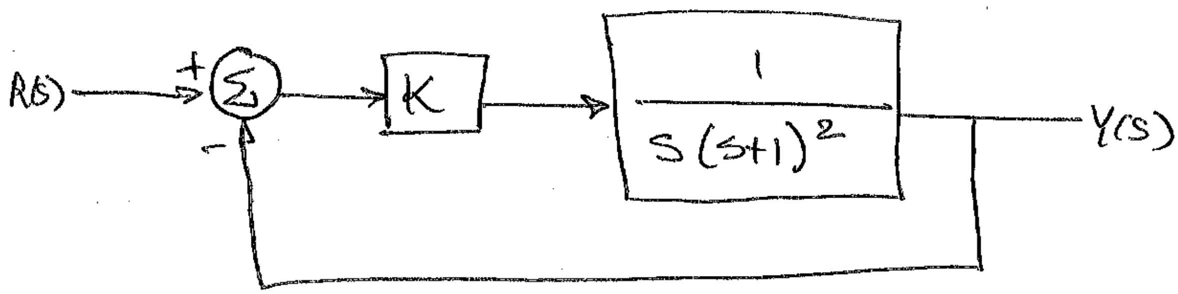
Stability Margins

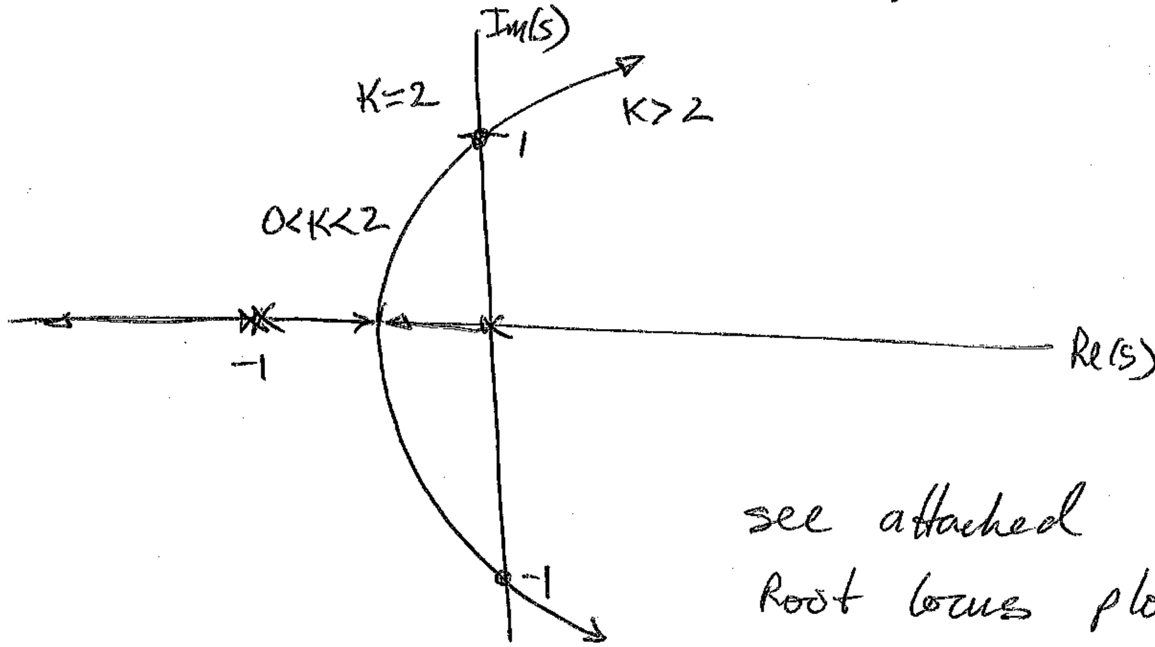
Goal: Analyze the range of gain and phase for stability of systems.

We know from using root locus techniques and Nyquist Method that the system can become unstable when we increase the gain.

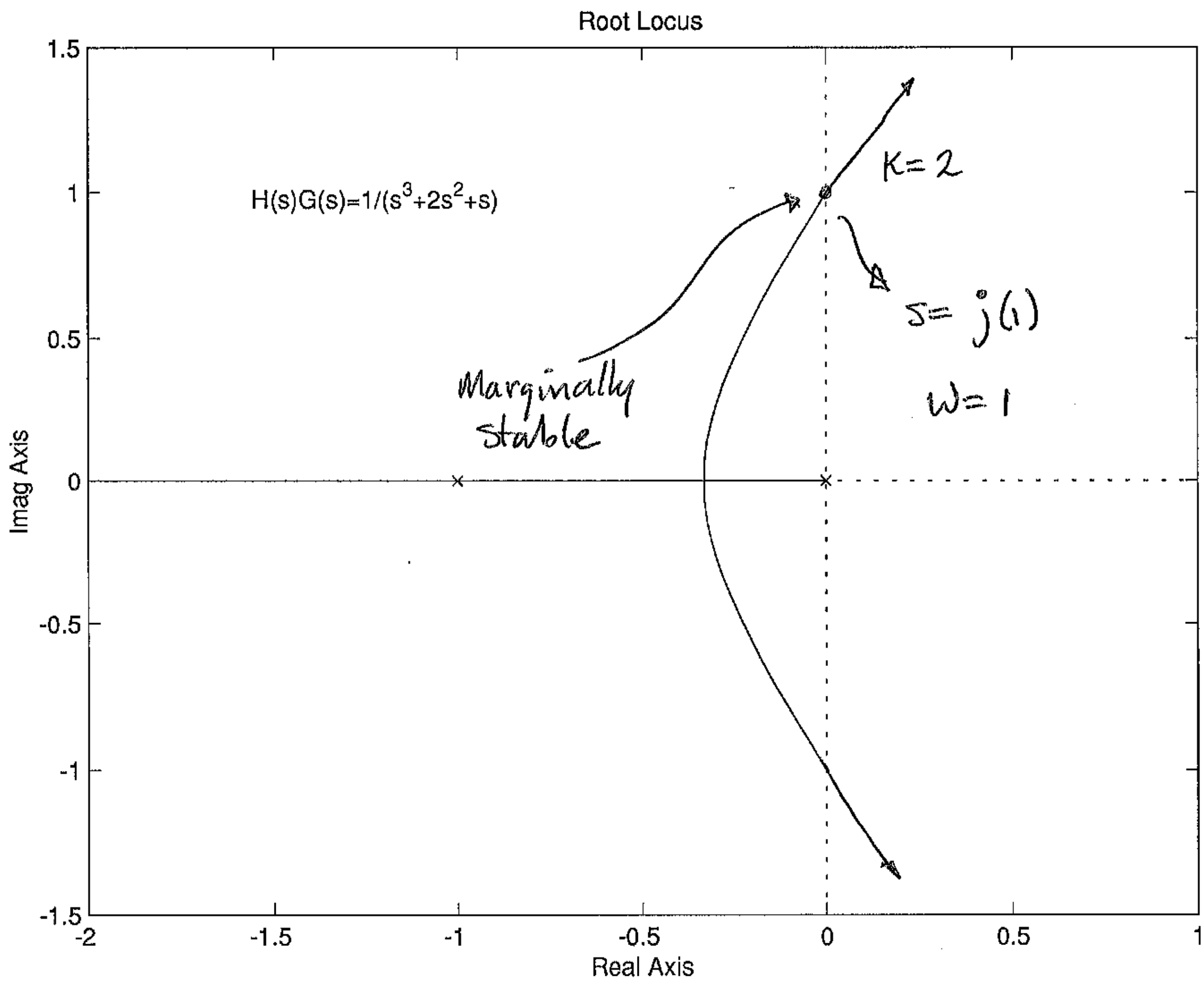
Take for example:



Where the root locus looks like:



see attached Root locus plot



Root Locus
H(s)G(s)=1/(s^3+2s^2+s)
K=2
s= j(1)
w=1
Marginally stable
Imag Axis
Real Axis
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-1.5
-1
-0.5
0
0.5
1

From Root-locus methods we have:

$$1 + KG(s)H(s) = 0$$

where

$$|KH(s)G(s)| = 1$$

$$\angle KH(s)G(s) = 180^\circ$$

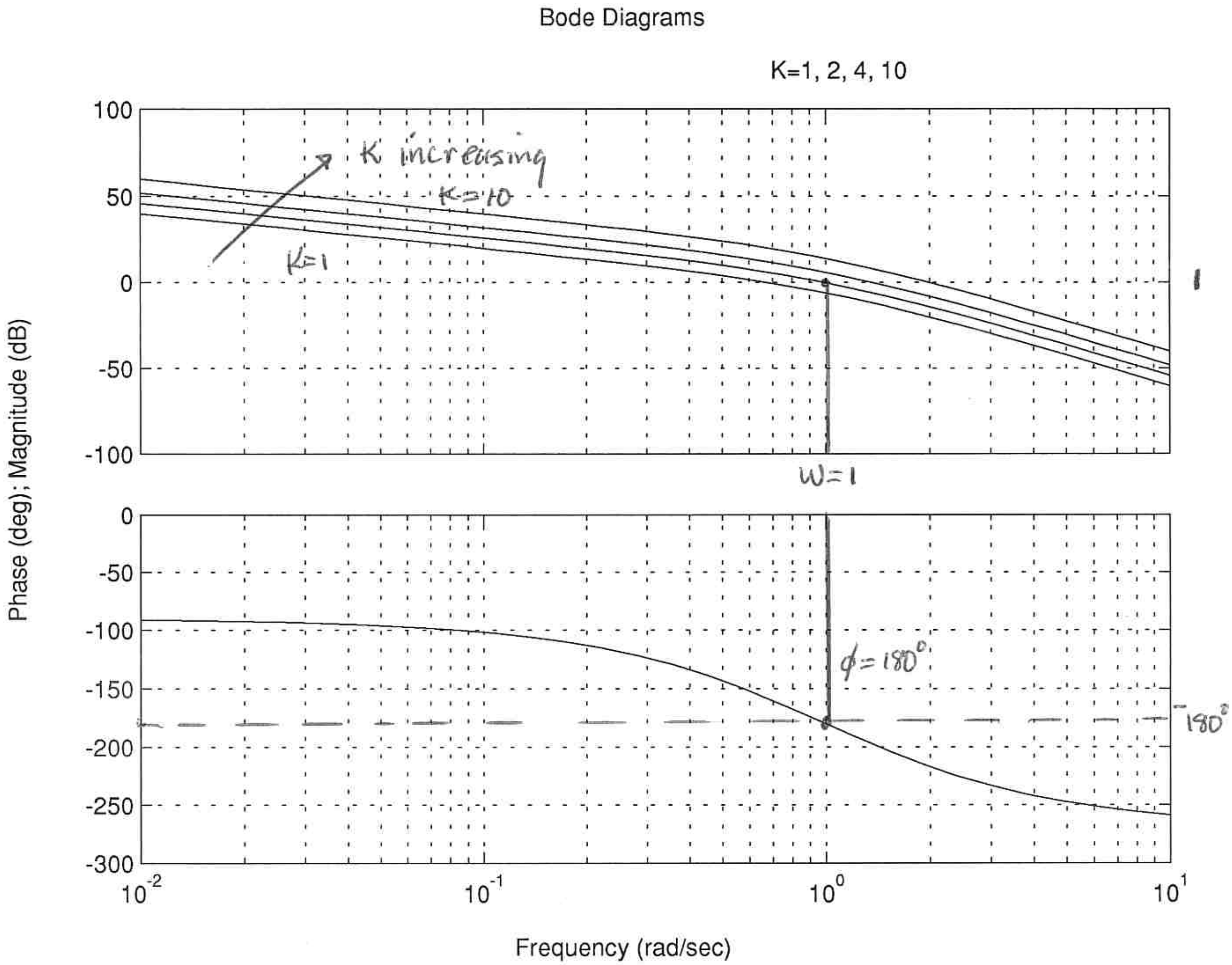
At the point of Marginal stability $(s = j\omega)$ the following applys:

$$|KH(j\omega)G(j\omega)| = 1$$

$$\angle KG(j\omega)H(j\omega) = 180^\circ$$

So let's look at Bode Plot to see if the conditions have been satisfied.

(see attached Matlab Bode Plot)



K=2 Marginally stable

$$|K H(s) G(s)| = 1$$

$$\angle K G(s) H(s) = -180°$$

Now question: Does increasing K make system stable or unstable?

when $K > 2 \Rightarrow$ unstable

$K < 2 \Rightarrow$ stable

looking at Bode Plot, when $\angle KH(s)G(s) = -180^\circ$

stable when $|KH(s)G(s)| < 1$

unstable when $|KH(s)G(s)| > 1$

This is only true for systems that go unstable when K increases and Magnitude of $KG(s)H(s)$ crosses Mag $= 0dB$ (1) once!!

The stability criterion is as follows:

$$|KG(s)H(s)| < 1 \quad \text{at} \quad \angle KH(s)G(s) = -180^\circ$$

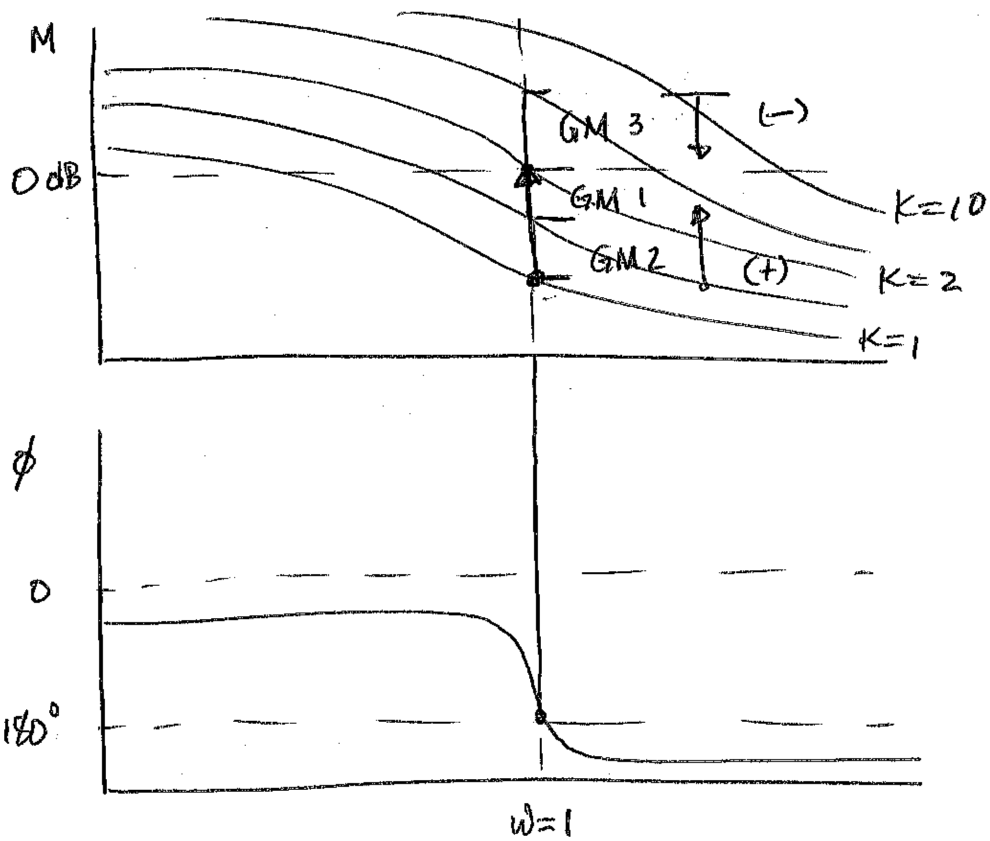
if increasing the gain leads to stability, then we apply:

$$|KG(s)H(s)| > 1 \quad \text{at} \quad \angle KH(s)G(s) = -180^\circ$$

Now we define the gain margin (GM) to be the factor by which the gain is less than the neutral stability value.

Marginally

From the bode Plot:



For our case of increasing gain results in instability:

when	$K=2$	$GM=0$	Marginally stable
	$K=1$	$GM= 6.02\,dB$	(stable)
	$K=10$	$GM= -13.98\,dB$	(unstable)

For this type of system, the system is unstable when the

$$GM_{dB} < 0\ dB$$

only true for systems where increasing K causes the system to go unstable and $|KH(s)G(s)|$ crosses mag $= 0\ dB$ once!!

OR we can get GM from root locus:

Ex. $K = 2$ (Marginally stable)

$$GM = \frac{K_{\text{marginally stable}}}{K_{\text{of interest}}}$$

or $GM|_{dB} = 20 \log (GM)$

take $K = 1$

$$GM = \frac{2}{1} = 2$$

$$GM_{dB} = 20 \log (2) = 6.02\ dB \quad \text{(stable)}$$

stable because $GM_{dB} > 0\ dB$

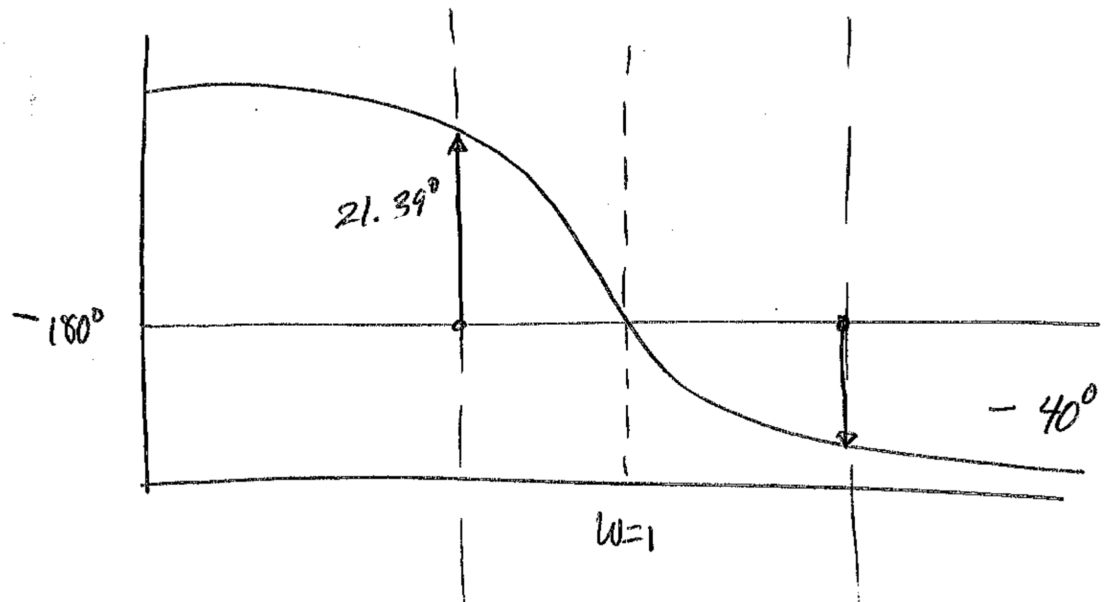
take $K = 10$

$$GM = \frac{K_{ms}}{K} = \frac{2}{10}$$

$$GM_{dB} = 20 \log (1/5) = -13.98\ dB$$

$\Rightarrow$ $GM_{dB} < 0dB$ (unstable)

Phase Margin: the amount by which the phase exceeds -180° when $|KG(s)H(s)| = 1$ (0dB)



PM when $K=1$ $PM = 21.39°$

PM when $K=10$ $PM = -40°$

For system to be unstable, then

$$PM < 0$$

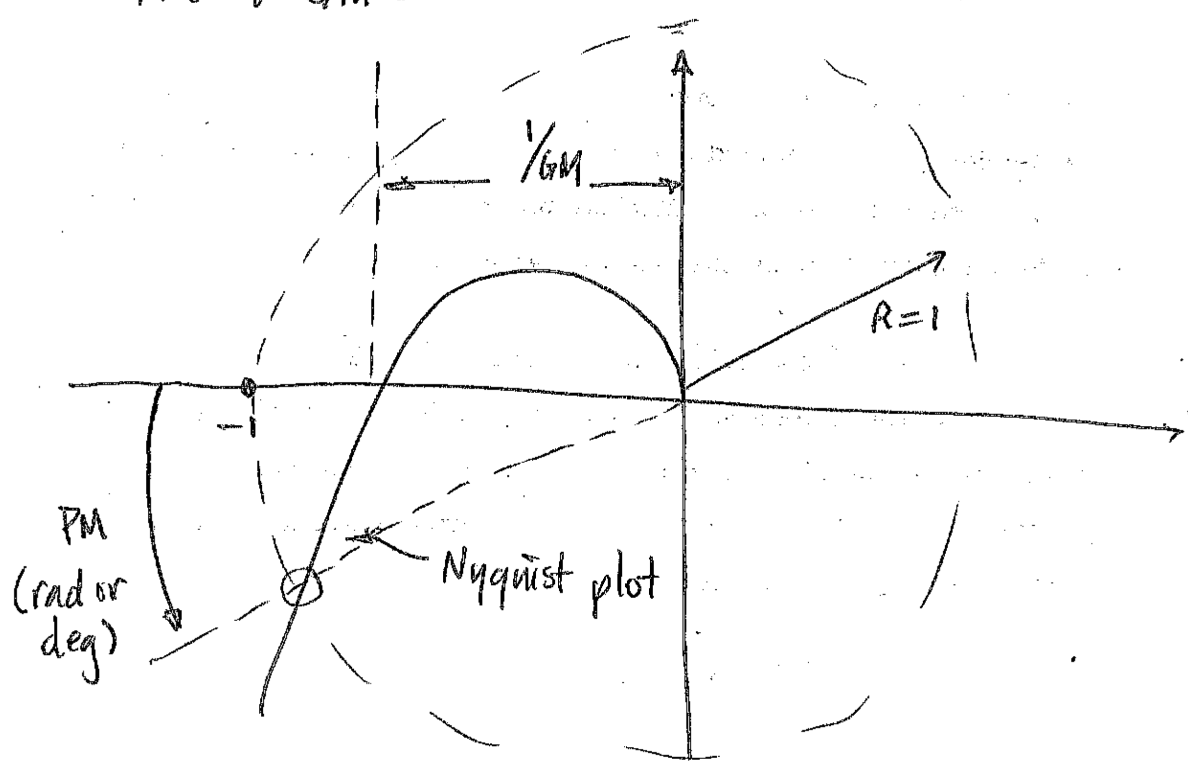
the PM + GM simply tell how far away $KH(j\omega)G(j\omega)$ is away from (-1)

remember

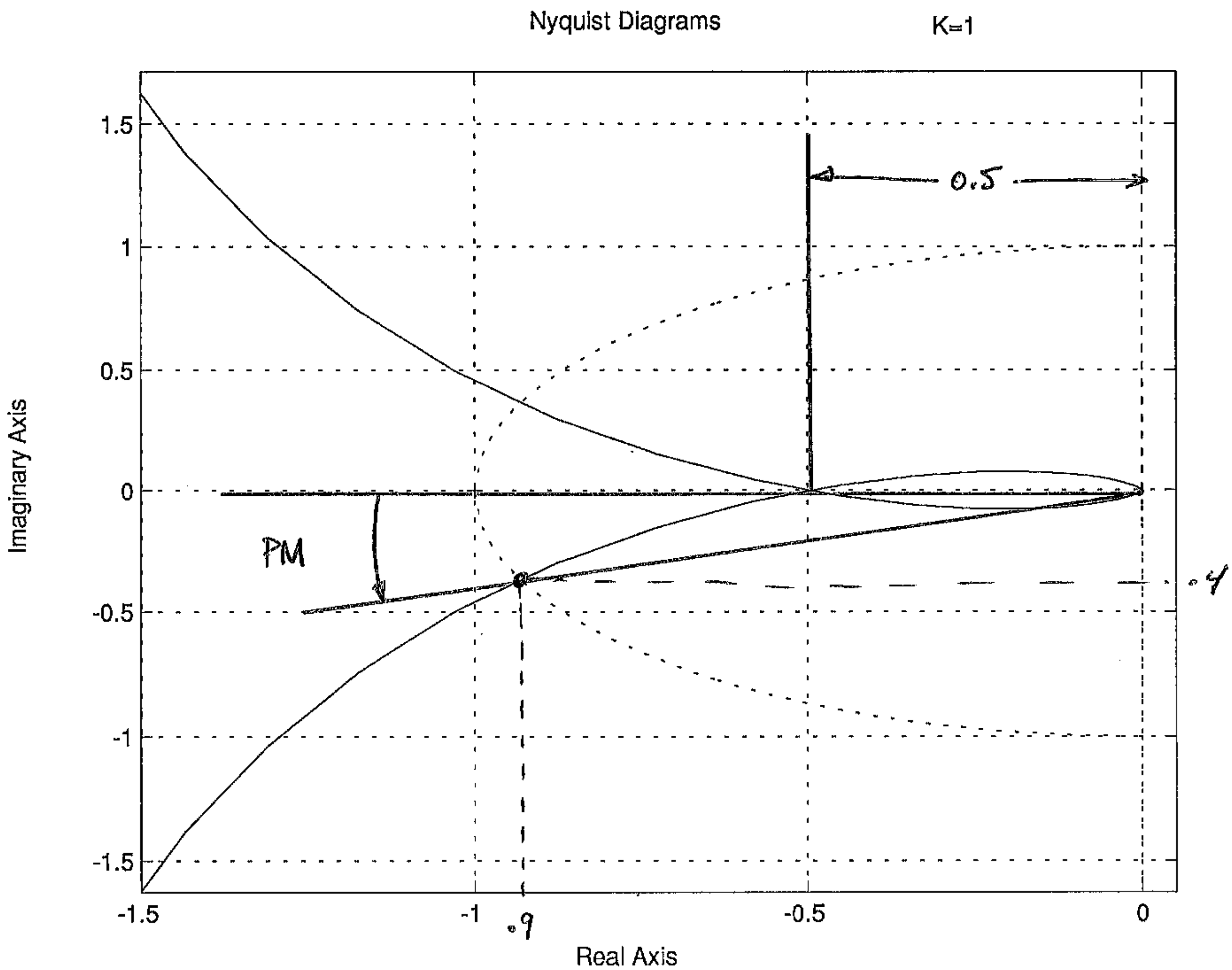
$$1 + KG(j\omega)H(j\omega) = 0$$

Next few* pages of notes talk about the use of Nyquist plots to get PM + GM - See discussion on Nyquist plots before reading further.

We can use Nyquist stability plots to get PM & GM:

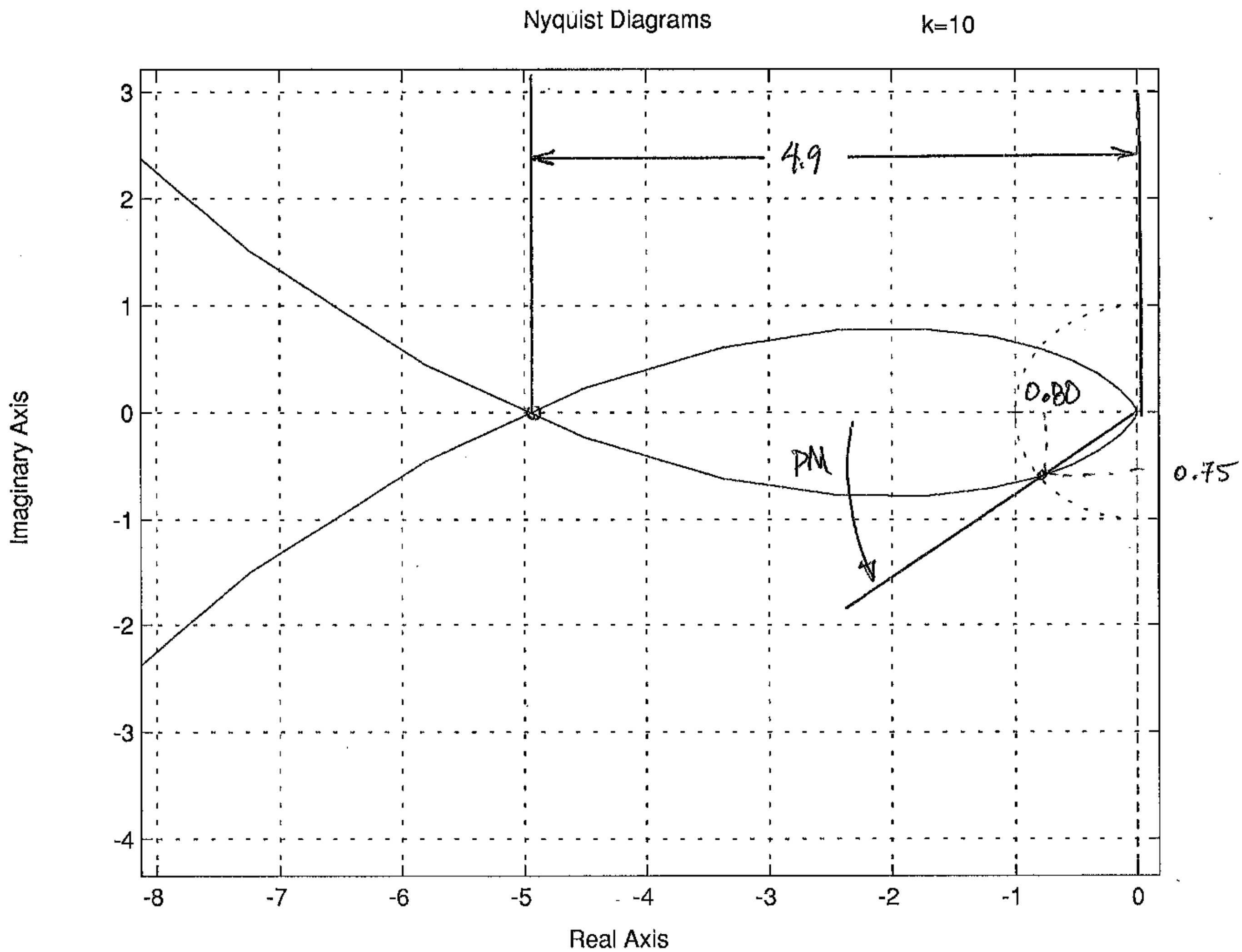


PM > 0 when there is no encirclement around (-1).



$$GM = \frac{1}{0.5} = 2 \quad \Rightarrow \quad GM_{dB} = 20 \log(2) = 6.02\,dB$$

$$PM \approx 23^\circ$$

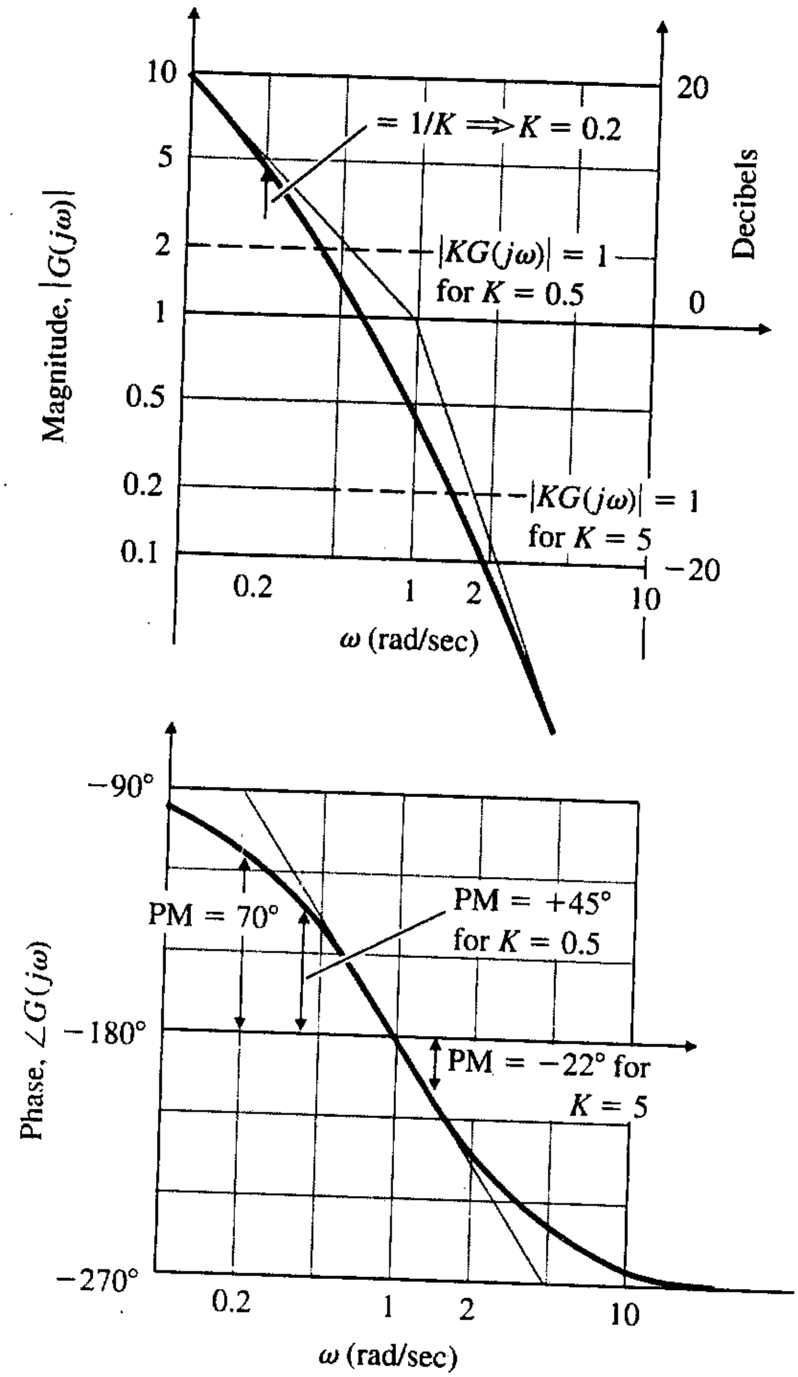


$\frac{1}{GM} = 4.9 \Rightarrow GM = 0.20 \Rightarrow GM_{dB} \cong -13.80\,dB$

PM < 0 because of −1 encirclement

$\Rightarrow PM = -\tan^{-1}\left(\frac{0.75}{0.80}\right) \cong -43.2^\circ$ <u>Unstable</u>

We can determine phase margin for any value of K as follows from the frequency Response curves:



Phase Margin is related to ζ (dampening ratio)

Consider a closed loop second order system:

$$T(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

then the PM and ζ are related as follows:

$$PM = \tan^{-1} \frac{2\zeta}{\sqrt{\sqrt{1+4\zeta^4} - 2\zeta^2}}$$

as an approximation between $0 \leq PM \leq 60^\circ$

$$PM \cong 100\zeta$$

Note: Only accurate for 2^{nd} order systems.

therefore it is sometimes appropriate to express design parameters in terms of phase margins instead of ζ!

What is GM for 2^{nd} systems?

$GM = \infty$ b/c no -180° crossing in phase plot!

For higher order systems when the magnitude plot cross Mag=1 more than once, we have to reevaluate the GM criteria for stability. The conditions that we previously developed are only applicable to systems w/ one Mag=1 crossing!