Intermediate Fluid Mechanics

Lecture 3: Graphical Representation of Flow Kinematics

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Chapter Overview

Lecture Objectives

In this lecture we will learn about:

- streamlines,
- pathlines,
- streaklines,

which are very useful tools in the study of flow kinematics.

Chapter Overview

Streamlines

Streamlines are lines we draw in the flow, such that the tangent at any point along the streamline is in the same direction than the velocity vector at that point.

$\implies$ In this way, streamlines are natural to the Eulerian framework.

At point (x, y) we can define the streamline and velocity vector as,

$$\text{Streamline: } d\vec{r} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k} \tag{1}$$

$$\text{Velocity: } \vec{U} = u\,\hat{i} + v\,\hat{j} + w\,\hat{k} \tag{2}$$

based on the fact that the streamline and velocity vector at a given point should be parallel, then

$$d\vec{r} \times \vec{U} = 0 \tag{3}$$

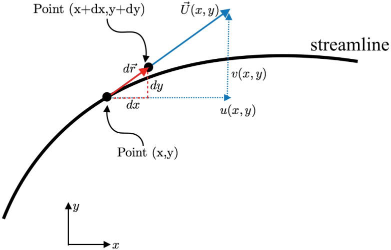


Figure: Representation of a streamline.

Streamlines (continued ...)

In a component-wise approach, the streamline condition,

$$d\vec{r} \times \vec{U} = 0 \tag{4}$$

can be rewritten as: $(dy\, w - v\, dz)\hat{i} + (dz\, u - dx\, w)\hat{j} + (dx\, v - dy\, u)\hat{k} = 0$,

$\Longrightarrow$ This means that each component must be zero individually,

$$\hat{i}: \quad dy\, w - v\, dz = 0 \quad \rightarrow \quad \frac{dy}{dz} = \frac{v}{w} \tag{5}$$

$$\hat{j}: \quad dz\, u - dx\, w = 0 \quad \rightarrow \quad \frac{dz}{dx} = \frac{w}{u} \tag{6}$$

$$\hat{k}: \quad dx\, v - dy\, u = 0 \quad \rightarrow \quad \frac{dx}{dy} = \frac{u}{v} \tag{7}$$

or equivalently, $\boxed{\dfrac{dx}{u} = \dfrac{dy}{v} = \dfrac{dz}{w} \quad \text{along a } \textbf{Streamline}}$

Note: Realize that if the velocity is unsteady, then the streamline pattern will change with time.

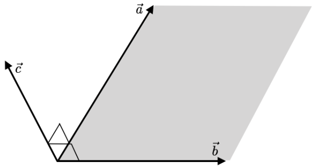
Note on Mathematics: the Cross-product

The cross product of $\vec{a}$ and $\vec{b}$ is a vector $\vec{c}$ having a direction orthogonal to the plane containing $\vec{a}$ and $\vec{b}$, and a magnitude $|\vec{c}|$ equivalent to the area of the parallelogram whose diagonal is $\vec{a} + \vec{b}$.

Mathematically, the magnitude of the cross product is

$$|\vec{c}| = |\vec{a} \times \vec{b}| = a\, b\, sin\, \theta. \qquad (8)$$

The direction of $\vec{c}$ results from the determinant of the matrix,



$$\vec{c} = \vec{a} \times \vec{b} = det \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix} = (a_2\, b_3 - a_3\, b_2)\hat{i} + (a_3\, b_1 - a_1\, b_3)\hat{j} + (a_1\, b_2 - a_2 b_1)\hat{k}.$$

Note: if two vectors are parallel $\vec{a} || \vec{b}$, then $\vec{a} \times \vec{b} = 0$.

Example Streamlines:

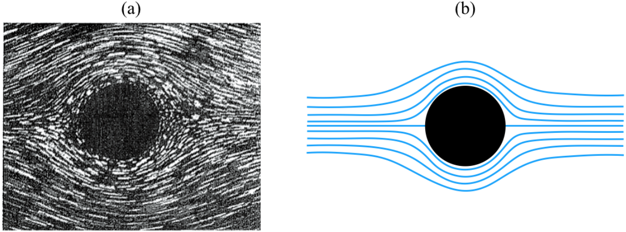


Figure: *(a) Photograph obtained experimentally using short time exposure of metal flakes suspended in fluid; (b) Resultant interpretation of the real streamlines at the same instant the snapshot was taken.*

Note:

Realize that streamlines do not give information about the magnitude of the velocity at any given point, nor do they convey which direction the flow is moving. However, streamlines are a nice visualization tool, giving you a qualitative overview of the flow pattern (useful to observe regions of separation, vortex structures, etc.)

Chapter Overview

Pathlines

- A **pathline** is a line that describes the trajectory of a particle in time.
- The pathline is the tangent curve to the velocity of the same fluid particle at different times.
- Therefore, Pathlines are a graphical representation of the flow natural to the Lagrangian framework.
- Mathematically, pathlines are given by:

$$\frac{d\vec{x}}{dt} = \vec{u}_p, \tag{9}$$

- The solution of the ODE, $\vec{x} = \vec{x}(t)$ gives the pathline.

Note: an initial condition is required for the solution, *e.g.* $\vec{x}(t=0) = x_0\,\hat{i} + y_0\,\hat{j} + z_0\,\hat{k}$.

Example Pathlines

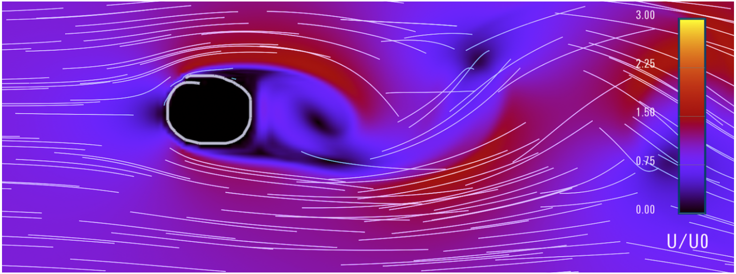


Figure: *Example of Pathlines.*

Particle trajectories illustrated in white as they travel around a cylinder. The trajectories are produced by a mean flow moving from left to right.

Comparison between Pathlines and Streamlines

- At the time of the snapshot, the pathline of a fluid particle $\vec{\xi}$ is tangent to the streamline at the place $\vec{x}$.
- By definition the velocity vector is tangential to the streamline at time t and simultaneously to its pathline.
- At another time, those same fluid particles may be associated with other streamlines.

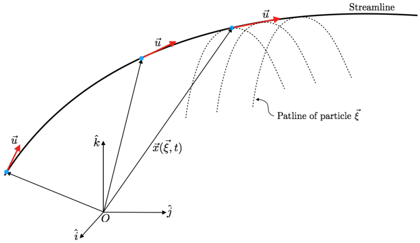


Figure: *Snapshot of the flow at one instance in time, with Pathlines for three different particles and the Streamline passing by a given set of points.*

Chapter Overview

Streaklines

A **streakline** is a line joining all the fluid particles that passed through a fixed point in the flow field at some previous time.

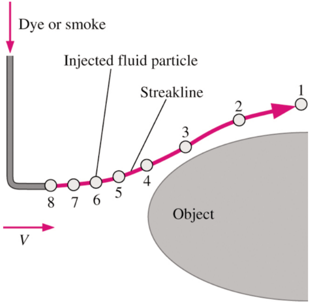


Figure: *Classic example of a streakline is the result of injecting dye at a fixed point into a flow. The numbers indicate the relative order at which the fluid particles were released from the origin location*

Practice Exercise:

Exercise

Can you draw the pathlines and streakline of the fluid elements exiting a sprinkler system that oscillates between 0 and 90° and releases water on a horizontal plane?

Steady/Unsteady Flows

- In a steady flow, none of the field variables (*e.g.* velocity, density, temperature, etc.) as viewed in the Eulerian framework change with time.
- In steady flow, the streaklines, pathlines and streamlines are coincident.
- For unsteady flows one must use caution. A flow may appear to be steady or unsteady depending on the reference frame used.

Example: flow induced by a sphere falling at a steady speed through a large tube.

- The flow will appear to be steady if we observe the flow in a reference frame that is translating at the same speed as the sphere.
- In this reference frame, the streamlines/pathlines are identical to those observed from a uniform flow around a stationary sphere.
- However, the flow will appear unsteady if we observe the motion of the falling sphere from a fixed point in space.

Example: Streamlines, Pathlines, Streaklines - a Mathematical study case

Consider a two-dimensional flow field in the x-y plane defined as

$$u = x(1 + 2t) \tag{10}$$

$$v = y \tag{11}$$

$$w = 0. \tag{12}$$

$\Longrightarrow$Compute the Streamlines, Pathlines, and Streaklines, considering that at t_0, the particle is at $(x_0, y_0) = (1, 1)$.

Example: Streamline

The streamlines are given by,

$$\frac{dx}{u} = \frac{dy}{v}, \text{ which upon substitution leads to, } \frac{dx}{x(1+2t)} = \frac{dy}{y} \tag{13}$$

Upon integration from an origin point (let's say (x_0, y_0)) to an actual location (x, y),

$$\int_{x_0}^{x} \frac{dx}{x(1+2t)} = \int_{y_0}^{y} \frac{dy}{y} \tag{14}$$

$$\frac{1}{(1+2t)}(ln\, x - ln\, x_0) = ln\, y - ln\, y_0 \tag{15}$$

$$ln\left(\frac{x}{x_0}\right)^{\frac{1}{1+2t}} = ln\left(\frac{y}{y_0}\right) \tag{16}$$

$$\left(\frac{x}{x_0}\right)^{\frac{1}{1+2t}} = \left(\frac{y}{y_0}\right) \tag{17}$$

for x_0, and $y_0 = 1$ at $t = 0$, then the equation of the streamlines is given by,

$$\boxed{x^{\frac{1}{1+2t}} = y} \tag{18}$$

Example: Pathline

The pathline will be given by,

$$\frac{dx}{dt} = u \quad \text{and} \quad \frac{dy}{dt} = v. \tag{19}$$

Upon substitution of u and v,

$$\frac{dx}{dt} = x(1+2t) \quad \text{and} \quad \frac{dy}{dt} = y, \tag{20}$$

and solving these equations using the initial conditions $x(t_0) = x_0$ and $y(t_0) = y_0$, it follows that,

$$\int \frac{dx}{x} = \int (1+2t)\, dt \quad \text{and} \quad \int \frac{dy}{y} = \int dt \tag{21}$$

$$ln\, x = t + t^2 + C_1^* \quad \text{and} \quad ln\, y = t + C_2^*, \tag{22}$$

$$x = C_1\, e^{t(1+t)} \quad \text{and} \quad y = C_2\, e^t. \tag{23}$$

Example: Pathline (continued)

- So far we have obtained that: $x = C_1 \, e^{t(1+t)}$ and $y = C_2 \, e^t$.

$\Longrightarrow$ In this case C_1, and C_2 represent constants of integration, which can be easily determined with the initial conditions. At $t = 0$,

$$x_0 = C_1 \, e^0 \quad \text{and} \quad y_0 = C_2 \, e^0, \tag{24}$$

hence $C_1 = x_0$ and $C_2 = y_0$. As a result the pathlines are given by,

$$x(t) = x_0 \, e^{t(1+t)} \tag{25}$$

$$y(t) = y_0 \, e^t. \tag{26}$$

Example: Streakline

The streaklines are defined by all the fluid particles that emanate from point (x_0, y_0) at some time τ, with $t \leq \tau$.

$$x = x(x_0, y_0, t, \tau) \quad \text{and} \quad y = (x_0, y_0, t, \tau), \tag{27}$$

- τ denotes the time at which the fluid particle passed through the common point (x_0, y_0);
- This time can not be after the current observation time t.

$\Longrightarrow$ Therefore, one proceeds in a similar manner as with the pathlines, *i.e.*

$$\frac{dx}{dt} = x(1 + 2t) \quad \text{and} \quad \frac{dy}{dt} = y, \tag{28}$$

which upon integration give

$$x = C_1 \, e^{t(1+t)} \quad \text{and} \quad y = C_2 \, e^{t}. \tag{29}$$

Example: Streakline (continued ...)

Now, using the initial condition that $x = x_0$ and $y = y_0$ at $t = \tau$, this gives

$$x_0 = C_1\, e^{\tau(1+\tau)} \rightarrow C_1 = x_0\, e^{-\tau(1+\tau)} \tag{30}$$

$$y_0 = C_2\, e^{\tau} \rightarrow C_2 = y_0\, e^{-\tau}. \tag{31}$$

Plugging into the pathline equations above yields the equations describing the streakline,

$$\boxed{x = x_0\, e^{t(1+t)-\tau(1+\tau)} \quad \text{and} \quad y = y_0\, e^{t-\tau}.} \tag{32}$$

Example: Conclusion

- The streamlines: $x^{\frac{1}{1+2t}} = y$.
- The Pathlines: $x(t) = x_0\, e^{t(1+t)}$ and $y(t) = y_0\, e^{t}$.
- The Streakline: $x = x_0\, e^{t(1+t)-\tau(1+\tau)}$ and $y = y_0\, e^{t-\tau}$.

Note:

Realize that in this case, the streakline, pathlines and streamlines are all different, which coincides with the fact that the flow is not steady.