ME 5200/6200 and ECE 5615/6615 Classical Controls

Homework 05 Solutions

Do the following problems and show all your work for full credit. Note: not all problems will be graded, but you must complete all problems to get full credit.

Problem 1

The system has the following characteristic equations:

(a) $s^2 + 5s + 2$
(b) $s^3 + 4s^2 + 8s + 4$
(c) $s^3 + 2s^2 - 6s + 20$
(d) $s^4 + s^3 + 2s^2 + 12s + 10$

(i) Determine whether the systems are stable using Routh-Hurwitz criterion
(ii) Using Matlab's 'roots' function, determine the roots of the characteristic equations above. Compare the location of the roots to the stability result from Part (a). Are the results consistent?

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Homework 05 Solutions

Problem 1

(a) $s^2 + 5s + 2$

R-H table

$$
\begin{array}{c|cc}
s^2 & 1 & 2 \\
s & 5 & 0 \\
1 & \dfrac{-\begin{vmatrix} 1 & 2 \\ 5 & 0 \end{vmatrix}}{5} = 2 &
\end{array}
$$

All positive, so sign change is 0

No poles in the ORHP, so system is stable b/c all poles in OLHP. Note we don't have a row of zeros so don't have roots on $j\omega$-axis!

(b)

R-H table

s^3	1	8
s^2	4	4
s	$\dfrac{-\begin{vmatrix} 1 & 8 \\ 4 & 4 \end{vmatrix}}{4} = 7$	
s^0	$\dfrac{-\begin{vmatrix} 4 & 4 \\ 7 & 0 \end{vmatrix}}{7} = 4$	

All positive, so no sign change! No poles in ORHP. Also, no rows of zeros so no poles on $j\omega$-axis, Stable

(c) $s^3 + 2s^2 \cancel{} - 6s + 20$

R-H table

s^3	$1 \quad -6$
s^2	$2 \quad 20$
s	$\dfrac{-\begin{vmatrix} 1 & -6 \\ 2 & 20 \end{vmatrix}}{2} = -4$
s^0	$\dfrac{-\begin{vmatrix} 2 & 20 \\ -4 & 0 \end{vmatrix}}{-4} = +5$

~~one~~ 2 sign changes, so 2 poles in ORHP. unstable

(d) $s^4 + s^3 + 2s^2 + 12s + 10$

s^4	1	2	10
s^3	1	12	0
s^2	-10	10	
s^1	13		
s^0	10		

2 sign change, so 2 poles in ORHP! unstable

(ii) Roots Using Matlab

s^2+5s+2

(a) >> roots([1 5 2])

Answer is: $s=-4.5616$, $s=-0.4384$) stable

Consistent w/ results from R-H table.

(b) $s^3+4s^2+8s+4=0$

$s_{1,2}=-1.6478 \pm 1.7214j$
$s=-0.7044$) stable

Consistent w/ R-H results!

(c) $s^3+2s^2-6s+20=0$

$s_{1,2}=1.1991 \pm 1.7634j$
$s_3=-4.3981$) unstable

Consistent w/ R-H table

(d) $s^4+s^3+2s^2+12s+10=0$

$s_1=-1$ $s_2=-1.8474$
$s_{3,4}=0.9237 \pm 2.1353j$) unstable

consistent w/ R-H table!

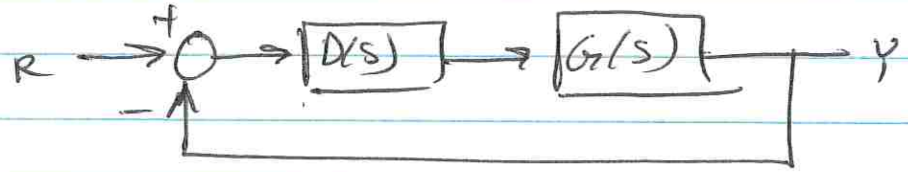
Problem 2

Consider the feedback system shown above. Suppose the plant and controller have the forms

$$D(s) = K_P + \frac{K_I}{s}$$
$$G(s) = \frac{1}{s^2 + 2s + 10}$$

Find the range of gains K_P and K_I for which the system is stable.

Problem 2



$D(s) = K_p + K_i/s \qquad G(s) = \dfrac{1}{s^2 + 2s + 10}$

poles: $1 + D(s)G(s) = 0$

$\Rightarrow 1 + \left(K_p + K_i/s\right)\left(\dfrac{1}{s^2 + 2s + 10}\right) = 0$

$s(s^2 + 2s + 10) + sK_p + K_i = 0$

$\Rightarrow s^3 + 2s^2 + (10 + K_p)s + K_i = 0$

R-H Table

s^3	1	$K_p + 10$
s^2	2	K_i
s	b_1	
s^0	c_1	

$b_1 = \dfrac{-\begin{vmatrix} 1 & (K_p + 10) \\ 2 & K_i \end{vmatrix}}{2}$

$b_1 = \dfrac{-(K_i - 2K_p + 20)}{2} > 0$

$\Rightarrow -\tfrac{1}{2}K_i + K_p + 10 > 0$

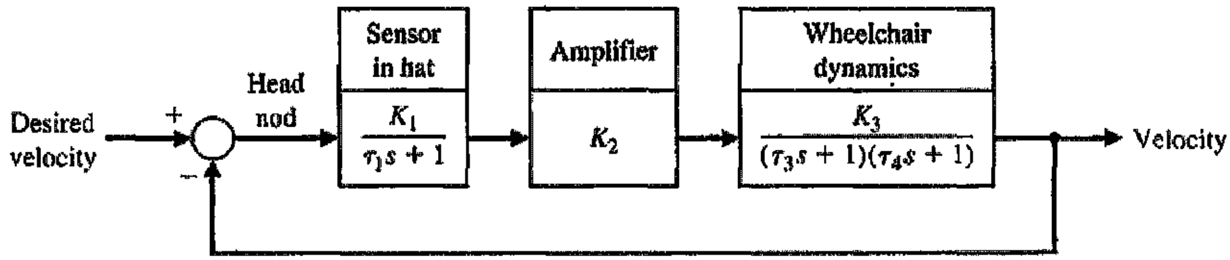
$\Rightarrow K_i < 2K_p + 20$

$c_1 = \dfrac{-\begin{vmatrix} 2 & K_i \\ b_1 & 0 \end{vmatrix}}{b_1} = \dfrac{-(0 - b_1 K_i)}{b_1} = K_i > 0$

$\boxed{\text{stable if } 0 < K_i < 2K_p + 20}$

Problem 3

Below is a velocity control system for a wheelchair, where velocity sensors are mounted in a headgear on the person in the wheelchair. We want to enable people paralyzed from the neck down to drive themselves in motorized wheelchairs. The headgear sensor provides an output proportional to the magnitude of the head movement. There is a sensor mounted at 90-degrees intervals so that forward, left, right, or reverse can be commanded. The typical values for the time constants are $\tau_1 = 0.5$ seconds; $\tau_3 = 1$ second; and $\tau_4 = 0.25$ seconds.



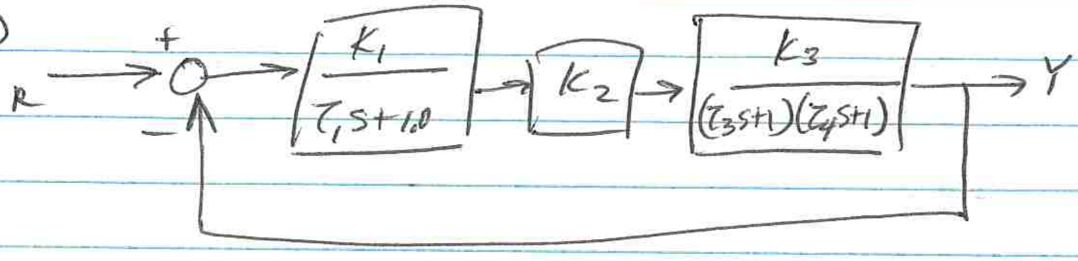
(a) Determine the limiting gain $K = K_1 K_2 K_3$ for a stable system

(b) When the gain K is set equal to one-third of the limiting value, determine whether the settling time (to within 2% of the final value of the system) is less than 4 seconds.

(c) Determine the value of the gain that results in a system with settling time of 4 seconds. Also, obtain the value of the roots of the characteristic equation when the settling time is 4 seconds.

Problem 3

(a)



$$K = K_1 K_2 K_3$$

$$\frac{Y}{R} = \frac{\left(\frac{K_1}{\tau_1 s+1}\right)(K_2)\left(\frac{K_3}{(\tau_3 s+1)(\tau_4 s+1)}\right)}{1+\left(\frac{K_1}{\tau_1 s+1}\right)(K_2)\left(\frac{K_3}{(s\tau_3+1)(\tau_4 s+1)}\right)}$$

$$\Rightarrow (\tau_1 s+1)(\tau_3 s+1)(\tau_4 s+1) + K_1 K_2 K_3 = 0$$

$$\Rightarrow \left(\tfrac{1}{2}s+1\right)(s+1)\left(\tfrac{1}{4}s+1\right) + K = 0$$

$$\left(\frac{s+2}{2}\right)\left(\frac{s+1}{1}\right)\left(\frac{s+4}{4}\right) + K = 0$$

$$\Rightarrow (s+2)(s+1)(s+4) + 8K = 0$$

$$(s^2+3s+2)(s+4) + 8K = 0$$

$$s^3 + 4s^2 + 3s^2 + 12s + 2s + 8 + 8K = 0$$

$$s^3 + 7s^2 + 14s + 8 + 8K = 0$$

R-H table

s^3	1	14
s^2	7	$8+8K$
s	b_1	
s^0	c_1	

$$b_1 = \frac{-\begin{vmatrix} 1 & 14 \\ 7 & 8+8K \end{vmatrix}}{7}$$

$$= \frac{-(8+8K - 14*7)}{7}$$

$$b_1 = -\frac{(8+8K)}{7} + 14 > 0$$

$$\Rightarrow \frac{8+8K}{7} < 14$$

$$\Rightarrow 8K < 98 - 8$$

$$\underline{K < 11.25}$$

$$c_1 = \frac{-\begin{vmatrix} 7 & 8+8K \\ b_1 & 0 \end{vmatrix}}{b_1}$$

$$c_1 = \frac{\cancel{b_1}(8+8K)}{\cancel{b_1}} > 0$$

$$\Rightarrow 8+8K > 0 \Rightarrow 8K > -8$$

$$\Rightarrow K > -1$$

range for K for stability: $\underline{-1 < K < 11.25}$

(b) Let $K = 1/3\,(11.25) = 3.75$

The closed-loop T.F. is then

$$\frac{Y(s)}{R(s)} = \frac{3.37}{s^3 + 7s^2 + 14s + 38}$$

Using Matlab to do a step response, the settling time is approx: $\underline{\underline{6\,s}}$

(c) To find K for $t_s = 4s$, we can either tune K using guess-and-check method, so doing this with Matlab step response, $\underline{\underline{K \simeq 1.5}}$

The other approach is to note that

$$t_s = \frac{4}{\zeta \omega_n} = 4 \Rightarrow \zeta \omega_n = 1$$

for a 3rd order system, the characteristic equation is:

$$\Rightarrow (s+b)(s^2 + \underbrace{2\zeta\omega_n}_{=1} s + \omega_n^2)$$

$$\Rightarrow s^3 + (2+b)s^2 + (\omega_n^2 + 2b)s + b\omega_n^2$$

But we know that from above:

$s^3 + 7s^2 + 14s + 8 + 8K$, so we equate

$$s^3 + 7s^2 + 14s + 8 + 8K = s^3 + (2+b)s^2 + (\omega_n^2 + 2b)s + b\omega_n^2$$

thus, $2 + b = 7$

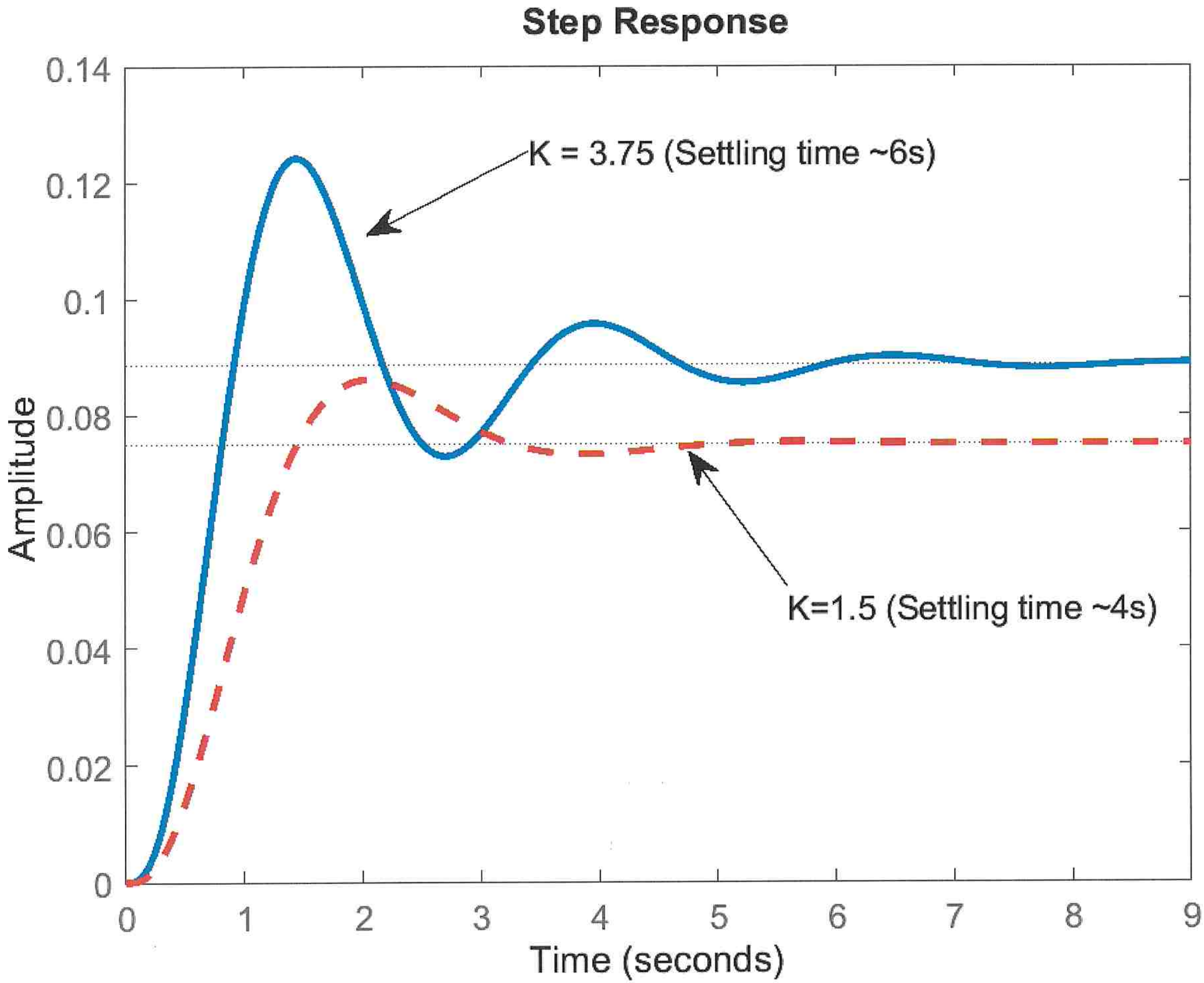
$\omega_n^2 + 2b = 14$

$b\omega_n^2 = 8 + 8K$

we solve all three to get:

$b = 5$, $\omega_n = 2$ and $\underline{\underline{K = 1.5}}$

This is the same as above!


Step Response
K = 3.75 (Settling time ~6s)
K=1.5 (Settling time ~4s)
Amplitude
0.14
0.12
0.1
0.08
0.06
0.04
0.02
0
0
1
2
3
4
5
6
7
8
9
Time (seconds)

Problem 4

A mass-spring-damper system may be represented by the following transfer function.

$$\frac{X(s)}{F(s)} = \frac{1}{ms^2 + bs + k}$$

- Find the range of system parameters (m, b, and k) that result in a stable system.

$$d(s) = s^2 + \frac{b}{m}s + \frac{k}{m}$$

R-H Table:

$$\begin{array}{c|cc} s^2 & 1 & k/m \\ s^1 & b/m & 0 \\ s^0 & \dfrac{-\begin{vmatrix} 1 & k/m \\ b/m & 0 \end{vmatrix}}{b/m} = k/m & \end{array}$$

for stability:

$$b/m > 0 \Rightarrow b > 0$$

$$k/m > 0 \Rightarrow k > 0$$

$$m \neq 0 \Rightarrow m > 0$$

Problem 5 (ME 6200/ECE 6615 Students ONLY!)

Consider the feedback system shown above. Suppose the plant and controller have the forms

$$D(s) = K_D s + K_P + \frac{K_I}{s}$$
$$G(s) = \frac{1}{s^2 + 2s + 10}$$

Find the range of gains K_D, K_P, and K_I for which the system is stable.

Closed-loop T.F. $\frac{Y(s)}{R(s)} = \frac{K_d s^2 + K_p s + K_i}{s^3 + (K_d+2)s^2 + (K_p+10)s + K_i}$

$d(s) = s^3 + (K_d+2)s^2 + (K_p+10)s + K_i$

R-H table:

s^3	1	K_p+10
s^2	K_d+2	K_i
s^1	b_1	
s^0	c_1	

$b_1 = \frac{-K_i}{K_d+2} + 10 + K_p > 0$

$\Rightarrow K_p > \frac{K_i}{K_d+2} - 10$

$c_1 = \frac{-\begin{vmatrix} K_d+2 & K_i \\ b_1 & 0 \end{vmatrix}}{b_1} = K_i > 0$

$\Rightarrow$ For stability:

$$\boxed{\begin{aligned} &K_i > 0 \\ &K_p > K_i/(K_d+2) - 10 \end{aligned}}$$