Homework 1

Vector Mathematics

Your answers to these questions, and what you learn from them, will be greatly enhanced through collaboration and discussion amongst your discussion group and in the recitation. This is actively encouraged. However, once you have decided how to answer these, the final solutions must be prepared individually.

1. A flow has a temperature field given by:

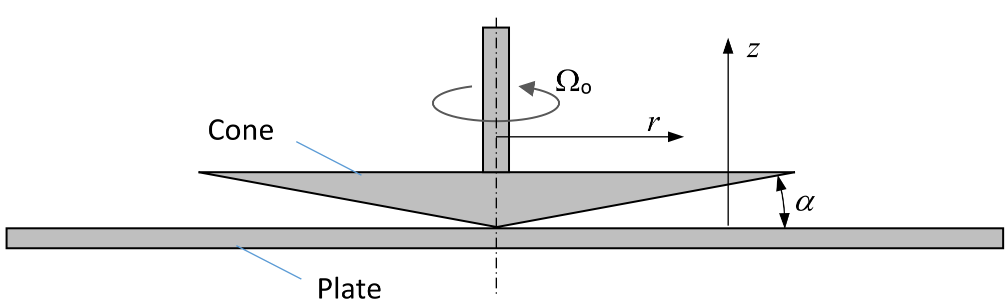
$$T = Ar^2 sin\theta + Bcos(\omega t)$$

and velocity field given by:

$$\boldsymbol{V} = \frac{C}{r^2}\boldsymbol{e_r} + Dsin(\omega t + \theta)\boldsymbol{e_\phi}$$

expressed using the spherical coordinates introduced in class and where *t* is time and *A, B, C, D* and ω are dimensional constants. Find, as functions of position and time, expressions for a) the time rate of change of temperature seen at a fixed point in space, b) the time rate of change of temperature seen by the fluid particles, c) the acceleration of the fluid material. Assuming values of 1 for constants *A, B, C* and *D*, compute values for d) the average angular velocity of the fluid at $(r, \theta, \phi)=(1, \pi/4, \pi/2)$, and e) The time rate of change of volume of fluid element as a fraction of its volume located at $(r, \theta, \phi)=(1, \pi/4, \pi/2)$.

2. The figure shows a plate and cone viscometer. The cone, is spinning at an angular velocity Ω, is placed adjacent to a stationary plate. The device measures the viscosity of the fluid filling the gap between the cone and plate by sensing the resulting torque on the plate.



The angle of the cone relative to the plate α is small and typically only a few degrees. In this circumstance, the flow between the plate and cone takes place in cylindrical stream surfaces, with axes coincident with the axis of rotation. Furthermore, in terms of the cylindrical coordinates (r, θ, z) the velocity field only has one component, and can be written as

$$\mathbf{V} = \frac{\Omega_o z}{\tan\alpha}\mathbf{e}_\theta$$

(a) Determine using these cylindrical coordinates, an equation for the circumferentially averaged angular velocity of the fluid particles, and
(b) Determine using these cylindrical coordinates, an equation for the acceleration experienced by the fluid particles.
(c) Using Newton's law of viscosity, find an equation for the viscous stress on the plate surface.

3. Consider Green's theorem in its second form:

$$\int_R \psi\nabla^2\phi - \phi\nabla^2\psi \; d\tau = \oint_S \psi\frac{\partial\phi}{\partial n} - \phi\frac{\partial\psi}{\partial n} \; dS$$

where the direction of distance n is along unit outward normal to surface S which encloses region R. In ideal flow (flow without compressibility or viscous effects) we have a scalar property of the flow ϕ known as the velocity potential, that satisfies Laplace's equation $\nabla^2\phi=0$. Green's theorem can be used to solve Laplace's equation by judicious choice of the function ψ. Specifically, we pick the function $\psi = -\frac{1}{4\pi|\boldsymbol{r}-\boldsymbol{r_0}|}$, where $\boldsymbol{r}$ is the position vector and $\boldsymbol{r_o}$ is a reference location which (for now) we will consider as constant.
(a) Determine the gradient of ψ.
(b) The Laplacian of ψ turns out to be the 'delta' function $\delta(\boldsymbol{r}-\boldsymbol{r_0})$. This function is zero everywhere, except when $\boldsymbol{r}=\boldsymbol{r_0}$, when it is infinity. The important and really useful thing about this delta function is how it behaves in integrals, picking out the value of whatever function it is multiplied by. For example for any function $f(\mathbf{r})$ in 3D space, the integral $\int_R \delta(\boldsymbol{r}-\boldsymbol{r_0})f(\boldsymbol{r})d\tau$ is simply equal to $f(\boldsymbol{r_0})$, as long as the region of the integration R contains $\mathbf{r_0}$. (The integral is zero if it doesn't.) By substituting ψ into Green's theorem, obtain an explicit expression for the velocity potential ϕ of a flow at the point $\boldsymbol{r_0}$ solely in terms of its value and derivative at the bounding surface of the flow S. This will be a very important result to us later in the course since it shows that you can get the right flow solution just by setting up and integrating the right boundary condition. Note that $-\frac{1}{4\pi|\boldsymbol{r}-\boldsymbol{r_0}|}$ is known as the 'free-space Greens function'.

4. A computational model of a fluid flow uses tetrahedral elements on which to discretize the equations of motion. The engineer writing the code wishes to evaluate the curl of the velocity vector in terms of the values of the velocity vector computed at the four vertices of a single element. Using the integral definition, find an approximate algebraic expression for the curl of velocity in terms of the position vectors of the four vertices $\mathbf{r_1}$, $\mathbf{r_2}$, $\mathbf{r_3}$, $\mathbf{r_4}$, and the velocities computed there, $\mathbf{V_1}$, $\mathbf{V_2}$, $\mathbf{V_3}$, $\mathbf{V_4}$.