Aerospace Propulsion

Lecture 9

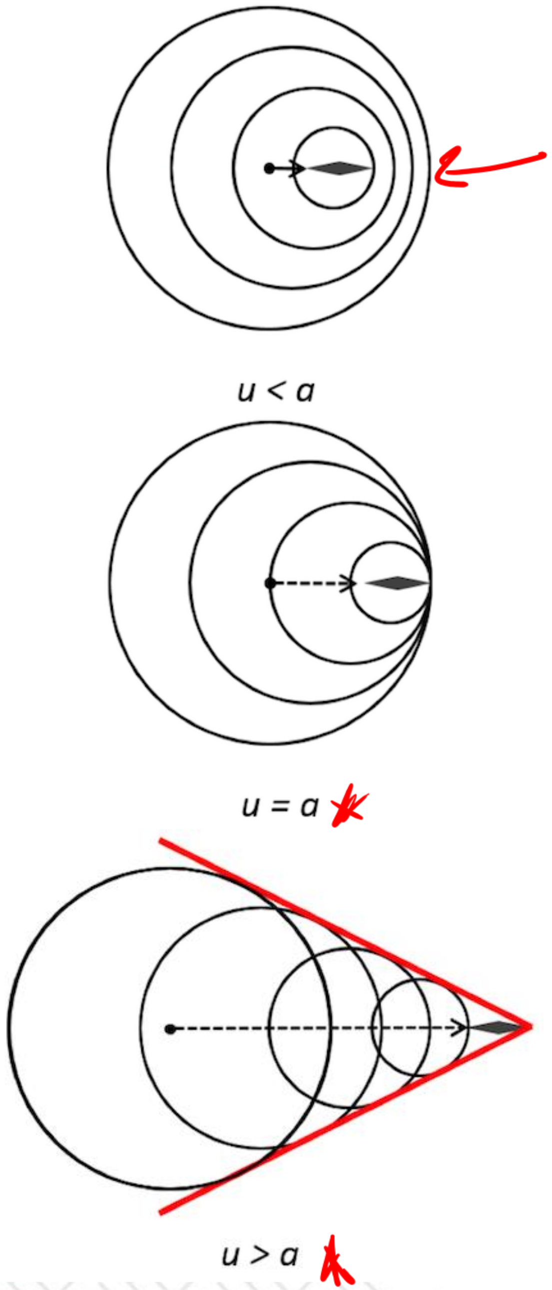
Compressible Flows: Part III

Compressible Flows: Part III

- What is a Shockwave?
- Normal Shocks
- Oblique Shocks
- Conical Shocks

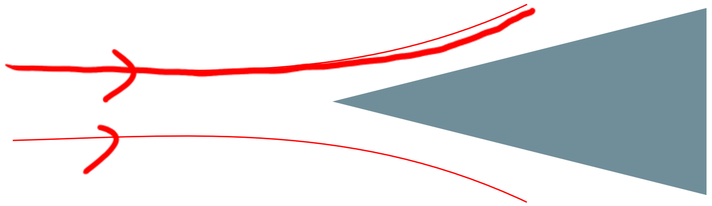
What is a Shockwave?

- Imagine an object moving in a fluid
  - As it moves, it creates pressure disturbances
    - These move at speed of sound
- Do the surroundings “know” about the object?
  - Only once pressure disturbances reach
- Subsonic
  - Pressure waves ahead of object
- Supersonic
  - Pressure waves behind object

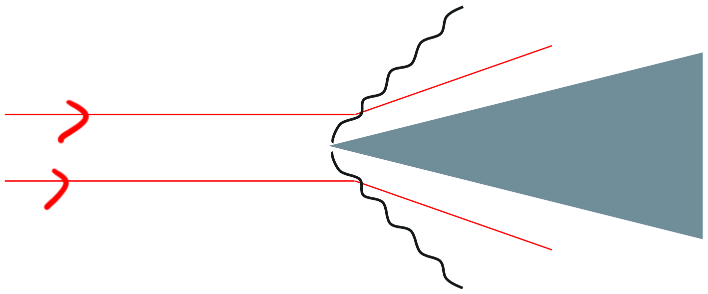


What is a Shockwave?

- Subsonic
  - Information (pressure waves) faster than object
  - Flow ahead of object “knows” it is arriving
  - Flow can gradually adjust

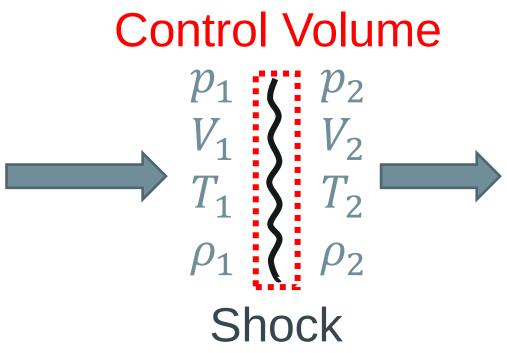


- Supersonic
  - Information slower than object
  - Flow ahead of object does not “know” it is arriving
  - Flow must adjust near-instantly (shock)



Normal Shocks

- Simplest analysis is flow normal to shock wave
- Assumptions
  - Upstream/downstream flow is isentropic
  - Shock itself is adiabatic but **not isentropic**
  - Shock is very thin (<100 nm), so area change negligible



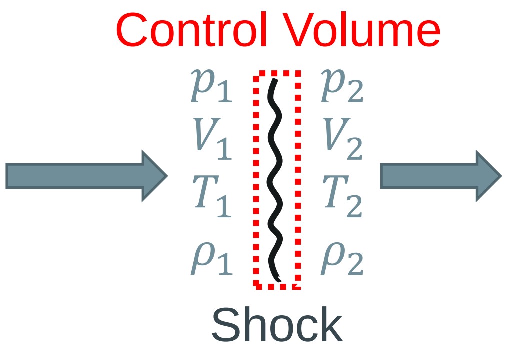
Normal Shocks

- Jump Conditions: Mass

$$\oiint \rho(\vec{V}\cdot\hat{n})\,dA = 0$$

$$A(\rho_2 V_2 - \rho_1 V_1) = 0$$

$$\rho_1 V_1 = \rho_2 V_2$$



- $\rho_1 V_1 = \rho_2 V_2$

Normal Shocks

- Jump Conditions: Momentum

$$\oiint \rho \vec{V}(\vec{V}\cdot\hat{n})\,dA = \sum F$$

$$A(\rho_2 V_2^2 - \rho_1 V_1^2) = A(p_1 - p_2)$$

$$p_1 + \rho_1 V_1^2 = p_2 + \rho_2 V_2^2$$

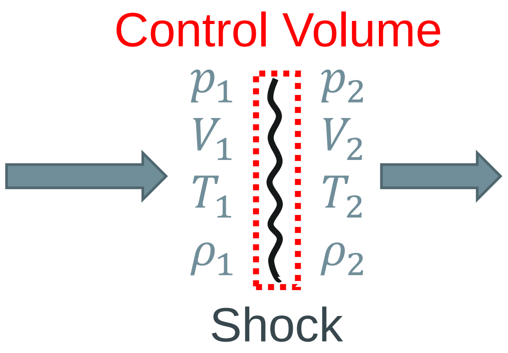
Recall for an ideal gas $a = \sqrt{\gamma p/\rho}$

So $p/\rho = a^2/\gamma$

$$p_1\left(1 + \frac{\rho_1 V_1^2}{p_1}\right) = p_2\left(1 + \frac{\rho_2 V_2^2}{p_2}\right)$$

$$p_1\left(1 + \frac{V_1^2}{a_1^2}\gamma\right) = p_2\left(1 + \frac{V_2^2}{a^2}\gamma\right)$$

$$p_1(1+\gamma M_1^2) = p_2(1+\gamma M_2^2)$$



- $p_1 + \rho_1 V_1^2 = p_2 + \rho_2 V_2^2$
- $p_1(1 + \gamma M_1^2) = p_2(1 + \gamma M_2^2)$

Normal Shocks

- Jump Conditions: Energy

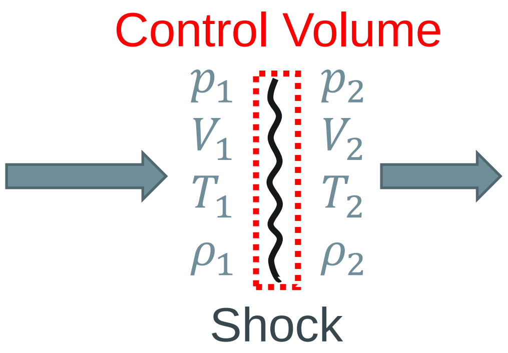
First law, adiabatic, no work, open system, steady

$dh_t = 0$

$d\left(h + \frac{1}{2}V^2\right) = 0$

$h_1 + \frac{1}{2}V_1^2 = h_2 + \frac{1}{2}V_2^2$

- $h_1 + \frac{1}{2}V_1^2 = h_2 + \frac{1}{2}V_2^2$
- $T_1 + \frac{1}{2c_p}V_1^2 = T_2 + \frac{1}{2c_p}V_2^2$
- $\underbrace{T_1\left(1 + \frac{\gamma-1}{2}M_1^2\right)}_{T_{t1}} = \underbrace{T_2\left(1 + \frac{\gamma-1}{2}M_2^2\right)}_{T_{t2}}$



Stagnation enthalpy & temperature conserved across shock!

(because adiabatic)

Normal Shocks

Ideal gas:

$p = \rho R T$

$\Rightarrow \rho = p/RT$

$\rho = \frac{p}{\sqrt{RT}\sqrt{RT}}$

$a = \sqrt{\gamma R T}$

$\frac{a}{\sqrt{\gamma}} = \sqrt{RT}$

$\rho = \frac{p}{\sqrt{RT}}\frac{\sqrt{\gamma}}{a}$

Recall momentum

$p_1 = p_2\frac{(1+\gamma M_2^2)}{(1+\gamma M_1^2)}$

- Downstream Mach number

Mass:

$\rho_1 V_1 = \rho_2 V_2$

$\frac{p_1\cancel{\sqrt{\gamma}}}{\sqrt{\cancel{R}T_1}}\frac{V_1}{a_1} = \frac{p_2\cancel{\sqrt{\gamma}}}{\sqrt{\cancel{R}T_2}}\frac{V_2}{a_2}$

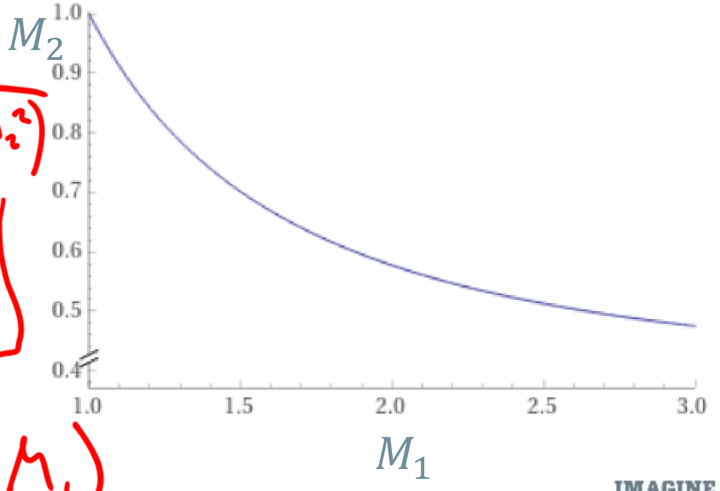
$\frac{p_1 M_1}{\sqrt{T_1}} = \frac{p_2 M_2}{\sqrt{T_2}}$

Stagnation T conserved across shock

$\frac{p_1 M_1}{\sqrt{T_1}} T_{t_1} = \frac{p_2 M_2}{\sqrt{T_1}} T_{t_2}$

$\frac{p_1 M_1}{\sqrt{\cancel{T_1}}}\sqrt{\cancel{T_1}\left(1+\frac{\gamma-1}{2}M_1^2\right)} = \frac{p_2 M_2}{\sqrt{\cancel{T_2}}}\sqrt{\cancel{T_2}\left(1+\frac{\gamma-1}{2}M_2^2\right)}$

  - $\boxed{M_2^2 = \frac{M_1^2 + \frac{2}{\gamma-1}}{\frac{2\gamma}{\gamma-1}M_1^2 - 1}}$
  - For $M_1 > 1$, $M_2 < 1$
  - From entropy, M_1 must be > 1 for a shock to exist



$p_1 M_1\sqrt{1+\frac{\gamma-1}{2}M_1^2} = p_2 M_2\sqrt{1+\frac{\gamma-1}{2}M_2^2}$

$\cancel{p_2}\frac{(1+\gamma M_2^2)}{(1+\gamma M_1^2)} M_1\sqrt{1+\frac{\gamma-1}{2}M_1^2} = \cancel{p_2} M_2\sqrt{1+\frac{\gamma-1}{2}M_2^2} \longrightarrow$ Solve for $M_2 = f(M_1)$

Normal Shocks

- Other downstream static properties
  - $$\frac{T_2}{T_1} = \frac{1+\frac{\gamma-1}{2}M_1^2}{1+\frac{\gamma-1}{2}M_2^2} = \frac{\left(1+\frac{\gamma-1}{2}M_1^2\right)\left(\frac{2\gamma}{\gamma-1}M_1^2-1\right)}{\left[\frac{(\gamma+1)^2}{2(\gamma-1)}\right]M_1^2}$$
  - $$\frac{p_2}{p_1} = \frac{1+\gamma M_1^2}{1+\gamma M_2^2} = \frac{2\gamma M_1^2}{\gamma+1} - \frac{\gamma-1}{\gamma+1}$$
  - $$\frac{\rho_2}{\rho_1} = \frac{(\gamma+1)M_1^2}{(\gamma-1)M_1^2+2}$$
  - For $M_1 > 1$, all these ratios are > 1

$f(M_1)$

Across a Shock

$M \downarrow$ T_t —

$T \uparrow$ P_t ?

$p \uparrow$

$\rho \uparrow$

Normal Shocks

- Normal shocks are not isentropic
  - How does stagnation pressure change?
    - $ds = c_p \frac{dT}{T} - R \frac{dp}{p} = c_p \frac{dT_t}{T_t} - R \frac{dp_t}{p_t}$
  - Stagnation temperature constant across shock
    - $s_2 - s_1 = -R \ln \frac{p_{t2}}{p_{t1}}$
    - Stagnation pressure decreases (non-isentropic loss)
  - $\frac{p_{t2}}{p_{t1}} = \left[\frac{\frac{\gamma+1}{2} M_1^2}{1+\frac{\gamma-1}{2} M_1^2}\right]^{\frac{\gamma}{\gamma-1}} \left[\frac{1}{\frac{2\gamma}{\gamma+1} M_1^2 - \frac{\gamma-1}{\gamma+1}}\right]^{\frac{1}{\gamma-1}}$
    - **Important:** Stronger shocks (larger M_1) experience larger stagnation pressure loss
    - Try to avoid strong normal shocks in practical systems

Normal Shocks

- Example:
  - Compute M_2, T_2, p_2 after a normal shock
    - $M_1 = 2$
    - $T_1 = 600$ K
    - $p_1 = 101325$ Pa

p_1 | p_2
M_1 | M_2
T_1 | T_2

$$M_2^2 = \frac{M_1^2 + \frac{2}{\gamma-1}}{\frac{2\gamma}{\gamma-1}M_1^2 - 1}$$

$$M_2^2 = \frac{2^2 + \frac{2}{1.4-1}}{\left[\frac{2(1.4)}{1.4-1}(2)^2\right] - 1}$$

$$M_2 = 0.577$$

$$T_2 = T_1\left(\frac{1 + \frac{\gamma-1}{2}M_1^2}{1 + \frac{\gamma-1}{2}M_2^2}\right)$$

$$T_2 = (600\text{K})\left(\frac{1 + \frac{1.4-1}{2}(2)^2}{1 + \frac{\gamma-1}{2}(0.577)^2}\right)$$

$$T_2 = 1012.58\text{K}$$

$$p_2 = p_1\left(\frac{2\gamma M_1^2}{\gamma+1} - \frac{\gamma-1}{\gamma+1}\right)$$

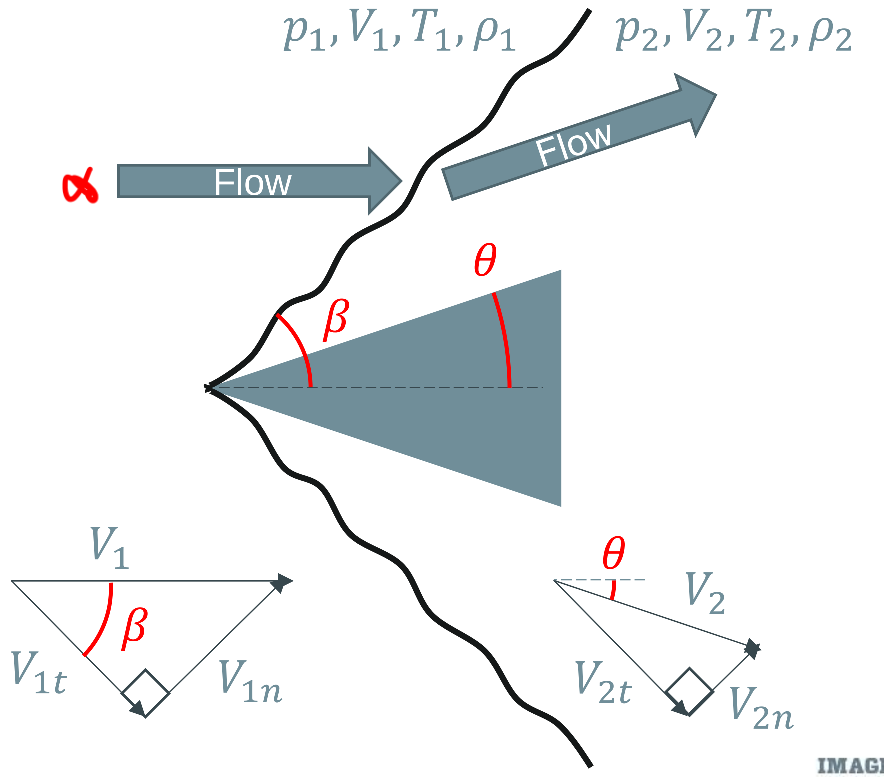
$$p_2 = (101325\text{Pa})\left(\frac{2(1.4)(2)^2}{1.4+1} - \frac{1.4-1}{1.4+1}\right)$$

$$\boxed{p_2 = 455{,}962.5\text{Pa}}$$

Oblique Shocks

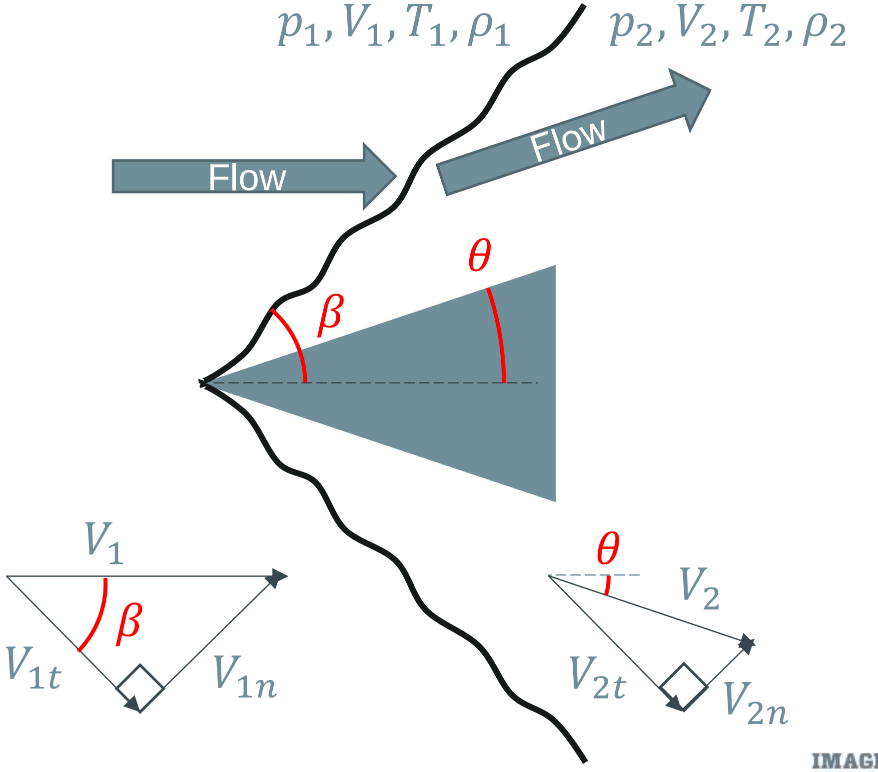
- What if the flow approaches the shockwave at an angle?
- New quantities
  - β = Shock Angle
  - θ = Deflection Angle (flow angle)
- How do we tackle this problem?
  - Decompose velocity to normal/tangent
  - Treat normal component as before
  - **Tangential velocity unchanged**

$V_{1t} = V_{2t}$



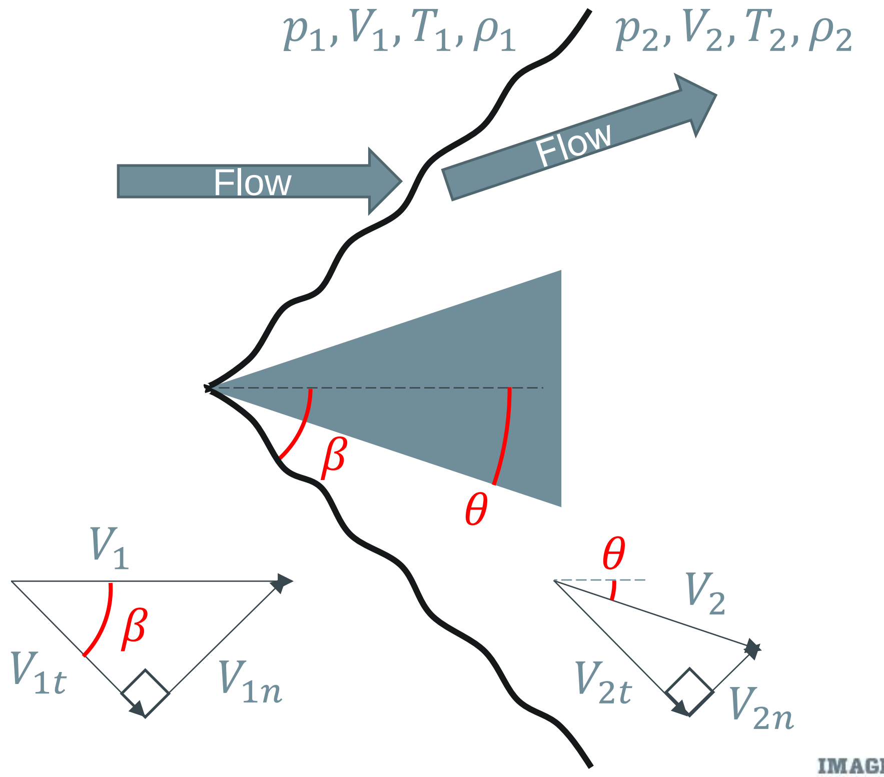
Oblique Shocks

- Jump Conditions
  - Mass:
    - $\rho_1 V_{n1} = \rho_2 V_{n2}$
  - N-Momentum:
    - $p_1 + \rho_1 V_{n1}^2 = p_2 + \rho_2 V_{n2}^2$
  - T-Momentum:
    - $V_{t2} = V_{t1}$
  - Energy:
    - $h_1 + \frac{1}{2} V_1^2 = h_2 + \frac{1}{2} V_2^2$
    - $h_1 + \frac{1}{2} V_{n1}^2 = h_2 + \frac{1}{2} V_{n2}^2$



Oblique Shocks

- Jump Conditions
  - Same conditions as normal shock
    - Just replace V with V_n
  - For static properties, use same shock relations with M_{n1} and M_{n2} replacing M_1 and M_2 as follows:
    - $M_{n1} = \frac{V_{n1}}{a_1} = \frac{V_1}{a_1} \sin\beta = M_1 \sin\beta$
    - $M_{n2} = M_2 \sin(\beta - \theta)$



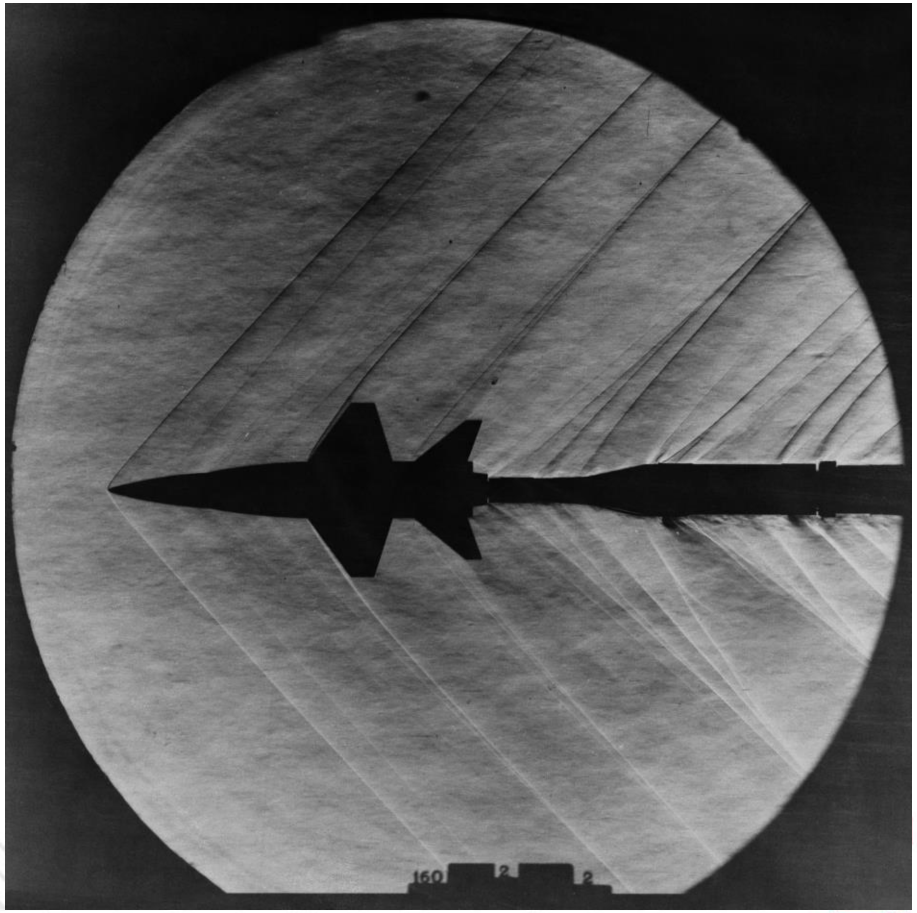
Oblique Shocks

N.S.

O.S.

While these equations exist, I highly recommend avoiding them and instead working with the normal velocity component and angles

- Downstream properties
  - $\frac{T_2}{T_1} = \frac{\left(1+\frac{\gamma-1}{2}M_1^2 \sin^2\beta\right)\left(\frac{2\gamma}{\gamma-1}M_1^2 \sin^2\beta-1\right)}{\left[\frac{(\gamma+1)^2}{2(\gamma-1)}\right]M_1^2 \sin^2\beta}$
  - $\frac{p_2}{p_1} = \frac{2\gamma M_1^2 \sin^2\beta}{\gamma+1} - \frac{\gamma-1}{\gamma+1}$
  - $\frac{\rho_2}{\rho_1} = \frac{(\gamma+1)M_1^2 \sin^2\beta}{(\gamma-1)M_1^2 \sin^2\beta+2}$
  - $\frac{p_{t2}}{p_{t1}} = \left[\frac{\frac{\gamma+1}{2}M_1^2 \sin^2\beta}{1+\frac{\gamma-1}{2}M_1^2 \sin^2\beta}\right]^{\frac{\gamma}{\gamma-1}} \left[\frac{1}{\frac{2\gamma}{\gamma+1}M_1^2 \sin^2\beta-\frac{\gamma-1}{\gamma+1}}\right]^{\frac{1}{\gamma-1}}$
  - $M_2^2 = \frac{1+\frac{\gamma-1}{2}M_1^2}{\gamma M_1^2 \sin^2\beta-\frac{\gamma-1}{2}} + \frac{M_1^2 \cos^2\beta}{1+\frac{\gamma-1}{2}M_1^2 \sin^2\beta}$



Oblique Shocks

- Example:
  - Compute M_2, p_2
  - $M_1 = 2$
  - $p_1 = 101325$ Pa
  - $\beta = 50$ degrees
  - $\theta = 18.13$ degrees

Mach #

$M_{n_1} = M_1 \sin\beta$

$M_{n_1} = (2)\sin(50°)$

$M_{n_1} = 1.532$

$$M_{n_2}^2 = \frac{M_{n_1}^2 + \frac{2}{\gamma-1}}{\frac{2\gamma}{\gamma-1}M_{n_1}^2 - 1}$$

$$M_{n_2}^2 = \frac{(1.532)^2 + \frac{2}{1.4-1}}{\frac{2(1.4)}{1.4-1}(1.532)^2 - 1}$$

$M_{n_2} = 0.69$

Weak oblique shock $(M_2 > 1)$

$M_{n_2} = M_2 \sin(\beta - \theta)$

$$M_2 = \frac{M_{n_2}}{\sin(\beta-\theta)} = \frac{0.69}{\sin(50° - 18.13°)}$$

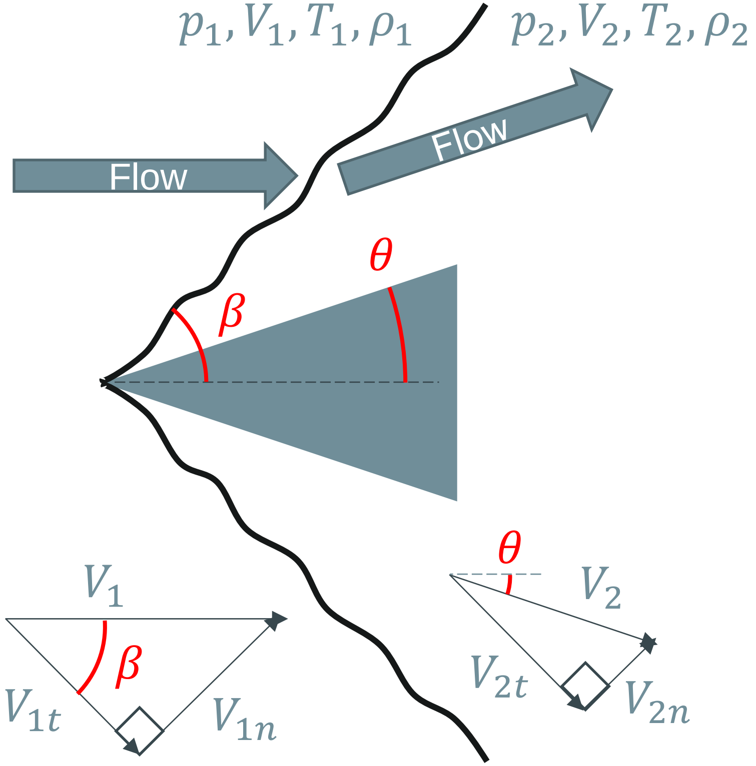
$$\boxed{M_2 = 1.306}$$

Pressure

$$\frac{p_2}{p_1} = \frac{1+\gamma M_{1n}^2}{1+\gamma M_{2n}^2}$$

$$p_2 = (101325\,\text{Pa})\left(\frac{1+1.4(1.532)^2}{1+1.4(0.69)^2}\right)$$

$p_2 = 260,547$ Pa



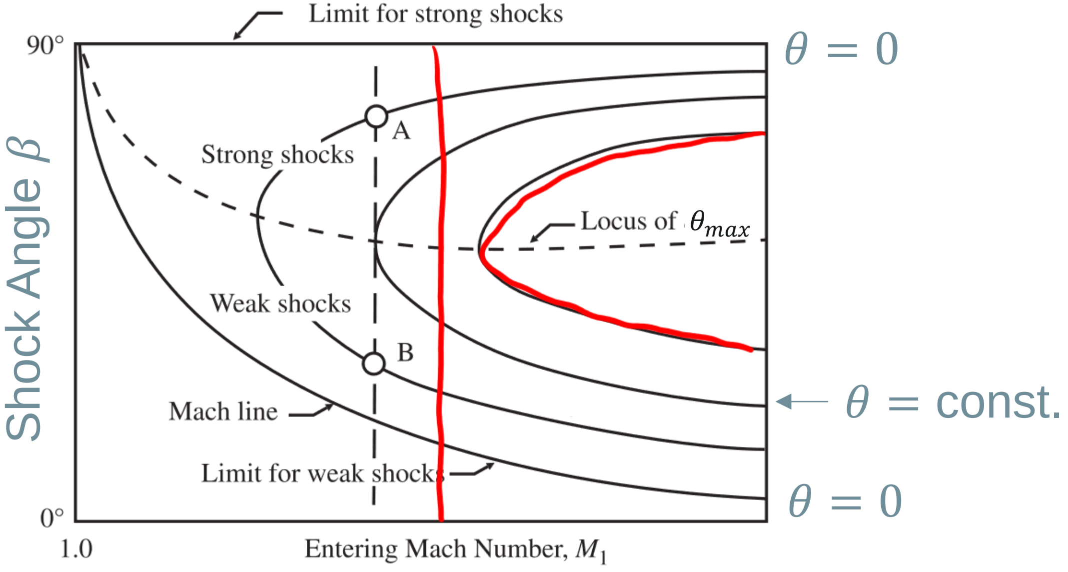
Oblique Shocks

- Shock Angle
  - If given the deflection angle θ, need to compute the shock angle β to evaluate the oblique shock jump conditions
  - $\tan\theta = \cot\beta \dfrac{\left(M_1^2 \sin^2\beta - 1\right)}{\left(\frac{\gamma+1}{2}M_1^2 - \left(M_1^2 \sin^2\beta - 1\right)\right)}$
  - No explicit expression for β in terms of θ
    - In this course, you will be given the shock angle

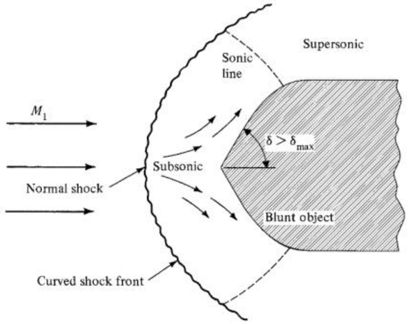
Oblique Shocks

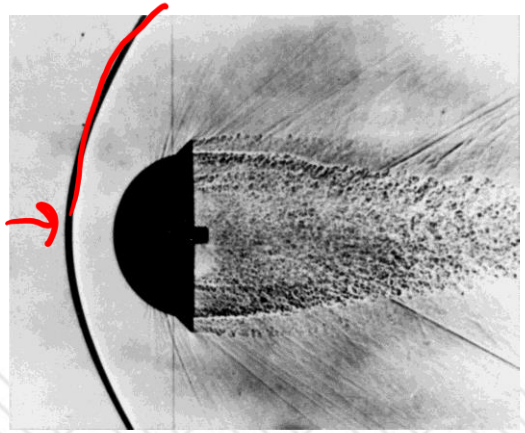
- Shock Angle
  - For given deflection angle and upstream Mach number, can be two, one, or zero possible shock angles

$$\tan\theta = \cot\beta \frac{\left(M_1^2 \sin^2\beta - 1\right)}{\left(\frac{\gamma+1}{2} M_1^2 - \left(M_1^2 \sin^2\beta - 1\right)\right)}$$



  - Zero angles
    - Deflection angle too large, curved *detached* shock forms





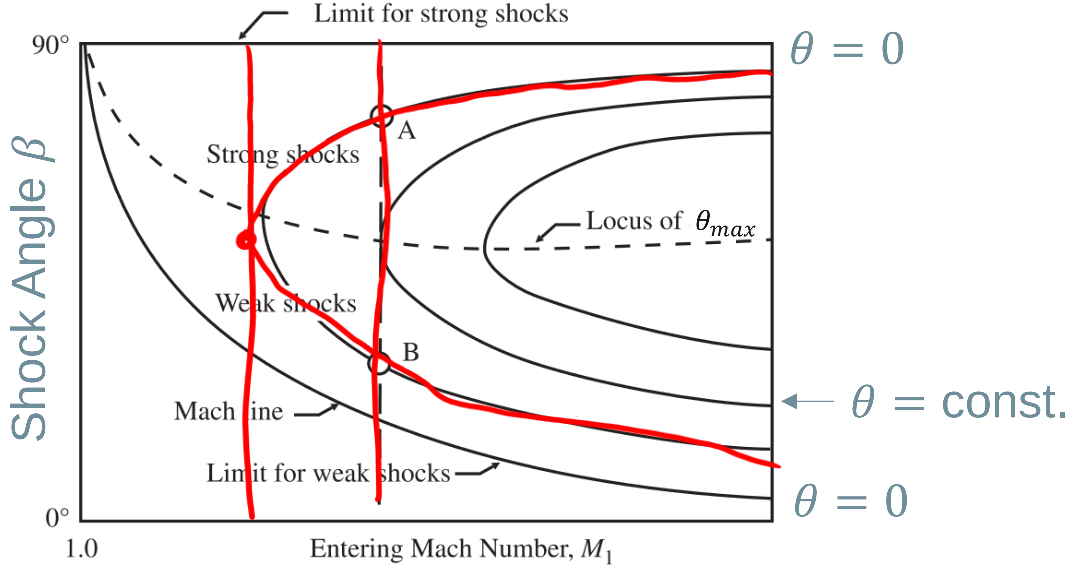
Oblique Shocks

Strong $\rightarrow$

weak $\rightarrow$

$$\tan\theta = \cot\beta \frac{\left(M_1^2 \sin^2\beta - 1\right)}{\left(\frac{\gamma+1}{2}M_1^2 - \left(M_1^2 \sin^2\beta - 1\right)\right)}$$

- Shock Angle
  - Two angles
    - Strong shock: Larger shock angle with subsonic M_2
    - Weak shock: Smaller shock angle with *generally* supersonic M_2
    - Exact results mainly depend on boundary conditions
    - Weak shocks are far more common in our applications – assume weak unless otherwise specified
  - One angle
    - Maximum deflection angle, strong/weak oblique shocks coincide



Conical Shocks

- Conical shocks occur in conical geometries
- Qualitatively similar to planar shocks
  - Additional considerations
- No time to discuss
- Read Farokhi if interested
  - Section 2.13