Intermediate Fluid Mechanics

Lecture 7: Basic Flow Fields

Marc Calaf

Department of Mechanical Engineering

University of Utah

ME 5700/6700

August 26, 2025

Chapter Overview

Lecture Objectives

In this lecture we will consider four different basic flow fields, namely

- Stagnation Point Flow,
- Solid Body Rotation,
- Linear Shear Flow,
- Vortex Line Induced Flow,

and investigate the kinematic behavior of the fluid particles in those flows.

Chapter Overview

Flow I: Stagnation Point flow

Let's consider the example flow illustrated below,

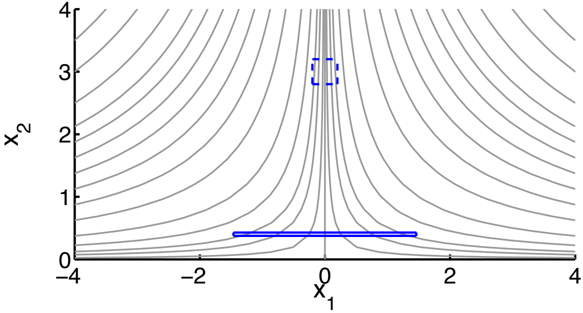

Figure: *Streamlines of a stagnation point flow. The square represents an initial fluid element, and the rectangle illustrates the same fluid element some time later.*

that is mathematically expressed as:

$$u_1 = c\, x_1, \qquad u_2 = -c\, x_2, \qquad u_3 = 0, \qquad \text{with } c > 0. \tag{1}$$

Flow I: Stagnation Point flow (continued ..)

To understand the kinematics of the fluid particles in this flow one needs to look at the strain-rate tensor e_{ij} and the spin tensor r_{ij}.

1) Let's start by first analyzing the **spin tensor**, recalling the definition of this one,

$$r_{ij} = -\frac{1}{2}\varepsilon_{ijk}\,\omega_k \qquad \text{with} \qquad \omega_k = \varepsilon_{ijk}\,\frac{\partial u_j}{\partial x_i}, \tag{2}$$

which in matrix form is written as,

$$r = \begin{pmatrix} 0 & -\frac{1}{2}\left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right) & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}\right) \\ \frac{1}{2}\left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right) & 0 & -\frac{1}{2}\left(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}\right) \\ -\frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}\right) & \frac{1}{2}\left(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}\right) & 0 \end{pmatrix}. \tag{3}$$

Flow I: Stagnation Point flow (continued ..)

Since the tensor is anti-symmetric, one only needs to evaluate three of the terms,

$$r_{13} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}\right) = \frac{1}{2}\left[\frac{\partial}{\partial x_3}(cx_1) - \frac{\partial}{\partial x_1}(0)\right] = 0 \tag{4}$$

$$r_{21} = \frac{1}{2}\left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right) = \frac{1}{2}\left[\frac{\partial}{\partial x_1}(-cx_2) - \frac{\partial}{\partial x_2}(cx_1)\right] = 0 \tag{5}$$

$$r_{32} = \frac{1}{2}\left(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}\right) = \frac{1}{2}\left[\frac{\partial}{\partial x_2}(0) - \frac{\partial}{\partial x_3}(-cx_2)\right] = 0. \tag{6}$$

- In this type of flow the spin tensor r is equal to zero $\Longrightarrow$ **the flow is irrotational**.
- Physically, this means that the fluid particles are not rotating about their centroid.

Flow I: Stagnation Point flow (continued ..)

2) Let's now analyze the strain-rate tensor, recalling the definition of this one in index notation,

$$e_{ij} = \frac{1}{2}\Big(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\Big), \tag{7}$$

and in matrix form,

$$e = \begin{pmatrix} \frac{\partial u_1}{\partial x_1} & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\right) \\ \frac{1}{2}\left(\frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2}\right) & \frac{\partial u_2}{\partial x_2} & \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\right) \\ \frac{1}{2}\left(\frac{\partial u_3}{\partial x_1} + \frac{\partial u_1}{\partial x_3}\right) & \frac{1}{2}\left(\frac{\partial u_3}{\partial x_2} + \frac{\partial u_2}{\partial x_3}\right) & \frac{\partial u_3}{\partial x_3} \end{pmatrix}. \tag{8}$$

Flow I: Stagnation Point flow (continued ..)

Since the tensor is symmetric, we need to evaluate six components, *i.e.* three diagonal and three off diagonal,

$$e_{11} = \frac{\partial u_1}{\partial x_1} = \frac{\partial}{\partial x_1}(cx_1) = c \tag{9}$$

$$e_{22} = \frac{\partial u_2}{\partial x_2} = \frac{\partial}{\partial x_2}(-cx_2) = -c \tag{10}$$

$$e_{33} = \frac{\partial u_3}{\partial x_3} = \frac{\partial}{\partial x_3}(0) = 0 \tag{11}$$

$$e_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) = \frac{1}{2}\left(\frac{\partial}{\partial x_2}(cx_1) + \frac{\partial}{\partial x_1}(-cx_2)\right) = 0 \tag{12}$$

$$e_{13} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\right) = \frac{1}{2}\left(\frac{\partial}{\partial x_3}(cx_1) + \frac{\partial}{\partial x_1}(0)\right) = 0 \tag{13}$$

$$e_{23} = \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\right) = \frac{1}{2}\left(\frac{\partial}{\partial x_3}(-cx_2) + \frac{\partial}{\partial x_2}(0)\right) = 0. \tag{14}$$

Flow I: Stagnation Point flow (continued ..)

Therefore, in this case the strain-rate tensor is not null, but

$$e = \begin{pmatrix} c & 0 & 0 \\ 0 & -c & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{15}$$

- Physically, this means that since $e_{11} = c$, line elements initially along the x_1-direction will elongate longitudinally in this direction.
- On the other hand, since $e_{22} = -c$, line elements initially along the x_2-direction will become compressed.
- Since all of the off-diagonal components are zero, the angle between material line elements never changes.
- Thus, two initially perpendicular line elements will remain perpendicular.

Flow I: Stagnation Point flow (continued ..)

Finally, one can further check for the **dilation property** (*i.e.* compressibility).

- For this purpose one needs to check whether the volume of the fluid particles changes in time.
- Recall that the divergence of the velocity represents the dilation,

$$\frac{\partial u_i}{\partial x_i} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} = \frac{\partial}{\partial x_1}(c\,x_1) + \frac{\partial}{\partial x_2}(-c\,x_2) + \frac{\partial}{\partial x_3}(0) = 0. \tag{16}$$

$\Longrightarrow$ the volume of the fluid particle cannot change. $\Longrightarrow$ This means that as material line elements elongate in the x_1-direction, line elements in the x_2-direction must proportionally compress.

$+$ Note: Realize that the flow is steady since the velocity $\vec{u}$ is independent of time.

Chapter Overview

Flow II: Solid Body Rotation

Let's consider now the flow induced by a solid body rotation about an axis through the origin of the coordinate system as shown in in the figure below,

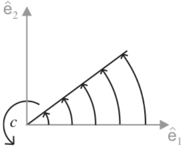

Figure: *Representation of the solid body rotation flow.*

This type of flow would occur, for example, in a large circular tube of water that was set spinning on a turn table with the axis of rotation through the center of the tube.

The velocity components for this flow are

$$u_1 = -c\, x_2, \tag{17}$$
$$u_2 = c\, x_1, \tag{18}$$
$$u_3 = 0, \qquad \text{with } c > 0. \tag{19}$$

Flow II: Solid Body Rotation (continued ...)

Proceeding similarly to what it was done for the stagnation point flow, one can easily compute the spin and strain-rate tensors.

1) For the **spin tensor** the off-diagonal terms are

$$r_{13} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}\right) = \frac{1}{2}\left[\frac{\partial}{\partial x_3}(-cx_2) - \frac{\partial}{\partial x_1}(0)\right] = 0 \tag{20}$$

$$r_{21} = \frac{1}{2}\left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right) = \frac{1}{2}\left[\frac{\partial}{\partial x_1}(cx_1) - \frac{\partial}{\partial x_2}(-cx_2)\right] = c \tag{21}$$

$$r_{32} = \frac{1}{2}\left(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}\right) = \frac{1}{2}\left[\frac{\partial}{\partial x_2}(0) - \frac{\partial}{\partial x_3}(cx_1)\right] = 0, \tag{22}$$

Flow II: Solid Body Rotation (continued ...)

2) For the **strain-rate tensor** the six different terms are

$$e_{11} = \frac{\partial u_1}{\partial x_1} = \frac{\partial}{\partial x_1}(-cx_2) = 0 \tag{23}$$

$$e_{22} = \frac{\partial u_2}{\partial x_2} = \frac{\partial}{\partial x_2}(cx_1) = 0 \tag{24}$$

$$e_{33} = \frac{\partial u_3}{\partial x_3} = \frac{\partial}{\partial x_3}(0) = 0 \tag{25}$$

$$e_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) = \frac{1}{2}\left(\frac{\partial}{\partial x_2}(-cx_2) + \frac{\partial}{\partial x_1}(cx_1)\right) = 0 \tag{26}$$

$$e_{13} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\right) = \frac{1}{2}\left(\frac{\partial}{\partial x_3}(-cx_2) + \frac{\partial}{\partial x_1}(0)\right) = 0 \tag{27}$$

$$e_{23} = \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\right) = \frac{1}{2}\left(\frac{\partial}{\partial x_3}(cx_1) + \frac{\partial}{\partial x_2}(0)\right) = 0. \tag{28}$$

Flow II: Solid Body Rotation (continued ...)

Therefore, in this type of flow, the matrix form the spin and strain-rate tensors are

$$r = \begin{pmatrix} 0 & -c & 0 \\ c & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \text{and} \qquad e = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{29}$$

- Physically, this means that there is neither longitudinal stretching of line elements, nor shear deformation in rigid body rotation.
- $\Longrightarrow$ material elements maintain both their shape and size (i.e. a sphere elements of radius a remains a sphere of radius a and does not deform into an ellipsoid).
- $\Longrightarrow$ the material element will rotate about the x_3-axis at an angular rotation rate of c.

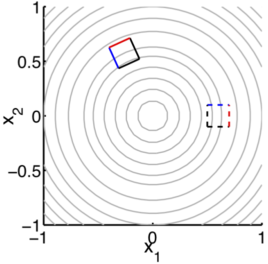

Figure: *Streamlines of a solid body rotation flow. The dotted square represents an initial fluid element that moves to anew location (solid lines). The gray circular lines illustrate the streamlines for this flow.*

Chapter Overview

Flow III: Linear Shear Flow

- The two previous flows exemplified either strain-rate or rotation but not both.
- Now, we consider linear shear flow that incorporates both characteristics.

In this case, the velocity components can be of the form,

$$u_1 = c\,x_2, \tag{30}$$

$$u_2 = 0, \tag{31}$$

$$u_3 = 0, \qquad \text{with } c > 0. \tag{32}$$

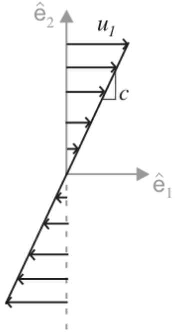

Figure: *Representation of a Linear Shear flow.*

Flow III: Linear Shear Flow (continued ...)

Let's now compute once more the spin and strain-rate tensors.

1) For the **spin tensor** the off-diagonal terms are:

$$r_{13} = \frac{1}{2}\Big(\frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}\Big) = \frac{1}{2}\Big[\frac{\partial}{\partial x_3}(cx_2) - \frac{\partial}{\partial x_1}(0)\Big] = 0 \tag{33}$$

$$r_{21} = \frac{1}{2}\Big(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\Big) = \frac{1}{2}\Big[\frac{\partial}{\partial x_1}(0) - \frac{\partial}{\partial x_2}(cx_2)\Big] = -\frac{c}{2} \tag{34}$$

$$r_{32} = \frac{1}{2}\Big(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}\Big) = \frac{1}{2}\Big[\frac{\partial}{\partial x_2}(0) - \frac{\partial}{\partial x_3}(cx_1)\Big] = 0, \tag{35}$$

Flow III: Linear Shear Flow (continued ...)

2) For the **strain-rate tensor** the six different terms are

$$e_{11} = \frac{\partial u_1}{\partial x_1} = \frac{\partial}{\partial x_1}(cx_2) = 0 \tag{36}$$

$$e_{22} = \frac{\partial u_2}{\partial x_2} = \frac{\partial}{\partial x_2}(0) = 0 \tag{37}$$

$$e_{33} = \frac{\partial u_3}{\partial x_3} = \frac{\partial}{\partial x_3}(0) = 0 \tag{38}$$

$$e_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) = \frac{1}{2}\left(\frac{\partial}{\partial x_2}(cx_2) + \frac{\partial}{\partial x_1}(0)\right) = \frac{c}{2} \tag{39}$$

$$e_{13} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\right) = \frac{1}{2}\left(\frac{\partial}{\partial x_3}(cx_2) + \frac{\partial}{\partial x_1}(0)\right) = 0 \tag{40}$$

$$e_{23} = \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\right) = \frac{1}{2}\left(\frac{\partial}{\partial x_3}(0) + \frac{\partial}{\partial x_2}(0)\right) = 0. \tag{41}$$

Flow III: Linear Shear Flow (continued ...)

Therefore, in this type of flow, the spin and strain-rate tensors are:

$$r = \begin{pmatrix} 0 & c/2 & 0 \\ -c/2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \text{and} \qquad e = \begin{pmatrix} 0 & c/2 & 0 \\ c/2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{42}$$

- In the linear shear flow, there are equal parts of shear-strain rate and rotation.
- Also, the diagonal components of e_{ij} are zero, so that one should not expect any longitudinal deformation of material.

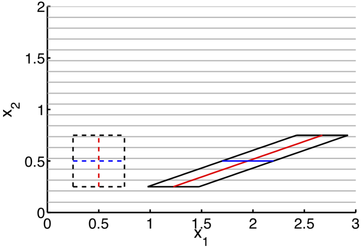

Figure: *Representation of the linear shear flow.*

One can decompose this motion into two parts, so that one is able to quantify it using the mathematical framework involving e_{ij} and r_{ij}.

Flow III: Linear Shear Flow (continued ...)

The motion of the fluid particle in shear flow can be obtained through a shear deformation and an instantaneous rotation about the centroid of the particle.

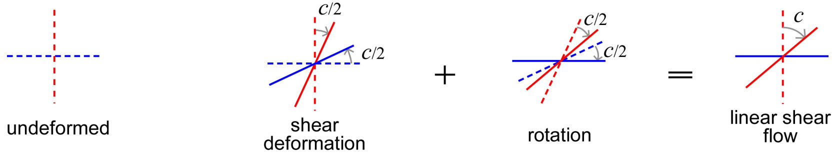

Figure: *Pictorial representation of the strain-rate and rotation decomposition associated with a linear shear flow.*

Note

- While there is vorticity in this process, the fluid element does not continuously spin about its centroid.
- In rigid body motion the fluid element rotates all the way through 360°.
- In linear shear flow, one should think about the vorticity as the instantaneous angular velocity necessary to achieve the observed fluid motion.

Chapter Overview

Flow IV: Vortex Line induced flow

Let's now consider the flow in the neighborhood of a single vortex line (similar to a streamline, a vortex line is a line that is everywhere tangent to the vorticity).

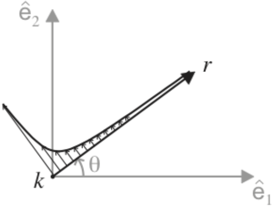

Figure: *The velocity field induced by the vortex line is independent of the polar angle* $\theta = tan^{-1}(x_2/x_1)$

The velocity components, assuming the vortex line passes through the origin of the coordinate system and is oriented along the $\hat{e}_3$-axis, are

$$u_1 = \frac{-k\, x_2}{x_1^2 + x_2^2}, \qquad u_2 = \frac{k\, x_1}{x_1^2 + x_2^2}, \qquad u_3 = 0. \tag{43}$$

Flow IV: Vortex Line induced flow (continued ...)

In this case:

- k denotes the strength of the vortex line, which is given by $k = \omega \, \Delta S$,
- ω is the magnitude of the vorticity along the line
- ΔS is the infinitesimal cross sectional area of the vortex line.
- Since k remains constant along the vortex line (by definition), it can be seen that as $\Delta S \longrightarrow 0$ then $\omega \longrightarrow \infty$.
- In this case, the origin is a singularity, so that the above velocity components are only valid for $x_1^2 + x_2^2 \neq 0$.

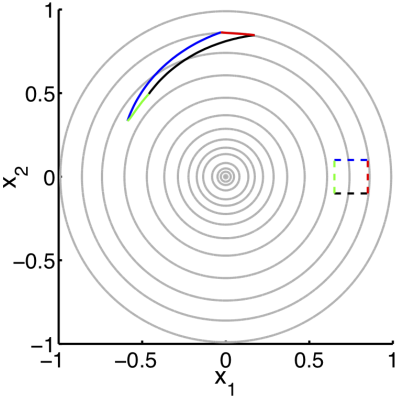

Figure: *Streamlines of steady vortex line flow and corresponding deformation of a fluid particle.*

Flow IV: Vortex Line induced flow (continued ...)

If we now proceed similarly to what it was done for the previous flows, one can easily compute the spin and strain-rate tensors.

1) For the **spin tensor** the off-diagonal terms are

$$r_{13} = \frac{\omega_2}{2} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}\right) = 0 \tag{44}$$

$$r_{21} = \frac{\omega_3}{2} = \frac{1}{2}\left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right) = 0 \tag{45}$$

$$r_{32} = \frac{\omega_1}{2} = \frac{1}{2}\left(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}\right) = 0, \tag{46}$$

Flow IV: Vortex Line induced flow (continued ...)

and for the strain-rate tensor the six different terms are

$$e_{11} = \frac{\partial u_1}{\partial x_1} = \frac{\partial}{\partial x_1}\Big(- kx_2(x_1^2 + x_2^2)^{-1}\Big) = \frac{2kx_1x_2}{(x_1^2 + x_2^2)^2} \tag{47}$$

$$e_{22} = \frac{\partial u_2}{\partial x_2} = \frac{-2kx_1x_2}{(x_1^2 + x_2^2)^2} \tag{48}$$

$$e_{33} = \frac{\partial u_3}{\partial x_3} = 0 \tag{49}$$

$$e_{12} = \frac{1}{2}\Big(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\Big) = \frac{k(x_2^2 - x_1^2)}{(x_1^2 + x_2^2)^2} \tag{50}$$

$$e_{13} = \frac{1}{2}\Big(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\Big) = 0 \tag{51}$$

$$e_{23} = \frac{1}{2}\Big(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\Big) = 0. \tag{52}$$

Flow IV: Vortex Line induced flow (continued ...)

Therefore, in this type of flow, the matrix form the spin and strain-rate tensors are

$$r = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \text{and} \qquad e = \begin{pmatrix} \frac{2kx_1x_2}{(x_1^2+x_2^2)^2} & \frac{k(x_2^2-x_1^2)}{(x_1^2+x_2^2)^2} & 0 \\ \frac{k(x_2^2-x_1^2)}{(x_1^2+x_2^2)^2} & \frac{-2kx_1x_2}{(x_1^2+x_2^2)^2} & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{53}$$

- Except for the origin, the flow is irrotational;
- However, the fluid particles follow circular paths.
- It can be seen from e_{11} and e_{22} that there is equal (but opposite) longitudinal stretch of material lines originally oriented along the $\hat{e}_1$ & $\hat{e}_2$ axes.
- One should also expect shear deformation since e_{12} (and e_{21}) are non-zero, *i.e.* we expect the angle between line elements initially along the $\hat{e}_1$ and $\hat{e}_2$ axes to decrease.

Flow IV: Vortex Line induced flow (continued ...)

Regarding the dilation of the flow,

$$\frac{\partial u_i}{\partial x_i} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} = 0, \tag{54}$$

and hence there is no change in the volume of the flow.

Flow IV: Vortex Line induced flow (continued ...)

Short-time Deformation: To interpret e_{ij} and r_{ij} by looking at the motion of a material line, the time between the initial and deformed states must be small.

- From the analysis of e_{ij} one expects the top and bottom lines to elongate while the right and left lines shorten.
- On the side Figure it can be observed that initially the right line has a length $l_r = 0.1$; at a short time later, $l_r = 0.0943$. For the bottom line, $l_b = 0.1$; however a short time later this one has a lenght of $l_b = 0.1122$.
- It can also be observed that the angle between the bottom and right line has decreased from 90° to about 66° in the short time shown.

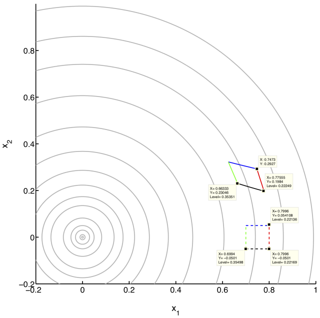

Figure: *Short-time deformation of a fluid element in the vortex line flow.*