TFES Lab (ME EN 4650)
Blackbody Radiation

Textbook Reference: Sections 12.1 – 12.4, 12.8, 13.1 from Incropera, DeWitt, Bergman, and Lavine, 6th ed., John Wiley & Sons

Objectives

(i) measure the radiation emitted from a blackbody using a pyroelectric detector,

(ii) examine the effects of detector location, blackbody temperature, and emitter source size on the measured radiation,

(iii) compare the experimental measurements to the theory for blackbody radiation using a 1:1 line plot and linear regression analysis.

Background

All matter at a nonzero absolute temperature, including gasses, solids, and liquids emit energy in the form of thermal radiation. The energy released is a result of oscillations and transitions of the electrons that constitute matter. For most solids and liquids, radiation is a surface phenomenon, with the emitted energy originating from molecules that are within a distance of about 1 μm from the exposed surface. Radiation is transported through electromagnetic waves that are characterized by their wavelength λ and frequency ν, which are related by the speed of light, $c = \lambda\nu$. Thermal radiation occurs over the wavelength band of 10^{-1}–10^{2} μm, and covers the entirety of the infrared and visible light ranges, as well as a portion of the ultraviolet range, as shown in Figure 1.

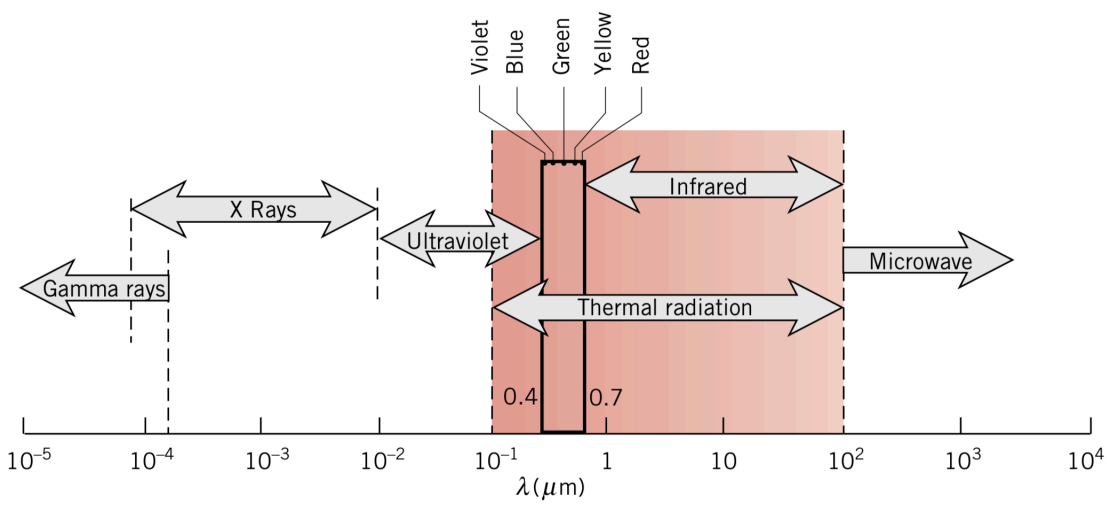

Figure 1. Electromagnetic wave spectrum, showing the wavelength range of thermal radiation (0.1–100 μm) relative to that of visible light (0.4–0.7 μm) and other types of waves.

Of particular interest in the present experiment, and in many engineering applications, is the radiation exchange between a surface at temperature T_s with its surroundings at temperature T_{sur}, as illustrated in Figure 2. In the figure, E represents the rate at which energy is emitted via radiation from the surface in W/m^2, and is referred to as surface *emissive power*; while, G represents the rate of radiation incident on a unit area of the surface and is referred to as the *irradiation*. The net radiation flux at the surface (q''_{rad}) is the difference between E and G, and is governed by the equation

$$q''_{\text{rad}} = \varepsilon\,\sigma(T_s^4 - T_{\text{sur}}^4), \tag{1}$$

where ε is the emissivity and $\sigma = 5.67\times10^{-8}$ $\text{W/m}^2\cdot\text{K}^4$ is the universal Stefan-Boltzmann constant. The emissivity is a radiative property of the surface, with a value in the range $0 \le \varepsilon \le 1$, that provides a measure of how efficiently a surface emits energy. The emissivity of a surface depends on the material, color, and type of surface finish (i.e., polished versus rough). As such, the method of fabrication of a part can effect its emissivity. Any body or surface that exhibits a value $\varepsilon = 1$ is called a blackbody. The net rate of radiation energy transfer in W from a body to its surroundings is given by

$$q_{\text{rad}} = A_s\, q''_{\text{rad}}, \tag{2}$$

where A_s represents the exposed surface area of the body.

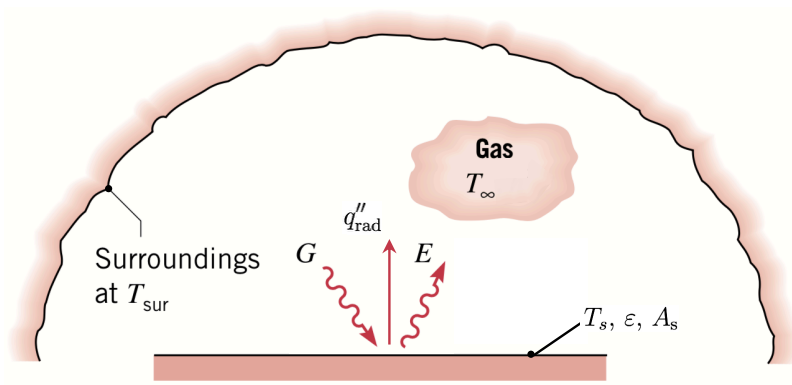

Figure 2. Schematic of a solid surface at a temperature T_s, and with surface area A_s, emitting thermal radiation through a gaseous medium, such as air, with a temperature T_∞ to the surroundings having a temperature T_{sur}. In most cases, T_∞ and T_{sur} are identical.

Experimental Setup

In this lab exercise, the radiative properties of a pseudo-blackbody will be examined. The radiative properties of a blackbody can be imitated in the laboratory using a heated cavity (see Appendix I). The temperature inside the cavity can be adjusted using a DC power supply and thermostat. Thermal radiation leaves the cavity through an aperture, the diameter of which can be controlled using a dial on the front face of the cavity. The thermal radiation emitted from the cavity is measured using a pyroelectric radiometer system (Oriel Model #7081), comprised of a pyroelectric detector, optical chopper wheel, and digital display unit.

A photograph and schematic of the experimental setup are shown in Figure 3. In order to simplify comparison with theory, the center of the aperture of the blackbody cavity (i.e., the source of emitted radiation) and the center of the detector sensing element are aligned along the unit normal to the aperture surface area (horizontal dashed line in Figure 3). The electromagnetic ray emanating from the blackbody cavity is assumed to be coincident with this unit normal. Here, "ray" defines the straight line along which the electromagnetic wave travels between two points. The quantities of interest in the experiment are: the blackbody temperature T, the area of the emitting source or aperture surface area

$$A_{\mathrm{s}} = \tfrac{\pi}{4}\, D^2, \tag{3}$$

where D is the aperture diameter, the distance from the source to the detector h, and the area of the sensing element of the detector A_{d} $(=\ell^2)$, where ℓ is the side length of the sensing element.

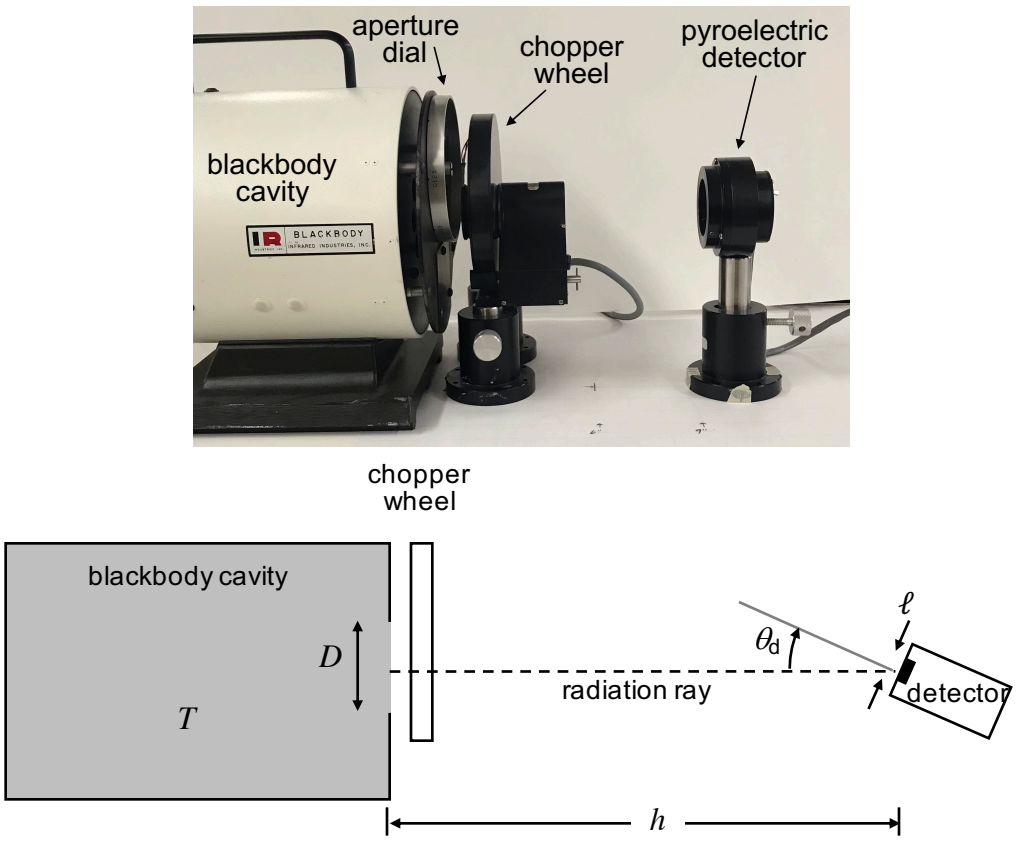

Figure 3. (top) Photograph of the blackbody radiation experiment showing the main components. (bottom) Schematic of the experimental setup from a top view. The dashed line represents the nominal path of the radiation ray emitted from the blackbody cavity and incident to the detector. The relevant quantities of interest are: T, the temperature inside the blackbody cavity; D, the aperture diameter on the blackbody cavity; h, the distance between the source and detector along the radiation ray; ℓ, the side length of the square sensing element on the detector; and, θ_d, the yaw angle between the unit normal to the detector surface area and the path of the ray.

Table 1. List of measurements acquired in the experiment with their native units.

Quantity	Symbol	Units	Instrument
Temperature of blackbody	T	°C	thermocouple
Separation distance	h	in	linear ruler
Angle of detector head	θ_{d}	deg	rotation table
Aperture diameter of source	D	in	markings on dial
Heat transfer rate	q	W	pyroelectric radiometer

In the lab, three sets of experiments will be performed. In each experiment, one quantity (T, D, or h) will be varied, while maintaining the others constant; and, the subsequent heat transfer rate to the detector (q) measured. Table 1 lists the quantities, along with their native units, that will be measured in the lab.

The pyroelectric radiometer system used in the lab is a highly sensitive instrument capable of measuring very small thermal radiation heat transfer rates in the range 10^{-2} to 10^{-6} W. A knob on the display panel is used to switch between different gain settings. The detector head is comprised of an infrared optoelectronic sensor chip (see Figure 4) that can detect electromagnetic radiation in the wavelength band from 2–14 μm. The surface area of the sensing chip is $A_{\mathrm{d}} = 1$ mm^2. The sensor chip is made of a thin pyroelectric crystal (e.g., gallium nitride, lithium tantalate, and tourmaline) that has an electric polarization, even in the absence of an applied voltage. Upon heating, the material expands and produces a change in polarization, which builds up a charge on opposite surfaces of the crystal layer. If the crystal layer is sandwiched between two electrodes connected to a closed circuit, a current will flow across the crystal layer. Since it is the change in temperature that produces the current, pyroelectric detectors respond only to pulsed or modulated radiation. They respond to variations in radiation much more rapidly than thermocouples, and are unaffected by steady background radiation.

Because the charge across the pyroelectric crystal is small, sophisticated electronics (including a low-noise, low-leakage operational amplifier) are needed to convert the charge to a voltage signal. In addition, because the pyroelectric crystal is only sensitive to changes in

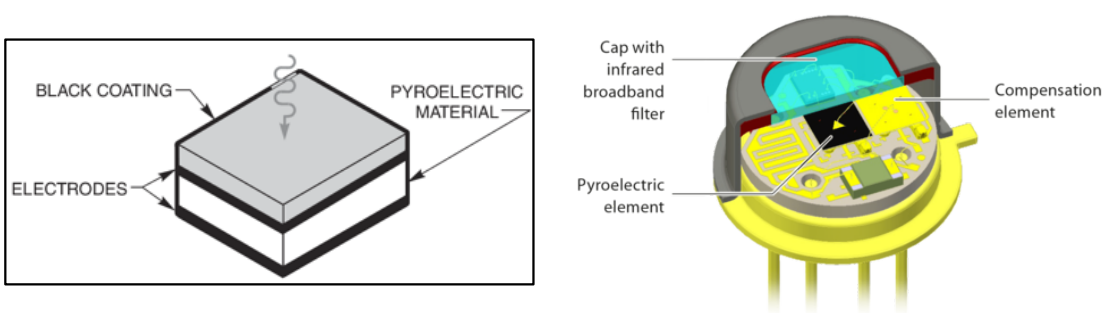

Figure 4. (left) Illustration of a pyroelectric sensing element. (right) Typical configuration of a pyroelectric detector with the sensing element mounted on an integrated circuit board and covered with a infrared filter.

temperature, a chopper wheel (also known as a fixed-frequency beam modulator) is used to temporarily block the aperture on the blackbody cavity, thereby effectively creating a square wave pulse of thermal radiation on the detector sensor chip. The chopper wheel is driven by a synchronous motor and can be set to either a frequency of 18.667 Hz or 93.337 Hz. Note, the detector head toggle switch should be in the proper position to match the chopper frequency (HI for 93.337 Hz, or LO for 18.667 Hz).

Laboratory Procedure

1. Plug in the power supply for the fan to the blackbody cavity.
2. Verify that the chopper wheel frequency is set to the same value as the selection switch on the detector head.
3. Perform Experiment #1 (vary h, with all other variables fixed):
   a. Set the temperature of the blackbody cavity to a value around 460°C.
   b. Wait for the temperature on the display panel of the power supply to reach steady state (about 10 minutes). Note, the temperature control unit is not very precise and will likely overshoot the desired temperature by about 10°C. This is fine, simply record the actual temperature on your data sheet once the system reaches steady state.
   c. Set the angular orientation of the detector so that $\theta_d = 0°$, i.e., the unit normal to the detector face is colinear with the unit normal to the blackbody cavity face.
   d. Set the aperture on the blackbody cavity to a diameter of $D = 0.6$ in.
   e. Slide the detector along the rail to a position $h = 6$ in. from the blackbody cavity aperture.
   f. Adjust the gain/multiplier knob on the radiometer instrument box to ensure that the display reading is between ± 0.000 – 1.999. Record the measured q value on your data sheet. Be sure to note the gain/multiplier setting.
   g. Slide the detector to a new separation distance. Repeat the measurements for h values up to 12 in. in increments of 1 in.
4. Perform Experiment #2 (vary T, with all other variables fixed):
   a. Set the detector to a position $h = 9$ in., the detector angular orientation to $\theta_d = 0°$, and the aperture diameter to $D = 0.6$ in.
   b. Set the temperature of the blackbody cavity to a value around 700°C.
   c. As the blackbody cavity temperatures increases, record the radiation heat transfer rate reading on the radiometer display panel every 20°C increments. Be sure to note the gain/multiplier setting. Note, the temperature increases very quickly initially.

5. Perform Experiment #3 (vary D, with all other variables fixed):
   a. Set the detector to a position $h = 9$ in., and the detector angular orientation to $\theta_d = 0°$.
   b. Wait for the temperature of the blackbody cavity to reach a steady-state value around 700°C.
   c. Record the actual steady-state temperature on your data sheet.
   d. Set the aperture on the blackbody cavity to a diameter of $D = 1$ in.
   e. Adjust the gain/multiplier knob on the radiometer instrument box to ensure that the display reading is between ± 0.000 – 1.999. Record the measured q value on your data sheet. Be sure to note the gain/multiplier setting.
   f. Rotate the aperture dial to a new diameter setting. Repeat the measurements for the following aperture settings: D=0.6, 0.4, 0.2, 0.1, and 0.05 in.

6. When finished with the last experiment, set the temperature controller back to 300°C and leave the temperature controller switch ON, in order to allow the blackbody cavity to cool to room temperature for the next lab section.

Data Analysis

In the data analysis procedures for this lab, you will consider how well the measured data agree with the theory of blackbody radiation, and perform curve fits to assess the agreement.

Theory of Blackbody Radiation

Figure 5 shows the geometry for a configuration involving a source of radiation (drawn in red) with differential area dA_s and a detector (drawn in blue) with differential area dA_d located a distance h from the source. The orientation of the surfaces relative to the direction of radiation is an important parameter in governing the rate of heat transfer exchanged between the two surfaces. The orientation is defined in terms of the angles θ_d (for the detector) and θ_s (for the source) measured between the unit normal to the surface and the ray delineating the direction of propagation of the radiation (drawn as a dashed line).

In this lab, the source is assumed to mimic the behavior of a blackbody. A "blackbody" has the special properties that (i) the amount of emitted radiation depends only on the surface temperature T, and (ii) the emitted radiation is independent of direction. The latter means that a blackbody is a *diffuse emitter*. For a blackbody, the total emissive power, E_b (with units of W/m^2), is given by the Stefan-Boltzmann law,

$$E_b = \sigma T^4 , \tag{4}$$

where T is the temperature of the blackbody in K, and $\sigma = 5.670 \times 10^{-8}$ W/m^2·K^4 is the universal Stefan-Boltzmann constant. Note, (4) represents the amount of power per unit area emitted from the blackbody in ALL directions. The electromagnetic intensity, I, of the emitted thermal radiation is the amount of power per unit area per unit radian. Since a

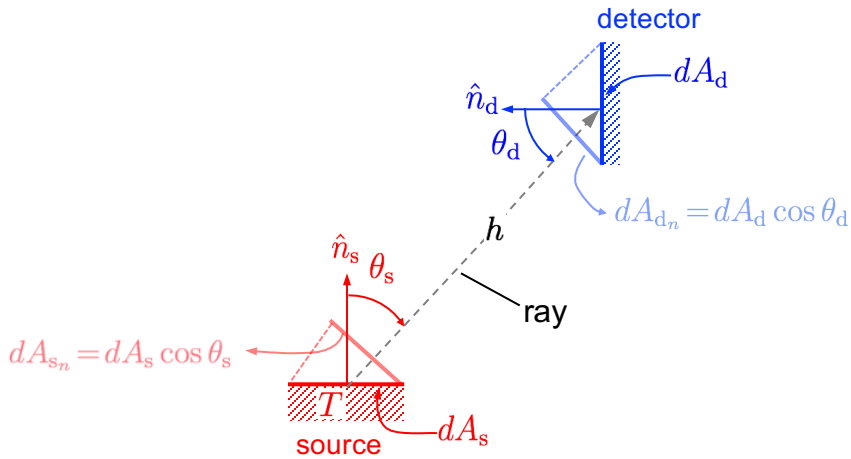

Figure 5. Typical geometry in a thermal radiation source-detector configuration. The ray delineating the direction of propagation of the electromagnetic wave transmitting the thermal radiation from the source to detector is denoted by the dashed line. The differential areas dA_{s} (for the source) and dA_{d} (for the detector) have unit normals $\hat{n}_{\mathsf{s}}$ and $\hat{n}_{\mathsf{d}}$, respectively. The differential areas dA_{d_n} and dA_{s_n} are the projections of dA_{d} and dA_{s}, respectively, perpendicular to the ray.

blackbody is a diffuse emitter, the emitted radiation is uniform with respect to angle. Thus, for a blackbody surface that radiates hemispherically from the surface to the surroundings, the total intensity of the emitted radiation is simply

$$I_b = \frac{E_b}{\pi} = \frac{\sigma\, T^4}{\pi}, \tag{5}$$

where the subscript b indicates "blackbody".

In general, the differential heat transfer rate at which thermal radiation emitted from the surface of a blackbody and absorbed by another surface some distance h away is given by

$$dq = I_b\, dA_{\mathrm{s}_n}\, d\omega \tag{6}$$

where $d\omega$ denotes the differential solid angle and dA_{s_n} represents the component of dA_{s} perpendicular to the electromagnetic ray emanating from the surface of the blackbody, as shown in Figure 5. A solid angle is the three-dimensional analog of a planar angle that measures how large an object appears to an observer looking from that point. Using the spherical coordinate system shown in Figure 6, we can say that the differential area dA_ω, located a distance h from the source and through which radiation passes, subtends the solid angle $d\omega$ when viewed from a point on the source dA_{s}. Mathematically, the solid angle is defined as $d\omega = dA_\omega/h^2$. Supplemental information about solid angles is provided in Appendix II.

In the present experiment, the solid angle indicates how large the detector appears to a point on the source. In terms of the source-detector geometry in Figure 5,

$$d\omega = \frac{dA_{\mathrm{d}_n}}{h^2}, \tag{7}$$

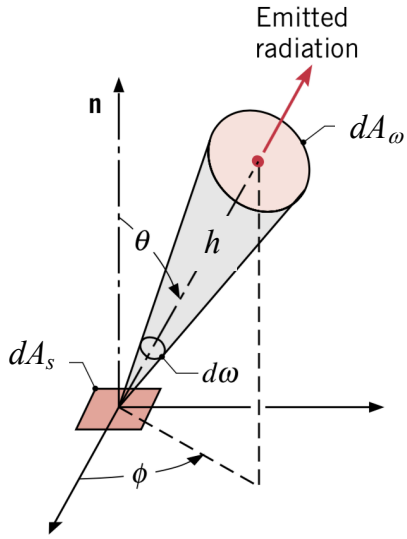

Figure 6. Coordinate system showing the definition of a differential solid angle $d\omega$. The area dA_ω through which the radiation passes subtends a differential solid angle $d\omega$ when viewed from a point on the dA_s of the emitting surface.

where dA_{d_n} is the differential area on the detector projected onto a plane perpendicular to the direction of radiation and h is the distance from the point of observation (i.e., the source) to the area of interest on the detector. From trigonometry, we can write the projected areas of the source and detector based on the actual areas as

$$dA_{\mathrm{d}_n} = dA_\mathrm{d}\,\cos\theta_\mathrm{d} \qquad \text{and} \qquad dA_{\mathrm{s}_n} = dA_\mathrm{s}\,\cos\theta_\mathrm{s}\,. \tag{8}$$

Therefore, the differential solid angle between the source and detector is

$$d\omega = \frac{dA_\mathrm{d}\,\cos\theta_\mathrm{d}}{h^2}. \tag{9}$$

Substituting (5), (8), and (9) into (6) yields an equation for the differential heat transfer rate from the blackbody source to the detector as

$$dq = \frac{\sigma\,T^4}{\pi\,h^2}\cos\theta_\mathrm{s}\,\cos\theta_\mathrm{d}\,dA_\mathrm{d}\,dA_\mathrm{s}\,. \tag{10}$$

In order to predict the rate of heat transfer from the blackbody to the detector, we need to integrate the above equation over both the source and detector surface area. Assuming the temperature of the blackbody is uniform over the area of the source yields

$$q = \sigma\,T^4 \underbrace{\int_{A_\mathrm{s}}\int_{A_\mathrm{d}} \frac{\cos\theta_\mathrm{s}\,\cos\theta_\mathrm{d}}{\pi\,h^2}\,dA_\mathrm{d}\,dA_\mathrm{s}}_{=A_\mathrm{s}\,F_{\mathrm{s}\to\mathrm{d}}}\,, \tag{11}$$

where A_s denotes the surface area of the source and $F_{\mathrm{s}\to\mathrm{d}}$ represents a <u>view factor</u> that is introduced, because the double integral can be very difficult to obtain analytically, even

for relatively simple configurations and shapes of the source and detector. Expressions and plots of $F_{\mathrm{s}\to\mathrm{d}}$ are tabulated for a wide range of configurations in textbooks and the primary literature. In the present laboratory experiment, since the detector chip is very small, it can be approximated as a differential area element (or patch); while, the source (i.e., blackbody cavity aperture) is essentially a coaxial disc in a plane parallel to the patch. For this configuration, we refer to the work of B. Chung and P. Sumitra, "Radiation shape factors from plane point sources," *J. of Heat Transfer*, v. 04, n. 3, 328–330 (1972), which yields a view factor expression of

$$F_{\mathrm{d}\to\mathrm{s}} = \frac{1}{1+\left(\frac{h}{r}\right)^2}, \tag{12}$$

where h is the distance between the source and detector along the ray, and r is the radius of the disc, as shown in Figure 7. Note, the expression in (12) is for $F_{\mathrm{d}\to\mathrm{s}}$, which assumes the patch is the source and the disc is the detector. To obtain the correct view factor when the source of radiation is the disc, as in the present experiments, we employ the following reciprocity relation

$$A_{\mathrm{s}}\, F_{\mathrm{s}\to\mathrm{d}} = A_{\mathrm{d}}\, F_{\mathrm{d}\to\mathrm{s}}\,.$$

Substituting the view factor expression for $F_{\mathrm{s}\to\mathrm{d}}$ from (12) and (13) into (11) and canceling the A_s quantities yields the theoretical equation for the heat transfer rate that we will compare against the measurements,

$$q_{\mathrm{theory}} = \frac{\sigma\, T^4\, A_d\, r^2}{(r^2+h^2)}\,. \tag{13}$$

Substituting the aperture diameter ($r = \frac{D}{2}$) into the above equation yields

$$q_{\mathrm{theory}} = \frac{\sigma\, T^4\, A_d\, D^2}{(D^2+4\,h^2)}\,. \tag{14}$$

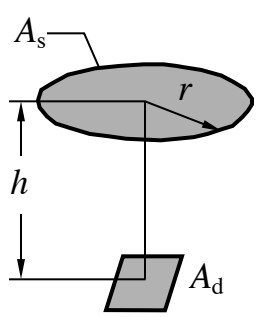

Figure 7. Schematic for the view factor formulation given in (12) between a differential planar element of area A_{d} parallel to a circular disc of area A_{s} and radius r, where the unit normal to the element passes through the center of the disc. The separation distance between the element and disc is h.

Curve Fitting

Based on the theoretical equation in (14), we expect that the measured heat transfer rate from the pyroelectric detector, q, should be linearly proportional to T^4 (cavity temperature), linearly proportional to D^2 (aperture diameter), and linearly proportional to h^{-2}. To test these expectations, we will plot the measured heat transfer rate, q_{data}, versus T^4, D^2, and h^{-2} on three separate plots. If q is indeed linearly proportional to these quantities, then the data should fall along a straight line in each plot. We will calculate the best fit linear regression line to the data for each case.

In its general form, the linear regression model can be written as

$$y = a_0 + a_1\, x \tag{15}$$

where $y = y_1, \ldots, y_N$ and $x = x_1, \ldots, x_N$ denote the data and a_0 (y-intercept) and a_1 (slope) are the curve fit parameters. For example, if we want to know whether q is proportional to T^4, then we would define $y_i = q_i$ and $x_i = T_i^4$, for $i = 1, \ldots, N$, and solve for the best-fit coefficients a_0, a_1 from (15). The linear regression analysis can be performed in Matlab using the polyfit command. To determine how well the data fit a straight line, we need to calculate the goodness-of-fit parameter, also referred to as the R^2 (R-squared) measure.

Recall that for linear regression, the R^2 value is defined as

$$R^2 = 1 - \frac{S_R}{S_T}, \tag{16}$$

where S_R denotes the sum of the square residuals and S_T denotes the sum of the total square variation. Note, R is limited to values between 0–1. A value of $R{=}1$ indicates that the data fit perfectly to a straight line. Whereas, a value of $R{=}0$ indicates that the best fit is simply the mean value of the data. Given a linear regression model of the general form as shown in (15), the mathematical expressions for S_R and S_T can be written as

$$S_R = \sum_{i=1}^{N} \left(y_i - a_0 - a_1\, x_i\right)^2 \qquad S_T = \sum_{i=1}^{N} \left(y_i - \overline{y}\right)^2 \,, \tag{17}$$

where N is the total number of data points and $\overline{y}$ represents the mean value of y.

Required Plots

1a. On a single figure, plot the measured heat transfer rate, q_{data}, from **all three experiments** versus the corresponding values predicted by theory, q_{theory}, as calculated from (14). Plot q_{theory} on the y-axis in units of μW, and q_{data} on the x-axis in units of μW. Use the following marker styles for each experiment: ○ Exp #1 (variable h); □ Exp #2 (variable T); ◇ Exp #3 (variable D). On the same figure, plot the 1:1 line as a solid black line. Include a legend.

Note, a 1:1 line is a line passing through the origin with a slope of 1. If the data agree exactly with the theory, then all data points should fall on the 1:1 line. Deviation of the data from this line indicates the accuracy and precision of the theory. However, this statement is only true if we are certain that the measurements have negligible error.

1b. Using only the data from **Experiment #1**, plot the measured heat transfer rate, q_{data}, versus the reciprocal of the squared separation distance between the source and detector, $1/h^2$. Plot q_{data} on the y-axis in units of μW, and h^{-2} on the x-axis in units of m^{-2}. Use ○ as the marker style. On the same figure, plot the linear regression line obtained from your analysis as a solid black line. Include a legend with the string "data" for the markers and contains the calculated R^2 value for the fit line in the following string format: "fit, R^2 =0.XXXX" where 0.XXXX is your R^2 value to within 4 decimal places.

Note, this type of plot allows one to see relatively quickly whether q is indeed proportional to h^{-2}. Deviation of the data points from the linear regression line reveals instances where the measurements do not agree with the derived theory of blackbody radiation. Note, this does not necessarily mean the theory is incorrect; it simply indicates an inconsistency.

1c. Using only the data from **Experiment #2**, plot the measured heat transfer rate, q_{data}, versus the measured blackbody temperature raised to the fourth power, T^4. Plot q_{data} on the y-axis in units of μW, and T^4 on the x-axis in units of K^4. Use □ as the marker style. On the same figure, plot the linear regression line obtained from your analysis as a solid black line. Include a legend that follows the format of plot 1b.

1d. Using only the data from **Experiment #3**, plot the measured heat transfer rate, q_{data}, versus the squared diameter of the aperture, D^2. Plot q_{data} on the y-axis in units of μW, and D^2 on the x-axis in units of mm^2. Use ◇ as the marker style. On the same figure, plot the linear regression line obtained from your analysis as a solid black line. Include a legend that follows the format of plot 1b.

Short-Answer Questions

2a. Quantify the agreement in q between the measurements and theory based on your results in plot 1a. State your answer in terms of an average percent relative difference, $\overline{\epsilon}$, according to the following relationship

$$\overline{\epsilon} = \frac{1}{N} \sum_{i=1}^{N} \frac{|q_{i_{\text{data}}} - q_{i_{\text{theory}}}|}{q_{i_{\text{theory}}}} \cdot 100\% \,, \tag{18}$$

where N is the total number of data points acquired (over all four experiments). Comment on whether the discrepancy between the measurements and theory seem to be random across the entire q range investigated, or whether there are particular q ranges where the agreement appears better/worse. [2–4 sentences]

2b. Given your results in plots in 1a–1d, including your calculated R^2 values for the curve fits, and using your engineering judgment, assess whether the theory adequately supports the measurements, in terms of the expected dependence of q on T, h, and D. Be specific in your explanation of why or why not. [2–3 sentences]

2c. Assume now that the blackbody cavity in the experiment is replaced with a diffuse "gray body", whose surface temperature can be adjusted uniformly with a heater and thermostat, in a manner similar to what was done in the actual experiment. Describe how you would modify the experimental procedures and/or data analysis in order to determine the unknown <u>emissivity</u> ε of the "gray body". [2–4 sentences]

APPENDIX I: Blackbody Cavity

A blackbody emitter can be simulated in the laboratory using a "blackbody cavity". Inside the cavity, heat is generated using a resistive heating fabric wrapped around the outside of a cylindrical tube. Because of the multiple internal reflections of the electromagnetic waves inside the tube, as illustrated in Figure 8, the spectral intensity I_λ of the radiation emitted from the cavity is independent of direction. So, the emission may be characterized as *diffuse*. The multiple internal reflections inside the cavity also produces an environment that is nearly isothermal with an effective emissivity very close to 1 (~ 0.995 typically). A technical drawing of the cavity used in the present experiments is shown in Figure 9 with relevant dimensions.

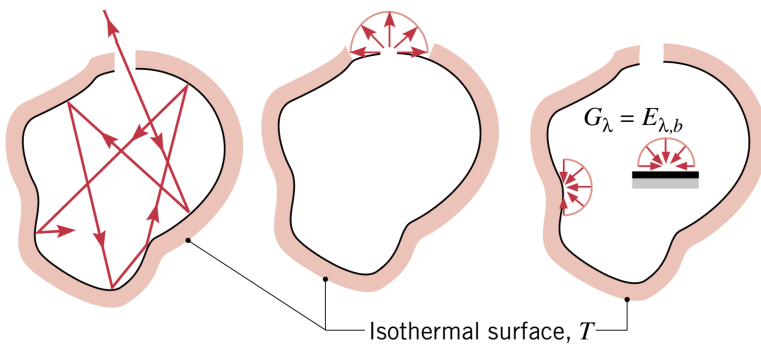

Figure 8. Due to the multiple internal reflections of the electromagnetic waves inside of the cavity, heat generated inside the cavity is emitted in a diffuse manner without any angular dependence, and closely mimics that expected from a blackbody.

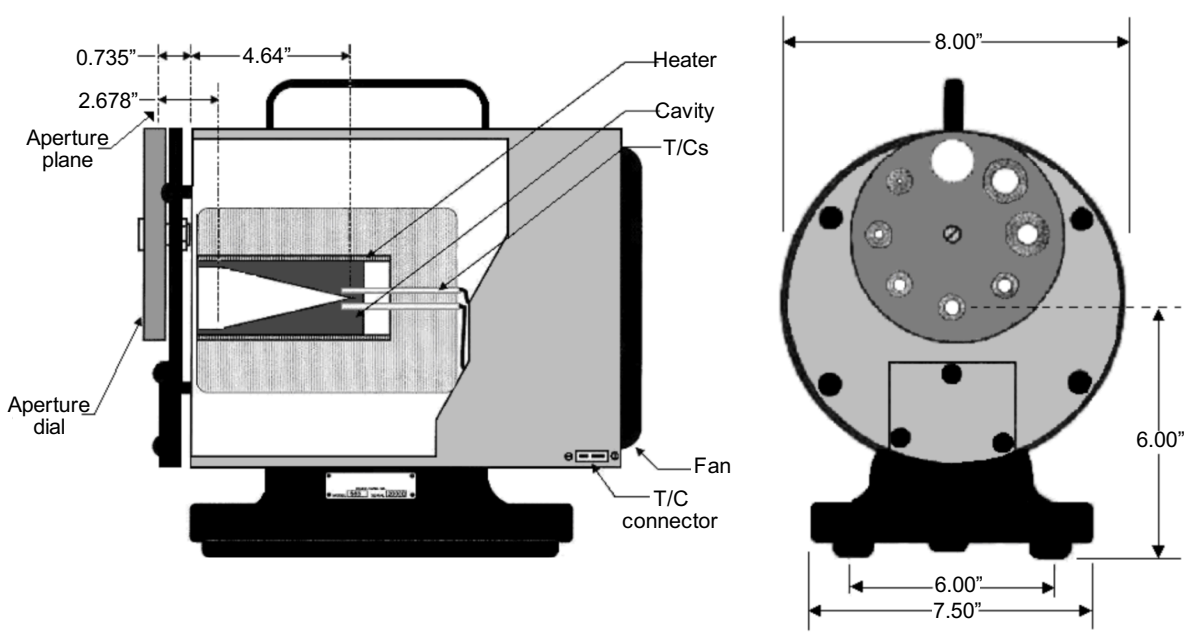

Figure 9. Technical drawing of the blackbody cavity apparatus used in the laboratory. The label "T/C" denotes "thermocouple".

APPENDIX II: Solid Angle

Due to the directional nature of thermal radiation, it is necessary to define the concept of a "solid angle. This is easiest done by comparing to a plane angle, as shown in Figure 10. We know from geometry that the length of a differential arc segment $d\ell$ along a circle is equal to the radius of the circle r times the differential central angle that intercepts the arc $d\alpha$, i.e., mathematically, $d\ell = r\,d\alpha$. Therefore, we can then define a plane angle as the arc length divided by the radius,

$$\text{plane angle:} \quad d\alpha = \frac{d\ell}{r} \quad \text{[radians]}. \tag{19}$$

The fundamental unit of a plane angle is radians. A differential solid angle $d\omega$ is similarly defined in three-dimensions as the differential surface area of a patch on a sphere dA_n divided by the squared radius of the sphere,

$$\text{solid angle:} \quad d\omega = \frac{dA_n}{r^2} \quad \text{[steradians]}. \tag{20}$$

The fundamental unit of a solid angle is steradians.

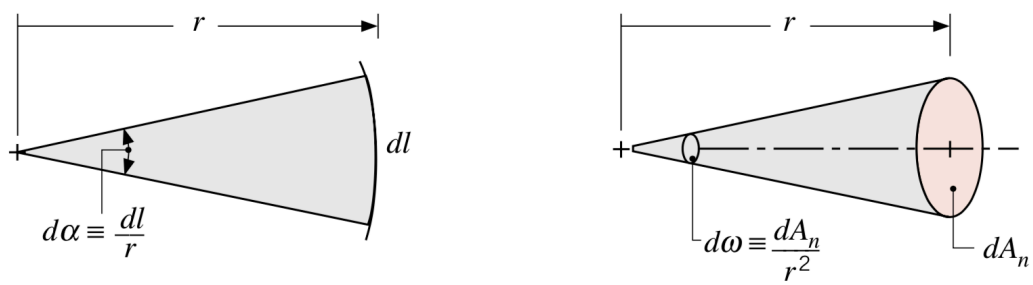

Figure 10. (left) Definition of a differential plane angle, $d\alpha$. (right) Definition of a differential solid angle, $d\omega$.

A solid angle provides a measure of how much radiation leaving a point (referred to as the apex) reaches an object a distance r away from the apex. Figure 11 illustrates the definition of a differential solid angle in spherical coordinates. Consider the case where radiation leaves a differential patch of area dA_s in the horizontal plane. Radiation is emitted from dA_s in all directions over the hemispherical space enclosing dA_s. The differential area dA_n through which radiation passes subtends a differential solid angle $d\omega$ when viewed from the center of dA_s. Note, the solid angle associated with the entire hemisphere is obtained by integration and is equal to 2π as shown below,

$$\omega_{\text{hemisphere}} = \int d\omega = \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi/2} \sin\theta\, d\theta\, d\phi = 2\pi \int_{\theta=0}^{\pi/2} \sin\theta\, d\theta = 2\pi \quad \text{[steradians]}. \tag{21}$$

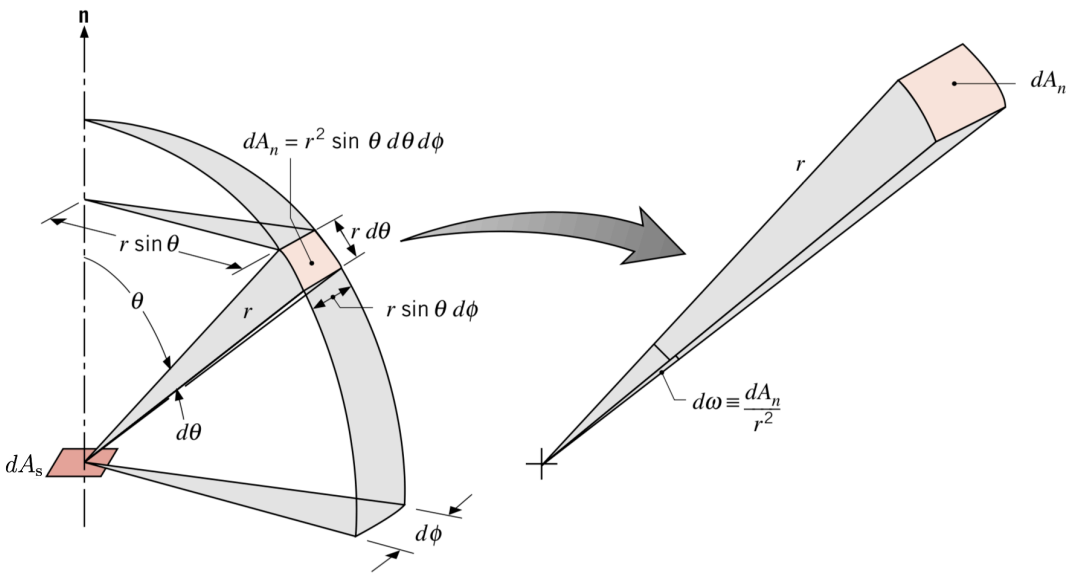

Figure 11. This drawing illustrates the definition of a solid angle, $d\omega$ subtended by the area dA_n at a point on the surface of dA_s.

A familiar example is the solid angle of the moon subtended from Earth, as illustrated in Figure 12. The radius of the moon is $R = 1.738 \times 10^3$ km. And the average distance from the Earth's surface to the moon's surface is $D = 3.844 \times 10^5$ km. Therefore, the solid angle is

$$\omega_{\text{moon}} = \frac{A_s}{D^2} = \frac{\pi R^2}{D^2} = \frac{\pi (1.738 \times 10^3)^2}{(3.844 \times 10^5)^2} = 6.4 \times 10^{-5} \text{ steradians.} \tag{22}$$

Since there are 2π steradians in the celestial hemisphere, the number of moons required to cover the entire night sky is $(2\pi)/(6.4 \times 10^{-5}) \approx 1 \times 10^5$. Therefore, using the concept of the solid angle, we can say that a full moon fills about 0.001% of the night sky.

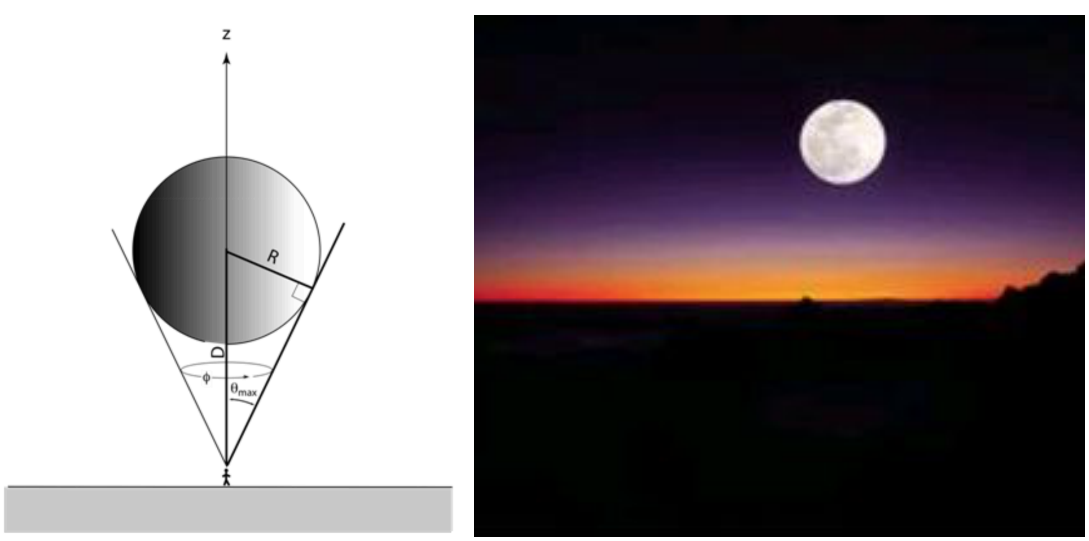

Figure 12. Calculating the solid angle of the moon subtended from a point on the Earth's surface.