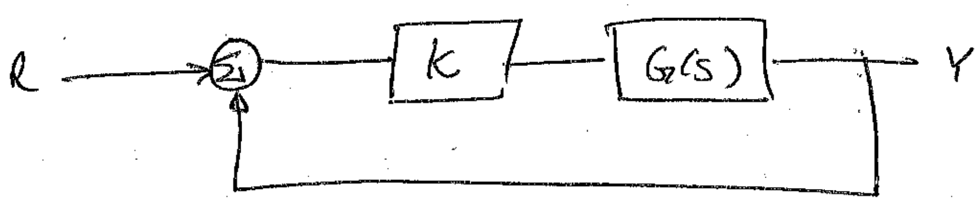
Frequency - Response Compensation

We will analyze various compensators using frequency response techniques such as:

1- PD, P
2- Lead
3- PI / Lag
4- PID compensators

Consider the system w/o compensation



the closed loop characteristic equation is:

$$1 + KG(s) = 0$$

When a compensator $D(s)$ is added, the closed-loop characteristic equation changes to:

$$1 + \underbrace{K D(s) G(s)}_{\text{new open-loop transfer function}} = 0$$

Recall that for a general system:

1. The open-loop stable system is stable in closed-loop if the open-loop magnitude frequency response has

$$|KG(j\omega)| < 1 \quad \text{at} \quad \angle G(j\omega) = -180^\circ$$

"Holds for systems when increasing the gain leads to instability and $|KG(j\omega)|$ crosses the mag = 1 once."

2. Percent overshoot for 2^{nd} order system can be reduced by increasing the phase margin

$$PM \cong 100\zeta$$

"Increasing PM, Increases ζ"

3. Increase the speed of response by increasing the bandwidth:

Recall: $$T_s = \frac{4}{\omega_{BW}\zeta}\sqrt{(1-2\zeta^2)+\sqrt{4\zeta^4-4\zeta^2+2}}$$

4. Steady-State error is improved by increasing the low-frequency magnitude response

Example: Recall for type 0 system

$$G(j\omega) = K_0 \frac{\prod_{i=1}^{n} (j\omega \tau_i + 1)}{\prod_{i=1}^{m} (j\omega \tau_i + 1)}$$

$$e_{ss} = \frac{1}{1+K_p} \quad \text{where} \quad K_p = K_0$$

The 4 basic realizations will serve as a design guideline to designing compensators in the frequency-domain (response).

These guidelines in conjunction with Bode plots and Nyquist diagrams only serve as basic guidelines and the designer can develop his/her techniques based on experience and concepts of frequency-response methods.

I. Proportional Control (Gain adjustment)

Compensator:

$$D(s) = K$$

We can increase/decrease K to obtain desired transient response.

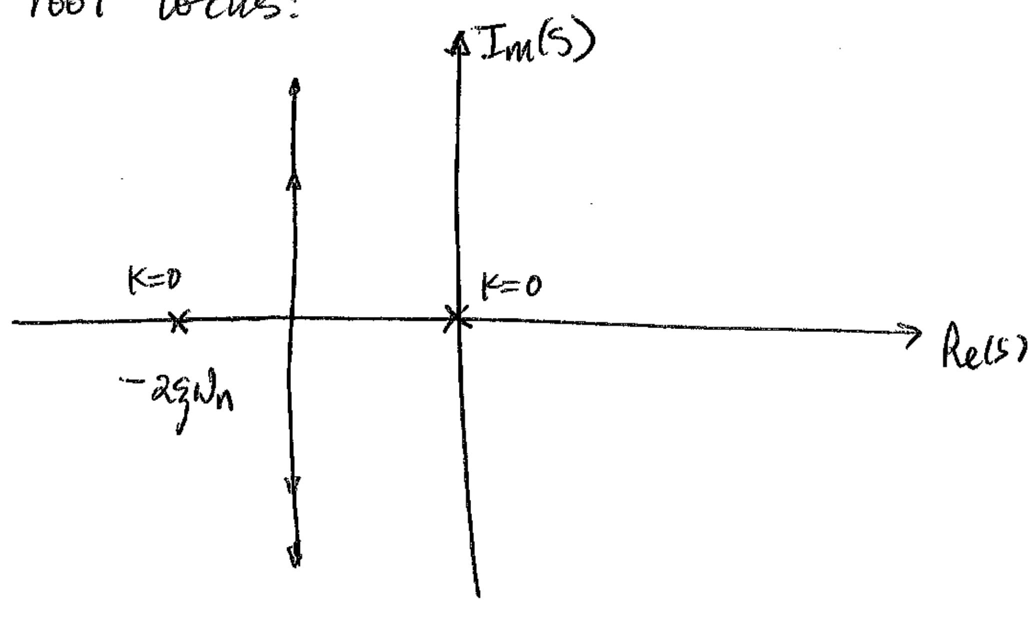
Consider a standard open-loop second order system

$$G(s) = \frac{\omega_n^2}{s(s+2\zeta\omega_n)}$$

then the closed loop t.f. is:

$$T(s) = \frac{\omega_n^2}{s(s+2\zeta\omega_n)+K\omega_n^2}$$

The root locus:



As K increases, it effects the dampening ratio ξ. In frequency-response approach we know that phase Margin is related to ξ by:

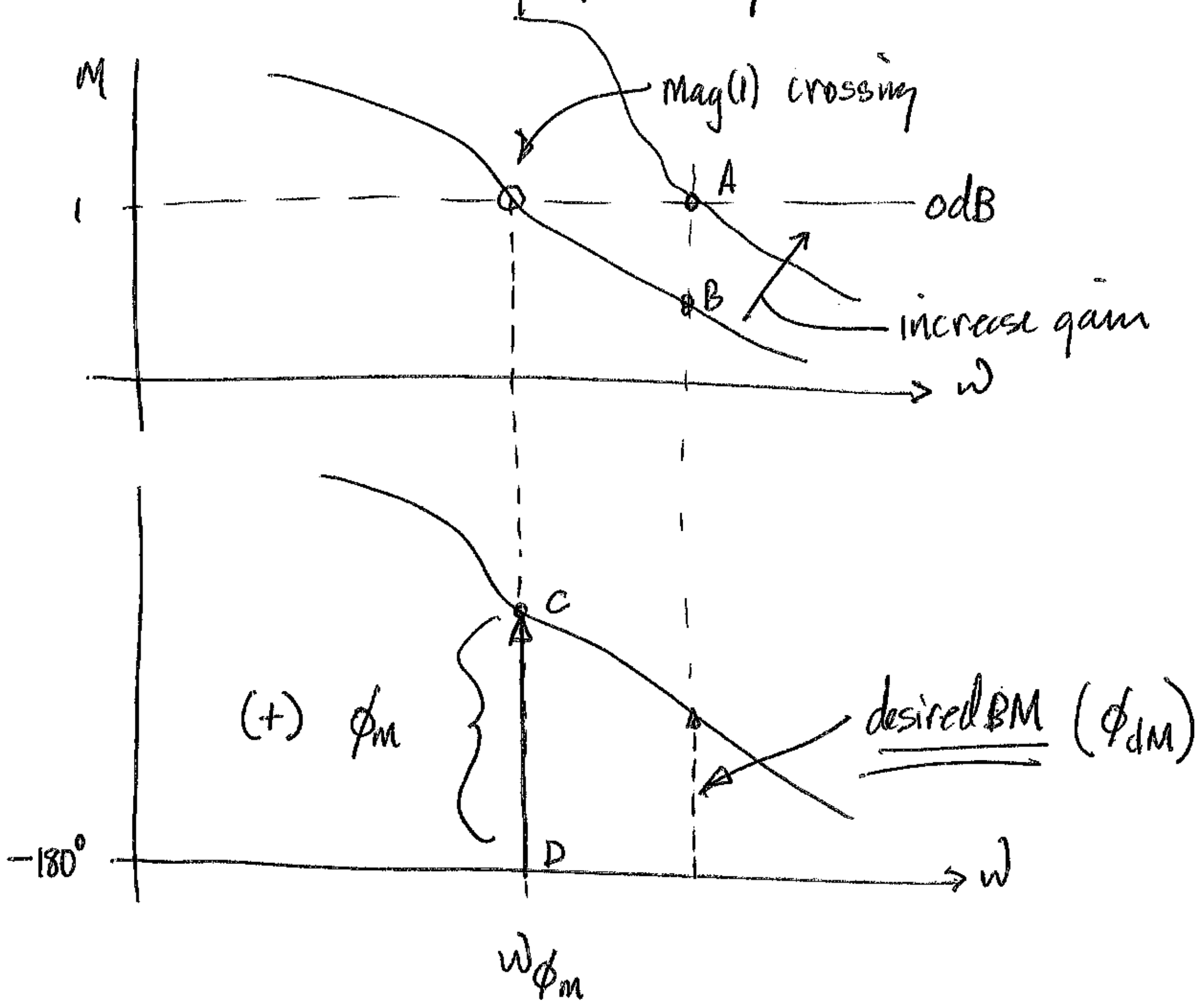
$$PM = \tan^{-1} \frac{2\xi}{\sqrt{\sqrt{1+4\xi^4} - 2\xi^2}}$$

for 2nd order system.

OR

$$PM \cong 100\xi \qquad 0 \le PM \le 60^\circ$$

But what is the PM graphically?



But what happens if the PM margin of the uncompensated system is not desirable based on ξ? Then we can increase/decrease K to change PM.

From plot, we need to increase by amount AB to shift Magnitude plot up by AB.

Note that shifting the magnitude plot up/down by varying the gain <u>does not</u> affect the phase!

<u>Procedures</u>

1. Draw bode Magnitude/phase plots for some convient gain, K.
2. Compute the required phase margin from desired ξ or %OS

$$\xi = \frac{-\ln(\%OS/100)}{\sqrt{\pi^2 + \ln^2(\%OS/100)}}$$

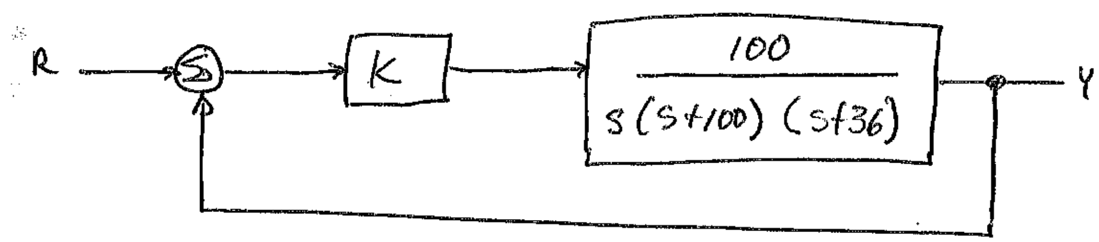
$$PM = \tan^{-1} \frac{2\xi}{\sqrt{-2\xi^2 + \sqrt{1+4\xi^4}}}$$

3. Find the frequency $\omega_{\phi m}$ on Bode phase diagram that yields the desired PM.

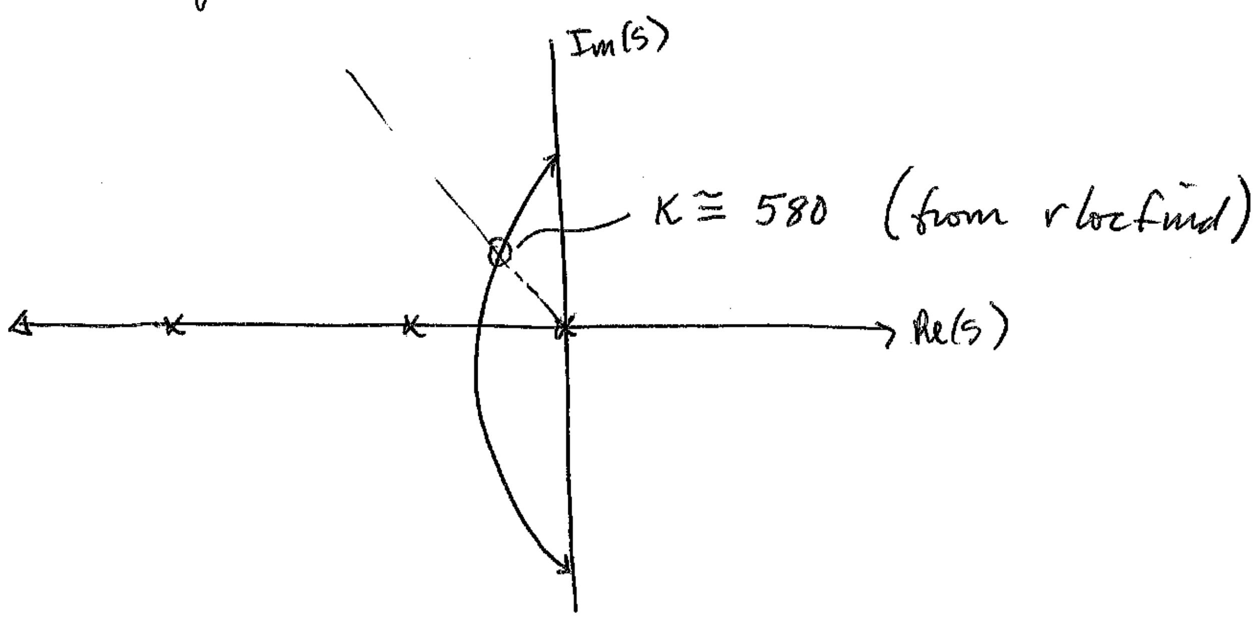
4. Change the gain by amount that corresponds to the appropriate PM so that new magnitude plot goes through 0 dB at $\omega_{\phi m}$.

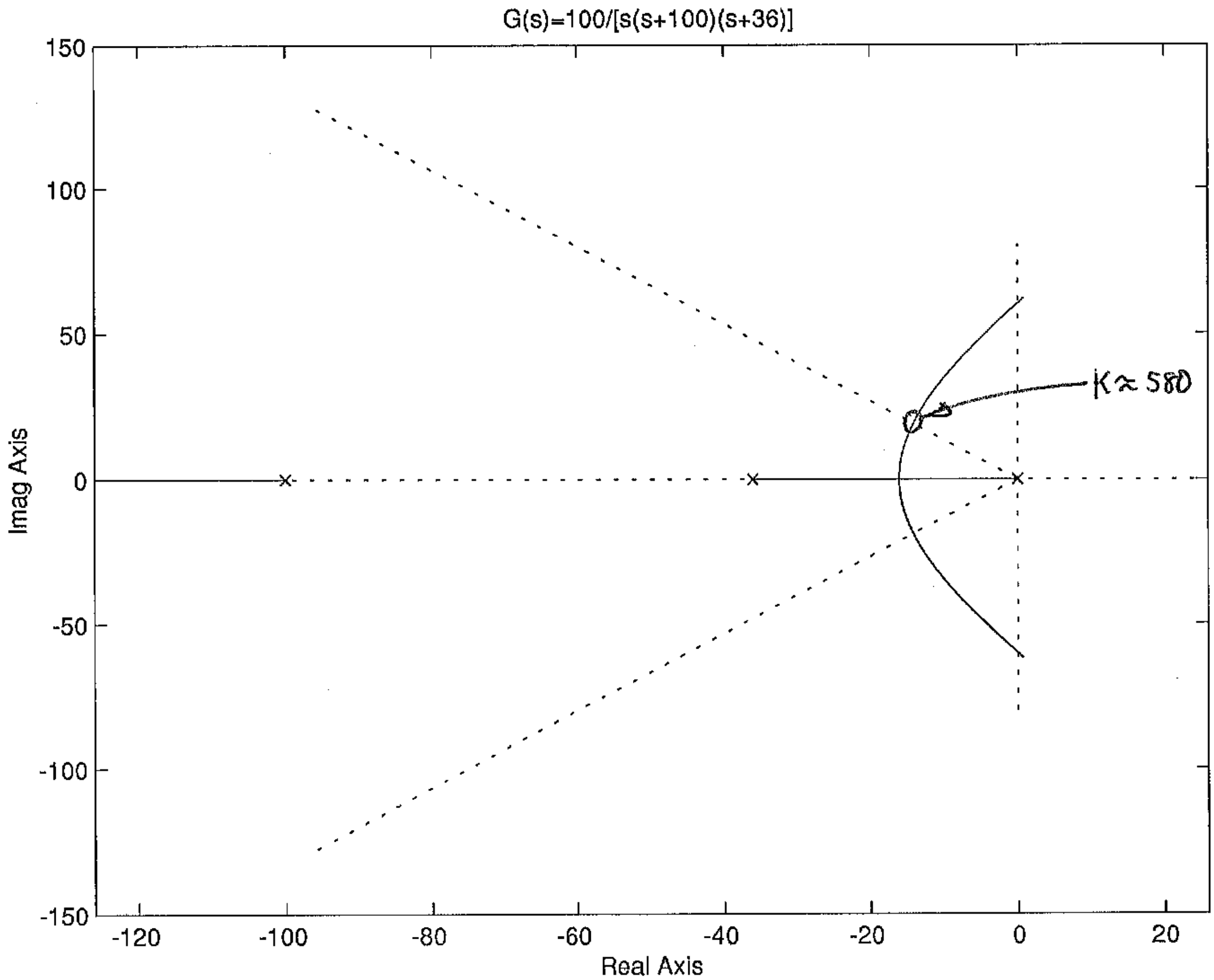
<u>Example</u>

Find K to yield 9.48% overshoot.



Recall from the Root locus method:





G(s)=100/[s(s+100)(s+36)]
150
100
50
0
-50
-100
-150
Imag Axis
-120
-100
-80
-60
-40
-20
0
20
Real Axis
K ≈ 580

Step1 Obtain ζ and PM

$$\zeta = \frac{-\ln(\%OS/100)}{\sqrt{\pi^2 + \ln^2(\%OS/100)}} = 0.6$$

$$\Rightarrow \quad PM = \tan^{-1}\frac{2\zeta}{\sqrt{\sqrt{1+4\zeta^4} - 2\zeta^2}} = \underline{\underline{59.19^\circ}}$$

Step2 Draw bode plot for arbitrary K

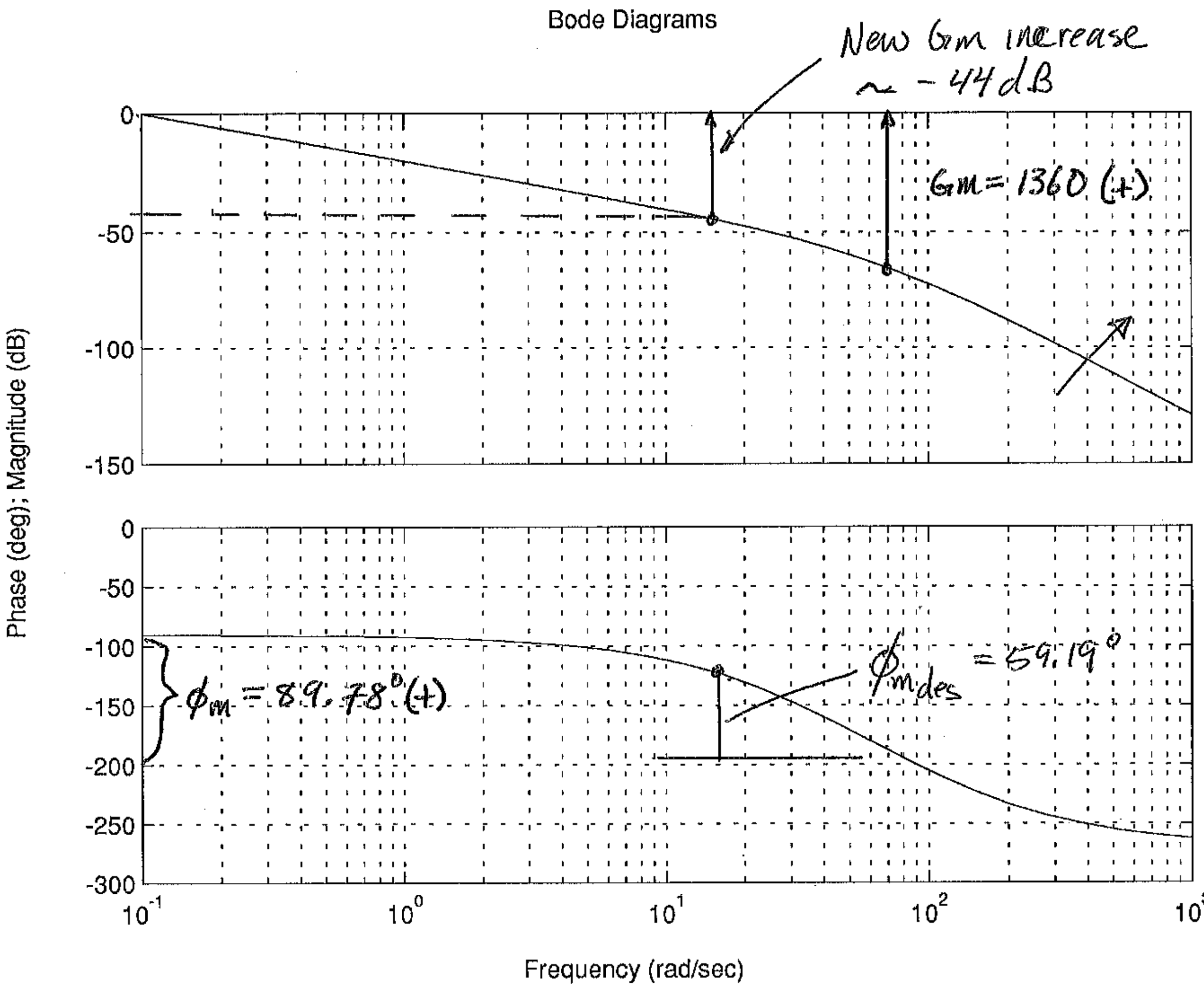
choose K= 3.6

Matlab commands:

```
>> K = 3.6
>> n= 100*K
>> d1 = [1 100 0]
>> d2 = [1 36 ]
>> d = conv (d1,d2)
>> sys = tf (n,d)
>> bode (sys)
>> [Gm, Pm, Wcg, Wcp] = margin (sys)

    Gm= 1360

    Pm= 89.78
```


Bode Diagrams
New Gm increase
~ -44 dB
Gm = 1360 (+)
$\phi_m = 89.78^\circ (+)$
$\phi_{m\,des} = 59.19^\circ$
Phase (deg); Magnitude (dB)
Frequency (rad/sec)
0
-50
-100
-150
0
-50
-100
-150
-200
-250
-300
10^{-1}
10^{0}
10^{1}
10^{2}
10^{3}

Step 3

Find $\omega_{\phi m_{des}}$ to get $PM = 59.19°$

$$\phi = -180° + 59.19° = -120.81°$$

$$\Rightarrow \quad \omega_{\phi m_{des}} \approx 14 \text{ rad/s}$$

Step 4

Compute the new K

Bode plot was for

$$K_1 = 3.6$$

Need an increase in 44 dB gain

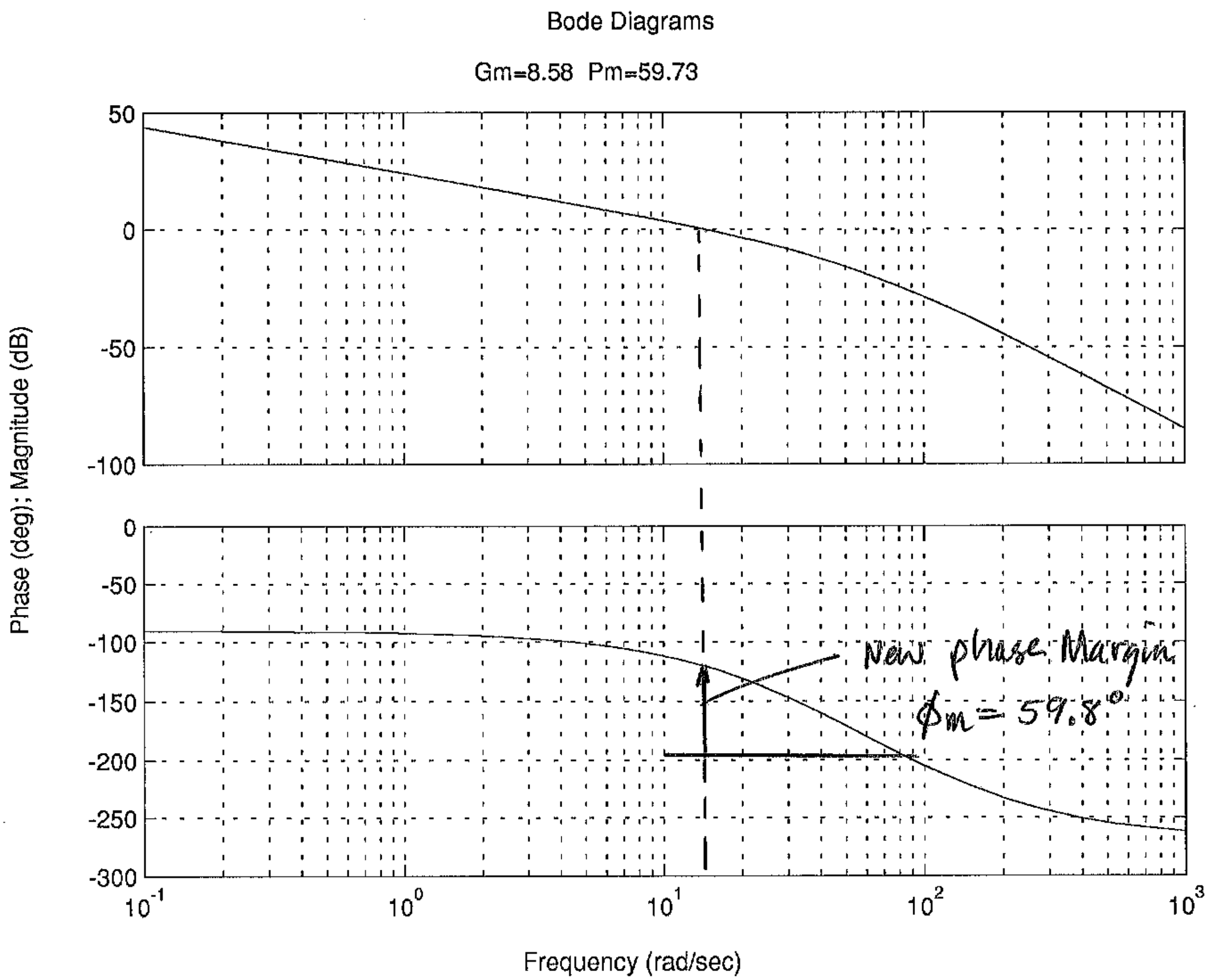
$$K_2 = 10^{(44/20)} = 158.5$$

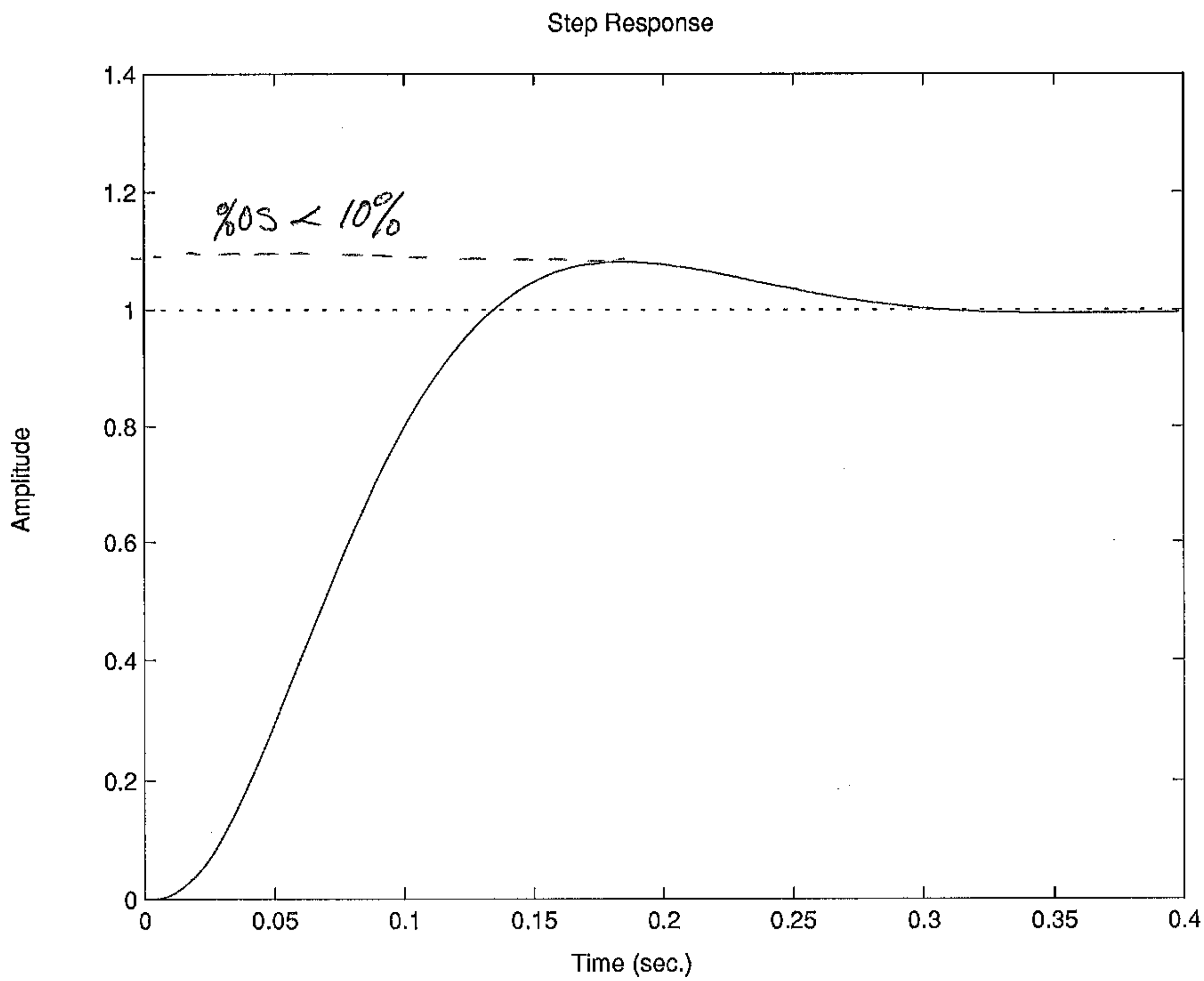
then new K

$$K = K_1 \times K_2$$

$$K = (3.6)(158.5)$$

$$\boxed{K \approx 570.6} \quad \Rightarrow \text{ verify w/ } \underline{\text{Matlab}}!$$


Bode Diagrams
Gm=8.58 Pm=59.73
Phase (deg); Magnitude (dB)
Frequency (rad/sec)
New phase Margin
$\phi_m = 59.8°$



Step Response
1.4
1.2
1
0.8
0.6
0.4
0.2
0
Amplitude
%OS < 10%
0
0.05
0.1
0.15
0.2
0.25
0.3
0.35
0.4
Time (sec.)

II PD Compensation

Compensator:

$$D(s) = K + K_d s$$

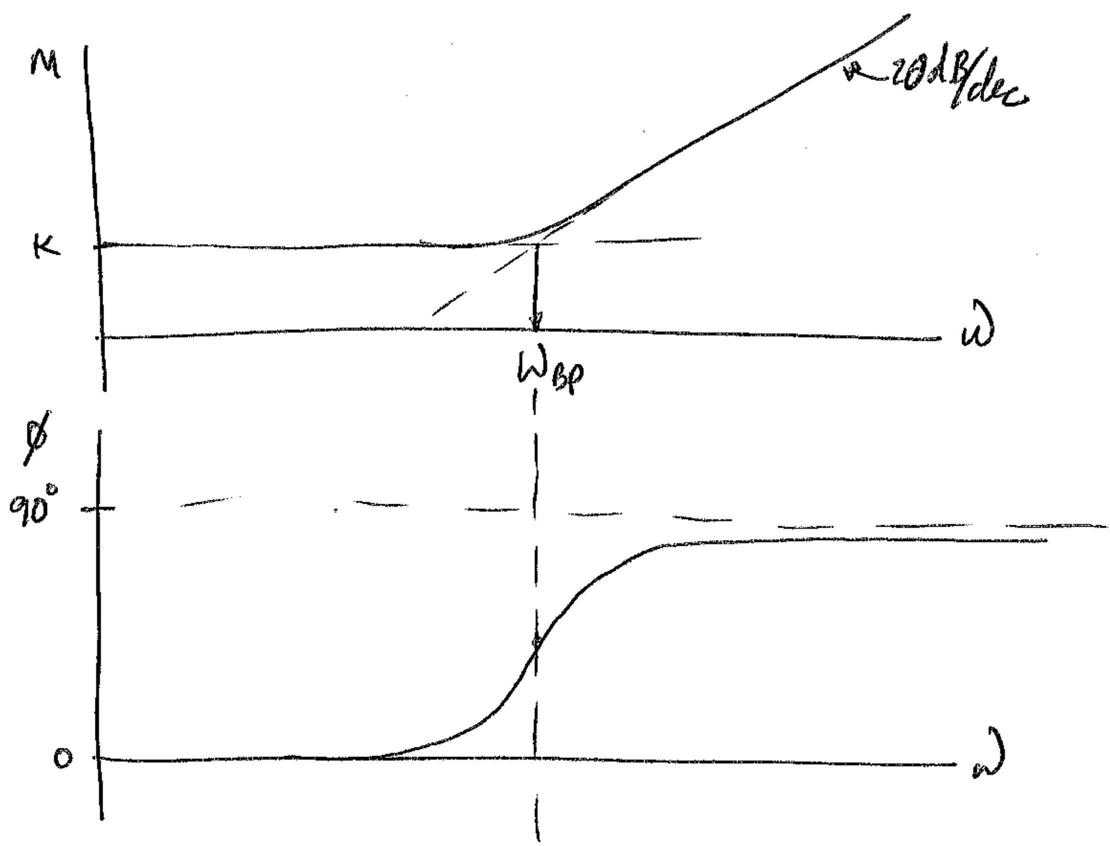
Was shown to have a stabilizing effect on the root locus of a 2nd order system.

Lets look at Bode Plot for D(s) only!

$$D(s) = K_d s + K = \left[\frac{K_d}{K}(j\omega) + 1\right] K$$

Break Point:

$$\omega_{BP} = \frac{K}{K_d}$$



So, what effects does adding a PD controller have on the closed-loop system?

From the Bode Plots, a PD controller will increase the phase Margin of the system by adding more gain beyond the break point frequency, hence add more dampening.

However, since the gain continues to grow as ω increases, this is undesirable when systems have noise.

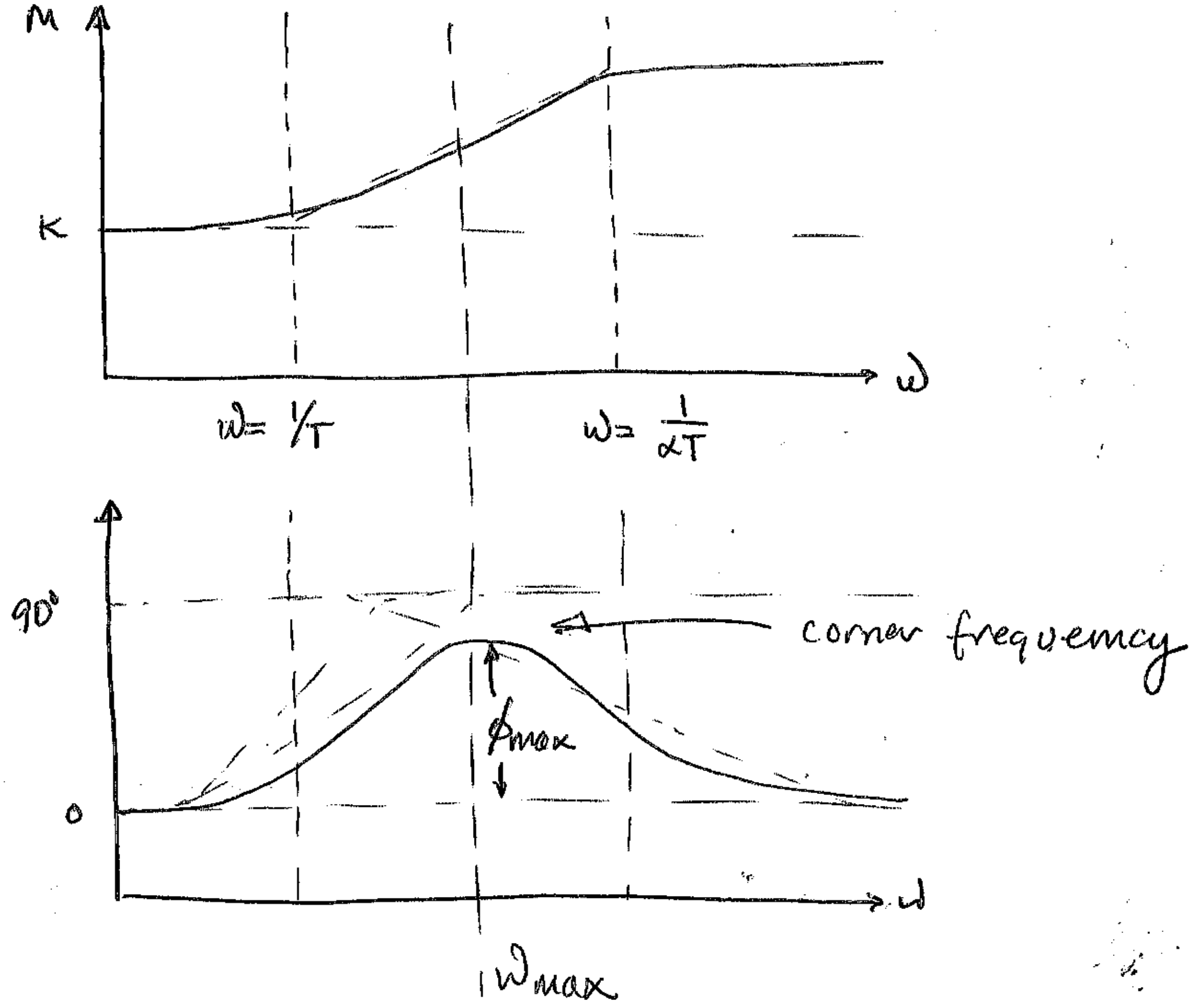
Lead Compensation

Recall the problem of increase amplification in PD compensator. To alleviate this problem, a pole is added at frequencies higher than the break point of a PD which results in a lead compensator:

$$D(s) = K \frac{Ts + 1}{\alpha Ts + 1} \quad \text{where } \alpha < 1$$

Note that since $\alpha < 1$, then the pole is further in the LHP than the zero.

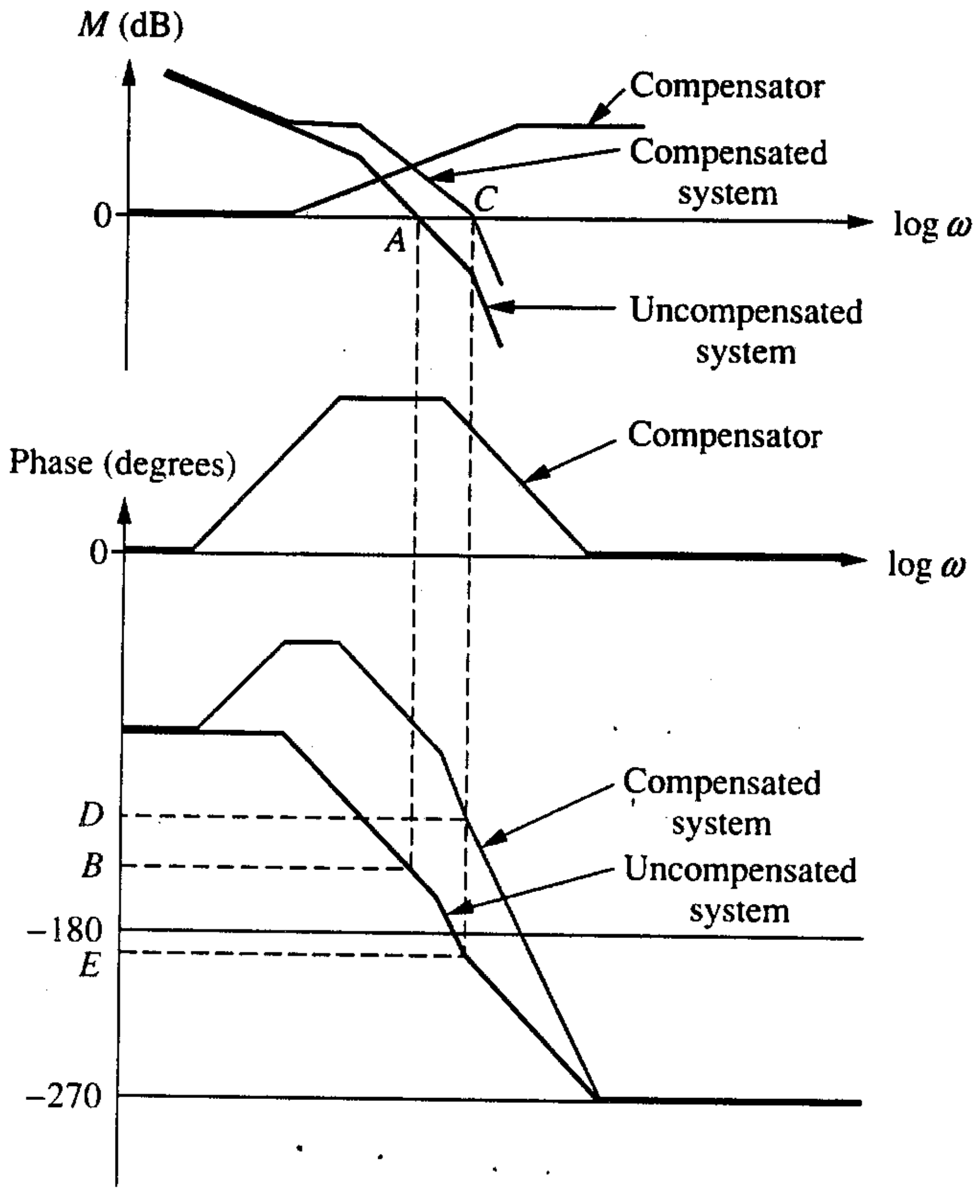
let's look at Bode Plot:



What effects Does lead have on a system?

1- Phase increase (or lead)

2- Improves dampening of the system



M (dB)
Compensator
Compensated system
0
C
A
log ω
Uncompensated system
Compensator
Phase (degrees)
0
log ω
Compensated system
D
B
Uncompensated system
−180
E
−270

From the equation:

$$D(s) = K \frac{Ts+1}{\alpha Ts+1} = K\left(\frac{1}{\alpha}\right)\left(\frac{s+1/T}{s+1/\alpha T}\right)$$

the phase contribution is:

$$\phi = \tan^{-1}(T\omega) - \tan^{-1}(\alpha T\omega)$$

the Maximum phase occurs at:

$$\omega_{max} = \frac{1}{T\sqrt{\alpha}}$$

where

$$\phi_{max} = \sin^{-1}\left(\frac{1-\alpha}{1+\alpha}\right)$$

OR

Maximum phase occurs between the 2 break point frequencies or also known as the corner frequency.

$$\alpha = \frac{1-\sin\phi_{max}}{1+\sin\phi_{max}}$$

Related to standard transfer function form:

$$D(s) = K \frac{(s+z)}{(s+p)} \qquad \text{where} \quad z = 1/T, \quad p = \frac{1}{\alpha T}$$

then we have

$$\omega_{max} = \sqrt{|z||p|}$$

and

$$\log \omega_{max} = \frac{1}{2}\left(\log|z| + \log|p|\right)$$

<u>Basic Design Guidelines:</u>

1. The amount of phase lead at ω_{max} only depend on α.
2. The maximum phase lead increase is limited to 90°.
3. Rule of thumb is to use <u>one</u> lead compensator to provide up to 60° of phase lead. More would result in too much amplification at higher frequencies.

Design Procedure for Lead Compensator:

Case 1: Design for low-frequency gain (e_{ss} error specs)

1. Determine K to satisfy e_{ss}

Case 2: Design for closed-loop bandwidth

1. Determine open-loop cross-over frequency to be a factor of 2 below bandwidth.

Continue ...

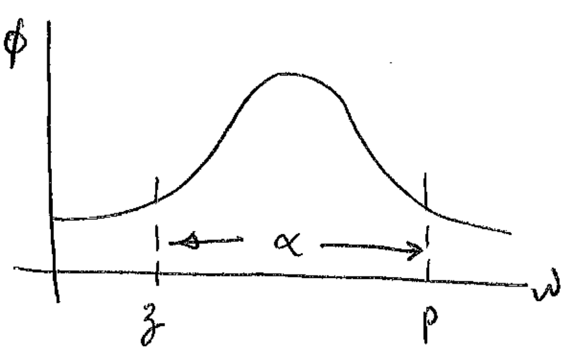
2. Find PM of uncompensated system with the gain K.

3. Find ϕ_{max} plus small amount (5° to 12°)

$$\phi_{max} = PM_{des} + \text{Extra} - PM_{\substack{\text{uncompensated}\\ \text{system}}}$$

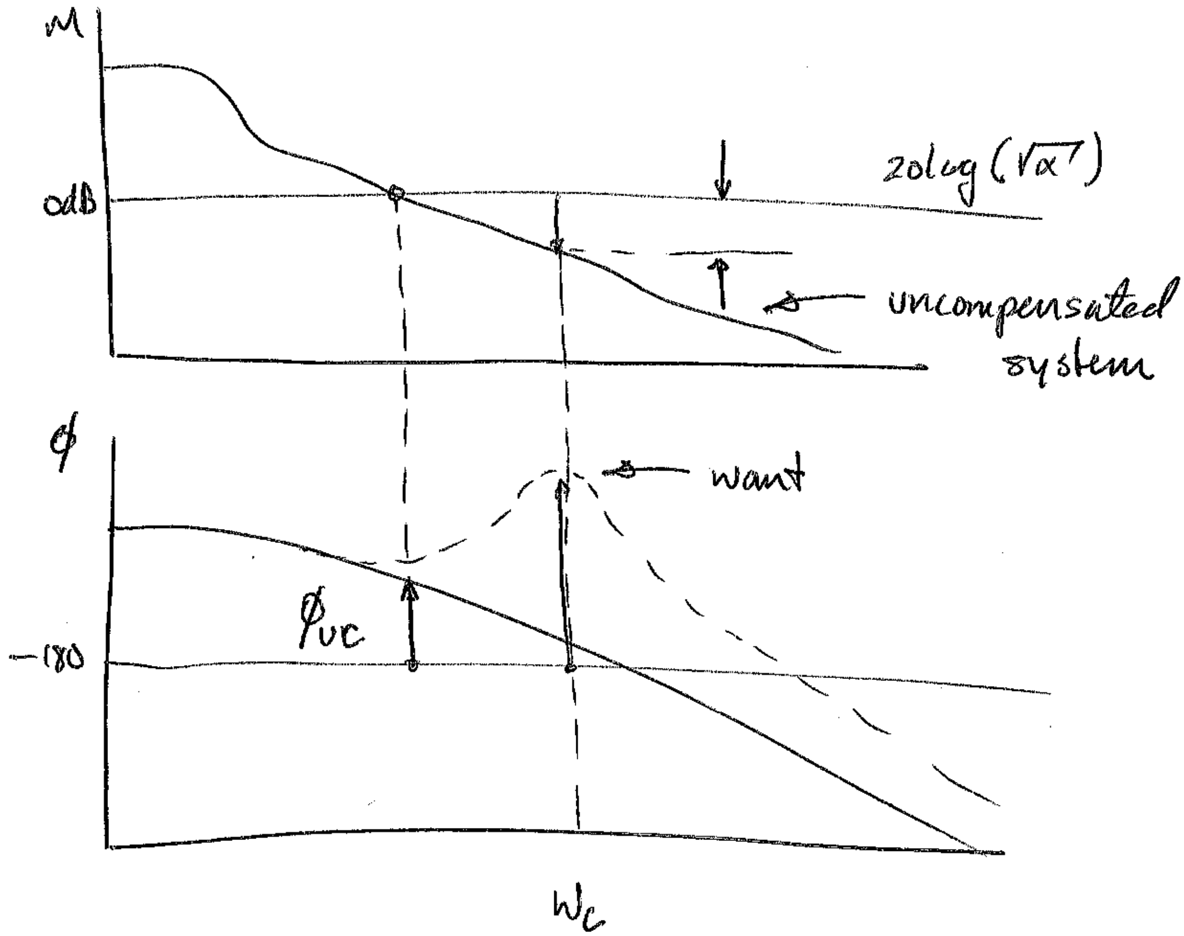
4. Determine α

$$\alpha = \frac{1-\sin\phi_{max}}{1+\sin\phi_{max}}$$



then

5. Determine ω_c (cross over frequency) to be ω_{max} (Trial Error, but you can start by):



Search for $20\log\sqrt{\alpha}$ on Mag. plot. find frequency at which this occurs then use it for ω_{max}

$$\omega_{max} \cong \omega_c$$

6. Solve for the zero and poles:

$$D(s) = K \left(\frac{1}{\alpha}\right) \frac{s + 1/T}{s + 1/\alpha T}$$

$$\frac{1}{T} = \omega_{max} \sqrt{\alpha}$$

$$\frac{1}{\alpha T} = \frac{\omega_{max}}{\sqrt{\alpha}}$$

7. Plot Bode diagram for new system and then check PM and ess.

8. Draw closed-loop bode plot to get closed-loop band width.

9. Iterate the design.

Example (lead compensator)

Consider the 3rd order System:

$$K\,G(s) = \frac{K}{(2s+1)(s+1)(0.5s+1)}$$

Design a lead compensator such that

$K_p = 9$ and $PM = 25^\circ$

solution

step 1: Find K given $K_p = 9$

since $D(s) = \frac{1}{\alpha}\left(\frac{s + 1/T}{s + 1/T\alpha}\right)$ with

$D(0) = 1$, then

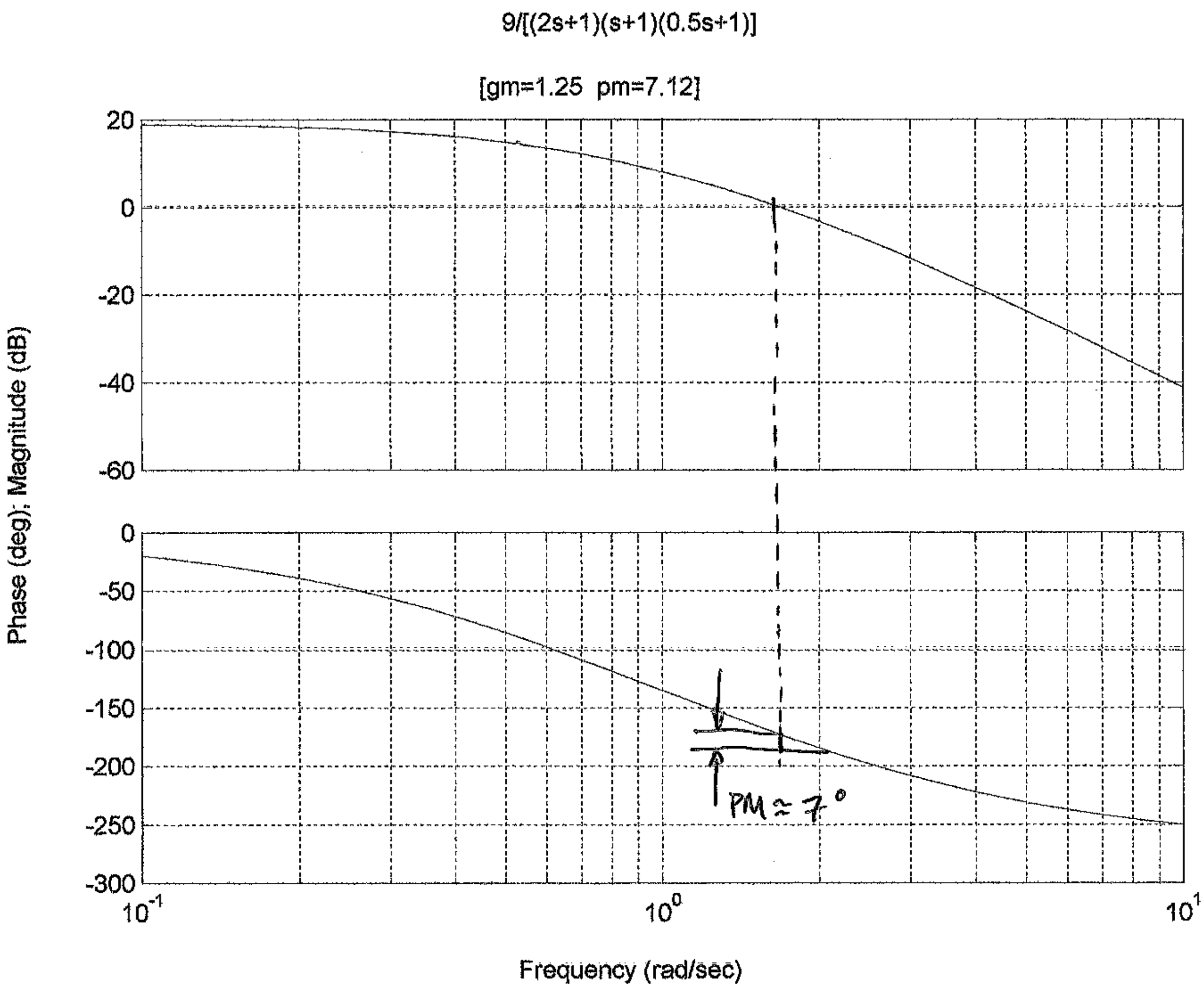
$$e_{ss} = \frac{1}{1+K_p} \Rightarrow K_p = \lim_{s \to 0} K\,D(s)\,G(s) = K$$

$$\Rightarrow \underline{K_p = K = 9}$$

step 2

Draw the bode plot with $K = 9$ and find the PM in Matlab.

$PM \approx 7^\circ$


9/[(2s+1)(s+1)(0.5s+1)]
[gm=1.25 pm=7.12]
Phase (deg); Magnitude (dB)
20
0
-20
-40
-60
0
-50
-100
-150
-200
-250
-300
PM ≈ 7°
10^-1
10^0
10^1
Frequency (rad/sec)

step 3

Allowing extra 5° of PM we find ϕ_{max}

$$\phi_{max} = PM_{des} + 5^\circ - PM_{uc}$$

$$\phi_{max} = 25^\circ + 5^\circ - 7^\circ$$

$\underline{\phi_{max} = 23^\circ}$ Want lead to contribute.

step 4

find α:

$$\alpha = \frac{1 - \sin 23^\circ}{1 + \sin 23^\circ} = 0.43$$

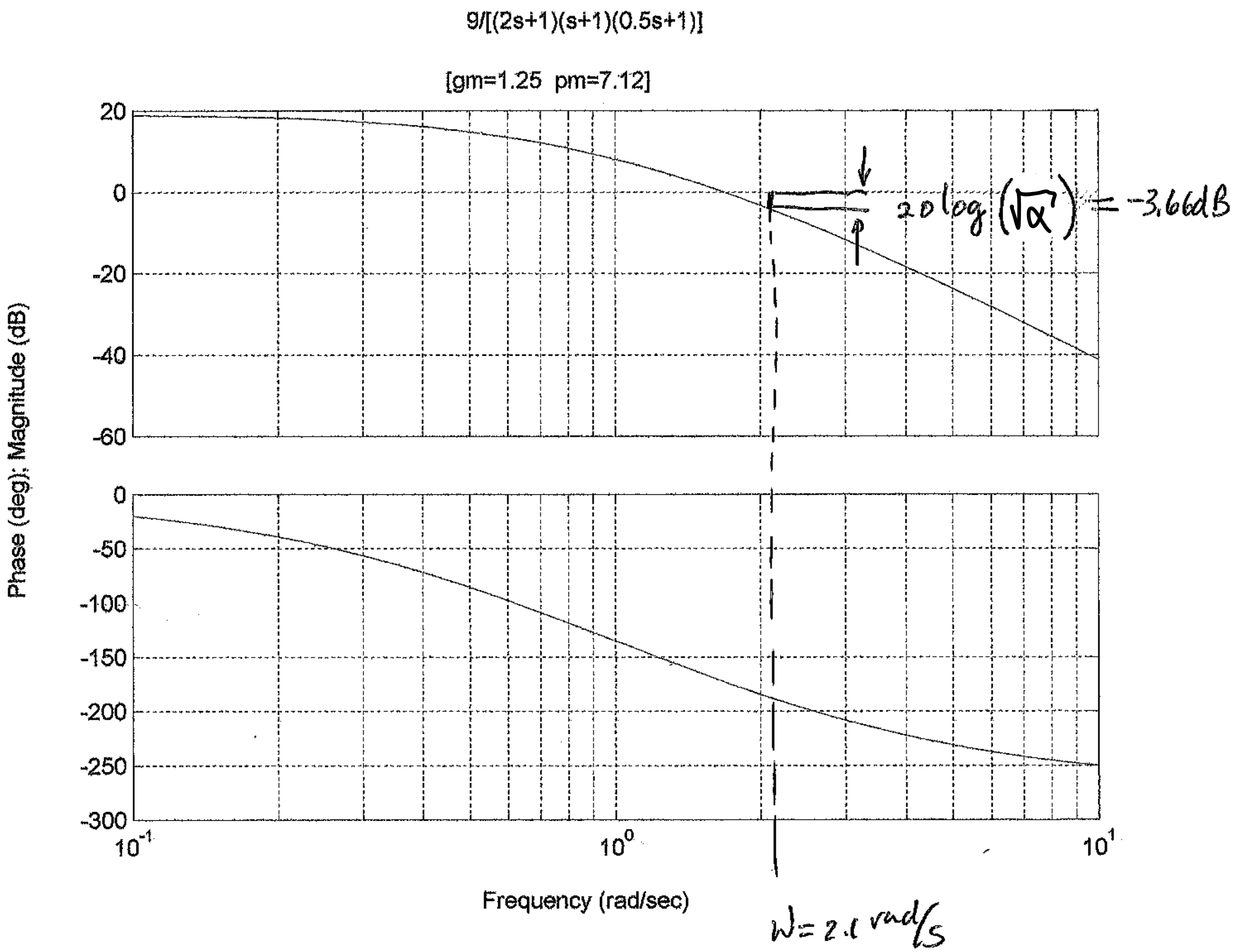
step 5

Determine the cross over frequency at which the new PM boost should occurs

$$Mag = 20 \log(\sqrt{\alpha}) = -3.66\,dB$$

Find $-3.66\,dB$ on Magnitude plot then find w_c, then

$$w_c \approx w_{max} = \underline{2.1\ rad/s}$$


9/[(2s+1)(s+1)(0.5s+1)]
[gm=1.25 pm=7.12]
20
0
-20
-40
-60
0
-50
-100
-150
-200
-250
-300
10^-1
10^0
10^1
Phase (deg); Magnitude (dB)
Frequency (rad/sec)
20log(√α) = -3.66dB
W = 2.1 rad/s

Step 6 Solve for the poles and zeros:

$$K\,D(s) = K\left(\frac{1}{\alpha}\right)\frac{s + 1/T}{s + 1/T\alpha}$$

$$\frac{1}{T} = \omega_{max}\sqrt{\alpha} \cong \omega_c\sqrt{\alpha}$$

$$\Rightarrow \quad \frac{1}{T} = (2.1)\sqrt{.43} \simeq 1.38$$

$$\frac{1}{T\alpha} = \frac{1}{\alpha}(1.38) = 3.2$$

so:

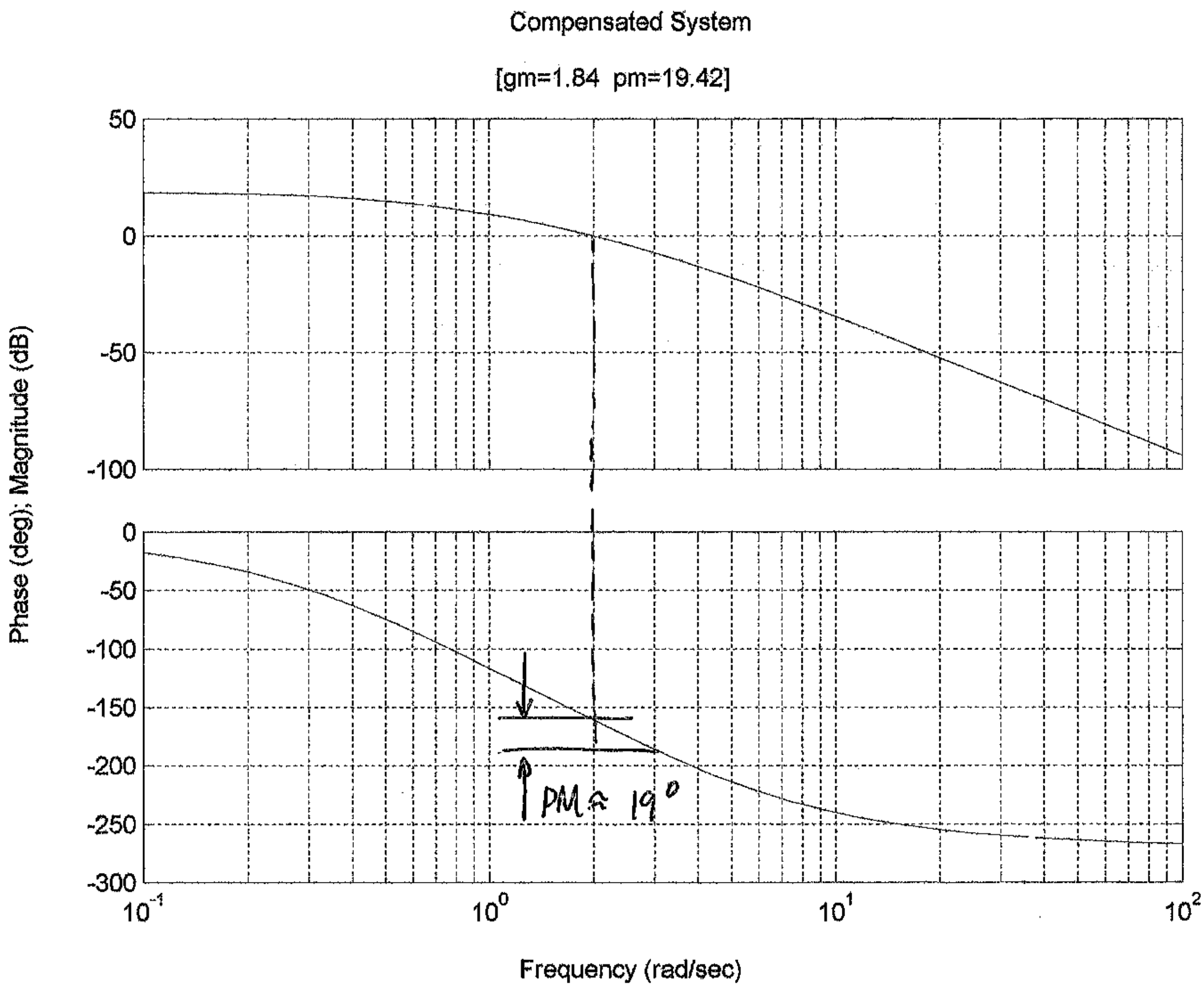
$$D(s) = 9\left(\frac{1}{0.43}\right)\left(\frac{s + 1.38}{s + 3.2}\right)$$

$$\boxed{D(s) = 20\left(\frac{s + 1.38}{s + 3.2}\right)}$$

Step 7 Check to see if conditions are met in Matlab.

$$PM = 19.4^{\circ}$$

Reiterate if necessary!



Compensated System
[gm=1.84 pm=19.42]
50
0
-50
-100
-150
-200
-250
-300
Phase (deg); Magnitude (dB)
PM ≈ 19°
10^{-1}
10^{0}
10^{1}
10^{2}
Frequency (rad/sec)

IV PI Compensation

Used to reduce bandwidth and minimize the steady state error.

$$D(s) = K + K_i(1/s)$$

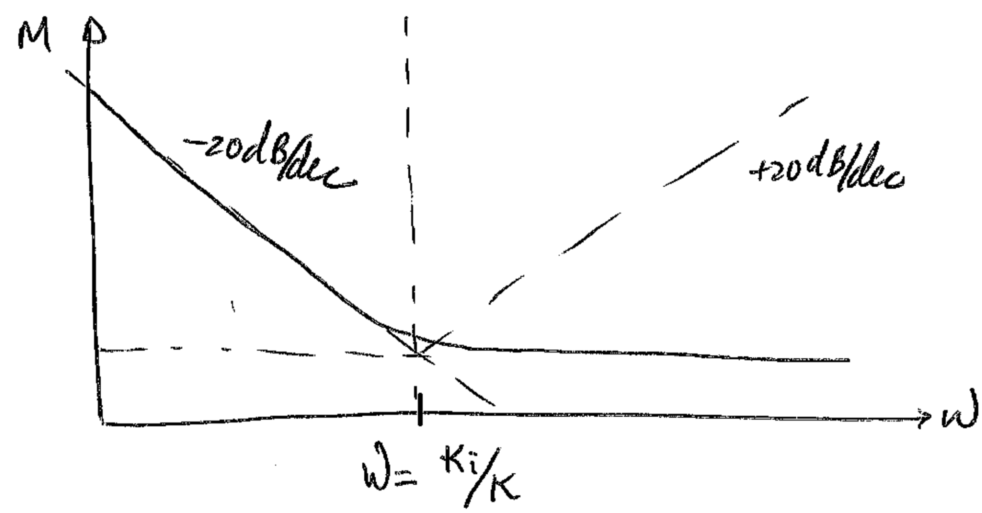
There is an infinite gain at zero frequency which reduces steady state error.

Look at the Bode Plots:

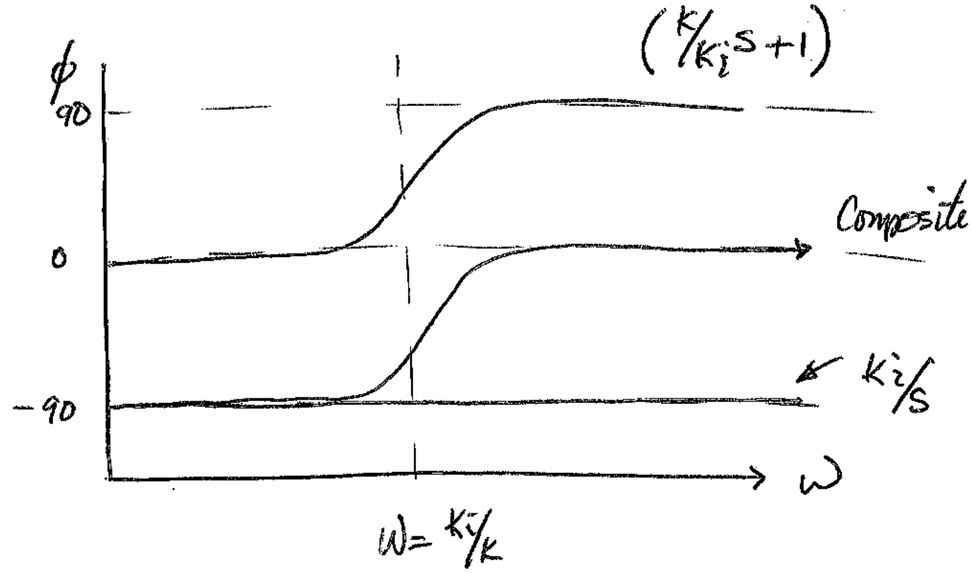
$$D(s) = \frac{sK + K_i}{s}$$

$$D(s) = \frac{K_i\left(\left(\frac{K}{K_i}\right)s + 1\right)}{s}$$

Magnitude:



Phase:



By examining the phase & Magnitude plot, we notice:

1. Gain increase below break point frequency
2. Phase decrease below break point frequency.

PI Design Guidelines:

Design PI such that the phase decrease occurs below the systems desired cross-over frequency to reduce the effects of decrease phase Margin!

V Lag Compensation

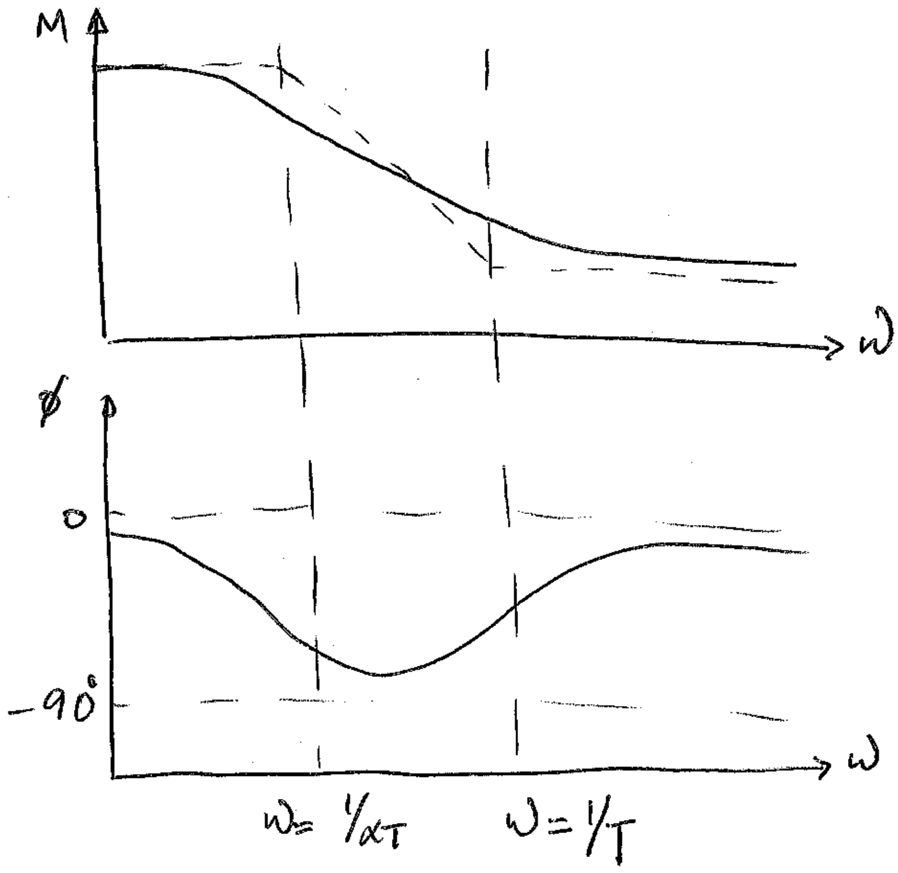
Lag compensation approximates a PI controller.

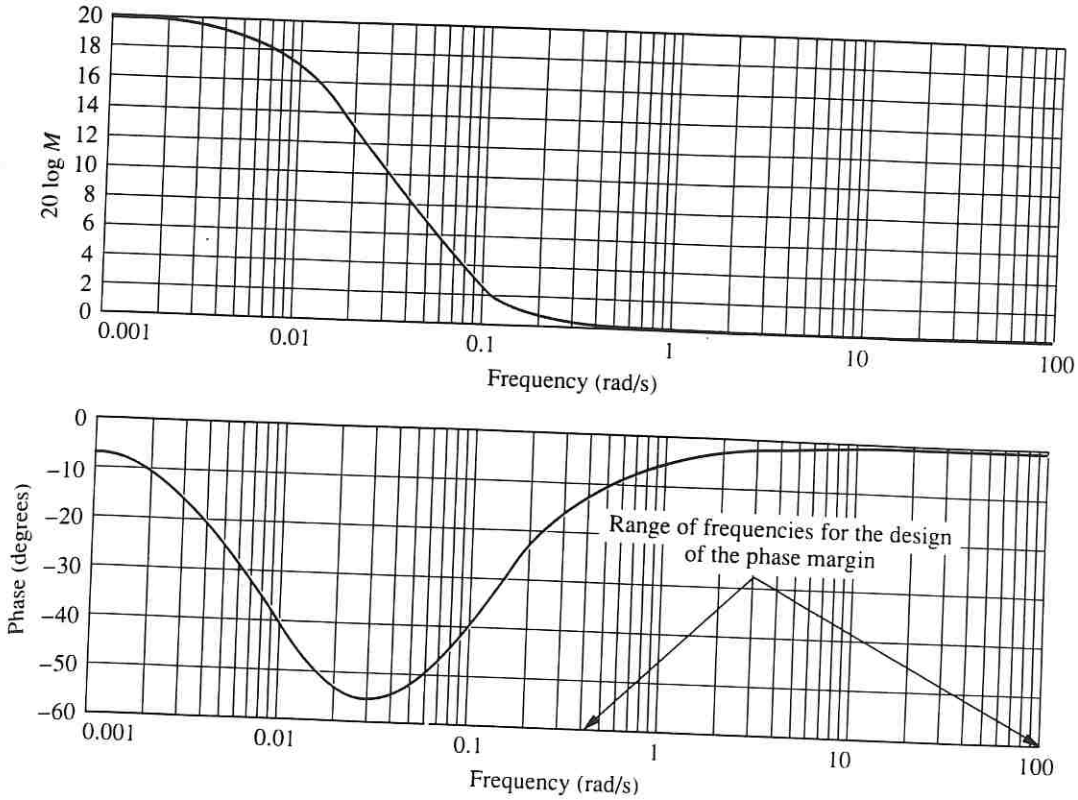
The transfer function:

$$D(s) = \alpha \frac{Ts+1}{\alpha Ts+1} \quad \text{where } \alpha > 1$$

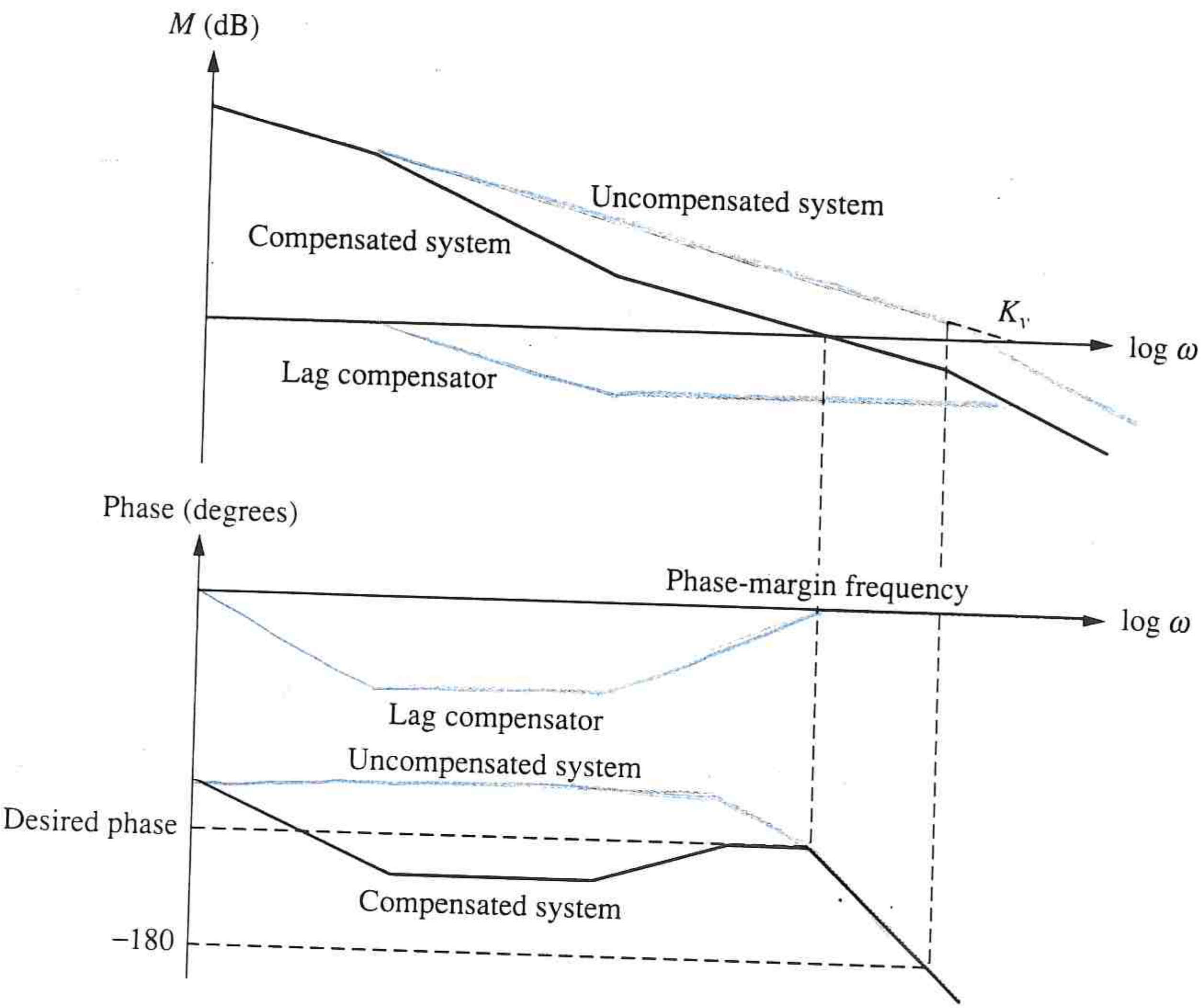
In this form we note that the pole has a lower break-point frequency than the zero. Which results in a low frequency increase in amplitude and phase decrease (lag).

Bode Plot for lag:





20 log M
20
18
16
14
12
10
8
6
4
2
0
0.001
0.01
0.1
1
10
100
Frequency (rad/s)
Phase (degrees)
0
−10
−20
−30
−40
−50
−60
Range of frequencies for the design
of the phase margin
0.001
0.01
0.1
1
10
100
Frequency (rad/s)



M (dB)
Uncompensated system
Compensated system
K_v
log ω
Lag compensator
Phase (degrees)
Phase-margin frequency
log ω
Lag compensator
Uncompensated system
Desired phase
Compensated system
−180

By examing the Bode Plot, the primary objective of lag include:

1. Provide additional $+20 \log(\alpha)$ gain at low frequency
2. Leave the system sufficient phase margin.
3. Improves steady state error, but does not greatly affect transient response.

Lag Design Guidelines

1. zero + pole location of lag are selected to be much smaller than the uncompensated system crossover frequency in order to affect PM to a minimum.
2. Reduce steady state error by increasing the low frequency gain, hence placing zero / poles close to $j\omega$-axis in s-plane.

Lag Design Procedure

Compensator:

$$D(s) = \alpha \frac{Ts+1}{\alpha Ts+1} \qquad \alpha > 1$$

1. Determine K (open-loop) that meets the phase margin requirements w/o compensation.
2. Draw Bode plot of uncompensated system with gain K, then find low frequency gain.
3. Determine α to meet low frequency gain condition.
4. Choose $\omega = \frac{1}{T}$ (zero of lag) to be one octave or decade below the new crossover frequency (ω_c)
5. The other corner frequency (pole) is then $\omega = 1/T\alpha$
6. Iterate on design.

Example (lag compensation)

Consider again the same 3rd order system as for the lead design.

$$G(s) = \frac{K}{(2s+1)(s+1)(0.5s+1)}$$

Design lag such that PM = 25° and $K_p = 9$

solution

step 1

Find K to satisfy PM = 25° : Plot Bode plot then determine K to give PM = 25°. we choose PM = 30° w/ fudge factor.

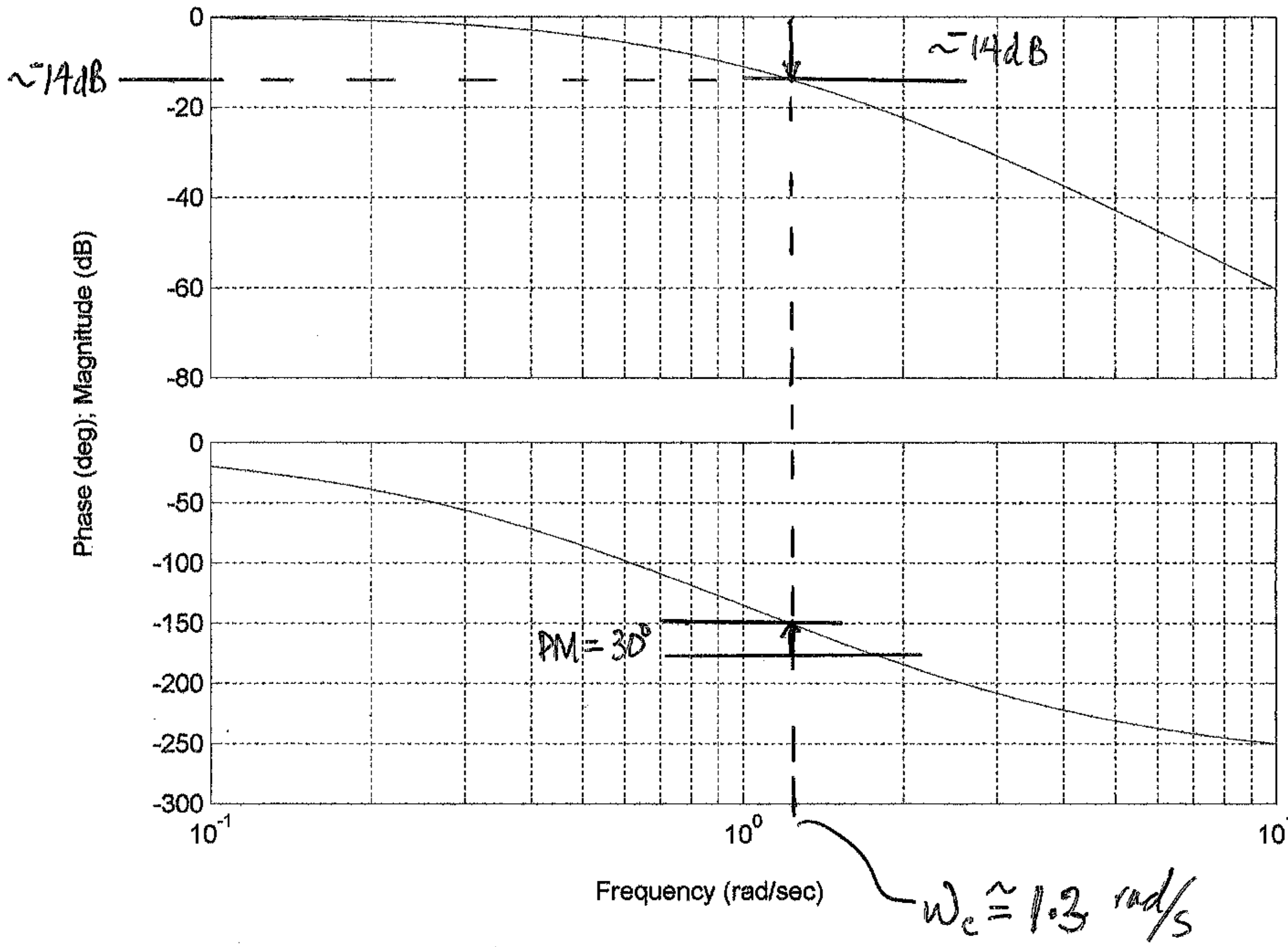
$$K \approx (1) * 10^{(14/20)}$$

$$K \approx 5.0$$

Plot Bode Again to verify PM ≅ 30°.

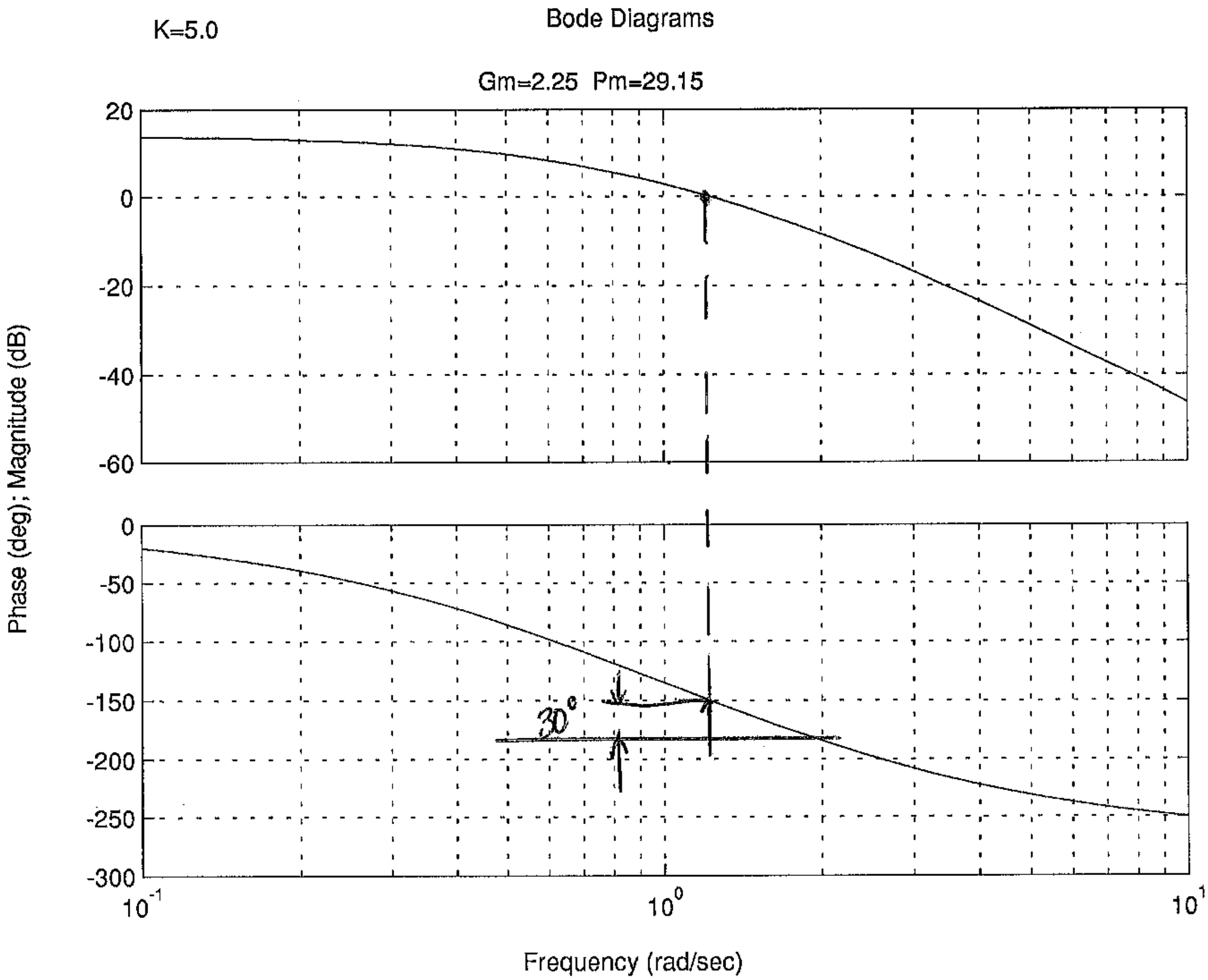
then we get $\omega_c \cong 1.3 \ rad/s$

G(s)=1/[(2s+1)(s+1)(0.5s+1)]



set $K \cong (1) * (10)^{(20/14)} \cong 5.0$

this will give us a PM $\approx 30^\circ$



K=5.0
Bode Diagrams
Gm=2.25 Pm=29.15
Phase (deg); Magnitude (dB)
20
0
-20
-40
-60
0
-50
-100
-150
-200
-250
-300
30°
10^{-1}
10^{0}
10^{1}
Frequency (rad/sec)

step 2

Low frequency gain is $K = 5.0$

at $\omega_c = 1.3$ rad/s

step 3

Determine α

Want $K_p = 9$

$$K_p = \lim_{s \to 0} D(s)G(s) = \lim_{s \to 0} \alpha \left(\frac{Ts+1}{\alpha T s+1}\right) \frac{K}{(2s+1)(s+1)(0.5s+1)}$$

$$\Rightarrow \alpha K = 9$$

$$\alpha(5.0) = 9 \Rightarrow \boxed{\alpha = 1.8}$$

step 4

Choose $\omega = 1/T$ to be at least 10 times less than $\omega_c = 1.3$ rad/s

let $\omega = 1/T = 0.2$

$$\Rightarrow T = 5$$

$$\Rightarrow \alpha T = 9$$

so compensator becomes:

$$\boxed{D(s) = 1.8\,\frac{5s+1}{9s+1}}$$

step 6

check the design

$$D(s)G(s) = 1.8\,\frac{5s+1}{9s+1} \cdot \frac{5.0}{(2s+1)(s+1)(0.5s+1)}$$

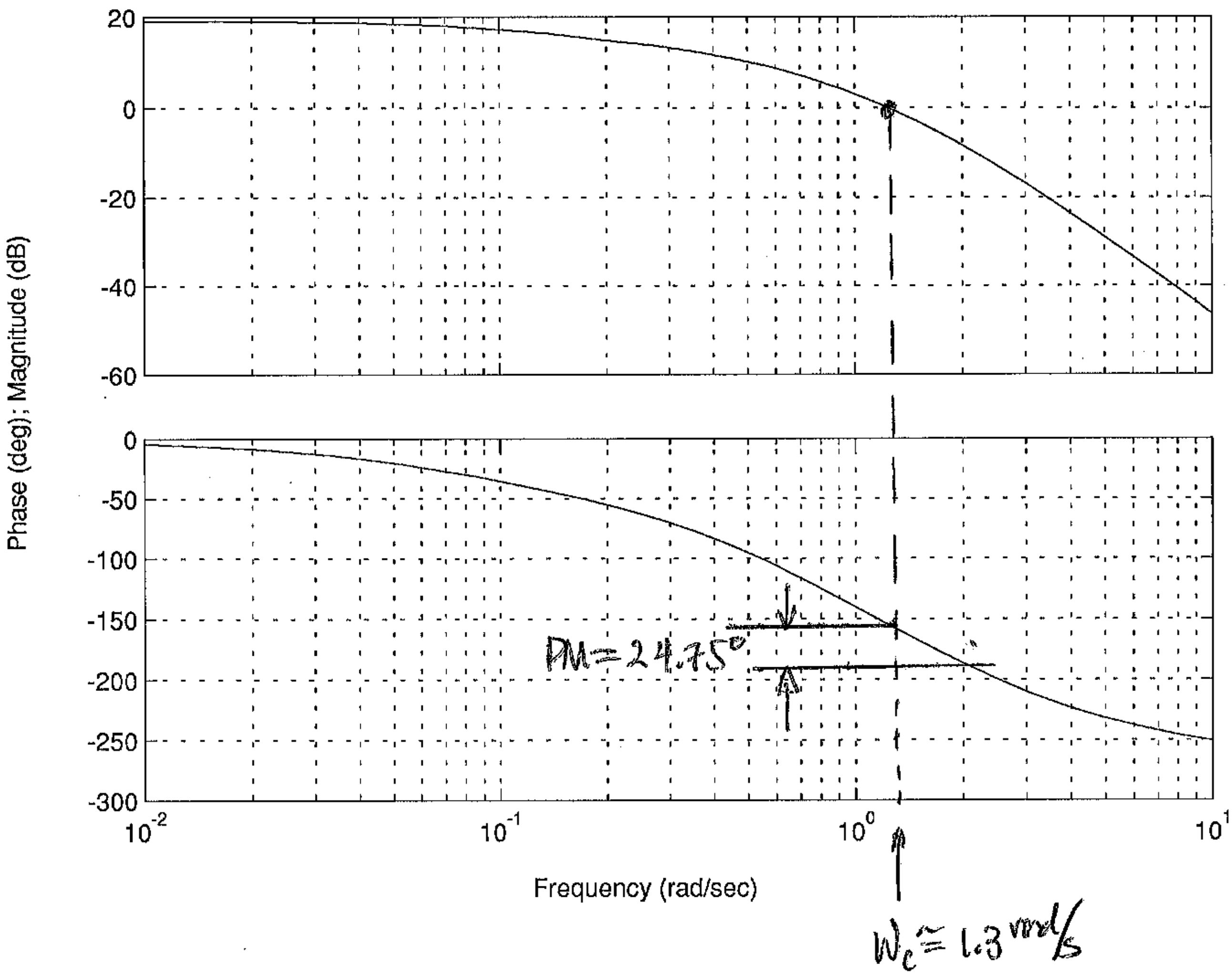
$$D(s)G(s) = 9\,\frac{(5s+1)}{(9s+1)(2s+1)(s+1)(0.5s+1)}$$

when we check with Matlab, we get

$$\underline{\underline{PM = 24.75^{\circ}}}$$

Bode Diagrams

with D(s)



VI PID Compensation

— the combination of a lead and lag compensator.

$$D(s) = K + K_d s + K_i (1/s)$$

Provides simultaneous improvement in transient and steady-state response.

Guidelines:

1. Design the lag portion to meet e_{ss} conditions
2. then Design the lead portion to take care of transients.