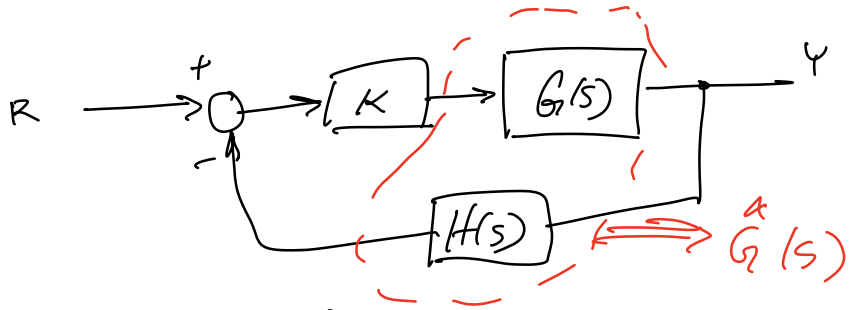
Root locus (Part II)

Closed-loop system:



$$\frac{Y(s)}{R(s)} = \frac{K\,G(s)}{\underbrace{1 + K\,G(s)H(s) = 0}_{\text{poles come from solving this.}}}$$

$$\Rightarrow \quad 1 + K\,\overset{\hat{G}(s)}{\boxed{G(s)\,H(s)}} = 0$$

<u>convert to this expression:</u>

$$\Rightarrow \quad 1 + K\,\hat{G}(s) = 0$$

if $K = 0 \Rightarrow$ open-loop case.

then we call $\hat{G}(s)$ open-loop system.

if $K \neq 0 \Rightarrow$ closed-loop

we get angle + magnitude ~~condition~~ from $1 + K\,\hat{G}(s) = 0$

(1) Angle condition

$$1 + K\hat{G}(s) = 0$$

$$\Rightarrow \hat{G}(s) = -\frac{1}{K}$$

$$\angle \hat{G}(s) = \angle -\frac{1}{K}$$

$$\angle \hat{G}(s) = 180^\circ + 360^\circ \ell \quad \ell = 0, \pm 1, \ldots$$

Look closer at $\angle \hat{G}(s)$ and what it means.

Reminder: we are working w/ complex numbers.

Consider

$$\hat{G}(s) = \frac{n(s)}{d(s)}$$

zeros of 0-loop T.F.

poles of 0-loop T.F.

$$\Rightarrow \hat{G}(s) = \frac{(s-z_1)(s-z_2)\cdots(s-z_m)}{(s-p_1)(s-p_2)\cdots(s-p_n)}$$

where $m = \#$ zeros of $\hat{G}(s)$

$n = \#$ poles of $\hat{G}(s)$

$$\angle \hat{G}(s) = \angle \frac{n(s)}{d(s)} = \angle \frac{(s-z_1)(s-z_2)\cdots(s-z_m)}{(s-p_1)(s-p_2)\cdots(s-p_n)}$$

so, $\angle \underbrace{\dfrac{(s-z_1)(s-z_2)\cdots(s-z_m)}{(s-p_1)(s-p_2)\cdots(s-p_n)}}$

$\Rightarrow$

$$\boxed{\angle \hat{G}(s) = \sum \psi_i - \sum \phi_j}$$

test point goes right here! s_0

Angle of a test point s_0 in the s-plane from a zero of $\hat{G}(s)$

Angle of a test point s_0 in the s-plane from a pole of $\hat{G}(s)$

<u>Ex</u>

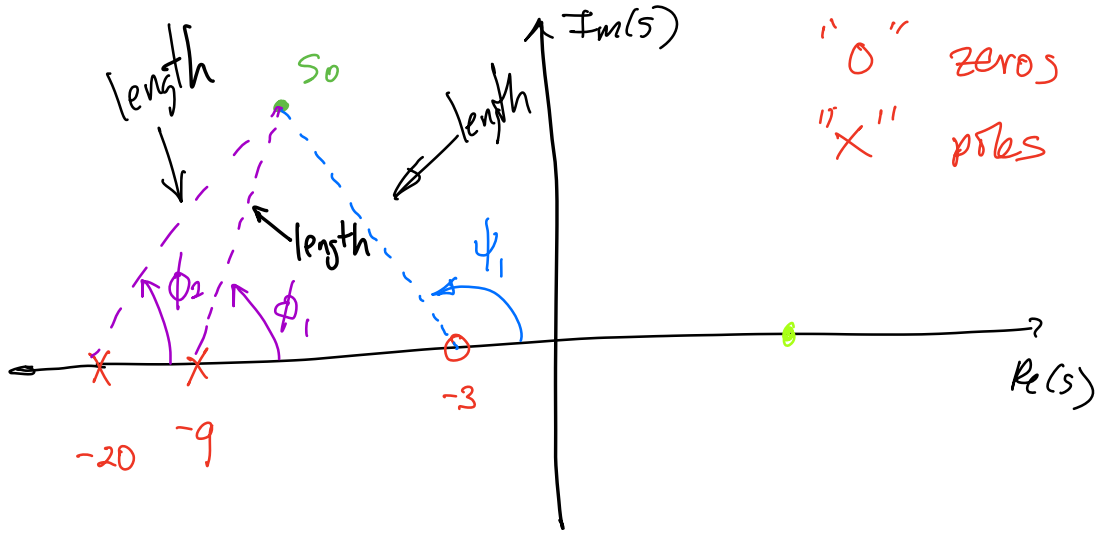
$$\hat{G}(s) = \frac{s+3}{(s+9)(s+20)}$$

Find $\angle \hat{G}(s)$ when $s = s_0$

① Identify o-loop zeros + poles:

zeros: -3

poles: $-9, -20$



$$\angle \hat{G}(s) = \sum \psi_i - \sum \phi_i$$

$$\angle \hat{G}(s) = \psi_1 - (\phi_1 + \phi_2)$$

(2) Magnitude

$$|\hat{G}(s)| = \left| \frac{-1}{K} \right|$$

we use magnitude of distance from zeros/poles to s_0 to find $|\hat{G}(s_0)|$.

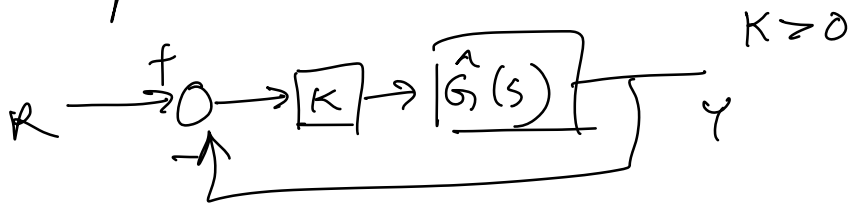
$$\Rightarrow |\hat{G}(s_0)| = \frac{|s_0 - z_1||s_0 - z_2| \cdots}{\underbrace{|s_0 - p_1|}_{\text{length}}|s_0 - p_2| \cdots}$$

Steps to draw root locus

Based on the following equation:

$$1 + K\hat{G}(s) = 0$$

Above expression is consistent with:



Step 1 Mark all poles of $\hat{G}(s)$ with "x" and zeros with an "o". These are the open-loop zeros + poles, i.e., when $K = 0$.

Step 2 Draw the root locus on the real axis to the left of an odd number of poles + zeros

Step 3 When $K \to \infty$, what happens?

$$1 + K\hat{G}(s) = 0$$

$$\hat{G}(s) = -\frac{1}{K} \to 0 \text{ when } K \to \infty$$

$$\Rightarrow \hat{G}(s) = 0 \quad \text{when } K = \infty$$

$$\hat{G}(s) = \frac{(s-z_1)(s-z_2)\cdots}{(s-p_1)(s-p_2)\cdots} = 0$$

z_1 z_2

(a) the closed-loop poles ended up at the zeros.

#m zeros and #n poles

#m poles goes to #m zeros

(b) Remaining n-m # poles go off to Infinity and they do so centered about α.

Asymptote: angle

$$\phi_h = \frac{180^\circ + 360^\circ (h-1)}{n-m} \quad \text{for } h = 1, 2 \ldots n-m$$

Center at α (on the real axis)

$$\alpha = \frac{\sum_i^n Re(P_i) - \sum_j^m Re(z_j)}{n-m}$$

Example of assymptote:

Let $1 + K\hat{G}(s) = 1 + \underbrace{\frac{K(s+1)}{s(s+2)(s+4)^2}}_{\hat{G}(s)} = 0$

$\rightarrow s = -1$

$\rightarrow s = 0,\ s = -2,\ s = -4,\ s = -4$

$\hat{G}(s) \leftarrow$ open-loop system

The open-loop system has

$m = 1$
$n = 4$ $\Big\}$ $n - m = 4 - 1 = 3$

tells us we have 3 asymptotes.

From our equations, $h = 1, 2, 3$

look for angle for each asymptote:

$$\phi_1 = \frac{180^\circ + 360^\circ(h-1)}{n-m} \qquad h = 1$$

$$= \frac{180^\circ + 360^\circ(1-1)}{3} = 60^\circ$$

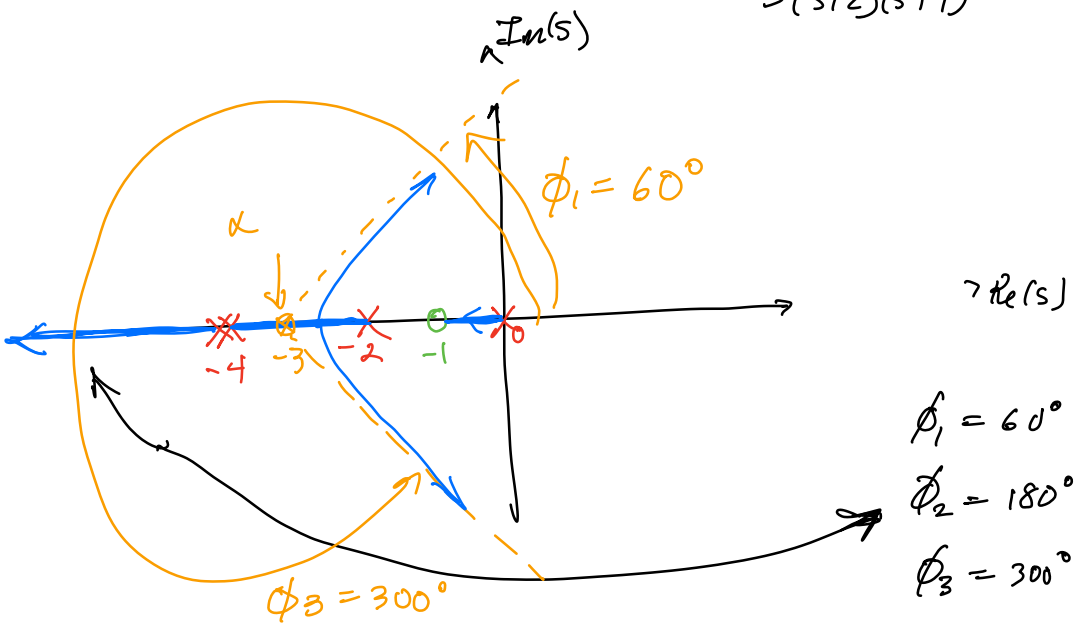
$$\phi_2 = \frac{180^\circ + 360^\circ(2-1)}{3} = 180^\circ$$

$$\phi_3 = \frac{180^\circ + 360^\circ(3-1)}{3} = 300^\circ$$

Asymptote will be centered about α:

$$\alpha = \frac{\overset{\text{all poles}}{\sum} Re(P_i) - \sum Re(z_i)}{n-m} = \frac{[0 + (-2) + 2(-4)] - [-1]}{3} = -3$$

Let's Draw root locus: $\hat{G} = \dfrac{s+1}{s(s+2)(s+4)^2}$



$\phi_1 = 60°$

$\phi_2 = 180°$

$\phi_3 = 300°$

<u>Step 4</u> Departure angles from poles / Arrive angle to zeros

Let q = # of order of pole or zero

($q > 1$, repeated poles case)

<u>Departure angles from poles</u>

$$\Rightarrow \quad q\,\phi_{dep} = \sum \psi_i - \sum \phi_i - 180° - 360°(h-1)$$

where $h = 1, 2 \ldots q$

Angle of departure for i^{th} pole

angle of all zeros to pole of interest

Angle of all poles to pole of interest.

Arrival angles for zeros:

q = order of zeros ($q > 1$, repeats of zeros)

$$\Rightarrow \quad q\,\psi_{arr} = \underbrace{\sum \phi_i}_{} - \underbrace{\sum \psi_i}_{} + 180^\circ + 360^\circ(h-1)$$

$h = 1, 2 \ldots q$

arrival angle for ith zero

Angle from all poles to zero of interest

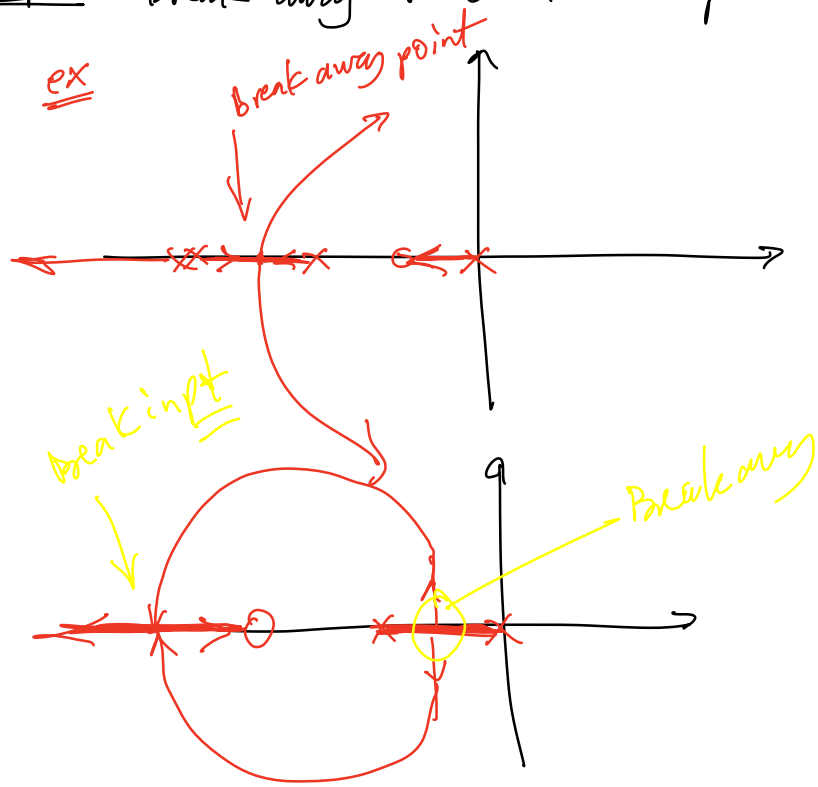
Angle from all other zeros to zero of interest.

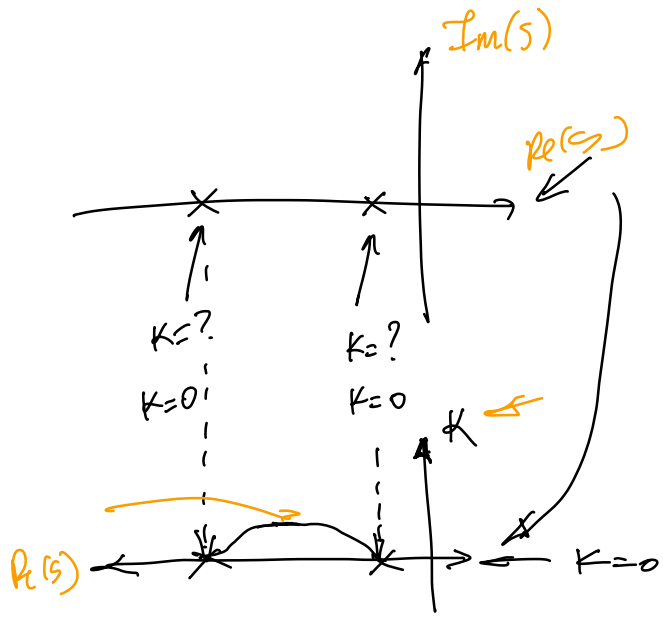
Step 5 (optional step) Computing point where the locus crossed the $j\omega$-axis for $K > 0$:

Use Routh-Hurwitz stability criterion to find K, then compute ω_d by substituting $s = j\omega_d$ into the closed-loop characteristic equation, then solve for $\underline{\omega_d}$.

Step 6 Break away + break in points.

ex





Find max/min for K:

$$\Rightarrow \quad \frac{d}{ds}(K) = \frac{d}{ds}\left(-\frac{1}{\hat{G}(s)}\right)\Bigg|_{s=s_0} = 0$$

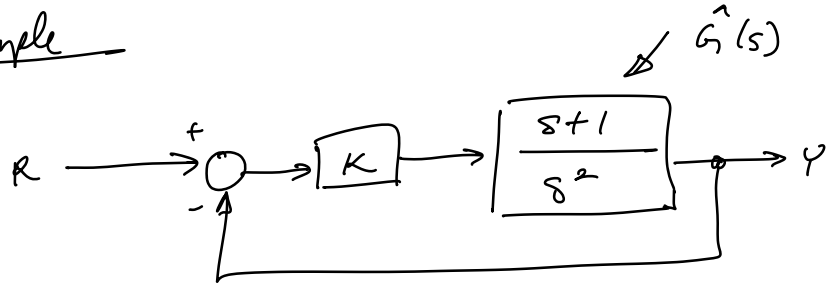
if $\hat{G}(s) = \dfrac{b(s)}{a(s)}$ then

$$\frac{d}{ds}\left(-\frac{1}{\hat{G}(s)}\right) = -\frac{1}{b^2}\left[b\frac{da}{ds} - a\frac{db}{ds}\right]\Bigg|_{s=s_0} = 0$$

* Basic rule is 2 segments come together at 180° separation and break away at ±90°.

3 locus segment approach each other at relative angle 120° and depart at angles rotated by 60°.

Example



Cloop T.F.

$$T(s) = \frac{K(s+1)}{s^2 + Ks + K}$$

← zeros of cloop

← poles of cloop

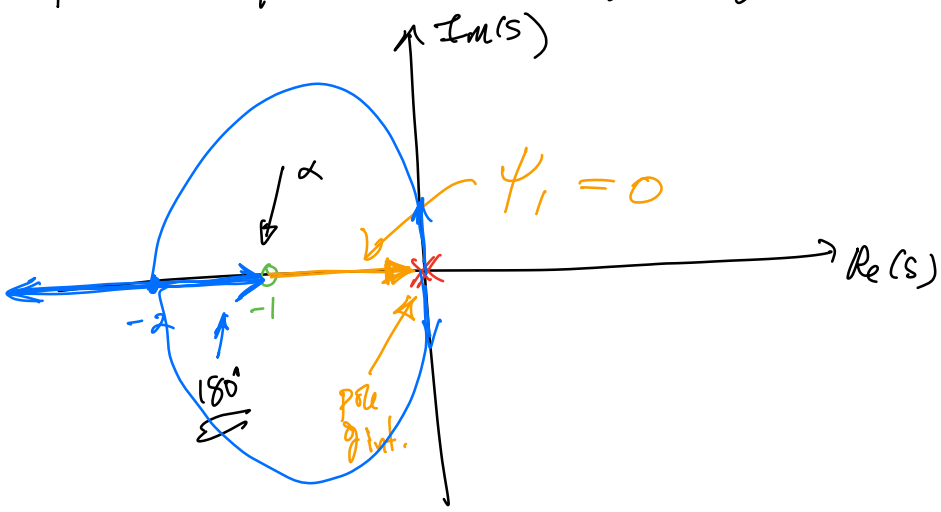
OLOP T.F.

$\frac{s+1}{s^2}$ ← zeros OLOOP $s = -1$

← poles OLOOP $s=0,\ s=0$

Assume $K > 0$

Step 1 put "x" poles and "o" zeros of the OLOOP.



Step 2 Draw root locus on real

Step 3 Find asymptotes for large values of s + K:

$$T(s) = \frac{K(s+1)}{s^2 + Ks + K}$$

$m = 1$ (1 zero)

$n = 2$ (2 poles

# of asymptotes $= h = n - m = 2 - 1 = 1$ asymptote

Angle of the asymptotes:

$$\phi_h = \phi_1 = \frac{180^\circ + 360^\circ(h-1)}{n-m} = \frac{180^\circ + 0}{1} = \underline{\underline{180^\circ}}$$

$$\alpha = \frac{\sum Re(p_i) - \sum Re(z_i)}{n-m} = \frac{+1}{1} = \underline{\underline{+1}}$$

<u>Step 4</u>: Departure angle from pole

$q = 2$ (b/c 2 poles at 0)

angle from zeros to pole of int.

angle from remaining pole to pole of int.

$$\Rightarrow q\phi_{dep} = \sum \psi_i - \sum \phi_i - 180^\circ - 360^\circ(l-1)$$

$l = 1 \ldots n-m$

$$\Rightarrow 2\phi_{dep} = 0 - 0 - 180^\circ - 360^\circ(l-1) \qquad l = 1$$

$$\boxed{2\phi_{dep} = -180^\circ} \Rightarrow \underline{\underline{\phi_{dep} = -90^\circ}}$$

By symmetry for 2nd pole: $\phi_{dep} = 90^\circ$

$$q\phi_{dep} = 0 - 0 - 180^\circ - 360(1)$$
$$= -540$$
$$= -\frac{540}{2} =$$

Step 5 Crossing the $j\omega$-axis

Apply R-H approach:

$$s^2 + Ks + K = 0$$

$$\begin{array}{c|cc} s^2 & 1 & K \\ s & K & 0 \\ s^0 & K & \end{array}$$

NO $j\omega$-axis crossing !!

Step 6 Break away + Break in point

$$\hat{G}(s) = \frac{s+1}{s^2} = \frac{b(s)}{a(s)} \qquad b(s) = s+1 \qquad a(s) = s^2$$

$$\frac{d}{ds}\left(-\frac{1}{G(s)}\right) = 0 \Rightarrow \frac{d}{ds}\left(-\frac{a(s)}{b(s)}\right) = -\frac{1}{b^2}\left(b\frac{da}{ds} - a\frac{db}{ds}\right) = 0$$

$$\frac{db}{ds} = 1 \qquad \frac{da}{ds} = 2s$$

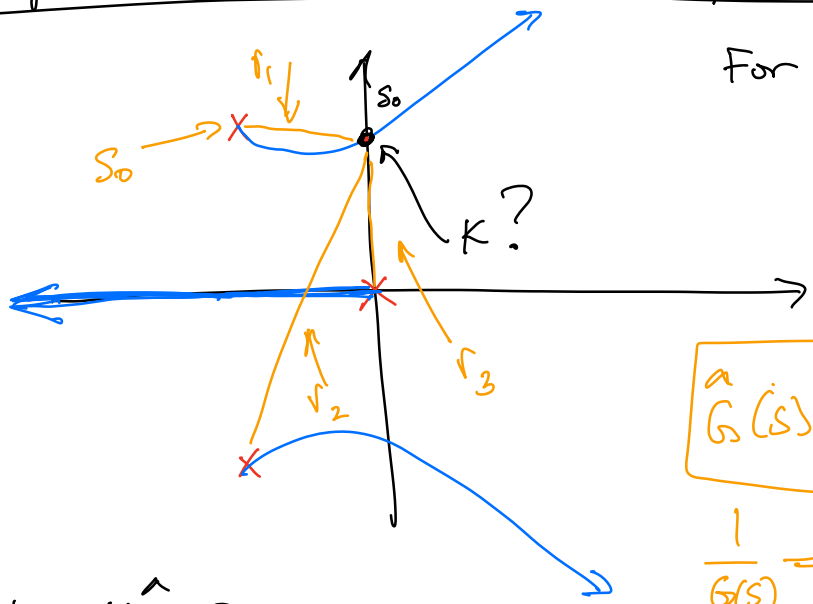
$$\Rightarrow -\frac{1}{(s+1)^2}\left((s+1)(2s) - s^2(1)\right) = 0$$

$$\Rightarrow -s^2 + (s+1)(2s) = 0$$

$$\Rightarrow s^2 + 2s = 0 \Rightarrow \left.\begin{array}{l} s = 0 \\ s = -2 \end{array}\right\} \begin{array}{l} \leftarrow \text{Break away} \\ \leftarrow \text{Break in} \end{array}$$

Selecting Gain from a root locus plot

For $K >$



$$\boxed{\hat{G}(s) = \frac{3}{s+2}}$$

$$\frac{1}{G(s)}\Big|_{s_0} = \frac{s_0+2}{3}$$

$$1 + K\hat{G}(s) = 0$$

$$\boxed{K = -\frac{1}{\hat{G}(s)} \Rightarrow K = \left|\frac{-1}{\hat{G}(s_0)}\right|}$$

$$K\Big|_{s_0} = \frac{1}{|\hat{G}(s_0)|} = \frac{1}{\left|\frac{z_i}{p_j}\right|} = \frac{|p_j|}{|z_i|} = \boxed{\frac{r_1 \cdot r_2 \cdot r_3}{c\, r_{z_1} r_{z_2} \cdots}}$$

for s_0

3 →