change $\mathbf{A}$ to $\mathbf{A} + \delta\mathbf{A}$, and let the value of N that keeps the tracking error at zero be N'. Then letting $T'(s)$ be the transfer function associated with the perturbed system,

$$\begin{aligned} N'^{-1} &= T'(0) = -\mathbf{C}(\mathbf{A} + \delta\mathbf{A} - \mathbf{B}\mathbf{K})^{-1}\mathbf{B}, \\ &= -\mathbf{C}[(\mathbf{A} - \mathbf{B}\mathbf{K})(\mathbf{I} - (\mathbf{A} - \mathbf{B}\mathbf{K})^{-1}\delta\mathbf{A})]^{-1}\mathbf{B}, \\ &= -\mathbf{C}(\mathbf{I} - (\mathbf{A} - \mathbf{B}\mathbf{K})^{-1}\delta\mathbf{A})^{-1}(\mathbf{A} - \mathbf{B}\mathbf{K})^{-1}\mathbf{B}. \end{aligned}$$

For $\delta\mathbf{A}$small,

$$(\mathbf{I}-(\mathbf{A}-\mathbf{BK})^{-1}\delta\mathbf{A})^{-1}=\mathbf{I}+(\mathbf{A}-\mathbf{BK})^{-1}\delta\mathbf{A}.$$

Hence,

$$N^{'-1}=\underbrace{-\mathbf{C}(\mathbf{A}-\mathbf{BK})^{-1}\mathbf{B}}_{N^{-1}}-\mathbf{H}(\mathbf{A}-\mathbf{BK})^{-1}\delta\mathbf{A}(\mathbf{A}-\mathbf{BK})^{-1}\mathbf{B}.$$

And for arbitrary $\delta\mathbf{A}$ we arrive at,

$$N^{'-1}\neq N^{-1}.$$

Therefore, small changes in the plant matrix $\mathbf{A}$ prevent the steady-state error from reaching zero. The control system is not robust with respect to changes in $\mathbf{A}$.

(d) Augmenting the system equations with an integrator state, x_I, the state equation become,

$$\begin{aligned}\begin{bmatrix}\dot{\mathbf{x}}\\ \dot{x}_I\end{bmatrix} &= \begin{bmatrix}\mathbf{A} & \mathbf{0}\\ -\mathbf{C} & 0\end{bmatrix}\begin{bmatrix}\mathbf{x}\\ x_I\end{bmatrix}+\begin{bmatrix}\mathbf{B}\\ 0\end{bmatrix}u+\begin{bmatrix}\mathbf{0}\\ 1\end{bmatrix}r,\\ y &= [\mathbf{C}\ \ \mathbf{0}]\begin{bmatrix}\mathbf{x}\\ x_I\end{bmatrix}.\end{aligned}$$

or with $\mathbf{z}=\begin{bmatrix}\mathbf{x}^T & x_I\end{bmatrix}^T$,

$$\begin{aligned}\dot{\mathbf{z}} &= \mathbf{A}_a\mathbf{z}+\mathbf{B}_a u+\mathbf{B}_r r,\\ y &= \mathbf{C}_a\mathbf{z}.\end{aligned}$$

Using feedback of the form $u=-\mathbf{Kx}-k_I x_I=-\mathbf{K}_a\mathbf{z}$, we have,

$$\det(s\mathbf{I}-\mathbf{A}_a+\mathbf{B}_a\mathbf{K}_a)=0 \quad \text{for} \quad s=-3,-2\pm j\sqrt{3},$$

when $\mathbf{K}_a=\begin{bmatrix}0.3 & 1.7 & -2.1\end{bmatrix}$. This result can be verified using the MATLAB place command.

(e) We can show that the closed-loop DC gain from r to y is independent of $\mathbf{A}$,

$$\begin{aligned}y_\infty &= T(0)r_\infty=[\mathbf{C}\ \ \mathbf{0}]\begin{bmatrix}-\mathbf{A}+\mathbf{BK} & \mathbf{G}k_I\\ \mathbf{C} & \mathbf{0}\end{bmatrix}^{-1}\begin{bmatrix}\mathbf{0}\\ 1\end{bmatrix}r_\infty\\ &= [\mathbf{C}\ \ \mathbf{0}]\begin{bmatrix}* & (\mathbf{A}-\mathbf{BK})^{-1}\mathbf{B}k_I[\mathbf{C}(\mathbf{A}-\mathbf{BK})^{-1}\mathbf{G}k_I]^{-1}\\ * & *\end{bmatrix}\begin{bmatrix}\mathbf{0}\\ 1\end{bmatrix}r_\infty\\ &= [\mathbf{C}(\mathbf{A}-\mathbf{BK})^{-1}\mathbf{B}k_I][\mathbf{C}(\mathbf{A}-\mathbf{BK})^{-1}\mathbf{G}k_1]^{-1}r_\infty=r_\infty \ \text{ independent of } \ \mathbf{A},\mathbf{B},\mathbf{C}\ .\end{aligned}$$

(f) The simulation of the closed-loop system in Simulink is shown on the next page. The closed-loop step response is shown next. The Bode plot of the controller, the sensitivity function ($\mathcal{S}$), as well as the complementary sensitivity function ($\mathcal{T}$), are also shown.

Simulink simulation for Problem 7.58.

Closed-loop step response for Problem 7.58.

Bode plot of the controller for Problem 7.58.

Bode plot of the sensitivity function for Problem 7.58.

59. Consider a servomechanism for following the data track on a computer-disk memory system. Because of various unavoidable mechanical imperfections, the data track is not exactly a centered circle, and thus the radial servo must follow a sinusoidal input of radian frequency ω_0 (the spin rate of the disk). The state matrices for a linearized model of such a system are

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}, \ \mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \ \mathbf{C} = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

The sinusoidal reference input satisfies $\ddot{r} = -\omega_0^2 r$.
a) Let $\omega_0 = 1$, and place the poles of the error system for an internal model design at,

$$\alpha_c(s) = (s + 2 \pm j2)(s + 1 \pm 1j),$$

and the pole of the reduced-order estimator at

$$\alpha_e(s) = (s + 6).$$

b) Draw a block diagram of the system, and clearly show the presence of the oscillator with

Figure 7.106: Bode plot of the complementary sensitivity function for Problem 7.58.

frequency ω_0 (the internal model) in the controller. Also verify the presence of the blocking zeros at $\pm j\omega_0$.
c) Use MATLAB (Simulink) software to plot the time response of the system to a sinusoidal input at frequency $\omega_0 = 1$.
d) Draw a Bode plot to show how this system will respond to sinusoidal inputs at frequencies different from but near ω_0.

Solution:

(a) The compensator design consists of two parts: a feedback design using an internal model approach, and a reduced-order estimator. Let $\mathbf{x}$ be the plant state vector where,

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u.$$

Since the reference input satisfies $\ddot{r} = -\omega_0^2 r$, we can write out the error-state equations (using $e = y - r$) as,

$$\dot{\mathbf{z}} = \mathbf{A_s}\mathbf{z} + \mathbf{B_s}\mu,$$

with,

$$\mathbf{z} = \begin{bmatrix} e \\ \dot{e} \\ \xi \\ \dot{\xi} \end{bmatrix}, \ \mathbf{A_s} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \ \mathbf{B_s} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

With $\mu = -\mathbf{K}\mathbf{z}$, we can find $\mathbf{K}$ from pole placement. In this case,

$$\det(s\mathbf{I} - \mathbf{A_s} + \mathbf{B_s}\mathbf{K}) = 0.$$

when $s = -2 \pm j2, -1 \pm j1$, for,

$$\begin{aligned} \mathbf{K} &= [K_2 \quad K_1 \quad K_{01} \quad K_{02}] \\ &= \begin{bmatrix} -1. & 18 & 17 & 5 \end{bmatrix}. \end{aligned}$$

This result can be verified using the MATLAB place command.

To design the estimator, consider the plant matrices given in the problem,

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_{aa} & \mathbf{A}_{ab} \\ \mathbf{A}_{ba} & \mathbf{A}_{bb} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} B_a \\ B_b \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

So that the equation for the estimate of x_2 only is,

$$\begin{aligned} \hat{x}_b &= \hat{x}_c + Ly, \\ \dot{\hat{x}}_c &= -(L+1)\hat{x}_b + u, \end{aligned}$$

where the value for L is chosen from the estimate error characteristic equation

$$\det(s - (\mathbf{A}_{bb} - L\mathbf{A}_{ab})) = s + 1 + L.$$

For an estimator pole at $s = -6$, we have $L = 5$.

(b) The block diagram for this compensator is given below. Note that the internal model of the oscillator is plainly seen.

Problem 7.59: Compensator structure for robust following of sinusoid using an internal model controller and reduced-order estimator.

To find the "blocking zeros", we compute the transfer function from r to e. The system equations from r to e are,

$$\begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\mathbf{x}}_I \\ \dot{\tilde{x}}_b \end{bmatrix} = \begin{bmatrix} \mathbf{A} - \mathbf{B}\mathbf{K}_{01} & \mathbf{B}\mathbf{C}_c & \mathbf{B}\mathbf{K}_{02} \\ \mathbf{B}_c\mathbf{C} & \mathbf{A}_c & 0 \\ 0 & 0 & -6 \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{x}_I \\ \tilde{x}_b \end{bmatrix} + \begin{bmatrix} \mathbf{B} \\ -\mathbf{B}_c \\ 0 \end{bmatrix} r,$$

$$e = \begin{bmatrix} \mathbf{C} & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{x}_I \\ \tilde{x}_b \end{bmatrix} - r.$$

where,

$$\mathbf{A}_c = \begin{bmatrix} 0 & 1 \\ -\omega_0^2 & 0 \end{bmatrix}, \ \mathbf{B}_c = \begin{bmatrix} -K_1 \\ -K_2 \end{bmatrix},$$
$$\mathbf{C}_c = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

The transmission zeros of this system realization are at $s = -6, -3 \pm j2.645, \pm j$ with the last two as the blocking zeros. The zero locations are computed using the MATLAB tzero command.

(c) The time response of the closed-loop system subjected to a sinusoid at a frequency of ω_0 is shown below using the MATLAB lsim command. The simulation of the closed-loop system in Simulink is shown on the next page.

Problem 7.59: Time history of closed-loop system with a sinusoidial input.

Simulink simulation for Problem 7.59.

Problem 7.59: Bode plot of closed-loop system for sinusoidal following.

(d) A Bode plot of the compensated system is given above.

60. Compute the controller transfer function (from $Y(s)$ to $U(s)$) in Example 7.38. What is the prominent feature of the controller that allows tracking and disturbance rejection?

Solution:

The related equations from the Text are,

$$\begin{aligned}
\dot{\hat{\rho}} &= l_1(e - \hat{x}), \\
\dot{\hat{x}} &= -3\hat{x} + \hat{\rho} + u + l_2(e - \hat{x}), \\
u &= -K\hat{x} - \hat{\rho}.
\end{aligned}$$

To find the transfer function from $Y(s)$ to $U(s)$, we re-write the equations as,

$$\begin{aligned}
\begin{bmatrix} \dot{\hat{\rho}} \\ \dot{\hat{x}} \end{bmatrix} &= \begin{bmatrix} 0 & -l_1 \\ 0 & -3 - K - l_2 \end{bmatrix} \begin{bmatrix} \hat{\rho} \\ \hat{x} \end{bmatrix} + \begin{bmatrix} l_1 \\ l_2 \end{bmatrix} y, \\
u &= \begin{bmatrix} -1 & -K \end{bmatrix} \begin{bmatrix} \hat{\rho} \\ \hat{x} \end{bmatrix}.
\end{aligned}$$

The controller transfer function is,

$$\frac{U(s)}{Y(s)} = \frac{-(l_1 + Kl_2)s - (3l_1 + Kl_1)}{s(s + 3 + K + l_2)} = \frac{-279(s + 4.0323)}{s(s + 32)},$$

and shows the presence of an *integrator*!

61. Consider the pendulum problem with control torque T_c and disturbance torque T_d:

$$\ddot{\theta} + 4\theta = T_c + T_d,$$

(Here $\frac{g}{\ell} = 4$.) Assume that there is a potentiometer at the pin that measures the output angle θ, but with a constant unknown bias b. Thus the measurement equation is $y = \theta + b$.
a) Take the "augmented" state vector tobe

$$\begin{bmatrix} \theta \\ \dot{\theta} \\ w \end{bmatrix},$$

where w is the input-equivalent bias. Write the system equations in state-space form. Give values for the matrices $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$.

b) Using state-variable methods, show that the characteristic equation of the model is $s(s^2+4) = 0$.
c) Show that w is observable if we assume that $y = \theta$, and write the estimator equations for

$$\begin{bmatrix} \hat{\theta} \\ \dot{\hat{\theta}} \\ \hat{w} \end{bmatrix}.$$

Pick estimator gains to place all the roots of the estimator-error characteristic equation at -10.
d) Using full-state feedback of the estimated (controllable) state-variables, derive a control law to place the closed-loop poles at $-2 \pm 2j$.
e) Draw a block diagram of the complete closed-loop system (estimator, plant, and controller) using integrator blocks.
f) Introduce the estimated bias into the control so as to yield zero steady-state error to the output bias b. Demonstrate the performance of your design by plotting the response of the system to a step change in b; that is, b changes from 0 to some constant value.

Solution:

(a) The system with equivalent input disturbance which replaces the actual disturbance, b, with the equivalent disturbance at the control input is,

$$\begin{aligned} \ddot{\theta} &= -4\theta + T_c + w + T_d, \\ \dot{w} &= 0, \\ y &= \theta. \end{aligned}$$

In state-space form,

$$\begin{bmatrix} \dot{\theta} \\ \ddot{\theta} \\ \dot{w} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -4 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \theta \\ \dot{\theta} \\ w \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} T_c + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} T_d,$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \theta \\ \dot{\theta} \\ w \end{bmatrix},$$

(b)

$$\det(s\mathbf{I}-\mathbf{A}) = \det\begin{bmatrix} s & -1 & 0 \\ 4 & s & -1 \\ 0 & 0 & s \end{bmatrix} = s(s^2+4) = 0.$$

(c) Forming the observability matrix, we have,

$$\mathcal{O} = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \mathbf{CA}^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}.$$

Clearly, $\det(\mathcal{O}) = 1 \neq 0$ so that w is observable. The estimator gains can be determined by solving,

$$\det(s\mathbf{I}-\mathbf{A}+\mathbf{LC}) = (s+10)^3 = s^3+30s^2+300s+1000,$$

We find that $\mathbf{L} = \begin{bmatrix} 30 & 296 & 1000 \end{bmatrix}^T$. This result can be verified using MATLAB's place command.

(d) The bias state variable w is not controllable, we cannot move the pole at 0. So state feedback should place the poles at $-2 \pm 2j$ and 0. Equating,

$$\det(s\mathbf{I}-\mathbf{A}+\mathbf{BK}) = s(s^2+4s+8),$$

we find that $\mathbf{K} = \begin{bmatrix} 4 & 4 & 0 \end{bmatrix}$. (Note: the zero was chosen arbitrarily).

(e) The equations shown in the figure are,

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{Ax} + \mathbf{B}T_c + \mathbf{C}T_d, \\ \dot{\hat{\mathbf{x}}} &= \mathbf{A}_{aug}\hat{\mathbf{x}} + \mathbf{B}_{aug}T_c + \mathbf{L}(y - \mathbf{C}\hat{\mathbf{x}}), \\ y &= \mathbf{Cx}, \\ T_c &= -\mathbf{K}\hat{\mathbf{x}} + r. \end{aligned}$$

Block diagram for Problem 7.61(e).

(f) The performance of the system is shown to a step change in the bias b from 0 to 1 at $t = 0$ sec and then another step change from 1 to 0.2 at $t = 5$ sec.

Problem 7.61: Step change in bias.

Problems for Section7.10.3: Model-following Design

62. Consider the servomechanism problem where we wish to track a ramp reference signal. The plant and the desired model equations are

$$\begin{aligned} \dot{\mathbf{x}} &= \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ y &= \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x} \end{aligned}$$

$$\begin{aligned} \dot{\mathbf{x}}_m &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \mathbf{x}_m \\ y_m &= \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x}_m \end{aligned}$$

Design a model-following control law and demonstrate its tracking performance. Place the closed-loop poles at $s = -2 \pm j2$.

Solution:

```
% Model-following Trankle & Bryson Example
% Reference Model
Am=[0 1;0 0];
Bm=[0;1];
Cm=[1 0];
Dm=[0];
A=[0 1;0 -1];
B=[0;1];
C=[1 0];
D=[0];
% desired poles
pc=[-2+sqrt(-1)*2;-2-sqrt(-1)*2];
K=place(A,B,pc);
% Model-following Design
[n,m]=size(B);
[p,n]=size(C);
J=zeros(p,m);
[nm,nm]=size(Am);
aa=[kron(eye(nm),A)-kron(Am',eye(n)) kron(eye(nm),B); kron(eye(nm),C) kron(eye(nm),D)];
bb=[zeros(nm*n,1);Cm(:)];
xx=aa\bb;
M=reshape(xx(1:n*nm),n,nm)
% closed-loop response
AAm=[Am zeros(2,2);B*(N+K*M) A-B*K];
BBm=[Bm;zeros(2,1)];
CCm=[zeros(1,2) C ];
DDm=[0];
sysmf=ss(AAm,BBm,CCm,DDm);
% x0=[0;1;0;0];
t=0:.01:5;
hold on;
[ymf,t]=impulse(sysmf,t);
plot(t,ymf)
hold on;
plot(t,t,'–')
xlabel('Time (sec)');
```

```
ylabel('r,y');
title('Problem 7.62');
grid on;
```

$$\mathbf{K} = \begin{bmatrix} 8.0 & 3.0 \end{bmatrix}$$

Solving the related equations we find

$$\mathbf{M} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \mathbf{N} = \begin{bmatrix} 0 & 1 \end{bmatrix}.$$

The response of the system is shown in the ensuing figure.

Model-following ramp response for Problem 7.62.

63. ▲ *Implicit Model-Following*: Suppose we wish the closed-loop system to behave like a desired model, called the implicit model

$$\dot{\mathbf{z}} = \mathbf{A}_m \mathbf{z}.$$

We may minimize a modified LQR performance index

$$\mathcal{J} = \int_0^\infty \left\{ (\dot{y} - \mathbf{A}_m y)^T \mathbf{Q}_1 (\dot{y} - \mathbf{A}_m y) + u^T \mathbf{R} u \right\} dt.$$

Show that this performance index is equivalent to the standard one with the addition of a cross-weighting term between the control and the state of the form

$$\mathcal{J} = \int_0^\infty \left\{ \mathbf{x}^T \widehat{\mathbf{Q}} \mathbf{x} + 2u^T \widehat{\mathbf{S}} \mathbf{x} + u^T \widehat{\mathbf{R}} u \right\} dt,$$

where

$$\hat{\mathbf{Q}} = (\mathbf{CA} - \mathbf{A}_m \mathbf{C})^T \mathbf{Q}_1 (\mathbf{CA} - \mathbf{A}_m \mathbf{C}),$$

$$\widehat{\mathbf{S}} = \mathbf{B}^T \mathbf{C}^T \mathbf{Q}_1 (\mathbf{CA} - \mathbf{A}_m \mathbf{C}),$$

$$\widehat{\mathbf{R}} = \mathbf{R} + \mathbf{B}^T \mathbf{C}^T \mathbf{Q}_1 \mathbf{CB}.$$

Solution:

$$\begin{aligned}
\mathcal{J} &= \int_0^\infty \left\{ (\mathbf{C}\dot{\mathbf{x}} - \mathbf{A}_m \mathbf{C}\mathbf{x})^T \mathbf{Q}_1 (\mathbf{C}\dot{\mathbf{x}} - \mathbf{A}_m \mathbf{C}\mathbf{x}) + u^T \mathbf{R} u \right\} dt, \\
&= \int_0^\infty \left\{ (\mathbf{CAx} + \mathbf{CB}u - \mathbf{A}_m \mathbf{C}\mathbf{x})^T \mathbf{Q}_1 (\mathbf{CAx} + \mathbf{CB}u - \mathbf{A}_m \mathbf{C}\mathbf{x}) + u^T \mathbf{R} u \right\} dt, \\
&= \int_0^\infty \left\{ \begin{array}{c} \mathbf{x}^T (\mathbf{CA} - \mathbf{A}_m \mathbf{C})^T \mathbf{Q}_1 (\mathbf{CA} - \mathbf{A}_m \mathbf{C}) \mathbf{x} + \mathbf{x}^T (\mathbf{CA} - \mathbf{A}_m \mathbf{C})^T \mathbf{Q}_1 \mathbf{CB} u + u^T (\mathbf{CB})^T \mathbf{Q}_1 \mathbf{CB} (\mathbf{CA} - \mathbf{A_m C}) \mathbf{x} \\ + u^T (\mathbf{CB})^{\mathbf{T}} \mathbf{Q_1} \mathbf{CB} u + u^T \mathbf{R} u \end{array} \right\} dt,
\end{aligned}$$

But since the second and third term in the integrand are both scalars and are equal

$$\begin{aligned}
\mathcal{J} &= \int_0^\infty \left\{ \mathbf{x}^T (\mathbf{CA} - \mathbf{A}_m \mathbf{C})^T \mathbf{Q}_1 (\mathbf{CA} - \mathbf{A}_m \mathbf{C}) \mathbf{x} + 2u^T (\mathbf{CB})^T \mathbf{Q}_1 \mathbf{CB} (\mathbf{CA} - \mathbf{A_m C}) \mathbf{x} + u^T (\mathbf{CB})^{\mathbf{T}} \mathbf{Q_1} \mathbf{CB} u + u^T \mathbf{R} u \right\} dt, \\
&= \int_0^\infty \left\{ \mathbf{x}^T \hat{\mathbf{Q}} \mathbf{x} + 2u^T \hat{\mathbf{S}} \mathbf{x} + u^T \hat{\mathbf{R}} u \right\} dt,
\end{aligned}$$

where

$$\begin{aligned}
\hat{\mathbf{Q}} &= (\mathbf{CA} - \mathbf{A}_m \mathbf{C})^T \mathbf{Q}_1 (\mathbf{CA} - \mathbf{A}_m \mathbf{C}), \\
\hat{\mathbf{S}} &= (\mathbf{CB})^T \mathbf{Q}_1 \mathbf{CB} (\mathbf{CA} - \mathbf{A_m C}), \\
\hat{\mathbf{R}} &= (\mathbf{CB})^{\mathbf{T}} \mathbf{Q_1} \mathbf{CB} + \mathbf{R}.
\end{aligned}$$

1. 64. ▲ *Explicit Model-Following*: Suppose in the LQR problem, we wish the closed-loop system to behave as close as possible to a system of the form

$$\dot{\mathbf{z}} = \mathbf{A}_m \mathbf{z},$$

which represents the model of desirable dynamics. We may choose a performance index of the form

$$\mathcal{J} = \int_0^\infty \left\{ (y - \mathbf{z})^T \mathbf{Q}_1 (y - \mathbf{z}) + u^T \mathbf{R} u \right\} dt.$$

(a) Show that this performance index can be converted to the standard one by augmenting the states of the plant and the model i.e. choose the augmented state vector, $\xi = \begin{bmatrix} \mathbf{x}^T & \mathbf{z}^T \end{bmatrix}^T$ and write down the system equations to show that

$$\mathcal{J} = \int_0^\infty \left\{ \xi^T \mathbf{Q}_1 \xi + u^T \mathbf{R} u \right\} dt.$$

where

$$\mathbf{Q} = \begin{bmatrix} \mathbf{C}^T \mathbf{Q}_1 \mathbf{C} & -\mathbf{C}^T \mathbf{Q}_1 \\ -\mathbf{Q}_1 \mathbf{C} & \mathbf{Q}_1 \end{bmatrix}.$$

(b) Which states of the system are uncontrollable? Is this result surprising?

(c) The optimal control is of the form

$$u = -\mathbf{K}_1 \mathbf{x} - \mathbf{K}_2 \mathbf{z},$$

which means that the model's equations must be implemented as part of the control law. Suppose we now drive the model as follows

$$\dot{\mathbf{z}} = \mathbf{A}_m \mathbf{z} + \mathbf{B}_p u_p,$$

where u_p may be the pilot input in an aircraft system. Show that

$$\frac{Y(s)}{U_p(s)} = \underbrace{-\mathbf{C}(s\mathbf{I} - \mathbf{A} + \mathbf{B}\mathbf{K}_1)^{-1}\mathbf{B}}_{Closed-loop\ dynamics} \underbrace{\mathbf{K}_2(s\mathbf{I} - \mathbf{A}_m)^{-1}\mathbf{B}_p}_{Feedforward\ dynamics}.$$

This indicates that the feedforward dynamics may be used to improve the transient response of the system.

(d) What are the transmission zeros of the overall system?

(e) What is a possible disadvantage of this scheme compared to the standard LQR, i.e., with no explicit model?

Solution:

(a)

$$\dot{\xi} = \begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\mathbf{z}} \end{bmatrix} = \begin{bmatrix} \mathbf{A} & 0 \\ 0 & \mathbf{A}_m \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{z} \end{bmatrix} + \begin{bmatrix} \mathbf{B} \\ \mathbf{0} \end{bmatrix} u$$

$$
\begin{aligned}
\mathcal{J} &= \int_0^\infty \left\{ (\mathbf{Cx}-\mathbf{z})^T \mathbf{Q}_1 (\mathbf{Cx}-\mathbf{z}) + u^T \mathbf{R} u \right\} dt, \\
&= \int_0^\infty \left\{ \mathbf{x}^T \mathbf{C}^T \mathbf{Q}_1 \mathbf{Cx} - \mathbf{x}^T \mathbf{C}^T \mathbf{Q}_1 \mathbf{z} - \mathbf{z}^T \mathbf{Q}_1 \mathbf{Cx} + \mathbf{z}^T \mathbf{Q}_1 \mathbf{z} + u^T \mathbf{R} u \right\} dt, \\
&= \int_0^\infty \left\{ \begin{bmatrix} \mathbf{x} & \mathbf{z} \end{bmatrix}^T \begin{bmatrix} \mathbf{C}^T \mathbf{Q}_1 \mathbf{C} & -\mathbf{C}^T \mathbf{Q}_1 \\ -\mathbf{Q}_1 \mathbf{C} & \mathbf{Q}_1 \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{z} \end{bmatrix} + u^T \mathbf{R} u \right\} dt.
\end{aligned}
$$

(b) The modes (eigenvalues) of $\mathbf{A}_m$ are uncontrollable. No surprise as control does not affect those modes.

(c)

$$
\begin{aligned}
\dot{\xi} &= \begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\mathbf{z}} \end{bmatrix} = \begin{bmatrix} \mathbf{A}-\mathbf{BK_1} & -\mathbf{BK}_2 \\ \mathbf{0} & \mathbf{A}_m \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{z} \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ \mathbf{B}_p \end{bmatrix} u_p, \\
y &= \begin{bmatrix} \mathbf{C} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{z} \end{bmatrix}.
\end{aligned}
$$

$$
\frac{Y(s)}{U_p(s)} = \begin{bmatrix} \mathbf{C} & \mathbf{0} \end{bmatrix} \begin{bmatrix} s\mathbf{I}-\mathbf{A}+\mathbf{BK_1} & +\mathbf{BK}_2 \\ \mathbf{0} & s\mathbf{I}-\mathbf{A}_m \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{0} \\ \mathbf{B}_p \end{bmatrix},
$$

$$
\begin{aligned}
\frac{Y(s)}{U_p(s)} &= \begin{bmatrix} \mathbf{C} & \mathbf{0} \end{bmatrix} \begin{bmatrix} (s\mathbf{I}-\mathbf{A}+\mathbf{BK_1})^{-1} & (s\mathbf{I}-\mathbf{A}+\mathbf{BK_1})^{-1}\mathbf{BK}_2(s\mathbf{I}-\mathbf{A}_m)^{-1}\mathbf{B}_p \\ \mathbf{0} & (s\mathbf{I}-\mathbf{A}_m)^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{0} \\ \mathbf{B}_p \end{bmatrix}, \\
&= \mathbf{C}(s\mathbf{I}-\mathbf{A}+\mathbf{BK_1})^{-1}\mathbf{BK}_2(s\mathbf{I}-\mathbf{A}_m)^{-1}\mathbf{B}_p.
\end{aligned}
$$

(d) The transmission zeros of the overall system are given by:

$$
\det \begin{bmatrix} s\mathbf{I}-\mathbf{A}+\mathbf{BK_1} & \mathbf{BK}_2 & \mathbf{0} \\ \mathbf{0} & s\mathbf{I}-\mathbf{A}_m & -\mathbf{B}_p \\ \mathbf{C} & \mathbf{0} & \mathbf{0} \end{bmatrix} = 0.
$$

(e) In the explicit model-following technique, the explicit model must be implemented as part of the control law, so it adds to implementation complexity.

Problems for Section7.13: Design for Systems with Pure Time Delay

65. ▲ Consider the system with the transfer function $e^{-Ts}G(s)$, where

$$
G(s) = \frac{1}{s(s+1)(s+2)}.
$$

1. The Smith compensator for this system is given by

$$
D_c'(s) = \frac{D_c(s)}{1+(1-e^{-sT})G(s)D_c(s)}.
$$

Plot the frequency response of the compensator for $T = 5$ and $D_c(s) = 1$, and draw a Bode plot that shows the gain and phase margins of the system.[2]

Solution:

This problem can be solved using a few different approaches. A computer tool such as MATLAB (see the MATLAB command pade) can be used to calculate the exact magnitude and phase of $D_c^{'}(s)$, or a pade approximation can be made for the e^{-sT} terms,

$$e^{-sT} \approx \frac{2 - sT + (-sT)^2/2! + (-sT)^3/3! + \ldots}{2 + sT + (sT)^2/2! + (sT)^3/3! + \ldots}.$$

We will show both the exact calculation and Bode plots using a fourth-order pade approximation. The Bode plot of the compensator

$$D_c(s) = \frac{1}{1 + [1 - e^{-sT}]G(s)},$$

appears in the figures, with,

$$G(s) = \frac{1}{s(s+1)(s+2)}, \quad \text{and } T = 5.$$

The Bode plot of the closed-loop system is also shown.

[2]This problem was given by Åström (1977).

Problem 7.65 Bode plot of compensator $D_c^{'}(s)$: exact.

Problem 7.65: Pade approximation.

Problem 7.65 Bode plot of closed-loop system: exact.

Problem 7.65 Bode plot of closed-loop system: Pade approximation.

Remark: Note that the Smith compensator, $D_c^{'}(s)$, is structured such that the closed-loop transfer function is,

$$\mathcal{T}(s) = \frac{G(s)}{1 + G(s)D_c(s)} e^{-Ts}.$$