Intermediate Fluid Mechanics

Lecture 18: Dimensional Analysis II

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Chapter Overview

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Lecture Objectives

- In this lecture we will dig deeper in the practice of non-dimensional analysis.
- We will consider simplified versions of the non-dimensional NS equation (valid when the appropriate assumptions are met).

Chapter Overview

Non-dimensional NS equations

Let's begin by considering a generic form of the non-dimensional NS equations,

$$\tilde{u}_i \frac{\partial \tilde{u}_i}{\partial \tilde{x}_j} = -\frac{1}{\tilde{\rho}} \frac{\partial \tilde{p}}{\partial \tilde{x}_i} + \frac{1}{R_e} \frac{\partial^2 \tilde{u}_i}{\partial \tilde{x}_j}. \tag{1}$$

Note that in order to have a complete problem, one also needs to non-dimensionalize the boundary conditions. If the problem was unsteady, the corresponding initial conditions should also be non-dimensionalized.

In the above equation, the only relevant non-dimensional parameter is the the Reynolds number.

$\Longrightarrow$ The question is, how does the magnitude of the Reynolds number affect the dynamics of the flow?

Non-dimensional NS equations (continued ...)

Answer:

Assuming that one has selected the characteristic scales appropriately, then each term in equation 1, namely:

$$\tilde{u}_i \frac{\partial \tilde{u}_i}{\partial \tilde{x}_j}, \quad -\frac{1}{\tilde{\rho}} \frac{\partial \tilde{p}}{\partial \tilde{x}_i}, \quad \frac{\partial^2 \tilde{u}_i}{\partial \tilde{x}_j^2}; \tag{2}$$

should all have an order of magnitude near unity.

Non-dimensional NS equations (continued ...)

$\Longrightarrow$ The significance of the viscous diffusion term will vary based on the value of the R_e.

- If $R_e \approx 1$, then all three terms in equation 1 have an equal contribution to the dynamics of the flow.
- If $R_e << 1$, the viscous diffusion becomes much more important than inertia.
- If $R_e >> 1$, then viscous diffusion becomes negligible compared to inertia.

In this way, the Reynolds number can be thought of as a dial. By turning the dial in different directions, the dynamics of the flow change, *e.g.* from laminar to turbulent.

The reduction of the flow dynamics to the value of relevant non-dimensional parameters is the fundamental element that gives rise to the concept of **Similarity**.

Chapter Overview

Similarity

Two flow fields will be identical if the flow in both cases satisfies the same governing equations with the same boundary/initial conditions. (This is in general true if the mathematical problem is well posed and one is convinced that there is a unique solution to the equations.)

Question: How can two flow fields with different geometries have the same solution to the Navier-Stokes equations?

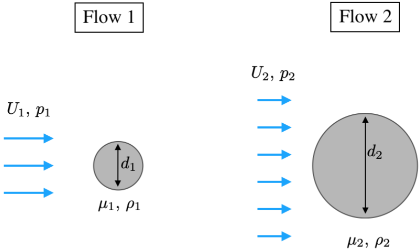

Figure: Representation of two problems with a-priori different constraints that lead to the same "similar" fluid dynamics.

Similarity (continued ...)

Answer:

1. if the governing equations in both cases are non-dimensionalized consistently.
2. if the non-dimensional boundary and initial conditions are the same.
3. if the values of all relevant non-dimensional parameters are the same.

Criteria (1) can be satisfied if the characteristic length and velocity scales are chosen consistently.

Criteria (1) and (2) are satisfied if the two flows exhibit geometric similarity.
$\Longrightarrow$This basically requires that the two geometries "look" identical except that one might be proportionally bigger than the other.

Similarity (continued ...)

Using the above example, one would non-dimensionalize $\vec{u}$ and $\vec{x}$,

$$\boxed{\text{Flow 1:}} \qquad \tilde{u} = \frac{u}{U_1}, \quad \tilde{v} = \frac{v}{U_1}, \quad \tilde{x} = \frac{x}{d_1} \quad \tilde{y} = \frac{y}{d_1} \tag{3}$$

$$\boxed{\text{Flow 2:}} \qquad \tilde{u} = \frac{u}{U_2}, \quad \tilde{v} = \frac{v}{U_2}, \quad \tilde{x} = \frac{x}{d_2} \quad \tilde{y} = \frac{y}{d_2} \tag{4}$$

Note, in the example given, there is only one length scale (the diameter) and one velocity scale (the approach flow speed), so it is relatively straightforward to perform the non-dimensionalization.

<u>Note:</u> This is not always the case!

Similarity (continued ...)

For example, consider the case of the flow around a ship hull.

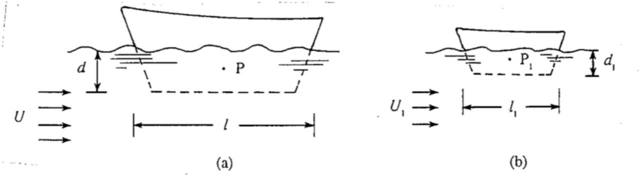

Figure: Flow around two different ship hulls.

In this case, one could choose either the depth d of the hull in the water or the length l of the hull as the characteristic length scale.

Similarity (continued ...)

The decision depends on the particular interest.

- For example, if one is interested in the drag, then one needs to know about the flow in the immediate vicinity of and around the hull.
- In this case, l would be the most appropriate characteristic length scale since the magnitude of x/l in the region around the ship is of order one.
- Similarly, the choice of velocity U as the characteristic velocity scale will guarantee that the maximum value of u/U is of order one.

Similarity (continued ...)

Going back to the example of the flow around a sphere, the non-dimensional form of the governing equations becomes,

Flow 1	Flow 2	Comments
$\tilde{u}_i \frac{\partial \tilde{u}_i}{\partial \tilde{x}_j} = -\frac{1}{\tilde{\rho}} \frac{\partial \tilde{p}}{\partial \tilde{x}_i} + \frac{1}{Re_1} \frac{\partial^2 \tilde{u}_i}{\partial \tilde{x}_j}$	$\tilde{u}_i \frac{\partial \tilde{u}_i}{\partial \tilde{x}_j} = -\frac{1}{\tilde{\rho}} \frac{\partial \tilde{p}}{\partial \tilde{x}_i} + \frac{1}{Re_1} \frac{\partial^2 \tilde{u}_i}{\partial \tilde{x}_j}$	Same Equation
$\tilde{u}_i(\tilde{x}_{b_i}) = 0$	$\tilde{u}_i(\tilde{x}_{b_i}) = 0$	No slip along boundary
$\tilde{u}_i(\tilde{x}_i \to \infty) = 0$	$\tilde{u}_i(\tilde{x}_i \to \infty) = 0$	Far field boundary condition
$\tilde{u}_i = u_i / U_1$	$\tilde{u}_i = u_i / U_2$	
$\tilde{x}_i = x_i / d_1$	$\tilde{x}_i = x_i / d_2$	
$\tilde{\rho} = \rho / \rho_1$	$\tilde{\rho} = \rho / \rho_2$	
$\tilde{p} = (p - p_1)/(\rho_1 U_1^2)$	$\tilde{p} = (p - p_2)/(\rho_2 U_2^2)$	
$Re_1 = \rho_1 U_1 d_1 / \mu_1$	$Re_2 = \rho_2 U_2 d_2 / \mu_2$	

Similarity (continued ...)

This result is significant for model testing because it states that (assuming geometric similarity has been satisfied), all one needs to do is match the Reynolds number.

Note:

Two flows could have different fluid properties, different diameters, and even different approach speeds; but none of these differences matter as long as the Reynolds number is the same in both flows.

Chapter Overview

Physical example:

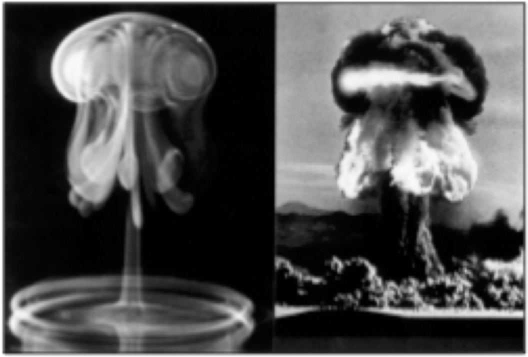

Figure: (a) Inverted image of a 2.6 mm drop of water after impacting a quiescent pool of water. (b) Above ground nuclear test in Nevada in 1957.

The similarities in the two flows is striking. Even though the respective length, time, and velocity scales are totally different, the flows definitely exhibit dynamic similarity.