Optimal Control HW2

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Problem 1)

Let $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$**, Find** $||A||_p$**, the p-norm of matrix A.**

$||A||_p =^{sup}_{x \neq 0} \frac{||Ax||_p}{||x||_p}, x \in R^n, A \in R^{mxn}$

$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$Ax = \begin{bmatrix} 2x_1 + x_2 \\ x_1 + 2x_2 \end{bmatrix}$

$||x||_p = (\sum_{i=1}^{n} |x_i|^p)^{1/p}$

$||x||_p = (|x_1|^p + |x_2|^p)^{1/p}$

$||Ax||_p = (\sum_{i=1}^{n} |(Ax)_i|^p)^{1/p}$

$||Ax||_p = (|2x_1 + x_2|^p + |x_1 + 2x_2|^p)^{1/p}$

$||A||_p = \frac{(|2x_1+x_2|^p+|x_1+2x_2|^p)^{1/p}}{(|x_1|^p+|x_2|^p)^{1/p}}$

let $|x_1|^p + |x_2|^p = 1$

$|x_1|^p + |x_2|^p - 1 = 0 = g(x)$

$J = |2x_1 + x_2|^2 + |x_1 + x_2|^2$

$\mathcal{L} = |2x_1 + x_2|^p + |x_1 + x_2|^p - \lambda(|x_1|^p + |x_2|^p - 1)$

$\frac{\partial \mathcal{L}}{\partial x_1} = p|2x_1 + x_2|^{p-1}(2) + p|x_1 + 2x_2|^{p-1} - \lambda(p|x_1|^p) = 0$

$\frac{\partial \mathcal{L}}{\partial x_2} = p|2x_1 + x_2|^{p-1} + p|x_1 + 2x_2|^{p-1}(2) - \lambda(p|x_2|^p) = 0$

$\frac{\partial \mathcal{L}}{\partial \lambda} = |x_1|^p + |x_2|^p - 1 = 0$

$\frac{\partial \mathcal{L}}{\partial x_1} = \frac{\partial \mathcal{L}}{\partial x_2}$

$p|2x_1 + x_2|^{p-1}(2) + p|x_1 + 2x_2|^{p-1} - \lambda(p|x_1|^p) = p|2x_1 + x_2|^{p-1} + p|x_1 + 2x_2|^{p-1}(2) - \lambda(p|x_2|^p)$

$\Rightarrow x_1^* = x_2^*$

$||A||_p = \frac{(|2x_1+x_1|^p+|x_1+2x_1|^p)^{1/p}}{(|x_1|^p+|x_1|^p)^{1/p}}$

$||A||_p = \frac{(|3x_1|^p+|3x_1|^p)^{1/p}}{(2|x_1|^p)^{1/p}}$

$||A||_p = \frac{(2*3^p|x_1|^p)^{1/p}}{(2|x_1|^p)^{1/p}}$

$||A||_p = (3^p)^{1/p}$

$||A||_p = 3$

Problem 2)
Let A and B be two real matrices of size n x n. Prove that if A and B are both positive definite, then so is the sum A+B. Note: this is a proof, not showing several examples with numbers and concluding that it's true for all matrices A and B of size n x n

$$Q_A = x^T Ax > 0$$

$$Q_B = x^T Bx > 0$$

$$\Rightarrow Q_A + Q_B = (x^T Ax + x^T Bx) > 0$$

$$Q_A + Q_B = x^T(A+B)x > 0$$

$$\Rightarrow (A+B) > 0, \text{ therefore, } A+B \text{ is positive definite}$$

Problem 3)
(a) Determine the definiteness of the following matrix:

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

(b) For what numbers b is the following matrix positive semi-definite?

$$\begin{bmatrix} 2 & -1 & b \\ -1 & 2 & -1 \\ b & -1 & 2 \end{bmatrix}$$

a)

$$A - \lambda I = \begin{bmatrix} 2-\lambda & -1 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 2-\lambda \end{bmatrix}$$

$$\det(A - \lambda I) = (2-\lambda)((2-\lambda)(2-\lambda) - 1) + 1(-1(2-\lambda))$$

$$= (2-\lambda)(4 - 2\lambda - 2\lambda + \lambda^2 - 1) + \lambda - 2$$

$$= (2-\lambda)(\lambda^2 - 4\lambda + 3) + \lambda - 2$$

$$= 2\lambda^2 - 8\lambda + 6 - \lambda^3 + 4\lambda^2 - 3\lambda + \lambda - 2$$

$$= -\lambda^3 + 6\lambda^2 - 10\lambda + 4 = 0$$

$$\lambda = 2 \Rightarrow +$$

$$\lambda = 2 + \sqrt{2} \Rightarrow +$$

$$\lambda = 2 - \sqrt{2} \Rightarrow +$$

All eigenvalues are positive, therefore the matrix is positive definite

b)
If A is nxn, ${}^{(r)}A$ denotes the rxr sub matricide in the upper left hand corner of A. If A is positive definite, so is all sub matrices of A.

$${}^{1}A = [2]$$

$$\det({}^{1}A) = 2 \Rightarrow +$$

$${}^{2}A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

$$\det({}^{2}A) = 4 - 1 = 3 \Rightarrow +$$

$${}^{3}A = \begin{bmatrix} 2 & -1 & b \\ -1 & 2 & -1 \\ b & -1 & 2 \end{bmatrix}$$

$$\det({}^{3}A) = 2(4-1) - (-1)(-2+b) + b)(1-2b)$$

$$= 6 + b - 2 + b - 2b^2$$

$$= -2b^2 + 2b + 4 \geq 0 \text{ (To be positive semi-definite)}$$

$$b = \tfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$b = \frac{-b \pm \sqrt{1-4(-2)}}{2}$$

$$b = \frac{1}{2} \pm \frac{3}{2}$$

$$b \in [-1, 2]$$

Problem 4)
For an inverted pendulum system, the objective is to maintain an upright position of the pendulum on a cart. The linearized state equations are:

$\dot{x_1}(t) = x_2(t)$
$\dot{x_2}(t) = -x_3(t) + 0.2u(t)$
$\dot{x_3}(t) = x_4(t)$
$\dot{x_2}4(t) = 10x_3(t) - 0.2u(t)$

where, $x_1(t)$ **is horizontal linear displacement of the cart,** $x_2(t)$ **is the linear velocity of the car,** $x_3(t)$ **is angular position of the pendulum from vertical line,** $x_4(t)$ **is angular velocity, and** $u(t)$ **is the horizontal force applied to the car. Formulate a performance index to keep the pendulum in the vertical position with as little energy as possible.**

$$J(u,x) = \int_{t_0}^{t_f} (x^T Q x + u^T R u) dt$$

$$u = [u(t)]$$

$$x = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix}$$

$$J(u,x) = \int_{t_0}^{t_f} (\begin{bmatrix} x_1(t) & x_2(t) & x_3(t) & x_4(t) \end{bmatrix} Q \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} + [u(t)]R[u(t)])dt$$

$$J(u,x) = \int_{t_0}^{t_f} \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix} Q \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + Ru^2 dt$$

We care about the angular position which, x_3, and the energy of the input, u^2

$\Rightarrow$ Build Q and R to weight the cost of x_3 and u^2

$$Q = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & C_1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$R = C_2$$

$$J(u,x) = \int_{t_0}^{t_f} \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & C_1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + C_2 u^2 dt$$

$$J(u,x) = \int_{t_0}^{t_f} (C_1 x_3^2 + C_2 u^2) dt$$

Problem 5)
(a) Consider the following function of two variables:

$$f(x,y) = x^2 + y^2$$

Determine the stationary points and classify them into maxima, minima and saddles.
(b) Consider the following function of two variables:

$$f(x,y) = 2x^3 + 6xy^2 - 3y^2 - 150x$$

Determine the stationary points and classify them into maxima, minima and saddles.
a)

$$\frac{\partial f}{\partial x} = 2x = 0 \Rightarrow x^* = 0$$

$$\frac{\partial f}{\partial y} = 2y = 0 \Rightarrow y^* = 0$$

$$\frac{\partial^2 f}{\partial x^2} = 2$$

$$\frac{\partial^2 f}{\partial y^2} = 2$$

$$\frac{\partial^2 f}{\partial x \partial y} = 0$$

$$\frac{\partial^2 f}{\partial y \partial x} = 0$$

$$f_{xx} f_{yy} - (f_{xy})^2 = (2)(2) - (0)^2 = 4 > 0$$

$$f_{xx}, f_{yy} > 0 \Rightarrow (x^* = 0, y^* = 0) \text{ is a minimum}$$

b)

$$\frac{\partial f}{\partial x} = 6x^2 + 6y^2 - 150 = 0 \Rightarrow x = \sqrt{25 - y^2}$$

$$\frac{\partial f}{\partial y} = 12xy - 9y^2 = 0$$

$$12(\sqrt{25 - y^2})y - 9y^2 = 0$$

$$(\sqrt{25 - y^2})y - \tfrac{3}{4}y^2 = 0$$

$$(25 - y^2)y^2 = 0.5625y^4$$

$$25y^2 - y^4 = 0.5625y^4$$

$$1.5625y^4 - 25y^2 = 0$$

$$y = -4, y = 4 \Rightarrow x = -3, 3$$

$$y = 0, y = 0 \Rightarrow x = -5, 5$$

$$\text{Stationary Points: } (-5,0), (5,0), (-3,-4), (3,4)$$

$$\frac{\partial^2 f}{\partial x^2} = 12x$$

$$\frac{\partial^2 f}{\partial y^2} = 12x - 18y$$

$$\frac{\partial^2 f}{\partial x \partial y} = 12y$$

$$\frac{\partial^2 f}{\partial y \partial x} = 12y$$

$f_{xx}f_{yy} - (f_{xy})^2 = 12x(12x - 18y) - (12y)^2 = 144x^2 - 216xy - 144y^2$

(5,0) = 3600, $f_{xx}, f_{yy} > 0 \Rightarrow$ minimum

(-5,0) = 3600, $f_{xx}, f_{yy} < 0 \Rightarrow$ maximum

(-3,-4) = -3600 $\Rightarrow$ saddle

(3,4) = -3600 $\Rightarrow$ saddle

Problem 6)
In this problem, you will show that the shortest distance between two points is a straight line connecting the two points. Let P1=(x1, y1) and P2=(x2, y2) be two points on a 2D plane. Find the third point P3=(x3, y3), such that d1=d2, where d1 is the distance from P3 to P1 and d2 is the distance from P3 to P2.

$g(x_3, y_3) = d_1 - d_2 = 0$, Using the Euclidean norm because it is a straight line between 2 points:

$$\sqrt{(x_3 - x_1)^2 + (y_3 - y_1)^2} = \sqrt{(x_3 - x_2)^2 + (y_3 - y_2)^2}$$

$$(x_3 - x_1)^2 + (y_3 - y_1)^2 = (x_3 - x_2)^2 + (y_3 - y_2)^2$$

$$x_3^2 - 2x_1x_3 + x_1^2 + y_3^2 - 2y_3y_1 + y_1^2 = x_3^2 - 2x_2x_3 + x_2^2 + y_3^2 - 2y_3y_2 + y_2^2$$

$$-2x_1x_3 + x_1^2 - 2y_3y_1 + y_1^2 = -2x_2x_3 + x_2^2 - 2y_3y_2 + y_2^2$$

$$g(x_3, y_3) = x_2^2 + y_2^2 - x_1^2 - y_1^2 + 2(x_1x_3 + y_1y_3 - x_2x_3 - y_2y_3) = 0$$

$$J(x_3, y_3) = d_1^2 + d_2^2 = (x_3 - x_1)^2 + (y_3 - y_1)^2 + (x_3 - x_2)^2 + (y_3 - y_2)^2$$

$$\mathcal{L}(x_3, y_3, \lambda) = J(x_3, y_3) - \lambda g(x_3, y_3)$$

$$\mathcal{L} = (x_3 - x_1)^2 + (y_3 - y_1)^2 + (x_3 - x_2)^2 + (y_3 - y_2)^2 - \lambda(x_2^2 + y_2^2 - x_1^2 - y_1^2 + 2(x_1x_3 + y_1y_3 - x_2x_3 - y_2y_3))$$

$$\frac{\partial \mathcal{L}}{\partial x_3} = 2(x_3 - x_1) + 2(x_3 - x_2) - \lambda(2x_1 - 2x_2) = 0$$

$$\frac{\partial \mathcal{L}}{\partial y_3} = 2(y_3 - y_1) + 2(y_3 - y_2) - \lambda(2y_1 - 2y_2) = 0$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = x_2^2 + y_2^2 - x_1^2 - y_1^2 + 2(x_1x_3 + y_1y_3 - x_2x_3 - y_2y_3) = 0$$

$$\lambda^* = 0$$

$$x_3* = \frac{x_1 + x_2}{2}$$

$$y_3* = \frac{y_1 + y_2}{2}$$

$$(x_3^*, y_3^*) = (\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2})$$

Problem 7)
You work in a box-making factory and you are the lead box designer. Your boss asks you to design a rectangular box with an open top that uses the least amount of metal and the box must have a capacity of 10 in3. Assume that the length, width, and height of the box are: x, y, and z, respectively. Determine the dimensions that meet these specs!

$$J(x,y,z) = 2(yz) + 2(xz) + xy$$

$$g(x,y,z) = xyz - 10 = 0$$

$$\mathcal{L}(x,y,z,\lambda) = J(x,y,z) - \lambda g(x,y,z)$$

$$\mathcal{L}(x,y,z,\lambda) = 2(yz) + 2(xz) + xy - \lambda(xyz - 10)$$

$$\frac{\partial \mathcal{L}}{\partial x} = 2z + y - \lambda yz = 0$$

$$\frac{\partial \mathcal{L}}{\partial y} = 2z + x - \lambda xz = 0$$

$$\frac{\partial \mathcal{L}}{\partial z} = 2y + 2x - \lambda xy = 0$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = xyz - 10 = 0$$

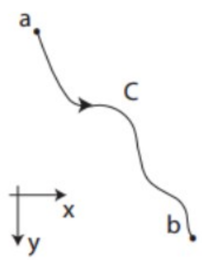
$$x^* = 2.7144$$

$$y^* = 2.7144$$

$$z^* = 1.3572$$

$$\lambda^* = 1.4736$$

Problem 8)
Given two points a and b in a vertical plane as shown below:



Assume that a metal wire connects points a and b with some shape. Next, let a bead start at point a. What shape should the wire take so that the bead traverses from a to b under gravity in the shortest time? Assume there's no friction.

$J = \int_{t_0}^{t_f} dt$

$ds = \sqrt{dx^2 + dy^2} = \sqrt{dx^2(1 + (\frac{dy}{dx})^2)} = \sqrt{(1 + (\frac{dy}{dx})^2)}dx$

$\frac{ds}{dt} = v \Rightarrow dt = \frac{ds}{v}$

$J = \int_{t_0}^{t_f} dt = \int \frac{ds}{v} = \int \frac{\sqrt{(1+(\frac{dy}{dx})^2)}}{v} dx$

From Conservation of energy: $mgy = \frac{1}{2}mv^2$

$v = \sqrt{2gy}$

$J = \int_{t_0}^{t_f} dt = \int \frac{ds}{v} = \int \frac{\sqrt{(1+(\frac{dy}{dx})^2)}}{\sqrt{2gy}} dx$

Integrand is a function of its own derivative, so we need to use the Euler-Lagrange Equation to solve. Although, the integrand is not explicitly dependent on x, so we can use the more simple Beltrami Identity:

$f - \frac{dy}{dx}\frac{\partial f}{\partial(\frac{dy}{dx})} = C$

$\frac{\sqrt{(1+(\frac{dy}{dx})^2)}}{\sqrt{2gy}} - \frac{dy}{dx}\frac{\frac{dy}{dx}}{\sqrt{2gy}\sqrt{(1+(\frac{dy}{dx})^2)}} = C$

$\frac{(1+(\frac{dy}{dx})^2)}{\sqrt{2gy}\sqrt{(1+(\frac{dy}{dx})^2)}} - \frac{(\frac{dy}{dx})^2}{\sqrt{2gy}\sqrt{(1+(\frac{dy}{dx})^2)}} = C$

$\frac{1}{\sqrt{2gy}\sqrt{(1+(\frac{dy}{dx})^2)}} = C$

$\frac{1}{2gy(1+(\frac{dy}{dx})^2)} = C^2$

$\frac{1}{2gC^2} = y + y(\frac{dy}{dx})^2$, let $\frac{1}{2gC^2} = k^2$

$k^2 = y + y(\frac{dy}{dx})^2$

$\frac{dy^2}{dx^2} = \frac{k^2-y}{y} \Rightarrow \frac{dy}{dx} = \sqrt{\frac{k^2-y}{y}}$

$\frac{dy^2}{dx^2} = \frac{k^2-y}{y} \Rightarrow \frac{dy}{dx} = \sqrt{\frac{k^2-y}{y}}$

$\frac{1}{\sqrt{\frac{k^2-y}{y}}}dy = dx$, Integrating gives:

$$x = -\sqrt{k^2 - y}\sqrt{y} - k^2 arctan(\frac{\sqrt{k^2 - y}}{\sqrt{y}}) + C$$

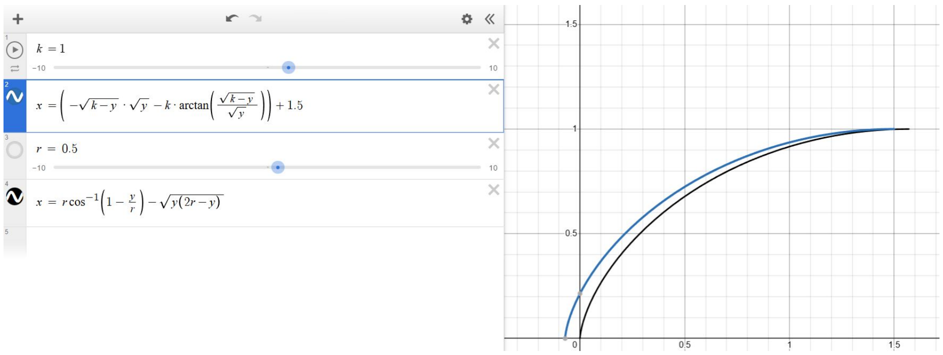


Figure 1: Wire curve to traverse from a to b under gravity in the shortest time from the derived expression (blue) and solution to the Brachistochrone problem given by a cycloid curve (black). Note: C was chosen here as 1.5 to offset the derived expression curve from the true cycloid curve for image comparison. When the constant is $\frac{\pi}{2}$, the curves are coincident for the chosen constants.