Intermediate Fluid Mechanics

Lecture 26: Boundary Layer Flows V

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Chapter Overview

Lecture Objectives

In this lecture, we will discuss the effect of the pressure gradient on the boundary layer.

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Chapter Overview

1 Chapter Objectives

2 A zero pressure gradient boundary layer in a wind tunnel

3 Comments on the effect of pressure gradient

4 Effect of pressure gradient on the Boundary Layer profile

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A zero pressure gradient boundary layer in a wind tunnel

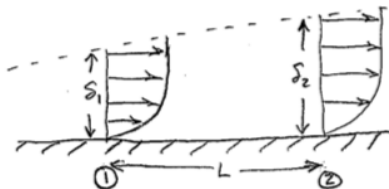Let's consider flow over a flat plate, as the one illustrated below,

Figure: In this boundary layer flow, $\delta_2 > \delta_1$ and $\delta_2^* > \delta_1^*$ due to the growth of the boundary layer down the plate.

A zero pressure gradient boundary layer in a wind tunnel

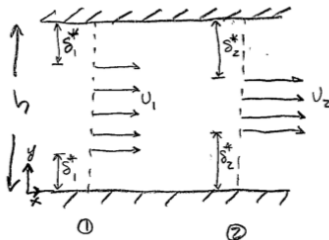Using the concept of the displacement thickness, one can represent the flow in a wind tunnel as drawn below,

Figure: Model the effect of the boundary layers using an equivalent uniform velocity that is displaced from the walls by a distance δ_1^* at section 1 and δ_2^* at section 2.

Note: In the case represented above, both the floor and ceiling of the wind tunnel, serve as flat plates over which boundary layers are forming.

A zero pressure gradient boundary layer in a wind tunnel

Because of boundary layer growth,

- $\implies \delta_2^* > \delta_1^*$, $\longrightarrow$ the effective cross sectional area of the uniform flow decreases as the flow moves down the plate, *i.e.* $(h - 2\delta_1^*) > (h - 2\delta_2^*)$.

- $\implies$ The flow U_2 will be different relative to U_1!

Since the flow is steady and density is constant, mass conservation reduces to the condition

$$\dot{m}_1 = \dot{m}_2 \tag{1}$$

$$U_1(h - 2\delta_1^*) = U_2(h - 2\delta_2^*) \tag{2}$$

$$\frac{U_2}{U_1} = \frac{(h - 2\delta_1^*)}{(h - 2\delta_2^*)} \tag{3}$$

A zero pressure gradient boundary layer in a wind tunnel

Therefore, since the right hand side is greater than one, we find that

$$U_2 > U_1, \tag{4}$$

$\implies$ This means that:

$$\frac{dU}{dx} > 0 \Rightarrow \frac{dP}{dx} < 0 \qquad \text{Favorable pressure gradient} \tag{5}$$

The accelerating flow translates into a favorable pressure gradient. So, the boundary layers forming on the floor and ceiling in a wind tunnel (where the floor and ceiling are parallel) actually experiences a favorable pressure gradient.

A zero pressure gradient boundary layer in a wind tunnel

From the Folker-Skan solution, laminar boundary layers exposed to a favorable pressure gradient ($m > 0$) have a larger wall shear stress (*i.e.*, steeper velocity gradient near the surface).

In the wind tunnel we would expect the laboratory data to exhibit characteristics similar to the Falker-Skan profile for $m > 0$,
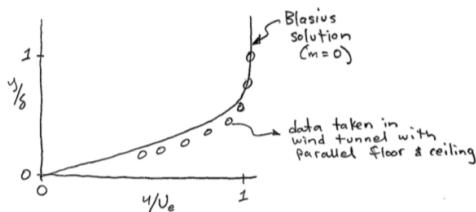

Figure: Velocity profile in the wind tunnel in comparison to the Blasius solution.

A zero pressure gradient boundary layer in a wind tunnel

<u>Question:</u> How could one achieve an actual zero pressure gradient boundary layer in practice?

A zero pressure gradient boundary layer in a wind tunnel

Answer: One needs to maintain the same effective cross-sectional area, *i.e.*
$h - 2\delta_1^* = h - 2\delta_2^*$.

$\implies$ One way to do this is to have a ceiling diverge slightly, so that the height of
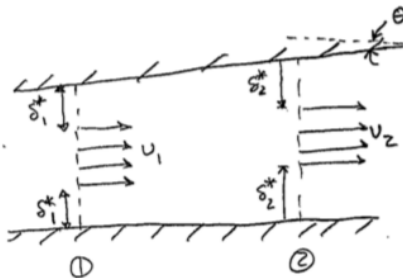the channel changes with x, as illustrated in Figure 4.

Figure: Flow in a wind tunnel with adjustable cross section. The divergence angle θ is selected appropriately to ensure that $h - 2\delta_1^* = h - 2\delta_2^*$ which will generate $U_1 = U_2$, thus yielding a zero pressure gradient, flat plate boundary layer along floor.

Chapter Overview

Comments on the effect of pressure gradient

Let's consider the flow over a curved surface illustrated below,
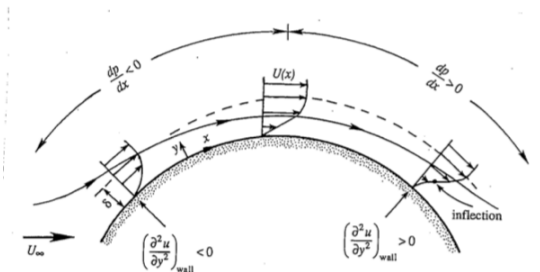

Figure: Flow over a curved surface with an evolving boudnary layer.

- As the flow goes around the body, it will accelerate, causing a negative pressure gradient force at the highest point on the surface.
- As the flow moves around toward the rear of the body, the streamlines will begin to diverge and the flow will decelerate causing a positive pressure gradient toward the aft of the body.

Streamline pattern

Recall that the volume flow rate between streamlines is equal to the difference in ψ along those streamlines.

- This means that for two streamlines with values ψ_1 and ψ_2,
  $\Delta\psi = \psi_2 - \psi_1 = \text{constant}$.

- Recall that by definition, ψ is constant along a streamline.

- Therefore, the volume flow rate between streamlines 1 and 2 will also remain constant, *i.e.* $Q_{1-2} = \text{constant}$.

Streamline pattern

Let's consider two locations a and b in the flow below, where a is fore of the apex and b is aft.
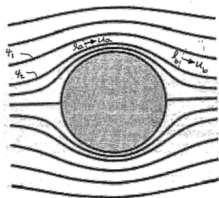

Figure: Streamlines around a sphere or cylinder.

The volume flow rate between streamlines 1 and 2 at locations a and b can be approximated as

$$Q_a \sim l_a \times u_a, \qquad Q_b \sim l_b \times u_b. \tag{6}$$

Streamline pattern

Also we have that

$$Q_a = Q_b = Q_{1-2}. \tag{7}$$

Therefore, since $l_b > l_a$, it is found that $u_b < u_a$.

This helps us interpret streamline patterns to understand how the pressure gradient is changing along the body, which we do by using the inviscid relation

$$U\frac{dU}{dx} = -\frac{1}{\rho}\frac{dP}{dx} \tag{8}$$

valid outside the boundary layer.

Chapter Overview

Effect of pressure gradient on the Boundary Layer profile

Notes about the second derivative:

For the following analysis it is worthwhile revisiting the mathematical interpretation of the second derivative of a given function.

Given $y = f(x)$,

- $f(x)$ is said to be concave up or convex when $d^2f/dx^2 > 0$.

- Alternatively, $f(x)$ is said to be concave down when $d^2f/dx^2 < 0$.
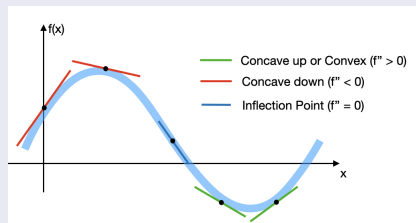
- When $d^2f/dx^2 = 0$ it represent an inflection point.

Graphically, this is represented as:

Figure: The tangent line is green where the curve is concave up, red where the curve is concave down, and blue at the inflection points

Effect of pressure gradient on the Boundary Layer profile

Let's consider the boundary layer equation

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial y}. \tag{9}$$

We now evaluate this equation at the wall $y = 0$.

Because of the non-slip condition, $u = v = 0$ at the wall, so both advection terms disappear, leaving

$$\frac{\partial p}{\partial x} = \mu\frac{\partial^2 u}{\partial y^2}\Big|_{\text{Wall}}. \tag{10}$$

Effect of pressure gradient on the Boundary Layer profile

For an accelerating flow, $\frac{\partial p}{\partial x} < 0$, leading to

$$\mu \frac{\partial^2 u}{\partial y^2}\Big|_{\text{Wall}} < 0. \tag{11}$$

- Also, we have that the velocity profile inside the boundary layer has to blend smoothly with the outer flow profile at the edge of the boundary layer.

- Near the edge of the boundary layer, but still within the boundary layer, $\frac{\partial u}{\partial y}$ decreases with y from a positive value (slightly below the edge of the boundary layer).

- Therefore, near the edge of the boundary layer $\frac{\partial^2 u}{\partial y^2} < 0$.

For accelerating flow, then, since $\frac{\partial^2 u}{\partial y^2} < 0$ at the wall and at the boundary layer edge, it is presumably negative throughout the boundary layer.

Effect of pressure gradient on the Boundary Layer profile

For decelerated flow, $\frac{\partial p}{\partial x} > 0$, it leads to

$$\mu \frac{\partial^2 u}{\partial y^2}\bigg|_{\text{Wall}} > 0. \tag{12}$$

- In this case, the quantity $\frac{\partial^2 u}{\partial y^2}$ changes sign from positive near the wall to negative near the boundary layer edge.

- This is significant because it tells us that for the case of decelerated flow, the velocity profile has an inflection point.

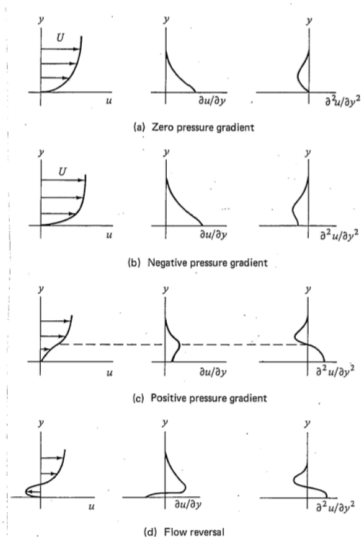# Effect of pressure gradient on the Boundary Layer profile

Figure: Different velocity profile shapes as a function of the external pressure gradient.

Chapter Overview

Separation

- For the case of decelerating flow, the existence of an inflection point implies a slowing down of the velocity in the region near the wall.

- Under a strong enough adverse pressure gradient, the flow near the wall actually reverses direction relative to the freestream flow outside the boundary layer.

- The separation point, s, is the point where the reversed flow meets the forward flow and is characterized by the condition that

$$\frac{\partial u}{\partial y}\bigg|_{\text{Wall}} = 0. \tag{13}$$

Separation
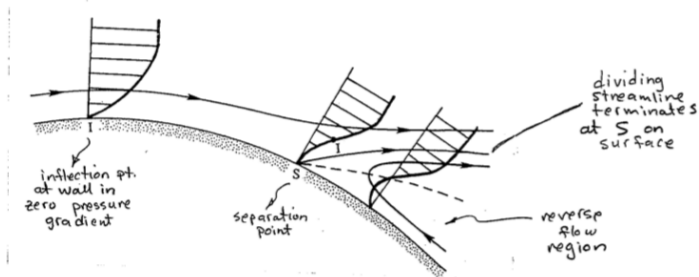
This is graphically represented as,

Figure: Graphical representation of boundary layer separation.

Separation

- At high enough Reynolds numbers, the existence of a separation point on the body leads to the formation of a wake behind the body.

- Experiments show that typically the pressure remains fairly uniform downstream of separation and has a lower value than the pressure on the forward face of the body.

- This difference in pressures results in a drag force that is termed form drag, because it depends on the shape of the body.

- The shape/geometry of a body has a significant influence on the pressure gradients experienced by the fluid as it flows around the body.

- Avoiding strong adverse pressure gradients is the key to preventing separation.

Separation

32. Laminar separation on a thin ellipse. A 6:1 elliptic cylinder is held at zero angle of attack in a wind tunnel. The Reynolds number is 4000 based 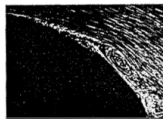on chord. Drops of titanium tetrachloride on the surface form white smoke, which shows the laminar boundary layer separating at the rear. *Bradshaw 1970*
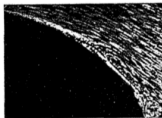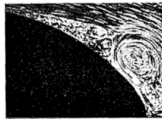

(a)   (b)

(c)   (d)

*change in the size of the recirculated flow region with increasing Reynolds number.

Evidence indicates that the point of separation is relatively insensitive to Reynolds number as long as the boundary layer remains laminar. However once the boundary layer transitions to turbulence, separation is delayed.

Figure: Examples of the interaction between body shape, pressure gradient and boundary layer separation.