Chapter 4

A First Analysis of Feedback

Problems and Solutions for Section 4.1: The Basic Equations of Control

1. If S is the sensitivity of the unity feedback system to changes in the plant transfer function and T is the transfer function from reference to output, show that $S + T = 1$.

   Solution:

$$
\begin{aligned}
S + T &= \frac{1}{1 + DG} + \frac{DG}{1 + DG}, \\
&= 1.
\end{aligned}
$$

   OK, this one was free!

2. We define the sensitivity of a transfer function G to one of its parameters K as the ratio of percent change in G to percent change in K.

$$
S_K^G = \frac{dG/G}{dK/K} = \frac{d \ln G}{d \ln K} = \frac{K}{G}\frac{dG}{dK}.
$$

   The purpose of this problem is to examine the effect of feedback on sensitivity. In particular, we would like to compare the topologies shown in Fig. 4.30 for connecting three amplifier stages with a gain of $-K$ into a single amplifier with a gain of -10.

   (a) For each topology in Fig.4.30, compute β_i so that if $K = 10$, $Y = -10R$.

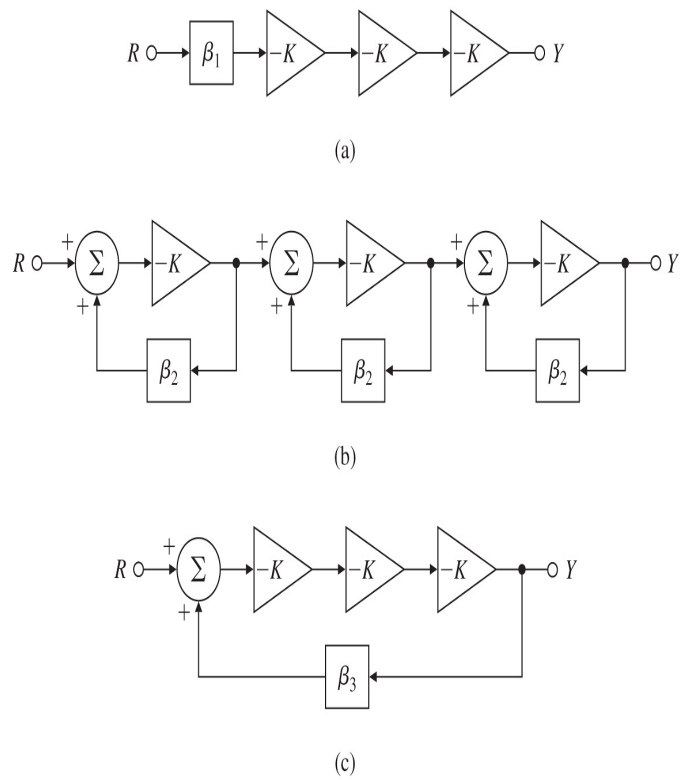


Figure 4.30: Three-amplifier topologies for Problem 4.2

(b) For each topology, compute S_k^G when $G = Y/R$. [Use the respective β_i values found in part(a).] Which case is the *least* sensitive?

(c) Compute the sensitivities of the systems in Fig.4.30(b, c) to β_2 and β_3. Using your results, comment on the relative need for precision in sensors and actuators.

Solution:

(a) For $K = 10$ and $y = -10r$, we have:

Case a:

$$\frac{Y}{R} = -\beta_1 K^3 \implies \beta_1 = 0.01.$$

Case b:

$$\frac{Y}{R} = \left(\frac{-K}{1+\beta_2 K}\right)^3 \implies \beta_2 = 0.364.$$

Case c:

$$\frac{Y}{R} = \frac{-K^3}{1+\beta_3 K^3} \implies \beta_3 = 0.099.$$

(b) Sensitivity S_K^G, $G = \dfrac{Y}{R}$,

Case a:

$$\frac{dG}{dK} = -3\beta_1 K^2.$$

$$\mathcal{S}_K^G = \frac{K}{G}\frac{dG}{dK} = \frac{K}{-\beta_1 K^3}(-3\beta_1 K^2) = 3.$$

Similarly:

Case b: $S_K^G = 0.646$

Case c: $S_K^G = 0.03$

Case c is the least sensitive.

(c) Sensitivities w.r.t. feedback gains:

Case b:

$$\mathcal{S}_{\beta_2}^G = -2.354$$

Case c:

$$\mathcal{S}_{\beta_3}^G = -0.99$$

The results indicate that the closed-loop system is much more sensitive to errors in the feedback path than in the forward path. It is 33 times as sensitive in case c. We conclude that sensors need to have much higher precision than actuators.

3. Compare the two structures shown in Fig. 4.31 respect to sensitivity to changes in the overall gain due to changes in the amplifier gain. Use the relation

$$S = \frac{d \ln F}{d \ln K} = \frac{K}{F}\frac{dF}{dK}.$$

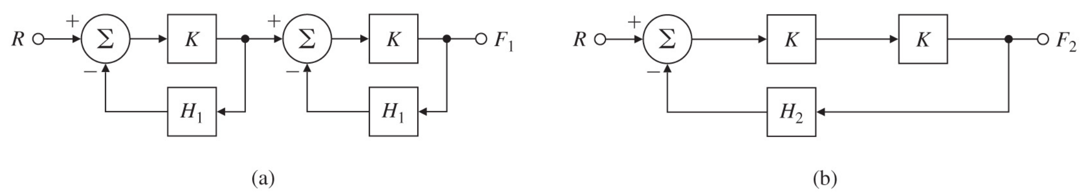


Figure 4.31: Control system for Problem 4.3

as the measure. Select H_1 and H_2 so that the nominal system outputs satisfy $F_1 = F_2$, and assume $KH_1 > 0$.

Solution:

$$F_1 = \frac{K}{1+KH_1}^{2}; \; F_2 = \frac{K^2}{1+K^2H_2}$$

$$\mathcal{S}_K^{F_1} = \frac{2}{1+KH_1}; \quad \mathcal{S}_K^{F_2} = \frac{2}{1+K^2H_2}$$

$$F_1 = F_2 \quad \Longrightarrow H_2 = H_1^2 + \frac{2H_1}{K}$$

$$\mathcal{S}_K^{F_2} = \frac{2}{(1+KH_1)^2} = \frac{\mathcal{S}_K^{F_1}}{1+KH_1}$$

System 2 is less sensitive. The conclusion is to put as much gain in the feedback loop as you can.

4. A unity feedback control system has the open-loop transfer function

$$G(s) = \frac{A}{s(s+a)}.$$

(a) Compute the sensitivity of the closed-loop transfer function to changes in the parameter A.

(b) Compute the sensitivity of the closed-loop transfer function to changes in the parameter a.

(c) If the unity gain in the feedback changes to a value of $\beta \neq 1$, compute the sensitivity of the closed-loop transfer function with respect to β.

Solution:

(a)

$$T(s) = \frac{G(s)}{1+G(s)} = \frac{\dfrac{A}{s(s+a)}}{1+\dfrac{A}{s(s+a)}} = \frac{A}{s^2+as+A},$$

$$\frac{dT}{dA} = \frac{(s^2 + as + A) - A}{(s^2 + as + A)^2},$$

$$\mathcal{S}_A^T = \frac{A}{T}\frac{dT}{dA} = \frac{A(s^2 + as + A)}{A}\frac{s^2 + as}{(s^2 + as + A)^2} = \frac{s(s + a)}{s(s + a) + A}.$$

(b)

$$\frac{dT}{da} = \frac{-sA}{(s^2 + as + A)^2}.$$

$$\frac{a}{T}\frac{dT}{da} = \frac{a(s^2 + as + A)}{A}\frac{-sA}{(s^2 + as + A)^2}.$$

$$\mathcal{S}_a^T = \frac{-as}{s(s + a) + A}.$$

(c) In this case,

$$T(s) = \frac{G(s)}{1 + \beta G(s)},$$

$$\frac{dT}{d\beta} = \frac{-G(s)^2}{(1 + \beta G(s))^2},$$

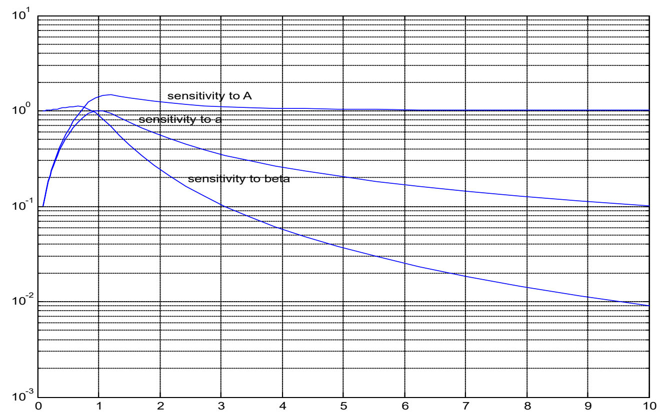
$$\frac{\beta}{T}\frac{dT}{d\beta} = \frac{\beta(1 + \beta G)}{G}\frac{-G^2}{(1 + \beta G)^2} = \frac{-\beta G}{1 + \beta G},$$

$$\mathcal{S}_\beta^T = \frac{\dfrac{-\beta A}{s(s + A)}}{1 + \dfrac{\beta A}{s(s + a)}} = \frac{-\beta A}{s(s + a) + \beta A}.$$

- If $a = A = 1$,the transfer function is most sensitive to variations in a and A near $\omega = 1$ rad/sec .
- The steady-state response is not affected by variations in A and a ($\mathcal{S}_A^T(0)$ and $\mathcal{S}_a^T(0)$ are both zero).
- The steady-state response *is* heavily dependent on β since $|\mathcal{S}_\beta^T(0)|$ $= 1.0$

See attached plots of sensitivities versus radian frequency for $a = A = 1.0$.

Sensitivity function frequency response follows.



Problem 4.4: Frequency response.

Frequency (rad/sec)

5. Compute the equation for the system error for the feedback system shown in Fig. 4.5.

Solution:

For this figure, the equation for the output is:

$$Y = \frac{D_c G}{1 + D_c GH} R + \frac{G}{1 + D_c GH} W - \frac{D_c GH}{1 + D_c GH} V$$

And the resulting equation for the error is:

$$\begin{aligned} E &= R - Y \\ &= \frac{1 + D_c G(H-1)}{1 + D_c GH} R - \frac{G}{1 + D_c GH} W + \frac{D_c GH}{1 + D_c GH} V \end{aligned}$$

Therefore, as we have seen, increasing the loop gain does not necessarily reduce the error as the result depends on the structure of the system.

Problems and Solutions for Section 4.2: Control of Steady-State Error

6. Consider the DC-motor control system with rate (tachometer) feedback shown in Fig. 4.32(a).

   (a) Find values for K' and k_t' so that the system of Fig. 4.32(b) has the same transfer function as the system of Fig. 4.32(a).

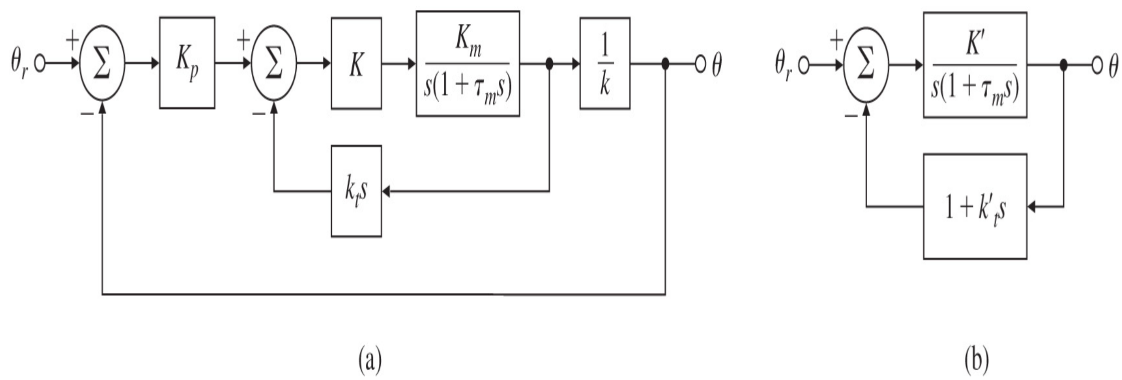


Figure 4.32: Control system for Problem 4.6

(b) Determine the system type with respect to tracking θ_r and compute the system K_v in terms of parameters K' and k'_t.

(c) Does the addition of tachometer feedback with positive k_t increase or decrease K_v?

Solution:

(a) Using block diagram reduction techniques:

- Move the pickoff point from the input of the $\frac{1}{k}$ to its output.
- Eliminate the second summer by absorbing K_p.

This will result in Figure 4.32(b) where

$$K' = \frac{K_p K K_m}{k},$$

$$k'_t = \frac{k k_t}{K_p}.$$

(b) The inner-loop in Fig. 4.32(a) may be reduced to

$$\frac{KK_m}{s(1 + \tau_m s + KK_m k_t)},$$

which means that the unity feedback system has a pure integrator in the forward loop and hence it is Type 1 with respect to the reference input (θ_r) and $K_v = \dfrac{KK_m}{(1 + KK_m k_t)}$.

(c) *We conclude that the introduction of k_t reduces the velocity constant and therefore makes the error to a ramp larger.*

7. A block diagram of a control system is shown in Fig. 4.33.

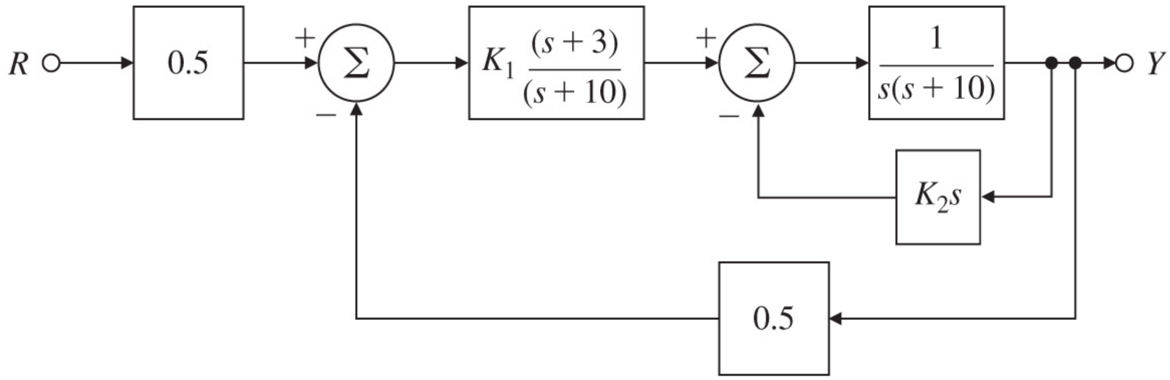


Figure 4.33: Closed-loop system for Problem 4.7

(a) If R is a step function and the system is closed-loop stable, what is the steady-state tracking error?

(b) What is the system type?

(c) What is the steady-state error to a ramp velocity 5.0 if $K_2 = 2$ and K_1 is adjusted to give a system step overshoot of 17%?

Solution:

(a) Using conversion to unity feedback rule, we obtain a controller of the form:

$$D_c(s) = 0.5K_1 \frac{(s+3)}{(s+10)},$$

The loop around the plant becomes:

$$\frac{1}{s(s+10+K_2)}.$$

(b) Therefore, we have a unity feedback system with an integrator in the forward path: Type I. Hence the steady-state tracking error is zero: $e_{\text{ step}}(\infty) = 0$.

(c)

Using MATLAB, we find that a value of $K_1 = 1200$ results in $M_p =$ 17%:

```
s=tf('s');
K1=1200;
sysDc=0.5*K1*(s+3)/(s+10);
sysP=1/(s*(s+12));
syscl=feedback(sysP*sysDc,1);
step(syscl)
```

```
grid on;
```

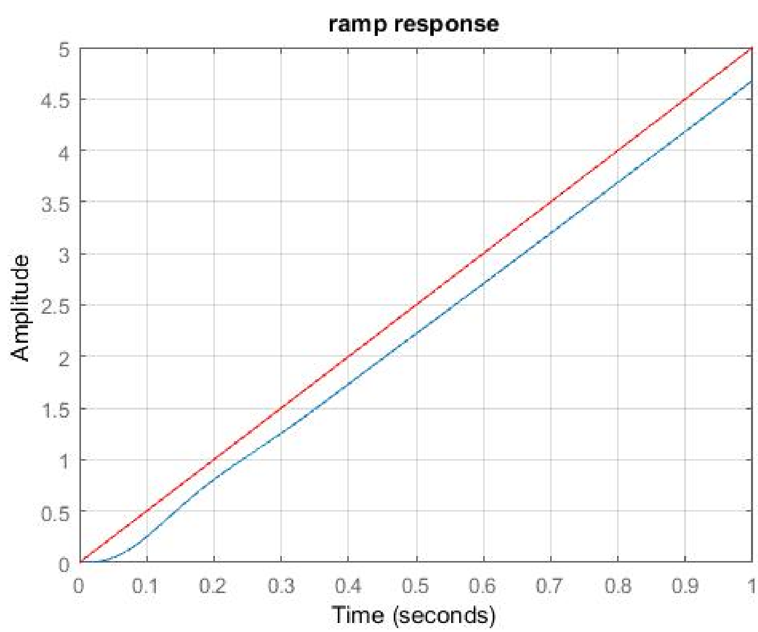
Using MATLAB to find the ramp response:

```
% Ramp response
t=0:.01:1;
step(5*syscl*1/s,t)
hold on;
plot(t,5*t,'red');
title('Ramp response for Problem 4.7 with a slope of size 5.')
```

$$e_{\text{ramp}}(\infty) = 0.32.$$

K_v is the ratio of input ramp rate to steady-state tracking error:

$$K_v = \frac{5}{0.32} = 15.625\,\mathrm{sec}^{-1}.$$



Problem 4.7: Ramp response.

8. A standard feedback control block diagram is shown in Figure 4.5 with the values

$$G(s) = \frac{1}{s};\ D_{cl}(s) = \frac{2(s+1)}{s};\ H(s) = \frac{100}{(s+100)}.$$

(a) Let $W = 0$ and compute the transfer function from R to Y.

(b) Let $R = 0$ and compute the transfer function from W to Y.

(c) What is the tracking error if R is a unit step input and $W \equiv 0$?

(d) What is the tracking error if R is a unit ramp input and $W \equiv 0$?

(e) What is the system type and the corresponding error coefficient?

Solution:

(a)

$$\frac{Y(s)}{R(s)} = \mathcal{T}(s) = F(s)\frac{D_{cl}(s)G(s)}{1 + D_{cl}(s)G(s)H(s)} = \frac{2(s+1)(s+100)}{s^3 + 100s^2 + 200s + 200}.$$

(b)

$$\frac{Y(s)}{W(s)} = \frac{G(s)}{1 + D_{cl}(s)G(s)H(s)} = \frac{s(s+100)}{s^3 + 100s^2 + 200s + 200}.$$

(c)

$$\frac{E(s)}{R(s)} = 1 - \mathcal{T}(s) = \mathcal{S}(s) = \frac{s^3 + 98s^2 - 2s}{s^3 + 100s^2 + 200s + 200}.$$

with $R(s) = \frac{1}{s}$,

$$e_{\text{ step}}(\infty) = \lim_{s \to 0} sE(s) = 0.$$

(d) with $R(s) = \frac{1}{s^2}$,

$$e_{\text{ ramp}}(\infty) = \lim_{s \to 0} sE(s) = -\frac{2}{200} = -\frac{1}{100}.$$

(e) System is Type I and $K_v = \frac{1}{|e_{\text{ ramp}}(\infty)|} = 100\,\text{sec}^{-1}$.

9. A generic negative feedback system with non-unity transfer function in the feedback path is shown in Figure 4.5.

   (a) Find the steady-state tracking error for this system to a ramp reference input.

   (b) If $G(s)$ has a single pole at the origin in the s-plane, what is the requirement on $H(s)$ such that the system will remain a Type I system?

   (c) Suppose,

   $$G(s) = \frac{1}{s(s+1)^2};\ D_{cl}(s) = 0.73;\ H(s) = \frac{2.75s + 1}{0.36s + 1}.$$

   showing a lead compensation in the feedback path. What is the value of the velocity error coefficient, K_v?

Solution:

(a)

$$\begin{aligned}
\frac{Y(s)}{R(s)} &= \mathcal{T}(s) = F(s)\frac{D_{cl}(s)G(s)}{1+D_{cl}(s)G(s)H(s)}.\\
\frac{E(s)}{R(s)} &= 1-\mathcal{T}(s) = \mathcal{S}(s) = \frac{1+D_{cl}(s)G(s)H(s)-F(s)D_{cl}(s)G(s)}{1+D_{cl}(s)G(s)H(s)}.\\
e_{\text{ ramp}}(\infty) &= \lim_{s\to 0} s\frac{1+D_{cl}(s)G(s)H(s)-F(s)D_{cl}(s)G(s)}{1+D_{cl}(s)G(s)H(s)}\frac{1}{s^2}.
\end{aligned}$$

(b) Let $G(s) = \frac{1}{s}G_1(s)$,

$$\begin{aligned}
e_{\text{ step}}(\infty) &= \lim_{s\to 0} s\frac{1+D_{cl}(s)\frac{1}{s}G_1(s)H(s)-F(s)D_{cl}(s)\frac{1}{s}G_1(s)}{1+D_{cl}(s)\frac{1}{s}G_1(s)H(s)}\frac{1}{s},\\
&= \lim_{s\to 0}\frac{s+0.73G_1(s)H(s)-0.73G_1(s)}{s+0.73G_1(s)H(s)}.
\end{aligned}$$

The DC gain of $H(s)$ must be unity, $H(0) = 1$ so that $e_{\text{ step}}(\infty) = 0$.

(c)

$$\begin{aligned}
e_{\text{ ramp}}(\infty) &= \lim_{s\to 0} s\frac{1+D_{cl}(s)G(s)H(s)-F(s)D_{cl}(s)G(s)}{1+D_{cl}(s)G(s)H(s)}\frac{1}{s^2},\\
&= \lim_{s\to 0}\frac{s(s+1)^2(0.36s+1)+0.73(2.75)s-0.73(0.36)s}{s(s+1)^2(0.36s+1)+0.73(2.75s+1)}\frac{1}{s},\\
&= \frac{1+0.73(2.75)-0.73(0.36)}{0.73} = 3.7599,\\
K_v &= 1/3.7599 = 0.266\,\text{sec}^{-1}.
\end{aligned}$$

10. Consider the system shown in Fig. 4.34, where

$$D_c(s) = K\frac{(s+\alpha)^2}{s^2+\omega_o^2}.$$

(a) Prove that if the system is stable, it is capable of tracking a sinusoidal reference input $r = \sin\omega_o t$ with zero steady-state error. (Look at the transfer function from R to E and consider the gain at ω_o.)

(b) Use Routh's criterion to find the range of K such that the closed-loop system remains stable if $\omega_o = 1$ and $\alpha = 0.25$.

Solution:

(a)

$$\begin{aligned}
D_c(s)G(s) &= \frac{K(s+\alpha)^2}{(s^2+\omega_o^2)s(s+1)},\\
\frac{E(s)}{R(s)} &= \frac{1}{1+D_cG},\\
&= \frac{s(s+1)(s^2+\omega_o^2)}{(s^2+\omega_o^2)s(s+1)+K(s+\alpha)^2}.
\end{aligned}$$

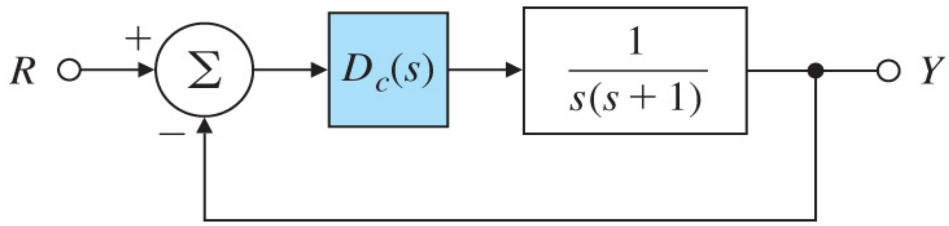


Figure 4.34: Control system for Problem 4.10

The gain of this transfer function is zero at $s = \pm j\omega_o$ and we expect the error to be zero if R is a sinusoid at that frequency. More formally, let $R(s) = \dfrac{\omega_n}{s^2+\omega_n^2}$ then

$$E(s) = \frac{s(s+1)(s^2+\omega_o^2)}{(s^2+\omega_o^2)s(s+1)+K(s+\alpha)^2}\,\frac{\omega_n}{s^2+\omega_n^2}.$$

Assuming the (closed-loop) system is stable, then if $\omega_n \neq \omega_o$, $E(s)$ has a pole on the imaginary axis and the FVT does not apply. The final error will NOT be zero in this case. However, if $\omega_n = \omega_o$ we *can* use the FVT and

$$e_{ss} = \lim_{s\to 0} sE(s) = 0$$

(b) To test for stability, the characteristic equation is,

$$s^4 + (K+\omega_o^2)s^2 + s^3 + (\omega_o^2 + 2\alpha K)s + K\alpha^2 = 0$$

Using the Routh array

$$\begin{array}{llll}
s^4: & 1 & \omega_o^2 + K & K\alpha^2 \\
s^3: & 1 & (\omega_o^2 + 2\alpha K) & \\
s^2: & K(1-2\alpha) & K\alpha^2 & \\
s^1: & \omega_o^2 + 2\alpha K - \dfrac{\alpha^2}{(1-2\alpha)} & & \\
s^0: & K\alpha^2 & &
\end{array}$$

If $\alpha = 0.25$, we must have $K > 0$, and $K > 0.25 - 2\omega_o^2$.

11. Consider the system shown in Fig. 4.35 which represents control of the angle of a pendulum that has no damping.

    (a) What condition must $D_c(s)$ satisfy so that the system can track a ramp reference input with constant steady-state error?

    (b) For a transfer function $D_c(s)$ that stabilizes the system and satisfies the condition in part(a), find the class of disturbances $w(t)$ that the system can reject with zero steady-state error.

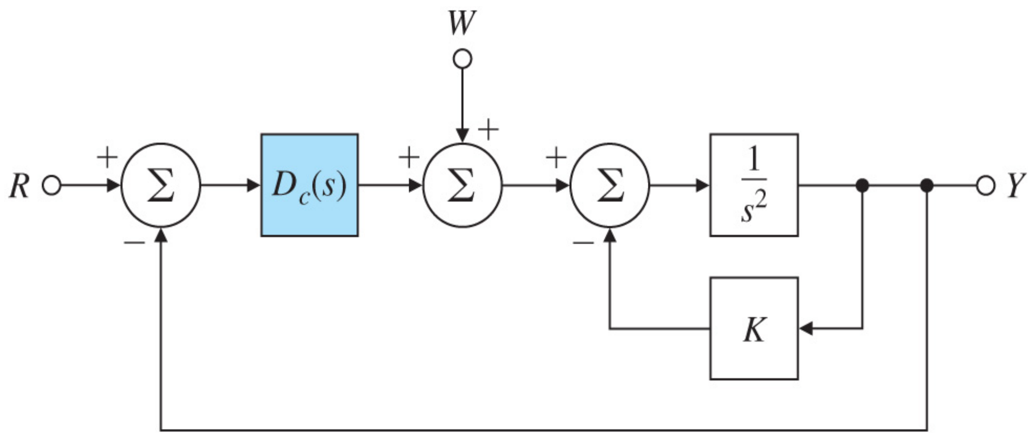


Figure 4.35: Control system for Problem 4.11

Solution:

(a) For a unity feedback system to be Type 1 the open-loop transfer function must have a pole at $s = 0$. Thus in this case, since G has no such pole, it is necessary for D_c to have a pole at $s = 0$.

(b)

$$Y(s) = \frac{1}{s^2 + D_c(s) + K} W(s)$$

$$\lim_{s \to 0} s \left(\frac{1}{s^2 + D_c(s) + K} \right) \frac{1}{s^\ell} = 0$$

iff

$$\lim_{s \to 0} s^{\ell - 1} D_c(s) = \infty$$

iff $\ell = 1$ since $D_c(s)$ has a pole at the origin. Therefore system will reject step disturbances with zero error.

12. A unity feedback system has the overall transfer function

$$\frac{Y(s)}{R(s)} = \mathcal{T}(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}.$$

Give the system type and corresponding error constant for tracking polynomial reference inputs in terms of ζ and ω_n.

Solution:

$$\frac{E(s)}{R(s)} = \frac{s^2 + 2\zeta\omega_n s}{s^2 + 2\zeta\omega_n s + \omega_n^2}.$$

Therefore the system is Type 1. The loop transfer function is:

$$L(s) = \frac{\mathcal{T}(s)}{1 - \mathcal{T}(s)} = \frac{\omega_n^2}{s(s + 2\zeta\omega_n)},$$

and the velocity constant is

$$K_v = \lim_{s \to 0} sL(s) = \frac{\omega_n}{2\zeta}.$$

13. Consider the second-order system

$$G(s) = \frac{1}{s^2 + 2\zeta s + 1}.$$

We would like to add a transfer function of the form $D_c(s) = K(s+a)/(s+b)$ in series with $G(s)$ in a unity-feedback structure.

(a) Ignoring stability for the moment, what are the constraints on K, a, and b so that the system is Type 1?

(b) What are the constraints placed on K, a, and$b$ so that the system is both stable and Type 1?

(c) What are the constraints on a and b so that the system is both Type 1 and remains stable for every positive value for K?

Solution:

(a) In a unity feedback structure, the error is $1/(1 + GD_c)$ and, as we saw, to be Type 1, there needs to be a pole at $s = 0$ in the product GD_c. Since there is no such pole in G, it must be supplied by D_c, thus, the answer is

$$b = 0$$

(b) To assure stability, all poles of the closed loop must be in the left half plane, for which the criterion is by Routh. Thus the characteristic equation is

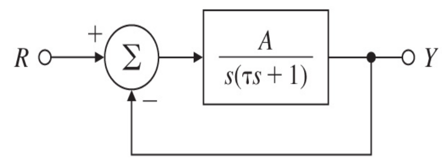
$$s(s^2 + 2\zeta s + 1) + K(s + a) = 0$$

and the Routh array is

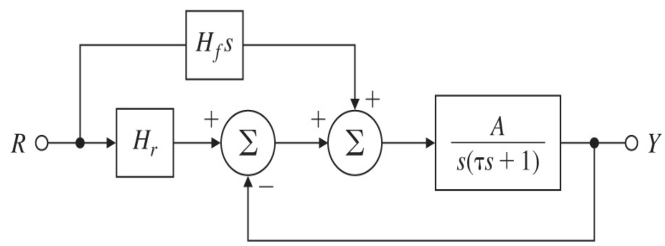
$$\begin{array}{lll} s^3: & 1 & 1+K \\ s^2: & 2\zeta & aK \\ s^1: & \frac{2\zeta(1+K)-aK}{2\zeta} & \\ s^0: & aK & \end{array}$$

Thus the requirements are

$$\begin{aligned} 2\zeta(1+K) - aK &> 0 \\ aK &> 0 \end{aligned}$$



(a)



(b)

Figure 4.36: Control system for Problem 4.14

(c) If we assume that $\zeta > 0$ and, for this part, that $a > 0$ also, the requirements can be reduced to

$$\begin{aligned} K &> 0 \\ 2\zeta + K(2\zeta - a) &> 0 \end{aligned}$$

If $a < 2\zeta$, inspection of these conditions shows that the system will be *stable for all positive values of* K. On the other hand, if $a > 2\zeta$, then the requirement is

$$0 < K < \frac{2\zeta}{a - 2\zeta}$$

Extra credit: work out the case for $\zeta < 0$. **Note to the Instructor**: you might come back to this problem in Chapter 5 and verify this point using the rule of asymptotes.

14. Consider the system shown in Fig. 4.36(a).

(a) What is the system type? Compute the steady-state tracking error due to a ramp input $r(t) = r_o t1(t)$.

(b) For the modified system with a feed forward path shown in Fig.4.36(b), give the value of H_f so the system is Type 2 for reference inputs and compute the K_a in this case.

(c) Is the resulting Type 2 property of this system robust with respect to changes in H_f i.e., will the system remain Type 2 if H_f changes slightly?

Solution:

(a) System is Type 1 since it is unity feedback and has a pole at $s = 0$in the forward path. Also,

$$\begin{aligned} E(s) &= [1 - \mathcal{T}(s)]R(s), \\ &= \left[\frac{1}{1 + G(s)}\right] R(s), \\ &= \frac{s(\tau s + 1)}{s(\tau s + 1) + A}\frac{r_o}{s^2}. \end{aligned}$$

The steady-state tracking error using the FVT (assuming stability) is,

$$e_{ss} = \lim_{s \to 0} sE(s) = \frac{r_o}{A}.$$

(b)

$$\begin{aligned} Y(s) &= \frac{A}{s(\tau s + 1)}U(s), \\ U(s) &= H_f s R(s) + H_r R(s) - Y(s), \\ Y(s) &= \frac{A(H_f s + H_r)}{s(\tau s + 1) + A}R(s). \end{aligned}$$

The tracking error is,

$$E(s) = R(s) - Y(s),$$

$$\begin{aligned} &= \frac{s(\tau s + 1) + A - A(H_f s + H_r)}{s(\tau s + 1) + K}R(s), \\ &= \frac{\tau s^2 + (1 - AH_f)s + A(1 - H_r)}{s(\tau s + 1) + A}. \end{aligned}$$

To get zero steady-state error with respect to a ramp, the numerator in the above equation must have a factor s^2. For this to happen, let

$$\begin{aligned} H_r &= 1, \\ AH_f &= 1. \end{aligned}$$

Then

$$E(s) = \frac{\tau s^2}{s(\tau s + 1) + A}R(s)$$

and, with $R(s) = \dfrac{r_o}{s^2}$, apply the FVT (assuming stability) to obtain

$$e_{ss} = 0.$$

Thus the system will be Type 2 with $K_a = \dfrac{\tau}{A}$.

(c) No, the system is not robust Type 2 because the property is lost if either H_r or H_f changes slightly.

15. A controller for a satellite attitude control with transfer function $G = 1/s^2$ has been designed with a unity feedback structure and has the transfer function $D_c(s) = \dfrac{10(s+2)}{s+5}$.

   (a) Find the system type for reference tracking and the corresponding error constant for this system.

   (b) If a disturbance torque adds to the control so that the input to the process is $u + w$, what is the system type and corresponding error constant with respect to disturbance rejection?

Solution:

(a) There are two poles at $s = 0$ so the system is Type 2 and the error constants are:

$$K_p = \lim_{s\to 0} D_c(s)G(s) = \infty.$$

$$e_{ss} = \frac{1}{1+K_p} = 0.$$

$$K_v = \lim_{s\to 0} sD_c(s)G(s) = \infty.$$

$$e_{ss} = \frac{1}{K_v} = 0.$$

$$K_a = \lim_{s\to 0} s^2 D_c(s)G(s) = 4.$$

$$e_{ss} = \frac{1}{K_a} = 0.25.$$

(b) For the disturbance input, the poles at $s = 0$ are *after* the input and therefore the system is Type 0. The error is

$$\begin{aligned}\frac{E(s)}{W(s)} &= -\frac{G}{1+GD_c}, \\ &= -\frac{s+5}{s^2(s+5)+10(s+2)}.\end{aligned}$$

The steady-state error to a step is thus $e_{ss} = 0.25 = \dfrac{1}{1+K_p}$. Therefore,

$$K_p = 3.$$

16. A compensated motor position control system is shown in Fig. 4.37. Assume that the sensor dynamics are $H(s) = 1$.

(a) Can the system track a step reference input r with zero steady-state error? If yes, give the value of the velocity constant.

(b) Can the system reject a step disturbance w with zero steady-state error? If yes, give the value of the velocity constant.

(c) Compute the sensitivity of the closed-loop transfer function to changes in the plant pole at -2.

(d) In some instances there are dynamics in the sensor. Repeat parts(a) to (c) for $H(s) = 20/(s+20)$ and compare the corresponding velocity constants.

Solution:

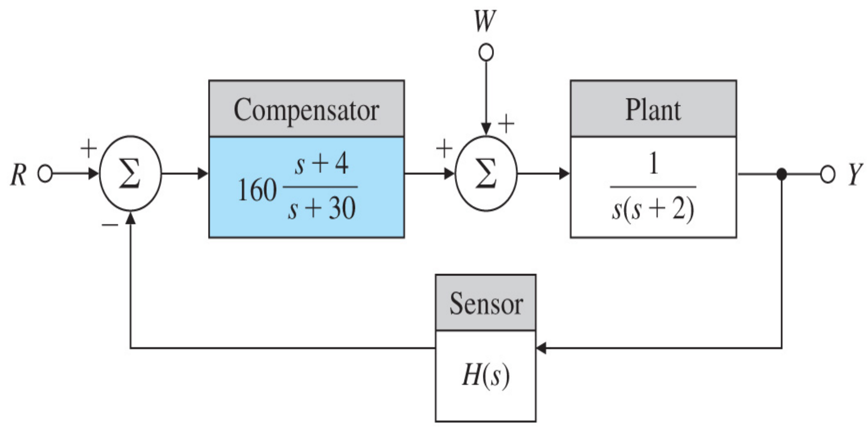


Figure 4.37: Control system for Problem 4.16

(a) The system is Type 1 with $H(s) = 1$.

$$E(s) = R(s) - Y(s) = \frac{s(s+2)(s+30)}{s(s+2)(s+30) + 160(s+4)}.$$

$$e_{ss} = \lim_{s \to 0} sE(s) = 0.$$

So the system can track a step input in the steady-state. The velocity constant is $K_v = \dfrac{4 \times 160}{2 \times 30} = 10.67\,\text{sec}^{-1}$.

(b) The system is Type 0 with respect to the disturbance and has the steady-state error.

$$y_{ss} = -\lim_{s \to 0} sY(s) = -\frac{s+30}{s(s+2)(s+30) + 160(s+4)},$$

$$= \frac{30}{640} = 0.046875.$$

So the system cannot reject a constant disturbance.

(c)

$$T(s) = \frac{160(s+4)}{s(s+A)(s+30) + 160(s+4)}.$$

where A was inserted for the pole at the nominal value of 2.

$$\mathcal{S}_A^T = \frac{A}{T}\frac{\partial T}{\partial A}.$$

But,

$$\frac{\partial T}{\partial A} = -\frac{160(s+4)(s)(s+30))}{[s(s+30)(s+A) + 160(s+4)]^2} = \frac{160(s+4)(s)(s+30))}{[*]^2},$$

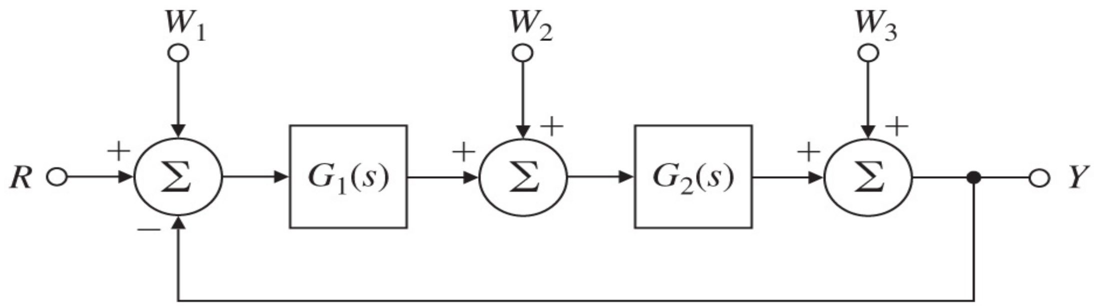


Figure 4.38: Single input-single output unity feedback system with disturbance inputs

therefore,

$$\mathcal{S}_A^T = -\frac{A[*]160(s+4)(s)(s+30)}{160(s+4)[*]^2}, \\ = -\frac{2s(s+30)}{s(s+30)(s+2)+160(s+4)}.$$

At $s = 0$ the sensitivity is zero.

(d) $T(s) = \frac{D_c(s)G(s)}{1+D_c(s)G(s)H(s)}, \mathcal{S}(s) = \frac{1+D_c(s)G(s)H(s)-D_c(s)G(s)}{1+D_c(s)G(s)H(s)}$. Because the system type is computed at $s = 0$ and at that value $H(0) = 1$, then the system remains Type 1 with respect to the reference input. However, the velocity error coefficient will change. The new expression for the error is

$$E(s) = \mathcal{S}(s)R(s) = \frac{s[(s+2)(s+30(s+20)-(160)(s+4)]}{s(s+2)(s+30)(s+20)+20(160)(s+4)}\frac{1}{s^2},$$

from which $e_{ss} = 0.0437, K_v = \frac{1}{e_{ss}} = 22.86\,\text{sec}^{-1}$. The system remains Type 0 with respect to the disturbance input with the same position error constant $K_p = 21.33$.

17. The general unity feedback system shown in Fig. 4.38 has disturbance inputs w_1, w_2 and w_3 and is asymptotically stable. Also,

$$G_1(s) = \frac{K_1 \prod_{i=1}^{m_1}(s+z_{1i})}{s^{l_1} \prod_{i=1}^{m_1}(s+p_{1i})}, \quad G_2(s) = \frac{K_2 \prod_{i=1}^{m_1}(s+z_{2i})}{s^{l_2} \prod_{i=1}^{m_1}(s+p_{2i})}.$$

(a) Show that the system is of Type 0, Type l_1, and Type $(l_1 + l_2)$ with respect to disturbance inputs w_1, w_2, and w_3 respectively.

Solution:

(a)

$$Y(s)=\frac{G_1(s)G_2(s)}{1+G_1(s)G_2(s)}W_1(s)=\frac{K_1K_2[\prod_i(s+z_i)]W_1(s)}{s^{l_1+l_2}\prod_i(s+p_i)+K_1K_2\prod_i(s+z_i)}\qquad\text{(i)}$$

(p_i, z_i are the poles and zeros of G_1 , G_2 not at the origin).

$$-e_{ss}=y_{ss}=\lim_{s\to 0}[sY(s)]=\lim_{s\to 0}[sW_1(s)]\qquad\text{Type 0}$$

$$Y(s)=\frac{G_2(s)}{1+G_1(s)G_2(s)}W_2(s)=\frac{K_2[\prod_i(s+z_{2i})]s^{l_1}\prod_i(s+p_{1i})}{\Delta(s)}W_2(s)\qquad\text{(ii)}$$

$\Delta(s)$ is the characteristic polynomial, same as in (i) (denominator in (i)).

$$y_{ss}=[\lim_{s\to 0}s.W_2(s).s^{l_1}]\frac{\prod_i p_{1i}}{\prod_i z_{1i}}\qquad\text{Type }\ell_1$$

$$Y(s)=\frac{W_3(s)}{1+G_1(s)G_2(s)}=\frac{s^{l_1+l_2}\prod_i(s+p_i)}{\Delta(s)}W_3(s)\qquad\text{(iii)}$$

$$y_{ss}=[\lim_{s\to 0}s.W_3(s).s^{l_1+l_2}]\frac{\prod_i p_i}{\prod_i z_i}\qquad\text{Type }\ell_1+\ell_2$$

$$Y_1=\frac{1}{s^2+s+1}R_1+\frac{s}{s^2+s+1}W_1+\frac{s(s+1)}{s^2+s+1}W_2.$$

For constant disturbances, $R_1=0,\ W_1(s)=\dfrac{W_{10}}{s}$, $W_2(s)=\dfrac{W_{20}}{s}$,

$$Y_1=\frac{W_{10}+(s+1)W_{20}}{s^2+s+1}.$$

Let u_2 be the signal coupling systems 1 and 2:

$$\begin{aligned}U_2 &= \frac{(s+1)(R_1-W_2)+s(s+1)W_1}{s^2+s+1}\\ Y_2 &= \frac{R_2}{s^2+3s+2}+\frac{(s+1)U_2}{s^2+3s+2}\\ &= \frac{(s+1)^2(-W_2)+s(s+1)^2W_2}{(s^2+3s+2)(s^2+s+1)}.\end{aligned}$$

The system Type w.r.t. disturbances:

y_1	w.r.t. W_1	Type 1
y_1	w.r.t. W_2	Type 1
y_2	w.r.t. W_1	Type 1
y_2	w.r.t. W_2	Type 0

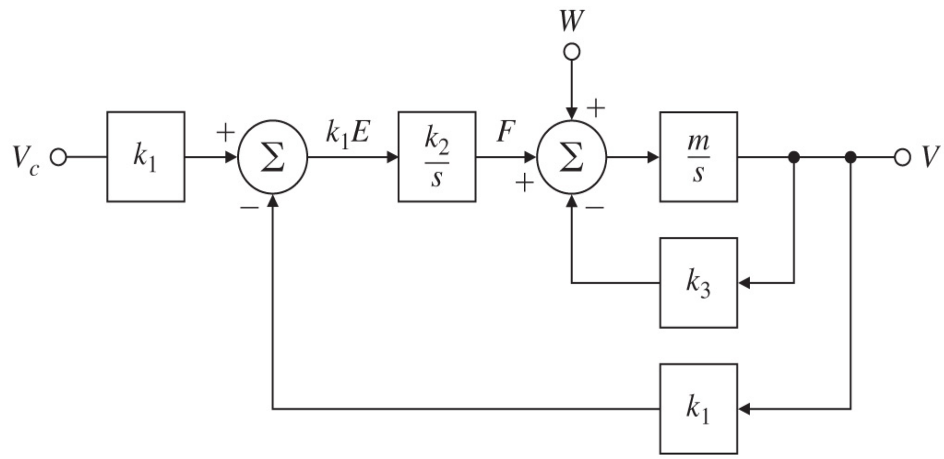


Figure 4.39: System using integral control

can be determined by applying FVT to Y_1 and Y_2 or by inspection following the rule of part (a).

18. One possible representation of an automobile speed-control system with integral control is shown in Fig. 4.39

    (a) With a zero reference velocity input ($v_c = 0$), find the transfer function relating the output speed v to the wind disturbance w.

    (b) What is the steady-state response of v if w is a unit ramp function?

    (c) What type is this system in relation to reference inputs? What is the value of the corresponding error constant?

    (d) What is the type and corresponding error constant of this system in relation to tracking the disturbance w?

    Solution:

(a)

$$\frac{V(s)}{W(s)} = \frac{ms}{s^2 + mk_3 s + mk_1 k_2}.$$

(b)

$$v_{ss} = \lim_{s\to 0} s \frac{V(s)}{W(s)} \frac{W_0}{s^2} = \frac{W_0}{k_1 k_2}.$$

where $W(s) = \dfrac{W_0}{s^2}$.

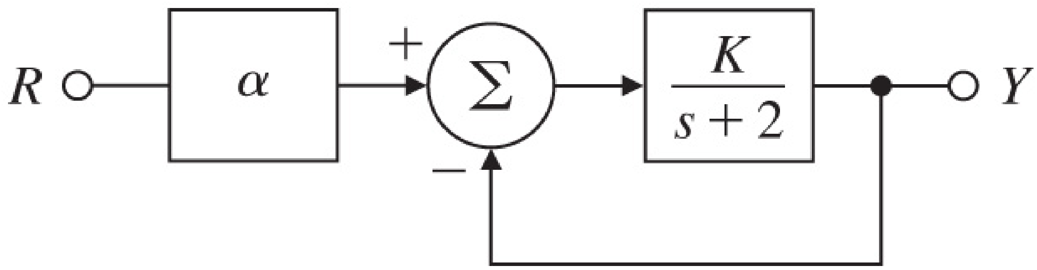


Figure 4.40: Control system for Problem 4.19

(c)

$$E = V_c - V = \left[1 - \frac{\frac{k_1k_2m}{s(s+mk_3)}}{1+\frac{k_1k_2m}{s(s+mk_3)}}\right] V_c = \frac{1}{1+\underbrace{\frac{mk_1k_2}{s(s+mk_3)}}_{G(s)}} V_c$$

$$e_{ss} = \lim_{s\to 0} \frac{sV_c}{1+G(s)}.$$

$$K_p = \lim_{s\to 0} G(s) = \infty \Longrightarrow e_\infty(\text{step input}) = 0.$$

$$K_v = \lim_{s\to 0} sG(s) = \frac{k_1k_2}{k_3} \Longrightarrow e_\infty(\text{ramp input}) = \frac{1}{K_v} = \frac{k_3}{k_1k_2}.$$

System is Type 1.

(d) For disturbances: If the disturbance is a ramp, the result of part (a) shows that the steady state error, which is V, will be $e_{ss} = \frac{1}{k_1k_2}$. Therefore, the system is Type 1 and the velocity constant is $K_v = k_1k_2$.

19. For the feedback system shown in Fig. 4.40, find the value of α that will make the system Type 1 for $K = 5$. Give the corresponding velocity constant. Show that the system is not robust by using this value of α and computing the tracking error $e = r - y$ to a step reference for $K = 4$ and $K = 6$.

Solution:

$$Y = \frac{\alpha K R}{s+2+K} \qquad E = R - Y = \frac{s+2+K(1-\alpha)}{s+2+K} R \bigg|_{K=5} = \frac{s+7-5\alpha}{s+7} R.$$

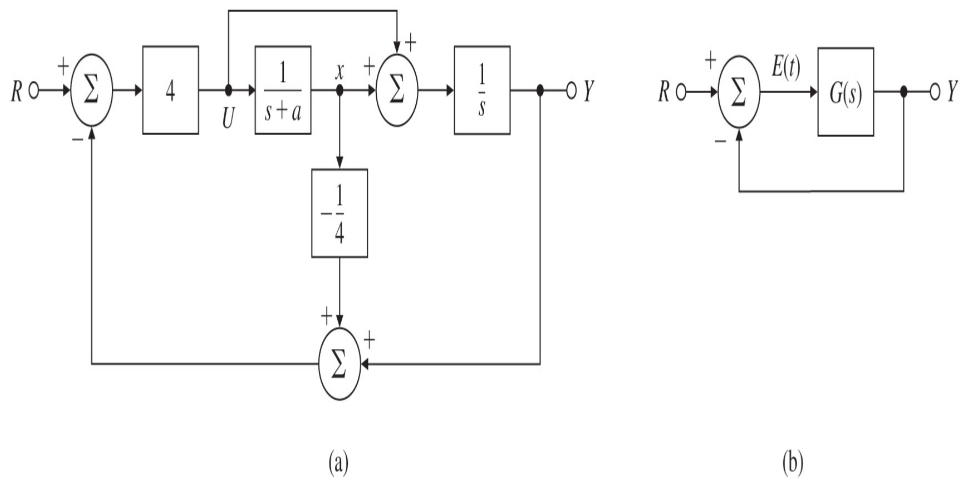


Figure 4.41: Control system for Problem 4.20

For $\alpha = \frac{7}{5}$ we have:

$$E = \frac{s}{s+7}R,$$

$$e_{ss,step} = \lim_{s\to 0} s\frac{s}{s+7}\frac{r_0}{s} = 0,$$

$$e_{ss,ramp} = \lim_{s\to 0} s\frac{s}{s+7}\frac{v_0}{s^2} = \frac{v_0}{7}.$$

So the system is Type 1.The system is not robust:

$$e_{ss,step}|_{K=4} = \lim_{s\to 0} s\frac{s+2+4(\frac{-2}{5})}{s+6}\frac{r_0}{s} = -\frac{r_0}{15}.$$

$$e_{ss,step}|_{K=6} = -\frac{r_0}{20}.$$

So the system is Type 0 if $K \neq 5$.

20. Suppose you are given the system depicted in Fig. 4.41(a), where the plant parameter a is subject to variations.

   (a) Find $G(s)$ so that the system shown in Fig.4.41(b) has the same transfer function from r to y as the system in Fig.4.41(a).

   (b) Assume that $a = 1$ is the nominal value of the plant parameter. What is the system type and the error constant in this case?

   (c) Now assume that $a = 1 + \delta a$, where δa is some perturbation to the plant parameter. What is the system type and the error constant for the perturbed system?

Solution:

(a)

$$\begin{aligned} Y(s) &= \frac{1}{s}\left(1+\frac{1}{s+a}\right)U(s), \\ &= \frac{1}{s}1+\frac{1}{s+a}\left(1+\frac{1}{s+a}\right)4(R(s)-Y(s)+\frac{1}{4(s+a)}U(s)). \qquad (1) \end{aligned}$$

$$\begin{aligned} U(s) &= 4R(s)-4Y(s)+\frac{1}{s+a}U(s), \\ \left(1-\frac{1}{s+a}\right)U(s) &= 4R(s)-4Y(s), \\ U(s) &= \frac{4(s+a)}{s+a-1}[R(s)-Y(s)]. \qquad (2) \end{aligned}$$

Combining Eqs. (1) and (2) gives

$$\begin{aligned} Y(s) &= \frac{1}{s}\left(\frac{s+a+1}{s+a}\right)\left(\frac{4(s+a)}{s+a-1}\right)(R(s)-Y(s)), \\ &= \frac{4(s+a-1)}{s(s+a-1)}[R(s)-Y(s)], \end{aligned}$$

which means

$$G(s)=\frac{4(s+a+1)}{s(s+a-1)}.$$

(b) $a=1$ therefore $G(s)=\dfrac{4(s+2)}{s^2}$

$$\frac{E(s)}{R(s)}=\frac{1}{1+G(s)}=\frac{s^2}{s^2+4s+8},$$

roots are in LHP so we can use the FVT,

$$e_{ss,step}=\lim_{\rightarrow 0} s(\frac{1}{s})\frac{s^2}{s^2+4s+8}=0,$$

therefore $K_p=\infty$

$$e_{ss,ramp}=\lim_{s\rightarrow 0} s(\frac{1}{s^2})\frac{s^2}{s^2+4s+8}=0,$$

and $K_v=\infty$. The error to acceleration is

$$e_{ss,parabola}=\lim_{\rightarrow 0} s(\frac{1}{s^3})\frac{s^2}{s^2+4s+8}=\frac{1}{8}.$$

therefore $K_a=\frac{1}{8}$ and the system is Type 2

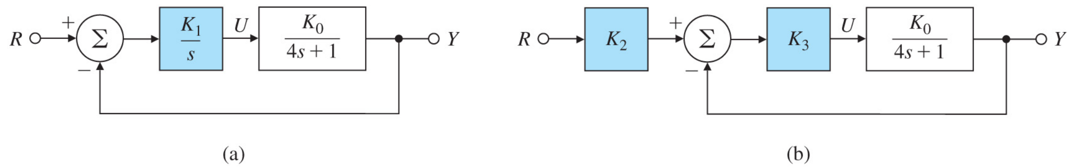


Figure 4.42: Two feedback systems for Problem 4.21

(c)

$$\frac{E(s)}{R(s)} = \frac{s(s+\delta a)}{s^2+(4+\delta a)s+4(2+\delta a)}$$

for small δa, roots remain in LHP.

$$e_{ss,step} = \lim_{s\to 0} s\left(\frac{1}{s}\right)\frac{s(s+\delta a)}{s^2+(4+\delta a)s+4(2+\delta a)} = 0$$

therefore $K_p = \infty$.

$$e_{ss,ramp} = \lim_{s\to 0} s\left(\frac{1}{s^2}\right)\frac{s(s+\delta a)}{s^2+(4+\delta a)s+4(2+\delta a)} = \frac{\delta a}{4(2+\delta a)}.$$

Therefore, $K_v = \dfrac{4(2+\delta a)}{\delta a}$. For parabolic input, $e(t)\to\infty$, therefore, $K_a = 0$. The system is now Type 1. Plant error (parameter variation) caused the change in system type.

21. Two feedback systems are shown in Fig. 4.42.

    (a) Determine values for K_1, K_2, and K_3 so that both systems:

        i. exhibit zero steady-state error to step inputs (that is, both are Type1), and

        ii. whose static velocity error constant $K_v = 1$ when $K_0 = 1$.

    (b) Suppose K_0 undergoes a small perturbation: $K_0 \to K_0 + \delta K_0$. What effect does this have on the system type in each case? Which system has a type which is robust? Which system do you think would be preferred?

    Solution:

    (a) System (a):

$$\begin{aligned} E &= R - Y = \frac{R(s)}{1+G(s)}, \\ E(s) &= \frac{s(4s+1)}{4s^2+s+K_0K_1}R(s). \end{aligned}$$

Applying FTV:

$$e_{ss,ramp} = \frac{1}{K_1} = \frac{1}{K_v} \implies K_1 = K_v = 1.$$

System (b):

$$\begin{aligned} E &= R - Y = \frac{1 + G - K_2 G}{1 + G} R, \\ E(s) &= \frac{4s + 1 + K_3 K_0 (1 - K_2)}{4s + 1 + K_3 K_0} R(s). \end{aligned}$$

Applying FVT:

$$e_{ss,step} = \frac{1 + K_3 K_0 (1 - K_2)}{1 + K_3 K_0} = 0,$$

for $K_0 = 1 \implies 1 + K_3(1 - K_2) = 0$

$$e_{ss,ramp} = \frac{4}{1 + K_3} = \frac{1}{K_v}$$

for $K_v = 1 \implies K_3 = 3$

$$\implies K_2 = \frac{4}{3}\ ,\ K_3 = 3.$$

(b) Let $K_0 = K_0 + \delta K_0$ In System (a):

$$e_{ss,step} = \lim_{s \to 0} s \frac{s(4s + 1)}{(4s^2 + s + K_0 + \delta K_0)} \frac{1}{s} = 0$$

regardless of K_0 value. In System (b):

$$e_{ss,step} = \frac{1 + K_3 (K_0 + \delta K_0)(1 - K_2)}{1 + K_3 (K_0 + \delta K_0)}|_{K_0 = 1} = \frac{-\delta K_0}{1 + 3(1 + \delta K_0} \neq 0.$$

Thus the system type of System (b) is not robust (it is a "calibrated" system type.) Control engineers prefer system (a) over (b) because it is more robust to parameter changes. (This can be expected for a closed-loop with feedback to the input while (b) has an open-loop stage to entering the feedback loop.)

22. You are given the system shown in Fig. 4.43, where the feedback gain β is subject to variations. You are to design a controller for this system so that the output $y(t)$ accurately tracks the reference input $r(t)$.

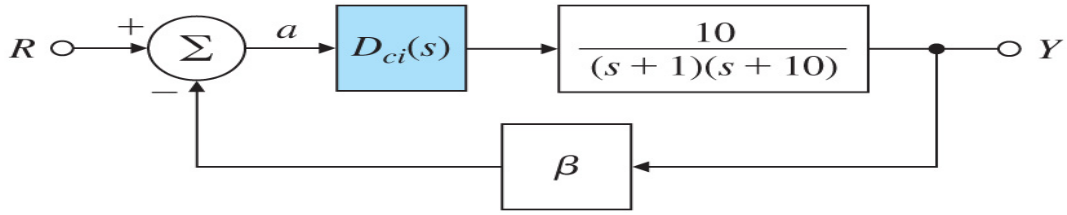


Figure 4.43: Control system for Problem 4.22

(a) Let $\beta = 1$. You are given the following three options for the controller $D_{ci}(s)$:

$$D_{c1}(s) = k_P, \quad D_{c2}(s) = \frac{k_P s + k_I}{s}, \quad D_{c3}(s) = \frac{k_P s^2 + k_I s + k_2}{s^2}.$$

Choose the controller (including particular values for the controller constants) that will result in a Type1 system with a steady-state error to a unit reference ramp of less than $\frac{1}{10}$.

(b) Next, suppose that there is some attenuation in the feedback path that is modeled by $\beta = 0.9$. Find the steady-state error due to a ramp input for your choice of $D_{ci}(s)$ in part(a).

(c) If $\beta = 0.9$, what is the system type for part(b)? What are the values of the appropriate error constant?

Solution:

(a) Need an integrator in the loop - choose $D_{c2}(s)$

$$T(s) = \frac{Y(s)}{R(s)} = \frac{\dfrac{10(k_P s + k_I)}{s(s+1)(s+10)}}{1 + \beta \dfrac{10(k_P s + k_I)}{s(s+1)(s+10)}},$$

$$E(s) = (1 - T(s))R(s) = \left[\frac{s(s+1)(s+10) + \ 10(k_P s + k_I)\beta - 10(k_P s + k_I)}{s(s+1)(s+10) + 10(k_P s + k_I)\beta}\right] \frac{1}{s^2}.$$

For $\beta = 1$,

$$e_{ss} = \lim_{s \to 0} s \left[\frac{s(s+1)(s+10)}{s(s+1)(s+10) + 10(k_P s + k_I)}\right] \frac{1}{s^2},$$

$$= \frac{10}{10 k_I} = \frac{1}{k_I}.$$

Therefore $k_I \geq 10$ will meet the steady-state specifications The closed-loop poles are the roots of $s^3 + 11s^2 + 10s + 10(k_P s + k_I) = 0$.

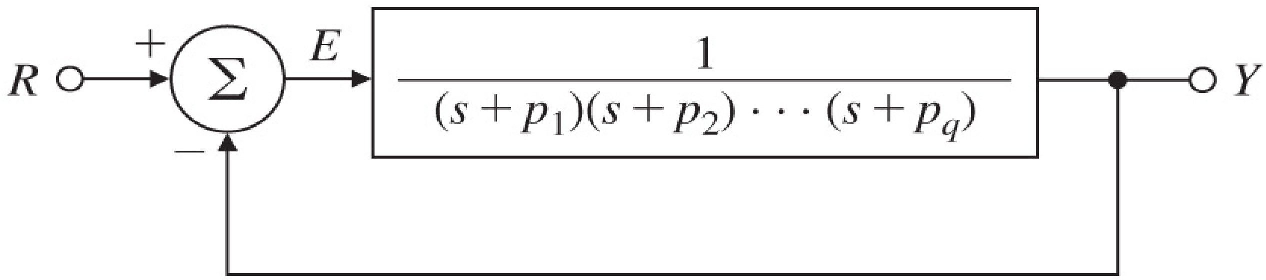


Figure 4.44: Control system for Problem 4. 23

The Routh's array is,

$$\begin{array}{lcc} s^3: & 1 & 10(1+k_P\ ) \\ s^2: & 11 & 10k_I \\ s^1: & \frac{110(1+k_P)-10k_I}{11} & \\ s^0: & 10k_I & \end{array}$$

which requires $k_I > 0$ and $11(1+k_P) - k_I > 0$ for stability.

(b) From above, with $\beta = 0.9$

$$\begin{aligned} E(s) &= \left[\frac{s(s+1)(s+10) + \ 9(k_P s + k_I) - 10(k_P s + k_I)}{s(s+1)(s+10) + 9(k_P s + k_I)}\right] R(s), \\ &= \frac{s(s+1)(s+10) - k_P - k_I}{s(s+1)(s+10) + 9(k_P s + k_I)} R(s). \end{aligned}$$

If k_P and k_I are chosen so that the system is stable, applying the FVT for $R(s) = \dfrac{1}{s^2}$ results in $e_{ss} \to \infty$. The system is no longer Type 1.

(c) Try $R(s) = \dfrac{1}{s}$

$$\begin{aligned} \lim_{s\to 0} sE(s) &= \lim_{s\to 0} \frac{s(s+1)(s+10) - k_P s - k_I}{s(s+1)(s+10) + 9(k_P s + k_I)}, \\ &= -\frac{k_I}{9k_I} = -\frac{1}{9}, \end{aligned}$$

and the system is Type 0. K_p is defined such that $|e_{ss}| = \dfrac{1}{1+K_p}$. Thus, $K_p = 8$. Without the magnitude an equivalent result is that $K_p = -10$.

23. Consider the system shown in Fig. 4.44.

(a) Find the transfer function from the reference input to the tracking error.

(b) For this system to respond to inputs of the form $r(t) = t^n 1(t)$ (where $n < q$) with zero steady-state error, what constraint is placed on the open-loop poles $p_1, p_2, \cdots, p_q$?

Solution:

(a)

$$\frac{E(s)}{R(s)} = \frac{1}{1+G(s)} = \frac{\prod_{i=1}^{q}(s+p_i)}{\prod_{i=1}^{q}(s+p_i)+1}.$$

(b)

$$r(t) = t^n \implies R(s) = \frac{n!}{s^{n+1}}.$$

$$e_{ss} = \lim_{s\to 0} s\frac{n!}{s^{n+1}}\frac{\prod_{i=1}^{q}(s+p_i)}{\prod_{i=1}^{q}(s+p_i)+1}.$$

If e_{ss} is to be zero the system must have at least $n+1$ poles at the origin:

$$e_{ss} = \lim_{s\to 0} s\frac{n!}{s^{n+1}}\frac{s^{n+1}\prod_{i=1}^{q}(s+p_i)}{s^{n+1}\prod_{i=1}^{q}(s+p_i)+1} = 0.$$

24. Consider the system shown in Fig. 4.45.

(a) Compute the transfer function from $R(s)$ to $E(s)$ and determine the steady-state error (e_{ss}) for a unit step reference input signal, and a unit ramp reference input signal.

(b) Determine the locations of the closed-loop poles of the system.

(c) Select the system parameters (k, k_p, k_I) such that the closed-loop system has damping coefficient $\zeta = 0.707$ and $\omega_n = 1$ What percent overshoot would you expect in $y(t)$ for unit step reference input?

(d) Find the tracking error signal as a function of time, $e(t)$, if the reference input to the system, $r(t)$, is a unit ramp.

(e) How can we select the PI controller parameters (k_p, k_I) to ensure that the amplitude of the transient tracking error, $|e(t)|$, from part (d) is small ?

(f) What is the transient behavior of the tracking error, $e(t)$, for a unit ramp reference input if the magnitude of the integral gain, k_I, is very large? Does the unit ramp response have an overshoot in that case?

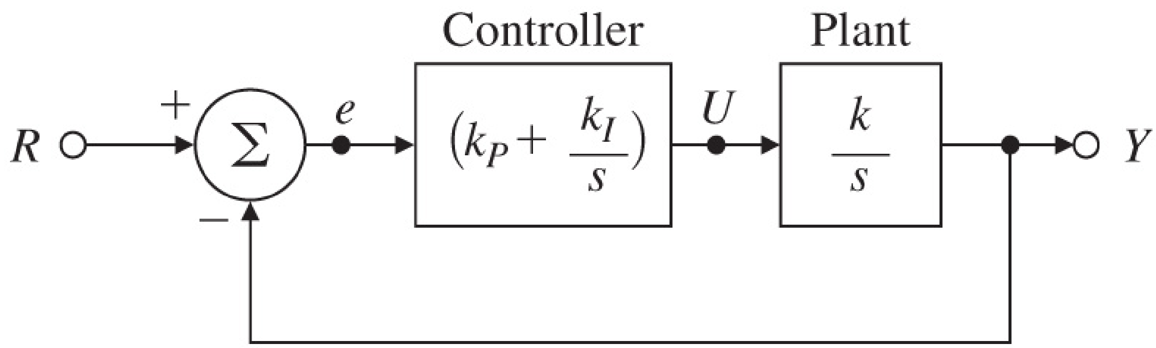


Figure 4.45: Control system diagram for Problem 4.24

Solution:

(a)

$$\begin{aligned}
\frac{E(s)}{R(s)} &= \frac{1}{1+G(s)D_c(s)} = \frac{s^2}{s^2 + k_p k\ s + kk_I}. \\
e_{ss} &= \lim_{s\to 0} sE(s) = \lim_{s\to 0} s\frac{s^2}{s^2 + k_p k\ s + kk_I}\frac{1}{s} = 0,\ \text{(step)} \\
e_{ss} &= \lim_{s\to 0} sE(s) = \lim_{s\to 0} s\frac{s^2}{s^2 + k_p k\ s + kk_I}\frac{1}{s^2} = 0.\ \text{(ramp)}
\end{aligned}$$

(b)

$$s^2 + k_p k\ s + kk_I = 0 \Rightarrow s_{1,2} = \frac{-k_p k \pm \sqrt{(k_p k)^2 - 4kk_I}}{2}.$$

(c)

$$s^2 + k_p k\ s + kk_I = s^2 + 2\zeta\omega_n\ s + \omega_n^2 = s^2 + 2(0.707)\ s + 1 \Rightarrow k_p k = 1.414, kk_I = 1.$$

Now the closed-loop transfer function becomes

$$\begin{aligned}
\frac{Y(s)}{R(s)} &= \frac{G(s)D_c(s)}{1+G(s)D_c(s)} = \frac{k(k_p s + k)}{s^2 + k_p k\ s + kk_I} = \frac{1.414s + 1}{s^2 + 1.414\ s + 1}, \\
&= \frac{\frac{s}{\alpha\zeta\omega_n} + 1}{s^2 + 1.414\ s + 1}.
\end{aligned}$$

So $\alpha = 1$. From Fig. 3.28 we find that $M_p \approx 20\%$.

(d) From part (a)

$$\begin{aligned} E(s) &= \frac{1}{s^2 + k_p k\ s + kk_I}, \\ &= \frac{1}{\left(s + \frac{kkp}{2}\right)^2 + \left(\frac{\sqrt{4kk_I - (kk_p)^2}}{2}\right)^2}. \end{aligned}$$

From the Table inside FPE8e back cover, we compute the inverse Laplace Transform to obtain

$$e(t) = \frac{2}{\sqrt{4kk_I - (kk_p)^2}} e^{-\frac{kkp}{2}t} \sin\left(\frac{\sqrt{4kk_I - (kk_p)^2}}{2} t\right).$$

(e) We have

$$|e(t)| = \frac{2}{\sqrt{4kk_I - (kk_p)^2}} e^{-\frac{kkp}{2}t}.$$

For $|e(t)|$ to be "small," we need

$$\begin{aligned} 4kk_I - (kk_p)^2 &\gg 0, \\ 4kk_I &\gg (kk_p)^2 \\ 4k_I &\gg kk_p^2 \text{ and } \frac{kkp}{2} > 0. \end{aligned}$$

(f) We know the system has an overshoot if there exists a time t_1 such that the tracking error $e(t_1) = 0$. Therefore $e(t)$ can only be zero when $sin(.) = 0$, and we have that

$$\begin{aligned} \sin\left(\frac{\sqrt{4kk_I - (kk_p)^2}}{2} t\right) &= 0, \\ \frac{\sqrt{4kk_I - (kk_p)^2}}{2} t &= n\pi;\ n = \text{integer} \\ t_1 &= \frac{\pi}{\sqrt{4kk_I - (kk_p)^2}} \quad (\text{ first crossing }). \end{aligned}$$

For overshoot, t_1 has to be finite and rather small for practical purposes since we want be small so

$$4k_I \gg kk_p^2.$$

25. A linear ODE model of the DC motor with negligible armature inductance ($L_a = 0$) and with a disturbance torque w was given earlier in the chapter; it is restated here, in slightly different form,as

$$\frac{JR_a}{K_t}\ddot{\theta}_m + K_e\dot{\theta}_m = \upsilon_a + \frac{R_a}{K_t} w,$$

where θ_m is measured in radians. Dividing through by the coefficient of $\ddot{\theta}_m$, we obtain

$$\ddot{\theta}_m + a_1\dot{\theta}_m = b_0 v_a + c_0 w,$$

where

$$a_1 = \frac{K_t K_e}{JR_a}, \quad b_0 = \frac{K_t}{JR_a}, \quad c_0 = \frac{1}{J}.$$

With rotating potentiometers, it is possible to measure the positioning error between θ and the reference angle θ_r or $e = \theta_{ref} - \theta_m$. With a tachometer we can measure the motor speed $\dot{\theta}_m$. Consider using feedback of the error e and the motor speed $\dot{\theta}_m$ in the form

$$v_a = K(e - T_D\dot{\theta}_m),$$

where K and T_D are controller gains to be determined.

(a) Draw a block diagram of the resulting feedback system showing both θ_m and $\dot{\theta}_m$ as variables in the diagram representing the motor.

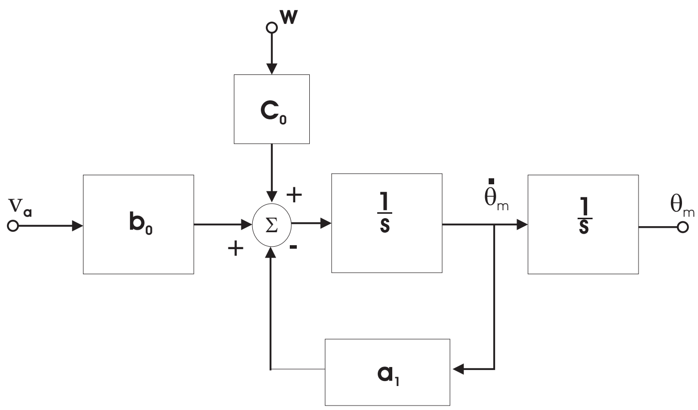
(b) Suppose the numbers work out so that $a_1 = 65$, $b_0 = 200$, and $c_0 = 10$. If there is no load torque ($w = 0$), what speed (in rpm) results from $v_a = 100$V?

(c) Using the parameter values given in part(b), let the control be $D = k_P + k_D s$ and find k_P and k_D so that, using the results of Chapter 3, a step change in θ_{ref} with zero load torque results in a transient that has an approximately 17% overshoot and that settles to within 5% of steady-state in less than 0.05sec.

(d) Derive an expression for the steady-state error to a reference angle input, and compute its value for your design in part(c) assuming $\theta_{ref} = 1$rad.

(e) Derive an expression for the steady-state error to a constant disturbance torque when $\theta_{ref} = 0$, and compute its value for your design in part(c) assuming $w = 1.0$.

Solution:



(a)

Block diagram for Problem 4.25

(b) If va= constant, the system in steady-state:

$$\dot{\theta} = \frac{b_0}{a_1} V_a = \frac{200 \times 100}{65} \frac{60}{2\pi} \quad \frac{rad.sec^{-1}}{rpm} = 2938 \text{ rpm.}$$

(c)

$$\frac{\Theta(s)}{\Theta_r(s)} = \frac{Kb_0}{s^2 + s(a_1 + T_D K b_0) + Kb_0} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2},$$

$M_p = 17\%$, $\Longrightarrow$: $\zeta = 0.5 \quad t_s = 0.05$ sec. to 5% :

$$\Longrightarrow e^{-\zeta\omega_n t_s} = 0.05 \Longrightarrow \zeta\omega_n = 60 \Longrightarrow \omega_n = 120.$$

Comparing coefficients:

$$K = 72 \quad , \quad T_D = 3.8 \times 10^{-3}$$

(d) Steady-state error:

$$E(s) = \Theta_r(s) - \Theta(s) = \frac{s(s + a_1 + T_D K b_0)}{s^2 + s(a_1 + T_D K b_0) + Kb_0} \theta_r.$$

For $\theta_r = \frac{1}{s}$:

$$e_{ss} = \lim_{s \to 0} sE(s) = 0 \quad \text{(Type 1).}$$

(e) Response to torque:

$$\frac{\Theta(s)}{Q_L(s)} = \frac{c_0}{s^2 + s(a_1 + T_D K b_0) + Kb_0}.$$

$$\theta_{ss} = \lim_{s \to 0} s.\theta(s) = \lim_{s \to 0} s \frac{c_0}{s^2 + \dots} \frac{1}{s} = \frac{c_0}{Kb_0} = \frac{1}{1440} \quad \text{rad.}$$

26. We wish to design an automatic speed control for an automobile. Assume that (1)the car has a mass m of 1000kg, (2)the accelerator is the control U and supplies a force on the automobile of 10 N per degree of accelerator motion, and (3)air drag provides a friction force proportional to velocity of $10\text{N}\cdot\text{sec/m}$.

    (a) Obtain the transfer function from control input U to the velocity of the automobile.

    (b) Assume the velocity changes are given by

    $$V(s) = \frac{1}{s+0.02}U(s) + \frac{0.05}{s+0.02}W(s),$$

    where V is given in meters per second, U is in degrees, and W is the percent grade of the road. Design a proportional control law $U = -k_P V$ that will maintain a velocity error of less than 1m/sec in the presence of a constant 2% grade.

    (c) Discuss what advantage (if any) integral control would have for this problem.

    (d) Assuming that pure integral control (that is, no proportional term) is advantageous, select the feedback gain so that the roots have critical damping ($\zeta = 1$).

    Solution:

    a.

    $$\begin{aligned} m\ddot{x} &= \sum F = K_a u - D\dot{x}, \\ \mathcal{L}\{m\dot{v} = K_a u - Dv\}, & \\ \frac{V(s)}{U(s)} &= \frac{K_a}{ms+D} = \frac{0.01}{s+0.01}. \end{aligned}$$

    b. Error:

    $$\begin{aligned} E(s) &= V_d - V = V_d - \frac{\frac{k_P}{s+0.02}}{1+\frac{k_P}{s+0.02}}V_d + \frac{0.05\frac{1}{s+0.02}}{1+\frac{k_P}{s+0.02}}G(s), \\ &= \frac{(s+0.02)V_d - 0.05G}{s+0.02+k_P}, \end{aligned}$$

    If we want error $<$ 1 m/sec in presence of grade, we in fact need $|e_{ss}(\text{step})| < 1$. Assume no input : ($V_d = 0$)

    $$e_{ss}(\text{step}) = \lim_{s\to 0} s\left(\frac{-0.05}{s+0.02+k_P}\right)\frac{2}{s} = \frac{-0.1}{0.02+k_P}$$

    $$\left|\frac{-0.1}{0.02+k_P}\right| < 1$$

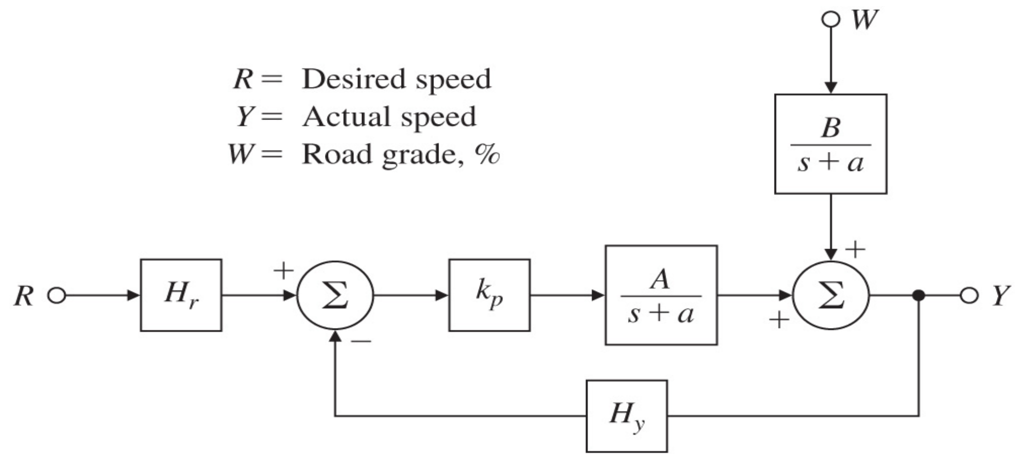


Figure 4.46: Automobile speed-control system

While solving the inequality apply (or check) restriction that poles are in LHP.

$$\Longrightarrow k_P > 0.08.$$

c. The obvious advantage of integral control would be zero s.s. error for step input (Type 1 system would result).

d. Pure integral control: $k_P \rightarrow \dfrac{k_I}{s}$

$$E(s) = \frac{s(s+0.02)V_d - 0.05sG(s)}{s^2 + 0.02s + k_I}.$$

$$\zeta = 1 \Longrightarrow \omega_n = 0.01 \Longrightarrow k_I = 0.0001.$$

27. Consider the automobile speed control system depicted in Fig. 4.46.

(a) Find the transfer functions from $W(s)$ and from $R(s)$ to $Y(s)$.

(b) Assume that the desired speed is a constant reference r, so that $R(s) = r_o/s$. Assume that the road is level, so $w(t) = 0$. Compute values of the gains k_P, H_r, and H_y to guarantee that

$$\lim_{t\to\infty} y(t) = r_o.$$

Include both the open-loop (assuming $H_y = 0$) and feedback cases ($H_y \neq 0$) in your discussion.

(c) Repeat part(b) assuming that a constant grade disturbance $W(s) = w_o/s$ is present *in addition to* the reference input. In particular, find the variation in speed due to the grade change for both the feed forward and feedback cases. Use your results to explain (1)why feedback control is necessary and (2)how the gain k_P should be chosen to reduce steady-state error.

(d) Assume that $w(t) = 0$ and that the gain A undergoes the perturbation $A + \delta A$. Determine the error in speed due to the gain change for both the feed forward and feedback cases. How should the gains be chosen in this case to reduce the effects of δA?

Solution:

(a)

$$Y(s) = \frac{B}{s + a + Ak_P H_y} W(s) + \frac{Ak_P H_r}{s + a + Ak_P H_y} R(s).$$

(b) Feedforward (open-loop: $H_y = 0$):

$$\lim_{s\to 0} sY(s) = \lim_{s\to 0} \frac{Ak_P H_r}{s + a + 0} r_o = r_o,$$

therefore,

$$H_r = \frac{a}{Ak_P}.$$

Feedback ($H_y \neq 0$):

$$\lim_{t\to\infty} y(t) = r_o$$

results in

$$\frac{Ak_P H_r}{a + Ak_P H_y} r_o = r_o$$

Choose k_P for performance and H_y for sensor characteristics, and set

$$H_r = \frac{a + Ak_P H_y}{Ak_P}.$$

(c) Feedforward:

$$\lim_{t\to\infty} y_{ff}(t) = \frac{Bw_o}{a} + \frac{ar_o}{a},$$

$$= r_o + \frac{Bw_o}{a}.$$

Therefore,

$$\delta y_{ff}(\infty) = \frac{Bw_o}{a},$$

all quantities are fixed- so there is no way to reduce the effect of the disturbance.

Feedback:

$$\lim_{t\to\infty} y_{fb}(t) = \frac{B}{a + Ak_P H_y} w + \frac{Ak_P H_r}{a + Ak_P H_y} r,$$

$$= \frac{B}{a + Ak_P H_y} w_o + r_o,$$

(if H_r is chosen as in part **(b)**). Therefore,

$$\delta y_{fb}(\infty) = \frac{B}{a + Ak_P H_y} w_o.$$

Effect of disturbance can be made small by choosing k_P large.

(d) Feedforward: using $k_P = \dfrac{a}{AH_r}$ as derived in part (b),

$$y_{ff}(\infty) = \left(1 + \frac{\delta A}{A}\right) r,$$

therefore,

$$\delta y_{ff}(\infty) = \frac{\delta A}{A} r_o,$$

or

$$\frac{\delta y_{ff}(\infty)}{r_o} = \frac{\delta A}{A},$$

which means that 5% error in A results in 5% error in tracking.
Feedback:

$$y_{fb}(\infty) = \frac{(A + \delta A) k_P H_r}{a + (A + \delta A) k_P H_y} r_o$$

using value for H_r chosen in part **(b)** gives

$$y_{fb}(\infty) = \left[\frac{(A + \delta A)}{a + (A + \delta A) k_P H_y} \frac{a + Ak_P H_y}{A}\right] r_o,$$

$$= r_o + \frac{a\delta A}{aA + (A + \delta A) A k_P H_y} r,$$

$$\cong r_o + \frac{a}{a + Ak_P H_y} \frac{\delta A}{A}$$

$$\frac{\delta y_{fb}(\infty)}{r_o} = \frac{a}{a + Ak_P H_y} \frac{\delta A}{A}.$$

Tracking error due to parameter variation can be reduced by choosing k_P large.

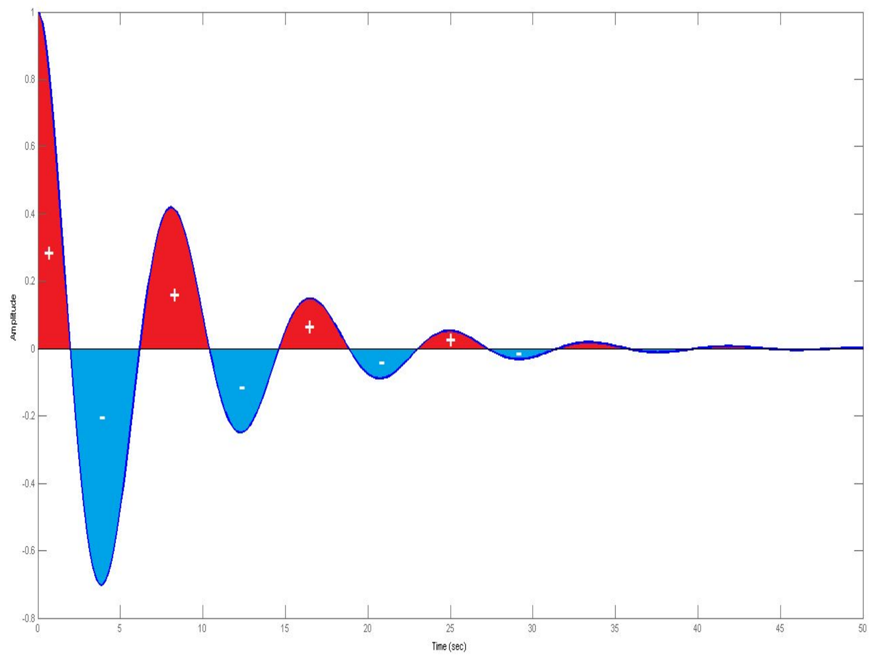
28. Prove that the step response of a Type II closed-loop stable system must *always* have a non-zero overshoot.

Solution: By definition, a Type II system can track a ramp reference input with zero steady-state error

$$e_{ss,ramp} = \lim_{s \to 0} sE(s) = \lim_{s \to 0} s\mathcal{S}(s) \frac{1}{s^2} = \frac{1}{K_v} \tag{4.1}$$

$$= \int_0^\infty e_{ss,step}(\tau) d\tau = 0 \tag{4.2}$$

This equation indicates that the area under the error curve to a step input over all time is equal to zero as shown below (the sum of the positive areas cancel the sum of the negative areas). This implies that there is at least one part of the time at which the error to a step input becomes negative, i.e., $y(t) > r(t)$. Thus there will always be a non-zero overshoot in the step response.



Problem 4.28: Tracking error response.

29. Consider the feedback control system shown in Figure 4.47.

(a) Assume $D_c(s) = K$. What values of K would make the closed-loop system stable? Explain all your reasoning.

(b) Now consider the controller of the form $D_c(s) = \frac{1}{s^n}$ with n being a non-negative integer. For what values of n is the closed-loop system stable? Explain all your reasoning.

Solution:

(a)

$1 + K\frac{(10s+5)}{(s+8)(s-1)} = 0$

$s^2 + (7 + 10K)s + 5K - 8 = 0$

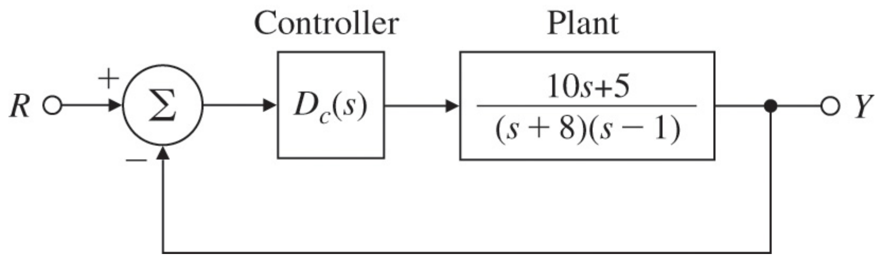


Figure 4.47: Unity feedback system

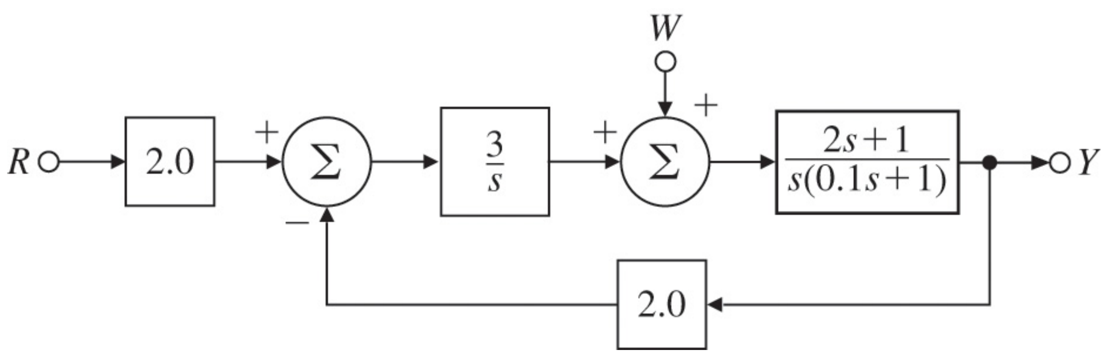


Figure 4.48: Feedback system for Problem 4.30

The Routh array is: $\begin{array}{lcc} s^2: & 1 & 5K-8 \\ s: & 7+10K & \\ s^0 & 5K & \end{array}$

Therefore, we need $K > -\frac{7}{10}$ and $K > \frac{8}{5}$.

(b) If $n = 1$,characteristic equation is $s^3 + 7s^2 + 2s + 5 = 0$ which has all its poles in the LHP as can be verified via the Routh test. If $n \geq 2$ then some of the coefficients in the characteristic polynomial will be negative or missing and the system will not be stable.

30. A feedback control system is shown in Figure 4.48.

    (a) Determine the system Type with respect to the reference input.

    (b) Compute the steady-state tracking errors for unit step and ramp inputs.

    (c) Determine the system Type with respect to the disturbance input, w.

    (d) Compute the steady-state errors for unit step and ramp disturbance inputs.

    Solution:

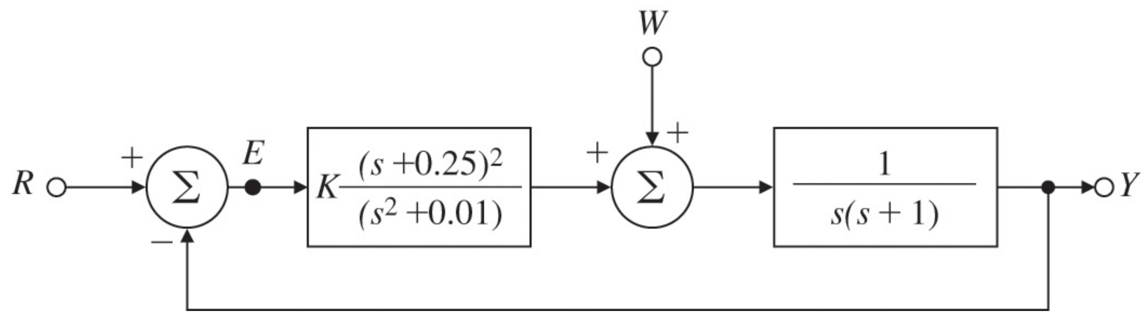


Figure 4.49: Unity feedback system for Problem 4.31

(a) Since the system can be readily transferred to unity feedback, the system Type can be determined by inspection: there are two pure integrators in the forward path so the system is Type II with respect to the reference input.

(b) They will be both zero as the system is Type II.

(c) Again by inspection, there is one pure integrator preceding the point where the disturbance comes in so the system is Type I with respect to the disturbance.

(d) $y_{step}(\infty) = 0$

$\frac{Y}{W} = \frac{s(2s+1)}{s(0.1s+1)+6(2s+1}$

$y_{ramp}(\infty) = \lim_{s\to 0} sY(s)\frac{1}{s^2} = \frac{1}{6}$.

31. Consider the closed-loop system shown in Figure 4.49.

(a) What is the condition on the gain, K, for the closed-loop system to be stable?

(b) What is the system Type with respect to the reference input?

(c) What is the system Type with respect to the disturbance input, w.

(d) Prove that the system can track a sinusoidal input, $r = \sin(0.1t)$, with zero steady-state error.

Solution:

(a) $s(s+1)(s^2+0.01) + K(s+0.25)^2 = 0$

The Routh array is:
$$\begin{array}{llll} s^4: & 1 & K+0.01 & 0.0625K \\ s^3: & 1 & 0.5K+0.01 & \\ s^2: & 0.5K & 0.0625K & \\ s: & 0.5K-0.115 & & \\ s^0: & 0.0625K & & \end{array}$$

So $K > 0$ and $0.5K - 0.115 > 0$, therefore $K > 0.23$.

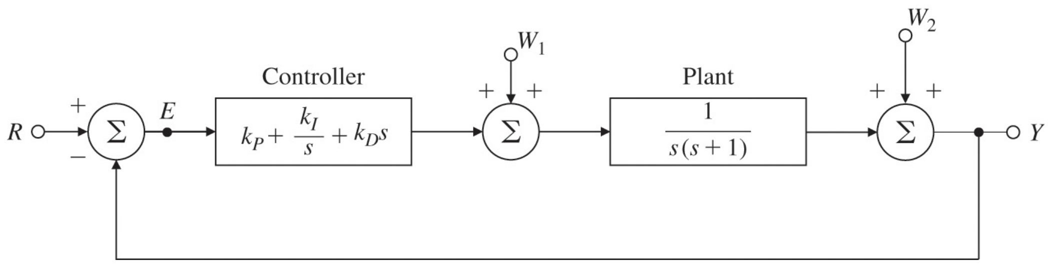


Figure 4.50: Servomechanism system for Problem 4.32

(b) There is one pure integrator in the forward path, so the system is Type I.

(c) There is no pure integrator preceding the point where the disturbance input comes in, so the system is Type 0 with respect to the disturbance.

(d) $T(s) = \frac{K(s+0.25)^2}{s(s+1)(s^2+0.01)+K(s+0.25)^2}$

$E(s) = (1 - T(s))R(s) = \frac{s(s+1)(s^2+0.01)}{s(s+1)(s^2+0.01)+K(s+0.25)^2} \frac{0.1}{s^2+0.01}$

Using the FVT $e(\infty) = \lim_{s\to 0} sE(s) = 0$.

(e)

$\mathcal{S}_K^T = \frac{K}{T}\frac{\partial T}{\partial K} = 1 - T(s) = \frac{s(s+1)(s^2+0.01)}{s(s+1)(s^2+0.01)+K(s+0.25)^2}$.

32. A servomechanism system is shown in Figure 4.50.

(a) Determine the conditions on the PID gain parameters to guarantee closed-loop stability.

(b) What is the system Type with respect to the refrence input?

(c) What is the system Type with respect to the disturbance inputs w_1 and w_2 ?

Solution:

(a) $\frac{Y}{R} = \frac{k_d s^2 + k_p s + k_I}{s^3 + (1+k_d)s^2 + k_p s + k_I}$

The Routh array is $\begin{array}{lcc} s^3: & 1 & k_p \\ s^2: & 1+k_d & k_I \\ s: & k_p - \frac{k_I}{1+k_d} & \\ s^0: & k_I & \end{array}$

Therefore $k_d > -1, k_p(1+k_d) > k_I, k_I > 0$.

(b) System is Type II since there are two pure integrators in the forward path.

(c) Type I with respect to w_1 and Type II with respect to w_2.

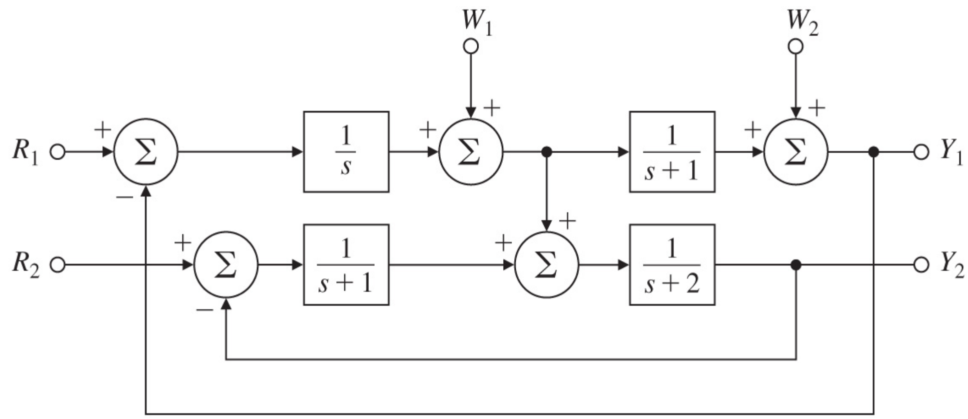


Figure 4.51: Multivariable control system for Problem 4.41

33. Consider the multivariable system shown in Fig. 4.52. Assume that the system is stable. Find the transfer functions from each disturbance input to each output and determine the steady-state values of y_1 and y_2 for constant disturbances. We define a multivariable system to be Type k with respect to polynomial inputs at w_i if the steady-state value of *every* output is zero for any combination of inputs of degree less than k and at least one input is a non-zero constant for an input of degree k. What is the system type with respect to disturbance rejection at w_1? At w_2?

Solution:

(a)

$$Y_1 = \frac{1}{s^2+s+1}R_1 + \frac{s}{s^2+s+1}W_1 + \frac{s(s+1)}{s^2+s+1}W_2.$$

For constant disturbances, $R_1 = 0,\ W_1(s) = \dfrac{W_{10}}{s}\ ,\ W_2(s) = \dfrac{W_{20}}{s}$

$$Y_1 = \frac{W_{10}+(s+1)W_{20}}{s^2+s+1}.$$

Let u_2 be the signal coupling systems 1 and 2:

$$\begin{aligned} U_2 &= \frac{(s+1)(R_1-W_2)+s(s+1)W_1}{s^2+s+1}, \\ Y_2 &= \frac{R_2}{s^2+3s+2} + \frac{(s+1)U_2}{s^2+3s+2} \\ &= \frac{(s+1)^2(-W_2)+s(s+1)^2W_2}{(s^2+3s+2)(s^2+s+1)}. \end{aligned}$$

The system type w.r.t. disturbances:

y_1 w.r.t. W_1 Type 1

y_1	w.r.t.	W_2	Type 1
y_2	w.r.t.	W_1	Type 1
y_2	w.r.t.	W_2	Type 0

can be determined by applying FVT to Y_1 and Y_2 or by inspection.

Problems and Solutions for Section 4.3: The Three Term controller: PID control

34. For the system shown in Figure 4.47,

(a) Design a proportional controller to stabilize the system.

(b) Design a PD controller to stabilize the system.

(c) Design a PI controller to stabilize the system.

(d) What is the velocity error coefficient K_v for the system in part (c)?

Solution:

(a)

$1 + K_p \frac{(10s+5)}{(s+8)(s-1)} = 0$

$s^2 + (7 + 10K_p)s + 5K_p - 8 = 0$

The Routh array is: $\begin{array}{ccc} s^2: & 1 & 5K_p - 8 \\ s: & 7 + 10K_p & \\ s^0 & 5K_p & \end{array}$

Therefore, we need $K_p > -\frac{7}{10}$ and $K_p > \frac{8}{5}$. Let us choose $K_p = 2$.

(b)

$T(s) = \frac{(K_p + K_D s)(10s+5)}{(s+8)(s-1)+(K_p+K_D s)(10s+5)}$

We can choose $K_p + K_D s = s + 8$ to cancel out the stable pole at $s = -8$. The other closed-loop pole will be at $s = -\frac{4}{11}$ which is stable. (c)

$T(s) = \frac{(K_p s + K_I)(10s+5)}{s(s+8)(s-1)+(K_p s+K_I)(10s+5)}$

We can choose $K_p s + K_I = s + 8$ to cancel out the stable pole at $s = -8$. The other two poles are located at the roots of $s^2 + 9s + 5 = 0$ which are both in the LHP. Therefore, $K_p = 1$and $K_I = 8$.

(d)

$K_v = \lim_{s \to 0} s \frac{(K_p s + K_I)(10s+5)}{s(s+8)(s-1)} = -5$

Since K_v is negative it tells us that the steady-state response to a ramp input overshoots the desired ramp and stays above the desired ramp as time increases. The distance between the desired ramp and the actual response, i.e., the steady-state error is $e_{ramp}(\infty) = \frac{1}{K_v} = 0.2$.

35. Consider the feedback control system with the plant transfer function $G(s) = \frac{1}{(s+1)^2}$.

(a) Design a proportional controller so that the closed-loop system has damping of $\zeta = 0.707$. Under what conditions on k_P is the closed-loop system stable?

(b) Design a PI controller so that the closed-loop system has no overshoot. Under what conditions on (k_P, k_I) is the closed-loop system stable?

(c) Design a PID controller such that the settling time is less than two seconds.

Solution:

(a) $\frac{Y}{R} = \frac{k_P}{s^2+2s+k_P+1}$

$2\varsigma\omega_n = 2$,so $\omega_n = \frac{1}{0.707} = 1.4144$

$1 + k_P = \omega_n^2$,and $k_P = 1$.

The Routh array is $\begin{array}{ccc} s^3: & 1 & k_P+1 \\ s^2: & 2 & k_I \\ s: & \frac{2+2k_P+k_I}{2} & \\ s^0: & k_I & \end{array}$

(b) $T(s) = \frac{Y}{R} = \frac{k_P s+k_I}{s^3+2s^2+(1+k_P)s+k_I}$

The Routh array is: $\begin{array}{ccc} s^3: & 1 & 1+k_P \\ s^2: & 2 & k_I \\ s: & \frac{2+2k_P-k_I}{2} & \\ s^0: & k_I & \end{array}$

$k_I > 0$ and $k_P > \frac{k_I}{2} - 1$. If we choose to cancel one of the (stable) poles of the plant, we set $\frac{k_I}{k_P} = 1$. Then we have that

$T(s) = \frac{k_P}{s^2+s+k_P}$.

Since we want no overshoot, let $\varsigma = 1$.Then $2\varsigma\omega_n = 1$ and $\omega_n = 0.5$.

$\omega_n^2 = k_P$ and $k_P = \frac{1}{4} = k_I$.

(c) $T(s) = \frac{k_D(s^2+\frac{k_P}{k_D}s+\frac{k_I}{k_D})}{s(s+1)^2+k_D\left(s^2+\frac{k_P}{k_D}s+\frac{k_I}{k_D}\right)}$

Therefore if we choose $\frac{k_P}{k_D} = 2$ and $\frac{k_I}{k_D} = 1$ the closed-loop transfer function becomes

$T(s) = \frac{k_D}{s+k_D}$.We choose k_D to satisfy the settling time specifications:$t_s \approx 5\tau = \frac{5}{k_D} < 2$ so we need $k_D > \frac{5}{2}$. For example , if we choose $k_D = 3$,then $k_P = 6$ and $k_I = 3$.

36. Consider the liquid level control system with the plant transfer function $G(s) = \frac{7}{s^2+8s+7}$.

(a) Design a proportional controller so that the damping ratio is $\zeta = 0.707$.

(b) Design a PI controller so that the settling time is less than 4 sec.

(c) Design a PD controller so that the rise time is less than 1 sec.

(d) Design a PID controller so that the settling time is less than 2 sec.

Solution:

(a) $T(s) = \frac{7k_P}{s^2+8s+7(1+k_P)}$

$\omega_n = \sqrt{7(1+k_P)}$ and $\varsigma = \frac{8}{2\omega_n}$. Therefore to have $\varsigma = 0.707$, $k_P = 3.57$.

(b) $T(s) = \frac{7k_P\left(s+\frac{k_I}{k_P}\right)}{s(s+1)(s+7)+7k_P\left(s+\frac{k_I}{k_P}\right)}$. If we choose $\frac{k_I}{k_P} = 1$ then we can cancel a stable pole of the plant at $s = -1$ to yield

$T(s) = \frac{7k_P}{s^2+7s+7k_P}$.

$t_s = \frac{4.6}{\varsigma\omega_n} = 1.3 < 4$. If we choose $\omega_n = 4.95$ then $k_I = k_P = 3.5$.

(c) $T(s) = \frac{7k_D\left(s+\frac{k_P}{k_D}\right)}{(s+1)(s+7)+7k_D\left(s+\frac{k_P}{k_D}\right)}$.

If we set $\frac{k_P}{k_D}$= 7,we cancel the stable pole of the plant at $s = -7$ and the resulting closed-loop transfer function is $T(s) = \frac{7k_D}{s+1+7k_D}$. We choose k_D to satisfy the rise time constraint $t_r \approx 2\tau = \frac{2}{1+7K_D} < 1$ which means $k_D > \frac{1}{7}$. For example if we choose $k_D = 0.5$ then $k_P = 3.5$.

(d) $T(s) = \frac{7k_D\left(s^2+\frac{k_P}{k_D}s+\frac{k_I}{k_D}\right)}{s(s^2+8s+7)+\left(7k_D\left(s^2+\frac{k_P}{k_D}s+\frac{k_I}{k_D}\right)\right)}$.

Therefore, if we choose $\frac{k_P}{k_D} = 8$ and $\frac{kI}{k_D} = 7$, the closed-loop transfer function becomes $T(s) = \frac{7k_D}{s+7k_D}$. We select k_D to satisy the settling time specifications $t_s \approx 5\tau = \frac{5}{7k_D} < 2$ or $k_D > \frac{5}{14}$. For example if we choose $k_D = 0.5$,then $k_P = 4$ and $k_I = 3.5$.

37. Consider the process control system with the plant transfer function $G(s) = \frac{0.9}{(s+0.4)(s+1.2)}$.

(a) Design a PI controller such that the rise time is less than 2 sec.

(b) Design a PID controller so that the system has no overshoot.

(c) Design a controller such that the peak time is less than 5 sec.

Solution:

(a) $D_c(s) = k_P + \frac{k_I}{s} = \frac{k_P}{s}(s+\frac{k_I}{k_P})$. The closed-loop characteristic equation is $s^2+1.2s+0.9k_P$. If we select $\frac{k_I}{k_P} = 0.4$ so as to cancel the stable pole of the plant at $s = -0.4$,and given that Routh requires $k_P > 0$, we may let $k_P = 1$,and $k_I = 0.4$. The closed-loop transfer function is$T(s) = \frac{0.9k_P}{s^2+1.2s+0.9k_P}$. $t_r = \frac{1.8}{\omega_n} = \frac{1.8}{\sqrt{0.9k_P}}$ =1.897 < 2.

(b)

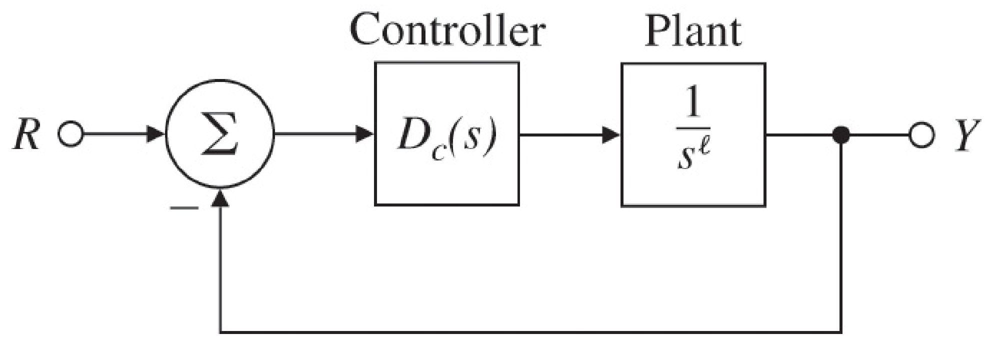


Figure 4.52: Multiple-integrator plant system

$D_c(s) = k_P + \frac{k_I}{s} + k_D s = \frac{k_D}{s}(s^2 + \frac{kP}{k_D}s + \frac{k_I}{k_D})$. If we choose to cancel both stable poles of the plant, we let $\frac{k_P}{k_D} = 1.6$ and $\frac{k_I}{k_D} = 0.48$. The closed-loop transfer function will be $T(s) = \frac{k_D}{s+k_D}$. Since the system is first order, there will be no overshoot. For stability, $k_D > 0$. If we pick $k_D = 1$,then $k_P = 1.6$ and $k_I = 0.48$.

(c) For the controller in part (a) we have $\varsigma = 0.632$,and $\omega_n = 0.949$.

$t_p = \frac{\pi}{\omega_d} = \frac{\pi}{0.949\sqrt{1-0.632^2}} = 4.27 < 5$.

38. Consider the multiple-integrator plant feedback control system shown in Figure 4.52, where ℓ is an integer. (a) Assume $\ell = 1$. Design a controller (P, I, D, or PID) to stabilize this system.

(b) Assume $\ell = 2$ (drone or satellite). Let $D_c(s) = K(s + 10)$. Prove that it is possible to stabilize the system with this dynamic controller. Use the Routh test to determine the range of the gain K for closed-loop stability.

(c) Assume $\ell = 3$ (hospital delivery robot or the Apollo Lunar Module). Let $D_c(s) = K(s + 10)^2$. Prove that it is possible to stabilize the system with this dynamic controller. Again use the Routh test to determine the range of the gain K for closed-loop stability.

(d) Assume $\ell \geq 4$. What form of controller will be required to stabilize the system?

Solution:

(a) We can use a proportional (P) controller $D_c(s) = K$. The closed-loop characteristic polynomial is $s + K$. Therefore $K > 0$ for stability.

(b) The characteristic equation is $s^2 + Ks + 10K$. The Routh array is

$$\begin{array}{rll} s^2: & 1 & 10K \\ s: & K & \\ s^0: & 10K & \end{array}$$

Therefore, we need to have $K > 0$.

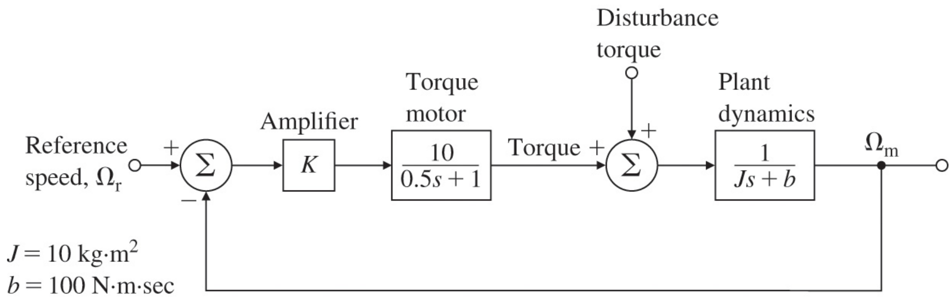


Figure 4.53: Generator speed-control problem: J=10, b=100

(c) The characteristic equation is $s^2 + Ks + 10K$. The Routh array is

s^3 :	1	$20K$
s^2 :	K	$100K$
s :	$\frac{20K^2-100K}{K}$	
s^0 :	$100K$	

which requires $K > 0$,and $K > 5$.

(d) Based on the previous parts, $D_c(s) = K(s+10)^{\ell-1}$.

39. The transfer functions for a generator speed control system are shown in Fig. 4.53. The speed sensor is fast enough that its dynamics can be neglected and the diagram shows the equivalent unity feedback system.

(a) Assuming the reference is zero, what is the steady-state error due to a step disturbance torque of 1 N · m? What must the amplifier gain K be in order to make the steady-state error $e_{ss} \leq 0.01$rad/sec.?

(b) Plot the roots of the closed-loop system in the complex plane, and accurately sketch the time response of the output for a step reference input using the gain K computed in part(a).

(c) Plot the region in the complex plane of acceptable closed-loop poles corresponding to the specifications of a 1% settling time of $t_s \leq$ 0.1sec. and an overshoot $M_p \leq 5\%$.

(d) Give values for k_P and k_D for a PD controller which will meet the specifications.

(e) How would the disturbance-induced steady-state error change with the new control scheme in part(d)? How could the steady-state error to a disturbance torque be eliminated entirely?

Solution: (a)

The TF from disturbance to output:

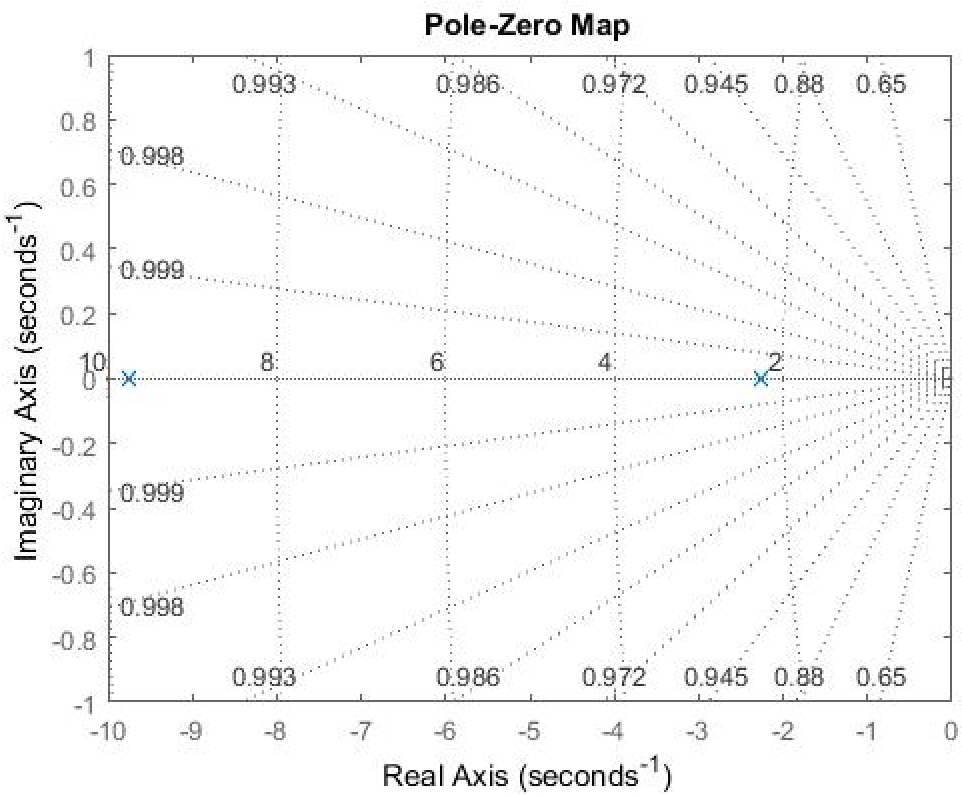
$$\frac{\Omega_m}{W} = \frac{\frac{1}{Js+b}}{1+\frac{1}{Js+b}.\frac{10k_P(k_Ds+1)}{(0.5s+1)}} = \frac{(0.5s+1)}{5s^2+60s+100+10K}.$$

$$e_{ss}(\text{step in } w) = \frac{1}{100+10K}.$$

$$e_{ss} \le 0.01, K \geqslant 0. \tag{4.3}$$

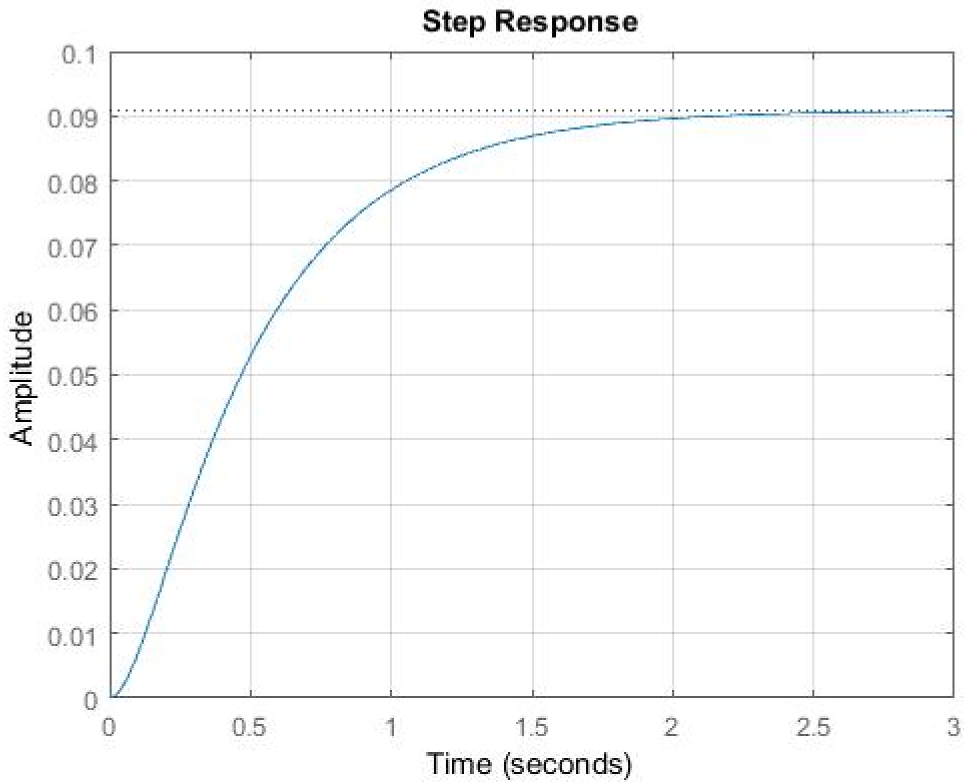
Let us choose $K = 1$.

(b)



Problem 4.39: Pole locations.

$$\frac{\Omega_m}{\Omega_r} = \frac{2}{s^2+12s+22} \tag{4.4}$$



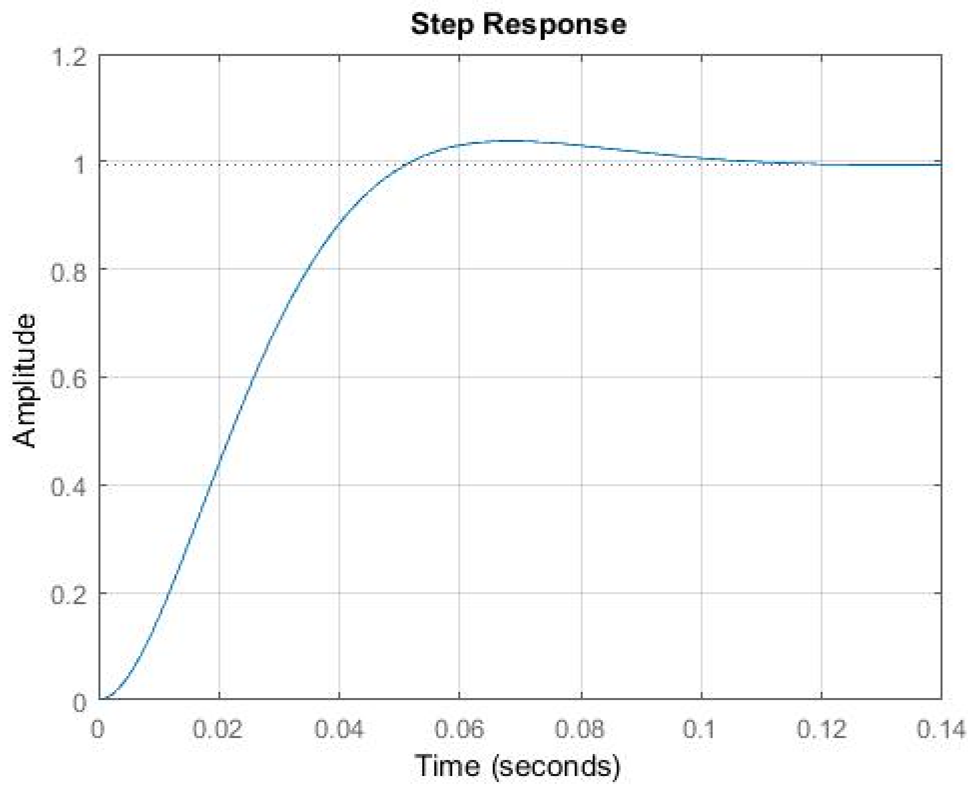
Problem 4.39: Step response.

(c) s-plane for part (c)

(a) d. We know that a larger ω_n is needed. This can be achieved by increasing k_P and adding derivative feedback $(k_D s + 1)$, in the feedback loop:

$$\frac{\Omega_m(s)}{\Omega_r(s)} = \frac{\frac{10k_P}{0.5s+1}.\frac{1}{Js+b}}{1 + \frac{10k_P(k_D s+1)}{(0.5s+1)(Js+b)}} = \frac{2k_P}{s^2 + (12 + 2k_P.k_D)s + 20 + 2k_P}.$$

By choosing k_P and k_D, any ζ and ω_n may be achieved.
Let choose poles at $-46 \pm j46$. Then $k_P = 2106$ and $k_D = 0.019$. $(\zeta = 0.707, \omega_n = 65)$.The step response is as shown below.



Problem 4.39: Closed-loop step response

e. The TF for disturbance to output with the new controller:

$$\frac{\Omega_m}{W} = \frac{\dfrac{1}{Js+b}}{1+\dfrac{1}{Js+b}.\dfrac{10k_P(k_Ds+1)}{(0.5s+1)}} = \frac{0.2(0.5s+1)}{s^2+(12+2k_Pk_D)s+20+2k_P}.$$

$$e_{ss}(\text{step in W}) = \frac{0.2}{20+2k_P}.$$

As before derivative feedback affects transient response only. To eliminate steady-state error we can add an integrator to the loop. This can be represented by replacing k_P with $k_P + \dfrac{k_I}{s}$ in the forward loop and still keeping PD control in the feedback loop to obtain

$$\frac{Y}{W} = \frac{0.2(0.5s+1)s}{s^3+(12+2k_Pk_D)s^2+(20+2k_P+2k_Ik_D)s+2k_I}.$$

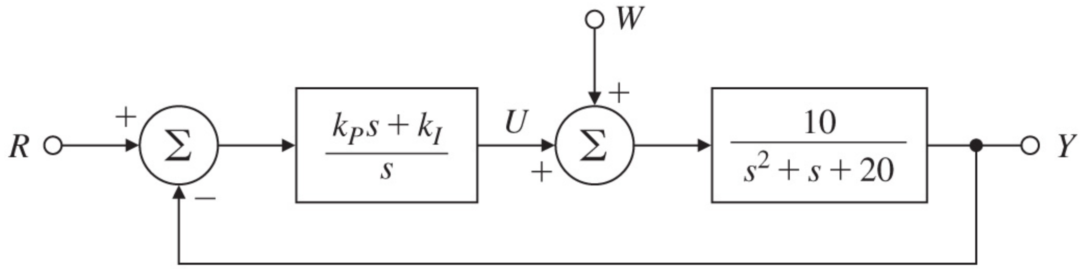


Figure 4.54: Control system for Problem 4.40

$$e_{ss}(\text{step in } w) = 0.$$

40. Consider the system shown in Fig. 4.54 with PI control..

(a) Determine the transfer function from R to Y.

(b) Determine the transfer function from W to Y.

(c) What is the system type and error constant with respect to reference tracking?

(d) What is the system type and error constant with respect to disturbance rejection?

Solution:

(a)
$$\frac{Y(s)}{R(s)} = \frac{10(k_I + k_P s)}{s[s(s+1)+20] + 10(k_I + k_P s)}.$$

(b)
$$\frac{Y(s)}{W(s)} = \frac{10s}{s[s(s+1)+20] + 10(k_I + k_P s)}.$$

(c) The characteristic equation is $s^3 + s^2 + (10k_P + 20)s + 10k_I = 0$. The Routh's array is

$$\begin{array}{lcc}
s^3: & 1 & 10k_P + 20 \\
s^2: & 1 & 10k_I \\
s^1: & 10k_P + 20 - 10k_I & \\
s^0: & 10k_I &
\end{array}$$

For stability we must have $k_I > 0$ and $k_P > k_I - 2$.

(d) System is Type 1 with respect to both r and w. The velocity constant with respect to reference tracking is $K_v = k_I/2$ and with respect to disturbance rejection is k_I.

41. Consider the second-order plant with transfer function

$$G(s) = \frac{1}{(s+1)(5s+1)}.$$

and in a unity feedback structure.

(a) Determine the system type and error constant with respect to tracking polynomial reference inputs of the system for P $[D = k_P]$, PD $[D = k_P + k_D s]$, and PID $[D = k_P + \frac{k_I}{s} + k_D s]$ controllers. Let $k_P = 19$, $k_I = 0.5$, and $k_D = \frac{4}{19}$.

(b) Determine the system type and error constant of the system with respect to disturbance inputs for each of the three regulators in part (a)with respect to rejecting polynomial disturbances $w(t)$ at the *input* to the plant.

(c) Is this system better at tracking references or rejecting disturbances? Explain your response briefly.

(d) Verify your results for parts (a)and (b)using MATLAB by plotting unit step and ramp responses for both tracking and disturbance rejection.

Solution:

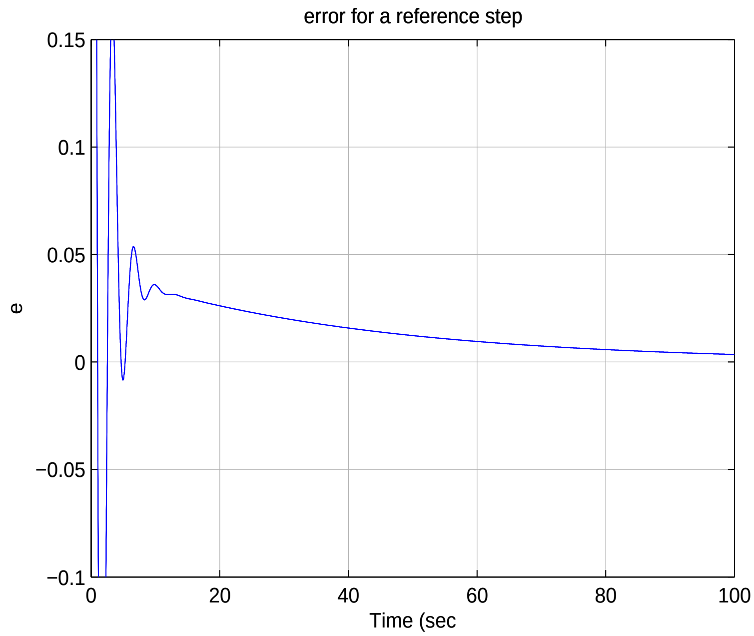
a. This plant has no pole at the origin and DC gain of 1 so, unless the controller has such a pole, the system will be Type 0.

Thus, we have: P and PD are Type 0, $K_p = k_P = 19$;PID is Type 1, $K_v = k_I = 0.5$

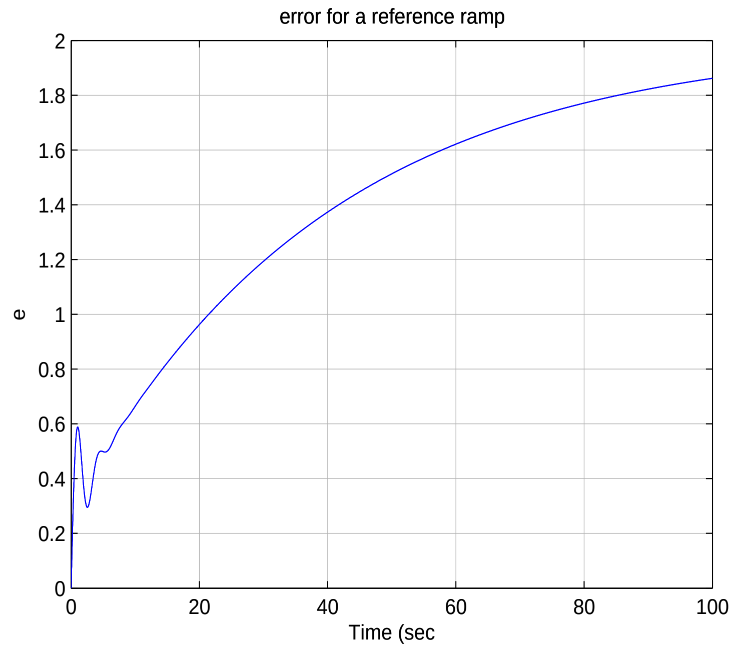
b. Again, P and PD are Type 0, $K_p = k_P = 19$;PID is Type 1, $K_v = k_I = 0.5$.

c. Because the Types and error constants are the same, this system does the same with references as with disturbances.

d. We expect the steady-state error to steps to be 0 and to unit ramps to be $1/k_v = 1/0.5 = 2.0$. Note that steady-state is after a long time



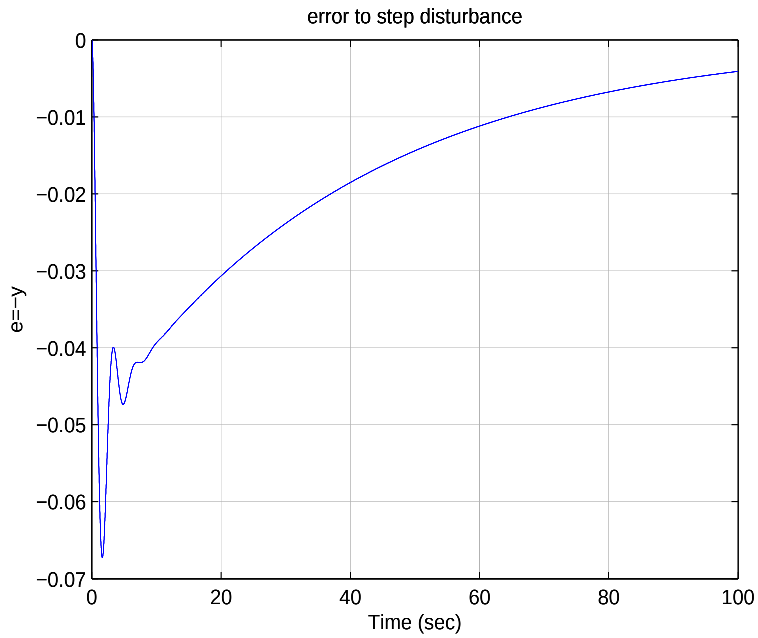
Problem 4.41: Error to a reference step for PID.



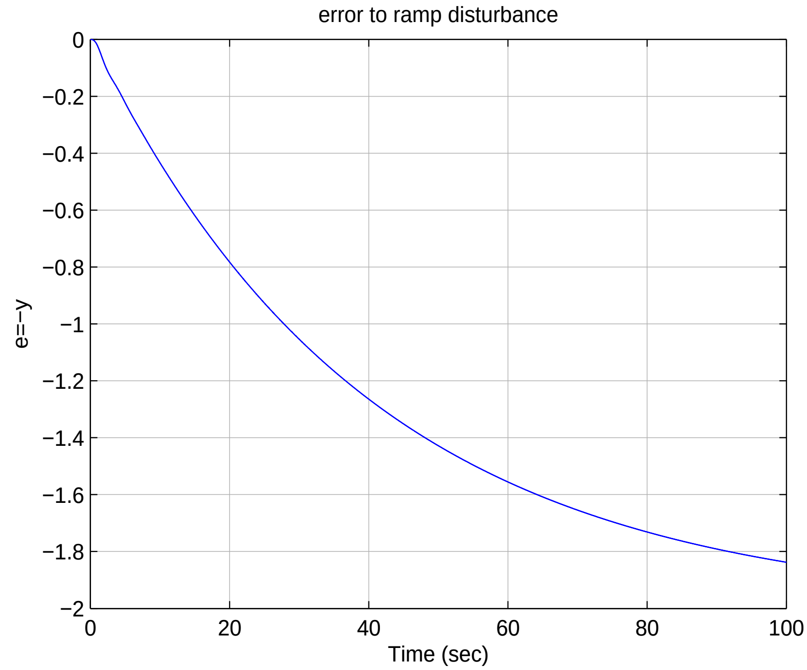
Problem 4.41: Error to a reference ramp for PID.

Notice that these transients are *very* slow. They are the consequence of a pole at $s = -0.0252$. A good rule of thumb is that a transient is over in 5 time constants.

In this case the time constant is $1/0.0252 = 39.68$. Therefore we would expect the transient to go on for about 200 seconds! The responses to disturbances are similar:



Problem 4.41: Error to a disturbance step for PID.



Problem 4.41: Error to a disturbance ramp for PID.

In this case, the disturbance ramp does not excite the fast roots very much at all.

42. The DC-motor speed control shown in Fig. 4.55 is described by the differential equation

$$\dot{y} + 60y = 600v_a - 1500w,$$

where y is the motor speed, v_a is the armature voltage, and w is the load torque. Assume the armature voltage is computed using the PI controllaw

$$v_a = -\left(k_P e + k_I \int_0^t edt\right).$$

where $e = r - y$.

(a) Compute the transfer function from W to Y as a function of k_P and k_I.

(b) Compute values for k_P and k_I so that the characteristic equation of the closed-loop system will have roots at $-60 \pm 60j$.

Solution:

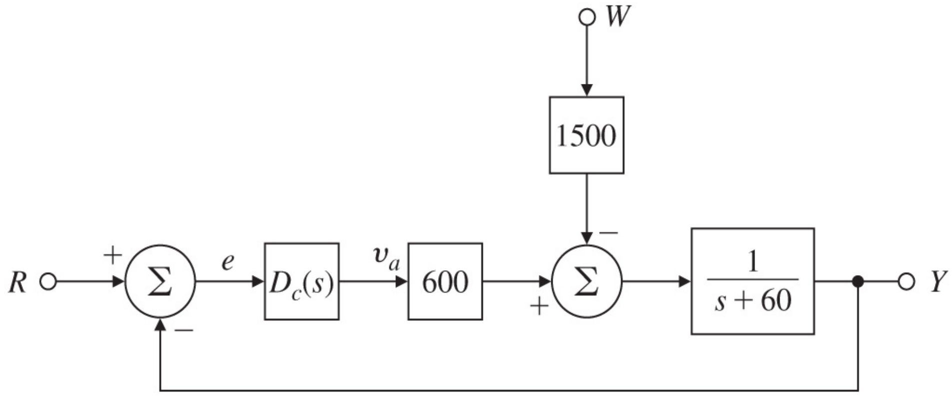


Figure 4.55: D.C. Motor speed control block diagram for Problems 4.42 and 4.43

(a) Transfer function: Set $R = 0$, then $E = -Y$

$$
\begin{aligned}
(s+60)Y(s) &= -600\left[k_P Y(s) + \frac{k_I}{s} Y(s)\right] - 1500W(s), \\
\frac{Y(s)}{W(s)} &= \frac{-1500s}{s^2 + 60(1+10k_P)s + 600k_I}.
\end{aligned}
$$

(b) For roots at $-60 \pm j60$: comparing to the standard form:

$$
s^2 + 2\zeta\omega_n s + \omega_n^2 = 0 \implies s = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2},
$$

$$
\begin{gathered}
\omega_n = 60\sqrt{2}\,, \quad \zeta = 0.707, \\
600k_I = (60\sqrt{2})^2 \implies k_I = 12, \\
60(1+10k_P) = 2 \times 0.707 \times 60\sqrt{2} \implies k_P = 0.1.
\end{gathered}
$$

43. For the system in Problem 4.55, compute the following steady-state errors:

(a) to a unit-step reference input;

(b) to a unit-ramp reference input;

(c) to a unit-step disturbance input;

(d) for a unit-ramp disturbance input.

(e) Verify your answers to (a) and (d) using MATLAB. Note that a ramp response can be generated as a step response of a system modified by an added integrator at the reference input.

Solution:

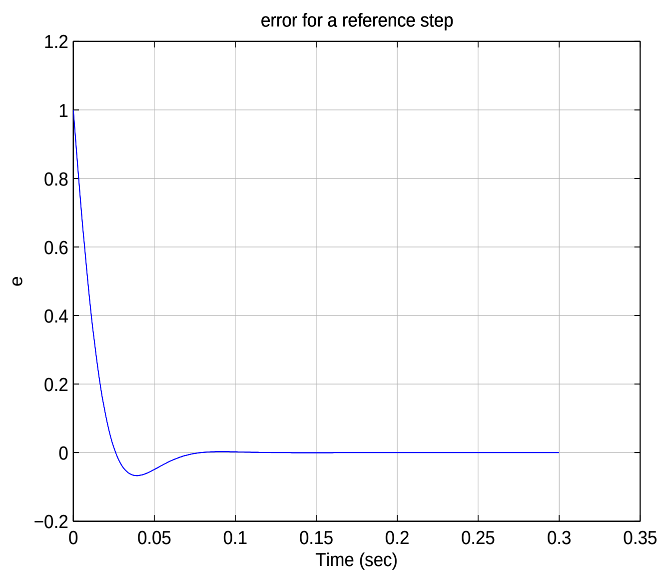
a. From Problem 33, $k_P = 0.1$ and $k_I = 12$. The DC gain of the plant is 10 so the $K_v = 10k_I$. The system is Type 1 so the error to a step is 0.

b. To a unit ramp, the error is $\dfrac{1}{K_v} = \dfrac{1}{10k_I} = \dfrac{1}{120}$.

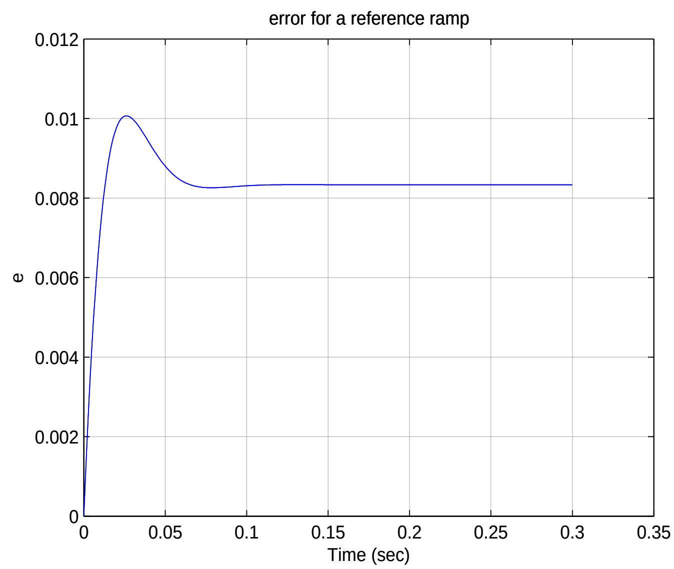
c. For a disturbance input, the system is also Type 1. The error to a step will be 0.

d. For a unit ramp disturbance input the error equals the output and is given by

$$\begin{aligned} E(s) &= -\frac{1500}{s+60+600D}W(s), \\ &= -\frac{1500s}{s^2+60s+600(k_P s+k_I)}W(s), \\ e_{ss} &= -\frac{5}{24} \qquad \text{for } W(s) = 1/s^2. \end{aligned}$$

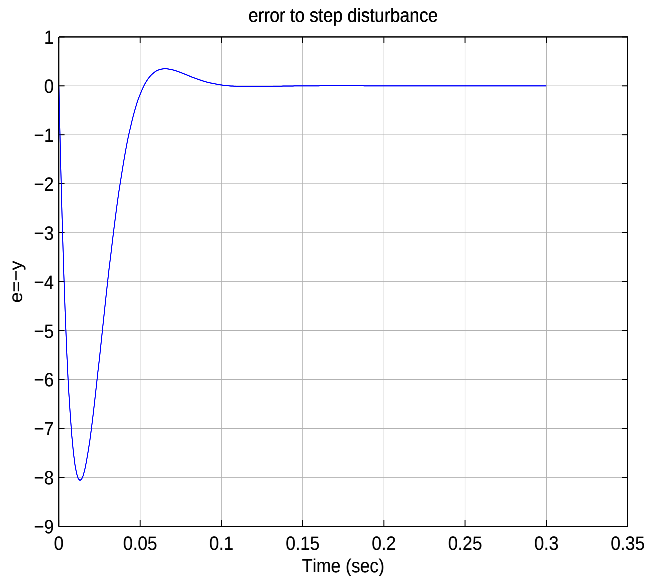


Problem 4.43: Error Response to a Reference Step

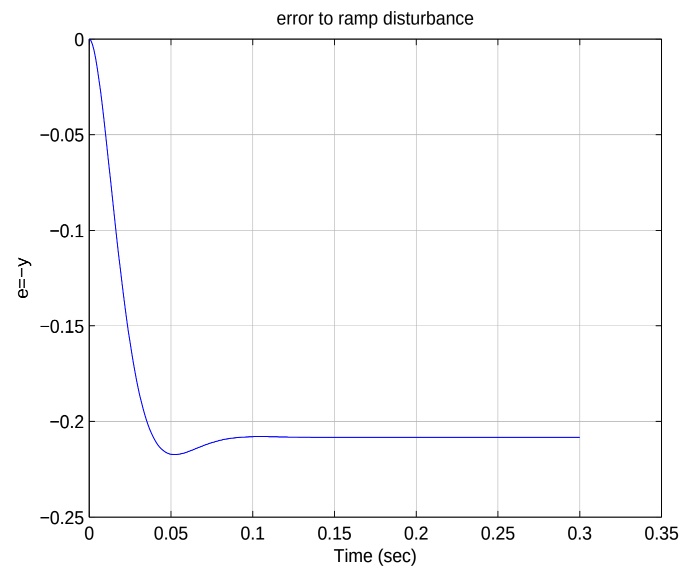


Problem 4.43: Error Response to a Reference Ramp.

As these figures show, the error to a step goes to zero and that to a ramp goes to $1/k_I = 1/120$.



Problem 4.43: Error to a Disturbance Step.



Problem 4.43: Error to a Disturbance Ramp.

And in this case, the error to a disturbance step goes to zero and the error to a disturbance ramp goes to $e_{ss} = 1/k_I = -0.208$.

44. Consider the satellite-attitude control problem shown in 4.56 where the normalized parameters are

$$\begin{aligned} J &= 10 \quad \text{spacecraft inertia, N-m-sec }^2/rad \\ \theta_r &= \text{reference satellite attitude, rad.} \\ \theta &= \text{actual satellite attitude, rad.} \\ H_y &= 1 \quad \text{sensor scale, factor volts/rad.} \\ H_r &= 1 \quad \text{reference sensor scale factor , volts/rad.} \\ w &= \text{disturbance torque. N-m} \end{aligned}$$

(a) Use proportional control, **P**, with $D_c(s) = k_P$, and give the range of values for k_P for which the system will be stable.

(b) Use **PD** control and let $D_c(s) = (k_P + k_D s)$ and determine the system type and error constant with respect to reference inputs.

(c) Use **PD** control, let $D_c(s) = (k_P + k_D s)$ and determine the system type and error constant with respect to disturbance inputs.

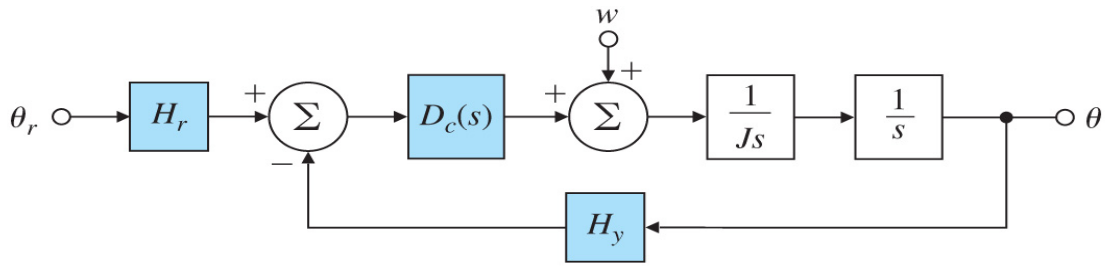


Figure 4.56: Satellite attitude control

(d) Use **PI** control, let $D_c(s) = (k_P + k_I/s)$, and determine the system type and error constant with respect to reference inputs.

(e) Use **PI** control, let $D_c(s) = (k_P + k_I/s)$, and determine the system type and error constant with respect to disturbance inputs.

(f) Use **PID** control, let $D_c(s) = (k_P + k_I/s + k_D s)$ and determine the system type and error constant with respect to reference inputs.

(g) Use **PID** control, let $D_c(s) = (k_P + k_I/s + k_D s)$ and determine the system type and error constant with respect to disturbance inputs.

Solution:

(a) $D_c(s) = k_P$; The characteristic equation is

$$1 + H_y D_c(s)\frac{1}{Js^2} = 0$$

$$Js^2 + H_y k_P = 0$$

or $s = \pm j\sqrt{\dfrac{H_y k_P}{J}}$ so that no additional damping is provided. The system cannot be made stable with proportional control alone.

(b) Steady-state error to reference steps.

$$\begin{aligned}\frac{\Theta(s)}{\Theta_r(s)} &= H_r\frac{D_c(s)\frac{1}{Js^2}}{1 + D_c(s)H_y\frac{1}{Js^2}},\\ &= H_r\frac{(k_P + k_D s)}{Js^2 + (k_P + k_D s)H_y}.\end{aligned}$$

The parameters can be selected to make the (closed-loop) system stable. If $\Theta_r(s) = \dfrac{1}{s}$ then using the FVT (assuming the system is stable)

$$\theta_{ss} = \frac{H_r}{H_y},$$

and there is zero steady-state error if $H_r = H_y$ (i.e., unity feedback).

(c) Steady-state error to disturbance steps

$$\frac{\Theta(s)}{W(s)} = \frac{1}{Js^2 + (k_P + k_D s)H_y}.$$

If $W(s) = \frac{1}{s}$ then using the FVT (assuming system is stable), the error is $\theta_{ss} = -\frac{1}{k_P H_y}$.

(d) The characteristic equation is

$$1 + H_y D_c(s)\frac{1}{Js^2} = 0.$$

With PI control,

$$Js^3 + H_y k_P s + H_y k_I = 0.$$

From the Hurwitz's test, with the s^2 term missing the system will always have (at least) one pole not in the LHP. Hence, this is not a good control strategy.

(e) See (d) above.

(f) The characteristic equation with PID control is

$$1 + H_y\left(k_P + \frac{k_I}{s} + k_D s\right)\frac{1}{Js^2} = 0,$$

or

$$Js^3 + H_y k_D s^2 + H_y k_P s + H_y k_I = 0.$$

There is now control over all the three poles and the system can be made stable.

$$\begin{aligned}
\frac{\Theta(s)}{\Theta_r(s)} &= H_r \frac{D_c(s)\frac{1}{Js^2}}{1 + D_c(s)H_y\frac{1}{Js^2}}, \\
&= \frac{H_r(k_P + \frac{k_I}{s} + k_D s)}{Js^2 + (k_P + \frac{k_I}{s} + k_D s)H_y}, \\
&= \frac{H_r(k_D s^2 + k_P s + k_I)}{Js^3 + (k_D s^2 + k_P s + k_I)H_y}.
\end{aligned}$$

If $\Theta_r(s) = \frac{1}{s}$ then using the FVT (assuming system is stable)

$$\theta_{ss} = \frac{H_r}{H_y},$$

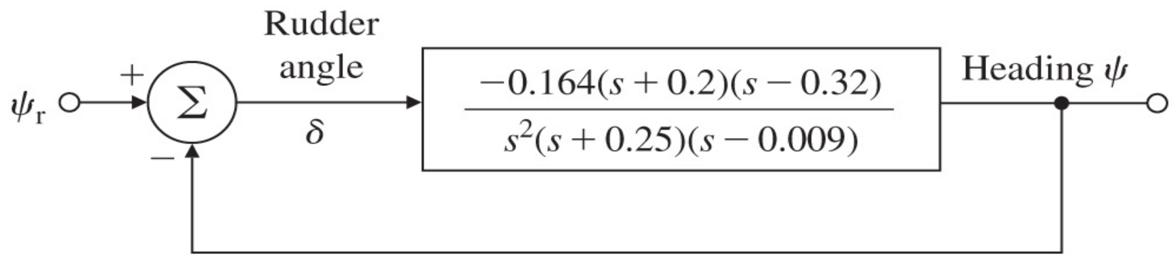


Figure 4.57: Ship-steering control system for Problem 4.45

and there is zero steady-state error if $H_r = H_y$ (i.e., unity feedback). In that case, the system is Type 3 and the (Jerk!) error constant is $K_J = \frac{k_I}{J}$.

(g) The error to a disturbance is found from

$$\frac{\Theta(s)}{W(s)} = \frac{s}{Js^3 + H_y(k_D s^2 + k_P s + k_I)}.$$

If $W(s) = \frac{1}{s}$ then using the FVT (assuming the system is stable), $\theta_{ss} = 0$, the system is Type 1 and the error constant is $K_v = H_y k_P$.

45. Automatic ship steering is particularly useful in heavy seas when it is important to maintain the ship along an accurate path. Such a control system for a large tanker is shown in 4.57, with the plant transfer function relating heading changes to rudder deflection in radians.

(a) Write the differential equation that relates the heading angle to rudder angle for the ship *without* feedback.

(b) This control system uses simple proportional feedback with the gain of unity. Is the closed-loop system stable as shown? (*Hint:* use Routh's criterion)

(c) Is it possible to stabilize this system by changing the proportional gain from unity to a lower value?

(d) Use MATLAB to design a dynamic controller of the form $D_c(s) = K\left(\frac{s+a}{s+b}\right)^2$ so that the closed-loop system is stable and in response to a step heading command it has zero steady-state error and less than 10% overshoot. Are these reasonable values for a large tanker?

Solution:

(a) Multiply out the numerator and denominator terms in the transfer function and take the inverse Laplace Transform:

$$\frac{d^4\psi}{dt^4} + 0.241\frac{d^3\psi}{dt^3} - 0.0025\frac{d^2\psi}{dt^2} = -0.164\left(\frac{d^2\delta}{dt^2} - 0.12\frac{d\delta}{dt} - 0.064\delta\right).$$

(b) s=tf('s');

den=s^2*(s+0.25)*(s-0.009)=s^4 + 0.241 s^3 - 0.00225 s^2;

num=-0.164*(s+0.2)*(s-0.32)=-0.164 s^2 + 0.01968 s + 0.0105;

$Den(G(s))+Num(G(s)) = s^4+0.241s^3-0.1663s^2+0.01968s+0.0105$. Not all the coefficients of the characteristic polynomial are positive: the system is unstable.

(c) The closed-loop characteristic polynomial is $1 + K_pG(s) = 0$ that leads to:

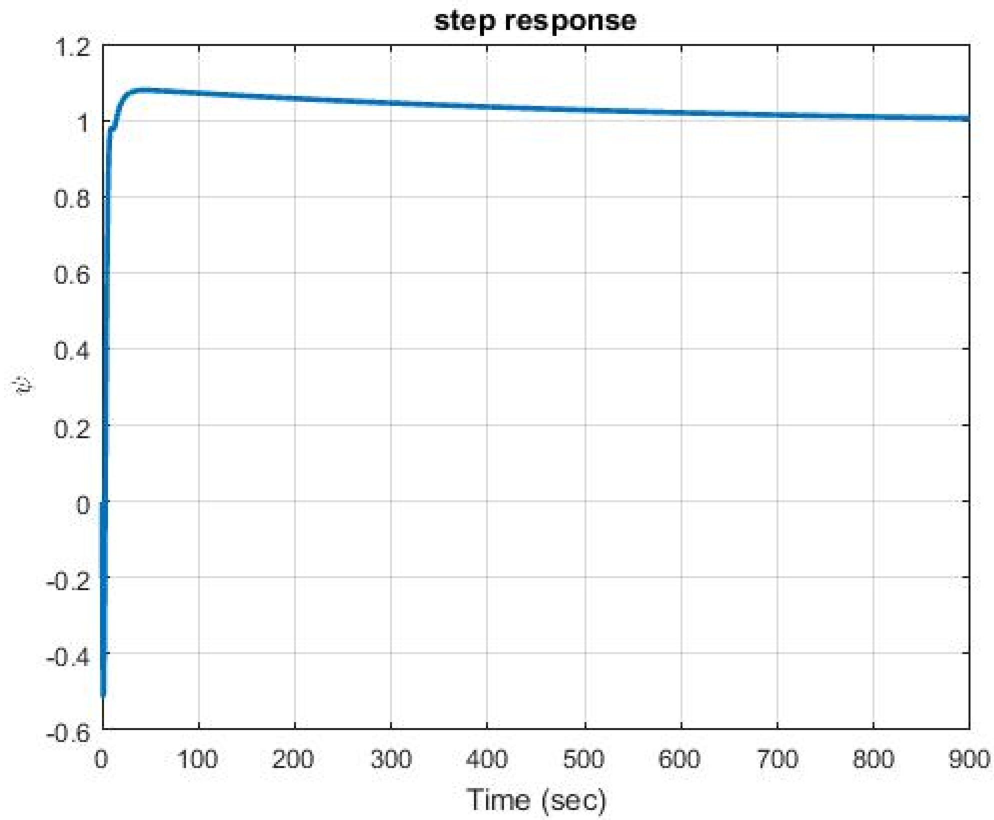
$$s^4+0.241s^3+(-0.00225-0.1663K_p)s^2+0.01968K_ps+0.0105K_p = 0$$

We observe that not all the coefficients of the closed-loop characteristics polynomial can have the same sign, regardless of K_p (be it positive or negative). No value of K_p stabilizes the system. Proportional control is just not sufficient.

(d) There are many solutions. One solution is to place the following dynamic controller with 2 poles and 2 zeros in front of the plant: $D_c(s) = 30\left(\frac{s+0.001}{s+3}\right)^2$.

```
% Ship-steering
s=tf('s');
numP=-0.164*(s+0.2)*(s-0.32);
denP=s^2*(s+0.25)*(s-0.009);
numDc=30*(s+0.001)^2;
denDc=(s+3)^2;
sys=tf(numP*numDc/(denP*denDc));
sysCL=feedback(sys,1);
step(sysCL);
grid on;
```

The closed-loop step response follows. Rise time is too short i.e. it is unrealistic for a large tanker. Note the *non-minimum phase* behavior of the system and the effect of the slow closed-loop poles resulting in a *long "tail"* on the transient response.



Problem 4.45: Step response.

46. The unit-step response of a paper machine is shown in Fig. 4.58(a) where the input into the system is stock flow onto the wire and the output is basis weight (thickness). The time delay and slope of the transient response may be determined from the figure.

(a) Find the proportional, PI, and PID-controller parameters using the Zeigler–Nichols transient-response method.

(b) Using proportional feedback control, control designers have obtained a closed-loop system with the unit impulse response shown in Fig.4.58(b). When the gain $K_u = 8.556$, the system is on the verge of instability. Determine the proportional-, PI-, and PID-controller parameters according to the Zeigler–Nichols ultimate sensitivity method.

Solution:

(a) From step response: $L = \tau_d \simeq 0.65$ sec

$$R = \frac{1}{\tau} \simeq \frac{0.2}{1.25 - 0.65} = 0.33 \quad \text{sec}^{-1}.$$

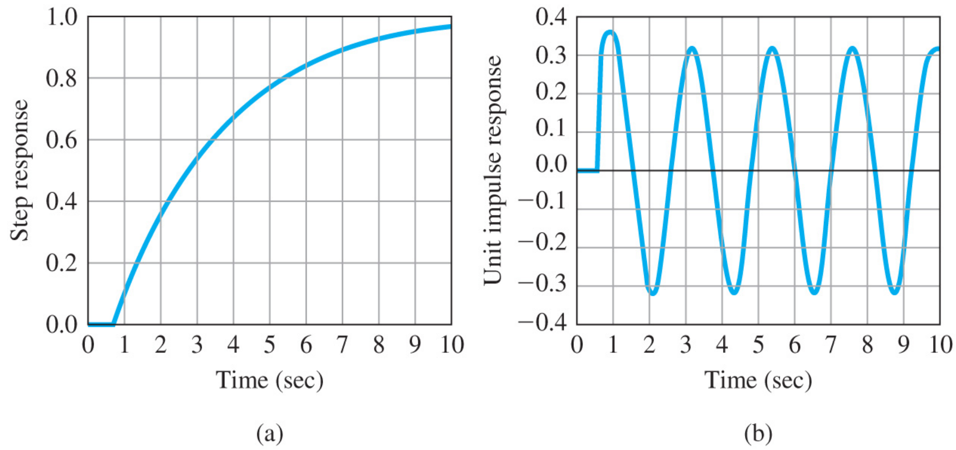


Figure 4.58: Paper-machine response data for Problem 4.46

From Table 4.1:

(b) From the impulse response: $P_u \simeq 2.33$ sec. and from Table 4.2:

47. A paper machine has the transfer function

$$G(s) = \frac{e^{-2s}}{3s+1},$$

where the input is stock flow onto the wire and the output is basis weight or thickness.

(a) Find the PID-controller parameters using the Zeigler–Nichols tuning rules.

(b) The system becomes marginally stable for a proportional gain of $K_u = 3.044$ as shown by the unit impulse response in Fig. 4.59. Find the optimal PID-controller parameters according to the Zeigler–Nichols tuning rules.

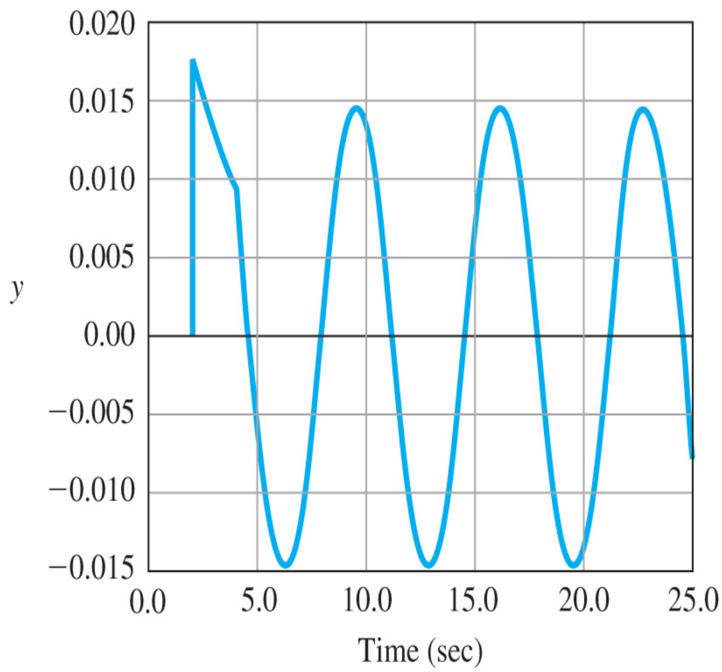


Figure 4.59: Unit impulse response for paper-machine in Problem 4.47

Solution:

(a) From the transfer function: $L = \tau_d \simeq 2$ sec

$$R = \frac{1}{3} \simeq 0.33 \quad \text{sec}^{-1}.$$

From Table 4.1:

$$\begin{array}{llll} \text{Controller Gain} & P : & K = \frac{1}{RL}1.5, & \\ & PI : & K = \frac{0.9}{RL} = 1.35 & T_I = \frac{L}{0.3} = 6.66, \\ & PID : & K = \frac{1.2}{RL} = 1.8 & T_I = 2L = 4 \ \ T_D = 0.5L = 1.0. \end{array}$$

(b) From the impulse response: $P_u \simeq 7$ sec From Table 4.2:

Problem and Solution for Section 4.4: Feedforward Control by Plant Model Inversion

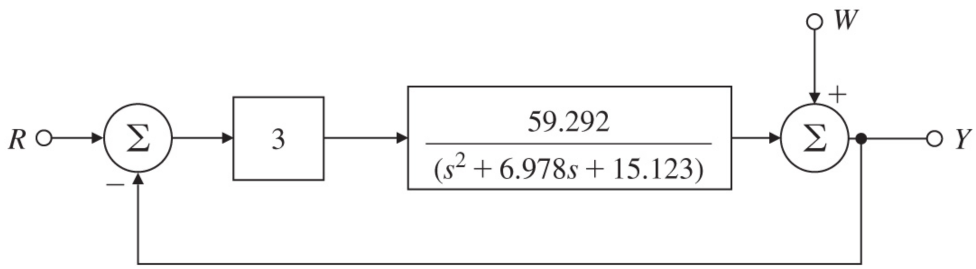


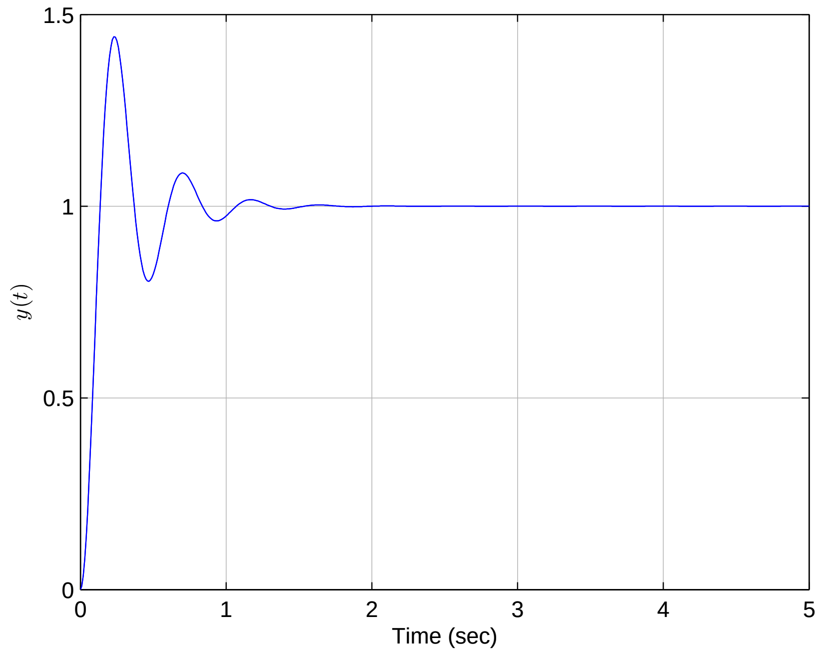
Figure 4.60: Block diagram for Problem 4.48

48. Consider the DC motor speed control system shown in Fig. 4.60 with proportional control. (a) Add feedforward control to eliminate the steady-state tracking error for a step reference input. (b) Also add feedforward control to eliminate the effect of a constant output disturbance signal on the output of the system.

Solution: (a) In this case the plant inverse DC gain is $G^{-1}(0) = \frac{15.123}{59.292} = 0.2551$. We implement the closed-loop system as shown in Figure 4.22 (a) with $D_c(s) = k_p = 3$. The closed-loop transfer function is

$$
\begin{aligned}
Y(s) &= G(s)[k_p E(s) + G^{-1}(0)R(s)], \\
E(s) &= R(s) - Y(s), \\
\frac{Y(s)}{R(s)} &= \mathcal{T}(s) = \frac{(G^{-1}(0) + k_p)G(s)}{1 + k_p G(s)}.
\end{aligned}
$$

Note that the closed-loop DC gain is unity ($\mathcal{T}(0) = 1$). The following figure illustrates the effect of feedforward control in eliminating the steady-state tracking error. The addition of feedforward control results in zero steady-state tracking error for a step reference input.

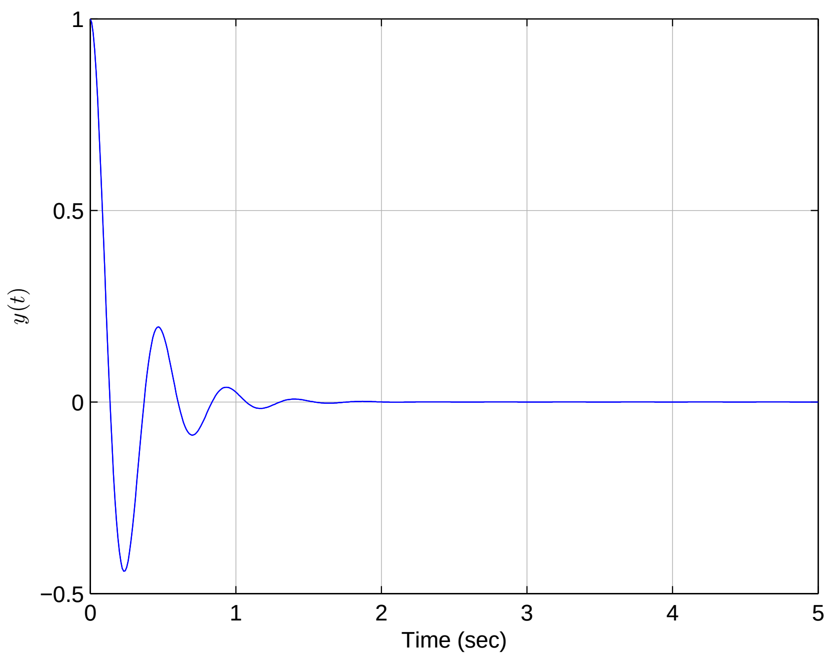


Problem 4.48: Tracking response with feedforward.

(b) Similarly, we implement the closed loop system as shown Figure 4.22 (b). The closed-loop transfer function is

$$\begin{aligned}
Y(s) &= W(s) + G(s)[k_p E(s) - G^{-1}(0)W(s)], \\
E(s) &= R(s) - Y(s) = 0 - Y(s), \\
\frac{Y(s)}{W(s)} &= \mathcal{T}_w(s) = \frac{1 - G^{-1}(0)G(s)}{1 + k_p G(s)}.
\end{aligned}$$

Note that the closed-loop DC gain is zero ($\mathcal{T}_w(s) = 0$). The following figure illustrates the effect of feedforward control in eliminating the steady-state error for a step output disturbance.



Problem 4.48: Disturbance rejection response with feedforward.

MATLAB code:

```
% FPE8e Problem 4.48
clf;
% Tracking
s=tf('s');
% plant
G=59.292/(s^2+6.978*s+15.123);
kp=3;
% Closed-loop Transfer function
dcgain1=dcgain(G);
T1=G*(1/dcgain1+kp)/(1+kp*G);
t=0:.01:5;
% Step response
y1=step(T1,t);
figure()
```

```
plot(t,y1);
xlabel('Time (sec)');
ylabel('$y(t)$','interpreter','latex');
grid on;
% Disturbance rejection
kp=3;
Tw1=(1-1/dcgain1*G)/(1+kp*G);
yw1=step(Tw1,t);
figure()
plot(t,yw1);
xlabel('Time (sec)');
ylabel('$y(t)$','interpreter','latex');
grid on;
```

Problems and Solutions for Section 4.5: Introduction to Digital Control

49. Compute the discrete equivalents for the following possible controllers using the trapezoid rule of Eq. (w14) in Appendix W4.5. Let $T_s = 0.05$ sec in each case.

(a) $D_{c1}(s) = (s+2)/2$,

(b) $D_{c2}(s) = 2\dfrac{s+2}{s+4}$,

(c) $D_{c3}(s) = 5\dfrac{(s+2)}{s+10}$,

(d) $D_{c4}(s) = 5\dfrac{(s+2)(s+0.1)}{(s+10)(s+0.01)}$.

Solution:

(a) Using the formula $s \leftarrow \dfrac{2}{T_s}\dfrac{z-1}{z+1}$ we find $D_{c1}(z) = \dfrac{21z-19}{z+1}$,

(b) $D_{c2}(z) = \dfrac{1.909z - 1.727}{z - 0.8182}$,

(c) $D_{c3}(z) = \dfrac{4.2z - 3.8}{z - 0.6}$,

(d) $D_{c4}(z) = \dfrac{4.209z^2 - 7.997z + 3.79}{z^2 - 1.6z + 0.5997}$.

50. Give the difference equations corresponding to the discrete controllers found in Problem 4.49 respectively

(a) part 1,

(b) part 2,

(c) part 3,

(d) part 4.

Solution:

(a) Reduce the $z-$ transforms to be in terms of z^{-1} if you want the equations in terms of past values. We have divided by the coefficient of the highest power if z in the denominator to get the coefficient of $u(k)$ to be 1.0 in each case. For part (a), $\frac{U}{E} = \frac{21z - 19}{z+1} = \frac{21 - 19z^{-1}}{1 + z^{-1}}$ and thus $[1 + z^{-1}]U(z) = [21 - 19z^{-1}]E(z)$ from which, as $z^{-1}U(z) \Longrightarrow u(k-1)$ we get
$u(k) = -u(k-1) + 21e(k) - 19e(k-1)$. We have suppressed the T_s in these equations. It should properly be $u(kT_s)$, $u([k-1]T_s)$, etc.

(b) $u(k) = 0.8182u(k-1) + 1.909e(k) - 1.727e(k-1)$.

(c) $u(k) = 0.6u(k-1) + 4.2e(k) - 3.8e(k-1)$.

(d) $u(k) = 1.6u(k-1) - 0.5997u(k-2) + 4.209e(k) - 7.997e(k-1) + 3.79e(k-2)$.