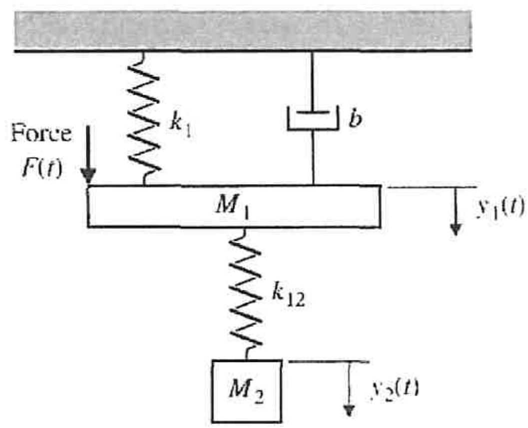
Classical Control Systems

Homework 01

Do the following problems and show all your work for full credit. Note: not all problems will be graded, however, you must complete all problems to get full credit.

Problem 1

Find the equations of motion (differential equations) that describes the behavior of the following mechanical system:

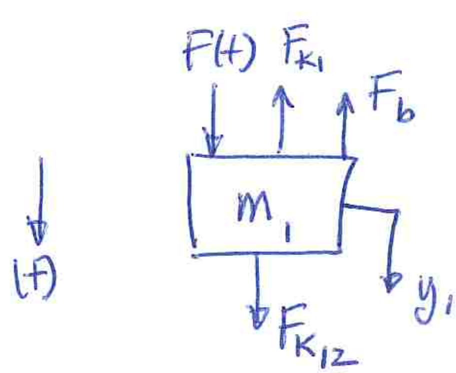


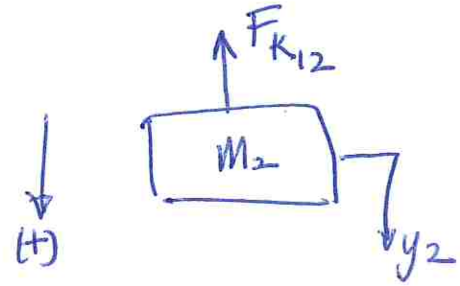
y (+) positive direction chosen same as direction of applied force.

* Start by assuming that $y_2(t) > y_1(t)$ and positive displacement direction is down (see arrow above).

* Because $y_2(t) > y_1(t)$, spring K_{12} is in tension!

* Now we draw FBD:





Note: gravity force is left out because it does not affect dynamics!

* Now we sum forces and apply Newton's 2nd Law:

for mass m_1:

$$\sum F = m_1 \ddot{y}_1(t)$$

$$\Rightarrow \quad F(t) - F_{k_1} - F_b + F_{k_{12}} = m_1 \ddot{y}_1(t)$$

sub in the forces for springs and damper:

$$F(t) - k_1 y_1(t) - b\dot{y}_1(t) + k_{12}\left(y_2(t) - y_1(t)\right) = m_1 \ddot{y}_1(t)$$

rearrange into standard O.D.E form:

$$\boxed{m_1 \ddot{y}_1(t) + b\dot{y}_1(t) + (k_1 + k_{12})y_1(t) - k_1 y_2(t) = F(t)}$$

for mass m_2:

$$\sum F = m_2 \ddot{y}_2(t)$$

$$\Rightarrow \quad -F_{k_{12}} = m_2 \ddot{y}_2(t)$$

$$-k_{12}\left(y_2(t) - y_1(t)\right) = m_2 \ddot{y}_2(t)$$

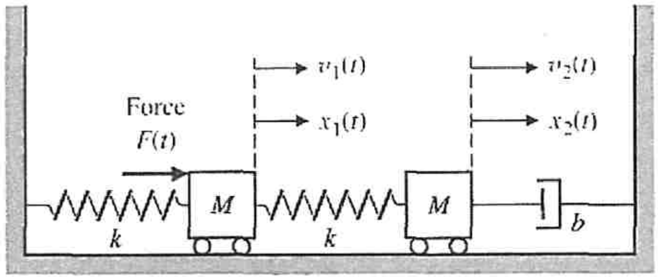
rearrange:

$$\boxed{m_2 \ddot{y}_2(t) + k_{12} y_2(t) - k_{12} y_1(t) = 0}$$

Both of these equations describe the dynamics of the two-mass system.

Problem 2

Find the equations of motion (differential equations)that describes the behavior of the following mechanical system (note, both masses are the same):



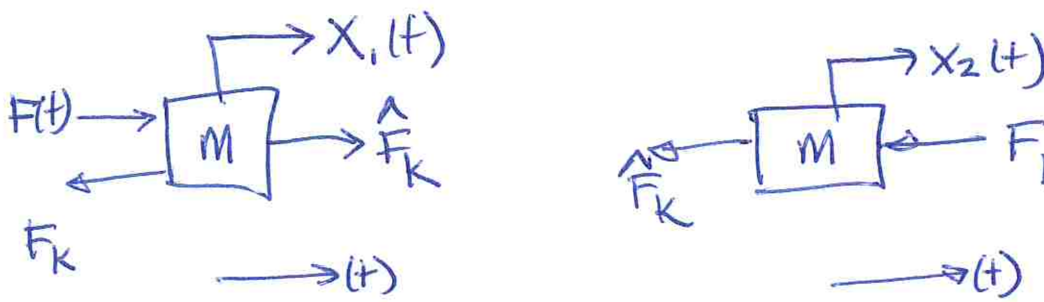
→ (+)

choose positive direction so it is consistent w/ applied force.

* Start by assuming that $X_2(t) > X_1(t)$ and the positive direction is to the right.

* Because $X_2(t) > X_1(t)$, middle spring is in tension!

* Now we draw FBD:



* Now apply Newton's 2nd Law and sum forces:

For the left mass:

$$\Sigma F = m\ddot{x}_1(t)$$

$$\Rightarrow F(t) - F_K + \hat{F}_K = m\ddot{x}_1(t)$$

sub in for the forces:

$$F(t) - kx_1(t) + k(x_2(t) - x_1(t)) = m\ddot{x}_1(t)$$

rearrange:

$$\boxed{m\ddot{x}_1(t) + 2kx_1(t) - kx_2(t) = F(t)}$$

For the right mass:

$$\sum F = m\ddot{x}_2(t)$$

$$-\hat{F}_k - F_b = m\ddot{x}_2(t)$$

$$-k(x_2(t) - x_1(t)) - b\dot{x}_2(t) = m\ddot{x}_2(t)$$

rearrange:

$$\boxed{m\ddot{x}_2(t) + b\dot{x}_2(t) + kx_2(t) - kx_1(t) = 0}$$

Both of the above equations describe the dynamics of the system.

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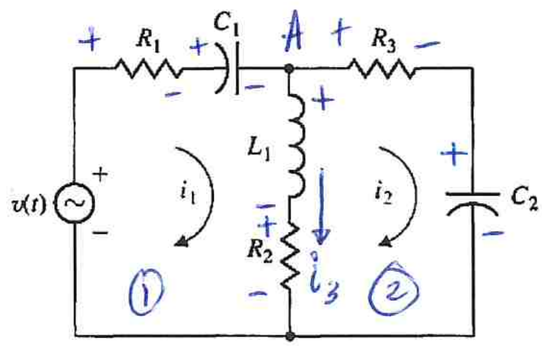
Problem 3

Find the integral-differential (integrodifferential) equations that governs the behavior of the following circuit. Note that the input is the voltage v(t) and the output variables are the currents i_1 and i_2. Your result should be differential equation(a), time-domain equations, that include these variables.



* Assume that i_1 flows through R_1 + C_1 and i_2 flows throw R_3 and C_2, so at node A, we have the following KCL:

$$i_1 = i_3 + i_2$$

* Next, we add "+" and "−" signs associated with current flow through each elment.

* Now, we apply the voltage loop equation (KVL):

Loop ①: $\Sigma V_i = 0$

$$\Rightarrow -V(t) + R_1 i_1 + \frac{1}{C_1}\int i_1 dt + L_1 \frac{di_3}{dt} + R_2 i_3 = 0$$

we can sub in for i_3 using KCL above, so

$$\Rightarrow -V(t) + R_1 i_1 + \frac{1}{C_1}\int i_1 dt + L_1 \frac{d}{dt}(i_1 - i_2) + R_2 (i_1 - i_2) = 0$$

rearrange to get:

$$\boxed{R_1 i_1 + \frac{1}{C_1}\int i_1 dt + L_1 \frac{d}{dt}(i_1 - i_2) + R_2(i_1 - i_2) = V(t)}$$

For loop 2:

$$\sum V_i = 0$$

$$\Rightarrow R_3 i_2 + \frac{1}{C_2}\int i_2 dt - R_2 i_3 - L_1 \frac{d}{dt} i_3 = 0$$

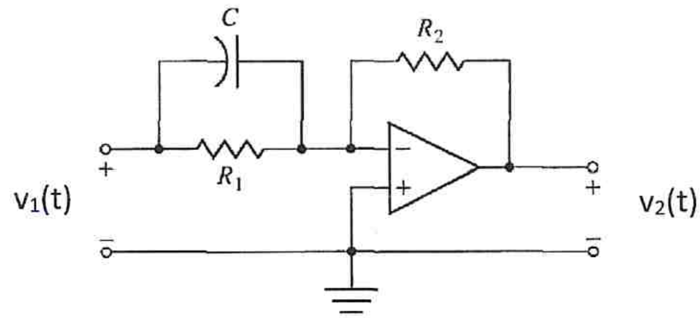
sub in for $i_3 = i_1 - i_2$ from KCL:

$$\Rightarrow \boxed{R_3 i_2 + \frac{1}{C_2}\int i_2 dt - R_2(i_1 - i_2) - L_1 \frac{d}{dt}(i_1 - i_2) = 0}$$

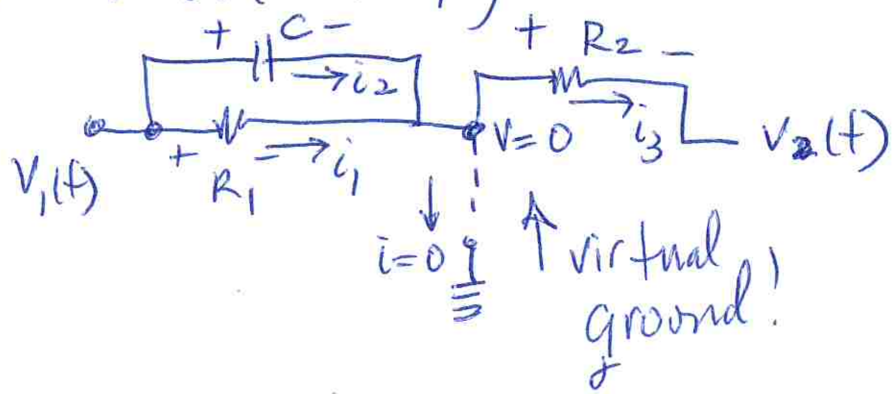
Both of these equations describe the dynamics of the system. They are integro-diff. equations in terms of i_1 and i_2 with $V(t)$ as the input!

Problem 4

For the following op-amp circuit, find the differential equation that relates the input $v_1(t)$ to the output voltage $v_2(t)$:



* First, this op-amp circuit is in negative feedback, so we can simply the circuit to:



Did this w/ the 2 rules for neg. FB op-amps!

* To find relationship between $V_1(t)$ and $V_2(t)$, we first notice that:

$$i_1 + i_2 = i_3 \quad \text{(KCL)}$$

* Next, we can find each current w/ element equations:

$$V_1(t) - 0 = R_1 i_1 \Rightarrow \underline{\underline{i_1 = \frac{V_1(t)}{R_1}}}$$

$$V_1(t) - 0 = \frac{1}{C}\int i_2\,dt$$

Take derivative of both sides to get

$$\dot{V}_1(t) = \frac{1}{C} i_2 \Rightarrow \underline{\underline{i_2 = C\dot{V}_1(t)}}$$

Finally: $0 - V_2(t) = R_2 i_3 \Rightarrow \underline{\underline{i_3 = -\frac{V_2(t)}{R_2}}}$

Now go back to KCL:

$$i_1 + i_2 = i_3$$

sub into this equation the currents:

$$\frac{V_1(t)}{R_1} + C\dot{V}_1(t) = -\frac{V_2(t)}{R_2}$$

rearrange into stand O.D.E form:

$$-\frac{1}{R_2} V_2(t) = C\dot{V}_1(t) + \frac{1}{R_1} V_1(t)$$

$$\Rightarrow \boxed{\frac{1}{R_2} V_2(t) = -C\dot{V}_1(t) - \frac{1}{R_1} V_1(t)}$$

Problem 5

Consider the following system where a human operator is part of the closed-loop control system. Sketch the block diagram of the valve control system and label the signals and blocks.