Chapter 9

Nonlinear Systems

Problems and Solutions for Section 9.2: Analysis By Linearization

1. An electric field transducer that can be used as a microphone or a speaker can be described by the following equations:

   Mechanical:
$$M\frac{d^2x}{dt^2} + B\frac{dx}{dt} + K(x-\ell) + \frac{\epsilon_o}{2}A\left(\frac{v}{x}\right)^2 = f(t) \tag{1}$$

   Electrical:
$$\frac{d}{dt}\left(\frac{\epsilon_o}{x}Av\right) = \frac{v_s - v}{R} \tag{2}$$

   where

   M = mass of the moving plate

   B = damping coefficient

   K = spring constant

   x = separation of plates

   ℓ = natural length of spring

   A = area of plate

   v = voltage across plates

   f = force on plate (input)

   V_s = bias voltage

   R = resistance

   The output is
$$y = -R\frac{d}{dt}\left(\frac{\epsilon_o Av}{x}\right) \tag{3}$$

   (a) Using the state variables $x_1 = x$, $x_2 = \dot{x}$, $x_3 = \frac{v}{x}$, write the equations in state variable form.

   (b) Find the equations which describe equilibrium points for x and v if $f(t) = 0$.

(c) Let the equilibrium points be X and V, find a set of linearized equations about X and V.

Solution:

(a) $x_1 = x$

$x_2 = \dot{x}$

$x_3 = \frac{v}{x}$

$\dot{x}_1 = \dot{x} = x_2$

$\dot{x}_2 = \ddot{x} = -\frac{B}{M}x_2 - \frac{K(x_1-\ell)}{M} - \frac{\epsilon_o A}{2M}(x_3)^2 + f(t)$

$\dot{x_3} = \frac{d}{dt}((\frac{v}{x})) = \frac{v_s - x_3 x_1}{RA\epsilon_o}$

$v = x_3 x = x_3 x_1$

(b)

$\dot{x_1} = x_2$

$\dot{x_2} = -\frac{K}{M}x_1 - \frac{B}{M}x_2 - \frac{\epsilon_o A}{2M}(x_3)^2 + \frac{K\ell}{M} + f(t)$

$\dot{x_3} = -\frac{x_1 x_3}{RA\epsilon_o} + \frac{v_s}{RA\epsilon_o}$

$Y = -R\epsilon_o A\frac{d}{dt}((\frac{v}{x})) = -R\epsilon_o A\frac{d}{dt}(x_3) = x_1 x_3 - v_s$

$\dot{x}_1 * = 0$

$\dot{x_1}^* = 0$

$-\frac{K}{M}x_1 * - \frac{B}{M}x_2 * - \frac{A\epsilon_o}{2M}(x_3 *)^2 + \frac{K\ell}{M} = 0$

$-\frac{x_1 * x_3 *}{R\epsilon_o A} + \frac{v_s}{R\epsilon_o A} = 0$

$v^* = x_1 * x_3 *$

$$\frac{\partial f}{\partial \mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 \\ -\frac{K}{M} & -\frac{B}{M} & \frac{\epsilon_o A}{M}x_3 \\ \frac{-x_3}{R\epsilon_o A} & 0 & \frac{-x_1}{R\epsilon_o A} \end{bmatrix} \tag{4}$$

$$\frac{\partial f}{\partial \mathbf{x}}(x^*, u) = \begin{bmatrix} 0 & 1 & 0 \\ -\frac{K}{M} & -\frac{B}{M} & \frac{\epsilon_o A}{M}X \\ \frac{-X}{R\epsilon_o A} & 0 & \frac{-X}{R\epsilon_o A} \end{bmatrix} \tag{5}$$

$$\frac{\partial f}{\partial u} = \begin{bmatrix} 0 \\ 0 \\ \frac{-1}{RA\epsilon_0} \end{bmatrix} \tag{6}$$

2. Figure 9.57 shows a simple pendulum system in which a cord is wrapped around a fixed cylinder. The motion of the system that results is described by the differential equation

$$(l + R\theta)\ddot{\theta} + g\sin\theta + R\dot{\theta}^2 = 0,$$

where

$$l = \text{length of the cord in the vertical (down) position,}$$
$$R = \text{radius of the cylinder.}$$

(a) Write the state-variable equations for this system.

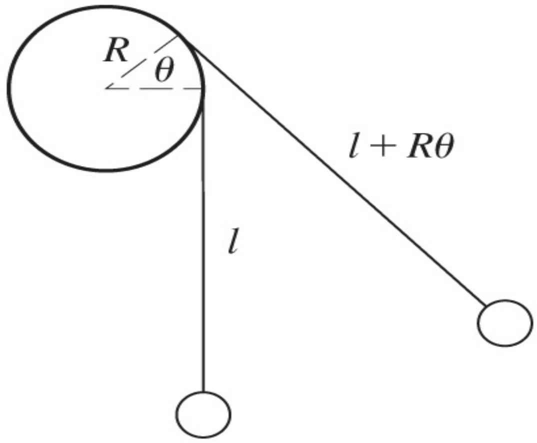


Figure 9.57: Motion of cord wrapped around a fixed cylinder

(b) Linearize the equation around the point $\theta = 0$, and show that for small values of θ the system equation reduces to an equation for a simple pendulum, thatis,

$$\ddot{\theta} + (g/l)\theta = 0.$$

Solution:

(a) This is a second order non-linear differential equation in θ . Let $\mathbf{x} = \begin{bmatrix} \dot{\theta} & \theta \end{bmatrix}^T$.

$$\begin{aligned} \dot{x}_1 &= \ddot{\theta} = -\frac{R\dot{\theta}^2 + g\sin\theta}{(l + R\theta)} = -\frac{Rx_1^2 + g\sin x_2}{(l + Rx_2)}, \\ \dot{x}_2 &= \dot{\theta} = x_1. \end{aligned}$$

(b) For small values of θ .

$$\begin{aligned} (l + R\theta) &\cong l, \\ \sin\theta &\cong \theta, \\ \dot{\theta}^2 &\cong 0. \end{aligned}$$

(a)

$$\begin{aligned} l\ddot{\theta} + g\theta &= 0 \\ \ddot{\theta} + \frac{g}{l}\theta &= 0 \end{aligned}$$

3. The circuit shown in Fig. 9.58 has a nonlinear conductance G such that $i_G = g(v_G) = v_G(v_G - 1)(v_G - 4)$. The state differential equations are

$$\begin{aligned} \frac{di}{dt} &= -i + v, \\ \frac{dv}{dt} &= -i + g(u - v), \end{aligned}$$

where i and v are the state variables and u is the input.

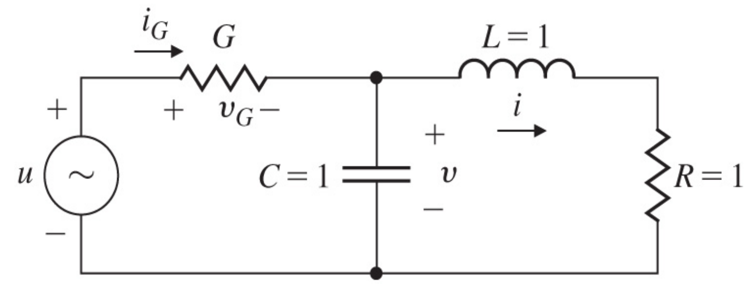


Figure 9.58: Nonlinear circuit for Problem 9.3

(a) One equilibrium state occurs when $u=1$ yielding $i_1=v_1=0$. Find the other two pairs of v and i that will produce equilibrium.

(b) Find the linearized model of the system about the equilibrium point $u=1$, $i=v_1=0$.

(c) Find the linearized models about the other two equilibrium points.

Solution:

(a) Equilibrium:

$$\begin{aligned}\frac{di}{dt} &= -i+v=0,\\ \frac{dv}{dt} &= -i+g(u-v)=0,\end{aligned}$$

$$g(u-v)-i=(u-v)\left[(u-v)-1\right]\left[(u-v)-4\right]-v=0,$$

as $u=1$,

$$(1-v)(-v)(-3-v)-v=v\left(v^2+2v-2\right)=0.$$

$$v=0,-1\pm\sqrt{3}.$$

So,

$$i=v=0,-1\pm\sqrt{3}.$$

(b) Let's replace u , v , and i by $1+\delta u$, δv , and δi .

$$\begin{aligned}\delta\dot{i} &= -\delta i+\delta v,\\ \delta\dot{v} &= -\delta i+g(1+\delta u-\delta v),\\ &= -\delta i+(1+\delta u-\delta v)((1+\delta u-\delta v)-1)((1+\delta u-\delta v)-4),\\ &= -\delta i+(1+\delta u-\delta v)\left(\delta u-\delta v\right)\left(\delta u-\delta v-3\right),\\ &\cong -\delta i-3\delta u+3\delta v.\end{aligned}$$

$$\frac{d}{dt}\begin{bmatrix}\delta i\\ \delta v\end{bmatrix}=\begin{bmatrix}-1 & 1\\ -1 & 3\end{bmatrix}\begin{bmatrix}\delta i\\ \delta v\end{bmatrix}+\begin{bmatrix}0\\ -3\end{bmatrix}\delta u.$$

(c) In general the linearized form will be,

$$\frac{d}{dt}\begin{bmatrix}\delta i\\ \delta v\end{bmatrix}=\begin{bmatrix}-1 & 1\\ -1 & \frac{\partial g}{\partial v}\end{bmatrix}\begin{bmatrix}\delta i\\ \delta v\end{bmatrix}+\begin{bmatrix}0\\ \frac{\partial g}{\partial u}\end{bmatrix}\delta u,$$

As $u=1$,

$$\begin{aligned} g(u,v) &= g(1,v)=v\left(v^2+2v-2\right),\\ \frac{\partial g}{\partial v} &= \left(v^2+2v-2\right)+v\left(2v+2\right),\\ &= 5\mp 2\sqrt{3}\text{ when }v=-1\pm\sqrt{3}. \end{aligned}$$

Also

$$\frac{\partial g\left(u-v\right)}{\partial v}=-g'\left(u-v\right),$$

$$\frac{\partial g\left(u-v\right)}{\partial u}=g'\left(u-v\right)=-\frac{\partial g}{\partial v}=-5\pm 2\sqrt{3}.$$

So,

$$\frac{d}{dt}\begin{bmatrix}\delta i\\ \delta v\end{bmatrix}=\begin{bmatrix}-1 & 1\\ -1 & 5\mp 2\sqrt{3}\end{bmatrix}\begin{bmatrix}\delta i\\ \delta v\end{bmatrix}+\begin{bmatrix}0\\ -5\pm 2\sqrt{3}\end{bmatrix}\delta u.$$

4. Consider the circuit shown in Fig. 9.59; u_1 and u_2 are voltage and current sources, respectively, and R_1 and R_2 are nonlinear resistors with the following characteristics:

$$\begin{aligned} \text{Resistor 1}: i_1 &= G(v_1)=v_1^3\\ \text{Resistor 2}: v_2 &= r(i_2), \end{aligned}$$

where the function r is defined in Fig. 9.60.

(a) Show that the circuit equations can be written as

$$\begin{aligned} \dot{x}_1 &= G(u_1-x_1)+u_2-x_3\\ \dot{x}_2 &= x_3\\ \dot{x}_3 &= x_1-x_2-r(x_3). \end{aligned}$$

Suppose we have a constant voltage source of 1Volt at u_1 and a constant current source of 27 Amps; i.e., $u_1^{\circ}=1$, $u_2^{\circ}=27$. Find the *equilibrium state* $\mathbf{x}^{\circ}=[x_1^{\circ}, x_2^{\circ}, x_3^{\circ}]^T$ for the circuit. For a particular input u°, an equilibrium state of the system is defined to be any constant state vector whose elements satisfy the relation

$$\dot{x}_1=\dot{x}_2=\dot{x}_3=0.$$

Consequently, any system started in one of its equilibrium states will remain there indefinitely until a different input is applied.

(b) Due to disturbances, the initial state (capacitance, voltages, and inductor current) is slightly different from the equilibrium and so are the independent sources; that is,

$$\begin{aligned} u(t) &= u^{\circ}+\delta u(t)\\ x(t_0) &= x^{\circ}(t_0)+\delta x(t_0). \end{aligned}$$

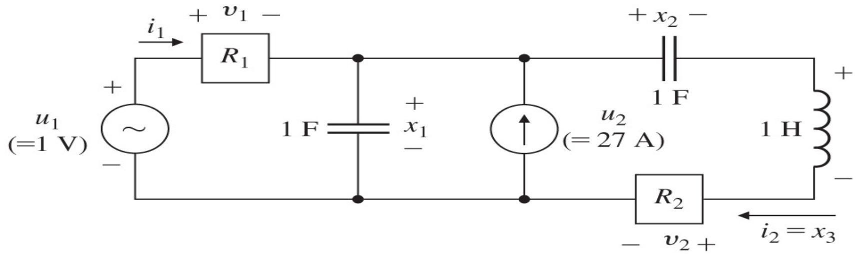


Figure 9.59: A nonlinear circuit

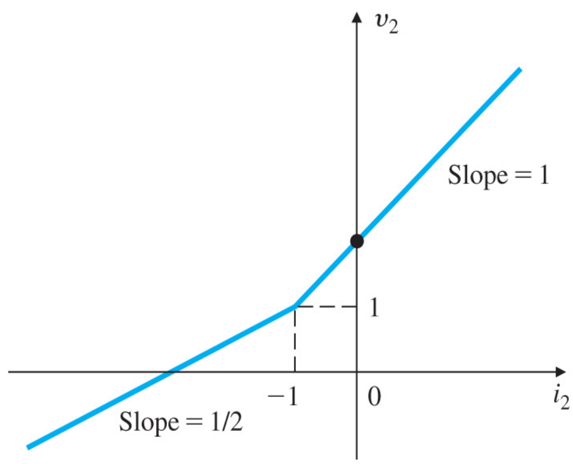


Figure 9.60: Nonlinear resistance

Do a small-signal analysis of the network about the equilibrium found in (a), displaying the equations in the form

$$\delta\dot{x}_1 = f_{11}\ \delta x_1 + f_{12}\ \delta x_2 + f_{13}\ \delta x_3 + g_1\ \delta u_1 + g_2\ \delta u_2.$$

(c) Give the values of the elements.

Solution:

(a)

$$\begin{aligned}
i_1 &= G\left(u_1 - x_1\right), \\
i_1 - i_2 + u_2 &= C\frac{d}{dt}x_1, \\
i_2 &= C\frac{d}{dt}x_2 = x_3, \\
v_2 &= r\left(i_2\right), \\
0 - x_2 + x_1 - v_2 &= L\frac{d}{dt}i_2.
\end{aligned}$$

$$
\begin{aligned}
C\dot{x}_1 &= G\left(u_1 - x_1\right) - x_3 + u_2, \\
C\dot{x}_2 &= x_3, \\
L\dot{x}_3 &= x_1 - x_2 - r\left(x_3\right).
\end{aligned}
$$

As $C = L = 1$,

$$
\begin{aligned}
\dot{x}_1 &= G\left(u_1 - x_1\right) - x_3 + u_2, \\
\dot{x}_2 &= x_3, \\
\dot{x}_3 &= x_1 - x_2 - r\left(x_3\right).
\end{aligned}
$$

(b) Equilibrium state around $u_1^0 = 1$, $u_2^0 = 27$.

$$
\begin{aligned}
0 &= G\left(u_1^0 - x_1^0\right) - x_3^0 + u_2^0, \\
&= G\left(1 - x_1^0\right) - x_3^0 + 27, \\
0 &= x_3^0, \\
0 &= x_1^0 - x_2^0 - r\left(0\right).
\end{aligned}
$$

$$
\begin{aligned}
G\left(1 - x_1^0\right) - x_3^0 + 27 &= \left(1 - x_1^0\right)^3 + 27 = 0, \\
x_1^0 &= 4, \\
x_2^0 &= x_1^0 - r\left(0\right) = 4 - 2 = 2, \\
x_3^0 &= 0.
\end{aligned}
$$

$$
\mathbf{x}^0 = \left[\begin{array}{ccc} 4 & 2 & 0 \end{array}\right]^T.
$$

$$
\begin{aligned}
\delta\dot{x}_1 &= G\left(\left(u_1^0 + \delta u_1\right) - \left(x_1^0 + \delta x_1\right)\right) - \left(x_3^0 + \delta x_3\right) + \left(u_2^0 + \delta u_2\right), \\
&= \left(-3 + \delta u_1 - \delta x_1\right)^3 - \delta x_3 + 27 + \delta u_2, \\
&\cong -27 + 3 \times 9\left(\delta u_1 - \delta x_1\right) - \delta x_3 + 27 + \delta u_2, \\
&= -27\delta x_1 - \delta x_3 + 27\delta u_1 + \delta u_2. \\
\delta\dot{x}_2 &= x_3^0 + \delta x_3, \\
&= \delta x_3. \\
\delta\dot{x}_3 &= \left(x_1^0 + \delta x_1\right) - \left(x_2^0 + \delta x_2\right) - r\left(x_3^0 + \delta x_3\right), \\
&= 4 + \delta x_1 - 2 - \delta x_2 - \left(\delta x_3 + 2\right), \\
&= \delta x_1 - \delta x_2 - \delta x_3.
\end{aligned}
$$

(c) Circuit diagram: Linear circuit model for Problem 9.4.
$L = 1\text{H}$, $R_1 = \frac{1}{27}\Omega$, $R_2 = 1\Omega$.

5. Consider the nonlinear system

$$\dot{x} = -x^2 e^{-\frac{1}{x}} + \sin u x(0) = 1$$

a) Assume $u^{\,o} = 0$ and solve for $x^{\,o}(t)$.

b) Find the linearized model about the nominal solution in part (a).

Solution:

(a)

$$\begin{aligned} \dot{x}(t) &= -x^2 e^{-\frac{1}{x}}, \\ \frac{dx}{dt} &= -x^2 e^{-\frac{1}{x}}, \\ -x^{-2} e^{\frac{1}{x}} dx &= dt. \end{aligned}$$

Integrate both sides:

$$\begin{aligned} \int_{x(0)}^{x(t)} -\frac{1}{x^2} e^{\frac{1}{x}} dx &= \int_0^t dt, \\ e^{\frac{1}{x(t)}} - e^{\frac{1}{1}} &= t, \\ x_o(t) &= \frac{1}{\log(t+e)}. \end{aligned}$$

(b)

$$\begin{aligned} \dot{x} &= -x^2 e^{-\frac{1}{x}} + \sin u, \\ \delta\dot{x} &= -\left[2xe^{-\frac{1}{x}} + e^{-\frac{1}{x}}\right] \delta x + \cos u \, \delta u \, . \end{aligned}$$

For the equilibrium conditions of part (a),

$$\delta\dot{x} = -\frac{1}{t+e}\left[1 + \frac{2}{\log(t+e)}\right] \delta x + \delta u \, .$$

6. *Linearizing effect of feedback.* We have seen that feedback can reduce the sensitivity of the input-output transfer function with respect to changes in the plant transfer function , and reduce the effects of a disturbance acting on the plant. In this problem we explore another beneficial property of feedback: it can make the input-output response *more linear* than the open-loop response of the plant alone. For simplicity let us ignore all the dynamics of the plant, and assume that the plant is described by the static nonlinearity

$$y(t) = \begin{cases} u & u \leq 1 \\ \frac{u+1}{2} & u > 1 \end{cases}$$

a) Suppose we use proportional feedback

$$u(t) = r(t) + \alpha(r(t) - y(t))$$

where $\alpha \geq 0$ is the feedback gain. Find an expression for $y(t)$ as a function of $r(t)$ for the closed-loop system (This function is called the *nonlinear characteristic* of the system.) Sketch the nonlinear transfer characteristic for $\alpha = 0$ (which is really open-loop), $\alpha = 1$, and $\alpha = 2$.

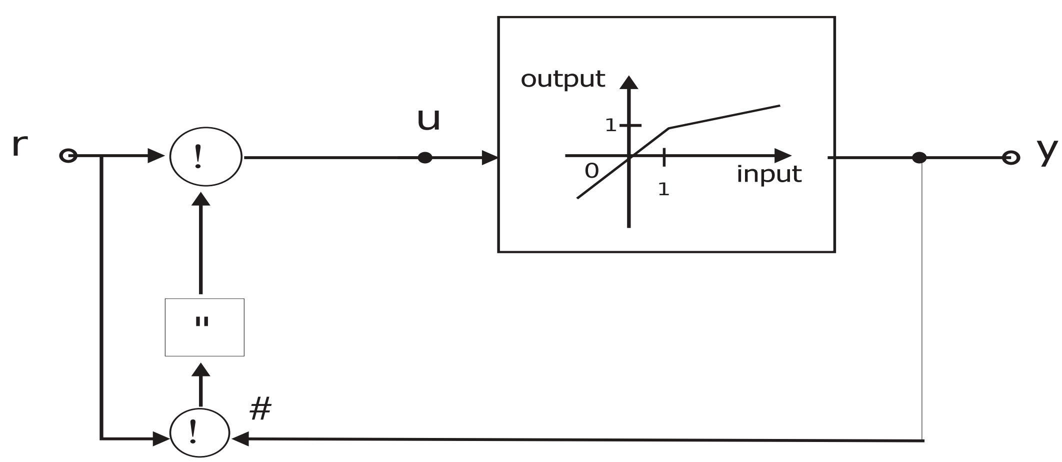
b) Suppose we use integral control,

$$u(t) = r(t) + \int_0^t (r(\tau) - y(\tau))d\tau$$

The closed-loop system is therefore nonlinear and dynamic. Show that if $r(t)$ is a constant, say r , then $\lim_{t\to\infty} y(t) = r$. Thus, the integral control makes the steady-state transfer characteristic of the closed-loop system *exactly linear*. Can the closed-loop system be described by a transfer function from r to y ?

Solution:

(a)



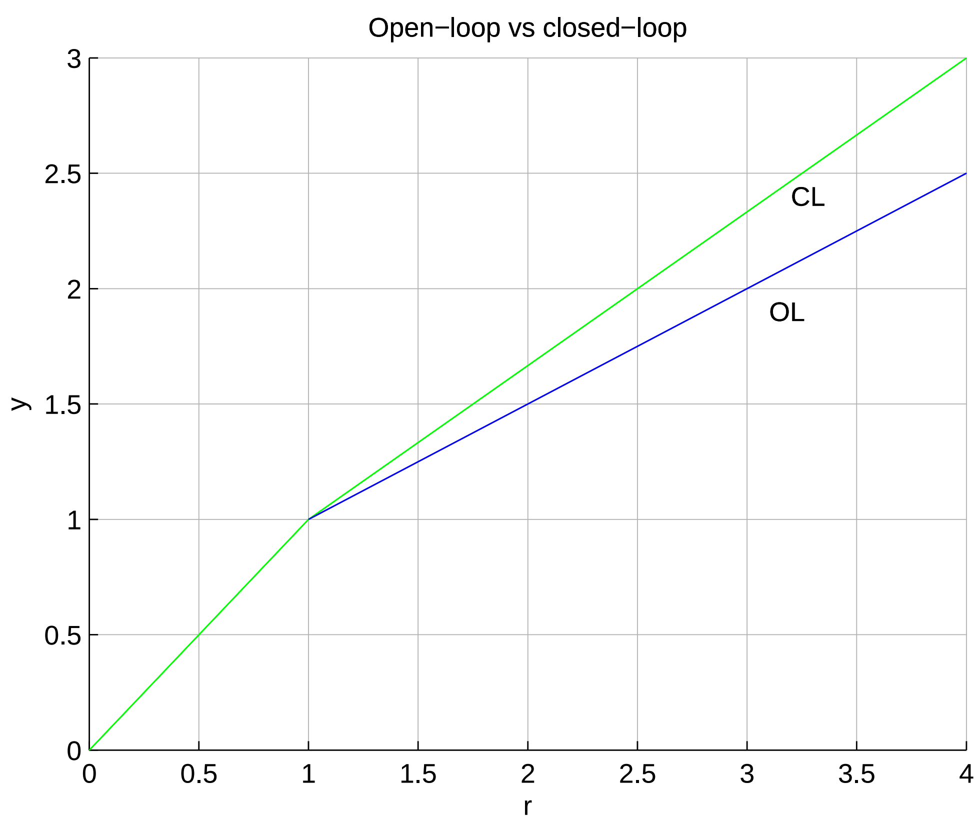
Problem 9.6. Nonlinear system with saturation: proportional control.

For $u \leq 1$

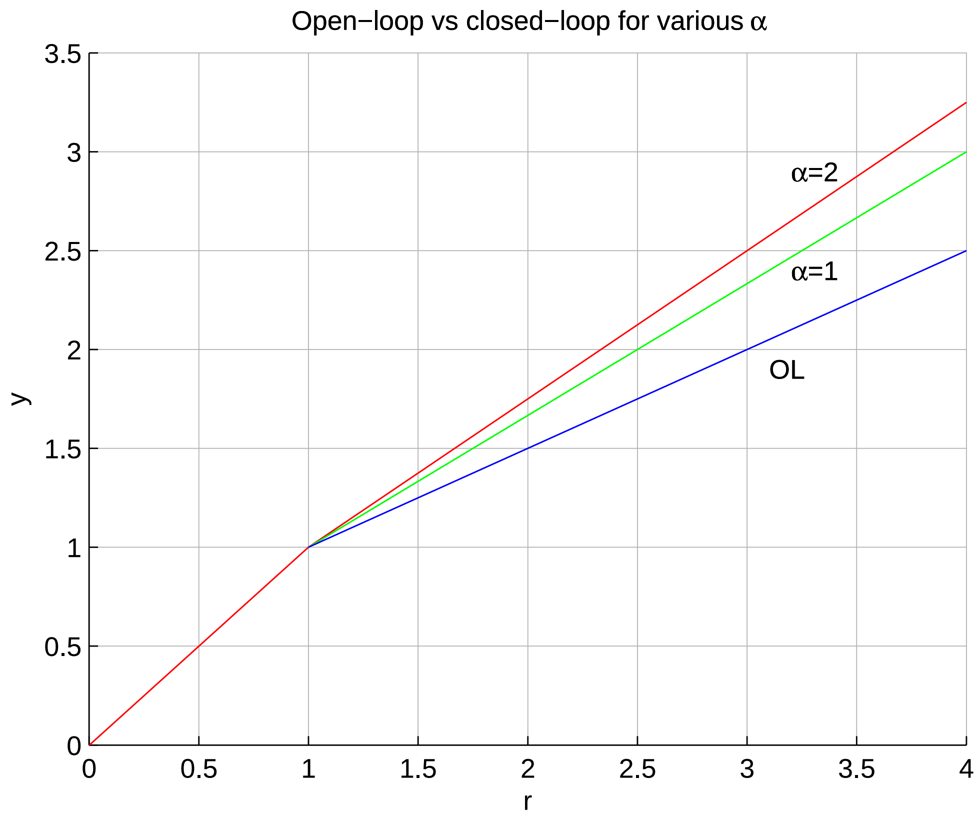
$$y = r + \alpha(r - y) = (1 + \alpha)y = (1 + \alpha)r;\ y = r = u.$$

For $u > 1$:

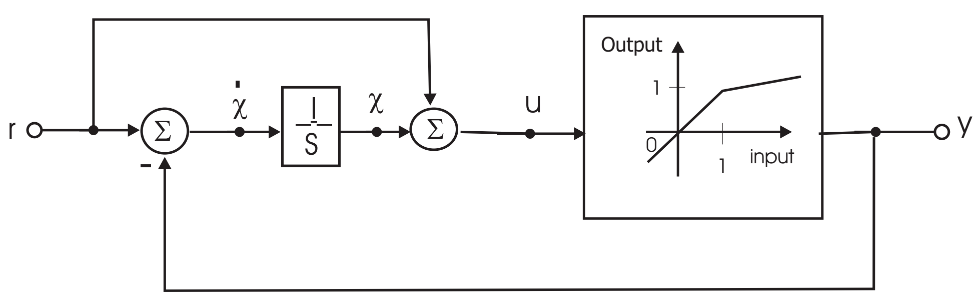
$$\begin{aligned} y &= \frac{u+1}{2} = \frac{1}{2} + \frac{1}{2}[r + \alpha(r-y)], \\ 2y &= 1 + r + \alpha r - \alpha y, \\ (2+\alpha)y &= 1 + (1+\alpha)r, \\ y &= \frac{1+(1+\alpha)r}{2+\alpha}. \quad \text{See Figure on top of the next page.} \\ \text{if } \alpha &= 0 \text{ then } \quad y = \frac{1+r}{2}. \\ \text{if } \alpha &= 1 \text{ then } \quad y = \frac{1}{3} + \frac{2}{3}r. \\ \text{if } \alpha &= 2 \quad \text{then} \quad y = \frac{1}{4} + \frac{3}{4}r. \quad \text{See middle Figure on the next page.} \end{aligned}$$



Problem 9.6. Open-loop vs closed-loop response.



Problem 9.6. Open-loop vs closed-loop response for various values of α.



(b) Problem 9.6. Nonlinear system with saturation: integral control.

$$u = r + \int_0^t (r - y)dt, \quad u < 1 \ ,$$
$$r = \text{ cons tan t} \ ,$$

$$\begin{aligned} y &= r + \int_0^t (r - y)dt, \\ Y &= R + \frac{R - Y}{s}, \\ (1 + s)Y &= (1 + s)R; Y = R \end{aligned}$$

Assume stable: y stays bounded,

$$\begin{aligned} y &\to y_\infty, \\ y &= r + \int_0^t (r - y_\infty)dt \to \infty \text{ if } y_\infty \neq r, \end{aligned}$$

$$\begin{aligned} &\Rightarrow y_\infty = r, \\ \dot{y} &= r - y, \\ \dot{y} + y &= r. \end{aligned}$$

$u > 1$,

$$\begin{aligned} y &= \frac{1}{2} + \frac{1}{2}(r + \int_0^t (r - y)dt, \\ y &= f(u), \\ \dot{x} &= r - y, \\ \dot{y} &= \frac{1}{2}(r - y), \\ 2\dot{y} + y &= r, \\ \frac{Y(s)}{R(s)} &= \frac{1}{2s+1}. \end{aligned}$$

In the steady-state: $y = r$.

7. This problem shows that linearization does not always work. Consider the system

$$\dot{x} = \alpha x^3 x(0) \neq 0$$

a) Find the equilibrium point and solve for $x(t)$.

b) Assume $\alpha = 1$. Is the linearized model a valid representation of the system?

c) Assume $\alpha = -1$. Is the linearized model a valid representation of the system?

Solution: (a) The equilibrium point is found from:

$$\begin{aligned} \dot{x} &= \alpha x^3 = 0, \\ &\Rightarrow x_e = 0. \end{aligned}$$

To determine $x(t)$ we re-write the system equation as,

$$\frac{dx}{x^3} = \alpha dt,$$

Integrating both sides:

$$\begin{aligned}
\int_{x(0)}^{x(t)} \frac{dx}{x^3} &= \alpha \int_0^t dt, \\
-\frac{1}{2}x^{-2}|_{x(0)}^{x(t)} &= \alpha t, \\
x^2 &= \frac{x(0)^2}{1-2\alpha x(0)^2 t}, \\
x(t) &= \frac{1}{\sqrt{x(0)^{-2}-2\alpha t}}.(*)
\end{aligned}$$

(b) If $\alpha = 1$ the linearized system is,

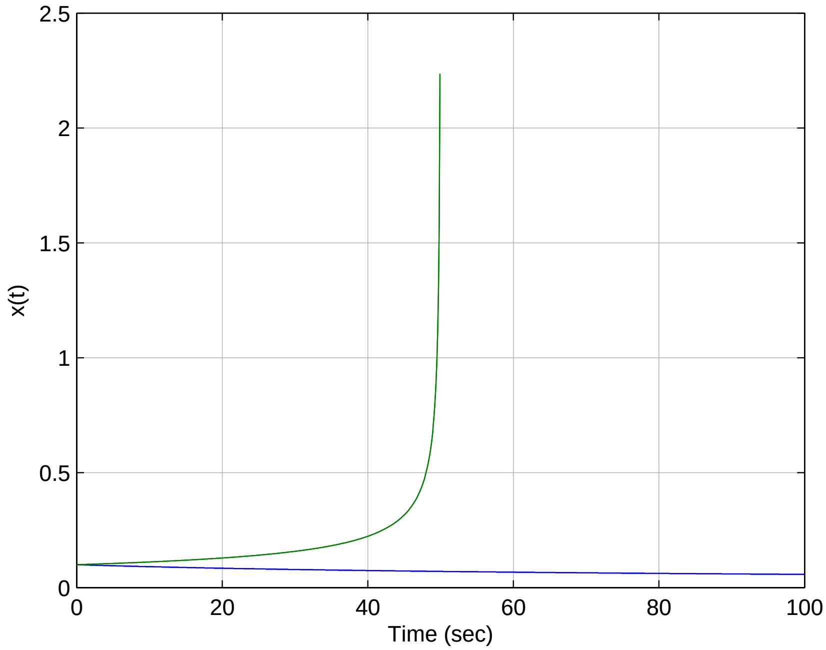
$$\begin{aligned}
\delta\dot{x} &= 3(1)x_e^2 = 0, \\
\delta x &= \text{cons tan t},
\end{aligned}$$

that is the linear system is not asymptotically stable (it is neutrally stable). However, we can see from the nonlinear solution given by Equation (*) that the system is unstable and exhibits a finite "escape-time" at $t = \frac{1}{2}x(0)^{-2}$ (i.e., the response of the nonlinear system tends to infinity in finite time;see Figure on the next page). The linear system does not predict qualitative behavior of the nonlinear system. So the linear model is not a valid representation of the system.

(c) If $\alpha = -1$ the linearized system is

$$\begin{aligned}
\delta\dot{x} &= 3(-1)x_e^2 = 0, \\
\delta x &= \text{cons tan t},
\end{aligned}$$

that is the linear system is not asymptotically stable (it is neutrally stable). However, we can see from the nonlinear solution given by Equation (*) that the system is asymptotically stable as x^2 starts off at x_o^2 but drops off to zero (see Figure on top of the next page). The linear system does not predict qualitative behavior of the nonlinear system. So the linear model is not a valid representation of the system. The two systems corresponding to $\alpha = +1$ and $\alpha = -1$ have the *same* linearized system but *very different* nonlinear behavior. The conclusion is that the linearized system *usually* gives a good idea of the system behavior around the equilibrium (x_e) *but not always.*



Problem 9.7: Behavior of nonlinear system.

8. Consider the object moving in a straight line with constant velocity shown in Figure 9.61. The only available measurement is the range to the object. The system equations are

$$\begin{bmatrix} \dot{x} \\ \dot{v} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ v \\ z \end{bmatrix}$$

where

$$\begin{aligned} z &= \text{cons tan t} \\ \dot{x} &= \text{cons tan t} = v_0 \\ r &= \sqrt{x^2 + z^2} \end{aligned}$$

Derive a linear model for this system.

Solution: This system has only an output nonlinearity,

$$\begin{aligned} y &= r = h(\mathbf{x}), \\ \delta r &= \frac{\partial h}{\partial \mathbf{x}} \delta \mathbf{x} = \mathbf{C} \delta \mathbf{x}, \\ \mathbf{C} &= \left[\begin{array}{ccc} \frac{\partial h}{\partial x} & \frac{\partial h}{\partial v} & \frac{\partial h}{\partial z} \end{array} \right], \\ &= \left[\begin{array}{ccc} \frac{x}{r} & 0 & \frac{z}{r} \end{array} \right]. \\ \dot{\mathbf{x}} &= \mathbf{A}\mathbf{x}, \\ \delta r &= \left[\begin{array}{ccc} \frac{x}{r} & 0 & \frac{z}{r} \end{array} \right] \mathbf{x}. \end{aligned}$$

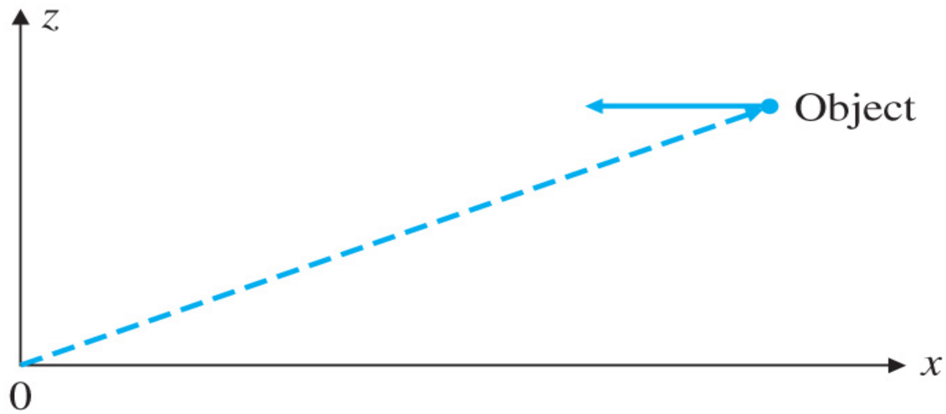


Figure 9.61: Diagram of moving object for Problem 9. 8

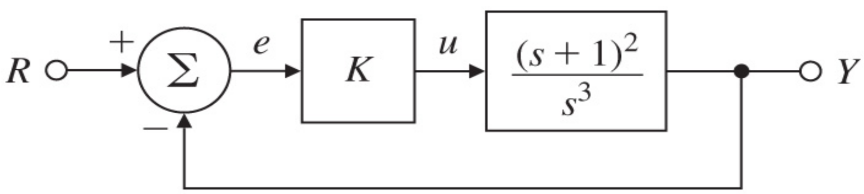


Figure 9.62: Control system for Problem 9. 9

Problems and Solutions for Section 9.3: Equivalent Gain Analysis Using Root Locus

9. Consider the third-order system shown in Fig. 9.62.

   (a) Sketch the root locus for this system with respect to K , showing your calculations for the asymptote angles, departure angles, and so on.

   (b) Using graphical techniques, locate carefully the point at which the locus crosses the imaginary axis. What is the value of K at that point?

   (c) Assume that, due to some unknown mechanism, the amplifier output is given by the following saturation non linearity (instead of by a proportional gain K):

$$u = \begin{cases} e, & |e| \leq 1; \\ 1, & e > 1; \\ -1, & e < -1. \end{cases}$$

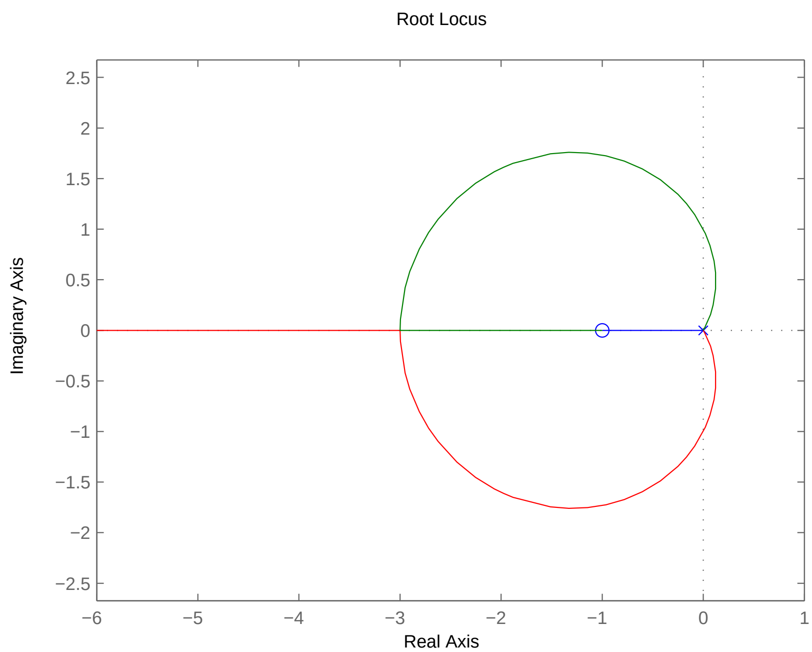
   Qualitatively describe how you would expect the system to respond to a unit step input.

   Solution:

(a) The locus branches leave the origin at angles of $180°$ and $\pm 60°$. Two break in at angles of $\pm 90°$ near $s = -3$. See root locus plot.

(b) The locus crosses the imaginary axis at $\omega = 1$ for $K = 0.5$.

(c) The system is conditionally stable and with saturation would be expected to be stable for small inputs and go unstable for large inputs.



Root locus for Problem 9.9.

10. Consider the system with the plant transfer function

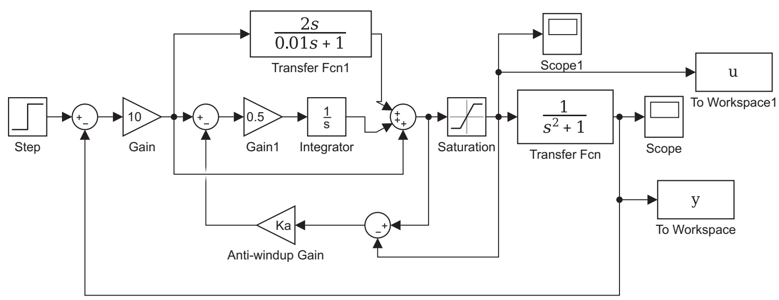
$$G(s) = \frac{1}{s^2 + 1}.$$

We would like to use PID control of the form

$$D_{c(}s) = 10\left(1 + \frac{1}{2s} + 2s\right),$$

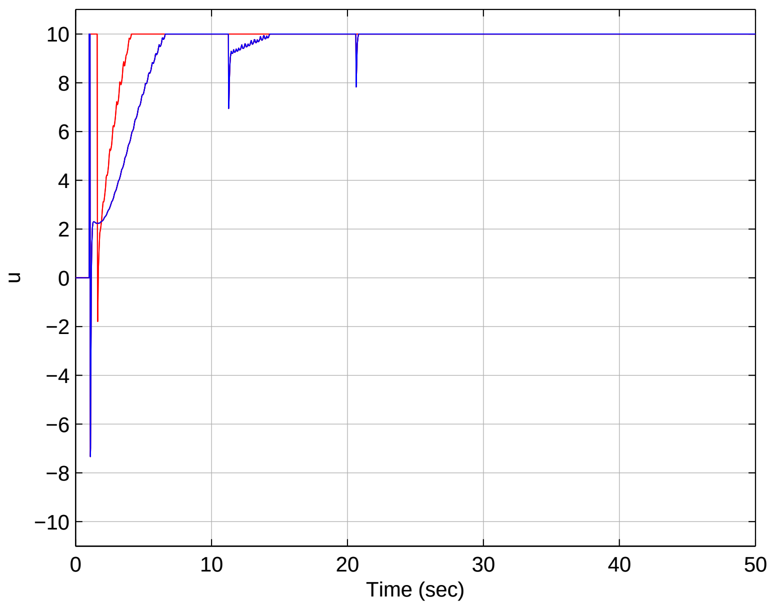
to control this system. It is known that the system's actuator is a saturation nonlinearity with a slope of unity and $|u| \leq 10$. Compare the system response for a step input of size 10 with and without antiwindup circuit. Plot both the step response and the control effort using Simulink. Qualitatively describe the effect of the antiwindup circuit.

Solution:

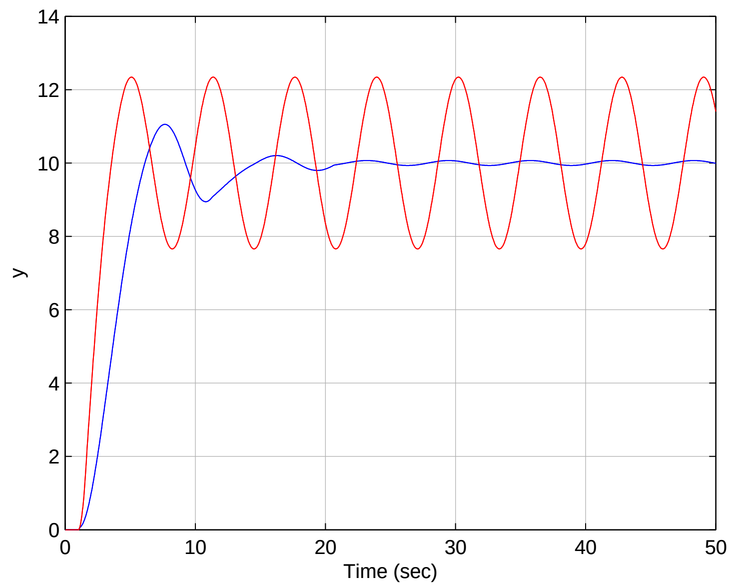


Simulink diagram for Problem 9.10.

The system's response with and without anti-windup circuit ($K_a = 1, K_a = 0$) are shown in the following figures. It is seen that the system has sustained oscillations without the anti-windup. $K_a = 1.2$ yields slightly better results. The system with anti-windup yields better transient response and uses less control effort as expected.



Control efforts for Problem 9.10.



Step responses for Problem 9.10.

11. Consider the system with the open-loop plant

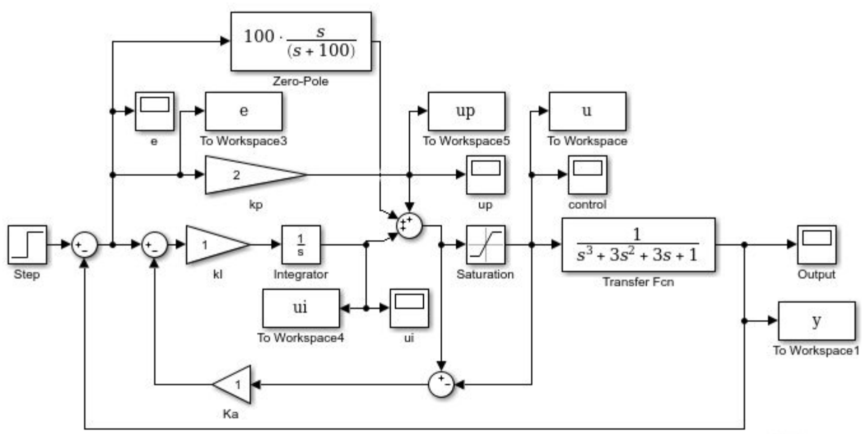
$$G(s) = \frac{1}{(s+1)^3} \tag{8}$$

and the nominal PID controller

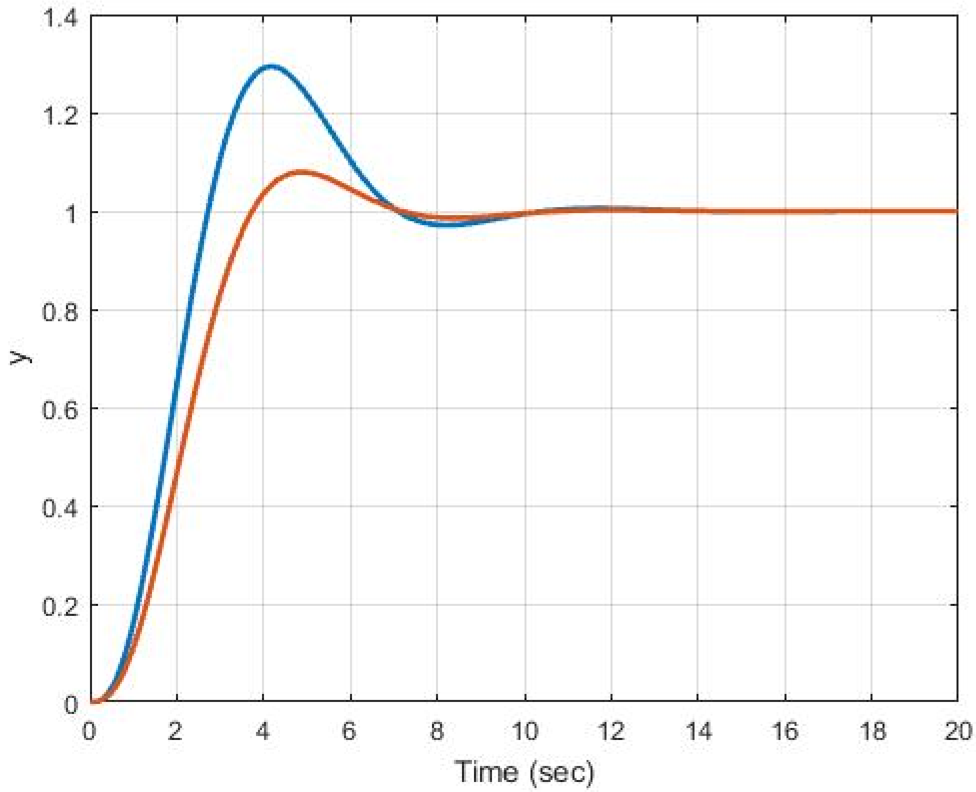
$$D_c(s) = 2 + \frac{1}{s} + \frac{s}{0.01s + 1} \tag{9}$$

The system has an actuator saturation nonlinearity with the slope of one and $|u| \leq 2$. Design an anti-windup scheme for this feedback system. Compare the step responses and control efforts for the system with and without anti-windup.

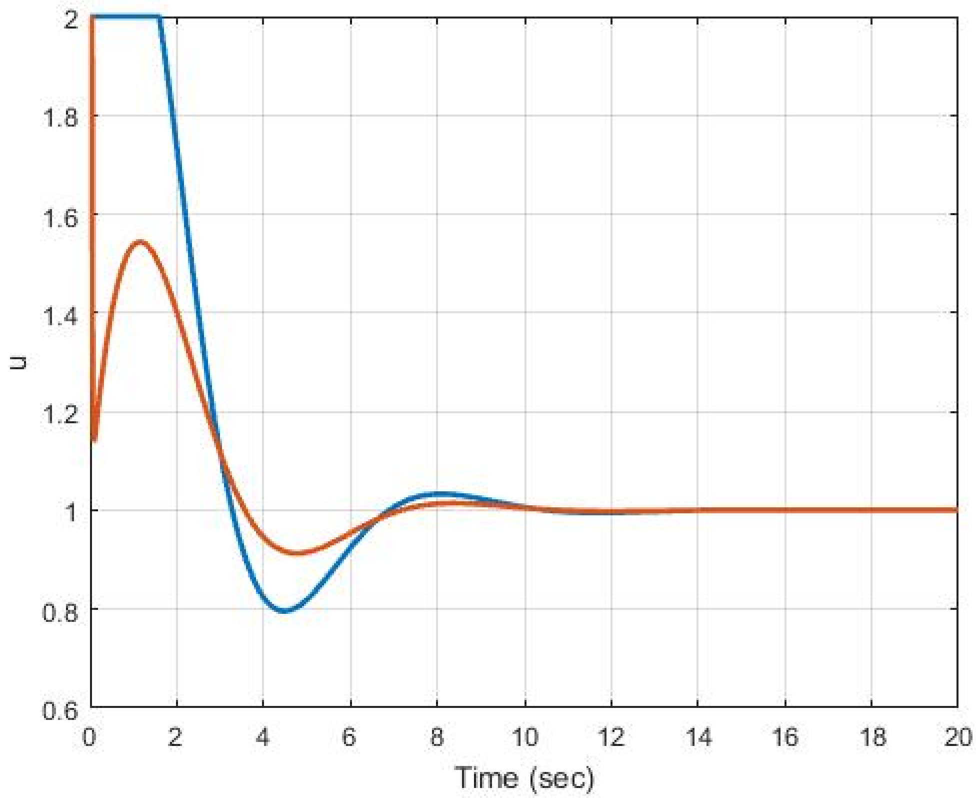
Solution: The Simulink implementation is shown in the ensuing figure. Simulink diagram for Problem 9.11**.** The system's response with and without anti-windup circuit ($K_a = 1, K_a = 0$) are shown in the following figures.



Problem 9.11: Simulink diagram



(a) Step response for Problem 9.11.



(b) Control effort for Problem 9.11.

The system with anti-windup yields better transient response and uses less control effort as expected.

(a) Problems and Solutions for Section 9.4: Equivalent Gain Analysis Using Frequency Response: Describing Functions

12. Compute the describing function for the relay with deadzone nonlinearity shown in Figure 9.6 (c).

Solution:

$$\begin{aligned}
Y_1 &= \frac{1}{\pi}\int_0^{2\pi} y(t)\sin(\omega t)d(\omega t)\\
&= \frac{4}{\pi}\int_0^{\frac{\pi}{2}} y(t)\sin(\omega t)d(\omega t)\\
&= \frac{4N}{\pi}\int_{\omega t_1}^{\frac{\pi}{2}} \sin(\omega t)d(\omega t)\\
&= \frac{4N}{\pi}\cos(\omega t_1).
\end{aligned}$$

Si nce $\omega t_1 = \frac{h}{a}$ then $\cos(\omega t_1) = \sqrt{1-\left(\frac{h}{a}\right)^2}$.
The describing function is then given by,

$$DF = \frac{Y_1}{a} = \frac{4N}{\pi a}\sqrt{1-\left(\frac{h}{a}\right)^2}.$$

13. Compute the describing function for gain with dead zone nonlinearity shown in Figure 9.6 (d).

    Solution: This is an odd nonlinearity so that all the cosine terms are zeros and the DF is real:

$$\begin{aligned}
Y_1 &= \frac{1}{\pi}\int_0^{2\pi} y(t)\sin(\omega t)d(\omega t),\\
&= \frac{4}{\pi}\int_0^{\frac{\pi}{2}} y(t)\sin(\omega t)d(\omega t),\\
&= \frac{4K_o}{\pi}\int_0^{\frac{\pi}{2}} (A\sin(\omega t)-a)\sin(\omega t)d(\omega t).
\end{aligned}$$

Si nce $h = a\sin(\omega t_1)$ then $\omega t_1 = \sin^{-1}\left(\frac{h}{\Delta}\right)$,

$$\begin{aligned}
Y_1 &= \frac{4aK_o}{\pi}\left[\int_{\omega t_1}^{\frac{\pi}{2}} (\sin^2(\omega t)-\sin(\omega t_1)\sin(\omega t))d(\omega t)\right],\\
&= \frac{2aK_o}{\pi}\left[\frac{\pi}{2}-\sin^{-1}\left(\frac{a}{A}\right)-\frac{a}{A}\sqrt{1-\left(\frac{a}{A}\right)^2}\right].
\end{aligned}$$

The describing function is then given by,

$$DF = \frac{Y_1}{a} = \frac{2K_o}{\pi}\left[\frac{\pi}{2}-\sin^{-1}\left(\frac{h}{a}\right)-\frac{h}{a}\sqrt{1-\left(\frac{h}{a}\right)^2}\right].$$

14. Compute the describing function for the preloaded spring or Coulomb plus viscous friction nonlinearity shown in Figure 9.6 (e).

    Solution: This is a combination of a gain, K_0 , plus a relay nonlinearity (see Example 9.11). Therefore,

$$DF = \frac{K_0 a}{a} + \frac{4N}{\pi a} = K_0 + \frac{4N}{\pi a}.$$

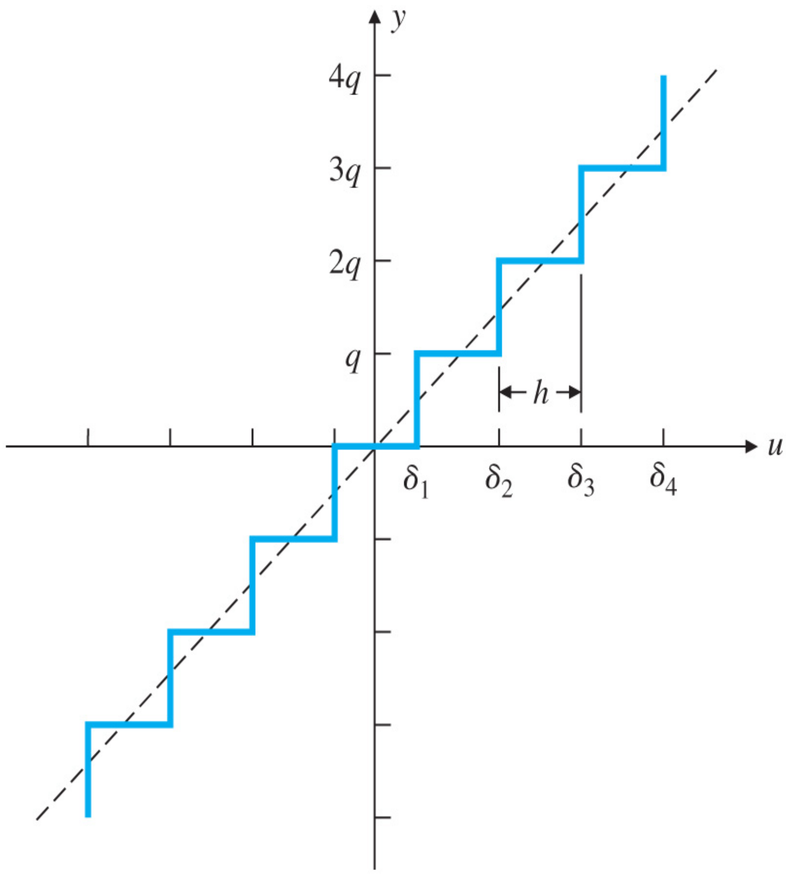


Figure 9.63: Quantizer nonlinearity for Problem 9.15.

15. Consider the quantizer function shown in Figure 9.63 that resembles a staircase. Find the describing function for this nonlinearity and write a MATLAB .m function to generate it.

Solution. The abscissa breakpoints are denoted by δ_i . From Eq. 9.23,

$$
\begin{aligned}
b_1 &= \frac{4}{\pi}\int_0^{\frac{\pi}{2}} f\left(a\sin\omega t\right)\sin\omega t \; d(\omega t) \\
&= \frac{4}{\pi}\int_0^{\varphi_1} 0\cdot\sin\omega t\; d(\omega t) + \int_0^{\varphi_2} q\cdot\sin\omega t\; d(\omega t) + \ldots + \int_0^{\varphi_n} nq\cdot\sin\omega t\; d(\omega t) \\
&= \frac{4}{\pi}(\cos\varphi_1 + \cos\varphi_2 + \ldots + \cos\varphi_n),
\end{aligned}
$$

where,

$$
\psi_i = \sin\left(\frac{\delta_i}{a}\right) i = 1, \ldots, n.
$$

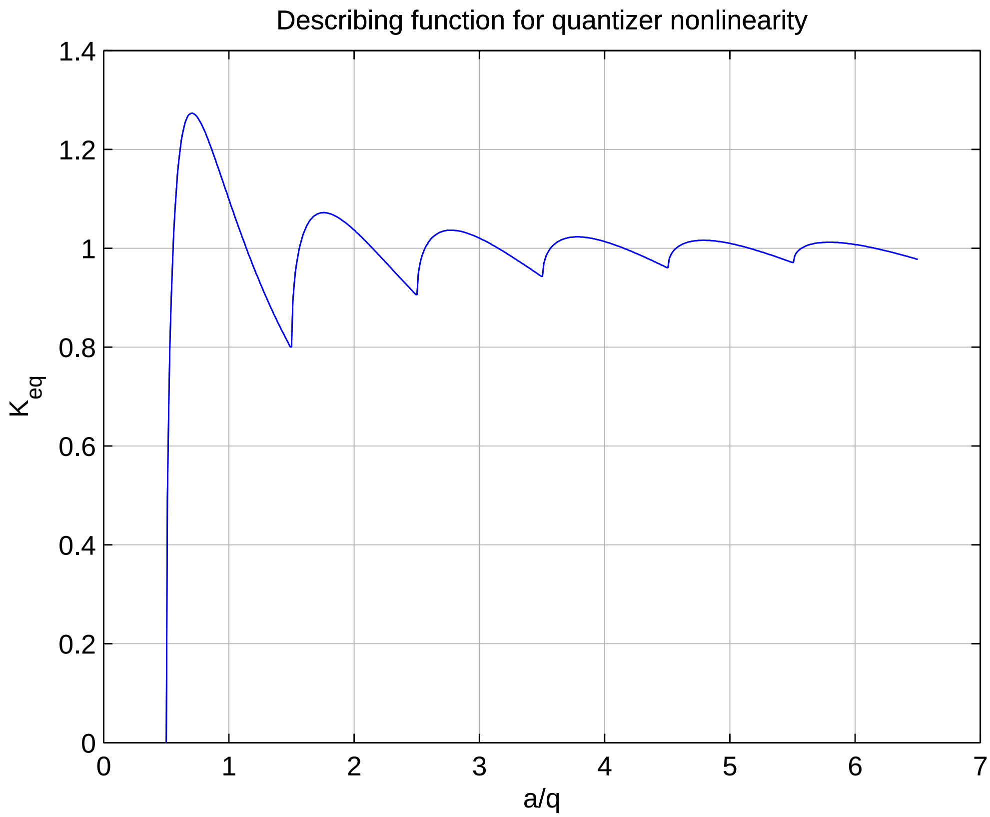
The describing function is then given by,

$$K_{eq}(a) = \frac{b_1}{a} = \begin{cases} 0 & 0 < a < \frac{q}{2} \\ \frac{4q}{\pi a}\sum_{i=1}^{n}\sqrt{1-\left(\frac{2i-1}{2a}q\right)^2} & \frac{2n-1}{2}q < a < \frac{2n+1}{2}q \end{cases}$$

The following shows the MATLAB .m function:

```
% Feedback Control of Dynamic Systems, 8e
% Franklin, Powell, Emami
% Problem 9.15
clear all;
close all;
na = 99;
nn = 6;
Keq=zeros(1,nn*na);
for n=1:nn
ai = linspace((2*n-1)/2, (2*n+1)/2, na);
for ni=1:na
for k=1:n
Keq((n-1)*na+ni) = Keq((n-1)*na+ni) + (4/(pi*ai(ni)))*sqrt(1- ((2*k-1)/(2*ai(ni)))^2);
end;
end;
end;
plot(linspace(1/2,(2*nn+1)/2,na*nn),Keq);
title('Describing function for quantizer nonlinearity')
xlabel('a/q')
ylabel('K_{eq}')
grid on;
hold off
```

The describing function is plotted in the figure as a function of $\frac{a}{q}$. The maximum of the DF occurs at $K_{eq} = \frac{4}{\pi} = 1.27$ corresponding to $\frac{a}{q} = 0.7$. Since the staircase can be approximated by a straight line, it is seen that the DF will in the limit approach the slope of the linear approximation, that is one.



Describing function for quantizer nonlinearity.

16. Derive the describing function for the ideal contactor controller shown in Figure 9.64. Is it frequency dependent? Would it be frequency dependent if it had a time delay or hysteresis? Graphically, sketch the time histories of the output for several amplitudes of the input and determine the describing function values for those inputs.

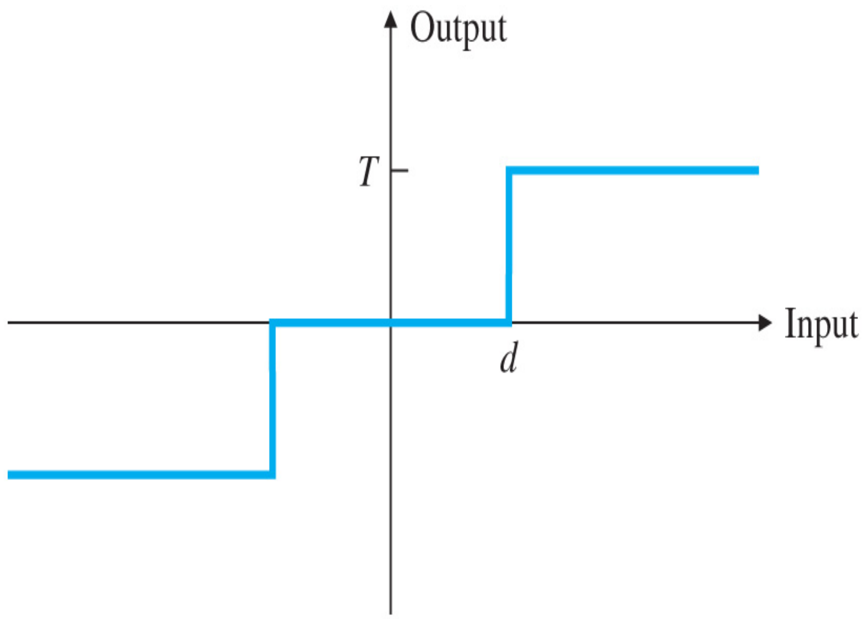


Figure 9.64: Contactor for Problem 9.16

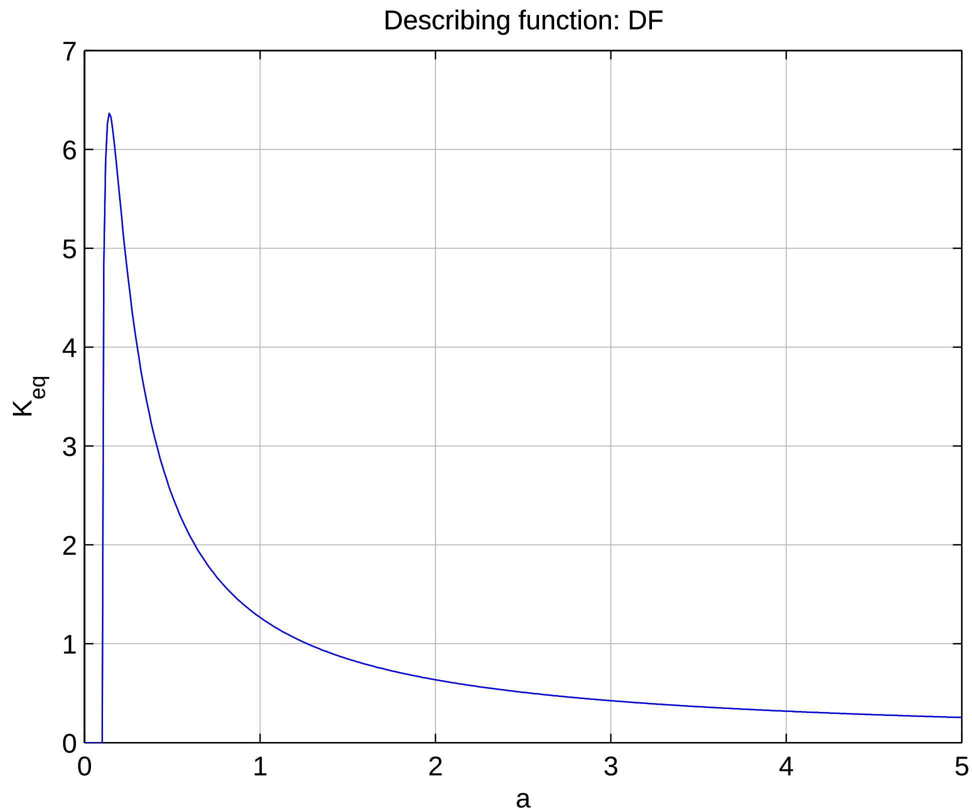
Solution:

$$\begin{aligned}
Y_1 &= \frac{1}{\pi}\int_0^{2\pi} y(t)\sin(\omega t)d(\omega t) \\
&= \frac{4}{\pi}\int_0^{\frac{\pi}{2}} y(t)\sin(\omega t)d(\omega t) \\
&= \frac{4T}{\pi}\int_{\omega t_1}^{\frac{\pi}{2}} \sin(\omega t)d(\omega t) \\
&= \frac{4T}{\pi}\cos(\omega t_1). \\
\text{Si nce } \omega t_1 &= \frac{d}{a} \text{ then } \cos(\omega t_1) = \sqrt{1-\left(\frac{d}{a}\right)^2}.
\end{aligned}$$

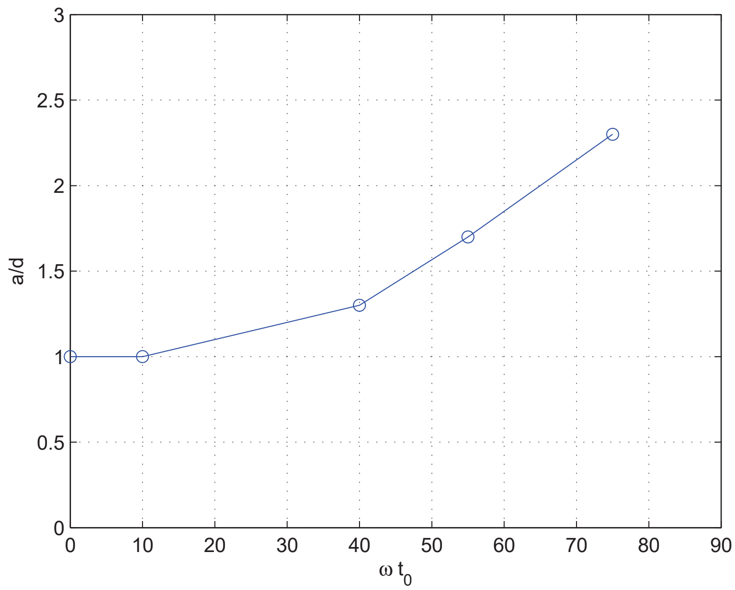
The describing function is then given by,

$$DF = \begin{cases} \frac{Y_1}{a} = \frac{4T}{\pi a}\sqrt{1-\left(\frac{d}{a}\right)^2} & d < a \\ 0 & a < d \end{cases}$$

and is *not* frequency dependent. See Figure on top of the next page. Frequency dependence will be introduced with a delay.



Problem 9.16. DF for $d = 0.1$, $T = 1.0$.



Problem 9.16. DF values for several different input frequencies.

17. A contactor controller of an inertial platform is shown in Figure 9.65 where

$$\begin{aligned} I &= 0.1 \text{ kgm }^2 \\ \frac{I}{B} &= 10\,\text{sec} \\ \frac{h}{c} &= 1 \\ \frac{J}{c} &= 0.01\,\text{sec} \\ \tau_L &= 0.1\,\text{sec} \\ \tau_f &= 0.01\,\text{sec} \\ d &= 10^{-5} \text{ rad} \\ T &= 1 \text{ Nm} \end{aligned}$$

The required stabilization resolution is approximately 10^{-6} rad

$$K\varphi_m > d \;\text{ for }\; \varphi_m > 10^{-6} \text{ rad}$$

Discuss the existence, amplitude and frequency of possible limit cycles as a function of the gain K and the DF of the controller. Repeat the problem for a deadband with hysteresis.

Solution: Limit cycles depend on the natural behavior of the closed-loop part. The DF of the switch $= K_{eq}$. Characteristic equation is:

$$KG + 1 = 0$$

$$\frac{K}{B}\frac{h}{c} = \frac{K_{eq}}{s\left[\left(\frac{J}{c}\right)s+1\right]}\left(\frac{\tau_L s+1}{\tau_f s+1}\right)\frac{1}{\left[\left(\frac{I}{B}\right)s+1\right]} + 1 = 0.$$

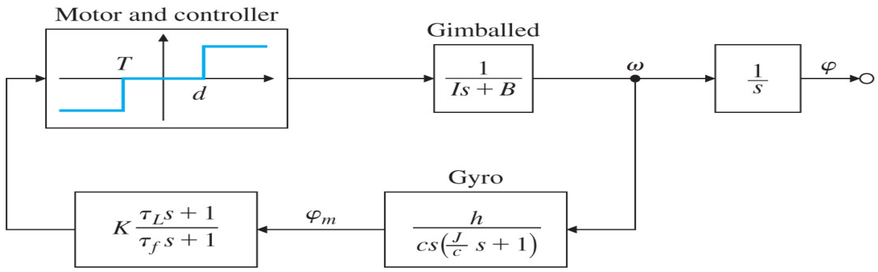
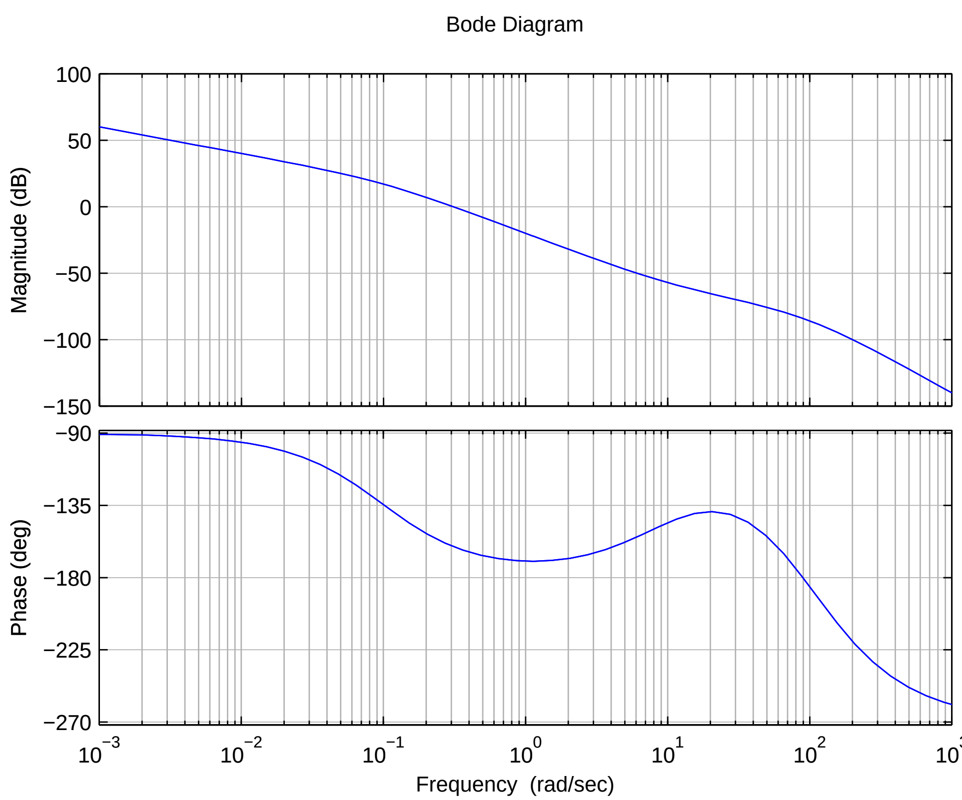
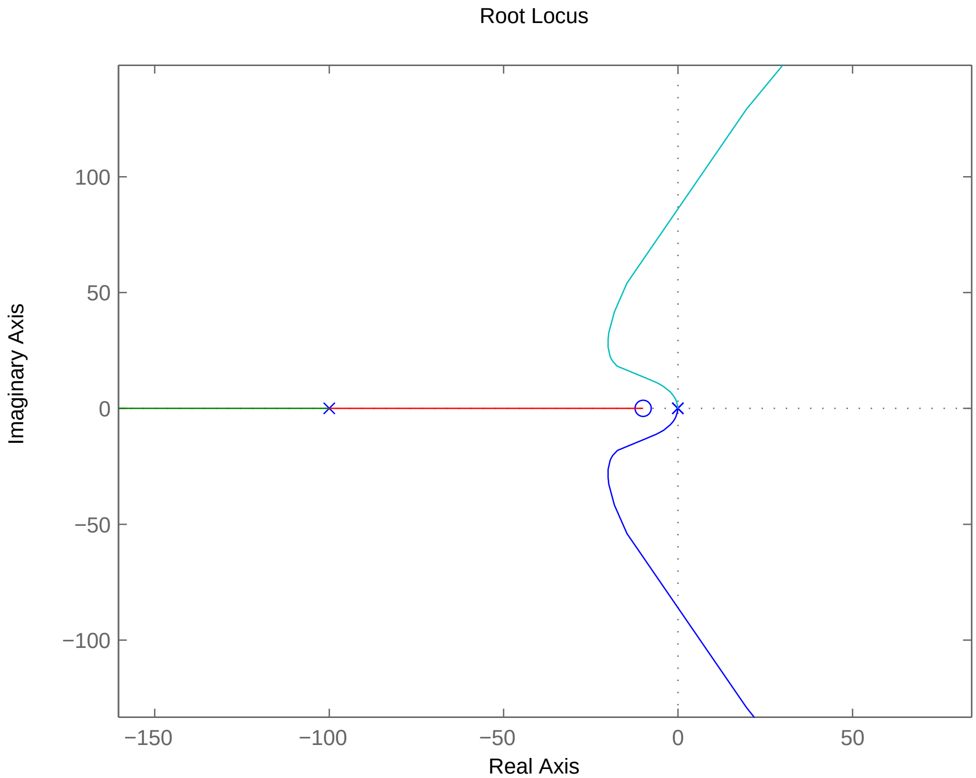


Figure 9.65: Block diagram of the system for Problem 9.17

$$\left(\frac{K_{eq}K}{0.01}\right)\left[\frac{0.1s+1}{s(10s+1)(0.01s+1)^2}\right] = -1.$$



Bode frequency response for Problem 9.17.



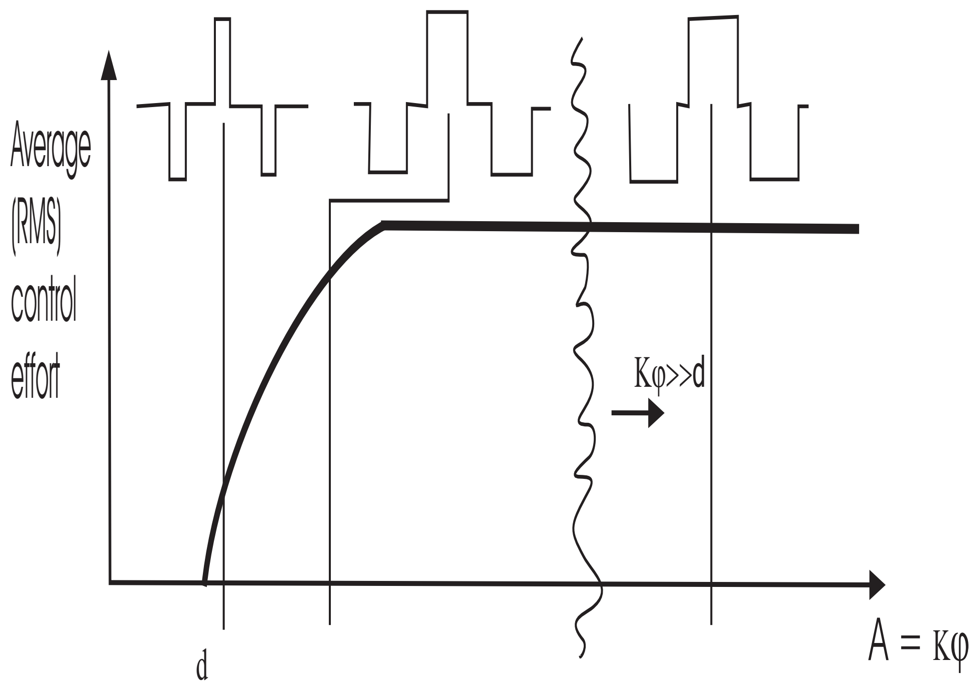
Root locus for Problem 9.17.

Condition for limit cycle is $\angle = -180°$ which occurs at ~ 95 rad/sec and magnitude $= 1$ (see Figure) but

$$\begin{aligned} \frac{KK_{eq}}{0.01} &\approx \frac{1}{10 \times 10^{-5}} (\text{ the value at } \omega \simeq 95 \text{ rad/ sec}), \\ KK_{eq} &= 100(\text{rad/ sec})(\text{nm sec /rad}) = 100 \text{ nm}, \end{aligned}$$

$$K_{eq} = \frac{4T}{\pi d}\left(\frac{d}{a}\right)\sqrt{1 - \left(\frac{d}{a}\right)^2}.$$

with maximum at $\frac{4T}{\pi d}$. But note from the figure below,

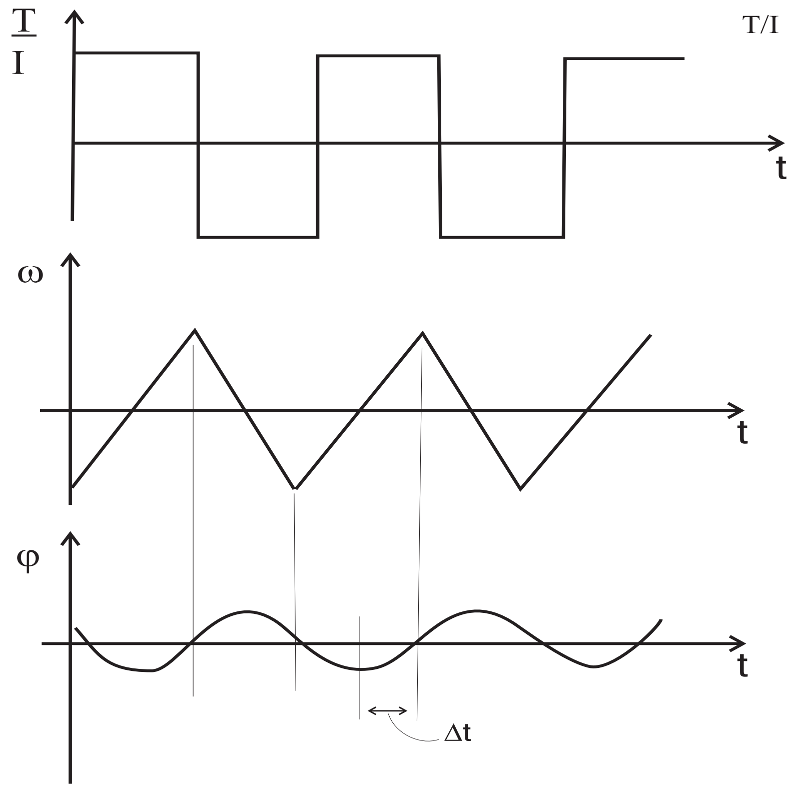


Response for nonlinear system for Problem 9.17.

that for levels of say $a > 3d$, the output is a constant. The frequency of the limit cycle is *fixed* by the *phase*! So φ is constant. The result is *independent of* K , for large enough K to insure that $K\varphi \gg d$. This is consistent with $KK_{eq} = \text{cons}\tan\text{t}$ for $K_{eq} = \frac{\text{cons}\tan\text{t}}{a}$, $a = K\varphi$,

$$K_{eq}K = \left(\frac{\text{cons tan t}}{K\varphi}\right)K = \frac{\text{cons tan t}}{\varphi} = \text{cons tan t} \quad \text{for} \quad K\varphi >> d\,.$$

The amplitude is:



See above figures:

$$\begin{aligned}
\frac{1}{2}\frac{T}{I}\Delta t^2 &= \varphi = 5\frac{rad}{\sec^2}(\Delta t)^2, \\
\Delta t &= \frac{P}{4}, \omega = \frac{2\pi}{P} \cong 100\frac{rad}{\sec}, \\
\Delta t &= \frac{2\pi}{4\omega} = \frac{3.14}{200} = 1.57 \times 10^{-2}\sec, \\
\varphi &= 12 \times 10^{-4}\,\text{rad.}
\end{aligned}$$

If the resolution of platform pickoff should be $\sim 10^{-6}$ rad and "short" term sensor noise $\ll 10^{-6}$ rad, then $K\varphi_m >> d$ is satisfied, say,

$$\begin{aligned}
K \times 10^{-6} &= d = 10^{-5}\,\text{rad,} \\
&\Rightarrow K = 10. \\
KK_{eq} &= 100\,\text{nm,} \\
&\Rightarrow K_{eq} = 10\,\text{nm.} \\
K_{eq} &= \frac{4T}{\pi d}\left(\frac{d}{a}\right)\sqrt{1-\left(\frac{d}{a}\right)^2} \approx \frac{4 \times 1}{\pi \times 10^{-5}}\left(\frac{d}{a}\right) = 10. \\
\frac{d}{a} &= 0.8 \times 10^{-5} \times 10 = 8 \times 10^{-5}.
\end{aligned}$$

To check:

$$a = \frac{d}{8 \times 10^{-5}} = 0.125 \times 10^5 \times 10^{-5} = 0.125 \text{ rad.}$$

Here we must consider the mid frequency model because the limit cycle is at $\omega = 100$ rad/sec. ω gets integrated in the gyro below its break frequency but φ_m goes through the lead circuit for a gain of 10 and $K = 10$ so $a = 12.5 \times 10^{-4}$ rad $\times 100 = 0.125$ rad as before, so it checks.

18. *Nonlinear Clegg Integrator* There have been some attempts over the years to improve upon the linear integrator. A linear integrator has the disadvantage of having a phase lag of 90° at all frequencies. In 1958, J. C. Clegg suggested that we modify the linear integrator to reset its state, x , to zero whenever the input to the integrator, e , crosses zero (i.e., changes sign). The Clegg integrator has the property that it acts like a linear integrator whenever its input and output have the same sign. Otherwise, it *resets* it output to zero. The Clegg integrator can be described by

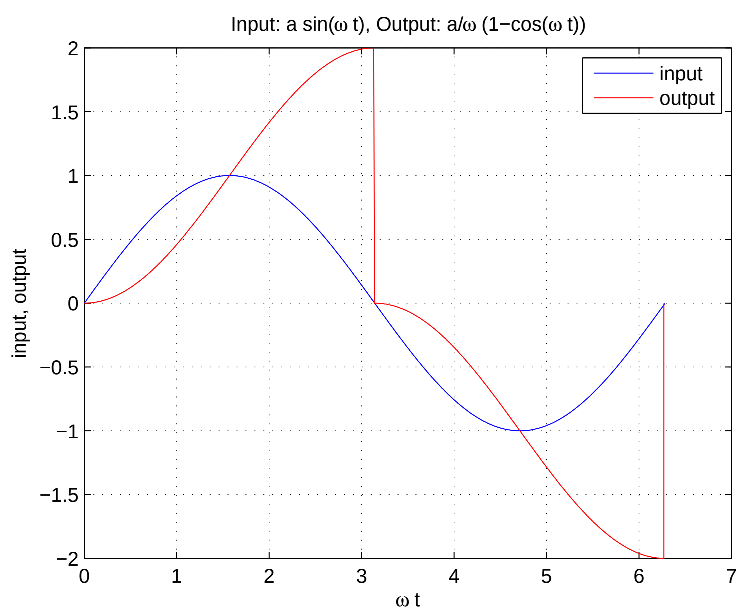
$$\begin{aligned} x(t) &= e(t) \quad \text{if } e(t) \neq 0, \\ x(t+) &= 0 \quad \text{if } e(t) = 0, \end{aligned}$$

where the latter equation implies that the state of the integrator, x , is reset to zero immediately after e changes sign. The Clegg integrator can be implemented with op-amps and diodes. A potential disadvantage of the Clegg integrator is that it may induce oscillations. (a) Sketch the output of the Clegg integrator if the input is $e = a\sin(\omega t)$(b) Prove that the DF for the Clegg integrator is

$$N(a,\omega) = \frac{4}{\pi\omega} - j\frac{1}{\omega}.$$

and this amounts to a phase lag of only 38° .

Solution: (a) See Figure below.



Plots of the input and output signals for Problem 9.18.

(b)

$$\begin{aligned}
a_1 &= \frac{2}{\pi}\int_0^{\pi} x(t)\cos(\omega t)d(\omega t) \\
&= \frac{2}{\pi}\int_0^{\pi} \frac{a}{\omega}\cos(\omega t)d(\omega t) \\
&= \frac{2a}{\pi\omega}\int_0^{\pi} \cos(\omega t)d(\omega t) - \int_0^{\pi}\cos^2(\omega t)d(\omega t) \\
&= \frac{2a}{\pi\omega}\left(0+\frac{\pi}{2}\right) = -\frac{a}{\omega}.
\end{aligned}$$

$$\begin{aligned}
b_1 &= \frac{2}{\pi}\int_0^{\pi} x(t)\sin(\omega t)d(\omega t) \\
&= \frac{2}{\pi}\int_0^{\pi} \frac{a}{\omega}(1-\cos(\omega t))\sin(\omega t)d(\omega t) \\
&= \frac{2a}{\pi\omega}\int_0^{\pi} \sin(\omega t)d(\omega t) - \int_0^{\pi}\cos(\omega t)\sin(\omega t)d(\omega t) \\
&= \frac{2a}{\pi\omega}(2+0) = \frac{4a}{\pi\omega}.
\end{aligned}$$

The describing function is then given by,

$$\begin{aligned}
DF &= \frac{Y_1}{a} = \frac{4}{\pi\omega} - j\frac{1}{\omega} \\
&= \frac{1}{\omega}\sqrt{1+\left(\frac{4}{\pi}\right)^2}\arctan\left(\frac{-\pi}{4}\right) \\
&= \frac{1.619}{\pi}\angle -38.15^\circ.
\end{aligned}$$

■ Problems and Solutions for Section 9.5: Analysis and Design Based on Stability

19. Compute and sketch the optimal reversal curve and optimal control for the minimal time control of the plant

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -x_2 + u \\
|u| &\leq 1
\end{aligned}$$

Use the reverse time method and eliminate the time.

Solution: Use the reverse time method and eliminate the time:

$$\begin{aligned}
\dot{x}_1 &= -x_2, \\
\dot{x}_2 &= x_2 - \mathrm{u}.
\end{aligned}$$

For $u = +1$, time reversal means that we let $\tau = -t$, and that changes the sign on the system and the input matrices,

$$\frac{d\mathbf{x}}{d\tau} = -\mathbf{A}\mathbf{x} - \mathbf{B}u.$$

In our case,

$$\begin{aligned} \dot{x}_1 &= -x_2, \\ \dot{x}_2 &= x_2 - 1, \\ \frac{dx_2}{d\tau} &= x_2 - 1, \\ \frac{dx_2}{(x_2 - 1)} &= d\tau. \end{aligned}$$

Integrate both sides:

$$\int_0^{x_2} \frac{dx_2}{(x_2 - 1)} = \int_0^{\tau} d\tau.$$
$$\ln(x_2 - 1) = \tau + C_1.$$

Since $x_2(0) = 0$ then,

$$\begin{aligned} C_1 &= \ln(-1), \\ \ln(x_2 - 1) - \ln(-1) &= \tau, \\ \ln(1 - x_2) &= \tau, \\ x_2 &= 1 - e^{\tau}, \\ \tau &= \ln(1 - x_2). \end{aligned}$$

Now,

$$\begin{aligned} \dot{x}_1 &= -x_2, \\ \dot{x}_1 &= -1 + e^{\tau}, \\ dx_1 &= (-1 + e^{\tau})d\tau. \end{aligned}$$

Integrate both sides:

$$\begin{aligned} \int_0^{x_1} dx_1 &= \int_0^{\tau} (-1 + e^{\tau})d\tau. \\ x_1 &= e^{\tau} - \tau - 1. \end{aligned}$$

Eliminate τ to get,

$$x_1 = -x_2 - \ln(1 - x_2).$$

This is the reversal curve for $u = 1$, $x_2 < 0$.
For $u = -1$,

$$\begin{aligned} \dot{x}_2 &= x_2 + 1, \\ \frac{dx_2}{x_2 + 1} &= d\tau. \end{aligned}$$

Integrate both sides:

$$\begin{aligned} \int_0^{x_2} \frac{dx_2}{x_2+1} &= \int_0^{\tau} d\tau. \\ \ln(x_2+1) &= \tau + C_1. \end{aligned}$$

Since $x_2(0) = 0$ then

$$\begin{aligned} \ln(x_2+1) &= \tau + C_1, \\ &\Rightarrow C_1 = \ln(1), \\ \ln(x_2+1) &= \tau, \\ x_2 + 1 &= e^{\tau}, \\ x_2 &= e^{\tau} - 1, \\ &\Rightarrow \tau = \ln(x_2+1). \end{aligned}$$

Now,

$$\begin{aligned} \dot{x}_1 &= -x_2 = 1 - e^{\tau}, \\ dx_1 &= (1-e^{\tau})d\tau. \end{aligned}$$

Integrate both sides:

$$\begin{aligned} \int_0^{x_1} dx_1 &= \int_0^{\tau} (1-e^{\tau})d\tau, \\ x_1 &= -e^{\tau} + \tau + C_2, \\ x_2(0) &= 0, C_2 = 1, \\ x_1 &= -e^{\tau} + \tau + 1. \end{aligned}$$

Eliminate τ to get,

$$x_1 = \ln(1+|x_2|) - x_2.$$

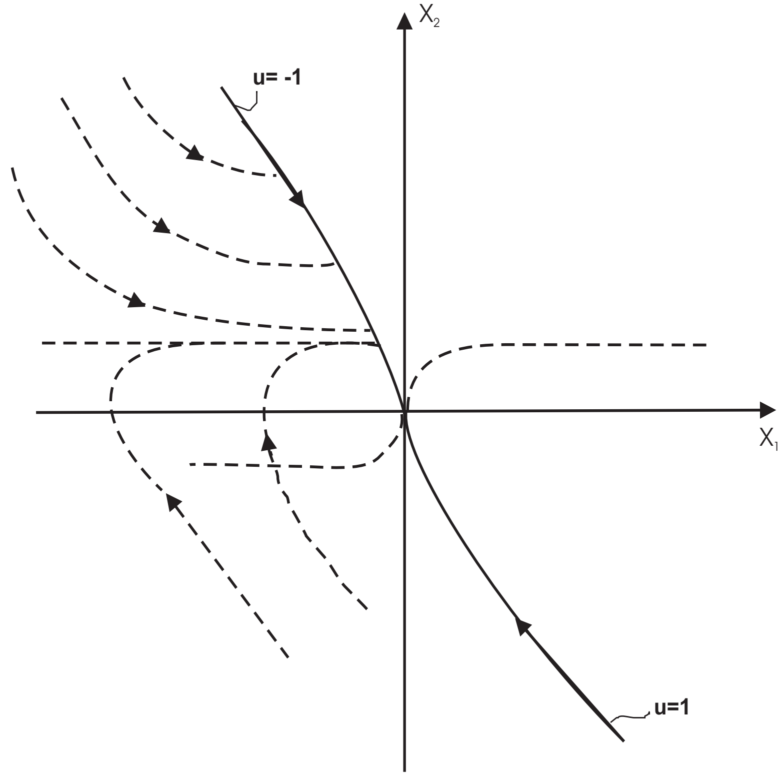
This is the reversal curve for $u = -1$, $x_2 > 0$.

We can then write in general, for all x_2 ,

$$x_1 = sgn(x_2)\ln(1+|x_2|) - x_2$$

Therefore, the control law is:

$$u = -sgn[x_1 + x_2 - sgn(x_2)\ln(1+|x_2|)].$$



Optimal reversal curve for Problem 9.19.

20. Sketch the optimal reversal curve for the minimal time control with $|u| \leq 1$ of the linear plant

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -2x_1 - 3x_2 + u.\end{aligned}$$

Solution: We reverse time that means $\tau = -t$, and that changes the sign on the system and the input matrices,

$$\frac{d\mathbf{x}}{d\tau} = -\mathbf{A}\mathbf{x} - \mathbf{B}u.$$

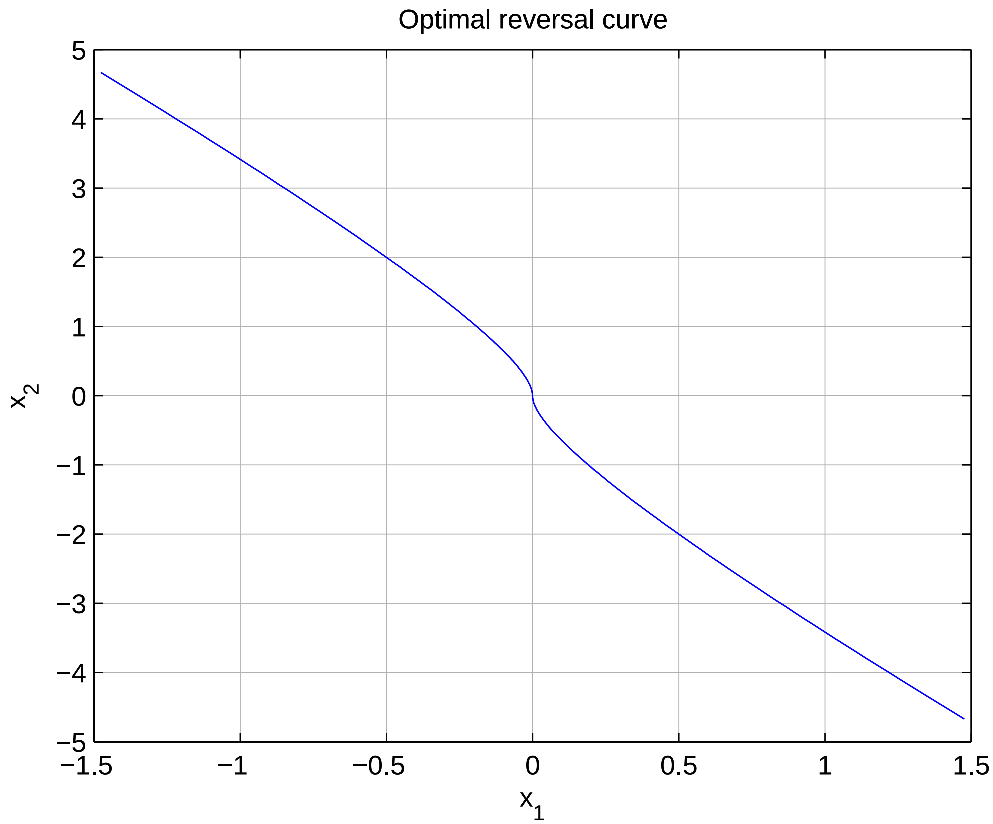
In our case,

$$\begin{aligned}\dot{x}_1 &= -x_2 \\ \dot{x}_2 &= +2x_1 + 3x_2 - u.\end{aligned}$$

We simulate the system using the MATLAB lsim function with $u = +1$ and store x_1 and x_2 , and repeat with $u = -1$ and store x_1 and x_2 and plot the results to obtain the optimal reversal curve shown in the plot on the next page:

```
% Franklin, Powell, Emami 8e
% Problem 9.20
t=0:.01:1;
```

```
A=[0 1;-2 -3];

B=[0;1];

C=[1 0];

D=[0];

% Using the reverse time method

sys=ss(-A,-B,C,D);

%u=+1;

[yp,t,xp]=lsim(sys,ones(101,1),t);

plot(xp(:,1),xp(:,2));

hold on;

%u=-1

[ym,t,xm]=lsim(sys,-1*ones(101,1),t);

plot(xm(:,1),xm(:,2));

grid on;

xlabel('x_1');

ylabel('x_2');

title('Optimal reversal curve');
```



Optimal reversal curve for Problem 9.20.

21. Sketch the time optimal control law for

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -x_1 + u\\ |u| &\leq 1\end{aligned}$$

and show a trajectory for $x_1(0) = 3$, and $x_2(0) = 0$.

Solution: $u = +1$,

$$\begin{aligned}sX_1(s) &= -X_2(s),\\ sX_2(s) &= X_1(s) - \frac{1}{s},\\ X_2(s) &= -\frac{1}{s^2+1},\\ x_2(t) &= -\sin(t),\\ \dot{x}_2 &= -\cos(t) = x_1 - 1,\\ x_1 &= +1 - \cos(t).\end{aligned}$$

We see that,

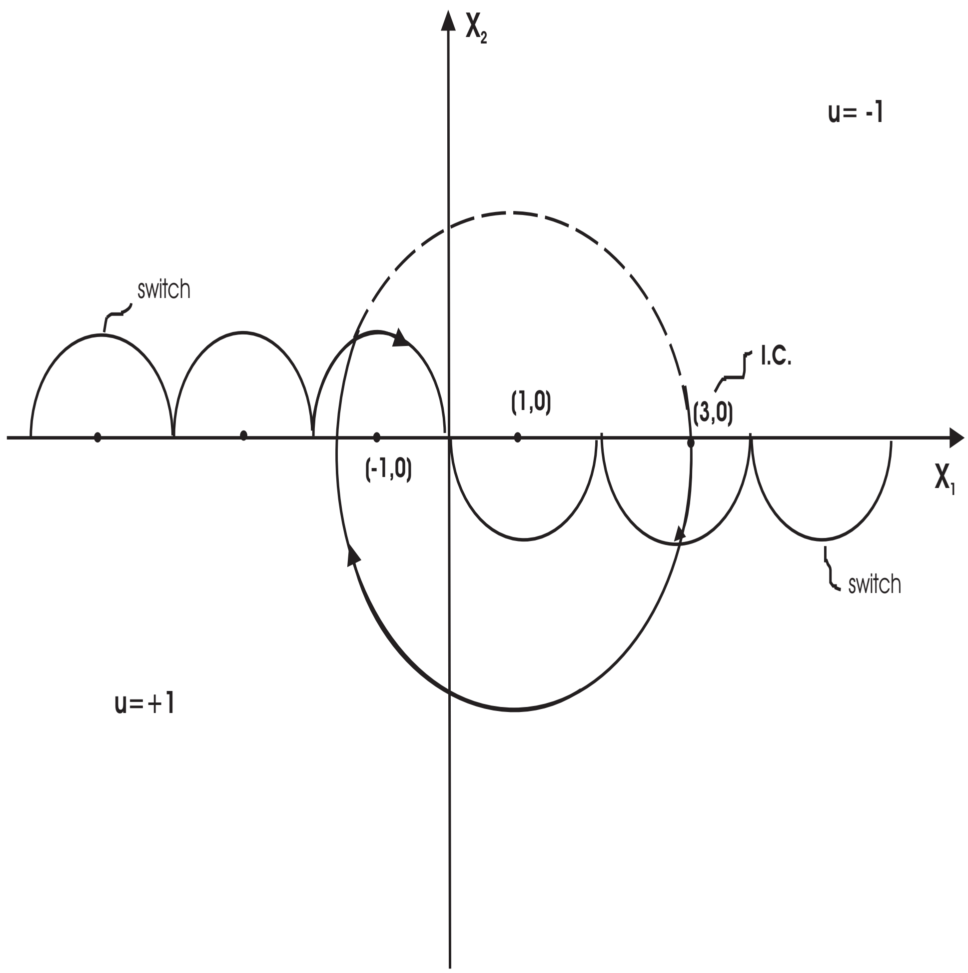
$$(x_1 - 1)^2 + x_2^2 = 1$$

that is a circle with center at $(1, 0), x_2 < 0$.

Similarly for $u = -1$ we get,

$$\begin{aligned} x_2(t) &= \sin(t) \\ x_1(t) &= \cos(t) - 1 \\ (x_1 + 1)^2 + x_2^2 &= 1, \end{aligned}$$

that is a circle with center at $(-1, 0),\ x_2 > 0$.



Reversal curves for Problem 9.21.

The trajectories for this system are circles centered at ($\pm 1, 0$). This is called the Bushaw problem in optimal control literature.

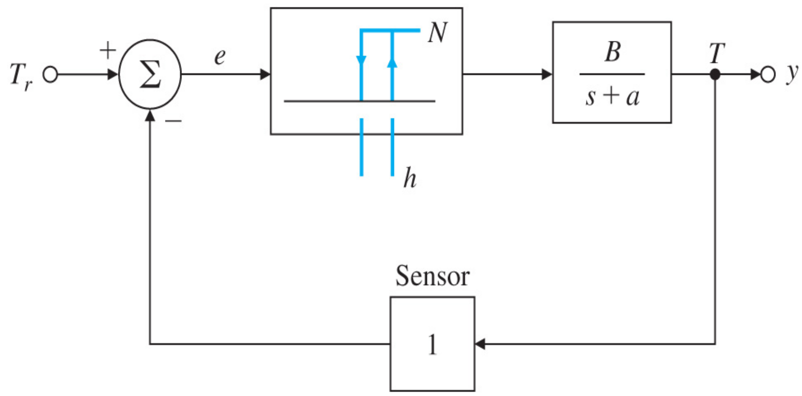


Figure 9.66: Thermal system for Problem 9.22

22. Consider the thermal control system shown in Figure 9.66. The physical plant can be a room, an oven, etc.

    (a) What is the limit cycle period?

    (b) If T_r is commanded as a slowly increasing function, sketch the output of the system, T . Show the solution for T_r "large".

    Solution:

    (a) This is a first order system so use (T, t) plot. For an oven, it is piecewise linear

$$\dot{T} + aT = BN\ sgn(e) \quad \text{with hysteresis}$$

with

$$T_r = 800°C, \frac{BN}{a} \approx 1000°C \text{above} T = 0 \text{ (say room temperature)}$$

$$a \simeq 0.01\,\text{sec}^{-1},\ \text{plot vs } at,\ h \sim 100°C,\ T_o = 0.$$

(b)

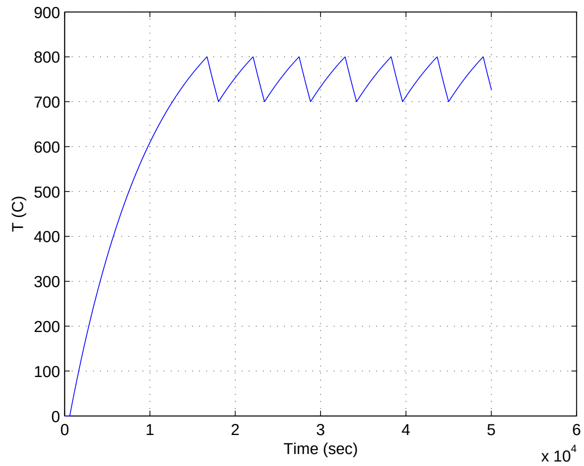
$$\begin{aligned} \text{limit cycle period} &\triangleq P = t_{on} + t_{off}, \\ T_r e^{-at} &= (T_r - h) \end{aligned}$$

gives t_{off} ,

$$\left[\frac{BN}{a} - (T_r - h)\right] e^{-at} = \left[\frac{BN}{a} - T_r\right]$$

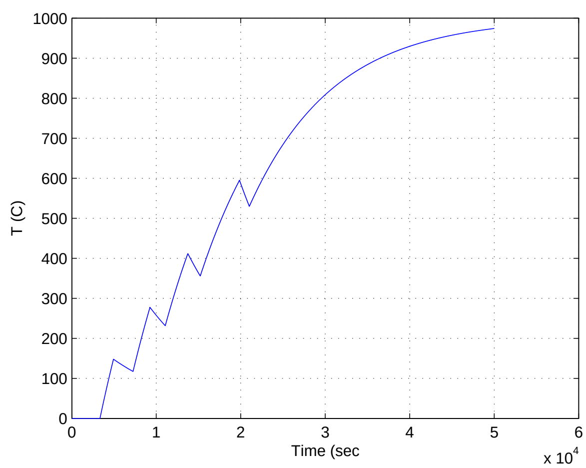
gives t_{on} ,

$$\begin{aligned} aP &= -\ln\left(\frac{700}{800}\right) - \ln\left(\frac{200}{300}\right), \\ P &= 100(0.058 + 0.176) = 23.4\,\text{sec}. \end{aligned}$$



Temperature output for Problem 9.22.

(c) See Figure below.



Temperature output for reference input $T_r = 3t$ for Problem 9.22.

23. Several systems such as spacecraft, spring-mass system with resonant frequency well below the frequency of switching, a large motor driven load with very small friction, etc. can be modeled as just an inertia. For an ideal switching curve, sketch the phase portraits of the system. The switching function is $e = \theta + \tau\omega$. Assume $\tau = 10$ *sec* , and the control signal $= 10^{-3}$ rad/sec ^{2} . Now sketch the results with,

a) deadband,

b) deadband plus hysteresis,

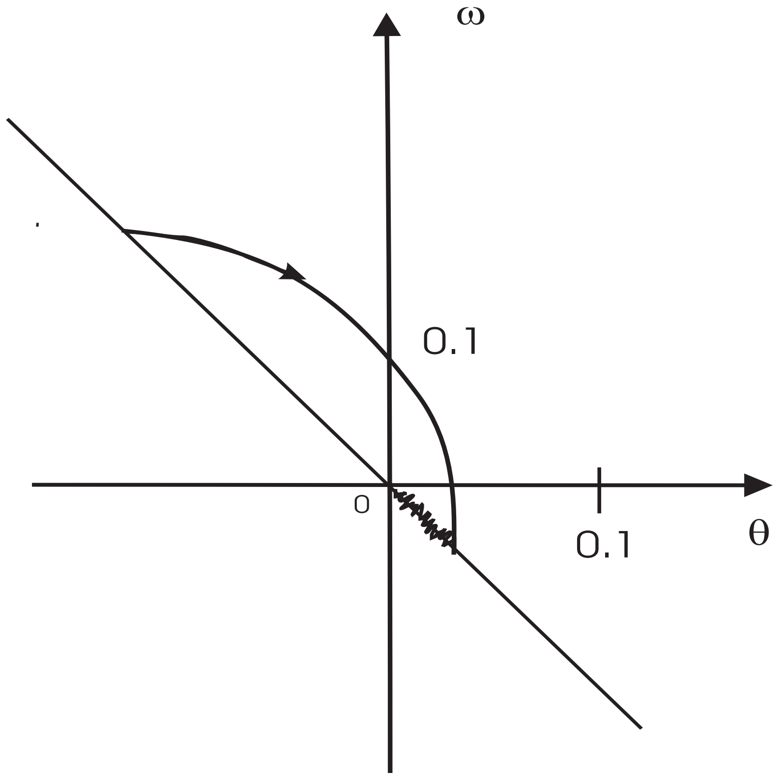
c) deadband plus time delay T ,

d) deadband plus a constant disturbance.

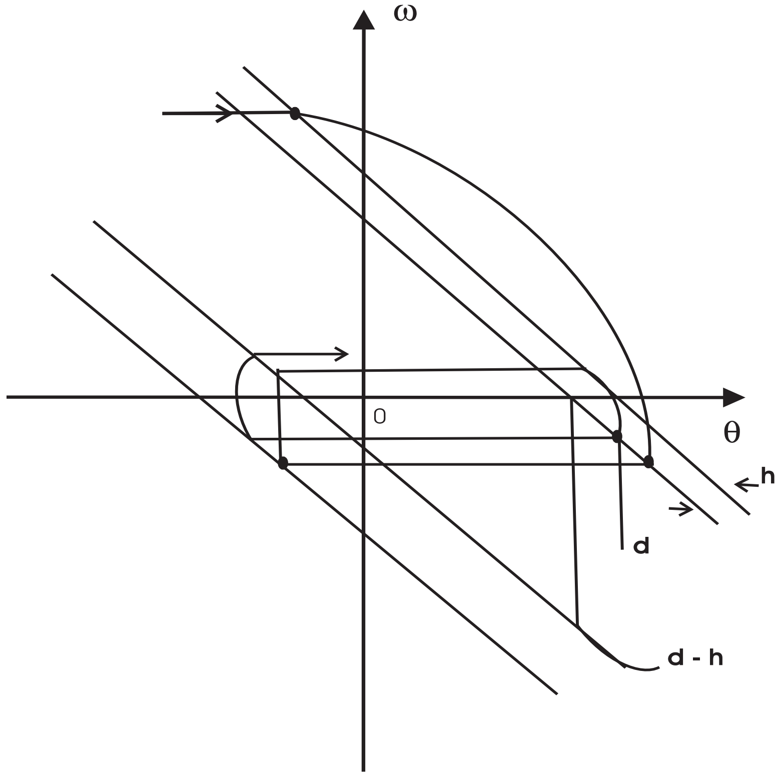
Solution:

(a)

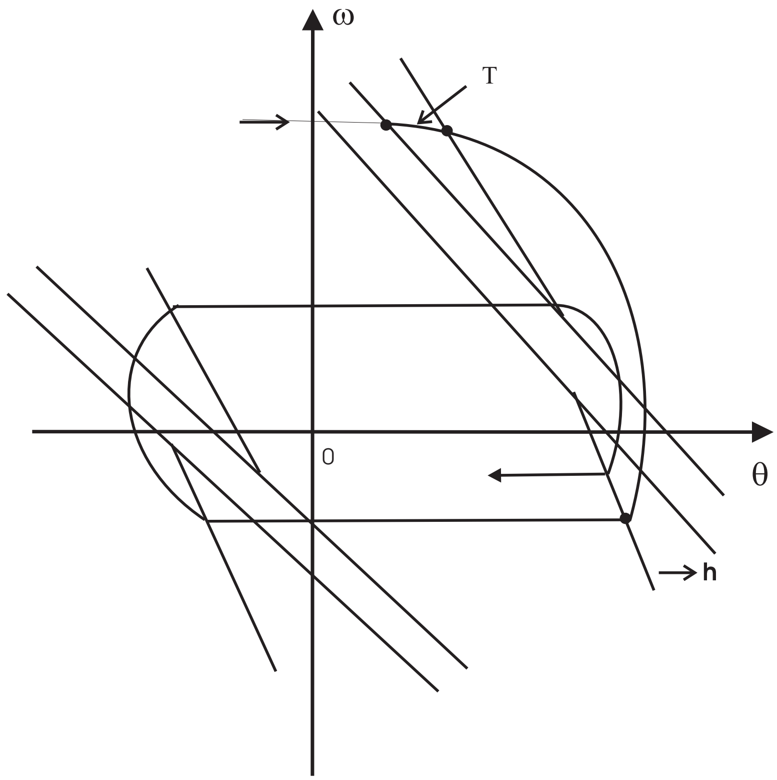
$$\begin{aligned}
\frac{d\omega}{d\theta} &= \frac{u}{\omega} = \frac{10^{-3}\text{rad/}\sec^2}{\omega}, \\
\frac{\omega^2}{2} &= u\,\theta, \\
\theta &= 500\,\omega^2, \\
\omega &= 10^{-2} \rightarrow \theta = 0.05.
\end{aligned}$$



(a) Phase portraits for deadband for Problem 9.23.



(b) Phase portraits for deadband plus hysteresis for Problem 9.23.



(c) Phase portraits for deadband plus plus time delay T for Problem 9.23.

(d)

$$\begin{aligned}
T\omega &= \Delta\theta, \\
\Delta\theta + \tau\omega &= 0, \\
\text{changes slope to} \quad \Delta\theta &= -\tau\omega + T\omega = -(\tau - T)\omega, \\
\dot{\theta} &= \omega u \to 0 \quad \text{at} \quad t \quad \text{for} \quad (\theta - d + h + \tau\omega = 0) + T, \\
\dot{\omega} &= u + Dt_s \longleftrightarrow \theta_s = -\tau\omega_s + (d - h), \\
\omega &= (u + D)t + \omega_o, \\
&\Rightarrow \theta = (u + D)\frac{t^2}{2} + \omega_s t + \theta_s, \\
\text{for } \ t &= T, \theta = \omega_s \underbrace{(T - \tau)}_{\text{new slope}} + \underbrace{(d - h) + \left(\frac{u + D}{2}\right)T^2}_{\text{new intercept}}.
\end{aligned}$$

Reference:

[1] D. Graham and D. McRuer, *Analysis of Nonlinear Control Systems*, John Wiley & Sons, 1961.

24. Compute the amplitude of the limit cycle in the case of satellite attitude control with delay

$$I\,\ddot{\theta} = N\,u(t - \Delta)$$

using

$$u = -sgn(\tau\dot{\theta} + \theta)$$

Sketch the phase plane trajectory of the limit cycle and time history of θ giving the maximum value of θ .

Solution: Since the delay is Δ seconds, $\dot{\theta}$ must travel $\frac{\Delta N}{I}$ units during the delay. We can obtain the following relations:

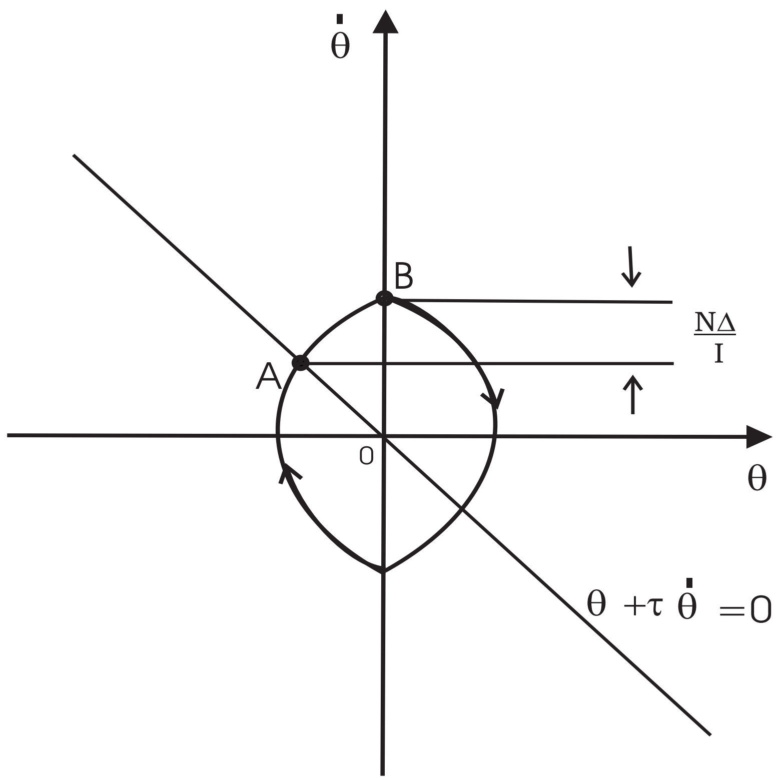
$$\begin{aligned} \dot{\theta} &= \frac{N}{I}(t-t_0) + \dot{\theta}_0, \\ \theta &= \frac{1}{2}\frac{N}{I}(t-t_0)^2 + \dot{\theta}_0(t-t_0) + \theta_0. \end{aligned}$$

Eliminating $(t - t_0)$, we get

$$\begin{aligned} \theta - \theta_0 &= \frac{I}{2N}(\dot{\theta}^2 - \dot{\theta}_0^2) = \frac{I}{2N}(\dot{\theta}^2 - 0), \\ \theta &= \frac{2N}{I}(\theta - \theta_0) + \frac{I}{2N}\dot{\theta}^2), \end{aligned}$$

and

$$\begin{aligned} \theta + \tau\dot{\theta} &= 0, \\ \tau &= \frac{1}{a}. \end{aligned}$$



Phase-plane trajectory of limit cycle for Problem 9.24.

From the geometry of the limit cycle (see above Figure),

$$\dot{\theta}_A + \frac{\Delta N}{I} = \dot{\theta}_B, \tag{10}$$

At point A,

$$\theta_A - \theta_0 = \frac{I}{2N}\dot{\theta}_A^2, \tag{11}$$

$$\theta_A + \tau\dot{\theta}_A = 0, \tag{12}$$

At point B,

$$\dot{\theta}_B^2\left(\frac{I}{2N}\right) = -\theta_0, \tag{13}$$

We need to solve the above four equations for θ_0 (for $\theta_{\max} = \theta_0$, $\theta_0 = 0$).
It seems to be easiest to first eliminate θ_0 using Eqs. (11) and (13) to get

$$\theta_A + \frac{I}{2N}\left[\dot{\theta}_B^2 - \dot{\theta}_A^2\right] = 0.$$

If we eliminate $\dot{\theta}_B$ using Eq. (10), then,

$$\theta_A = -\frac{\Delta}{2}\left[2\dot{\theta}_A + \frac{\Delta N}{I}\right] = 0.$$

If we use Eq. (12) to eliminate θ_A ,

$$\tau\dot{\theta}_A = \frac{\Delta}{2}\left[2\dot{\theta}_A + \frac{\Delta N}{I}\right] = 0.$$

Solve for $\dot{\theta}_A$,

$$\dot{\theta}_A = \frac{\Delta^2 N}{2I}\left(\frac{1}{\tau - \Delta}\right).$$

From Eq. (10),

$$\dot{\theta}_B + \frac{N\Delta}{I} = \frac{\Delta^2 N}{2I}\left(\frac{1}{\tau - \Delta}\right).$$

Then using Eq. (13),

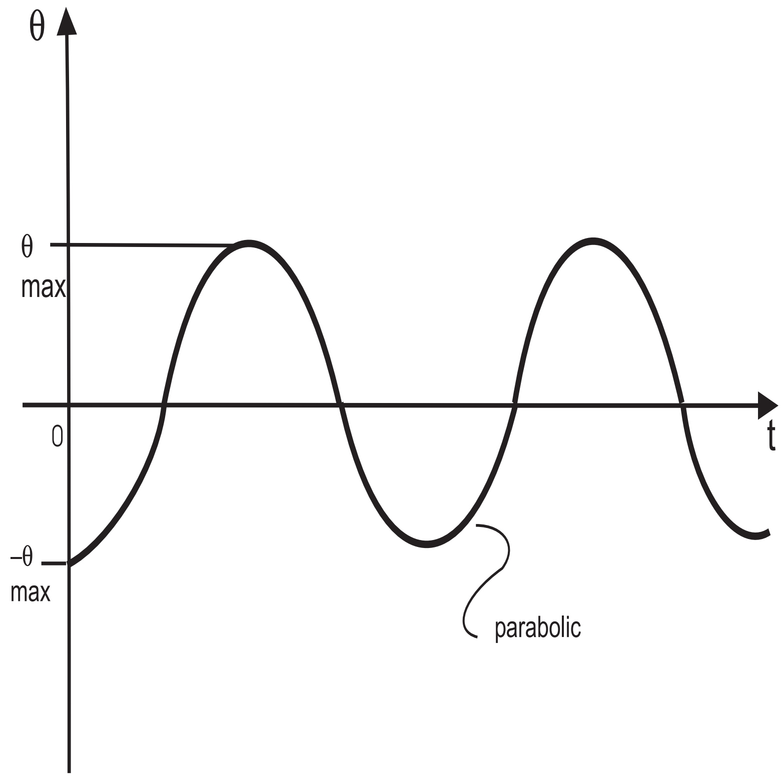
$$-\theta_0 = \frac{I}{2N}\left[\frac{\Delta^2 N}{2I}\left(\frac{1}{\tau - \Delta}\right) + \frac{\Delta N}{I}\right]^2,$$

or,

$$|\theta_0| = \frac{I}{2N}\left[\frac{\Delta N(\Delta + 2(\tau - \Delta))}{2I(\tau - \Delta)}\right]^2,$$

$$|\theta_0| = \frac{N\Delta^2}{8I}\left[\frac{2\tau-\Delta}{\tau-\Delta}\right]^2 = |\theta_{\max}|.$$

Time history shown in the Figure below and shows a “nonlinear oscillator.”



Nonlinear oscillator for Problem 9.24.

25. Consider the point mass pendulum with zero friction as shown in Figure 9.67. Using the method of isoclines as a guide, sketch the phase-plane portrait of the motion. Pay particular attention to the vicinity of $\theta = \pi$. Indicate a trajectory corresponding to spinning of the bob around and around rather than oscillating back and forth.

    (a) **Solution:** The equations are:

$$\begin{aligned} I\ddot{\theta} &= mgl\sin\theta, \\ ml^2\ddot{\theta} &= mgl\sin\theta, \\ \ddot{\theta} &= \frac{g}{l}\sin\theta, \end{aligned}$$

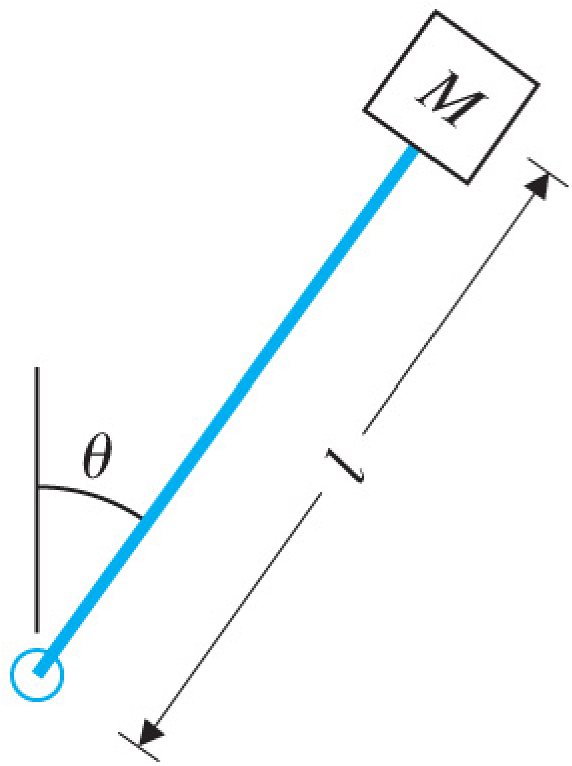


Figure 9.67: Pendulum for Problem 9.25

For $\theta \sim 0$, $\sin\theta \sim \theta$,

$$\begin{aligned} \ddot{\theta} &= \frac{g}{l}\theta, \\ s^2 &= \frac{g}{l}, \\ s &= \pm\sqrt{\frac{g}{l}}. \end{aligned}$$

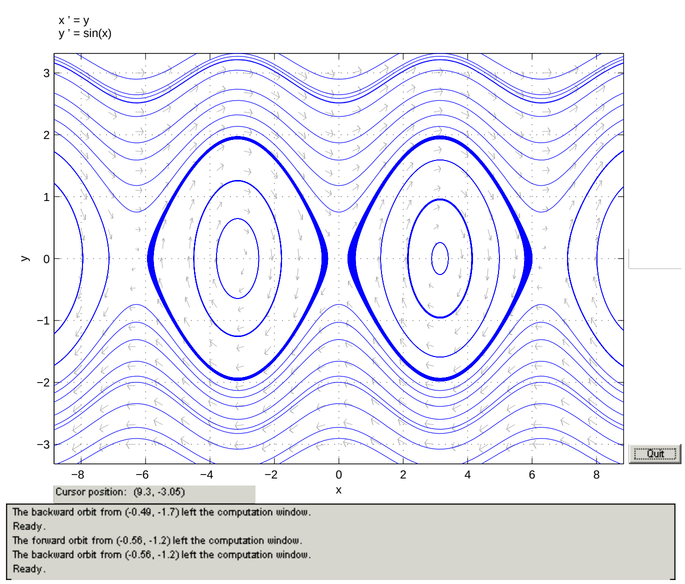
There is a "saddle" point at $\theta = 0, 2\pi, \ldots$.
For $\theta \sim \pi$, $\sin\theta \sim -\theta$,

$$\begin{aligned} \ddot{\theta} &= -\frac{g}{l}\theta, \\ s^2 &= -\frac{g}{l}, \\ s &= \pm j\sqrt{\frac{g}{l}}. \end{aligned}$$

There is a "center" at $\theta = \pm\pi, \pm 3\pi, \ldots$.
Using isoclines:

$$\begin{aligned} \ddot{\theta} &= \dot{\theta}\frac{d\dot{\theta}}{d\theta} = \frac{g}{l}\sin\theta \\ \alpha &= \frac{d\dot{\theta}}{d\theta} = \frac{g}{l}\sin\theta. \end{aligned}$$

The isoclines are sinusoidal curves. The phase portraits are shown in the Figure below. The upper and lower portraits correspond to the "whirling motion" with pendulum going round and round.



Phase portraits for Problem 9.25.

NOTE: The phase portraits can be generated using the ODE Software in Matlab pplane7.m (for Matlab Version 7.7) by Professor John C. Polking at Rice University available on the Web at:

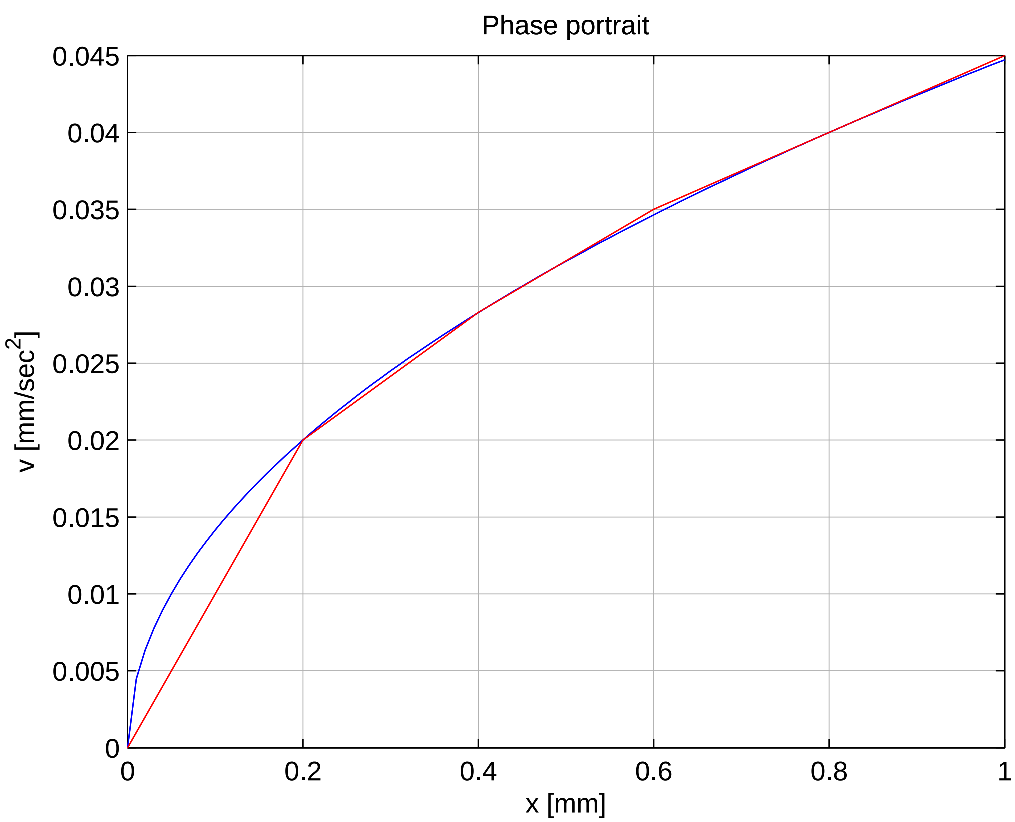
http://math.rice.edu/ ∼ dfield/

26. Draw the phase trajectory for a system

$$\ddot{x} = 10^{-6} \ \text{m/ sec}^{2}$$

between $\dot{x}(0) = 0$, $x(0) = 0$ and $x(t) = 1$ mm. Find the transition time, t_f , by graphical means from the parabolic curve by comparing your solution with several different interval sizes and the exact solution.

Solution: The phase portrait is shown in the figure below.



Phase portrait for Problem 9.26.

$$
\begin{aligned}
\ddot{x} &= a, \frac{\dot{v}}{\dot{x}} = \frac{a}{v}, \\
\frac{v^2}{2} &= ax \text{ and } a = 10^{-6}, \\
v &= \sqrt{2ax} = at + v_o = at, \\
x(t) &= \frac{a}{2}t^2 + v_o t + x_o = 5 \times 10^{-7} t^2 = 1 \times 10^{-3}, \\
t^2 &= \left(\frac{1}{5}\right) \times 10^4 \Rightarrow t_f = 45\,\text{sec}.
\end{aligned}
$$

$$v = \frac{\Delta x}{\Delta t}.$$

To obtain t graphically, i.e., by graphical integration, we write,

$$\Delta t = \frac{\Delta x}{\hat{v}},$$

where $\hat{v}$ is the "average" v in a given interval of Δx . In the figure above, we divide x into five intervals and find (going from left to the right),

$$
\begin{aligned}
t_f &= \Delta t_1 + \Delta t_2 + \Delta t_3 + \Delta t_4 + \Delta t_5 \\
&= \frac{0.2}{0.01} + \frac{0.2}{0.025} + \frac{0.2}{0.0315} + \frac{0.2}{0.037} + \frac{0.2}{0.0425} \\
&= 20 + 8 + 6 + 5.4 + 4.7 = 44.1\,\text{sec}.
\end{aligned}
$$

which compares well with the exact answer of $t = 45$ sec . Better approximation can be found by finer division of x . Alternatively we can compute the time from,

$$t_f = \int_0^x \frac{1}{v} dx$$

which means the time can also be found by finding the area under the $v(x)^{-1}$ plot.

27. Consider the system with equations of motion,

$$\ddot{\theta} + \dot{\theta} + \sin\theta = 0$$

a) What physical system does this correspond to?

b) Draw the phase portraits for this system.

c) Show a specific trajectory for $\theta_0 = 0.5$ rad $\dot{\theta} = 0$.

Solution:

(a) Physical system is a pendulum with a hinge damping.

$$
\begin{aligned}
I\,\ddot{\theta} + b\,\dot{\theta} + mgl\sin\theta &= 0, \\
\dot{\theta} &= \omega, \\
\dot{\omega} &= -\frac{b}{I}\omega - \frac{mgl}{I}\sin\theta.
\end{aligned}
$$

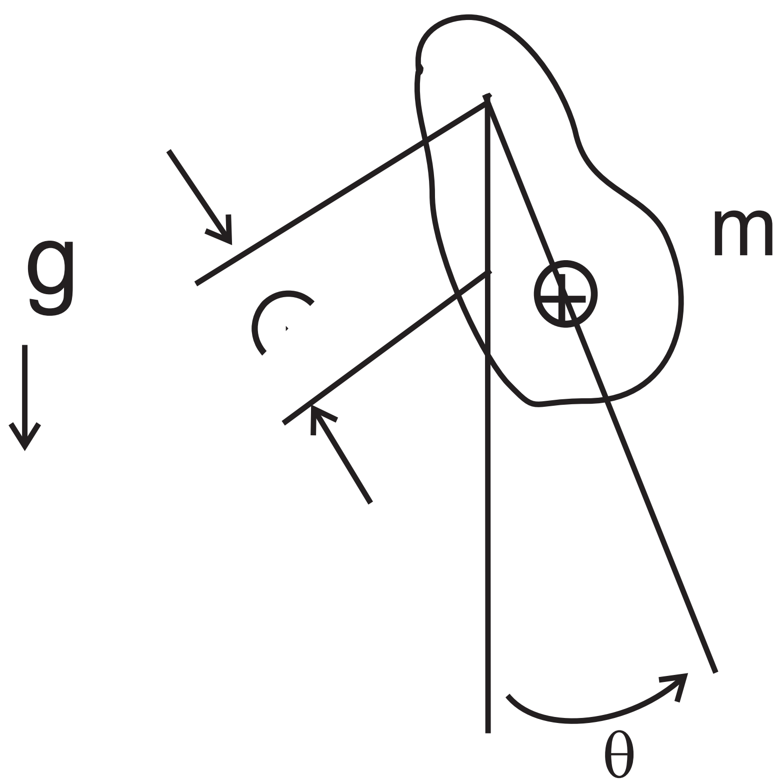
Independent variable scaled for $\tau \triangleq \frac{t}{T}$ where $T^2 = \frac{I}{mgl}$,

$$\frac{d\theta}{T\frac{dt}{T}} = \frac{1}{T}\frac{d\theta}{d\tau} = \omega.$$

Let us define $\Omega \triangleq T\omega$.

$$\begin{aligned}\frac{d\theta}{d\tau} &= \Omega,\\ \frac{d\Omega}{d\tau} &= -\left(\frac{Tb}{I}\right)\Omega - \sin\theta.\end{aligned}$$

.



Pendulum free body diagram for Problem 9.27.

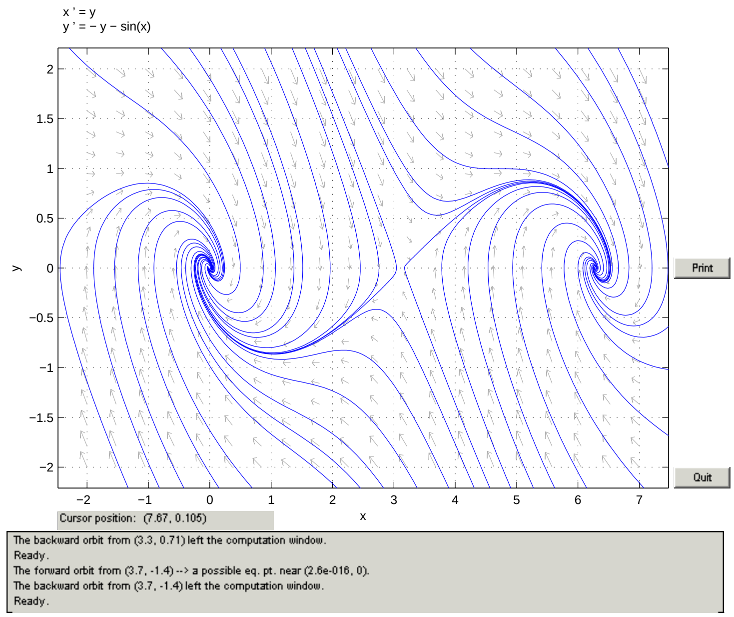
So the problem is a special case with $\frac{Tb}{I} = \sqrt{\frac{b^2}{\mathrm{Im}\, gl}} = 1$.

$$\frac{d\Omega}{d\theta} = -1 - \frac{\sin\theta}{\Omega}\ ,$$

isoclines with slope $M = -1 - \frac{\sin\theta}{\Omega}$, where M is the slope.

(b) See phase portrait figure shown in the Figure below. Note the unstable equilibrium corresponding to $\theta = \pi$ and the stable equilibrium corresponding to the origin and $\theta = 2\pi$.

(c) See trajectory corresponding to ($x = \theta_0 = 0.5\text{rad},\ y = \dot{\theta}_0 = 0$) in the phase portrait in the Figure below using pplane7.m software. .



Phase portraits for pendulum with damping for Problem 9.27.

28. Consider the nonlinear upright pendulum with a motor at its base as an actuator. Design a feedback controller to stabilize this system.

Solution:

$$\ddot{\theta} = \sin\theta + u.$$

U sin g a lead network: $U(s) = -\frac{4(s+1)}{(s+3)}\Theta(s)$

$$\begin{aligned} x_1 &= \theta, \\ x_2 &= \dot{\theta}, \\ x_3 &= x_c. \end{aligned}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} x_2 \\ \sin x_1 - 4x_1 - 4x_3 \\ -3x_3 - 2x_1 \end{bmatrix}.$$

The linearized system is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -3 & 0 & -4 \\ -2 & 0 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

The system has poles at

$$\det(s\mathbf{I} - \mathbf{F}) = (s+1)^3.$$

The system is asymptotically stable near the origin so if the system starts near the upright position, it will be balanced.

29. Consider the system
$$\dot{x} = -\sin x$$
Prove that the origin is an asymptotically stable equilibrium point.

Solution: We wish to show that
$$\dot{V}(x) \le -x^T Q x \ .$$
Select the Lyapunov function for
$$\begin{aligned} P &= 1: \\ V(x) &= x^2, \end{aligned}$$
then
$$\begin{aligned} \dot{V} &= 2x\dot{x} = -2x\sin x, \\ \text{and} \quad \dot{V}(x) &\le -x^2 \qquad \text{for } |x| \le 1. \\ \text{Si nce } \sin x &\ge \frac{1}{2}x \text{ for } 0 \le x \le 1, \\ \text{we choose } Q &= 1, \end{aligned}$$
and conclude that the origin is an asymptotically stable equilibrium point.

30. A first-order nonlinear system is described by the equation $\dot{x} = -f(x)$, where $f(x)$ is a continuous and differentiable nonlinear function that satisfies the following:
$$\begin{aligned} &f(0) = 0, \\ &f(x) > 0 \text{for} x > 0, \\ &f(x) < 0 \text{for} x < 0. \end{aligned}$$
Use the Lyapunov function $V(x) = x^2/2$ to show that the system is stable near the origin $(x = 0)$.

Solution:
$$\begin{aligned} \dot{x} &= -f(x), \\ V(x) &= \frac{1}{2}x^2, \\ \dot{V}(x) &= x\dot{x} = -xf(x), \\ \text{For } x &> 0 \text{ and } f(x) > 0 \Longrightarrow \dot{V}(x) < 0, \\ \text{For } x &< 0 \text{ and } f(x) < 0 \Longrightarrow \dot{V}(x) < 0, \\ \text{For } x &= 0 \text{ and } f(x) = 0 \Longrightarrow \dot{V}(x) = 0. \end{aligned}$$
Thus, for all $x \ne 0$, $\dot{V} < 0$. So applying Lyapunov's stability criterion, we conclude that the system is stable.

31. Use the Lyapunov equation
$$\mathbf{A}^T\mathbf{P} + \mathbf{PA} = -\mathbf{Q} = -\mathbf{I}$$
to find the range of K for which the system in Fig. 9.68 will be stable. Compare your answer with the stable values for K obtained using Routh's stability criterion.

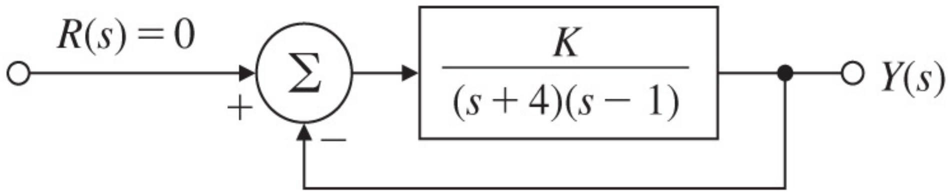


Figure 9.68: Control system for Problem 9.31

Solution:

Our approach is to set up the continuous Lyapunov equation and check that $\mathbf{P}$ is a positive definite matrix, i.e., $\mathbf{P} > 0$. Let,

$$\mathbf{P} = \begin{bmatrix} p & q \\ q & r \end{bmatrix}.$$

From the figure, the *closed-loop* system matrix $\mathbf{A}$ in controller canonical form is,

$$\mathbf{A} = \begin{bmatrix} -3 & 4-k \\ 1 & 0 \end{bmatrix}.$$

Solving $\mathbf{A}^T\mathbf{P} + \mathbf{P}\mathbf{A} = -\mathbf{I}$ yields,

$$\begin{aligned}
\begin{bmatrix} p & q \\ q & r \end{bmatrix}\begin{bmatrix} -3 & 1 \\ 4-k & 0 \end{bmatrix} + \begin{bmatrix} p & q \\ q & r \end{bmatrix}\begin{bmatrix} -3 & 4-k \\ 1 & 0 \end{bmatrix} &= \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \\
2q - 6p &= -1, \\
2q(4-k) &= -1, \\
p(4-k) + r - 3q &= 0.
\end{aligned}$$

Hence,

$$\begin{aligned}
q &= \frac{-1}{2(4-k)}, \\
p &= \frac{1}{6}\left(\frac{3-k}{4-k}\right), \\
r &= \frac{-k^2 + 7k - 21}{6(4-k)}.
\end{aligned}$$

The two conditions for $\mathbf{P} > 0$ are $p > 0$ and $pr - q^2 > 0$, or,

$$p > 0 \Longrightarrow k > 3 \;\text{ or }\; k > 4,$$

and,

$$\begin{aligned}
pr - q^2 &= \frac{k^3 - 10k^2 + 42k - 72}{36(k-4)^2} > 0, \\
&= \frac{(k-4)(k^2 - 6k + 18)}{36(k-4)^2} > 0.
\end{aligned}$$

which is satisfied when $k > 4$, since $(k^2 - 6k + 18)$ is always positive. Thus, $k > 4$ for stability. Forming the Routh array, we have,

$$\begin{array}{cc} 1 & k-4 \\ 3 & 0 \\ 3(k-4) & \end{array}$$

Recall that the condition for stability is that all of the coefficients in the first column must be positive, which agrees with our previous answer above, namely $k > 4$.

Remark: Back of the envelope calculations using the Routh array are handy when the order of the system is low (as in this example). However for higher order systems, use of Lyapunov equation solvers, such as MATLAB's `lyap` command, are recommended.

32. Consider the system

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 u \\ x_2(x_2 + u) \end{bmatrix}, y = x_1.$$

Find all values of α and β for which the input $u(t) = \alpha y(t) + \beta$ will achieve the goal of maintaining the output $y(t)$ near 1.

Solution:

(a) It is desired to maintain the output $y(t)$ of the system,

$$\begin{aligned} \frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= \begin{bmatrix} x_1 + x_1 + x_2 u \\ x_2(x_2 + u) \end{bmatrix}, \\ y &= x_1. \end{aligned}$$

near 1. Find all values of α and β for which the input $u(t) = \alpha y(t) + \beta$ will achieve this goal. The problem has two parts: First, we investigate the equilibrium points; Next, we investigate the stability of the system by linearizing the nonlinear state equations near these equilibria. The nonlinear, closed-loop system equations are,

$$\begin{aligned} \dot{x}_1 &= x_1 + x_2(\alpha x_1 + \beta), \\ \dot{x}_2 &= x_2(x_2 + \alpha x_1 + \beta). \end{aligned}$$

To find the equilibrium points for the desired output of $y = 1$, we set $x_1 = 1$, $\dot{x}_1 = \dot{x}_2 = 0$, to get,

$$\begin{aligned} 0 &= 1 + x_2(\alpha + \beta), \\ 0 &= x_2(x_2 + \alpha + \beta), \end{aligned}$$

which can be solved for the equilibrium values of x_2 and the necessary relationship between α and β . Simultaneous solution yields,

$$x_2 = -\frac{1}{\alpha + \beta}.$$

and,

$$0 = x_2^2 + x_2(\alpha + \beta) = x_2^2 - 1 \Longrightarrow x_2 = \pm 1.$$

Consider the two equilibrium cases:

$$x_1 = 1,\ x_2 = 1 : \ \text{Let}\ y_1 = x_1 - 1,\ y_2 = x_2 - 1,\ \text{and}\ \alpha + \beta = -1.$$

Substituting these into the nonlinear closed-loop equations, we get,

$$\begin{aligned} \dot{y}_1 &= (1+\alpha)y_1 - y_2 + \alpha y_1 y_2, \\ \dot{y}_2 &= \alpha y_1 + y_2 + \alpha y_1 y_2 + y_2^2. \end{aligned}$$

The characteristic equation of the linearized system is,

$$s^2 - (\alpha + 2)s + (2\alpha + 1) = 0.$$

There are no values of α which produce stable roots. So we conclude $x_1 = 1$ and $x_2 = 1$ is an unstable equilibrium point.

$$x_1 = 1,\ x_2 = -1 : \ \text{Let}\ \ y_1 = x_1 - 1,\ y_2 = x_2 + 1,\ \ \text{and}\ \ \alpha + \beta = 1.$$

Then,

$$\begin{aligned} \dot{y}_1 &= (1-\alpha)y_1 + y_2 + \alpha y_1 y_2, \\ \dot{y}_2 &= -\alpha y_1 - y_2 + \alpha y_1 y_2 + y_2^2. \end{aligned}$$

The characteristic equation of the linearized system is,

$$s^2 + \alpha s + (2\alpha - 1) = 0.$$

So the system is stable for small signals near the equilibrium point if,

$$\alpha > 1/2 \ \ \text{and}\ \ \alpha + \beta - 1 = 0.$$

33. Consider the nonlinear autonomous system

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_2(x_3 - x_1) \\ x_1^2 - 1 \\ -x_1 x_3 \end{bmatrix}.$$

a) Find the equilibrium point(s).
b) Find the linearized system about each equilibrium point.
c) For each case in part(b), what does Lyapunov theory tell us about the stability of the nonlinear system near the equilibrium point?

Solution:

(a) Setting $\dot{x}_1 = \dot{x}_2 = \dot{x}_3 = 0$ and solving the nonlinear equations, we obtain $[1, 0, 0]^T$ and $[-1, 0, 0]^T$ as the equilibrium points.

(b) We linearize the nonlinear state equations around the two equilibrium points from the first part.

(i)

$$\mathbf{x} = [1, 0, 0]^T : \ \text{Let} \quad y_1 = x_1 - 1,\ y_2 = x_2,\ \ \text{and}\ \ y_3 = x_3.$$

Then the nonlinear equations become,

$$\begin{aligned}\dot{y}_1 &= -y_2 + y_2 y_3 - y_1 y_2,\\ \dot{y}_2 &= 2y_1 + y_1^2,\\ \dot{y}_3 &= -y_3 - y_1 y_3.\end{aligned}$$

Thus, the linearized system is $\dot{\mathbf{y}} = \mathbf{A}\mathbf{y}$ where,

$$\mathbf{A} = \begin{bmatrix} 0 & -1 & 0 \\ -2 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

(ii)

$$\mathbf{x} = [-1, 0, 0]^T : \text{Let } \ y_1 = x_1 + 1, \ y_2 = x_2, \text{ and } \ y_3 = x_3.$$

Then the nonlinear equations become,

$$\begin{aligned}\dot{y}_1 &= y_2 + y_2 y_3 - y_1 y_2,\\ \dot{y}_2 &= -2y_1 + y_1^2,\\ \dot{y}_3 &= y_3 - y_1 y_3.\end{aligned}$$

Thus, the linearized system is $\dot{\mathbf{y}} = \mathbf{A}\mathbf{y}$ where,

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ -2 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

(c) We can use the linearization from the previous part to determine the stability of the system near the two equilibria.

(i) The characteristic equation is $(s^2+2)(s+1) = 0$. The linear system is neutrally (marginally) stable with two poles on the $j\omega$ axis. So Lyapunov theory does not tell us whether this system is stable, and the nonlinear terms will affect the stability at the equilibrium point $[1, 0, 0]$.

(ii) The characteristic equation is $(s^2 + 2)(s - 1) = 0$. Thus the system at the equilibrium point $[-1, 0, 0]$ is unstable.

34. Consider the circuit shown in Figure 9.69. For what diode characteristics will this system be stable?

Solution:

The system equations are:

$$\text{Device: } C\frac{dv}{dt} = i_C; \ L\frac{di_L}{dt} = v; \ i_D = f(v), \ f(0) = 0.$$

$$\begin{aligned}\text{KCL} \quad &: \quad i_C + i_L + i_D = 0,\\ C\frac{dv}{dt} + i_L + f(v) &= 0,\\ L\frac{di_L}{dt} &= v.\end{aligned}$$

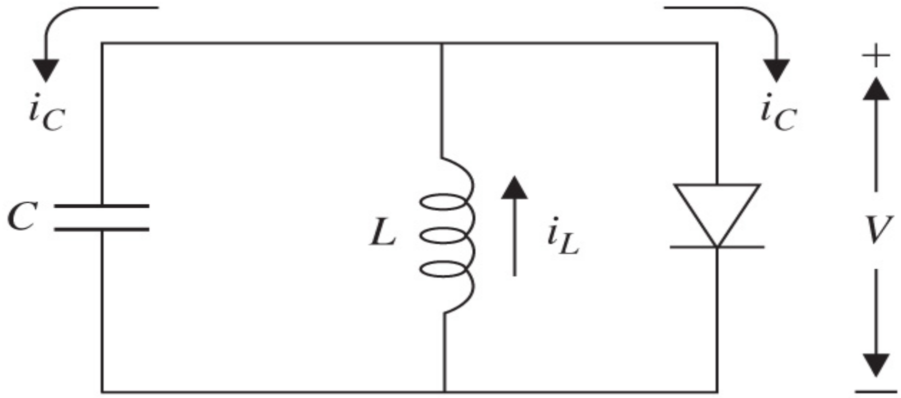
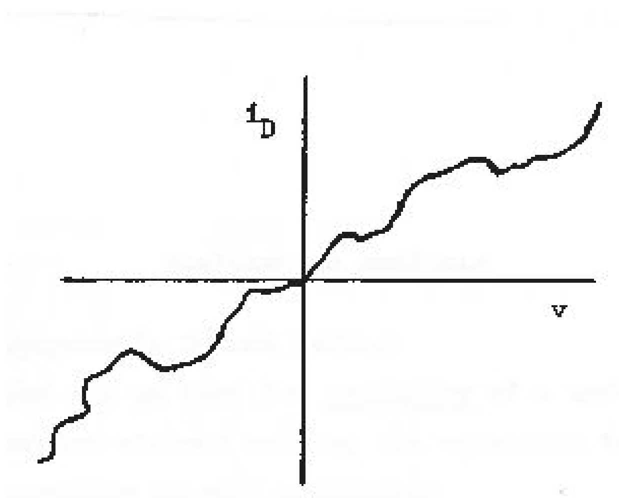


Figure 9.69: Circuit diagram for Problem 9.34

$$
\begin{aligned}
\text{Energy} \quad &: \quad V = \frac{1}{2}cv^2 + \frac{1}{2}Li_L^2, \\
\dot{V} &= Cv\dot{v} + Li_L\frac{di_L}{dt} \\
&= Cv\left[\frac{-iL}{C} - \frac{f(v)}{C}\right] + Li_L\left[\frac{v}{L}\right] \\
&= -vf(v).
\end{aligned}
$$



Diode characteristics.

So the system will be stable for any positive D.C. diode characteristics.

35. Van der Pol's equation: Consider the system described by the nonlinear equation

$$\ddot{x} + \varepsilon(1 - x^2)\dot{x} + x = 0$$

with the constant $\varepsilon > 0$.

(a) Show that the equations can be put in the form [Liénard or (x, y) plane]:

$$
\begin{aligned}
\dot{x} &= y + \varepsilon\left(\frac{x^3}{3} - x\right) \\
\dot{y} &= -x\,.
\end{aligned}
$$

(b) Use the Lyapunov function $V = \frac{1}{2}(x^2 + \dot{x}^2)$ and sketch the *region* of stability as predicted by this V in the Liénard plane.

(c) Plot the trajectories of part (b) and show the initial conditions that tend to the origin. Simulate the system in Simulink® using various initial conditions on $x(0)$ and $\dot{x}(0)$. Consider two cases with $\varepsilon = 0.5$, and $\varepsilon = 1.0$.

Solution:

(a) If we differentiate the first Liénard equation, we obtain

$$
\begin{aligned}
\ddot{x} &= \dot{y} + \varepsilon\left(\frac{3x^2}{3}\dot{x} - \dot{x}\right) = -x + \varepsilon\left(x^2\dot{x} - \dot{x}\right) \\
&= -x - \varepsilon(1 - x^2)\dot{x}.
\end{aligned}
$$

which is the same as van der Pol's equation. Hence the two representations are equivalent. The two coordinate systems are related by the transformation,

$$
\begin{bmatrix} x \\ \dot{x} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \varepsilon(\frac{x^2}{3} - 1) & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}
$$

or,

$$
\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\varepsilon(\frac{x^2}{3} - 1) & 1 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \end{bmatrix}
$$

(b) If we linearize the system, we obtain

$$
\ddot{x} + \varepsilon\dot{x} + x = 0
$$

which has both roots inside the LHP at $\frac{-\varepsilon \pm \sqrt{\varepsilon^2 - 4}}{2}$. Therefore, there is a region of stability around the origin. Define $x_1 \triangleq x,\ x_2 \triangleq \dot{x}$,

$$
\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -\varepsilon(1 - x_1^2)x_2 - x_1.
\end{aligned}
$$

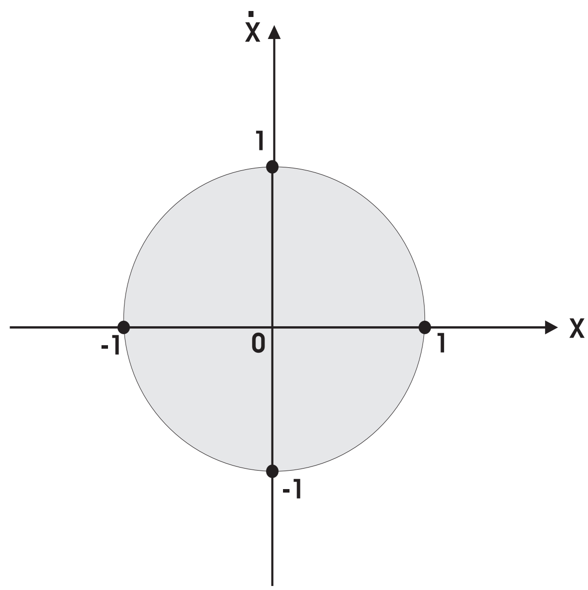
Then,

$$
\begin{aligned}
\dot{V} &= x\,\dot{x} + \dot{x}\,\ddot{x} = \dot{x}\left[x - \varepsilon(1 - x^2)\dot{x} - x\right] \\
&= -\varepsilon\dot{x}^2(1 - x^2).
\end{aligned}
$$

Now,

$$
\dot{V} \le 0 \Rightarrow (1 - x^2) \geqslant 0 \quad \Rightarrow |x| \le 1.
$$

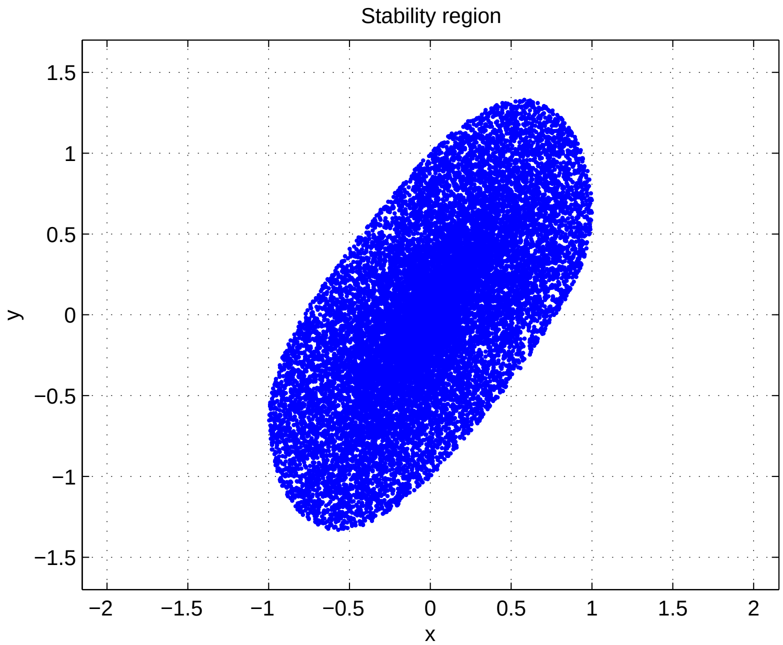
The level curves where $V = c =$ constant are circles centered at the origin of the (x , $\dot{x}$) plane. The theory requires that the region of stability be *inside* a level curve where $V < c$. Therefore, all trajectories starting inside a circle of radius one centered at the origin (see Figure below) converge to the origin of the (x , $\dot{x}$) plane. This means that the origin is Lyapunov-stable. It also means that the limit cycle must lie outside a circle of radius one centered at the origin.



The stability region may be mapped into the Liénard plane. The circular boundary in the (x , $\dot{x}$) plane can be mapped into the (x , y) plane:

$$y = -\varepsilon\left(\frac{x^3}{3} - x\right) + \sqrt{1 - x^2}$$

which resembles an ellipsoidal curve as shown in the Figure on top of the next page.



Stability region in the Liénard (x , y) plane.

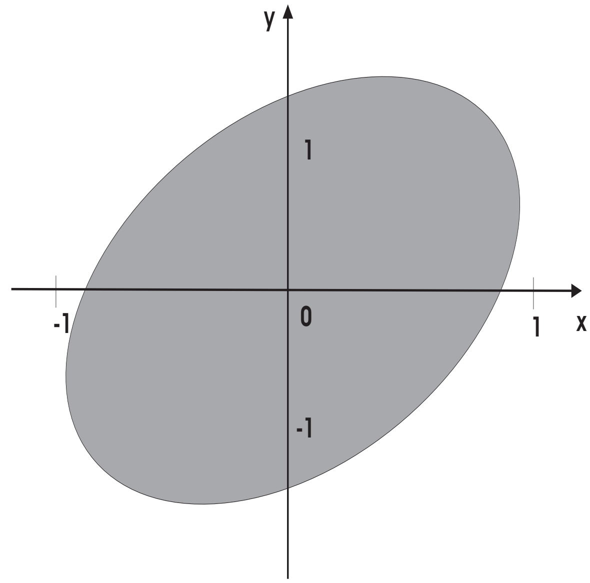
An approximate analytical answer is also possible. The stability region in the Liénard (x , y) plane is roughly a rotated ellipse. This can be seen as follows.

$$\begin{aligned} V &= \frac{1}{2}(x^2+\dot{x}^2) = c \\ &= \frac{1}{2}\left(x^2+\left[y+\varepsilon\left(\frac{x^3}{3}-x\right)\right]^2\right) \\ &= \frac{1}{2}\left(x^2+y^2+\varepsilon^2\left(\frac{x^3}{3}-x\right)^2+2y\varepsilon\left(\frac{x^3}{3}-x\right)\right) \end{aligned}$$

Assuming that ε^2 and x^3 are small and may be neglected,

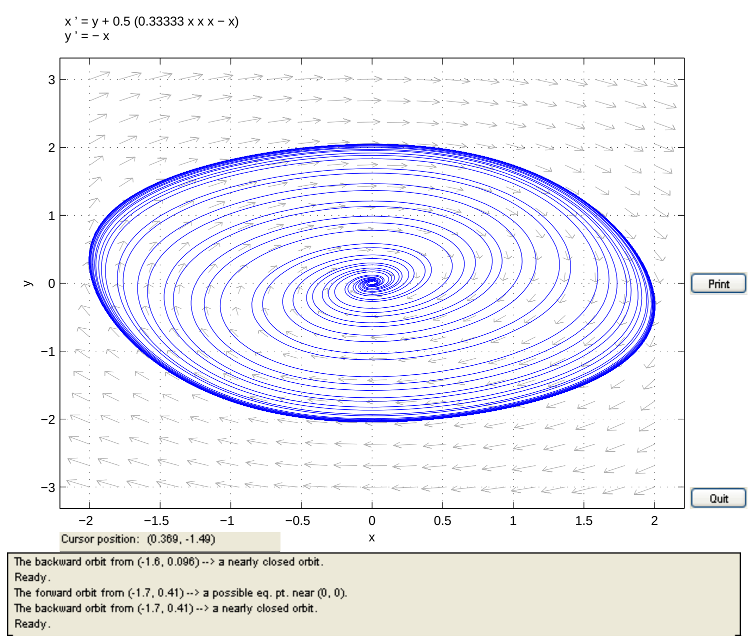
$$V \approx \frac{1}{2}\left(x^2+y^2-2\varepsilon yx\right) \approx c$$

It is seen that the level curves are roughly ellipses that are rotated by an angle of $+45°$ in the Liénard (x , y) plane.

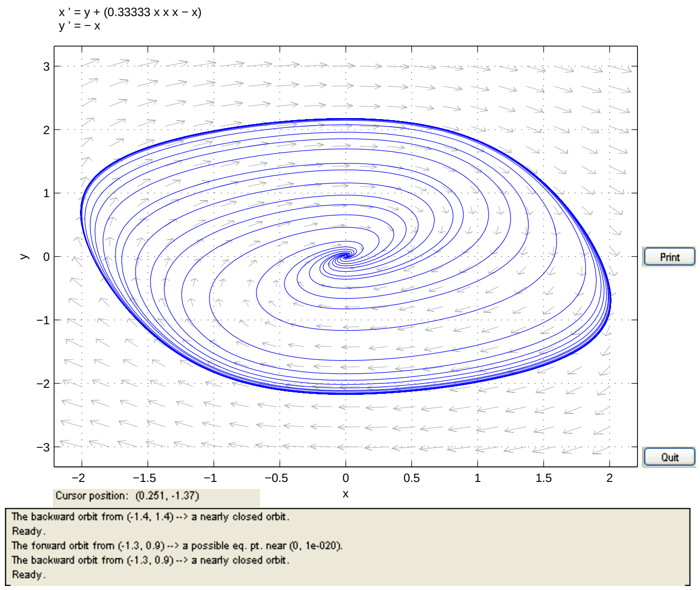


Approximation of stability region in the Liénard (x , y) plane.

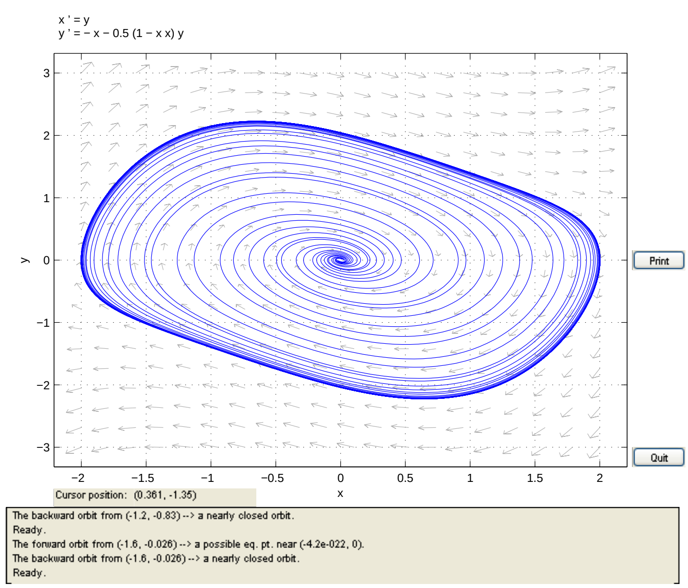
(c) Using pplane7.m software we see that the limit cycle is *nearly circular* with radius 2:



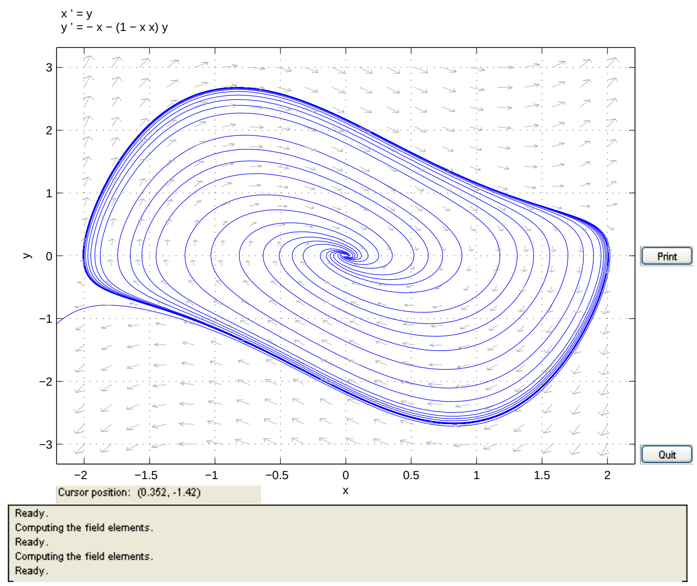
Phase portraits for van der Pol equation in Liénard (x , y) form for $\varepsilon = 0.5$.



Phase portraits for van der Pol equation in the Liénard (x , y) plane for $\varepsilon = 1.0$.

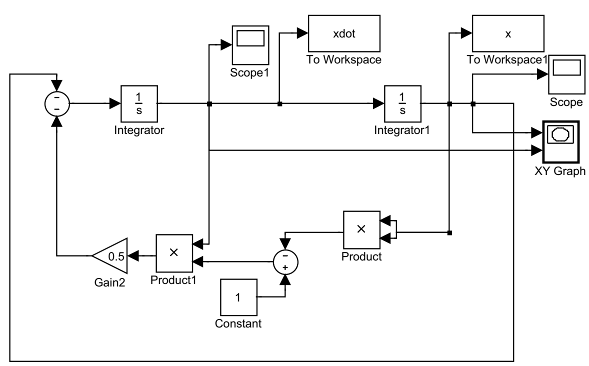


Phase portraits for van der Pol equation in the (x , $\dot{x}$) plane for $\varepsilon = 0.5$.

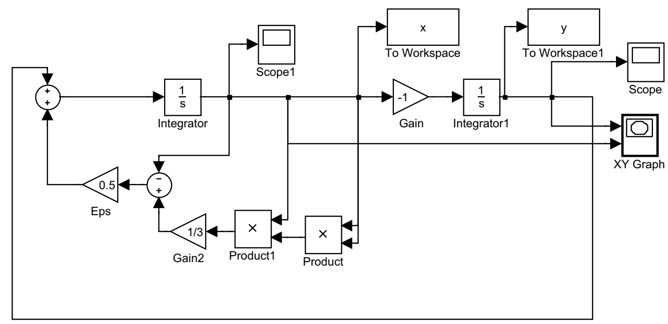


Phase portraits for van der Pol equation in the (x , $\dot{x}$) plane for $\varepsilon = 1.0$.

The Simulink simulations are shown in the following figures.

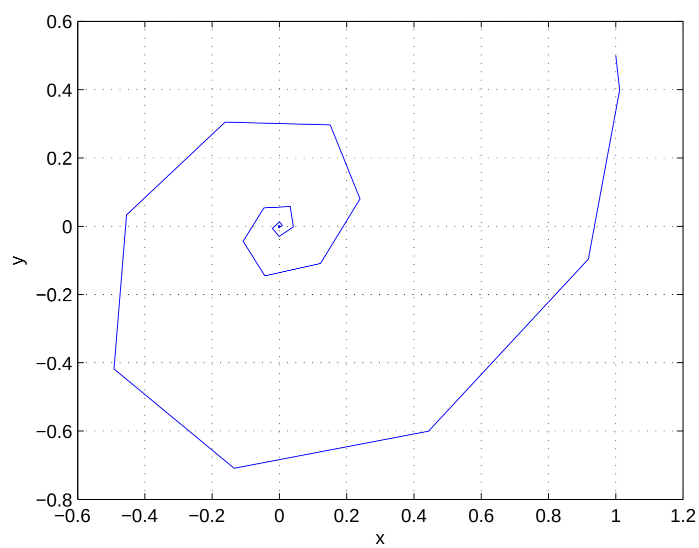


Simulink simulation diagram for van der Pol's equation for (x , $\dot{x}$) plane.

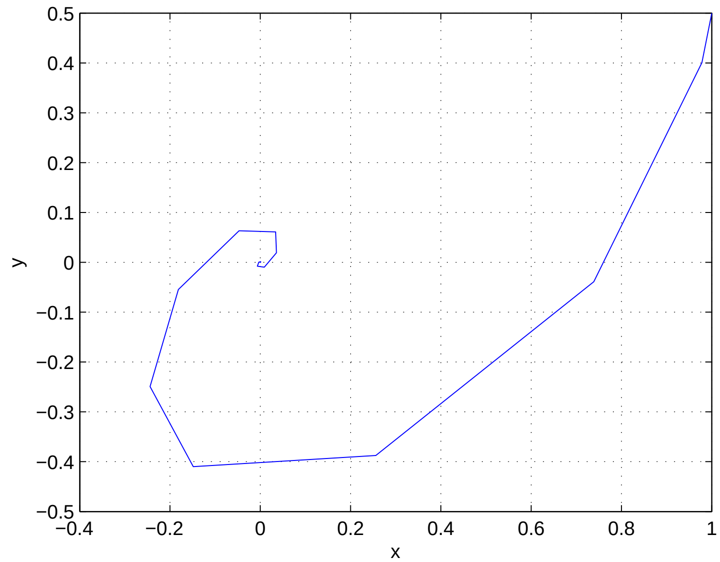


Simulink simulation of van der Pol's equation in the Liénard (x , y) plane.

Sample trajectories are shown in the following figures.

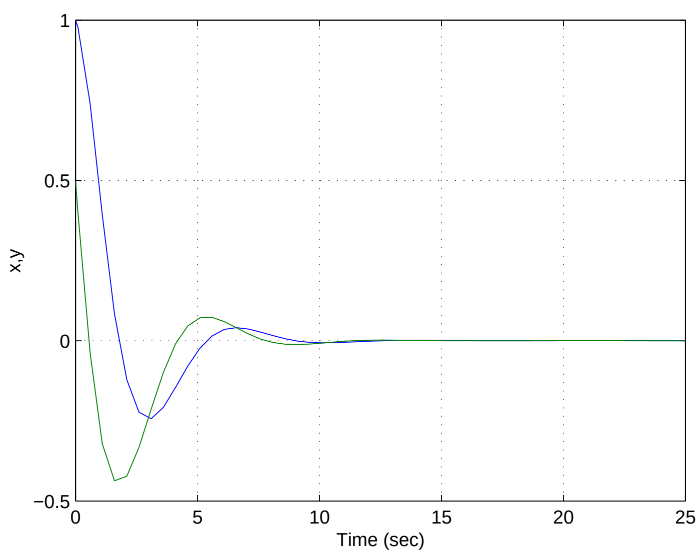


Trajectory for $x(0) = 1$, $y(0) = 0.5$ in the Liénard (x , y) plane for $\varepsilon = 0.5$.

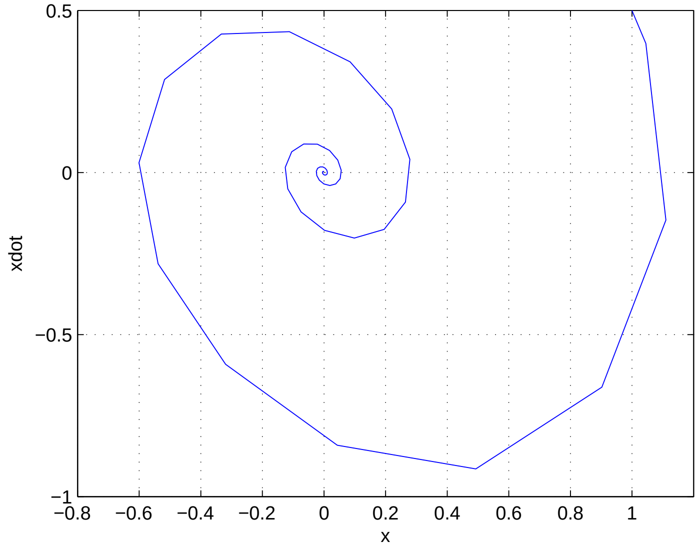


Trajectory for $x(0) = 1$, $y(0) = 0.5$ in the Liénard (x , y) plane for $\varepsilon = 1.0$.

The corresponding time domain responses are shown in the following figures.

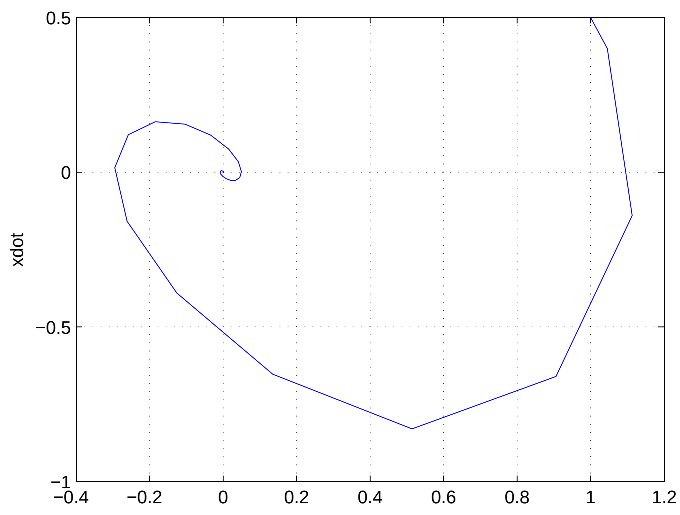


State variables for $x(0) = 1$, $y(0) = 0.5$ in the Liénard plane for $\varepsilon = 1.0$.

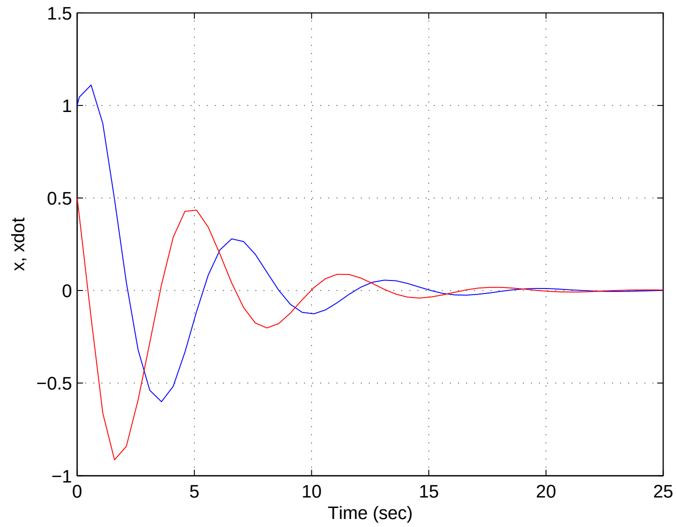


Trajectory for $x(0) = 1$, $\dot{x}(0) = 0.5$ in the (x , $\dot{x}$) plane for $\varepsilon = 0.5$.

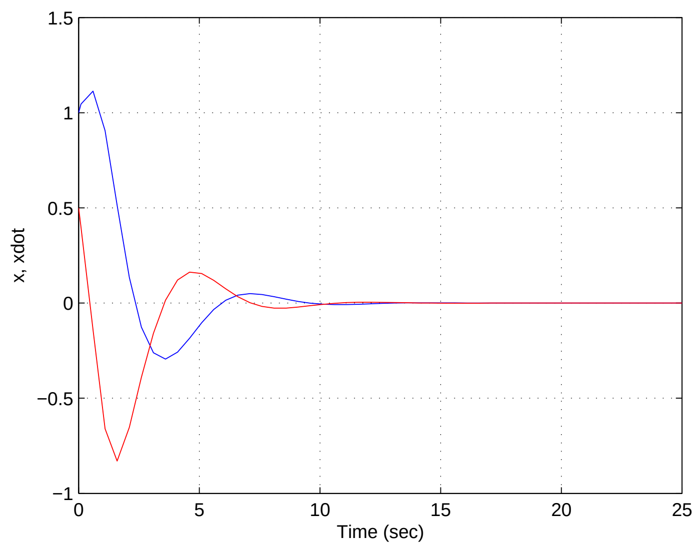
.



Trajectory for $x(0) = 1$, $\dot{x}(0) = 0.5$ in the (x , $\dot{x}$) plane for $\varepsilon = 1.0$



State variables for $x(0) = 1$, $\dot{x}(0) = 0.5$ in the (x , $\dot{x}$) plane for $\varepsilon = 0.5$.



State variables for $x(0) = 1$, $\dot{x}(0) = 0.5$ in the (x , $\dot{x}$) plane for $\varepsilon = 1.0$.

36. *Duffing's equation*: Consider the system described by the nonlinear differential equation

$$\ddot{x} + k\dot{x} + \varepsilon x^3 = u$$

where $u = A\cos(t)$. This equation represents the model of a hard spring where k is the spring constant and if $\varepsilon > 0$, the spring gets stiffer as the displacement increases. Let $k = 0.05$, $\epsilon = 1$, and $A = 7.5$.

(a) Build a simulation of the system in SIMULINK. Show that the system response can be very sensitive to slight perturbations on the initial conditions $x(0)$, $\dot{x}(0)$ (the system is said to be *chaotic*). Simulate the response of the system with $x(0) = 3$ and $\dot{x}(0) = 4$ for $t = 30$ sec. Repeat the simulation for slightly perturbed initial conditions $x(0) = 3.01$ and $\dot{x}(0) = 4.01$. Compare the two results.

(b) Consider the unforced Duffing equation ($u = 0$). Plot the time response of the system for $x(0) = 1$, $\dot{x}(0) = 1$ for $t = 200$ sec. Draw the phase-plane plot for the system. Show that the origin is an equilibrium point.

(c) Now consider the forced Duffing equation ($u \neq 0$). Find the solution to the Duffing equation for $x(0) = -1$, $\dot{x}(0) = 1$ for $t = 30$ sec. Draw the phase-plane plot ($\dot{x}(t)$ vs. $x(t)$) for this case.

(d) Repeat part (c) for $k = 0.25$, $\epsilon = 1$, and $A = 8.5$.

(e) Repeat part (c) for $k = 0.1$, $\epsilon = 1$, and $A = 11$.

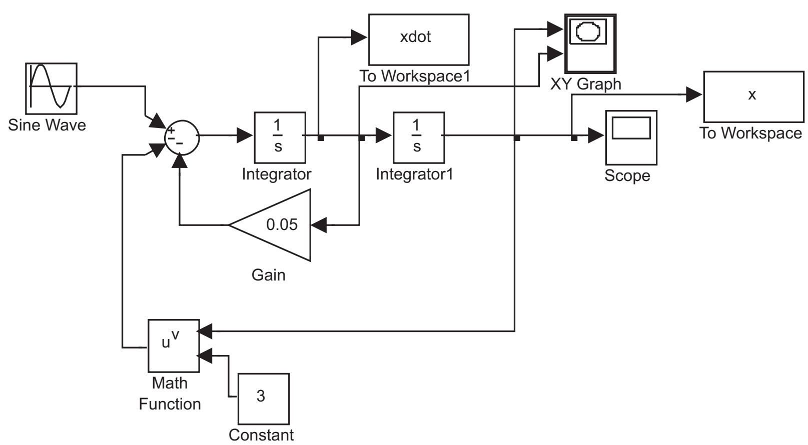
(f) We can get more insight into the system by plotting $\dot{x}(t_j)$ vs. $x(t_j)$ at several hundred points at 2π periodic observation times. In other words rather than looking at the system continuously, we "strobe" the system and plot the behavior at strobe times only. Show that unlike the parametric plots in parts (c)-(e), the points fall on a well-structured plot referred to as a Poincaré section (also called a strange attractor). Plot the Poincaré sections for parts (c)-(e). Simulate the system using the initial conditions $x(0) = -1$ and $\dot{x}(0) = 1$ for $t = 10,000$ sec in order to plot the Poincaré sections.

(g) What can you conclude about the nature of the solution of the Duffing equation from the results of the previous parts?

(h) Characterize the system behavior in terms of the ranges of the system parameters k , ϵ , and A .

Solution:

(a) The SIMULINK implementation of the system is shown in the following figure.



Simulation of the Duffing equation.

We see that the two responses become quite different especially over a longer period of time indicating the chaotic behavior of the Duffing equation.

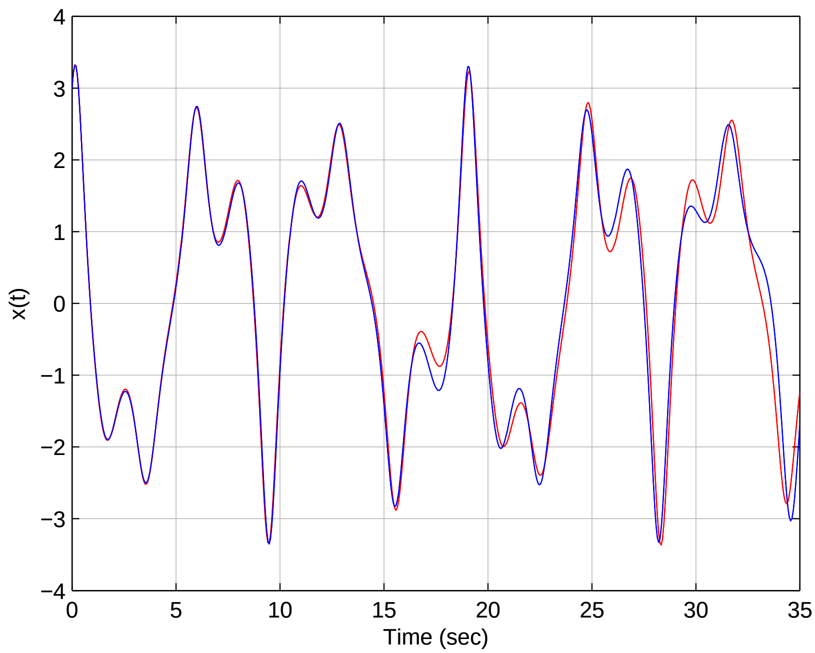
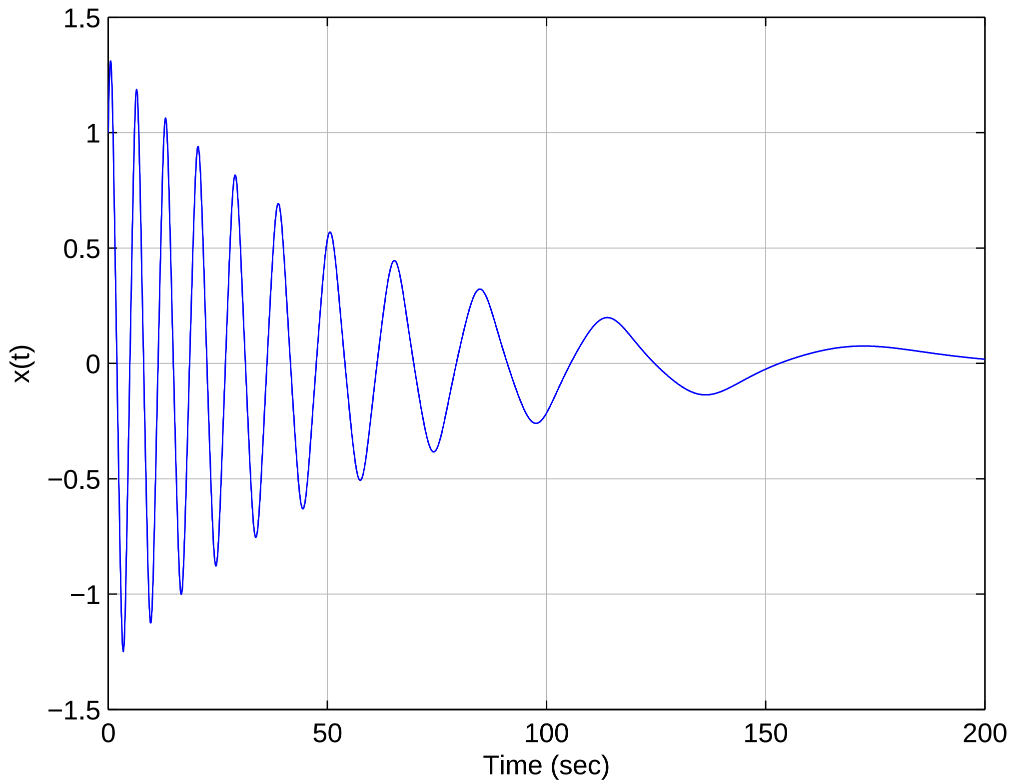
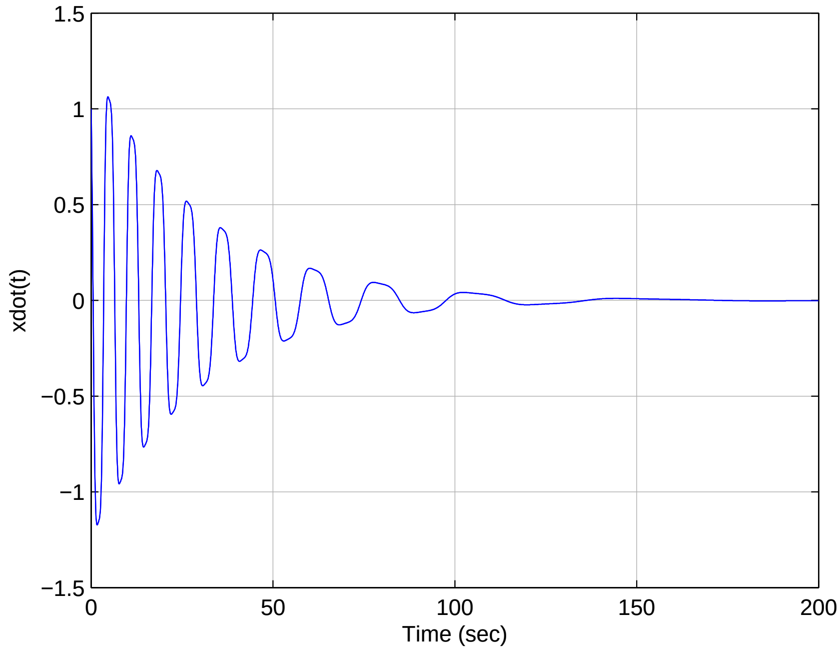


Figure 9.70: Siumulation of the duffing equation (blue) and the solution with slightly perturbed initial conditions (red).

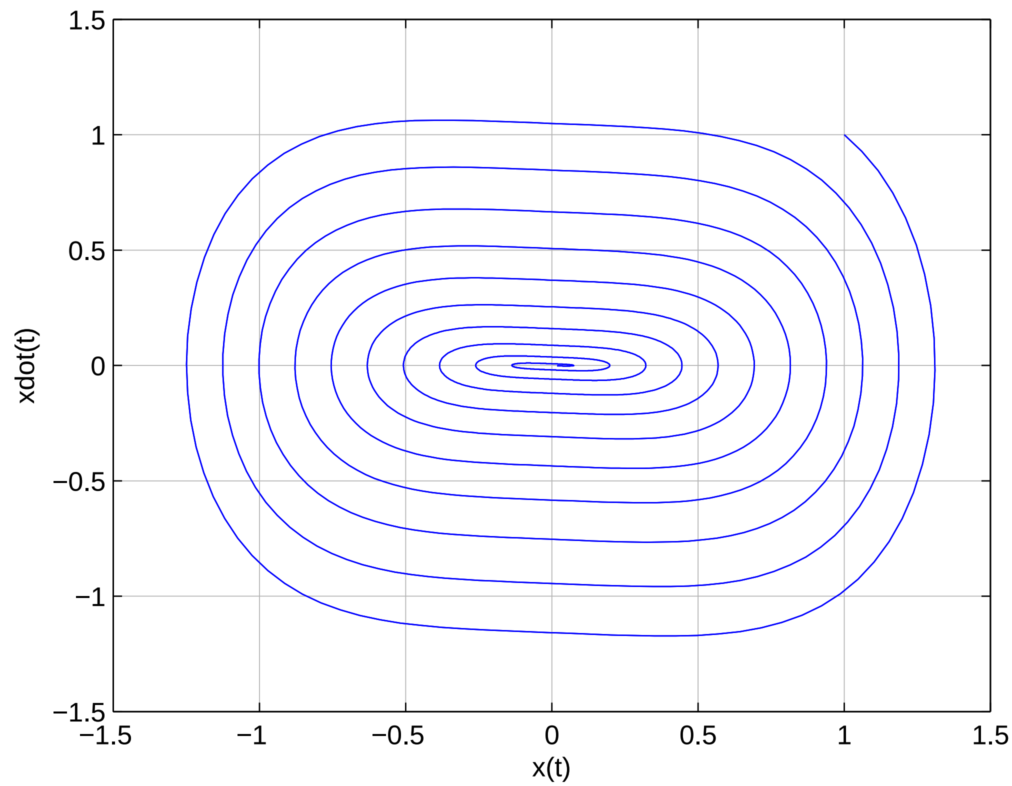
(b) A plot of $x(t)$ and $\dot{x}(t)$ are shown in the following figures along with the phase plane plot that shows the origin is an equilibrium point.



Plot of $x(t)$ for $k = 0.05$, $\epsilon = 1$ and $A = 0$ (with $u = 0$)

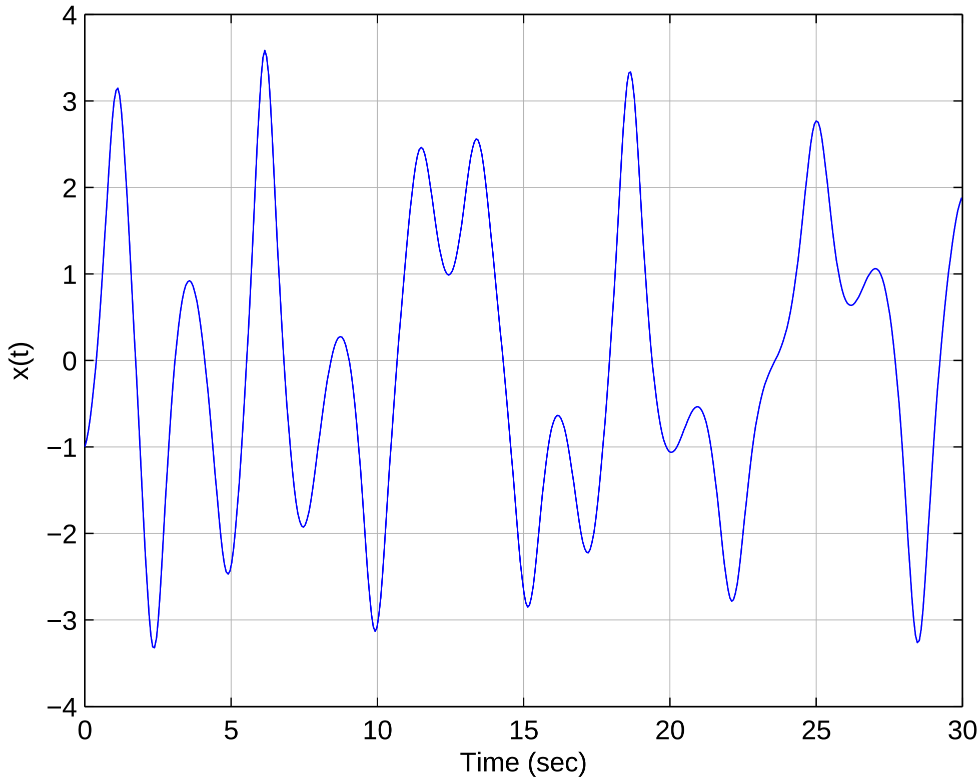


Plot of $\dot{x}(t)$ for $k = 0.05$, $\epsilon = 1$ and $A = 0$ (with $u = 0$)

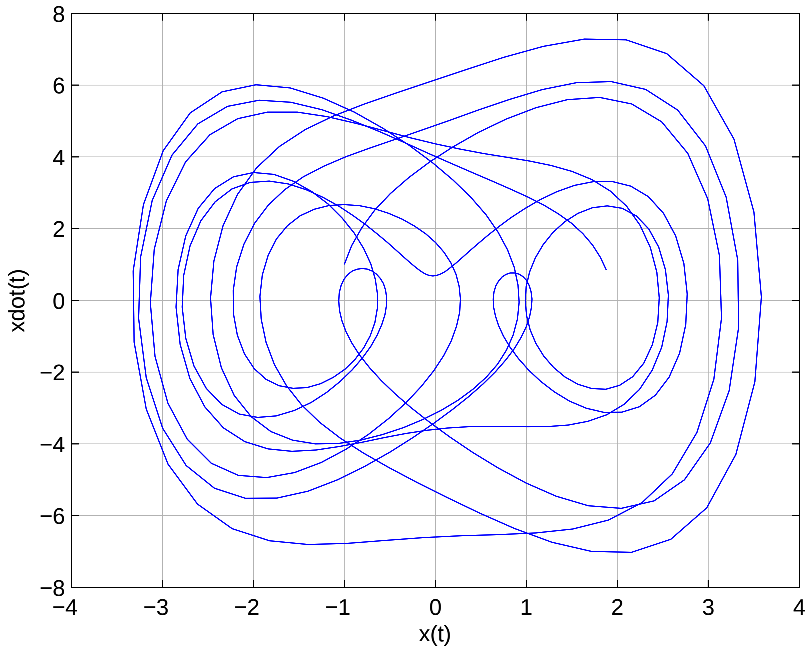


Phase plane plot for $k = 0.05$, $\epsilon = 1$ and $A = 0$ (with $u = 0$)

(c) The solution and the phase-plane plot are shown below.

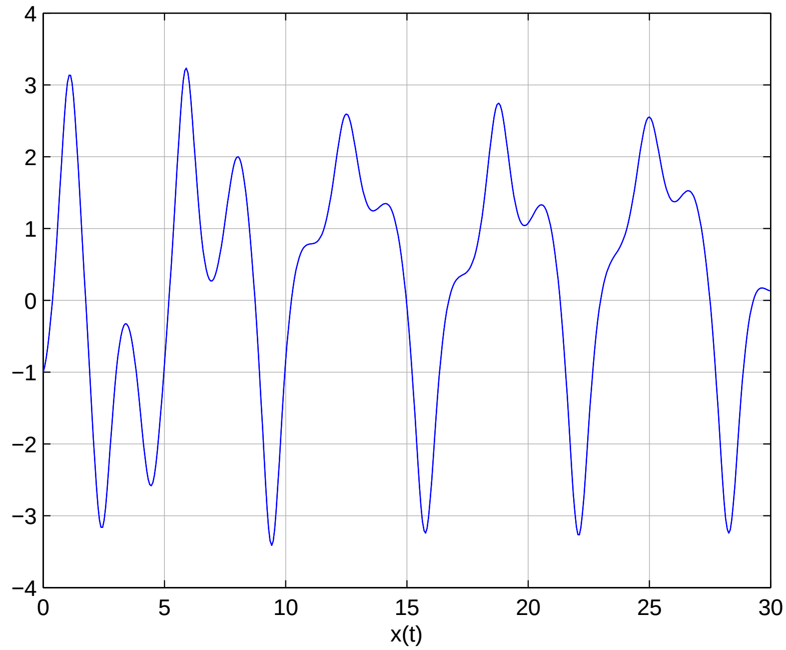


Solution for $x(t)$ with $k = 0.05$, $\epsilon = 1$ and $A = 7.5$.

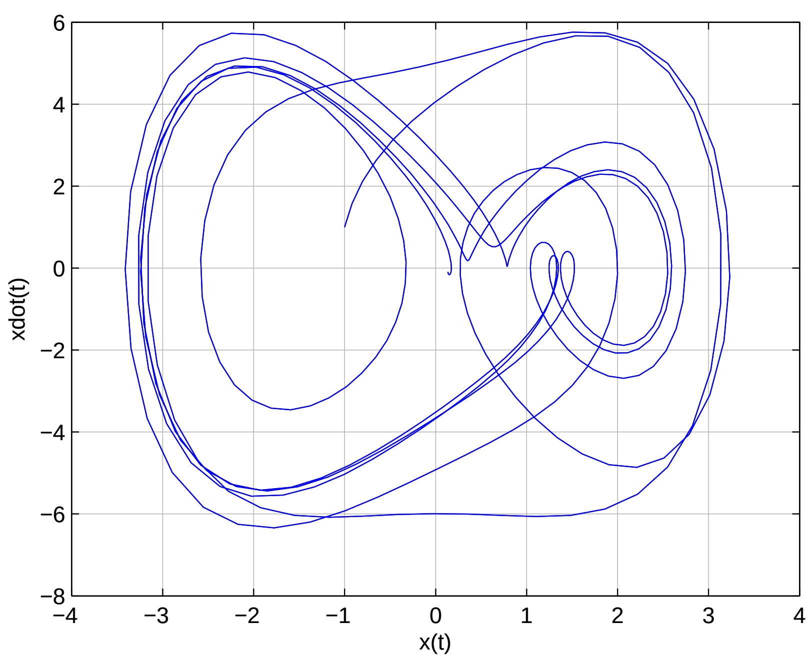


Phase-plane plot for $k = 0.05$, $\epsilon = 1$ and $A = 7.5$.

(d) The solution and the phase-plane plot are shown below.

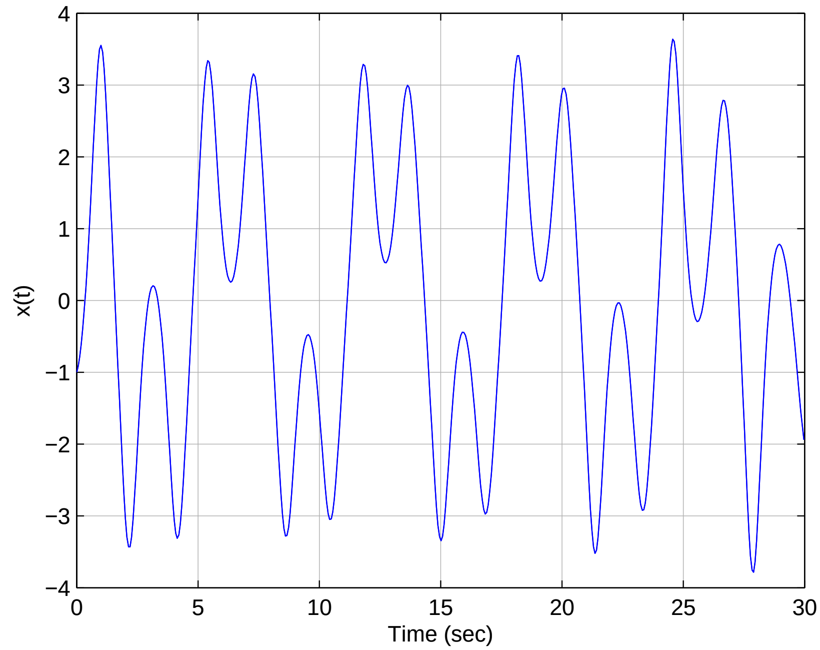


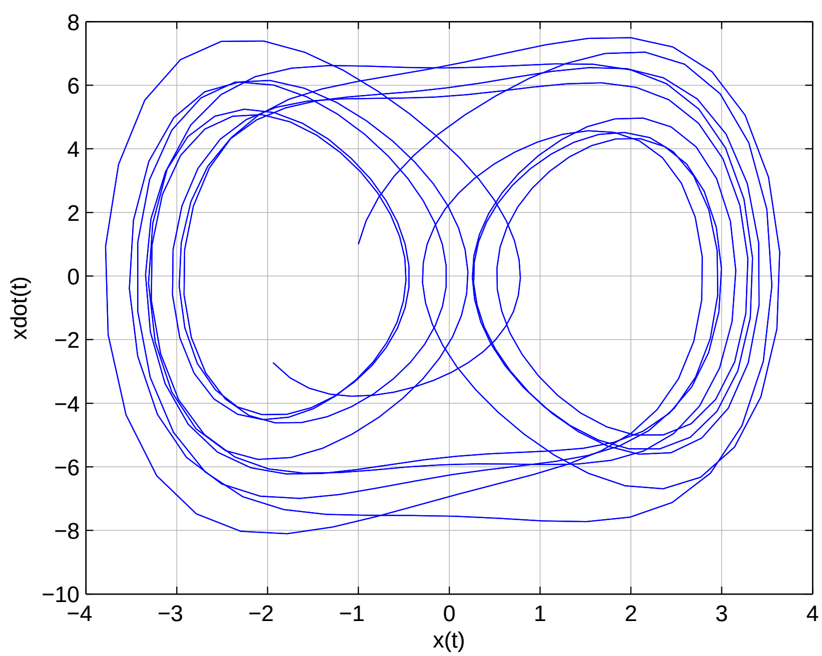
Solution for $x(t)$ with $k = 0.25$, $\epsilon = 1$ and $A = 8.5$.



Phase-plane plot with $k = 0.25$, $\epsilon = 1$ and $A = 8.5$.

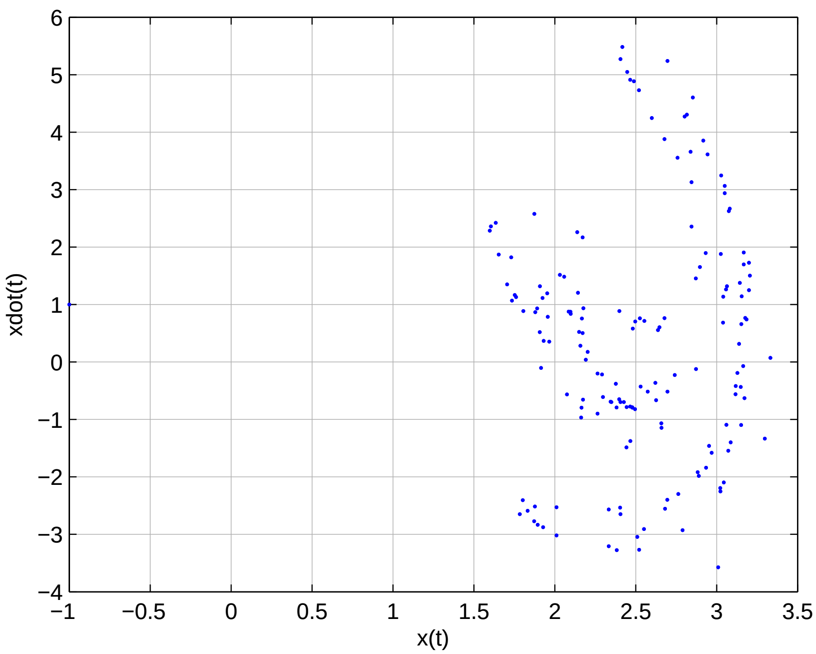
(e) The solution and the phase-plane plot are shown below.



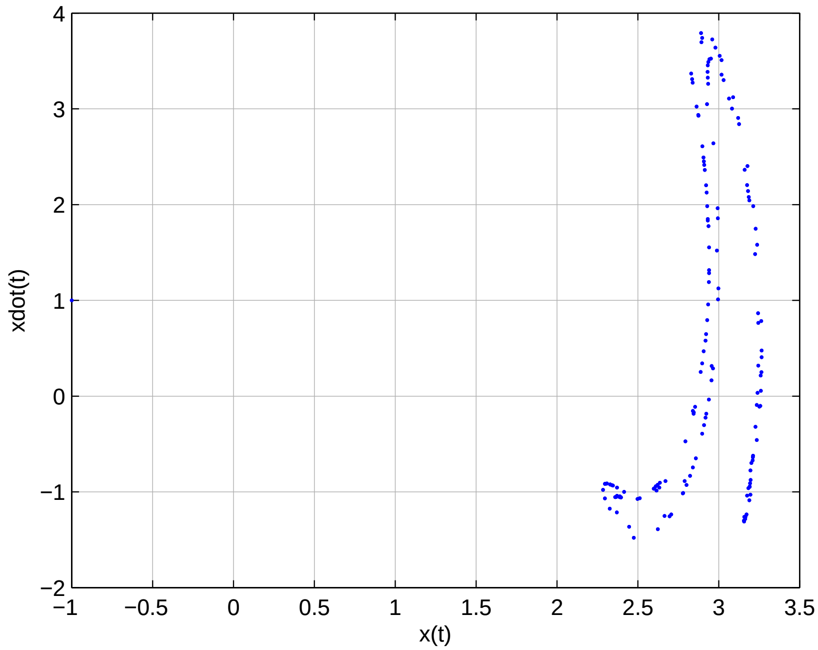


Phase-plane plot with $k = 0.1$, $\epsilon = 1$ and $A = 12$.

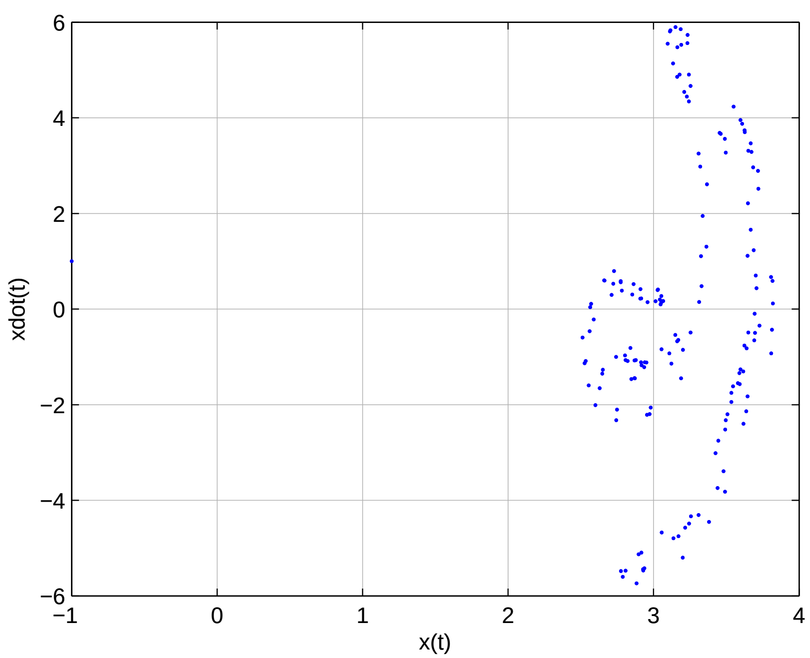
(f) Strange attractors are shown below.



Poincaré section ($\dot{x}(t_j)$ vs $x(t_j)$) for $k = 0.05$, $\epsilon = 1$, and $A = 7.5$.



Poincaré section ($\dot{x}(t_j)$ vs $x(t_j)$) for $k = 0.25$, $\epsilon = 1$, and $A = 8.5$.



Poincaré section ($\dot{x}(t_j)$ vs $x(t_j)$) for $k = 0.1$, $\epsilon = 1$, and $A = 11$.

(g) We can conclude from the above results that the exact solution to the Duffing equation is periodic.

(h) For $\epsilon = 1$ and $0 < k < 0.7$ and $0 < A < 15$ the system is stable. If $\epsilon < 0$, then the system is unstable. For $k = 0.05$, and $A = 7.5$, and the range of $0 < \epsilon < 568$, the system is stable.