Determine stability for linear systems

There are many definitions of stability.

<u>BIBO stable</u>: Bounded-Input bounded output stable

Recall from your Diff. Equations class the notion of the convolution:

Convolution is a fancy way of finding time response for a linear system through the impulse behavior.

$$u \longrightarrow \boxed{G(s)} \longrightarrow y$$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\{G(s)U(s)\}$$

$h(\tau)$

$$\Rightarrow y(t) = \mathcal{L}^{-1}\{G(s)\} \text{ w/ } u(t) = \text{impulse.}$$

$$\boxed{y(t) = \int_{-\infty}^{\infty} h(\tau)\, u(t-\tau)\, d\tau} \quad (\#1)$$

impulse response

convolution theorem.

If $u(t)$ is bounded, then there exists a constant $M$ s.t.

$$\boxed{|u| \le M < \infty \quad \forall t} \quad (\#2)$$

So:

$$|y| = \underbrace{\left|\int h(\tau)\,u(t-\tau)\,d\tau\right|}\leq$$

$$\Rightarrow |y| = \left|\int h(\tau)\,u(t-\tau)\,d\tau\right| \leq \int |h(\tau)u(t-\tau)|\,d\tau$$

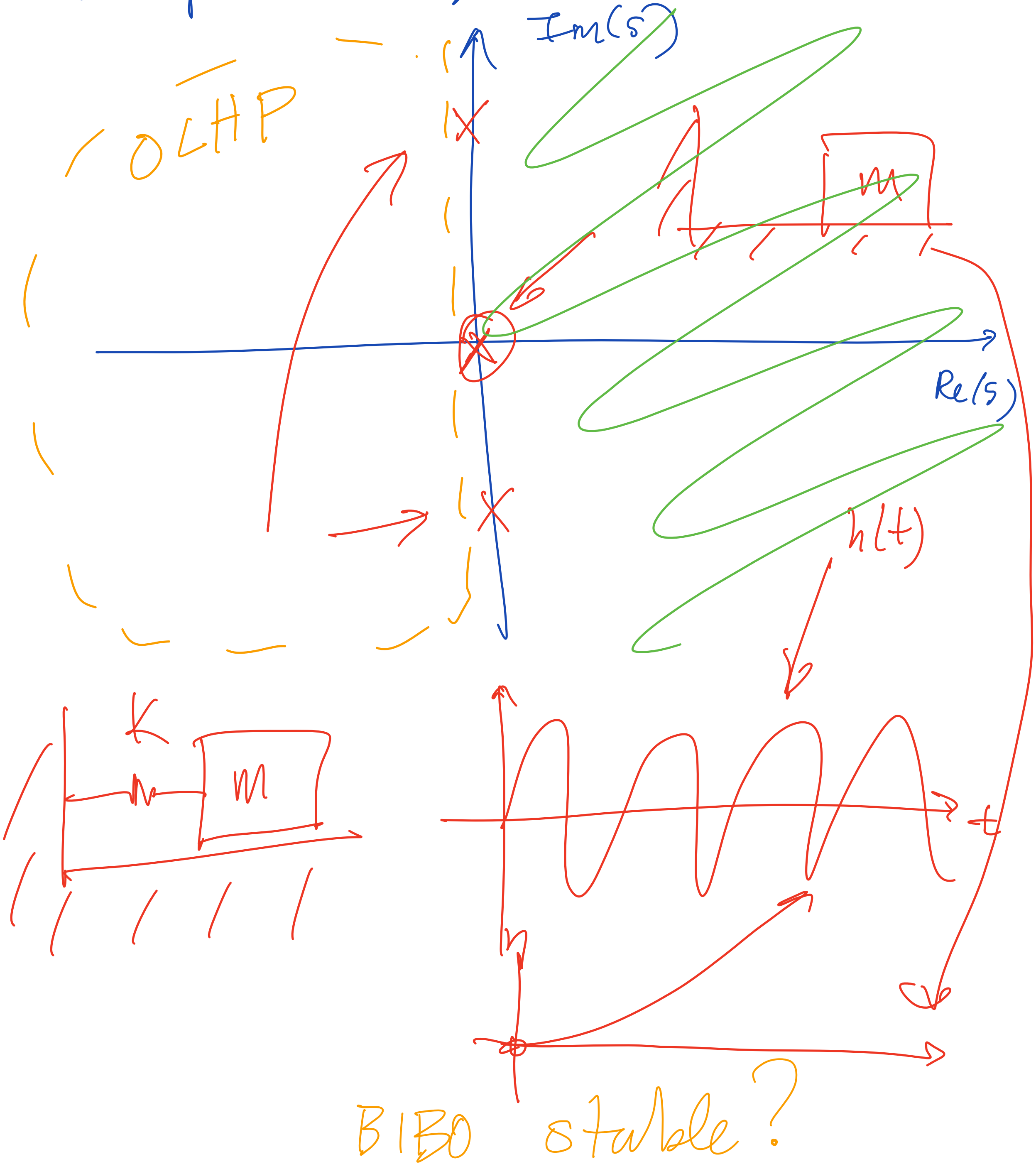
$$\Rightarrow |y| \leq \int |h(\tau)u(t-\tau)\,d\tau| \leq \int |h|\,|u(t-\tau)|\,d\tau$$

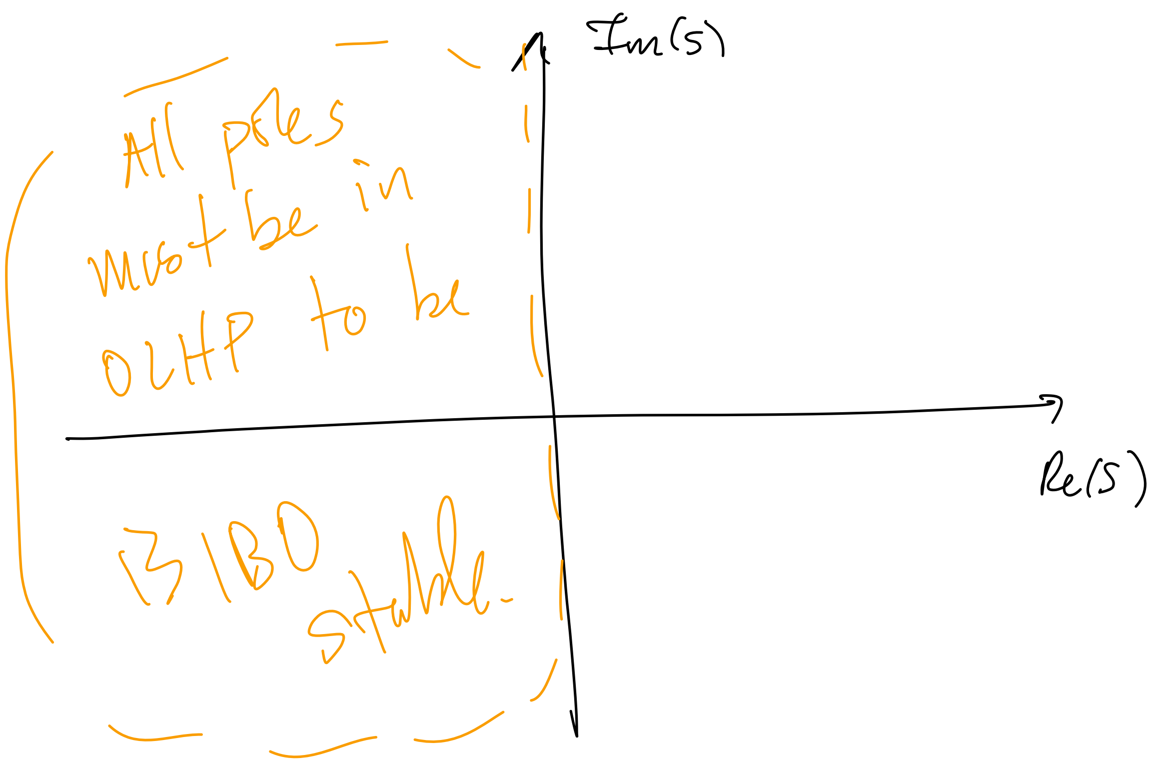
$$\Rightarrow |y| \leq \int |h|\,\underbrace{|u(t-\tau)|}_{M}\,d\tau \leq M\int |h|\,d\tau$$

$$\Rightarrow |y| \leq M \int |h|\,d\tau < \infty$$

$$\Rightarrow \boxed{\int_{-\infty}^{+\infty} |h(\tau)|\,d\tau < \infty}$$

How does this relate back to poles + zeros?




Im(s)
Re(s)
All poles
must be in
OLHP to be
BIBO
stable.

## Determining stability

$$\text{Given} \quad G(s) = \frac{n(s)}{d(s)} \begin{matrix} \leftarrow \text{numerator} \\ \leftarrow \text{denominator} \end{matrix}$$

To determine stability, we find poles of $G(s)$, or equivalent the roots of $d(s)$ $\Rightarrow$ $d(s) = 0$

To find poles of $G(s)$:

Solve $d(s) = 0$

~~*~~ Let $s = \sigma \pm j\omega$ be a root of $d(s)$ or pole of $G(s)$. Then, System is BIBO stable when:

$$Re(s) = \sigma < 0$$

* System is marginally stable when $\sigma = 0$.

Ex Let $G(s) = \dfrac{2}{(s^2+3s+2)}$

is this system stable?

Need to find poles:

$$d(s) = s^2 + 3s + 2 = 0$$

$$(s+1)(s+2) = 0$$

$$\Rightarrow s = -1$$
$$s = -2$$

stable?

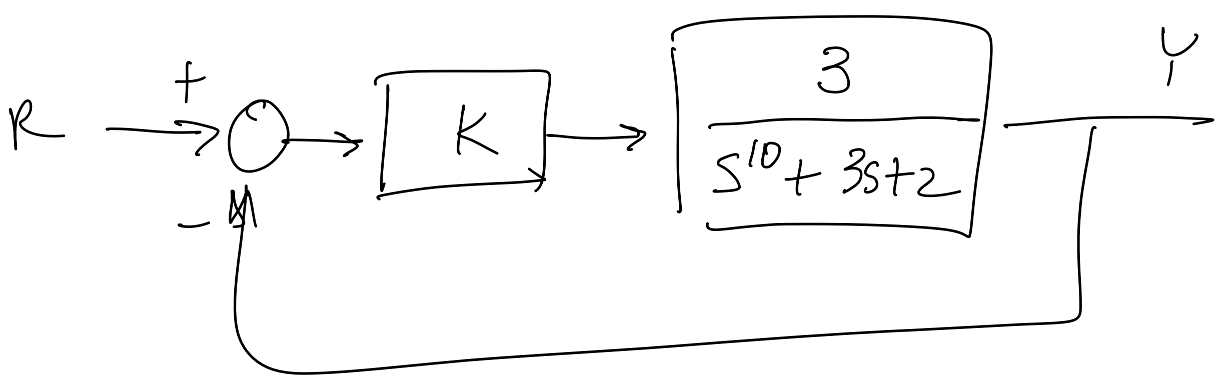
Yes!

Ex $\quad G(s) = \dfrac{s^2+2s+4}{s^{50}+3s+2}$

$$\Rightarrow s^{50}+3s+2=0$$

solve using Matlab.

is it that easy??




$$\frac{Y}{R} = \frac{3/s^{10}+3s+2}{1+K\dfrac{3}{s^{10}+3s+2}} = \frac{3}{s^{10}+3s+2+3K}$$

$$\boxed{s^{10}+3s+2+K=0}$$

## Stability

Given a system (LTI systems):

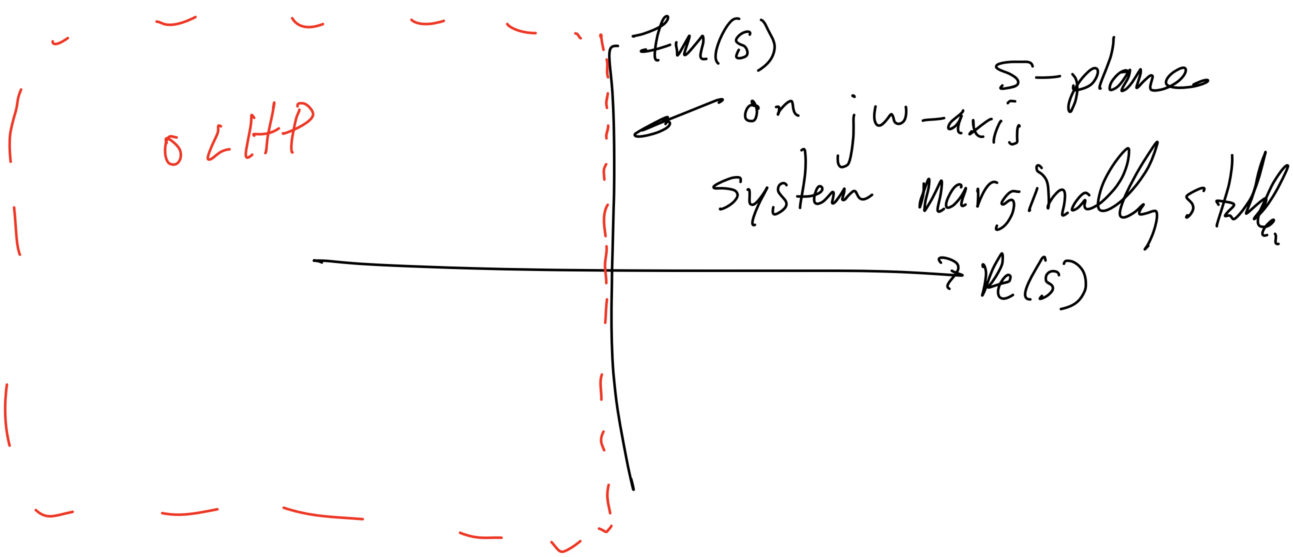
$$U(s) \longrightarrow \boxed{G(s)} \longrightarrow Y(s)$$

$$G(s) = \frac{n(s)}{d(s)}$$

Determine stability: $d(s) = 0$ challenging to solve

solve for $s$ (roots of $d(s)$)

stable requires all roots $s$ to be in the OLHP:



# Routh-Hurwitz Stability Criterion

Consider the system:

$$R(s) \longrightarrow \boxed{\frac{n(s)}{s^n + a_1 s^{n-1} + a_2 s^{n-2} + \dots + a_0}} \longrightarrow Y(s)$$

where $d(s) = s^n + a_1 s^{n-1} + a_2 s^{n-2} + \dots + a_0$

is a monic polynomial in $s$.

* For stability (necessary & sufficient cond.) we want the real parts of roots of $d(s)$ to be negative (i.e., roots in OLHP).

<u>Note</u>: on $j\omega$-axis is <u>not</u> in OLHP.

<u>Note</u>: If any coefficients of $d(s)$ are missing, then the system will have

poles located outside of the OLHP.

By examining the monic polynomial:

$$d(s) = s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_0$$

we can make two statements:

1) If there is at least one sign change, i.e., "+" to "−", then there will be at least one pole not in OLHP.

ex: $s^2 - 2s + 1 = d(s)$
$\Rightarrow$ at least one pole outside OLHP.

2) If all the coefficients of $d(s)$ are positive, then we cannot say anything about location of the poles. $s^2 + 2s + 1$

# Routh - Hurwitz Method (1905)

1. Generate the Routh - table
2. Interpret the Routh - table

**Step 1** Generate the R - table

start with $d(s) = \underline{1} s^n + a_1 s^{n-1} + \underline{a_2} s^{n-2} + \dots + a_0$

Coefficients

$$
\begin{array}{c|cccc}
s^n & 1 & a_2 & a_4 & \cdots \\
s^{n-1} & a_1 & a_3 & a_5 & \cdots \\
\hline
s^{n-2} & b_1 & b_2 & b_3 & \cdots \\
\vdots & c_1 & c_2 & c_3 & \cdots \\
 & \vdots & \vdots & \vdots & \ddots \\
s^0 & \vdots & \vdots & \vdots & 
\end{array}
$$

Important !!

where :

$$b_1 = \frac{-\begin{vmatrix} 1 & a_2 \\ a_1 & a_3 \end{vmatrix}}{a_1} \qquad b_2 = \frac{-\begin{vmatrix} 1 & a_4 \\ a_1 & a_5 \end{vmatrix}}{a_1} \qquad b_3 = \frac{-\begin{vmatrix} 1 & a_6 \\ a_1 & a_7 \end{vmatrix}}{a_1}$$

$$c_1 = \frac{-\begin{vmatrix} a_1 & a_3 \\ b_1 & b_2 \end{vmatrix}}{b_1} \qquad c_2 = \frac{-\begin{vmatrix} a_1 & a_5 \\ b_1 & b_3 \end{vmatrix}}{b_1} \quad \cdots$$

<u>Step 2</u> Interpreting R-Table

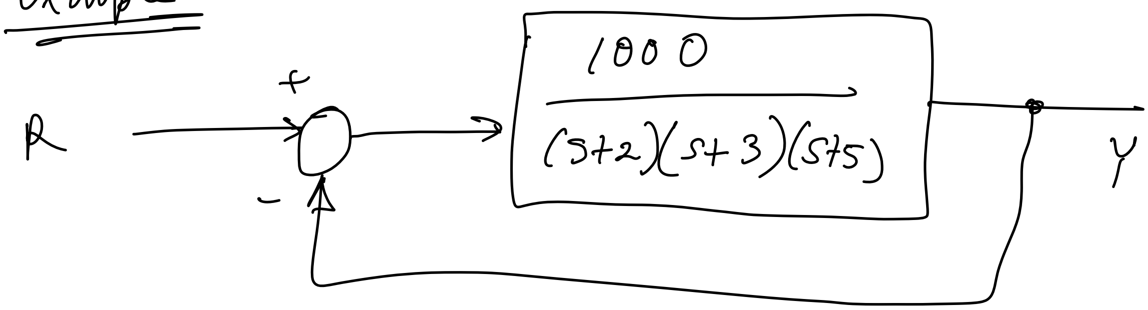
The base table will only tell us the number of poles in OLHP and ORHP. Poles on the $j\omega$-axis will be covered in special cases.

* Routh-Hurwitz Criterion declares that the number of roots (poles) of $d(s)$ that are in the <u>ORHP</u> is equal to the number of sign changes in the first column of R-H Table

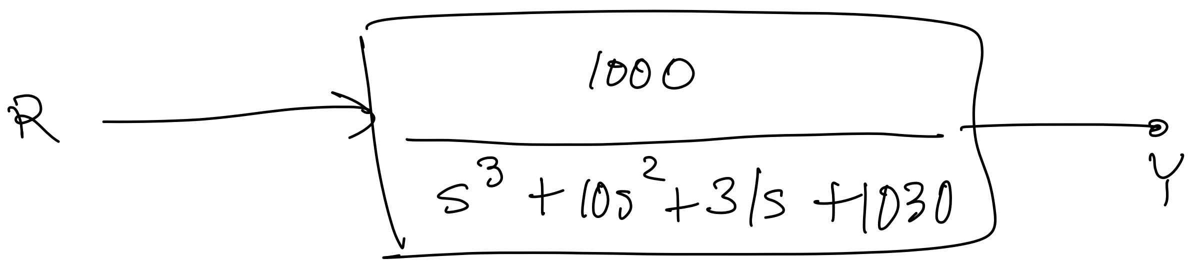
Hence, if the system is stable (can be marginally stable), all poles are in LHP, thus there should be no sign changes in first column!!

$$G(s) = \frac{1000}{\underbrace{(s+2)(s+3)(s+5)}}$$

Example




Is this system stable?




① R-Table $\quad d(s) = s^3 + 10s^2 + 31s + 1030$

| | | | |
|---|---|---|---|
| $s^3$ | 1 | 31 | 0 |
| $s^2$ | 10 | 1030 | 0 |
| $s^1$ | -72 | 0 | |
| $s^0$ | 1030 | | |

$$b_1 = \frac{-\begin{vmatrix} 1 & 31 \\ 10 & 1030 \end{vmatrix}}{10} = -72 \qquad b_2 = \frac{-\begin{vmatrix} 1 & 0 \\ 10 & 0 \end{vmatrix}}{10} = 0$$

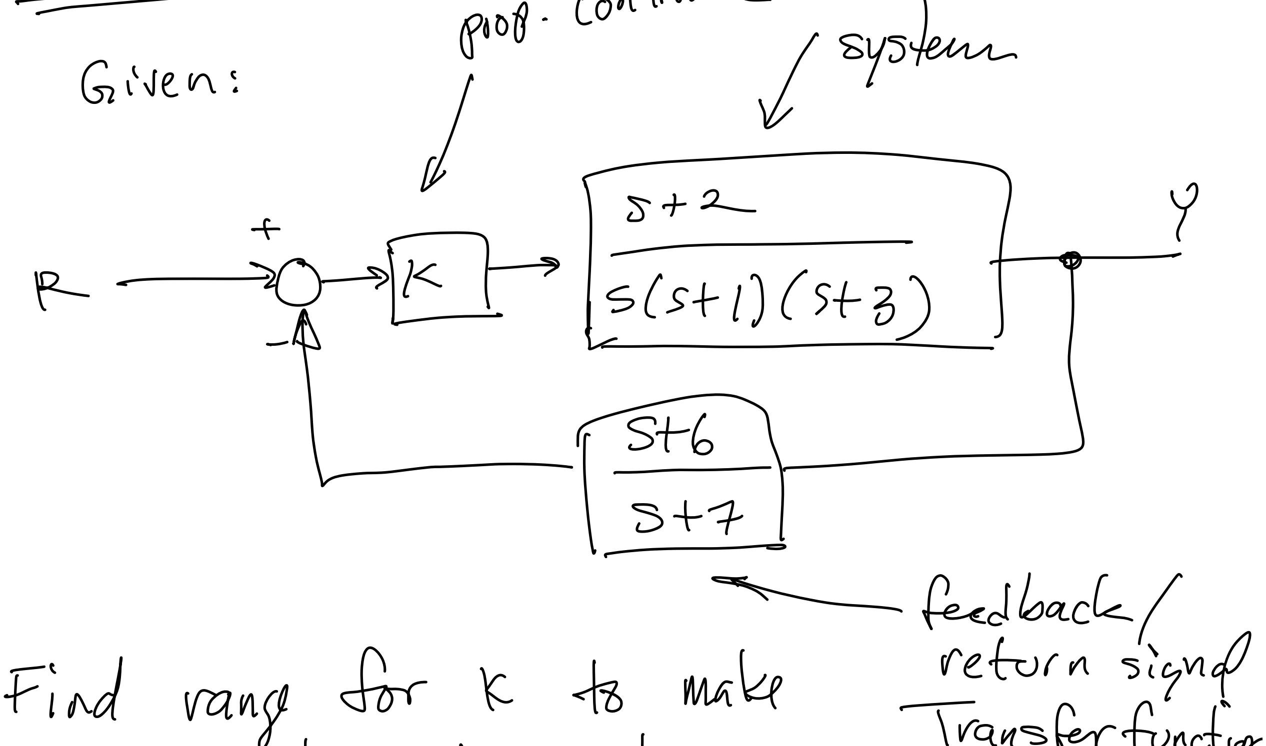
$$c_1 = \frac{-\begin{vmatrix} 10 & 1030 \\ -72 & 0 \end{vmatrix}}{-72} = 1030$$

(2) Count sign changes in $1^{\underline{st}}$ column: 2 $\Rightarrow$ 2 pole in ORHP

$\therefore$ system is unstable (!)

Ex 2

Given:




Find range for K to make this system stable!

Step 1 Need C Loop d(s):

$$T(s) = \frac{Y(s)}{R(s)} = \frac{(s+2)(s+7)}{s^4 + 11s^3 + (K+3)s^2 + (8K+21)s + 12K}$$

Next, Create R-Table

$$d(s) = s^4 + 11s^3 + (K+3)s^2 + (8K+21)s + 12K$$

$$
\begin{array}{c|ccc}
s^4 & 1 & K+31 & 12K \\
s^3 & 11 & 8K+21 & 0 \\
s^2 & b_1 & b_2 & b_3 \\
s^1 & c_1 & c_2 & c_3 \\
s^0 & d_1 & & \\
\end{array}
$$

$$
b_1 = \frac{-\begin{vmatrix} 1 & K+31 \\ 11 & 8K+21 \end{vmatrix}}{11} = \frac{3K+320}{11} > 0
$$

$$
b_2 = 12K
$$

$$
c_1 = \frac{24K^2 + 1171K + 6720}{3K+320} \qquad c_2 = 0
$$

$$
d_1 = 12K
$$

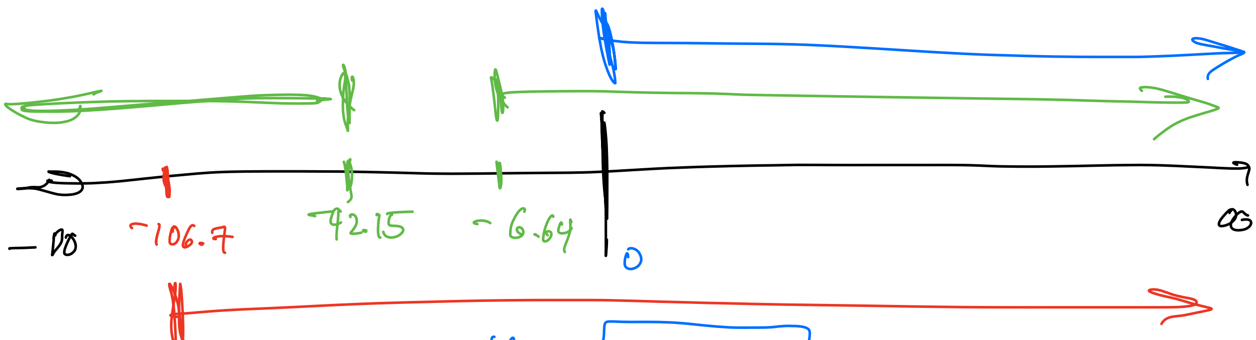
we want $b_1 > 0$ ; $c_1 > 0$ ; $d_1 > 0$

$$b_1 = \frac{3K+320}{11} > 0 \Rightarrow K > -106.7$$

$$c_1 = \frac{24K^2 + 1171K + 6720}{3K+320} > 0$$

$$K < -42.15 \quad \text{or} \quad K > -6.64$$

$$d_1 = 12K > 0 \Rightarrow K > 0$$




Blue pill : $\boxed{K > 0}$