Intermediate Fluid Mechanics

Lecture 28: Vorticity Dynamics II

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Chapter Overview

Lecture Objectives

In this lecture we will:

- Continue to learn about the generation of vorticity at a wall.
- Write down the differential equation for Vorticity Dynamics
- ...

Chapter Overview

Vorticity on a 2D Boundary Layer flow over a Flat Plate

Let's consider now the case of the two-dimensional, laminar, flat-plate boundary layer.

$\longrightarrow$ Since $dp/dx = 0$ in this case there is no generation of either sign vorticity at the wall.

• However it is clear that vorticity exists in the boundary layer developing a flat plate, no?

Question: Where did this vorticity come?

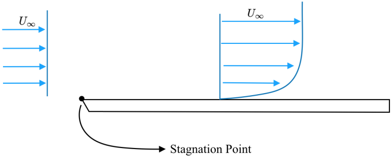

Figure: Sketch of the flow over a flat plate, including the stagnation point.

Vorticity on a 2D Boundary Layer flow over a Flat Plate

Answer:

In reality, the length of the plate must be finite. $\Longrightarrow$ A stagnation point must exist on the leading edge of the plate.

In the neighborhood of this stagnation point the streamlines will be curved dramatically as a result of the strong favorable pressure gradient in this area.

$\Longrightarrow$ All the vorticity on a flat plate boundary layer is generated at the leading edge stagnation point. This vorticity then simply advects downstream and at the same time diffuses outward away from the wall.

Vorticity generated by an accelerating wall

There may be an additional contribution to the wall-normal flux vorticity if the boundary is accelerating tangentially.

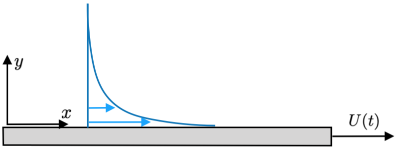

Figure: Flow over an instantaneously accelerated plate.

One can see this by evaluating the boundary layer equations at the wall, while retaining the unsteady term.

$\Longrightarrow$ For illustrative purposes let's consider the case of an impulsively started wall, ...

Vorticity generated by an accelerating wall

- The velocity of the wall is zero initially, *i.e.* $u(t=0,y)=0$.
- The plate undergoes an unsteady acceleration so that $u(t, y=0) = U(t)$ for all $t > 0$.
- Considering the x-momentum boundary layer equation evaluated at the wall,

$$\underbrace{\rho \frac{\partial u}{\partial t}\Big|_{y=0} + \rho\left(u\frac{\partial u}{\partial x}\right)\Big|_{y=0}}_{\substack{\rho\frac{Du}{Dt}\text{: material derivative}\\ \text{of wall momentum}}} + \underbrace{\rho\left(v\frac{\partial u}{\partial y}\right)\Big|_{y=0}}_{\substack{=0\text{: since}\\ v(y=0)=0}} = -\frac{\partial p}{\partial x} + \underbrace{\mu\frac{\partial^2 u}{\partial y^2}\Big|_{y=0}}_{\substack{-\mu\frac{\partial\omega_z}{\partial y}\text{: Vorticity}\\ \text{flux at the wall}}} \tag{1}$$

This equation, reduces to

$$-\nu\frac{\partial \omega_z}{\partial y}\Big|_{y=0} = \frac{Du}{Dt} + \frac{1}{\rho}\frac{\partial p}{\partial x}. \tag{2}$$

Vorticity generated by an accelerating wall

1. If the boundary is horizontal then, $\partial p/\partial x$ and the vorticity flux at the wall is only due to the acceleration of the wall,

$$-\nu \frac{\partial \omega_z}{\partial y}\Big|_{y=0} = \frac{Du}{Dt}. \tag{3}$$

2. Furthermore, if the wall is moving at a constant velocity, then

$$-\nu \frac{\partial \omega_z}{\partial y}\Big|_{y=0} = 0. \tag{4}$$

$\Longrightarrow$ In this scenario, the vorticity flux at the wall is zero!

• However, we know there is vorticity in the flow because $\partial u/\partial y \neq 0$. The question is gain, from where did the vorticity come?

Vorticity generated by an accelerating wall

Answer:

All of the vorticity was generated instantaneously at $t = \epsilon$ (where ϵ represents a very samll number in the neighborhood of zero) due to an infinite acceleration of the plate from a stationary position to a constant velocity of U.

Chapter Overview

Cross-Diffusive annihilation of vorticity

- Earlier we discussed that vorticity is not lost by diffusion to the boundaries.
- Instead, the decay of vorticity occurs due to vorticity of opposite sign being generated at a boundary, and diffusing out into the interior of the boundary layer, and suffering cross-diffusive annihilation with pre-existing vorticity.

This can be better seen if we revisit the solenoidal condition of vorticity ($\vec{\nabla} \cdot \vec{\omega} = 0$).

$\implies$ If we integrate this over an arbitrary volume and apply the divergence theorem, once finds that,

$$0 = \int_{\forall} \vec{\nabla} \cdot \vec{\omega} \, d\forall = \int_{S} \vec{\omega} \cdot \vec{n} \, dA \tag{5}$$

This relation implies that each vortex line must cut the surface an even number of time, typically twice (entry and exit) to ensure that $\int_S \vec{\omega} \cdot \vec{n} \, dA = 0$.

Cross-Diffusive annihilation of vorticity

$$0 = \int_{\forall} \vec{\nabla} \cdot \vec{\omega} \, d\forall = \int_S \vec{\omega} \cdot \vec{n} \, dA \quad (6)$$

This relation implies that each vortex line must cut the surface an even number of time, typically twice (entry and exit) to ensure that $\int_S \vec{\omega} \cdot \vec{n} \, dA = 0$.

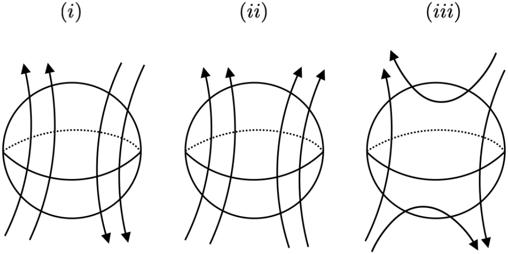

Figure: Vortex lines cutting different surfaces that enclose a given random volume.

Cross-Diffusive annihilation of vorticity

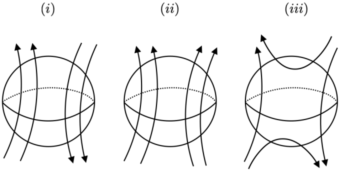

Figure: Vortex lines cutting different surfaces that enclose a given random volume.

- Both, (i) and (ii) satisfy the conservation equation, but will be quite differently affected by diffusion.
- (i) can suffer cross-diffusive annihilation of the vorticity in the interior (if the vorticity lines with opposite sign get close enough to each other) and reconnection near the poles as shown in (iii).
- Scenario (ii) will suffer little immediate change due to diffusion except to lose (or gain) some vorticity magnitude by transfer of vorticity to neighboring vortex lines.

Cross-Diffusive annihilation of vorticity

- The process of cross-diffusive anihilation and reconnection is shown below, which illustrates two colliding vortex rings.
- The time between frames is $1/24$ s, indicating this process of cutting and reconnecting is very rapid.

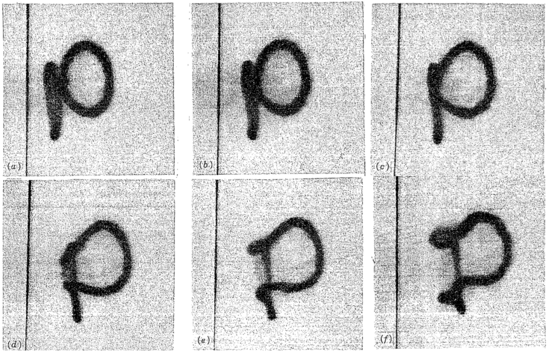

Figure: Anihilation and reconnection of two colliding vortex rings.

Cross-Diffusive annihilation of vorticity

Another example of the process of cross-diffusive annihilation and reconnection of vortex lines is the interaction of wing-tip vortices behind an aircraft.

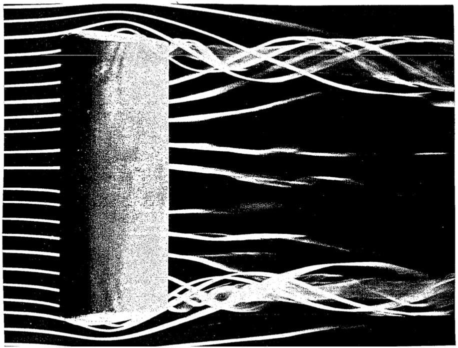

Figure: Visualization of the generation of wing-tip vortices from finite-length airfoil.

Cross-Diffusive annihilation of vorticity

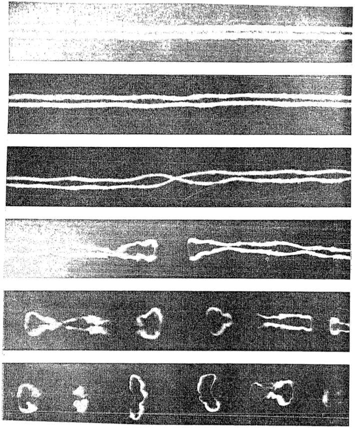

Figure: Cross-diffusive annihilation and reconnection of wing-tip vortex lines as visualized in the sky by water vapor condensation. As time progresses the initially straight lines begin to wiggle, and approach each other in certain locations. When the two lines get close enough, cross-diffusive annihilation and reconnection takes place so that the two lines begin to form vortex rings that appear to be spaced uniformly by a characteristic distance.

Velocity induced by a vortex line: Biot-Savart Law

Biot-Savart Law:

We will not prove this law, but merely state that the velocity in the neighborhood of a vortex line is

$$\vec{u}(\vec{x}, t) = \frac{1}{4\pi} \int_{\forall} \frac{\vec{r} \times \vec{\omega}(\vec{x}', t)}{|\vec{r}|^3} \, d\forall \quad (7)$$

where $\vec{r} = \vec{x} - \vec{x}'$, with $\vec{x}$ being the location of interest and $\vec{\omega}(\vec{x}', t)$ defines the vorticity at location $\vec{x}'$ along the vortex line.

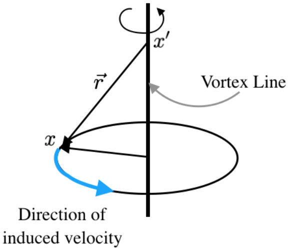

Figure: Illustration of Biot-Savart's law.

Chapter Overview

Vorticity Transport Equation

The last topic on vorticity dynamics is the derivation and interpretation of the vorticity transport equation.

Let's start with the incompressible Navier-Stokes equation,

$$\frac{\partial \vec{u}}{\partial t} + \left(\vec{u} \cdot \vec{\nabla}\right)\vec{u} = -\frac{1}{\rho}\vec{\nabla} p + \vec{\nabla}\phi + \nu \vec{\nabla}^2 \vec{u} \tag{8}$$

where we have assumed body forces can be written in terms of a potential function ϕ.

Using a vector identity one can rewrite the advection term as,

$$\left(\vec{u} \cdot \vec{\nabla}\right)\vec{u} = \frac{1}{2}\vec{\nabla}\left(\vec{u} \cdot \vec{u}\right) - \vec{u} \times \left(\vec{\nabla} \times \vec{u}\right) \tag{9}$$

$$= \frac{1}{2}\vec{\nabla}\left(\vec{u} \cdot \vec{u}\right) - \vec{u} \times \vec{\omega} \tag{10}$$

$\implies$ The Navier-Stokes equations can be rewritten as

$$\underbrace{\vec{\nabla} \times \frac{\partial \vec{u}}{\partial t}}_{\text{Term I}} + \underbrace{\vec{\nabla} \times \frac{1}{2}\vec{\nabla}\left(\vec{u}\cdot\vec{u}\right)}_{\text{Term II}} \underbrace{- \vec{\nabla} \times \vec{u} \times \vec{\omega}}_{\text{Term III}} = \underbrace{-\vec{\nabla} \times \frac{1}{\rho}\vec{\nabla} p}_{\text{Term IV}} + \underbrace{\vec{\nabla} \times \vec{\nabla}\phi}_{\text{Term V}} + \underbrace{\vec{\nabla} \times \nu \vec{\nabla}^2 \vec{u}}_{\text{Term VI}}. \tag{11}$$

where some of these terms can be rewritten as follows:

- $\vec{\nabla} \times \frac{\partial \vec{u}}{\partial t} = \frac{\partial \vec{\omega}}{\partial t}$
- $\vec{\nabla} \times \frac{1}{2}\vec{\nabla}\left(\vec{u}\cdot\vec{u}\right) = 0$ because $\vec{u}\cdot\vec{u}$ is a scalar and there is a vector identity which tells us that the curl of the gradient of a scalar quantity is zero.

Vorticity Transport Equation

$$\underbrace{\vec{\nabla} \times \frac{\partial \vec{u}}{\partial t}}_{\text{Term I}} + \underbrace{\vec{\nabla} \times \frac{1}{2}\vec{\nabla}\left(\vec{u} \cdot \vec{u}\right)}_{\text{Term II}} \underbrace{- \vec{\nabla} \times \vec{u} \times \vec{\omega}}_{\text{Term III}} = \underbrace{- \vec{\nabla} \times \frac{1}{\rho}\vec{\nabla} p}_{\text{Term IV}} + \underbrace{\vec{\nabla} \times \vec{\nabla}\phi}_{\text{Term V}} + \underbrace{\vec{\nabla} \times \nu \vec{\nabla}^2 \vec{u}}_{\text{Term VI}}. \tag{12}$$

as well as,

- $\vec{\nabla} \times \vec{u} \times \vec{\omega} = \vec{u}\left(\vec{\nabla} \cdot \vec{\omega}\right) - \vec{\omega}\left(\vec{\nabla} \cdot \vec{u}\right) + \left(\vec{\omega} \cdot \vec{\nabla}\right)\vec{u} - \left(\vec{u} \cdot \vec{\nabla}\right)\vec{\omega}$ from a vector identity.

  From the solenoidal condition on vorticity the first term on the right hand side vanishes.

  Furthermore, assuming incompressible flow, the second term on the right hand side also vanishes. Therefore one ends up having, $\vec{\nabla} \times \vec{u} \times \vec{\omega} = \left(\vec{\omega} \cdot \vec{\nabla}\right)\vec{u} - \left(\vec{u} \cdot \vec{\nabla}\right)\vec{\omega}$.

Vorticity Transport Equation

- $\vec{\nabla} \times \frac{1}{\rho}\vec{\nabla} p = \frac{1}{\rho^2}\vec{\nabla}\rho \times \vec{\nabla} p$. One cas see this if we expand out the curl operator as shown,

$$\begin{bmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{1}{\rho}\frac{\partial p}{\partial x} & \frac{1}{\rho}\frac{\partial p}{\partial y} & \frac{1}{\rho}\frac{\partial p}{\partial z} \end{bmatrix} = \Big[\frac{\partial}{\partial y}\Big(\frac{1}{\rho}\frac{\partial p}{\partial z}\Big) - \frac{\partial}{\partial z}\Big(\frac{1}{\rho}\frac{\partial p}{\partial y}\Big)\Big]\hat{i}$$

$$+ \Big[\frac{\partial}{\partial z}\Big(\frac{1}{\rho}\frac{\partial p}{\partial x}\Big) - \frac{\partial}{\partial x}\Big(\frac{1}{\rho}\frac{\partial p}{\partial z}\Big)\Big]\hat{j} \tag{13}$$

$$+ \Big[\frac{\partial}{\partial x}\Big(\frac{1}{\rho}\frac{\partial p}{\partial y}\Big) - \frac{\partial}{\partial y}\Big(\frac{1}{\rho}\frac{\partial p}{\partial x}\Big)\Big]\hat{k} \tag{14}$$

$$= \Big[-\frac{1}{\rho^2}\frac{\partial \rho}{\partial y}\frac{\partial p}{\partial z} + \frac{1}{\rho^2}\frac{\partial \rho}{\partial z}\frac{\partial p}{\partial y}\Big]\hat{i} + \cdots \tag{15}$$

By inspection one can verify the equality written above.

Vorticity Transport Equation

$$\underbrace{\vec{\nabla} \times \frac{\partial \vec{u}}{\partial t}}_{\text{Term I}} + \underbrace{\vec{\nabla} \times \frac{1}{2}\vec{\nabla}\left(\vec{u} \cdot \vec{u}\right)}_{\text{Term II}} \underbrace{- \vec{\nabla} \times \vec{u} \times \vec{\omega}}_{\text{Term III}} = \underbrace{-\vec{\nabla} \times \frac{1}{\rho}\vec{\nabla} p}_{\text{Term IV}} + \underbrace{\vec{\nabla} \times \vec{\nabla}\phi}_{\text{Term V}} + \underbrace{\vec{\nabla} \times \nu \vec{\nabla}^2 \vec{u}}_{\text{Term VI}}. \tag{16}$$

continuing with the term by term analysis,

- $\vec{\nabla} \times \vec{\nabla}\phi = 0$ due to the same reason for which term II vanishes.
- $\vec{\nabla} \times \nu \vec{\nabla}^2 \vec{u} = \nu \vec{\nabla}^2 \vec{\omega}$, by definition of $\vec{\omega}$ and assuming constant viscosity.

Vorticity Transport Equation

Therefore, the vorticity equation reduces to the following expression,

$$\underbrace{\frac{\partial \vec{\omega}}{\partial t}}_{\text{Term 1}} + \underbrace{\left(\vec{u} \cdot \vec{\nabla}\right)\vec{\omega}}_{\text{Term 2}} = \underbrace{\left(\vec{\omega} \cdot \vec{\nabla}\right)\vec{u}}_{\text{Term 3}} + \underbrace{\frac{1}{\rho^2}\vec{\nabla}\rho \times \vec{\nabla}p}_{\text{Term 4}} + \underbrace{\nu\vec{\nabla}^2\vec{\omega}}_{\text{Term 5}} \tag{17}$$

where,

1. Local time rate of change of vorticity.
2. Advection of vorticity.
3. Vortex stretching and reorientation.
4. Baroclinic torque.
5. Viscous difussion of voritcity.

Vorticity Transport Equation

Note that:

- The material derivative of the vorticity, $\frac{\partial \vec{\omega}}{\partial t} + \left(\vec{u} \cdot \vec{\nabla}\right)\vec{\omega} = \frac{D\vec{\omega}}{Dt}$, just tells us how fast the vorticity is changing following individual fluid particles.
- Alternatively, the baroclinic torque results in a generation of vorticity as a result of non-aligned density and pressure gradients.
- This term obviously only plays a role in flows where the density is non-homogeneous. $\longrightarrow$ Consider the example illustrated in the side figure, where $\vec{\nabla}\rho$ and $\vec{\nabla}p$ are perpendicular.

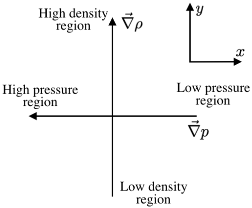

Figure: Case in which $\vec{\nabla}\rho$ and $\vec{\nabla}p$ are perpendicular.

Vorticity Transport Equation

- Since $\vec{F} = m\vec{a}$, or $\vec{a} = \vec{F}/m$, low density fluid will accelerate faster for a given $\vec{\nabla} p$.
- In the above example, all fluid is exposed to the same force (*i.e.* pressure gradient).
- Since the low density fluid has smaller mass, it accelerates more than the higher density fluid which generates a shear flow and causes a net circulation in the clockwise direction:

$$\vec{\nabla}\rho = |\vec{\nabla}\rho|\hat{j} \qquad \text{and} \qquad \vec{\nabla} p = -|\vec{\nabla} p|\hat{i} \tag{18}$$

therefore

$$\vec{\nabla}\rho \times \vec{\nabla} p = -\omega\hat{k} \quad \text{Generation of vorticity directed to the page.} \tag{19}$$

Vorticity Transport Equation

In regards to the viscous diffusion,

- This is the process through which non-vortical fluid attains vorticity.
- The redistribution of existing vorticity due to the molecular interactions and the exchange of linear momentum between neighboring fluid particles.

Vorticity Transport Equation

The term for vortex stretching and reorientation, is arguably the most important mechanism in substaining turbulence.

It is the mechanism by which turbulent energy is transferred from large to small scales. Let's write out the term in full form,

$$\left(\vec{\omega}\cdot\vec{\nabla}\right)\vec{u} = \left[\underbrace{\omega_x\frac{\partial u}{\partial x}}_{*} + \omega_y\frac{\partial u}{\partial y} + \omega_z\frac{\partial u}{\partial z}\right]\hat{i} + \left[\omega_x\frac{\partial v}{\partial x} + \underbrace{\omega_y\frac{\partial v}{\partial y}}_{*} + \omega_z\frac{\partial v}{\partial z}\right]\hat{j} \tag{20}$$

$$+ \left[\omega_x\frac{\partial w}{\partial x} + \omega_y\frac{\partial w}{\partial y} + \underbrace{\omega_z\frac{\partial w}{\partial z}}_{*}\right]\hat{k}. \tag{21}$$

The $*$ terms are **vortex stretching** terms, while the other terms represent **vortex reorientation**.

To better understand the vortex stretching process, let's consider a couple of examples ...

Vorticity Transport Equation

Example 1: Amplification of Streamwise vorticity (Figure 9)

$$\frac{D\omega_x}{Dt} = \omega_x \frac{\partial u}{\partial x} + \omega_y \frac{\partial u}{\partial y} + \omega_z \frac{\partial u}{\partial z} \quad (22)$$

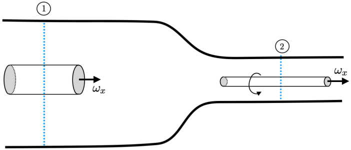

Figure: Vorticity in a pipe contraction.

- From mass conservation $\dot{m}_1 = \dot{m}_2$, therefore $u_2 > u_1$ and $\frac{\partial u}{\partial x} > 0$.
- If ω_x already exists in the flow at location 1, then the $\frac{\partial u}{\partial x}$ velocity gradient serves to increase the magnitude of ω_x at location 2.
- Kinematically, this would result in a stretching of the vortex tube at location 2 relative to location 1.

Vorticity Transport Equation

Example 2: Attenuation of transverse vorticity (Figure 10)

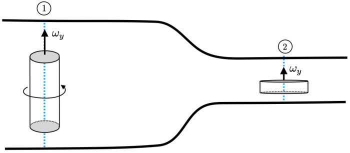

Figure: Vorticity in a pipe contraction.

- From continuity $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$. Since $\frac{\partial u}{\partial x} > 0$ then $\frac{\partial v}{\partial y} < 0$.
- If ω_y already exists in the flow at 1, then the gradient $\frac{\partial v}{\partial y}$ serves to reduce the magnitude of the ω_y vorticity at 2.
- The reorientation terms serve to redistribute vorticity between the different components.

Vorticity Transport Equation

For example, consider the x-component of vorticity,

$$\frac{D\omega_x}{Dt} = \omega_x \frac{\partial u}{\partial x} + \omega_y \frac{\partial u}{\partial y} + \omega_z \frac{\partial u}{\partial z} \tag{23}$$

If we already have ω_y in the flow and the fluid encounters a region in the flow having non-zero $\partial u/\partial y$, then ω_x will be generated by reorienting the original ω_y vorticity in the x-direction.

$+$<u>Note:</u> The term $\left(\vec{\omega} \cdot \vec{\nabla}\right)\vec{u}$ cannot generate new vorticity, it only acts to reorient and stretch existing vorticity that is already in the flow.