Optimal Control HW4

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Problem 1)
Read the two papers by Arechavaleta et al., on human walking and Mombaur et al., paper on optimal control for humanoids. Briefly answer the following questions about the two papers:
a) What is the main contribution of the paper?
b) How did they chose the objective (cost) function for their work? Please be very specific about this and describe in your own words the motivation and reasoning behind their choice. Also, you want to clearly articulate the variables or parameters they chose for their cost function. Basic hand-waving discussions or arguments will not be accepted.
c) How did they implement and validate their work?
d) What kind of results did they get and how well did it work?

An Optimality Principle Governing Human Walking

a) The main contribution of this paper was showing that human trajectory planning from a start position to a goal position in forward constrained motion could be accurately approximated by an optimal control algorithm that penalized the linear velocity and time derivative of path curvature inputs using a cost function.

b) These researchers assumed that the observed path that humans take from a start position to the goal position is the optimal path planned from a trajectory rather than on the fly (step by step). The experimental setup limited human motion backwards or side to side to force more direct trajectory planning with linear and angular velocities to create a path. The first input to the system model was then the linear velocity forwards and the second input to the system was added as an extension which was the time derivative of path curvature. This input was added to avoid sharp turns or discontinuities in the trajectory where true continuous curvature had been modeled in human motion as a critical input in other research studies. The cost function was then given as a penalty on the inputs meaning if the linear velocity increased dramatically, the cost would as well, and if the curvature increased dramatically, so would the cost. The optimal solution would then come from minimizing both the linear velocity and the curvature of the path.

c) To first implement their work, the did an observational study to track the trajectories that humans take from a start position to a goal position. Using motion capture tools, a model of the system was obtained. These trajectories were assumed to be the optimal trajectories and were geometrically averaged from trial to trial. They then solved for optimal trajectories using the obtained system model and cost function that penalized inputs linear velocity and derivative of path curvature .

d) When plotting the numerical solution against the observed trajectories, the geometric paths aligned very well, especially with the additional input of time derivative of curvature. Using the system model that did not include the time derivative of curvature yielded inaccurate trajectories. They found that 90% of the trajectories had an average error of less than 10cm and a maximum error of less than 20cm. When looking at the optimally calculated path vs each observational trial path, the optimally calculated path fell within the middle of the geometric distribution of the observed paths showing that the optimally calculated path was very accurate.

An Optimal Control-Based Formulation to Determine Natural Locomotor Paths for Humanoid Robots

a) The main contribution of the paper is locomotor paths formulated through optimal control with a cost function that considers both non-holonomic and holonomic inputs rather than choosing between dynamic modes like previous research. The research focuses on incorporating a cost function that minimizes both linear and angular velocity from non-holonomic dynamics with orthogonal velocity from holonomic dynamics with input penalties that reflect dominant dynamic modes in path planning.

b) This research aims at finding a general locomotion based on complete dynamics rather than choosing from modes of dynamic motion, non-holonomic and holonomic as seen in other humanoid locomotion research. To do this, they include the inputs of both non-holonomic and holonomic modes which are forward and rotational velocity for non-holonomic and orthogonal velocity for holonomic. By adding the third input of orthogonal motion in comparison to Arechavaleta's approach, the cost function is now a general formulation of forward and sideways motion which can be minimized to find the general best overall path. The benefit of this formulation is that the robot now does not need to "choose" between dynamic modes based on some type of decision making. All of the dynamic modes can be seen in the cost function. The cost function is then an integration of 3 acceleration components from the three inputs of non-holonomic and holonomic dynamics where a greater penalty weight is placed on the holonomic input based on the distance to the target to ensure that orthogonal motion is only considered when the target is evidently closer to the robot.

c) To implement and validate the work, the cost function was numerically solved and implemented into 6 different experimental trials where either non-holonomic, holonomic or both dynamic modes would be favored to test whether the cost function was accurately minimizing the cost of holonomic or non-holonomic inputs. With these numerical solutions, different penalties on the inputs were used to find the most well behaved solution through qualitative comparison.

d) The researchers achieved good results. Test case 1 favored strong non-holonomic dynamics and the cost function resulted in zero orthogonal input. Test case 2 favored non-holonomic dynamics with slight holonomic dynamics through a rotation which was achieved. Test case 3 favored holonomic motion which was achieved through pure orthogonal translation. Test case 4 was a larger distance rendition of test case 3 where more penalty should have been placed on the holonomic motion through the cost function through a greater distance which occurred resulting in dominant non-holonomic input. Test case 5 was a diagonal trajectory that required dominant holonomic input to keep orientation the same which was the dominant input. Test case 6 was a curved trajectory which required more forward motion which prefers non-holonomic inputs which was achieved.

Problem 2)
Consider the extremization of a functional which is dependent on derivatives higher than the first derivative $\dot{x}(t)$ such as

$$J(x(t),t) = \int_{t_o}^{t_f} V(x(t),\dot{x}(t),\ddot{x}(t),t)dt$$

with fixed-end point conditions. Show that the corresponding Euler-Lagrange equation is given by

$$\frac{\partial V}{\partial x} - \frac{d}{dt}(\frac{\partial V}{\partial \dot{x}}) + \frac{d^2}{dt^2}(\frac{\partial V}{\partial \ddot{x}}) = 0$$

Step 1)

$$\begin{aligned} \text{let } x_a(t) &= x^*(t) + \delta x(t) \\ \delta x_{t_o} &= x_a(t_o) - x^*(t_o) = 0 \\ \delta x_{t_f} &= x_a(t_f) - x^*(t_f) = 0 \end{aligned}$$

Step 2)

$$\Delta J(x(t),\delta x(t))) = \int_{t_o}^{t_f} V(x^*(t)+\delta x(t),\dot{x}^*(t)+\delta\dot{x}^*(t),\ddot{x}^*(t)+\delta\ddot{x}^*(t),t) - V(x^*(t),\dot{x}^*(t),\ddot{x}^*(t),t)dt$$

Step 3) Expand using T.S. Expansion and only retain linear terms

$$= \int_{t_o}^{t_f} (\frac{\partial v(x(t),\dot{x}(t),\ddot{x}(t),t)}{\partial x}\delta x + \frac{\partial v(x(t),\dot{x}(t),\ddot{x}(t),t)}{\partial \dot{x}}\delta \dot{x} + \frac{\partial v(x(t),\dot{x}(t),\ddot{x}(t),t)}{\partial \ddot{x}}\delta \ddot{x})dt \tag{1}$$

Integrate by parts

$$\text{let } u = \frac{\partial v}{\partial \dot{x}}, v = \delta x$$

$$\int \frac{\partial v(x(t),\dot{x}(t),\ddot{x}(t),t)}{\partial \dot{x}}\delta\dot{x} = \underbrace{\frac{\partial v}{\partial \dot{x}}\delta x}_{0} - \int_{t_0}^{t_f} \delta x \frac{d}{dt}(\frac{\partial v}{\partial \dot{x}})dt$$

$$\text{let } u = \frac{\partial v}{\partial \ddot{x}}, v = \delta \dot{x}$$

$$\int \frac{\partial v(x(t),\dot{x}(t),\ddot{x}(t),t)}{\partial \ddot{x}}\delta\ddot{x} = \frac{\partial v}{\partial \ddot{x}}\delta \dot{x} - \int_{t_0}^{t_f} \delta \dot{x} \frac{d}{dt}(\frac{\partial v}{\partial \ddot{x}})dt$$

$$\text{let } u = \frac{d}{dt}\frac{\partial v}{\partial \ddot{x}}, v = \delta x$$

$$\underbrace{\frac{\partial v}{\partial \ddot{x}}\delta\dot{x}}_{0} - \left[\underbrace{\frac{d}{dt}\frac{\partial v}{\partial \ddot{x}}\delta x}_{0} - \int_{t_0}^{t_f} \delta x \frac{d^2}{dt^2}(\frac{\partial v}{\partial \ddot{x}})dt\right]$$

Equation (1) then becomes:

$$= \int_{t_o}^{t_f} (\frac{\partial v(x(t),\dot{x}(t),\ddot{x}(t),t)}{\partial x} - \frac{d}{dt}\frac{\partial v(x(t),\dot{x}(t),\ddot{x}(t),t)}{\partial \dot{x}} + \frac{d^2}{dt^2}\frac{\partial v(x(t),\dot{x}(t),\ddot{x}(t),t)}{\partial \ddot{x}})\delta x dt$$

Step 4) Apply F.T.C.V

$$\int_{t_o}^{t_f} (\frac{\partial v(x(t),\dot{x}(t),\ddot{x}(t),t)}{\partial x} - \frac{d}{dt}\frac{\partial v(x(t),\dot{x}(t),\ddot{x}(t),t)}{\partial \dot{x}} + \frac{d^2}{dt^2}\frac{\partial v(x(t),\dot{x}(t),\ddot{x}(t),t)}{\partial \ddot{x}})\delta x dt = 0$$

Step 5) Simplify using lema

If for every function g(t) which is continuous

$$\int_{t_o}^{t_f} g(t)\delta x(t)dt = 0$$

where $\delta x(t)$ is continuous on $[t_o, t_f]$, g(t) must be zero on $[t_o, t_f]$

Step 6)

$$\frac{\partial v}{\partial x} - \frac{d}{dt}(\frac{\partial v}{\partial \dot{x}}) + \frac{d^2}{dt^2}(\frac{\partial v}{\partial \ddot{x}}) = 0$$

Problem 3)
For a second order system

$$\dot{x}_1(t) = x_2(t)$$
$$\dot{x}_2(t) = -2x_1(t) + 3u(t)$$

with performance index

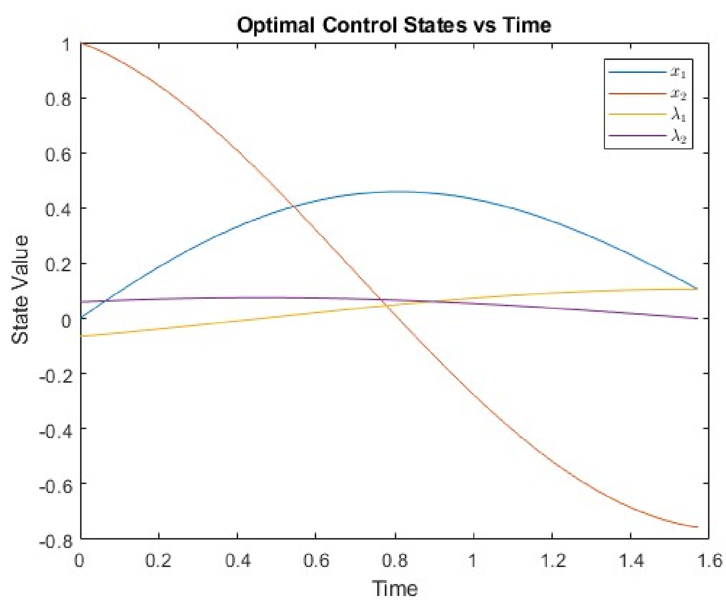
$$J = 0.5x_1^2(\pi/2) + \int_0^{pi/2} 0.5u^2(t)dt$$

and boundary conditions x(0) = [0 1]' and x(tf) is free, find the optimal control.

Fixed final time and free final state (Type C)

$$V(x,u,t) = 0.5u^2(t)$$
$$f(x,u,t) = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \begin{bmatrix} x_2(t) \\ -2x_1(t) + 3u(t) \end{bmatrix}$$
$$H(x_1, x_2.u, \lambda_1, \lambda_2) = V(u + \lambda f(x,u)) = 0.5u^2 + \lambda_1 x_2 - 2\lambda_2 x_1 + 3\lambda_2 u$$
$$\frac{\partial H}{\partial u} = 0 \rightarrow u^* + 3\lambda_2^* = 0$$
$$u^* = -3\lambda_2^*$$
$$H^*(x_1^*, x_2^*, \lambda_1^*, \lambda_2^*) = 4.5\lambda_2^{*2} + \lambda_1^* x_2^* - 2\lambda_2^* x_1^*$$
$$\dot{x_1}^*(t) = +(\frac{\partial H}{\partial \lambda_1})_* = x_2^*$$
$$\dot{x_2}^*(t) = +(\frac{\partial H}{\partial \lambda_2})_* = -9\lambda_2^* - 2x_1^*$$
$$\dot{\lambda_1}^*(t) = -(\frac{\partial H}{\partial x_1})_* = 2\lambda_2^*$$
$$\dot{\lambda_2}^*(t) = -(\frac{\partial H}{\partial x_2})_* = -\lambda_1^*$$

Solved numerically in Matlab, code given in the appendix



Sufficient condition:

$$\pi = \begin{bmatrix} \frac{\partial^2 H}{\partial x_1^2} & \frac{\partial^2 H}{\partial x_1 \partial x_2} & \frac{\partial^2 H}{\partial x_1 \partial u} \\ \frac{\partial^2 H}{\partial x_2 \partial x_1} & \frac{\partial^2 H}{\partial x_2^2} & \frac{\partial^2 H}{\partial x_2 \partial u} \\ \frac{\partial^2 H}{\partial u \partial x_1} & \frac{\partial^2 H}{\partial u \partial x_2} & \frac{\partial^2 H}{\partial u^2} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

π is positive semi-definite

$\rightarrow$ Weak minimum, further investigation is needed.

Appendix

```
clear, clc, close all

% Time mesh (integral bounds of time to evaluate solution over)
tmesh = linspace(0, pi/2, 100);

% Initial guess mapped onto time mesh (Can be anything, better guesses will just save computation resources)
solinit = bvpinit(tmesh, [0 0 0 0]);

% Solve BVP
sol = bvp4c(@bvpfcn, @bcfun, solinit);

% ODE Function
function dxdt = bvpfcn(t,x)
    A = [ 0  1  0  0;
        -2  0  0 -9;
         0  0  0  2;
         0  0 -1  0];
    dxdt = A*x;
end

%Boundary Condition Function
function res = bcfun(xa,xb)
    % Boundary Condition vector
    res = zeros(4,1);
    % Fromatted via matlab guidelines for bvp4c (Boundary conditions set equal to zero)
    % Initial conditions
    res(1) = xa(1);          % x1(0) = 0
    res(2) = xa(2) - 1;      % x2(0) = 1
    % Final conditions
    res(3) = xb(1) - xb(3);  % l1(tf) = x1(tf) - l1(tf) = 0
    res(4) = xb(4);          % l2(tf) = 0
end

figure
plot(sol.x,sol.y)
legend("$x_1$","$x_2$", "$\lambda_1$", "$\lambda_2$", "Interpreter","latex")
xlabel("Time"); ylabel("State Value"); title("Optimal Control States vs Time")
```