ME 5700/6700 – Intermediate Fluid Dynamics
Exam 1 - Fall 2020
Instructor: Eric Pardyjak

Time: start Monday October 19, 2020 at 2 pm (24-hour take-home exam, the exam should take about 2-3 hours).

Allowed: Open-book, open-notes exam; specifically, you may use your textbook, a calculator, a computer, and the internet

Not allowed: You are not allowed to communicate with any other human being about the exam except the instructor, Eric Pardyjak. No texting, no emailing, no talking, posting, etc.

Exam Process: Download the exam from Canvas at 2pm on Monday October 19, 2020; you will have 24-hours to complete the exam; It must be uploaded onto Canvas by 2 pm Tuesday October 20, 2020. You don't have to physically download the exam, you may choose to use your own paper and then upload a pdf or photo of your work. Please write as neatly and as organized as possible. Much of the work we do in this class involves derivations, please show all of your work and important steps. <u>You must sign the honor policy on the following page.</u>

ME EN 5700/6700 – Intermediate Fluid Mechanics Honor Policy for Exams
Fall 2020

In order to protect the integrity of this course, I am requiring all students taking this exam to sign a pledge that you will not cheat on this exam (Exam 1 ME EN 5700/6700). Please note that this exam is open book and open notes, however, all of the work that you submit must be your own. You may consult material and problems in the book, in your notes, or online, but you are not allowed to discuss the material on this exam with your peers in the class or anyone else prior to noon on Tuesday October 20, 2020. Furthermore, you are not allowed to post questions to online forums. Anyone found to have cheated on this exam will automatically fail the exam.

I affirm that I, ____________________ (write your name) am submitting my own work for this exam. I have neither given nor received help on this exam. I have not consulted with anyone (other than the instructor) on questions nor looked at or copied another student's work. I understand the penalties for academic dishonesty on this exam is failure.

____
Signature

____
Date/Time

ME 5700/6700: Intermediate Fluid Dynamics
EXAM - I
(October 19, 2020)

Feel free to add pages if there is not sufficient room to do your work.

NAME: KEY

1. [**5 points**] Explain the difference between Lagrangian and Eulerian descriptions of fluid motion.

For a Lagrangian description, individual fluid elements are followed over time ($x_i(t)$), while for Eulerian description flow is considered while passing through a fixed region of space (variable are fields $T = T(x_i, t)$)

2. [**5 points**] Given the following 2D velocity field for a fluid flow: $u = 2x + 1$ and $v = -2y + 0.5x$, compute the acceleration components of a fluid element.

$$\frac{D\vec{V}}{Dt} = \frac{Du}{Dt}\hat{i} + \frac{Dv}{Dt}\hat{j} = \left(\cancel{\frac{\partial u}{\partial t}} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right)\hat{i} + \left(\cancel{\frac{\partial v}{\partial t}} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right)\hat{j} \quad [2\text{pts}]$$

$$\frac{\partial u}{\partial t} = 0, \ \frac{\partial v}{\partial t} = 0$$

$$\frac{\partial u}{\partial x} = 2, \ \frac{\partial u}{\partial y} = 0$$

$$\frac{\partial v}{\partial x} = 0.5, \ \frac{\partial v}{\partial y} = -2$$

$$= \left[(2x+1)2 + \cancel{(-2y + \tfrac{1}{2}x)0}\right]\hat{i} + \left[(2x+1)\tfrac{1}{2} + (-2y + \tfrac{1}{2}x)(-2)\right]\hat{j}$$

$$= (4x+2)\hat{i} + (\cancel{x} + \tfrac{1}{2} + 4y \cancel{- x})\hat{j}$$

$$\frac{D\vec{V}}{Dt} = (4x+2)\hat{i} + (4y + \tfrac{1}{2})\hat{j} \quad [3\text{pts}]$$

3. [**5 points**] What is the purpose of the Reynolds Transport Theorem?

The R.T.T. allows one to mathematically transform conservation equations from a Lagrangian description to an Eulerian description.

4. [**5 points**] Consider the following simple 2D velocity field: $u = a$ and $v = bt$, where a and b are constants.
   a. Determine the equation of the streamlines and plot the resulting streamline that goes through the point ($x_0 = 1, y_0 = 1$) at $t = 1$.
   b. Determine the equation of pathlines and plot the pathline that starts at the point ($x_0 = 1, y_0 = 1$) from t = 0 to t = 2 using dt steps of 0.1.
   c. Are the streamlines and pathlines the same? Why?

Use Matlab, Python, etc. to make your plots. Do not hand draw.

a. Equation of a streamline: $\frac{dx}{u} = \frac{dy}{v}$ $\quad u = a$, $v = bt$

$$\frac{dx}{a} = \frac{dy}{bt}$$

$$\int dy = \left(\frac{b}{a}t\right)\int dx$$

Apply $x_0 = 1, y_0 = 1$

$$y = \frac{bt}{a}x + C \Rightarrow C = 1 - \frac{bt}{a}$$

$$y = \frac{bt}{a}x - \frac{bt}{a} + 1$$

$$\boxed{y = \frac{bt}{a}(x-1) + 1}$$

b. For a pathline

$$u = \frac{dx}{dt} \qquad v = \frac{dy}{dt}$$

$$\frac{dx}{dt} = a \Rightarrow \int_{x_0}^{x} dx = \int_{t=0}^{t} a\,dt$$

$$x = x_0 + at$$

$$\frac{dy}{dt} = bt \Rightarrow \int_{y_0}^{y} dy = \int_{t=0}^{t} bt\,dt$$

$$y = y_0 + \frac{bt^2}{2}$$

for $x_0 = 1, y_0 = 1$

$$x = 1 + at$$
$$y = 1 + \frac{bt^2}{2} \Rightarrow @t=2 \quad x = 1 + 2a, \quad y = 1 + 2b$$

5. **[3 points]** Answer the following questions:

   a. What is meant by the term dilatation rate?

   Rate of volumetric strain or rate of volume change

   b. Write down an equation that quantifies the rate of dilatation.

   $$\frac{\partial u_i}{\partial x_i} = \vec{\nabla} \cdot \vec{V} \text{ or } S_{ii}$$

   c. Compute the dilatation rate for the following velocity field:

   $$\vec{V} = x\hat{\imath} + (x^2y + x^2 - y)\hat{\jmath} - zx^2\hat{k}$$

   $$\frac{\partial u}{\partial x} = 1;\quad \frac{\partial v}{\partial y} = x^2 - 1;\quad \frac{\partial w}{\partial z} = -x^2$$

   $$\vec{\nabla} \cdot \vec{V} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 1 + x^2 - 1 - x^2 = 0$$

   $$\boxed{\vec{\nabla} \cdot \vec{V} = 0}$$

6. **[4 points]** Consider the vector: $\vec{A} = \nabla \phi$, where ϕ is a scalar potential function.
   use index notation to determine: $\nabla \cdot (\nabla \times \vec{A})$

7. [**3 points**] Write the following vector quantity in index notation and evaluate the result:

1

2

3

$$\nabla \times \vec{x} = \varepsilon_{ijk} \frac{\partial x_j}{\partial x_i}$$

$$= \varepsilon_{ij1} \frac{\partial x_j}{\partial x_i} + \varepsilon_{ij2} \frac{\partial x_j}{\partial x_i} + \varepsilon_{ij3} \frac{\partial x_j}{\partial x_i}$$

$$\boxed{\begin{aligned} &= \varepsilon_{231} \frac{\partial x_3}{\partial x_2} + \varepsilon_{321} \frac{\partial x_2}{\partial x_3} + \\ &\varepsilon_{132} \frac{\partial x_3}{\partial x_1} + \varepsilon_{312} \frac{\partial x_1}{\partial x_3} + \\ &\varepsilon_{123} \frac{\partial x_2}{\partial x_1} + \varepsilon_{213} \frac{\partial x_1}{\partial x_2} = 0 \end{aligned}}$$

all derivative are equal to zero. for example $x_3 \neq f(x_2)$

$$\therefore \frac{\partial x_2}{\partial x_3} = 0$$

8. [**4 points**] Consider Cauchy's equation written in index notation.

$$\frac{\partial \rho u_j}{\partial t} + \frac{\partial \rho u_j u_k}{\partial x_k} = \frac{\partial \tau_{ij}}{\partial x_i} + \rho f_j$$

Determine and write out the components of the vector:

$$\frac{\partial \tau_{ij}}{\partial x_i} = T_j$$

let T_j be a vector with the free index j.

3 components of vector $\vec{T}$

$$\begin{cases} T_1 = \frac{\partial \tau_{11}}{\partial x_1} + \frac{\partial \tau_{21}}{\partial x_2} + \frac{\partial \tau_{31}}{\partial x_3} \\ T_2 = \frac{\partial \tau_{12}}{\partial x_1} + \frac{\partial \tau_{22}}{\partial x_2} + \frac{\partial \tau_{32}}{\partial x_3} \\ T_3 = \frac{\partial \tau_{13}}{\partial x_1} + \frac{\partial \tau_{23}}{\partial x_2} + \frac{\partial \tau_{33}}{\partial x_3} \end{cases}$$

9. [**4 points**] List and explain the key assumptions required in the derivation of a constitutive equation (i.e., stress tensor closure) for a Newtonian fluid.

- $\sigma_{ij} \propto S_{ij}$, where σ_{ij} is the deviatoric stress tensor
- $\sigma_{ij} = K_{lmnp} S_{ij}$
- σ_{ij} is symmetric
- K is isotropic

10. [**6 points**] Consider conservation of mass:

a. Write the conservation of mass equation for a fluid from a Lagrangian description. What does it physically mean?

$$\frac{D M_{sys}}{Dt} = 0$$

- the mass of a collection of fluid particles being followed does not change in time.

b. Write the differential form of the conservation of mass equation for a fluid in the Eulerian view. Explain the physical meaning of each of the terms.

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u_j}{\partial x_j} = 0$$

$$\overbrace{\frac{\partial \rho}{\partial t} + u_j \frac{\partial \rho}{\partial x}}^{\frac{D\rho}{Dt}} + \rho \frac{\partial u_j}{\partial x_j} = 0$$

local time rate of change of the fluid density

Change in density due to the advection of mass by the bulk flow

Volumetric strain term. change in density due to the linear expansion or contraction of the fluid

11. [**15 points**] Consider a steady, stably stratified flow (i.e., the density increases with height, *z* coordinate) that is described by the following density and velocity fields:

$$\rho = \rho_0(1 - az)$$

$$u = x$$

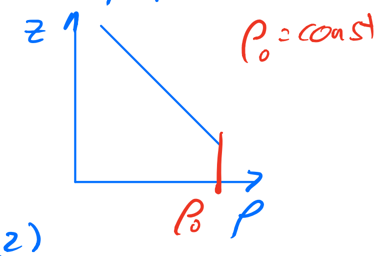
$$v = -y$$

Here, *a* is a positive constant and the vertical velocity has the boundary condition *w(z=0) = 0*.

a. [12 points] Determine the *w*-component of the velocity field.

b. [3 points] Is this this flow incompressible?

density profile



(a) Start with conservation of mass:

$$\cancel{\frac{\partial \rho}{\partial t}} + u\cancel{\frac{\partial \rho}{\partial x}} + v\cancel{\frac{\partial \rho}{\partial y}} + w\frac{\partial \rho}{\partial z} + \rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = 0 \quad (1)$$

Steady $\qquad \frac{\partial \rho}{\partial x} = 0 \qquad \frac{\partial \rho}{\partial y} = 0$

$$w\frac{\partial \rho}{\partial z} + \rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = 0 \quad (2)$$

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(x) = 1; \quad \frac{\partial v}{\partial y} = \frac{\partial}{\partial y}(-y) = -1 \quad (3)$$

$$w\frac{\partial \rho}{\partial z} + \rho\left(1 - 1 + \frac{\partial w}{\partial z}\right) = 0$$

- Eq (2) Simplifies to

$$w\frac{\partial \rho}{\partial z} + \rho\frac{\partial w}{\partial z} = 0 \quad (4)$$

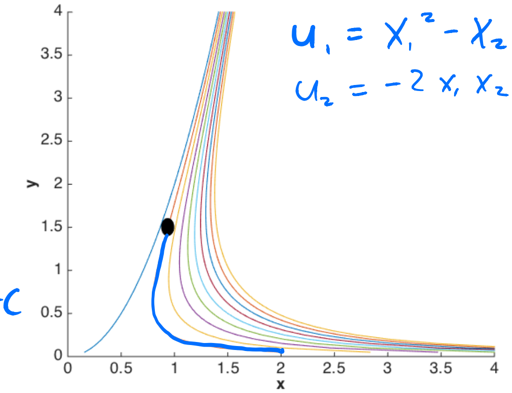
Note that $\frac{\partial(\rho w)}{\partial z} = \rho\frac{\partial w}{\partial z} + w\frac{\partial \rho}{\partial z} = 0$

if $\frac{\partial(\rho w)}{\partial z} = 0$

$$\rho w = C$$

$$\boxed{w = \frac{C}{\rho_0(1 - az)}}$$

12. [**20 points**] Consider the steady 2D flow associated with the streamlines shown in the figure. The horizontal and vertical velocity components associated with the flow are: $u = x^2 - y$ and $v = -2xy$



$u_1 = x_1^2 - x_2$

$u_2 = -2x_1 x_2$

a. Determine the equation of the stream function

$u_1 = \frac{\partial \psi}{\partial x_2} = x_1^2 - x_2 \qquad u_2 = -\frac{\partial \psi}{\partial x_1} = -(-2x_1 x_2)$

$\int \partial \psi = \int x_1^2 - x_2 \, dx_2 \qquad \int \partial \psi = 2x_1 x_2 \, dx_1$

$\psi = x_1^2 x_2 - \frac{x_2^2}{2} + f(x_1) \qquad \psi = \frac{2x_1^2}{2} x_2 + f(x_2) + C$

$$\boxed{\psi = x_1^2 x_2 - \frac{x_2^2}{2}}$$

b. Is the flow incompressible? Show.

if $\frac{\partial u_i}{\partial x} = 0$

$\left.\begin{aligned} \frac{\partial u_1}{\partial x_1} &= 2x_1 \\ \frac{\partial u_2}{\partial x_2} &= -2x_1 \end{aligned}\right\} \quad \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} = \boxed{2x_1 - 2x_1 = 0 \text{ Yes!}}$

c. Determine the components of the strain-rate tensor, S_{ij}

$S_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$

$S_{ij} = \begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{bmatrix}$

$S_{11} = \frac{\partial u_1}{\partial x_1} = 2x_1$

$S_{22} = \frac{\partial u_2}{\partial x_2} = -2x_1$

$S_{33} = \frac{\partial u_3}{\partial x_3} = 0$

$S_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right)$

$= \frac{1}{2}(-1 - 2x_2) = (-\frac{1}{2} - x_2)$

$S_{13} = 0$

$S_{21} = S_{12}$

$S_{23} = \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\right)$

$S_{23} = 0$

$$S_{ij} = \begin{bmatrix} 2x_1 & -\frac{1}{2} - x_2 & 0 \\ -\frac{1}{2} - x_2 & -2x_1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

d. Determine the components of the vorticity vector, ω_i

$$\vec{\nabla} \times \vec{v} = \vec{\omega} \quad = \quad \omega_i = \varepsilon_{ijk} \frac{\partial u_j}{\partial x_i}$$

$$\omega_1 = \frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3} = 0$$

$$\omega_2 = \frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1} = 0$$

$$\omega_3 = \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}$$

$$\boxed{\omega_3 = -2x_2 + 1}$$

e. On the figure, draw the pathline of the fluid particle that originates at the point indicated in the figure at time $t = 0$.

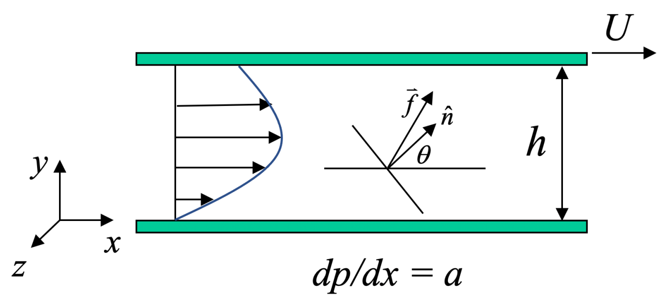
Flow is steady so it follows the streamline as indicated in the sketch.

13. [**10 points**] Consider fully developed, incompressible, laminar flow of a Newtonian fluid (with dynamic viscosity μ and density ρ) between two infinite plates with the upper plate moving at a velocity, U and the horizontal plate stationary. Furthermore, there is a pressure gradient in streamwise direction given by $dp/dx = a$. The solution to the velocity field is given by the following equation:

See lecture 14-15

$$u(y) = \frac{a}{2\mu}(y^2 - yh) + \frac{Uy}{h}$$

Find the magnitude and direction of the force due to viscous shear stresses per unit area on an element in the fluid whose outward pointing normal points a 35° as shown in the figure.



$\theta = 35°$

Since the flow is incompressible, the stress tensor is given by:

$$\tau_{ij} = \overbrace{-p\delta_{ij}}^{①} + \overbrace{\mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)}^{②}$$

the force due to the viscous shear stresses, is the deviatoric part of τ_{ij} or term ②.

$$\text{term } 2 = 2\mu S_{ij}$$

Compute S_{ij} terms: $u_1 = \frac{a}{2\mu}(x_2^2 - x_2 h) + \frac{U x_2}{h}$

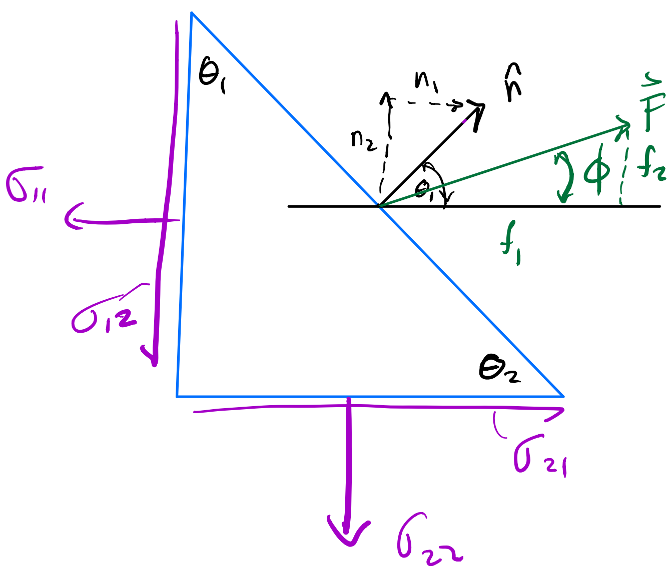
$$S_{11} = \frac{\partial u_1}{\partial x_1} = 0$$

$$S_{22} = \frac{\partial u_2}{\partial x_2} = 0 \quad (u_2 = 0)$$

$$S_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) = \frac{1}{2}\left(\frac{a}{2\mu}(2x_2 - h) + \frac{U}{h}\right) = S_{21}$$

$$2\mu S_{ij} = \begin{bmatrix} 0 & \frac{9}{2}(2x_2 - h) + \frac{U\mu}{h} & 0 \\ \frac{9}{2}(2x_2 - h) + \frac{U\mu}{h} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

This is <u>only</u> a 2D problem



call $\sigma_{ij} = 2\mu S_{ij}$

Force per unit are:

$$f_i = \sigma_{ij} n_j$$

$$f_1 = \cancelto{0}{\sigma_{11}} n_1 + \sigma_{12} n_2$$

$$f_2 = \sigma_{21} n_1 + \cancelto{0}{\sigma_{22}} n_2$$

$$n_1 = \cos\theta_1 = \cos(35) = 0.8192$$

$$n_2 = \cos\theta_2 = \cos(55) = 0.5736$$

$$f_1 = n_2 \sigma_{12} = 0.5736\left(\frac{9}{2}(2x_2 - h) + \frac{U\mu}{h}\right)$$

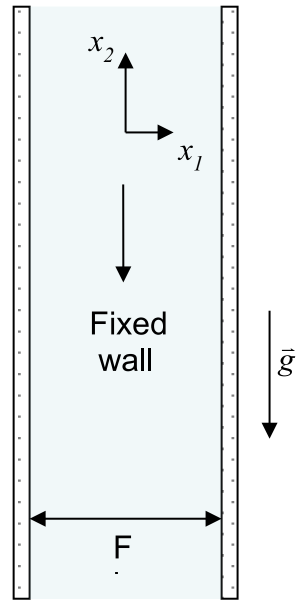
$$f_2 = n_1 \sigma_{21} = 0.8192\left(\frac{9}{2}(2x_2 - h) + \frac{U\mu}{h}\right)$$

$$|f| = \left(f_1^2 + f^2\right)^{1/2}$$

$$\sin\phi = \frac{f_2}{|f|} \qquad \cos\phi = \frac{f_1}{|f|}$$

14. [**25 points**] Consider steady, incompressible, parallel, laminar flow of a viscous fluid falling between two infinite vertical walls (see adjacent figure). There is no applied pressure force driving the flow (i.e., the fluid falls by gravity alone). The pressure is constant everywhere in the flow field.

   a. [10 points] Start with the appropriate equations of motion and simplify; state all assumptions and identify how the assumptions cancel out particular terms.
   b. [5 points] What is the resulting force balance?
   c. [10 points] Calculate the velocity field for u_2.



Assumptions

1. laminar — ½ pt each
2. steady
3. incompressible
4. 2D ($u_3 = 0$) $\left\{\frac{\partial()}{\partial x_3} = 0\right.$
5. parallel flow ($u_1 = 0$)
6. $\frac{\partial p}{\partial x_i} = 0$ (no external pressure grad)

a

Continuity Equation — 2pt

$$(1)\quad \frac{\partial u_i}{\partial x_i} = 0 = \underbrace{\cancel{\frac{\partial u_1}{\partial x_1}}}_{(5)} + \frac{\partial u_2}{\partial x_2} + \overbrace{\cancel{\frac{\partial u_3}{\partial x_3}}}^{(4)} = 0 \Rightarrow \frac{\partial u_2}{\partial x_2} = 0,$$

this indicates the flow is fully developed

X_2-momentum eq. — 3 ½ pts

$$(2)\quad \underbrace{\cancel{\frac{\partial u_2}{\partial t}}}_{(2)} + u_1\underbrace{\cancel{\frac{\partial u_2}{\partial x_1}}}_{(5)} + u_2\underbrace{\cancel{\frac{\partial u_2}{\partial x_2}}}_{f.D.} + u_3\overbrace{\cancel{\frac{\partial u_2}{\partial x_3}}}^{(4)} = -\overbrace{\cancel{\frac{1}{\rho}\frac{\partial p}{\partial x_2}}}^{(6)} + g_2 + \nu\left(\frac{\partial^2 u_2}{\partial x_1^2} + \overbrace{\cancel{\frac{\partial^2 u_2}{\partial x_2^2}}}^{f.D.} + \overbrace{\cancel{\frac{\partial^2 u_2}{\partial x_3^2}}}^{(4)}\right)$$

based on coord. system $g_2 = -g$ ½ pt

$$\therefore (2) \text{ becomes } \boxed{\nu \frac{d^2 u_2}{dx_1^2} = \rho g} \quad (3)$$

1 pt

(b) Balance of forces:

2.5pts 2.5pts

Buoyancy and viscous shear

(c) Calculate the velocity field $u_2(x_1)$

Integrate (3) twice

$$\frac{d}{dx_1}\left(\frac{du_2}{dx_1}\right) = \rho g$$

$$\int \frac{du_2}{dx_1} dy = \int \rho g \, dx_1$$

2pt

$$\int \left[\frac{du_2}{dx_1} = \frac{\rho}{\mu} g x_1 + C_1\right] dx_1$$

2pts $$u_2(x_1) = \frac{\rho}{\mu} g \frac{x^2}{2} + C_1 x_1 + C_2 \quad (4)$$

Apply B.C.'s to Eq. (4)

$$u_2\left(\frac{F}{2}\right) = 0 \;\{\; u_2\left(-\frac{F}{2}\right) = 0$$ 2pts

$$u_2\left(\frac{F}{2}\right) = 0 = \frac{\rho g}{2\mu}(F/2)^2 + C_1\left(\frac{F}{2}\right) + C_2 = 0$$

$$u_2\left(-\frac{F}{2}\right) = 0 = \frac{\rho g}{2\mu}(F/2)^2 + C_1\left(-\frac{F}{2}\right) + C_2 = 0$$

$$C_1 F = 0$$

$$C_1 = 0$$ 1pt

$$\frac{\rho g}{\mu}\frac{F^2}{8} + C_2 = 0$$ 1pt

$$C_2 = -\frac{\rho g}{\mu}\frac{F^2}{8}$$

$$\boxed{u_2(x_1) = \frac{\rho g}{2\mu}\left(x^2 - \frac{F^2}{4}\right)}$$ 2pts