1. (a) $a = b_i c_{ij} d_j$ $\Rightarrow$ valid

(b) $a = b_i c_i + d_j$ $\Rightarrow$ Invalid, free index "j" in only one term

(c) $a_i = \delta_{ij} b_i + c_i$ $\Rightarrow$ Invalid as written since free index "j" only appears in one term

(d) $a_k = b_i c_{ki}$ $\Rightarrow$ valid

(e) $a_k = b_k c + d_i e_{ik}$ $\Rightarrow$ valid

(f) $a_i = b_i + c_{ij} d_{ji} e_i$ $\Rightarrow$ Invalid, "i" is repeated three times in last term

(g) $a_l = \varepsilon_{ijk} b_j c_k$ $\Rightarrow$ Invalid, free index "l" does not match free index "i" on RHS

(h) $a_{ij} = b_{ji}$ $\Rightarrow$ valid

(i) $a_{ij} = b_i c_j + e_{jk}$ $\Rightarrow$ Invalid, free index "k" appears in only one term

(j) $a_{kl} = b_i c_{ki} d_l + e_{ki}$ $\Rightarrow$ Invalid, free index "l" in first two terms does not appear in last term

2. (a) Prove $\delta_{ij}\delta_{ij} = 3$

ANSWER

$$\delta_{ij}\delta_{ij} = \sum_{i=1}^{3}\sum_{j=1}^{3}\delta_{ij}\delta_{ij} = \sum_{i=1}^{3}\left[\delta_{i1}^2 + \delta_{i2}^2 + \delta_{i3}^2\right] = \underbrace{\delta_{11}^2}_{=1} + \cancel{\delta_{12}^2}^{0} + \cancel{\delta_{13}^2}^{0} + \cancel{\delta_{21}^2}^{0} + \underbrace{\delta_{22}^2}_{=1} + \cancel{\delta_{23}^2}^{0} + \cancel{\delta_{31}^2}^{0} + \cancel{\delta_{32}^2}^{0} + \underbrace{\delta_{33}^2}_{=1}$$

Since $\delta_{ij} = 0$ for $i \neq j$ and $\delta_{ij} = 1$ for $i = j$,

$\rightarrow \delta_{ij}\delta_{ij} = 3$

2. (b) Prove $\varepsilon_{pqr}\varepsilon_{pqr} = 6$

ANSWER

$$\varepsilon_{pqr}\varepsilon_{pqr} = \sum_{p=1}^{3}\sum_{q=1}^{3}\sum_{r=1}^{3}\varepsilon_{pqr}\varepsilon_{pqr} = \sum_{p=1}^{3}\left[\sum_{q=1}^{3}\varepsilon_{pq1}^2 + \varepsilon_{pq2}^2 + \varepsilon_{pq3}^2\right]$$

$$= \sum_{p=1}^{3}\left[\cancel{\varepsilon_{p11}^2}^{0} + \varepsilon_{p12}^2 + \varepsilon_{p13}^2 + \varepsilon_{p21}^2 + \cancel{\varepsilon_{p22}^2}^{0} + \varepsilon_{p23}^2 + \varepsilon_{p31}^2 + \varepsilon_{p32}^2 + \cancel{\varepsilon_{p33}^2}^{0}\right]$$

$$= \cancel{\varepsilon_{112}^2}^{0} + \cancel{\varepsilon_{113}^2}^{0} + \underbrace{\varepsilon_{121}^2}_{=1} + \underbrace{\varepsilon_{123}^2}_{=1} + \cancel{\varepsilon_{131}^2}^{0} + \underbrace{\varepsilon_{132}^2}_{=1}$$
$$+ \cancel{\varepsilon_{212}^2}^{0} + \underbrace{\varepsilon_{213}^2}_{=1} + \cancel{\varepsilon_{221}^2}^{0} + \cancel{\varepsilon_{223}^2}^{0} + \underbrace{\varepsilon_{231}^2}_{=1} + \cancel{\varepsilon_{232}^2}^{0}$$
$$+ \underbrace{\varepsilon_{312}^2}_{=1} + \cancel{\varepsilon_{313}^2}^{0} + \underbrace{\varepsilon_{321}^2}_{=1} + \cancel{\varepsilon_{323}^2}^{0} + \cancel{\varepsilon_{331}^2}^{0} + \cancel{\varepsilon_{332}^2}^{0}$$

$\rightarrow \varepsilon_{pqr}\varepsilon_{pqr} = 6$

3. Prove $\varepsilon_{pqi}\varepsilon_{pqj} = 2\delta_{ij}$

ANSWER

Use epsilon-delta relation: $\varepsilon_{ijk}\varepsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$ (as given in textbook)

or, using the same indices as in the problem and realizing that $\varepsilon_{pqi} = \varepsilon_{qip}$,

$$\varepsilon_{pqi}\varepsilon_{pqj} = \varepsilon_{qip}\varepsilon_{pqj} = \underbrace{\delta_{qq}}_{=3}\delta_{ij} - \delta_{qj}\delta_{iq} = 3\delta_{ij} - \delta_{ij}$$

$$\varepsilon_{pqi} \xrightarrow{(-1)} \varepsilon_{qpi} \rightarrow \varepsilon_{qip}$$

$\rightarrow \varepsilon_{pqi}\varepsilon_{pqj} = 2\delta_{ij}$

4. (a) Prove $\vec{u} \times \vec{v} = -\vec{v} \times \vec{u}$

ANSWER

LHS: $\vec{u} \times \vec{v} \Rightarrow \varepsilon_{ijk} u_i v_j$

RHS: $-\vec{v} \times \vec{u} \Rightarrow -\varepsilon_{ijk} v_i u_j = -\varepsilon_{ijk} u_j v_i$

Change indices on v and u to match those used on LHS: $i \to j$, $j \to i$

$$\Rightarrow -\varepsilon_{jik} u_i v_j$$

Now use identity, $-\varepsilon_{jik} = \varepsilon_{ijk}$

$$\Rightarrow \varepsilon_{ijk} u_i v_j$$ ✓

4. (b) Show using index notation that

$$\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c}) \vec{b} - (\vec{a} \cdot \vec{b}) \vec{c}$$

ANSWER

Let $\vec{d} = \vec{b} \times \vec{c}$. Then the left hand side is

Left-hand side: $LHS = \varepsilon_{ijk} a_i d_j$ but $d_j = \varepsilon_{lmj} b_l c_m$

$$= \varepsilon_{ijk} \varepsilon_{lmj} a_i b_l c_m$$

Use epsilon-delta relation: $\varepsilon_{ijk} \varepsilon_{klm} = \delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}$ (as given in book)

However, the repeating index is "k"; whereas in our problem, the repeating index is "j". Therefore, we switch "k" and "j" above:

$$\varepsilon_{ikj} \varepsilon_{jlm} = \delta_{il} \delta_{km} - \delta_{im} \delta_{kl}$$

Now, we have to get the order of the indices to match that in our problem. We realize that

$$\varepsilon_{ikj} = -\varepsilon_{ijk} \quad \text{and} \quad \varepsilon_{jlm} = \varepsilon_{lmj}$$

Now, we go back to our problem ...

$$= -\varepsilon_{ikj} \varepsilon_{jlm} a_i b_l c_m = \underbrace{-(\delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}) a_i b_l c_m}_{\text{two terms}}$$

$$= \delta_{im} \delta_{jl} a_i b_l c_m - \delta_{il} \delta_{jm} a_i b_l c_m$$

$$= b_j \underbrace{a_i c_i}_{\vec{a} \cdot \vec{c}} - c_j \underbrace{a_l b_l}_{\vec{a} \cdot b}$$

$$= (\vec{a} \cdot \vec{c}) \vec{b} - (\vec{a} \cdot \vec{b}) \vec{c}$$

4.(c) Prove $\underbrace{(\vec{a}\times\vec{b})\cdot\vec{c}}_{\mathrm{I}} = \underbrace{\vec{a}\cdot(\vec{b}\times\vec{c})}_{\mathrm{II}} = \underbrace{(\vec{c}\times\vec{a})\cdot\vec{b}}_{\mathrm{III}}$

ANSWER

Term I: $(\vec{a}\times\vec{b})\cdot\vec{c} \Rightarrow (\varepsilon_{ijk}\, a_i b_j)\, c_k = \varepsilon_{ijk}\, a_i b_j c_k$

Term II: $\vec{a}\cdot(\vec{b}\times\vec{c}) \Rightarrow \varepsilon_{ijk}\, b_i c_j a_k$

now rewrite indices on b, c, a to match that of Term I, i.e., change $k\to i$, $j\to k$, $i\to j$

$$\Rightarrow \varepsilon_{jki}\, a_i b_j c_k = \varepsilon_{ijk}\, a_i b_j c_k \;\checkmark$$

use $\varepsilon_{jki} = \varepsilon_{ijk}$

Term III: $(\vec{c}\times\vec{a})\cdot\vec{b} \Rightarrow (\varepsilon_{ijk}\, c_i a_j)\, b_k$

change indices to match those above: $i\to k$, $j\to i$, $k\to j$

$$\Rightarrow \varepsilon_{kij}\, a_i b_j c_k = \varepsilon_{ijk}\, a_i b_j c_k \;\checkmark$$

$\varepsilon_{kij} = \varepsilon_{ijk}$

5. Prove $\vec{\nabla}\cdot(\vec{\nabla}\times\vec{u}) = 0$ for any arbitrary vector $\vec{u}$.

ANSWER

First we write the term in parenthesis using index notation

$$\vec{\nabla}\times\vec{u} \Rightarrow \varepsilon_{ijk}\frac{\partial u_k}{\partial x_j}$$

Not take divergence recognizing that $\vec{\nabla}\cdot\vec{a} = \frac{\partial a_i}{\partial x_i}$

$$\vec{\nabla}\cdot(\vec{\nabla}\times\vec{u}) \Rightarrow \frac{\partial}{\partial x_i}\left(\varepsilon_{ijk}\frac{\partial u_k}{\partial x_j}\right) = \varepsilon_{ijk}\frac{\partial^2 u_k}{\partial x_i \partial x_j}$$

However, since "i" and "j" are just dummy (repeated) indices, we can swap them without changing the result. Therefore,

$$\varepsilon_{ijk}\frac{\partial^2 u_k}{\partial x_i \partial x_j} \underbrace{=}_{\text{swap "i" \& "j"}} \varepsilon_{jik}\frac{\partial^2 u_k}{\partial x_j \partial x_i} \underbrace{=}_{\text{order of differentiation does not matter}} \varepsilon_{jik}\frac{\partial^2 u_k}{\partial x_i \partial x_j}$$

Now recall that $\varepsilon_{jik} = -\varepsilon_{ijk}$, so

$$\varepsilon_{ijk}\frac{\partial^2 u_k}{\partial x_i \partial x_j} = -\varepsilon_{ijk}\frac{\partial^2 u_k}{\partial x_i \partial x_j}.$$

→ The only way that a quantity can equal the negative value of itself is if that quantity is <u>zero</u>.