Intermediate Fluid Mechanics

Lecture 4: The Strain-Rate Tensor

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Chapter Overview

Lecture Objectives

In this lecture we will ...

- learn about how neighboring fluid elements move relative to each other, .
- How do they interact among themselves

Chapter Overview

Normal Strain Rate

Let's begin by considering how an infinitesimal rectangle of fluid having length δx_1 becomes stretched in the x_1-direction,

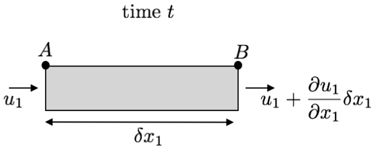

Figure: *At time t we mark a region of fluid, which results to be stretched at time $t + dt$.*

- At time t, the x_1-component of the velocity acting along the A-side is u_1.
- The velocity along the B-side can be written using Taylor's series expansion as long as the distance AB is small, *i.e.*

$$u_B = u_A + \left.\frac{\partial u}{\partial x_1}\right|_A \delta x_1 = u_1 + \frac{\partial u_1}{\partial x_1}\delta x_1. \tag{1}$$

Normal Strain Rate (continued)

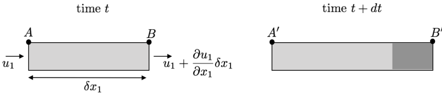

Figure: *At time t we mark a region of fluid, which results to be stretched at time t + dt.*

At the same time, the rate of change of the fluid element (AB) per unit length is

$$\frac{1}{\delta x_1}\frac{D}{Dt}(\delta x_1) = \frac{A'B' - AB}{AB}\left(\frac{1}{dt}\right). \tag{2}$$

- Note that in this case we use the material derivative because AB marks the element of fluid that we will follow as it's being stretched.
- Also, one can next recognize that $A'B' = AB + BB' - AA'$, where AA' represents the distance that the fluid particle A moved in time dt.
- Similarly for BB' with respect to the fluid particle B.

Normal Strain Rate (continued)

As long as dt is infinitesimal and the motion is one-dimensional along x_1, one can write

$$AA' = u_1\, dt \quad \text{and} \quad BB' = \left(u_1 + \frac{\partial u_1}{\partial x_1}\delta x_1\right)dt, \tag{3}$$

Note: the length traveled is the velocity at which the fluid particle is traveling multiplied by the total time traveled.

Substituting all these elements in equation 2,

$$\frac{1}{\delta x_1}\frac{D}{Dt}(\delta x_1) = \frac{1}{dt}\left(\frac{A'B' - AB}{AB}\right) \tag{4}$$

$$= \frac{1}{dt}\left(\frac{AB + BB' - AA' - AB}{AB}\right) \tag{5}$$

$$= \frac{1}{dt}\frac{1}{\delta x_1}\left[\left(u_1 + \frac{\partial u_1}{\partial x_1}\delta x_1\right)dt - u_1\, dt\right] \tag{6}$$

$$= \frac{\partial u_1}{\partial x_1}. \tag{7}$$

Normal Strain Rate (continued)

Since we had just found that in the streamwise direction:

$$\frac{1}{\delta x_1}\frac{D}{Dt}(\delta x_1) = \frac{\partial u_1}{\partial x_1}. \tag{8}$$

$\Longrightarrow$ This means that the <u>linear strain rate</u> or <u>normal strain rate</u> is given by the velocity gradient.

Following the same reasoning, one could go through similar procedures for stretching in the x_2- and x_3- directions to find that,

$$\frac{1}{\delta x_2}\frac{D}{Dt}\left(\delta x_2\right) = \frac{\partial u_2}{\partial x_2}, \quad \text{and} \quad \frac{1}{\delta x_3}\frac{D}{Dt}\left(\delta x_3\right) = \frac{\partial u_3}{\partial x_3}. \tag{9}$$

Chapter Overview

Rate of Change of Volume

Let's consider the following volume of fluid at time t, such that it is deformed with a different shape at $t + dt$,

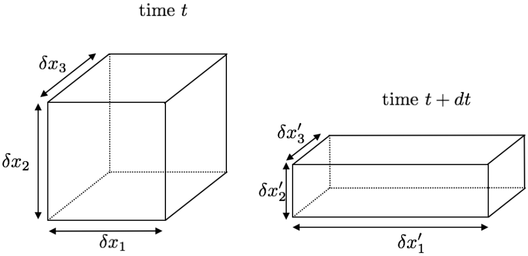

Figure: *Stretch of a fluid element.*

The rate of change of volume per unit volume (or volumetric strain rate) is given by

$$\frac{1}{\delta V}\frac{D}{Dt}(\delta V) = \frac{1}{\delta x_1\,\delta x_2\,\delta x_3}\frac{D}{Dt}(\delta x_1\,\delta x_2\,\delta x_3). \tag{10}$$

Rate of Change of Volume (continued ...)

By using the product rule it is found that,

$$\frac{1}{\delta V}\frac{D}{Dt}(\delta V) = \frac{1}{\delta x_1\,\delta x_2\,\delta x_3}(\delta x_2\,\delta x_3)\frac{D}{Dt}(\delta x_1) + \frac{1}{\delta x_1\,\delta x_2\,\delta x_3}(\delta x_1\,\delta x_3)\frac{D}{Dt}(\delta x_2) + \frac{1}{\delta x_1\,\delta x \cdots} \tag{11}$$

$$= \frac{1}{\delta x_1}\frac{D\delta x_1}{Dt} + \frac{1}{\delta x_2}\frac{D\delta x_2}{Dt} + \frac{1}{\delta x_3}\frac{D\delta x_3}{Dt}. \tag{12}$$

$\Longrightarrow$ Upon substitution of the previous results for the linear strain rates gives that

$$\frac{1}{\delta V}\frac{D}{Dt}(\delta V) = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}. \tag{13}$$

Rate of Change of Volume (continued ...)

Note that the right hand side of

$$\frac{1}{\delta \mathcal{V}} \frac{D}{Dt}(\delta \mathcal{V}) = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}, \tag{14}$$

represents the divergence of the velocity, which can be written as,

$$\frac{1}{\delta \mathcal{V}} \frac{D}{Dt}(\delta \mathcal{V}) = \vec{\nabla} \cdot \vec{u} = \frac{\partial u_i}{\partial x_i}. \tag{15}$$

Note:

If a flow is <u>incompressible</u>, then the volume of any fluid element does not change in time,

$$\frac{1}{\delta \mathcal{V}} \frac{D}{Dt}(\delta \mathcal{V}) = 0 \quad \Longrightarrow \quad \vec{\nabla} \cdot \vec{u} = 0 \tag{16}$$

Chapter Overview

Shear Strain Rate

- In addition to undergoing a normal strain rate (or stretch), a fluid element may also simply deform.
- The shear strain rate is defined as the rate of decrease of the angle formed by two initially perpendicular lines on the fluid element.

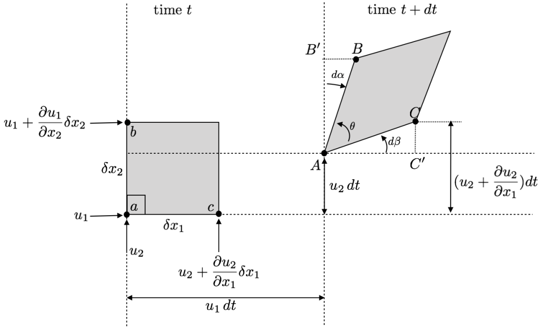

Figure: *Deformation of fluid element. Note that the velocity at b (time t) is determined by a Taylor's series expansion of velocity at a, similarly for the velocity at c.*

Shear Strain Rate (continued ...)

- During the time dt, the flow field has moved points a, b, c to the corresponding points A, B, C.
- Because the velocities at b and c are different than that at a, the angle θ between the lines AB and AC is less than $\pi/2$.
- Consequently, the rate of change of this angle θ is given by,

$$\frac{d\theta}{dt} = \frac{1}{dt}\Big[\underbrace{\frac{\pi}{2}}_{\text{Original } \theta} - \underbrace{\left(\frac{\pi}{2} - d\alpha - d\beta\right)}_{\text{final } \theta} \Big] = \frac{d\alpha + d\beta}{dt}. \tag{17}$$

Shear Strain Rate (continued ...)

If one assumes that dt is infinitesimal such that $d\alpha$ and $d\beta$ are small angles, one can then approximate the angles as

$$d\alpha = \frac{BB'}{AB'} \quad \text{and} \quad d\beta = \frac{CC'}{AC'}. \tag{18}$$

Here, the lengths BB' and CC' are determined by multiplying dt by the velocity at b (relative to a) and the velocity at c (relative to a), hence,

$$BB' = \left(u_1 + \frac{\partial u_1}{\partial x_2}\delta x_2 - u_1\right)dt \quad \text{and} \quad CC' = \left(u_2 + \frac{\partial u_2}{\partial x_1}\delta x_1 - u_2\right)dt. \tag{19}$$

Also, note that $AB' = \delta x_2$ and that $AC' = \delta x_1$.

Shear Strain Rate (continued ...)

Upon substitution, it is found that,

$$\frac{d\theta}{dt} = \frac{d\alpha + d\beta}{dt} \tag{20}$$

$$= \frac{1}{dt}\left[\frac{1}{\delta x_2}\left(\frac{\partial u_1}{\partial x_2}\delta x_2\right)dt + \frac{1}{\delta x_1}\left(\frac{\partial u_2}{\partial x_1}\delta x_1\right)dt\right] \tag{21}$$

$$= \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}. \tag{22}$$

A similar procedure can be used to determine the shear-strain rates in the other two planes (*i.e.* in the $x_1 - x_3$ and $x_2 - x_3$ planes).

Chapter Overview

Strain Rate Tensor

Based on the above calculations of the normal- and shear-strain rates of a generic fluid element, one can construct a useful quantity called the strain rate tensor,

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) = \begin{pmatrix} \frac{\partial u_1}{\partial x_1} & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\right) \\ \frac{1}{2}\left(\frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2}\right) & \frac{\partial u_2}{\partial x_2} & \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\right) \\ \frac{1}{2}\left(\frac{\partial u_3}{\partial x_1} + \frac{\partial u_1}{\partial x_3}\right) & \frac{1}{2}\left(\frac{\partial u_3}{\partial x_2} + \frac{\partial u_2}{\partial x_3}\right) & \frac{\partial u_3}{\partial x_3} \end{pmatrix}. \quad (23)$$

Note that the strain rate tensor is symmetric, and hence $e_{ij} = e_{ji}$.

Physical interpretation of the strain rate tensor

Hence, given

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) = \begin{pmatrix} \frac{\partial u_1}{\partial x_1} & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\right) \\ \frac{1}{2}\left(\frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2}\right) & \frac{\partial u_2}{\partial x_2} & \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\right) \\ \frac{1}{2}\left(\frac{\partial u_3}{\partial x_1} + \frac{\partial u_1}{\partial x_3}\right) & \frac{1}{2}\left(\frac{\partial u_3}{\partial x_2} + \frac{\partial u_2}{\partial x_3}\right) & \frac{\partial u_3}{\partial x_3} \end{pmatrix}. \quad (24)$$

- The sum of the diagonal components represents the **rate of change of the volume of an infinitesimal fluid element**.
- The off-diagonal components represent **half of the shear-strain rate**, which is the rate of decrease in the angle between two line elements originally oriented along the $\hat{e}_i$ and $\hat{e}_j$ unit vectors.

Chapter Overview

Note on Mathematics: The Divergence Operator

Mathematically the divergence is defined as,

$$\vec{\nabla} \cdot \vec{u} = \frac{\partial u_i}{\partial x_i} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}. \tag{25}$$

What does it mean physically ?

- Let's consider a vector function $F(\vec{x})$ that varies across the differential cube.
- Let's now recall Gauss's Theorem, which states that

$$\vec{\nabla} \cdot \vec{F} = \lim_{\forall \to 0} \int \int_S \vec{F} \cdot \hat{n}\, dS, \tag{26}$$

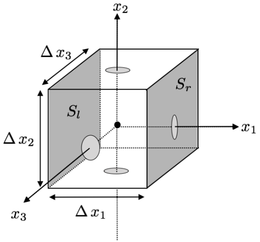

Figure: Small Cube

Note on Mathematics (continued ...)

On the equation,

$$\vec{\nabla} \cdot \vec{F} = \lim_{\mathcal{V} \to 0} \int \int_S \vec{F} \cdot \hat{n} \, dS, \tag{27}$$

- $\mathcal{V}$ is the volume of the box,
- S is the surface bounding the box,
- $\hat{n}$ is the outward-pointing unit vector that is everywhere perpendicular to the bounding surface.

The integral around the right face or surface S_r can be approximated as,

$$\int \int_{S_r} \vec{F} \cdot \hat{n} \, dS = \int \int_{S_r} \left[(F_1 \, \hat{i} + F_2 \, \hat{j} + F_3 \, \hat{k}) \cdot \hat{i} \right] dS \approx \underbrace{F_1(x_1 + \frac{\Delta x_1}{2}, x_2, x_3)}_{F_1 \text{ on the right-face}} \underbrace{\Delta x_2 \Delta x_3}_{\text{area of } S_r}. \tag{28}$$

Note on Mathematics (continued ...)

Note that as the size of the volume decreases to zero, the approximation becomes exact.

This approach can be done also for the left-face, where in this case $\hat{n}$ points in the negative x_1-direction, *i.e.*

$$\int\int_{S_l} \vec{F}\cdot\hat{n}\,dS = \int\int_{S_l} \left[(F_1\,\hat{i}+F_2\,\hat{j}+F_3\,\hat{k})\cdot(-\hat{i})\right] dS \approx -F_1(x_1-\frac{\Delta x_1}{2}, x_2, x_3)\Delta x_2\Delta x \tag{29}$$

At this point, one can use a Taylor series expansion about the value of $\vec{F}$ at the center of the differential cube,

$$F_1(x_1+\frac{\Delta x_1}{2}, x_2, x_3) = F_1(x_1, x_2, x_3) + \left.\frac{\partial F_1}{\partial x_1}\right|_{x_1,x_2,x_3} \frac{\Delta x_1}{2} + \ldots \tag{30}$$

$$F_2(x_1-\frac{\Delta x_1}{2}, x_2, x_3) = F_1(x_1, x_2, x_3) - \left.\frac{\partial F_1}{\partial x_1}\right|_{x_1,x_2,x_3} \frac{\Delta x_1}{2} + \ldots \tag{31}$$

Note on Mathematics (continued ...)

As a result, the total integral over the left and right surfaces is

$$\int\int_{S_r+S_l} \vec{F}\cdot\hat{n}\,dS \approx \left(F_1(x_1,x_2,x_3) + \frac{\partial F_1}{\partial x_1}\bigg|_{x_1,x_2,x_3}\frac{\Delta x_1}{2}\right)\Delta x_2\Delta x_3 \tag{32}$$

$$-\left(F_1(x_1,x_2,x_3) - \frac{\partial F_1}{\partial x_1}\bigg|_{x_1,x_2,x_3}\frac{\Delta x_1}{2}\right) \approx \frac{\partial F_1}{\partial x_1}\Delta x_1\Delta x_2\Delta x_3 \tag{33}$$

A similar procedure can be performed for the top-bottom surfaces and the front-back surfaces of the cube to yield,

$$\int\int_{S_t+S_b} \vec{F}\cdot\hat{n}\,dS \approx \frac{\partial F_2}{\partial x_2}\Delta x_1\Delta x_2\Delta x_3 \tag{34}$$

$$\int\int_{S_f+S_b} \vec{F}\cdot\hat{n}\,dS \approx \frac{\partial F_3}{\partial x_3}\Delta x_1\Delta x_2\Delta x_3. \tag{35}$$

Note on Mathematics (continued ...)

Summing the integrals over all of the faces and taking the limit as the volume goes to zero gives that,

$$\vec{\nabla} \cdot \vec{F} = \lim_{\forall \to 0} \int \int_S \vec{F} \cdot \hat{n} \, dS = \frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2} + \frac{\partial F_3}{\partial x_3}. \tag{36}$$

Hence:

- The divergence represents the net amount of $\vec{F}$ leaving a differential volume at the point (x_1, x_2, x_3).
- If one considers the case where $\vec{F}$ was the velocity field, then the divergence of the velocity will tell us how much the flow is expanding (or contracting) at every point in space.
- Does now $\vec{\nabla} \cdot \vec{u} = 0$ has a more clear physical meaning?