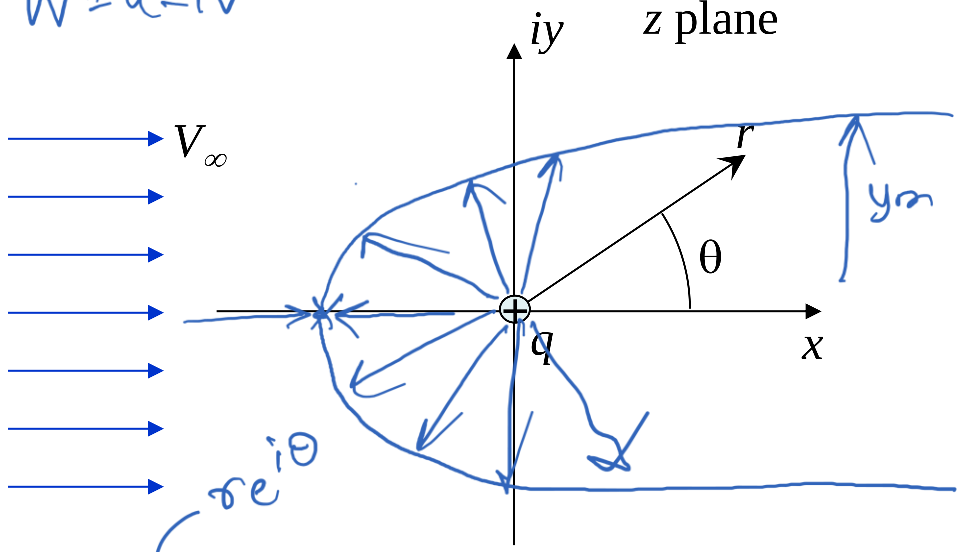
Example – The Rankine Halfbody

$W = u - iv$

Consider a source of strength q, placed at the origin in a uniform flow V_∞ in the positive x direction. This produces a flow solution for a semi-infinite body in a uniform stream. Determine (a) the complex velocity and potential (b) the body shape including its max thickness



a) $W(z) = V_\infty e^{-i\alpha} + \dfrac{q}{2\pi(z - z_1)} = V_\infty + \dfrac{q}{2\pi z}$

$F(z) = \int W\,dz = V_\infty z + \dfrac{q}{2\pi} \log_e(z)$

$re^{i\theta}$

b) Use streamfunction ψ: $\psi = \mathrm{Im}(F) = \mathrm{Im}\left(V_\infty z + \dfrac{q}{2\pi} \log_e z\right) = V_\infty y + \dfrac{q\theta}{2\pi}$ ☆

Streamlines: $\psi = V_\infty y + q\theta/2\pi = \text{CONST}$

STAGNATION POINT $\Rightarrow$ $W(z) = 0 = V_\infty + \dfrac{q}{2\pi z}$ so $z = -q/2\pi V_\infty$

AT STAGNATION, $\psi = q/2$, so Body shape: $V_\infty y + \dfrac{q\theta}{2\pi} = \dfrac{q}{2}$

$y = \dfrac{q}{2V_\infty}\left(1 - \dfrac{\theta}{\pi}\right)$, so max thickness $= q/V_\infty$

Pressure in Ideal Flows

$$P + \frac{1}{2}\rho V^2 = \text{CONST} = P_\infty + \frac{1}{2}\rho V_\infty^2$$

ABSOLUTE SPEED IS MEANINGLESS. WHAT IS IMPORTANT IS V/V_∞.

$$\Rightarrow \quad P - P_\infty = \frac{1}{2}\rho\left(V_\infty^2 - V^2\right)$$

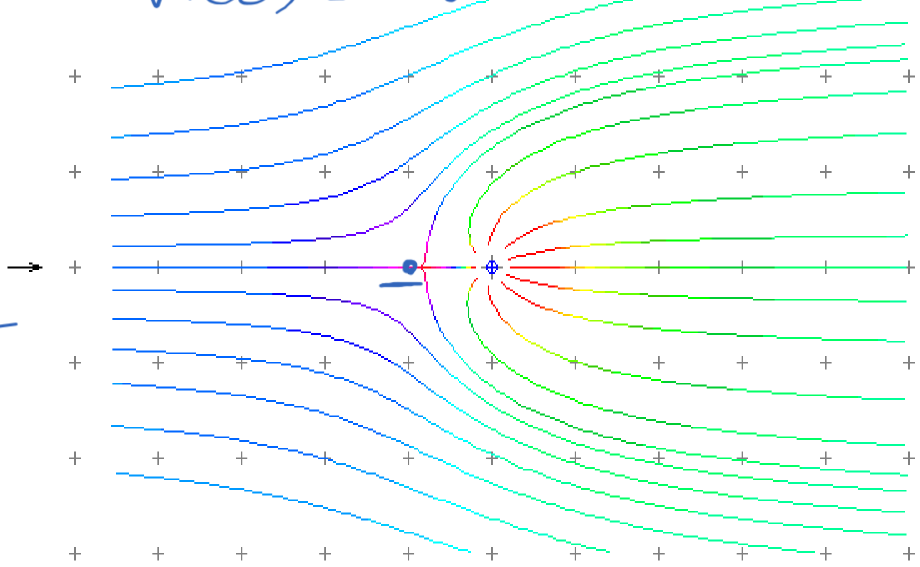
(OR)

$$\frac{P - P_\infty}{\frac{1}{2}\rho V_\infty^2} = 1 - \frac{V^2}{V_\infty^2} \equiv C_p$$

(PRESSURE COEFFICIENT)

$$C_p = 1 - \frac{|W(z)|^2}{V_\infty^2}$$

$$W(z) = V_\infty + q/2\pi z$$



$V_\infty = 2 \quad q = 2\pi, \quad z = -1$

SO $V = 1$

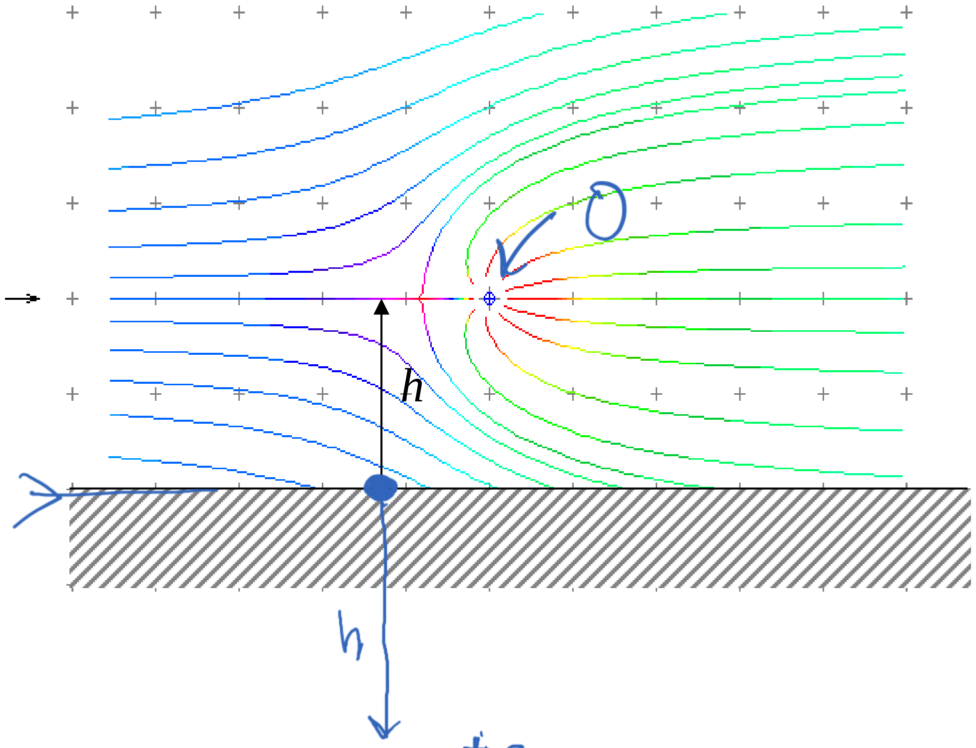
$$C_p\big|_\infty = 0$$

$$C_p\big|_0 = 1$$

Simulating Flat Walls

Using Method of Images (For Flat Walls)

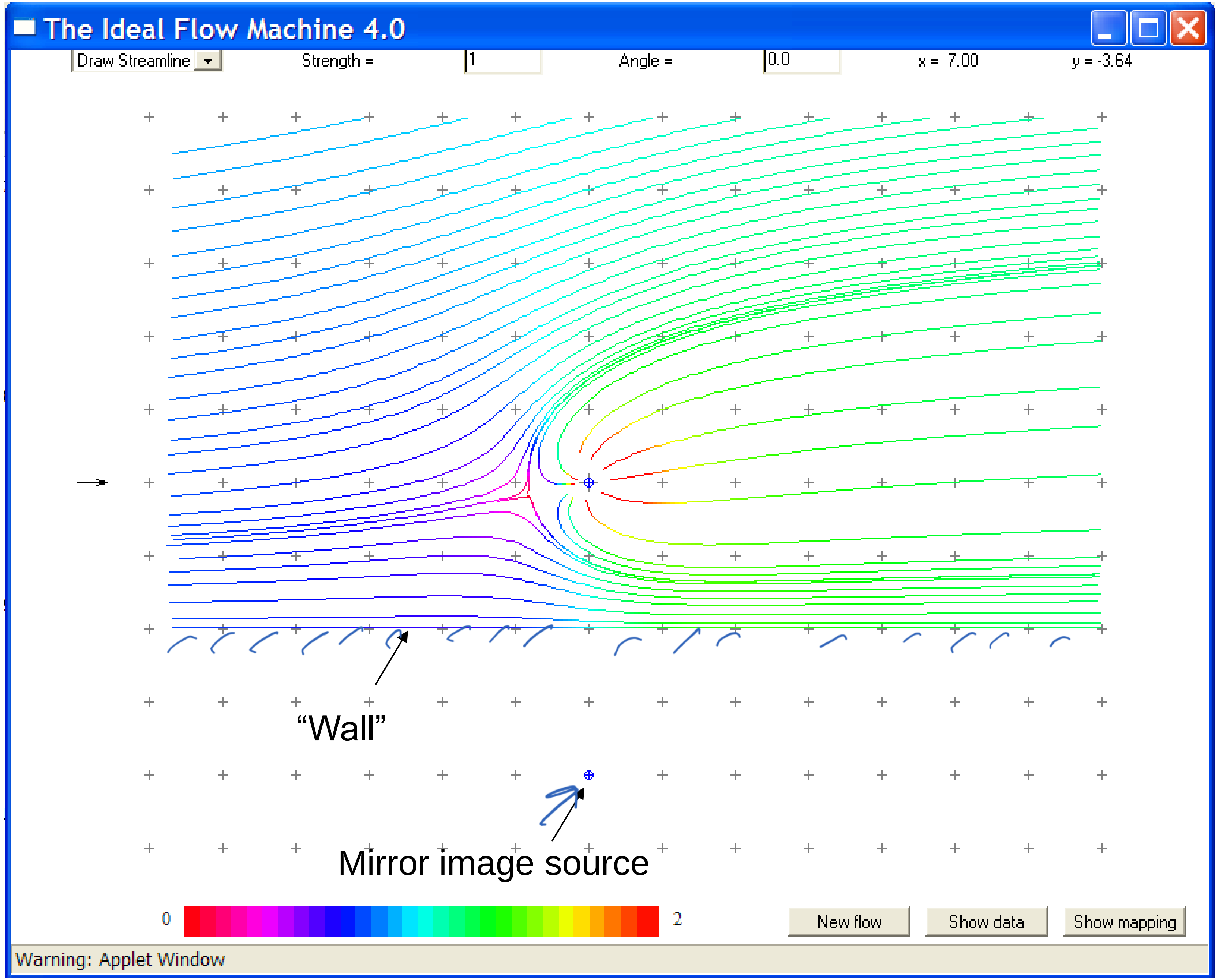
"Consider a flow produced by singularities. Placing a plane wall in the flow is equivalent to adding the mirror image of the singularities in the wall"



In this case ⇒ $W(z) = V_\infty + \frac{q}{2\pi z} + \frac{q}{2\pi(z+2hi)}$

At the wall, for example $z = -ih$

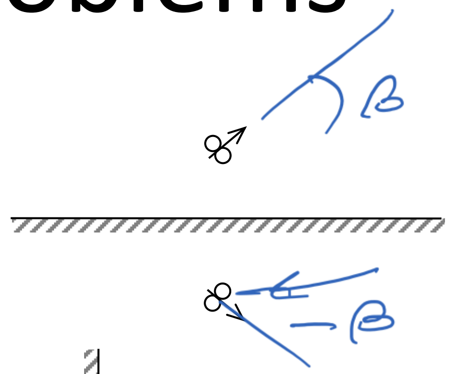
$$W(z) = V_\infty + \frac{q}{2\pi(-ih)} + \frac{q}{2\pi(-ih+2ih)} = V_\infty - \frac{q}{2\pi i h} + \frac{q}{2\pi i h} = V_\infty$$



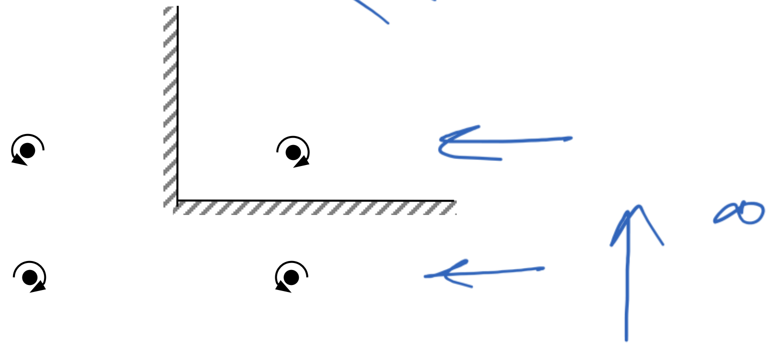
The Ideal Flow Machine 4.0
Draw Streamline
Strength =
1
Angle =
0.0
x = 7.00
y = -3.64
"Wall"
Mirror image source
0
2
New flow
Show data
Show mapping
Warning: Applet Window

Other Flat Wall Problems

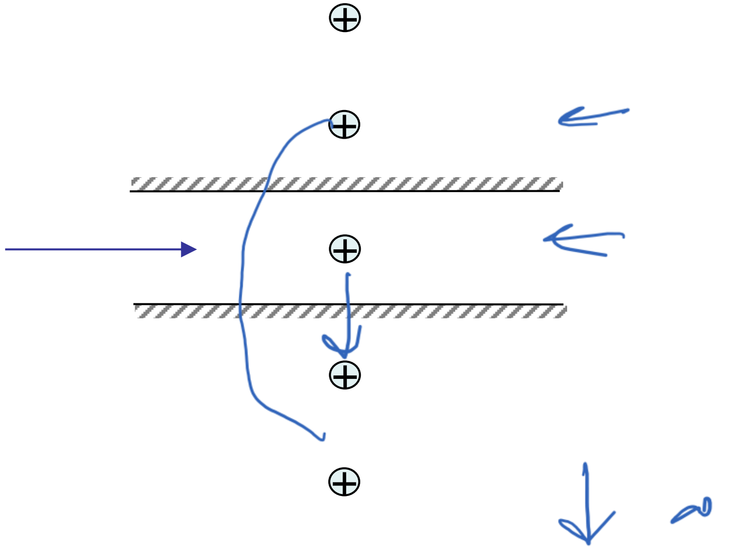
- Doublet near wall



- Vortex near corner



- Source in wind tunnel



No circulation

1. Acyclic Flow Past a Circular Cylinder

= Uniform flow + Opposing Doublet

Complex Velocity and Potential? $W(z) = V_\infty e^{-i\alpha} - \dfrac{\mu e^{i\alpha}}{2\pi(z - z_1)^2}$ $\quad F = \int W \, dz$

General case

Flow in x dir^n.
Cylinder at origin $\quad \left.\begin{array}{l}\alpha = 0 \\ z_1 = 0\end{array}\right\} \Rightarrow W(z) = V_\infty - \mu / 2\pi z^2$

Cylinder radius? $W(z) = 0 = V_\infty - \mu / 2\pi z^2$

$$z = \pm \sqrt{\frac{\mu}{2\pi V_\infty}} = a$$

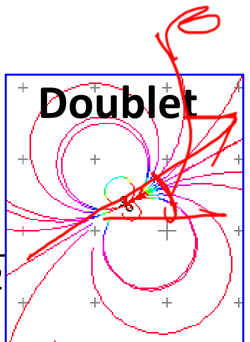
THIS IS THE CYLINDER RADIUS 'a'

$$\mu = 2\pi a^2 V_\infty$$

FOR A CYLINDER OF RADIUS 'a',

$$W(z) = V_\infty e^{-i\alpha} - \frac{a^2 V_\infty e^{i\alpha}}{(z - z_1)^2}$$

$$W(z) = \frac{\mu e^{i\beta}}{2\pi(z - z_1)^2}$$



1. Acyclic Flow Past a Circular Cylinder

$$F(z) = V_\infty e^{-i\alpha} z + \frac{V_\infty a^2 e^{i\alpha}}{z - z_1}$$

$$W(z) = V_\infty e^{-i\alpha} - \frac{V_\infty a^2 e^{i\alpha}}{(z - z_1)^2}$$

Prove it's circular?

$$W(z) = V_\infty e^{-i\alpha} - \frac{\mu e^{i\alpha}}{2\pi (z - z_1)^2} \text{ (or) } V_\infty - \frac{V_\infty a^2}{z^2}$$

ON CYLINDER: $z = a e^{i\theta}$; so $W(z) = V_\infty - \dfrac{V_\infty a^2}{a^2 e^{2i\theta}}$

$$V_r - iV_\theta = W \cdot e^{i\theta} = V_\infty e^{i\theta} - \frac{V_\infty \cancel{a^2}}{\cancel{a^2} e^{2i\theta}} \cdot e^{i\theta}$$

$$= V_\infty e^{i\theta} - V_\infty e^{-i\theta}$$

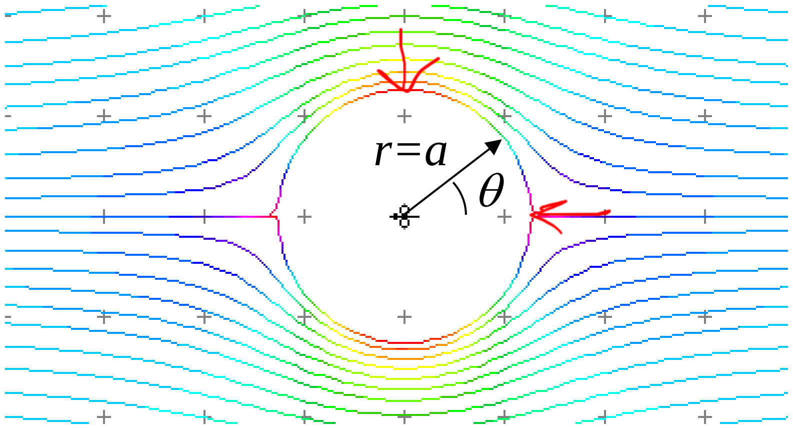
$$= -2iV_\infty \sin\theta$$

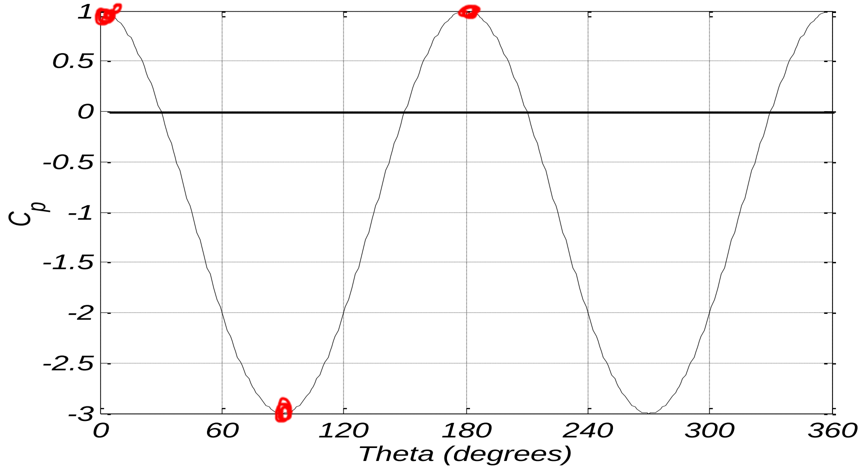
$\underline{V_r = 0}$; $V_\theta = -2V_\infty \sin\theta$

Pressure distribution on its surface?

$$C_p = 1 - \frac{|W(z)|^2}{V_\infty^2} \quad (V_\theta^2) \quad = 1 - 4\sin^2\theta$$

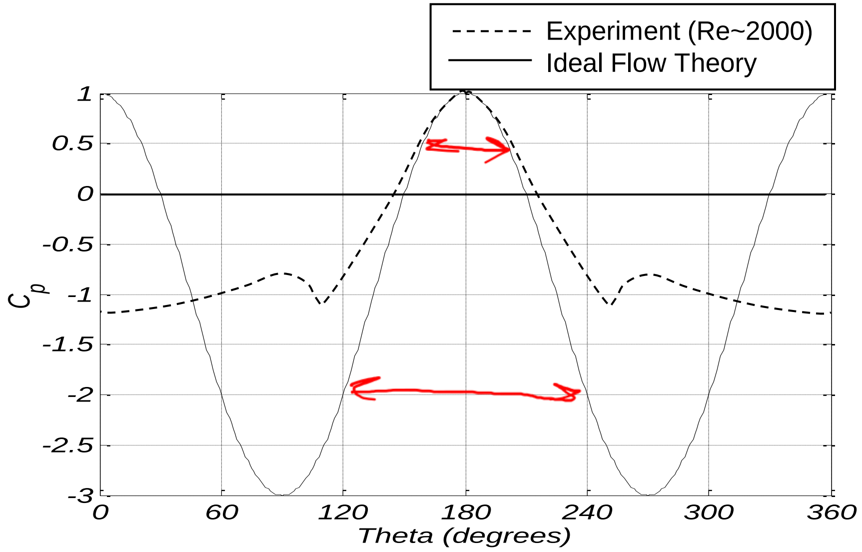
$$C_p = \frac{P - P_\infty}{\frac{1}{2}\rho U_\infty^2}$$

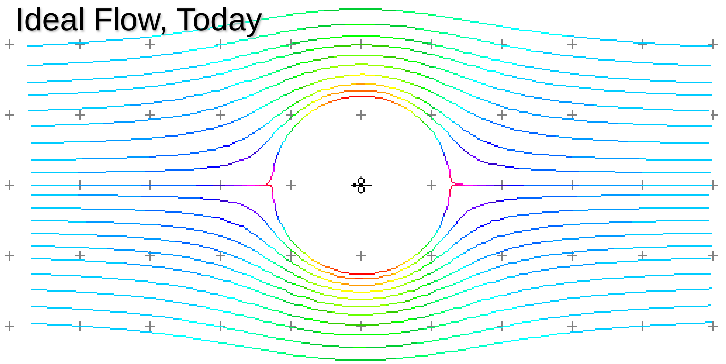


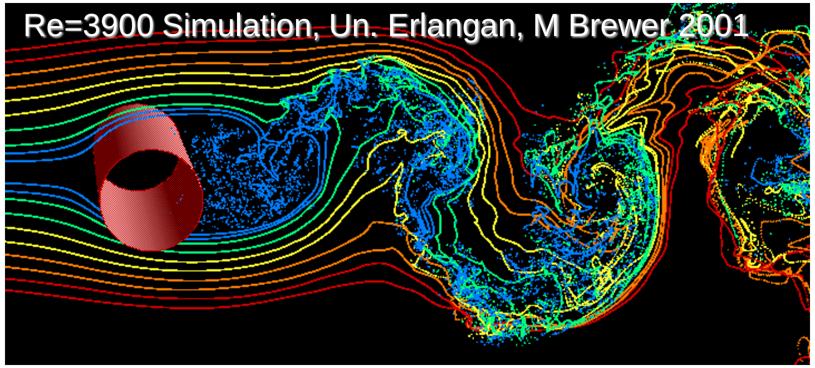


1. Acyclic Flow Past a Circular Cylinder

Comparison with real life







2. Circular Cylinder with Circulation

= <u>Uniform flow</u> + <u>Opposing Doublet</u> + <u>Vortex</u>

1 2 3

So, for a cylinder radius a, centered at z_1 in a free stream of velocity V_∞ at angle α to the x axis with circulation Γ :

$$F(z) = V_\infty e^{-i\alpha} z + \frac{V_\infty a^2 e^{i\alpha}}{z - z_1} - \frac{i\Gamma}{2\pi} \log_e(z - z_1)$$

$$W(z) = V_\infty e^{-i\alpha} - \frac{V_\infty a^2 e^{i\alpha}}{(z - z_1)^2} - \frac{i\Gamma}{2\pi(z - z_1)}$$

Velocity on cylinder surface

(take $\alpha = z_1 = 0$ and $z = ae^{i\theta}$)

$$W(z) = V_\infty - \frac{V_\infty a^2}{z^2} - \frac{i\Gamma}{2\pi z}$$

$$v_r - iv_\theta = V_\infty e^{i\theta} - \frac{V_\infty a^2 e^{i\theta}}{z^2} - \frac{i\Gamma e^{i\theta}}{2\pi z}$$

$$= V_\infty e^{i\theta} - \frac{V_\infty a^2 e^{i\theta}}{a^2 e^{2i\theta}} - \frac{i\Gamma e^{i\theta}}{2\pi a e^{i\theta}}$$

$$v_\theta = -2V_\infty \sin\theta + \frac{\Gamma}{2\pi a} \qquad v_r = 0$$

Pressure and Stagnation Points

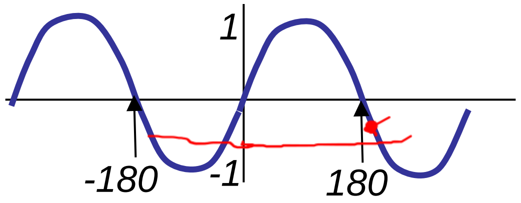
$$v_\theta = 0 = -2V_\infty \sin\theta_{stag} + \frac{\Gamma}{2\pi a}$$

$$\theta_{stag} = \arcsin\frac{\Gamma}{4\pi a V_\infty}$$

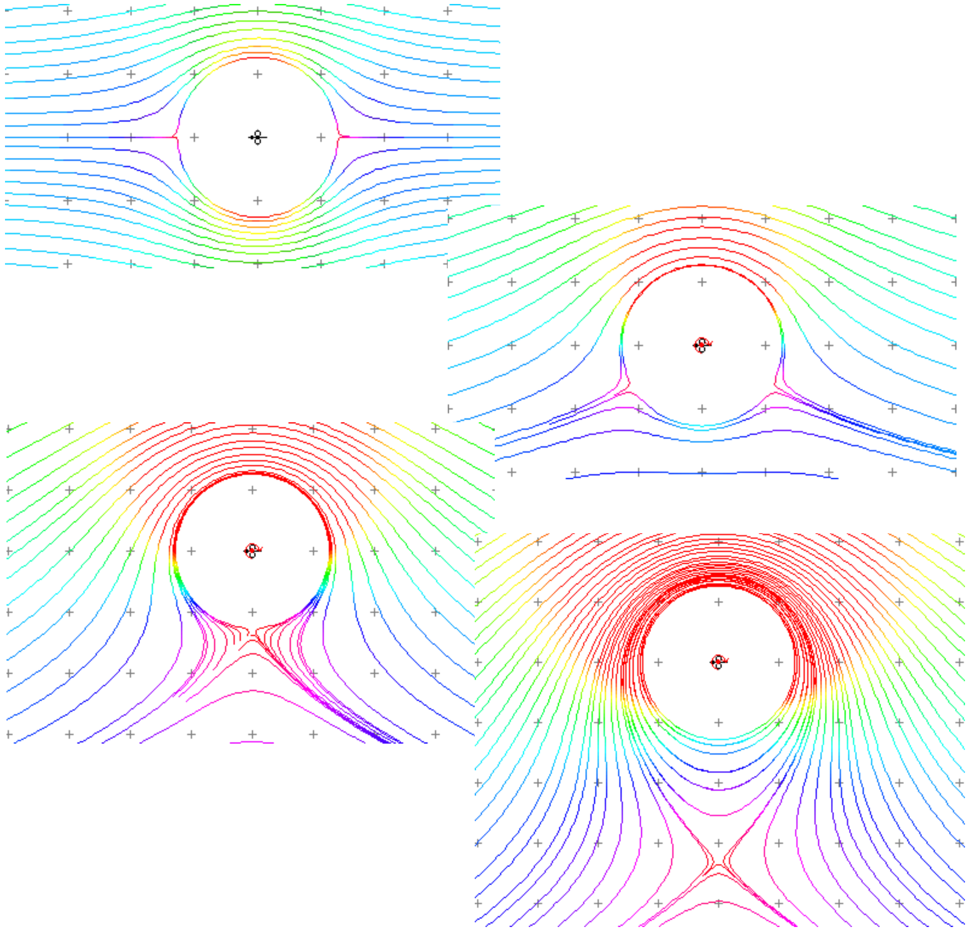
$$C_p\Big|_{surf} = 1 - v_\theta^2/V_\infty^2$$

$$= 1 - 4\sin^2\theta - 4\left[\frac{\Gamma}{4\pi a V_\infty}\right]^2 + 8\sin\theta\left[\frac{\Gamma}{4\pi a V_\infty}\right]$$

2. Circular Cylinder with Circulation



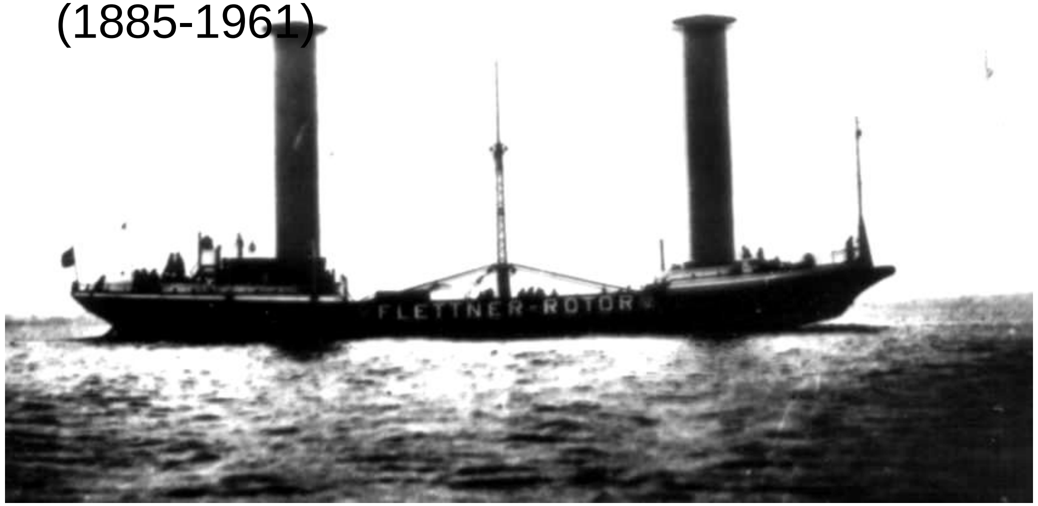
$\frac{\Gamma}{4\pi a V_\infty}$	$\theta_{stag} = \arcsin\frac{\Gamma}{4\pi a V_\infty}$
0	0°, 180°
-0.5	-30°, 210°
-1	-90°, 270°
-1.5	?



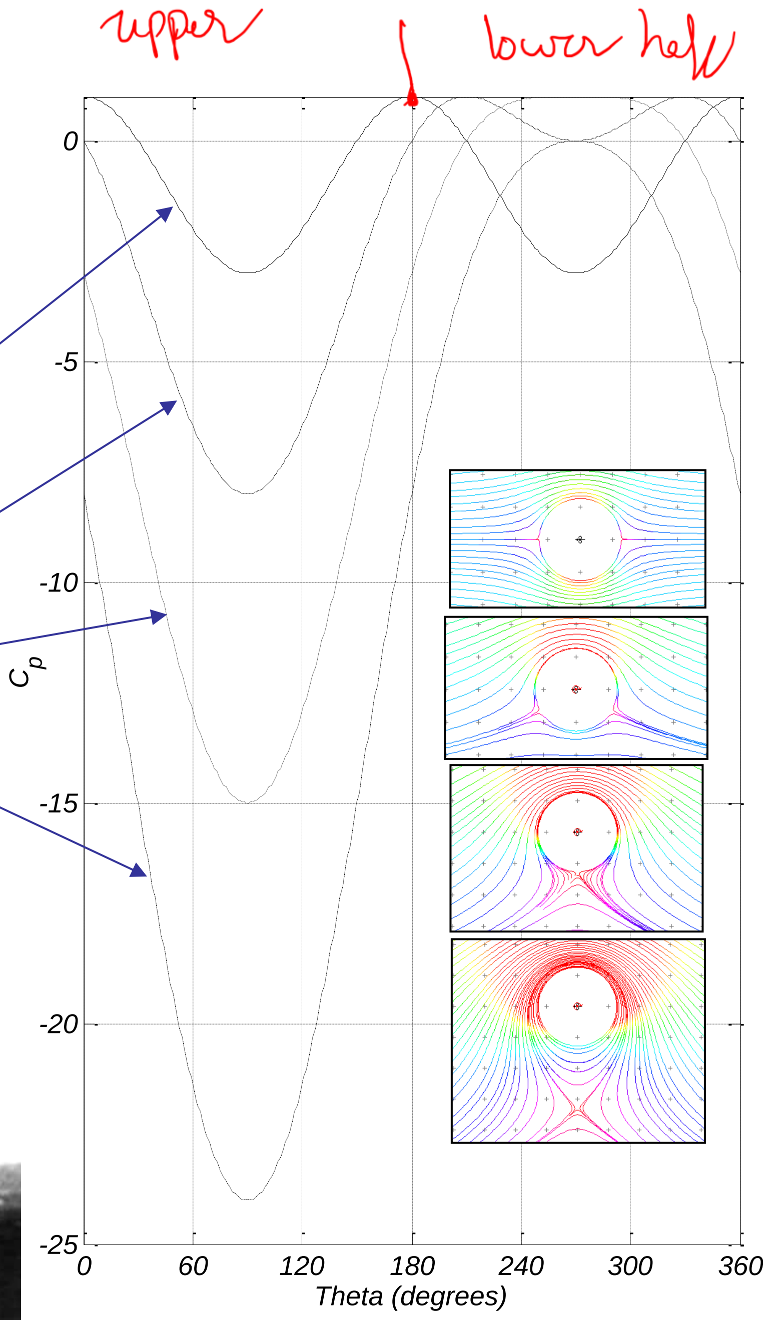
2. Circular Cylinder with Circulation

$$C_p\Big|_{surf} = 1 - 4\sin^2\theta - 4\left(\frac{\Gamma}{4\pi a V_\infty}\right)^2 + 8\sin\theta\left(\frac{\Gamma}{4\pi a V_\infty}\right)$$

Anton Flettner
(1885-1961)



$$\frac{\Gamma}{4\pi a V_\infty} = 0$$
$$= -0.5$$
$$= -1.0$$
$$= -1.5$$



Summary / Crib

1. Uniform flow of V_∞ at angle α to the x axis
   $W(z) = V_\infty e^{-i\alpha}$ $\quad$ $F(z) = V_\infty z e^{-i\alpha}$

2. Source at z_1 flow producing volume flowrate q
   $W(z) = \dfrac{q}{2\pi(z - z_1)}$ $\quad$ $F(z) = \dfrac{q}{2\pi}\log_e(z - z_1)$

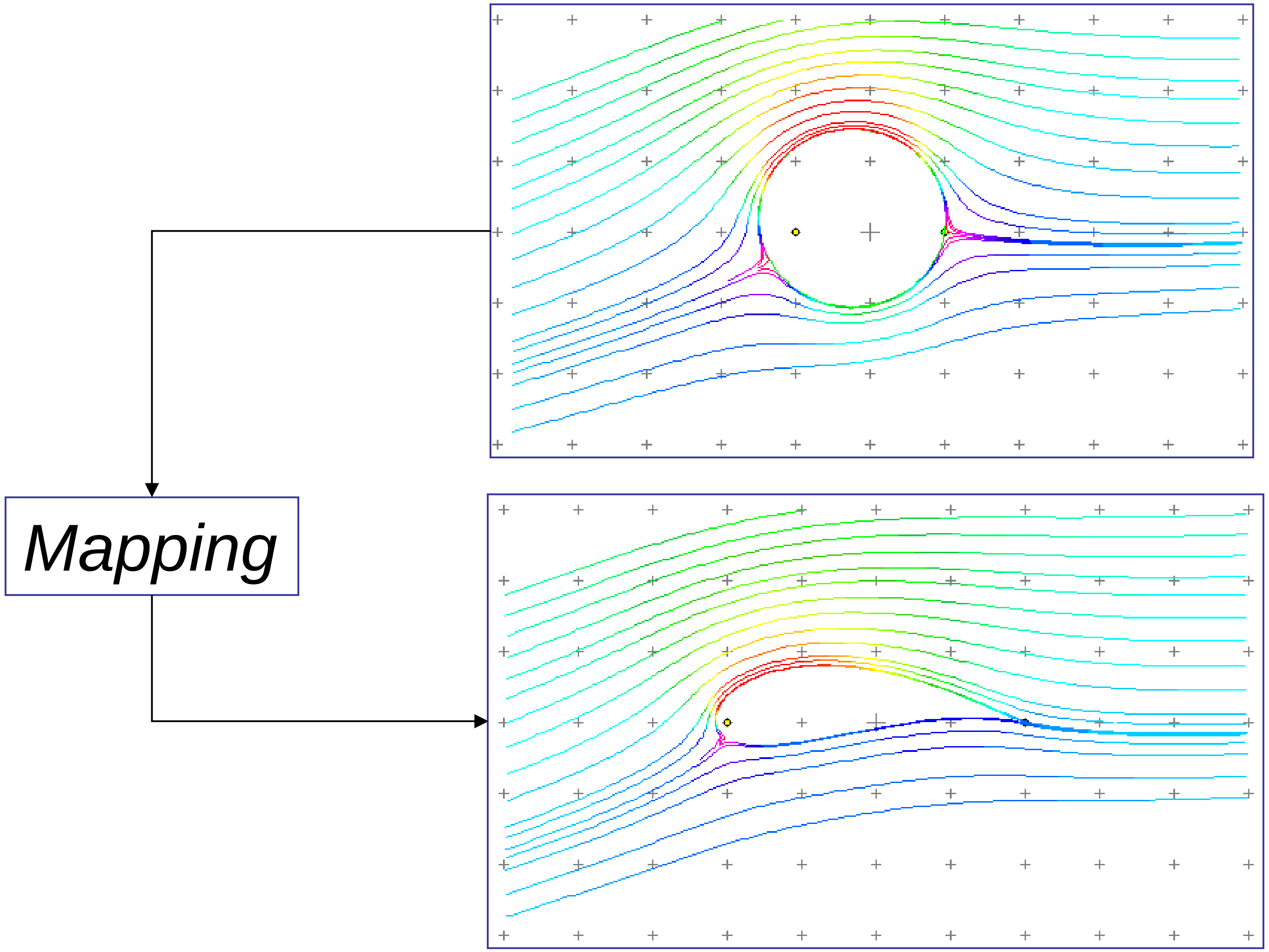
3. Vortex at z_1 producing circulation Γ
   $W(z) = -\dfrac{i\Gamma}{2\pi(z - z_1)}$ $\quad$ $F(z) = -\dfrac{i\Gamma}{2\pi}\log_e(z - z_1)$

4. Doublet at z_1 strength μ aligned at angle β to the x axis
   $W(z) = \dfrac{\mu e^{i\beta}}{2\pi(z - z_1)^2}$ $\quad$ $F(z) = -\dfrac{\mu e^{i\beta}}{2\pi(z - z_1)}$

5. Flow of velocity V_∞ at angle α past a circular cylinder of radius a at z_1 with circulation Γ

$$F(z) = V_\infty e^{-i\alpha} z + \frac{V_\infty a^2 e^{i\alpha}}{z - z_1} - \frac{i\Gamma}{2\pi}\log_e(z - z_1)$$

$$W(z) = V_\infty e^{-i\alpha} - \frac{V_\infty a^2 e^{i\alpha}}{(z - z_1)^2} - \frac{i\Gamma}{2\pi(z - z_1)}$$


Mapping

Mapping Functions

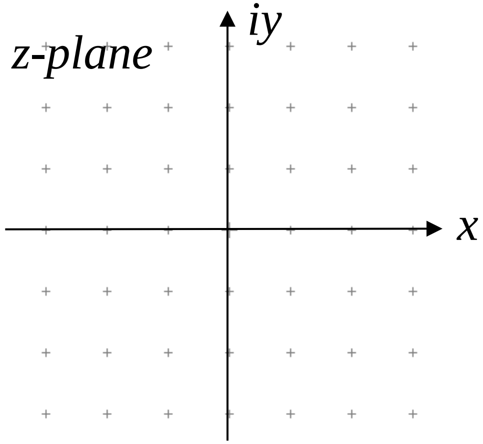
ANY FUNCTION THAT TRANSFORMS ONE SET OF COORDINATES TO ANOTHER

E.G: αz, z^2, $\sin(z)$, $\log_e(z)$

$$\zeta = \zeta(z)$$

$z = x + iy$ represents coordinate (x,y)

$\zeta = \xi + i\eta$ represents coordinate (ξ,η)

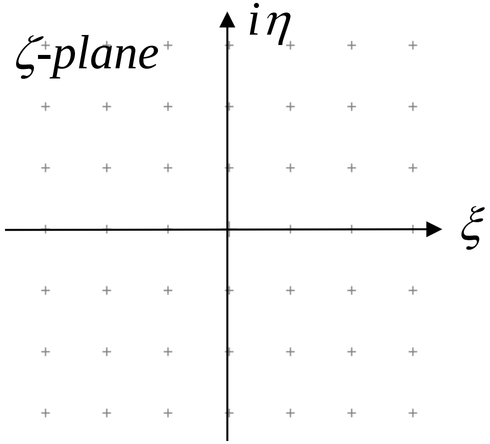


$$\zeta = 2z$$

$$z = \frac{1}{2}\zeta$$

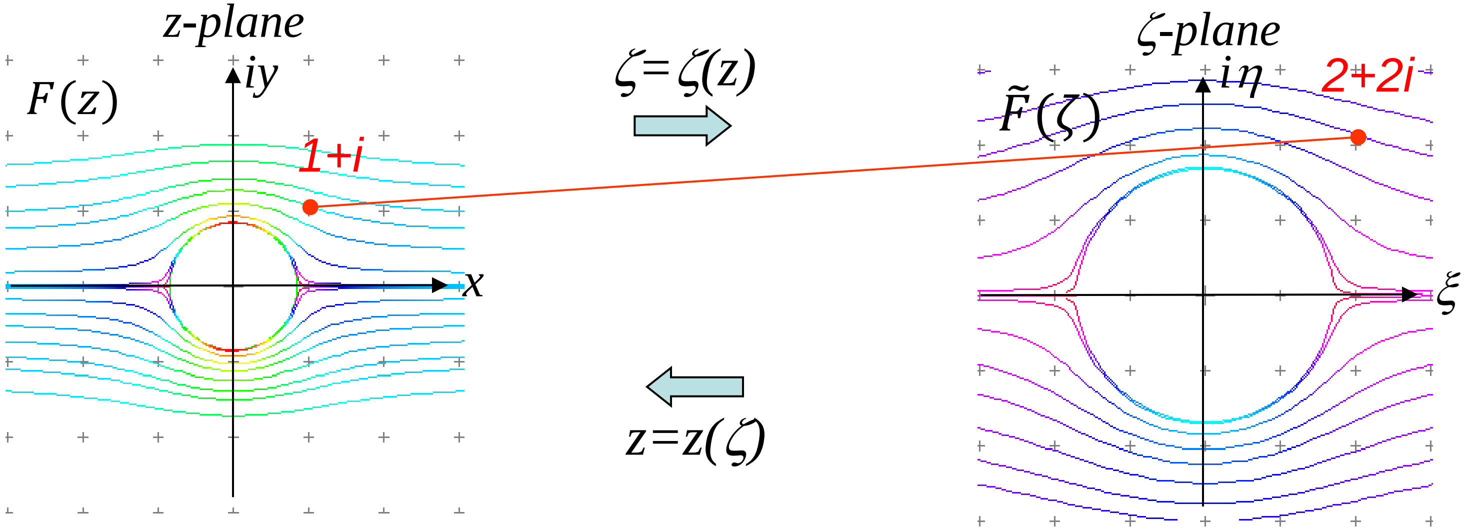
Inverse mapping

$z = z(\zeta)$



Mapping a Flow

Imagine the simple mapping $\zeta=2z$. We would want the mapping to transfer the flow at $1+i$, say, to the point $2+2i$



$$\tilde F(\zeta) = F(z(\zeta))$$

WE SHOULD HAVE $\tilde F(\zeta) = \tilde\phi + i\tilde\psi$

AND $\tilde W(\zeta) = \frac{d\tilde F}{d\zeta} = \frac{d\,F(z(\zeta))}{d\zeta} = \frac{dF}{dz}\cdot\frac{dz}{d\zeta} = W(z)\cdot\frac{dz}{d\zeta}$

Whe~~n~~re is a mapped flow valid?

$\tilde{F}(\zeta)$ IS A VALID FLOW WHEREVER ITS ANALYTIC $\quad \dfrac{d\tilde{F}}{d\zeta} = \dfrac{dF}{dz} \cdot \dfrac{dz}{d\zeta}$

—— || —— AS LONG AS $F(z)$ IS ANALYTIC AND $\dfrac{d\zeta}{dz} \neq 0$

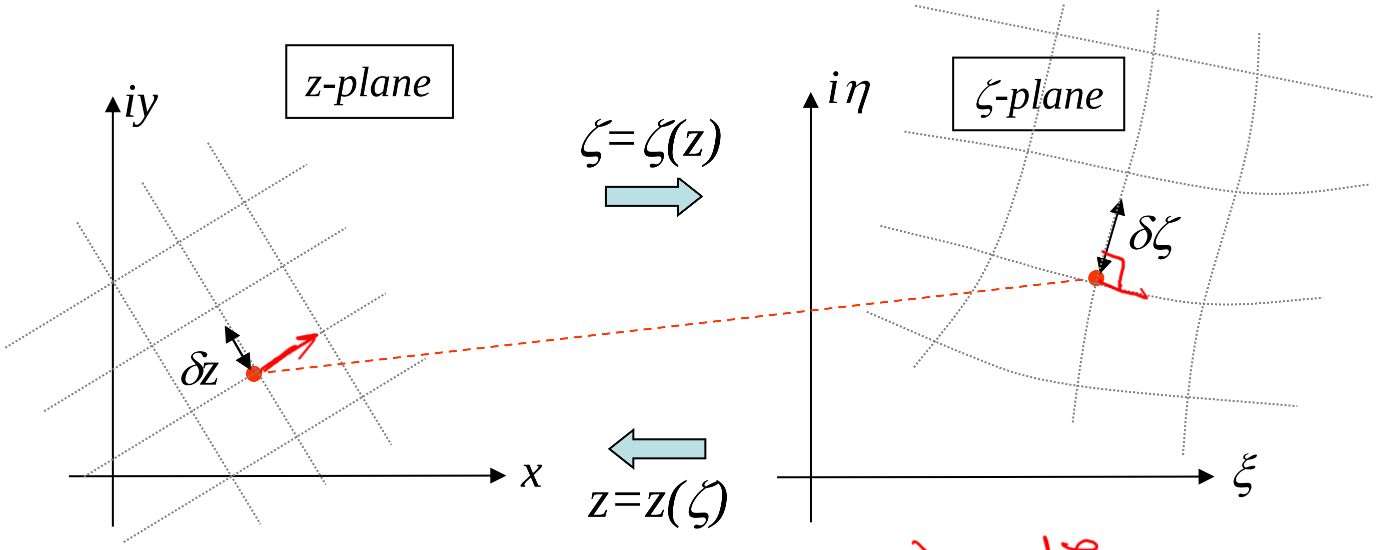
i) A VALID MAPPING IS CALLED ~~CO~~ "CONFORMAL" $\quad \dfrac{d\zeta}{dz} \neq 0$

i) POINTS WHERE $\dfrac{d\zeta}{dz} = 0$ ARE CALLED "CRITICAL POINTS"

Mapping complex potential or velocity?

We usually map the complex potential because it is easy to envisage moving or distorting the streamlines with the mapping function. One can map the complex velocity instead (hodograph mapping) but this is harder to visualize and in general <u>produces a different result</u>

Effects of Mapping on Microscopic Geometry



$\delta\zeta = \frac{d\zeta}{dz} \cdot \delta z$ (IF IT IS CONFORMAL). $\frac{d\zeta}{dz}$ IS A COMPLEX NUMBER, e.g. $c\,e^{i\beta}$

SO MULTIPLYING BY $d\zeta/dz$ MEANS MAGNIFYING BY c & ROTATING BY β

ANGLES OF INTERSECTION ARE PRESERVED UNDER CONFORMAL MAPPING

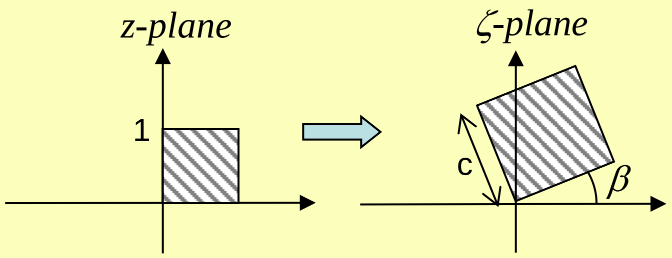
ANGLES OF INTERSECTION ARE NOT PRESERVED AT CRITICAL POINTS

Effects of Mapping on Macroscopic Geometry

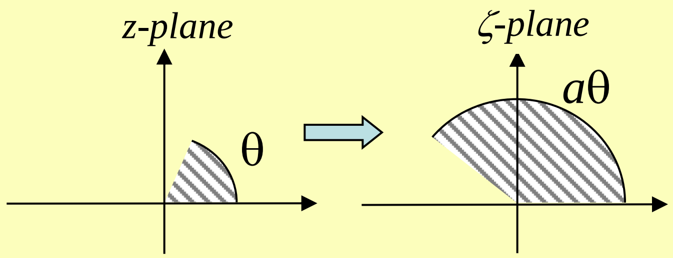
"To understand what a mapping does to a flow, one must first understand what it does to the space containing that flow"

1. Rotation and scaling $\zeta = Az$

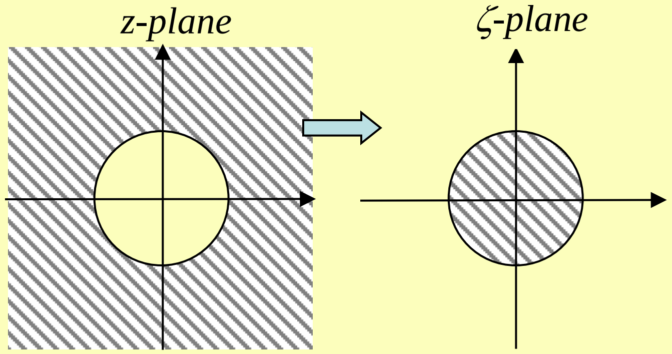
$A = const. = ce^{i\beta}$



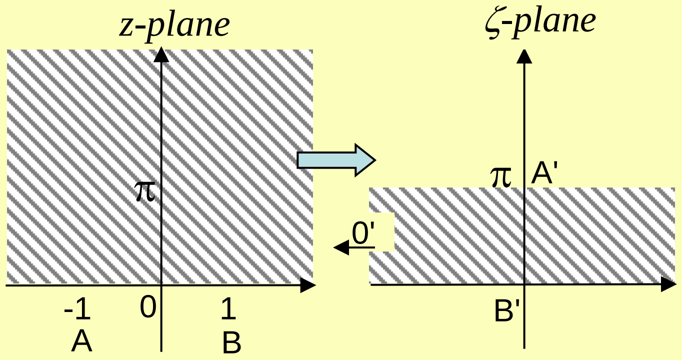
2. Power $\zeta = z^a$ $a = real > 0$



3. Inversion in a circle $\zeta = 1/z$

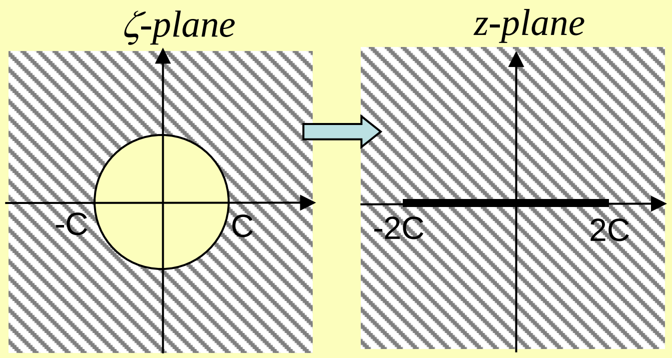


4. Logarithm $\zeta = \log_e z$



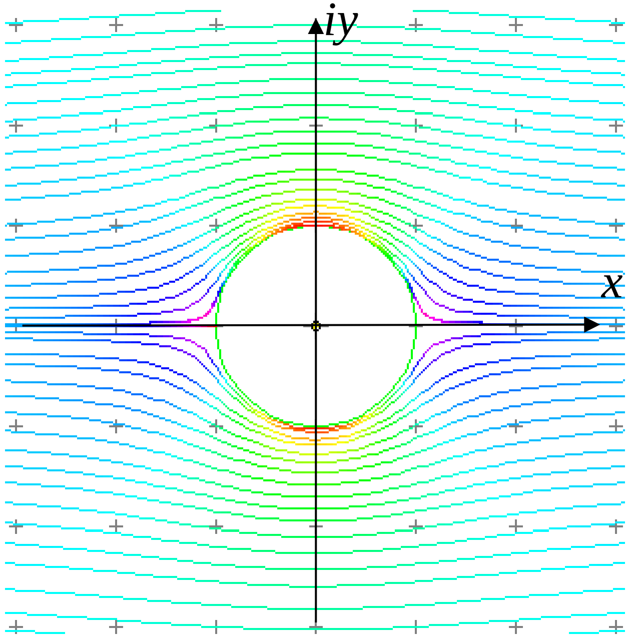
5. Joukowski $z = \zeta + C^2/\zeta$

$C = real > 0$



1. Magnification and Rotation

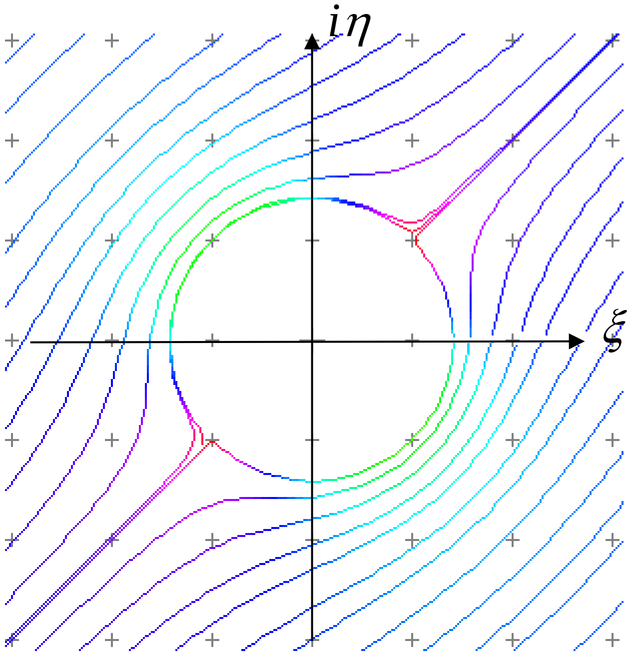
z-plane



$$\zeta = (1+i)z$$

$$z = \zeta/(1+i)$$

ζ-plane



Flow past a circular cylinder

Effects of Mapping on Macroscopic Geometry

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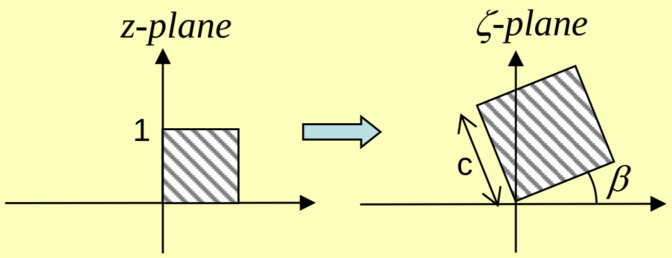
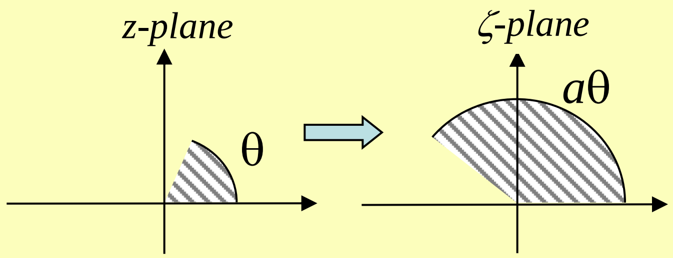
1. Rotation and scaling $\zeta = Az$

$A = const. = ce^{i\beta}$



2. Power $\zeta = z^a$ $a = real > 0$



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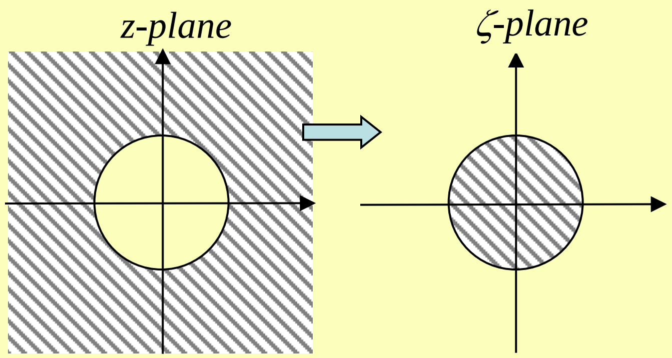
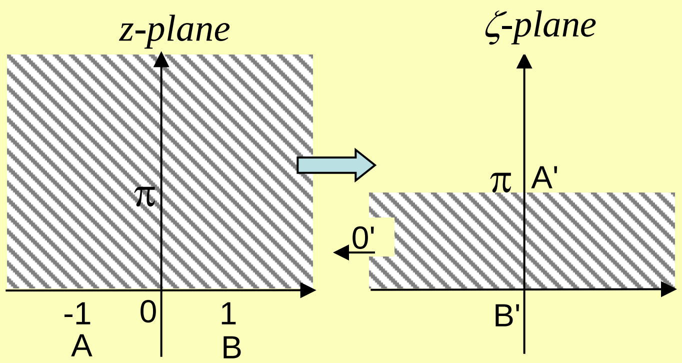
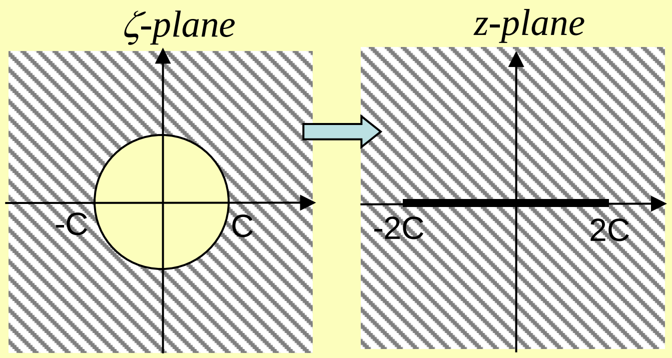


4. Logarithm $\zeta = \log_e z$



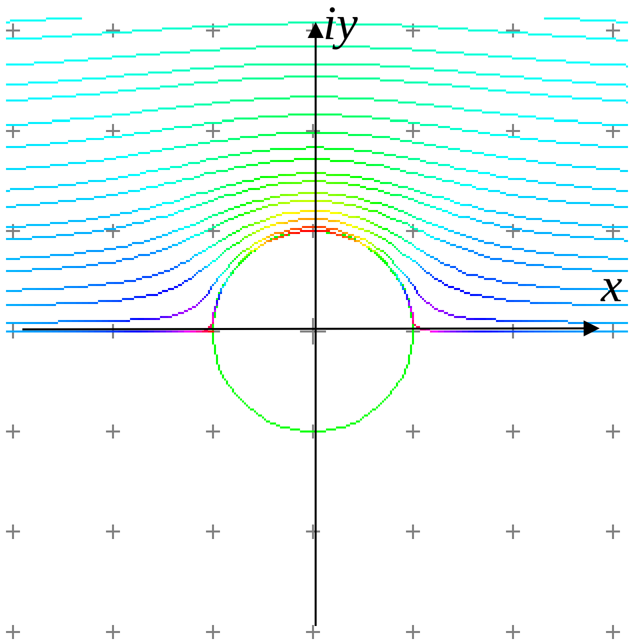
5. Joukowski $z = \zeta + C^2/\zeta$

$C = real > 0$



2. Power

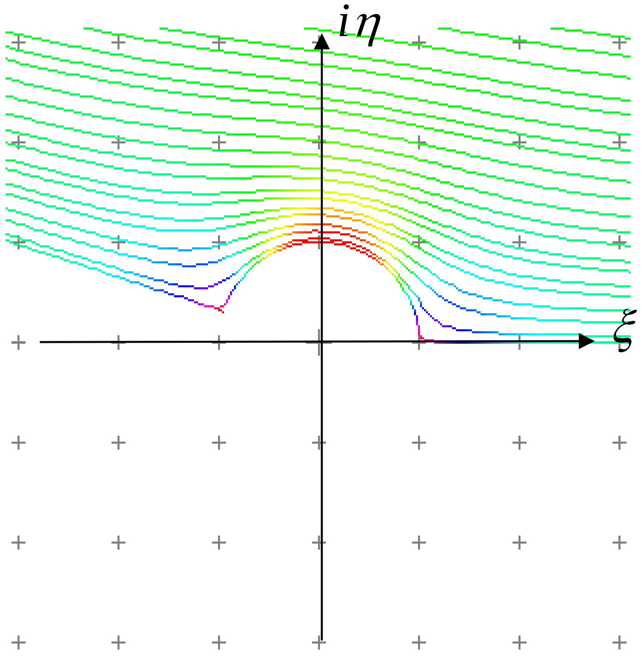
z-plane



$$\boxed{\zeta = z^{0.8}}$$

$$z = \zeta^{1.25}$$

ζ-plane



Flow past a circular cylinder

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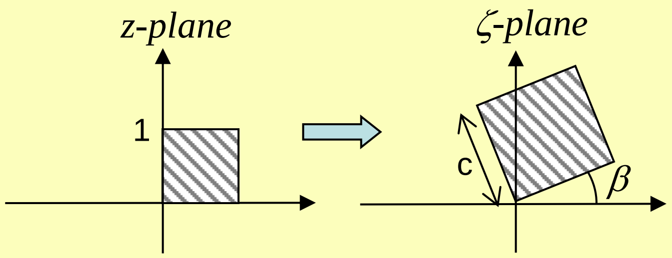
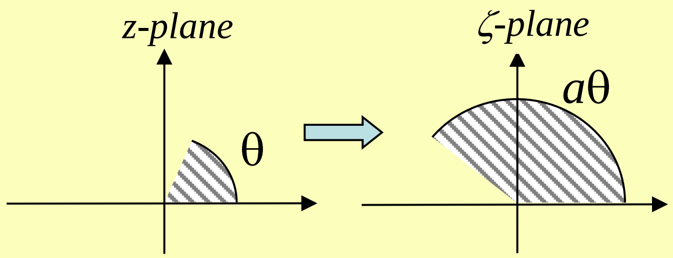


2. Power $\zeta = z^a$ $a = real > 0$



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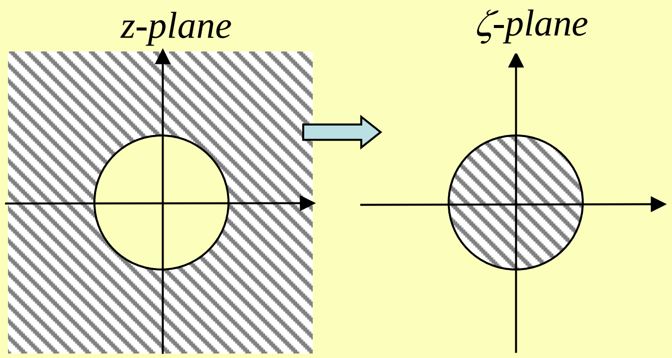
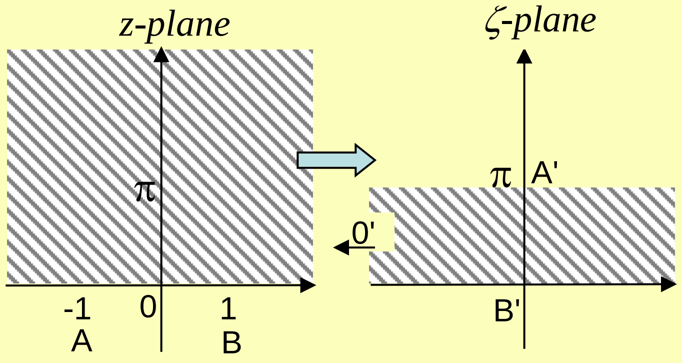
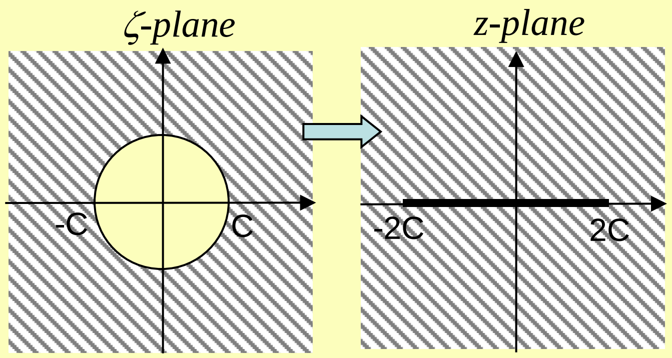


4. Logarithm $\zeta = \log_e z$



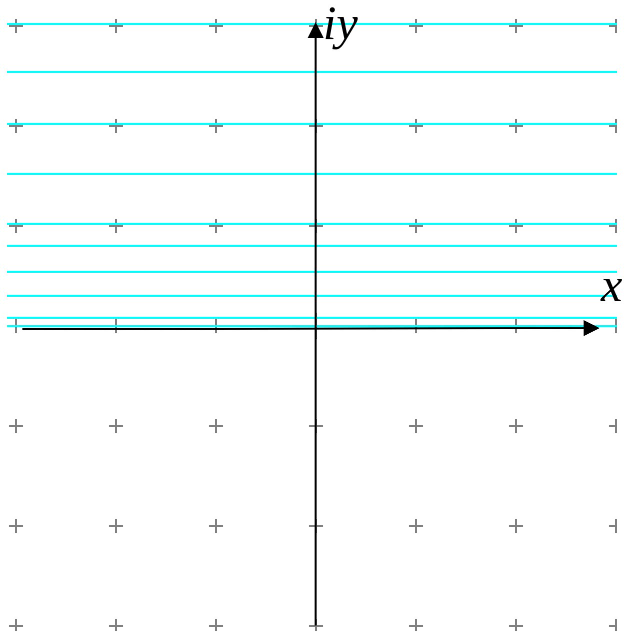
5. Joukowski $z = \zeta + C^2/\zeta$

$C = real > 0$



4. Logarithm Mapping

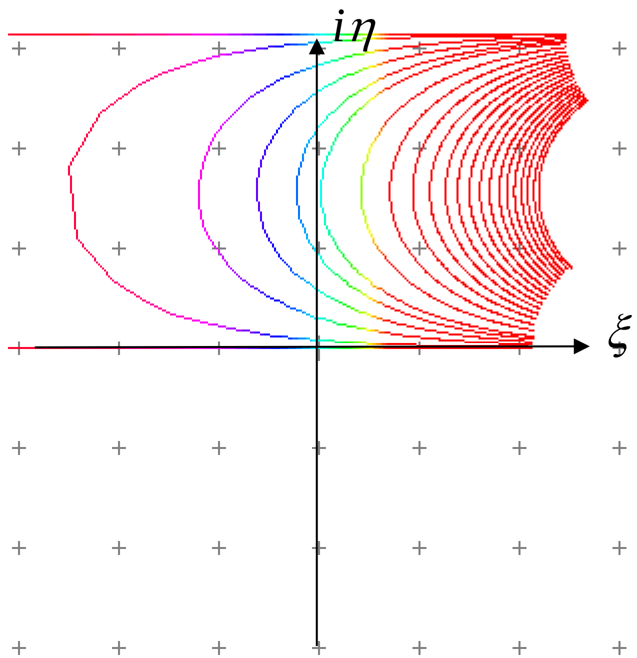
z-plane



$\zeta = \log_e z$

$z = e^{\zeta}$

ζ-plane



Uniform flow

$$\zeta = \log_e z = \log_e\left(re^{i\theta}\right)$$

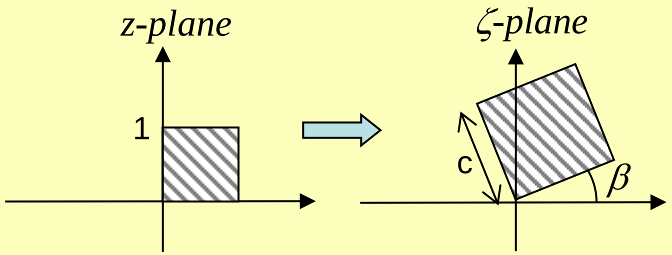
$$\xi + i\eta = \log_e r + i\theta$$

Effects of Mapping on Macroscopic Geometry

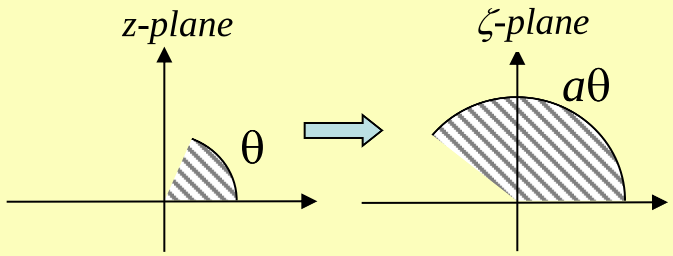
"To understand what a mapping does to a flow, one must first understand what it does to the space containing that flow"

1. Rotation and scaling $\zeta = Az$

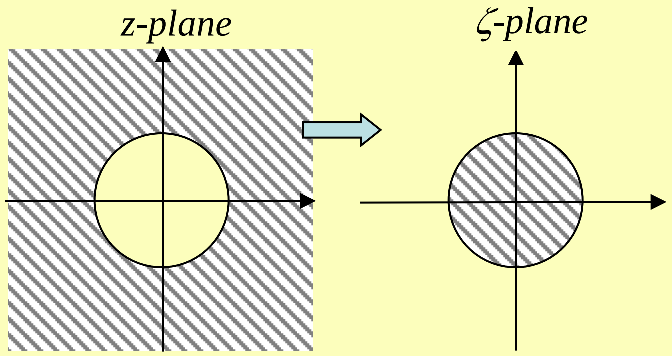
$A = const. = ce^{i\beta}$



2. Power $\zeta = z^a$ $a = real > 0$



3. Inversion in a circle $\zeta = 1/z$



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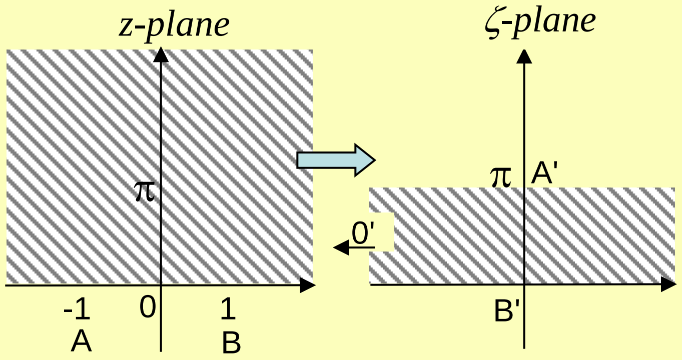

5. Joukowski $z = \zeta + C^2/\zeta$

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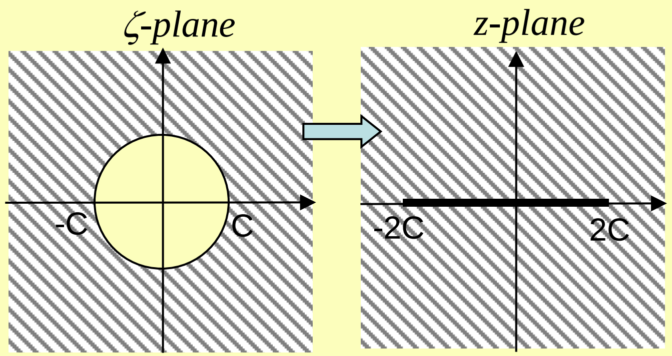
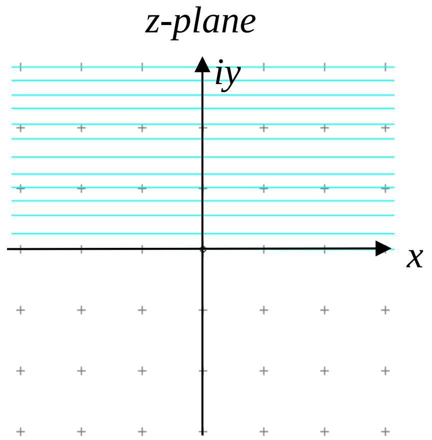


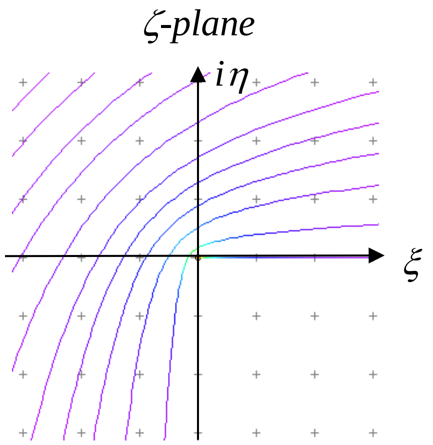
Example: Flow past a 90° Corner

Determine the complex potential and complex velocity for flow past an external 90 deg corner by mapping a uniform flow



$\zeta = z^{3/2}$

$z = \zeta^{2/3}$



$F(z) = V_\infty z$

$$\tilde{F}(\zeta) = F(z(\zeta)) = V_\infty \zeta^{2/3}$$

$$\tilde{W}(\zeta) = \frac{d\tilde{F}}{d\zeta} = \frac{2}{3} V_\infty \zeta^{-1/3}$$

~~Mapping~~? Critical Points?

$F(z)$? $\tilde{F}(\zeta)$? $\tilde{W}(\zeta)$?

CRITICAL POINTS:

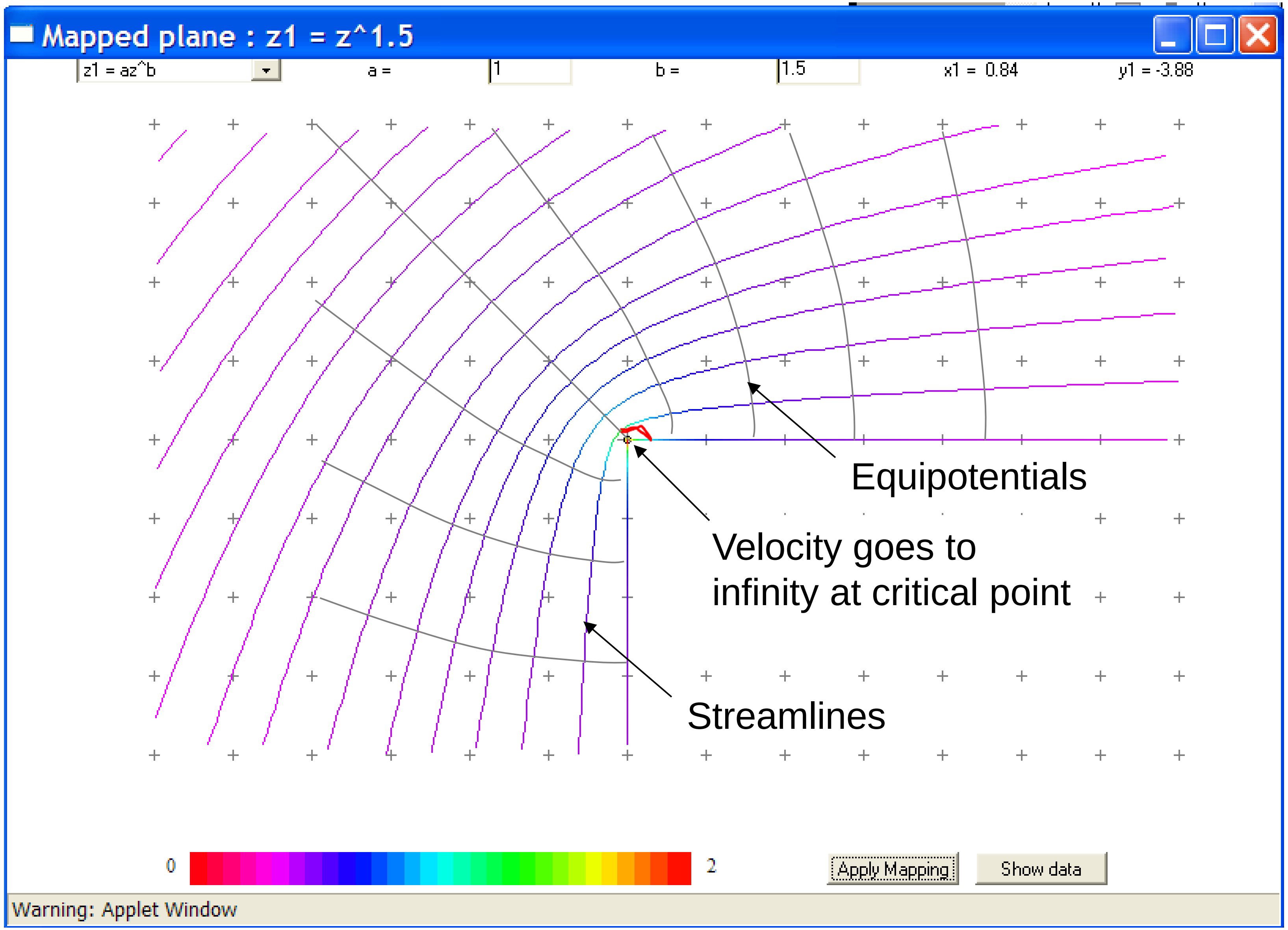
$$\frac{d\zeta}{dz} = \frac{3}{2} z^{1/2} = 0$$

$z = 0$ IS A CRITICAL POINT

$\zeta = z^{3/2}$; $\zeta = 0$ IS A CRITICAL POINT.

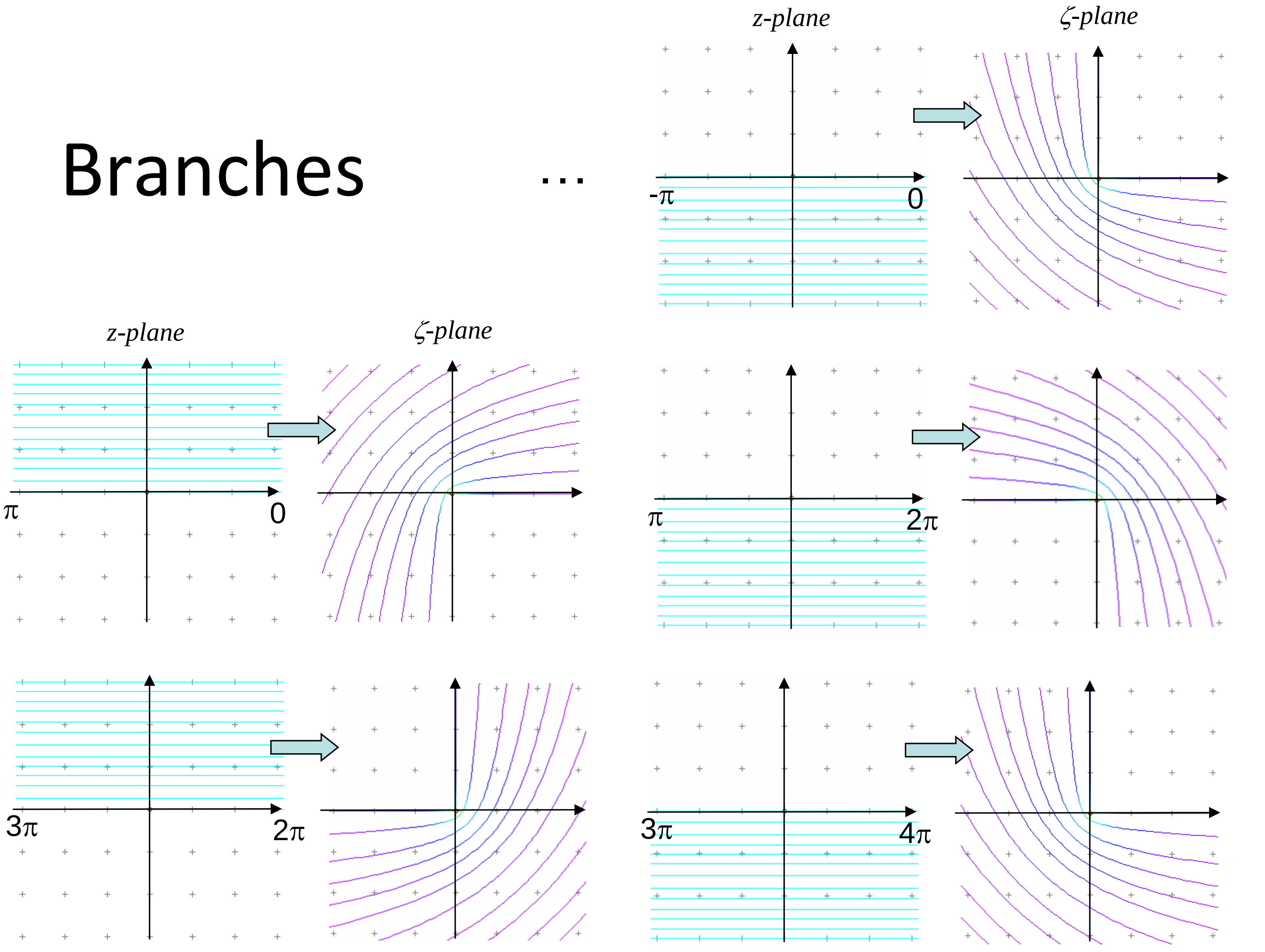
What happens at the critical point?

What happens if we consider the flow below the x axis?


Mapped plane : z1 = z^1.5
z1 = az^b
a =
1
b =
1.5
x1 = 0.84
y1 = -3.88
Equipotentials
Velocity goes to infinity at critical point
Streamlines
0
2
Apply Mapping
Show data
Warning: Applet Window

Branches

…



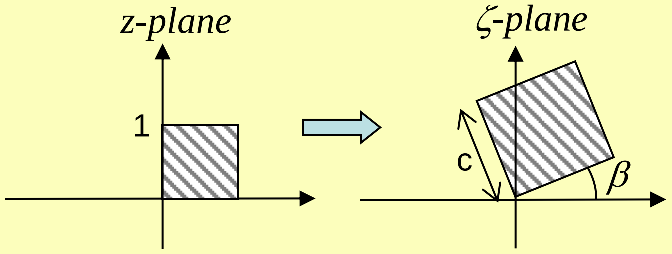
…

Effects of Mapping on Macroscopic Geometry

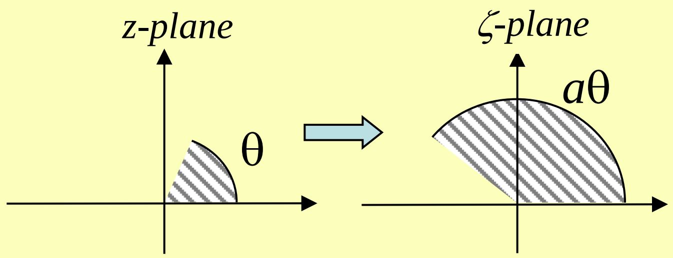
"To understand what a mapping does to a flow, one must first understand what it does to the space containing that flow"

1. Rotation and scaling $\zeta = Az$

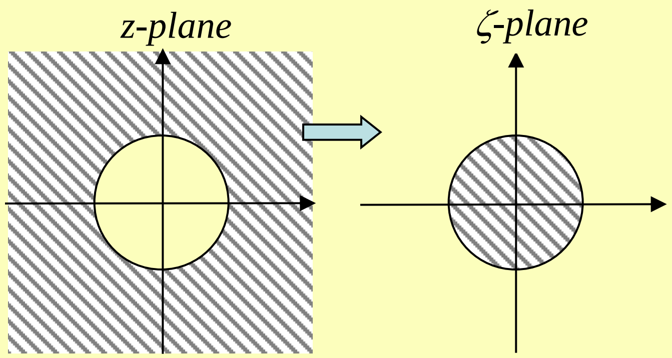
$A = const. = ce^{i\beta}$



2. Power $\zeta = z^a$ $a = real > 0$



3. Inversion in a circle $\zeta = 1/z$



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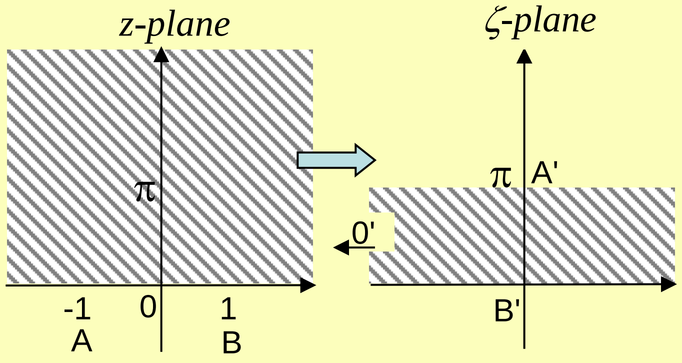
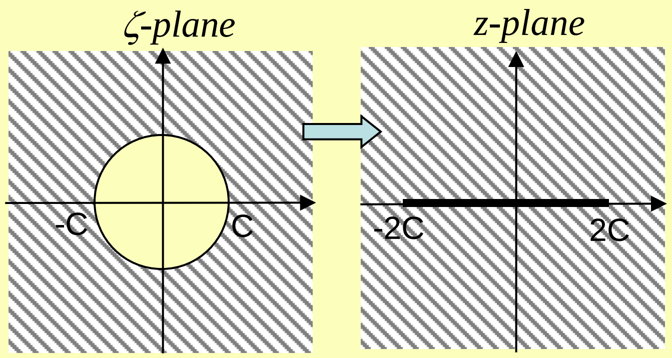


5. Joukowski $z = \zeta + C^2/\zeta$

$C = real > 0$



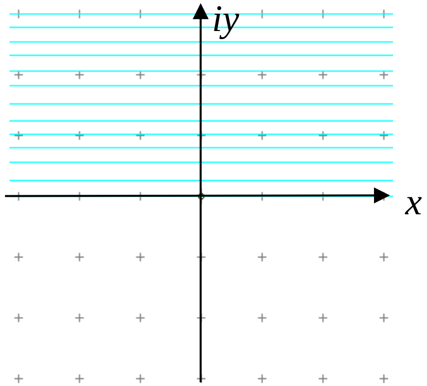
Example: Flow past a 90° Corner

Determine the complex potential and complex velocity for flow past an external 90 deg corner by mapping a uniform flow

~~Mapping?~~ Critical Points?

$F(z)$? $\tilde{F}(\zeta)$? $\tilde{W}(\zeta)$?

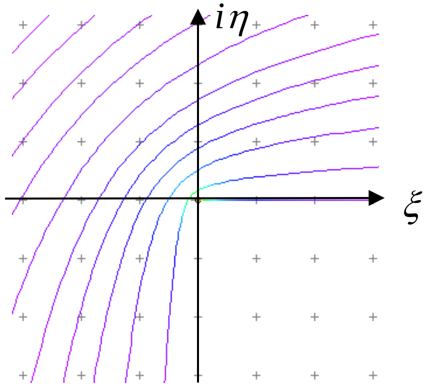
z-plane



$$\zeta = z^{3/2}$$

$$z = \zeta^{2/3}$$

ζ-plane



CRITICAL POINTS:

$$\frac{d\zeta}{dz} = \frac{3}{2} z^{1/2} = 0$$

$z = 0$ IS A CRITICAL POINT

$\zeta = z^{3/2}$; $\zeta = 0$ IS A CRITICAL POINT.

$$F(z) = V_\infty z$$

$$\tilde{F}(\zeta) = F(z(\zeta)) = V_\infty \zeta^{2/3}$$

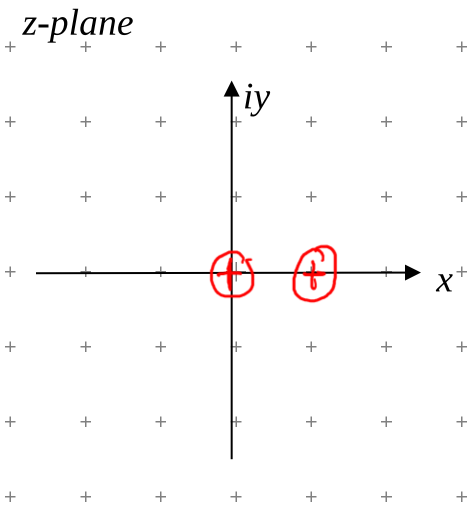
$$\tilde{W}(\zeta) = \frac{d\tilde{F}}{d\zeta} = \frac{2}{3} V_\infty \zeta^{-1/3}$$

What happens at the critical point?

What happens if we consider the flow below the x axis?

Example: Flow through a channel

Determine the complex potential and complex velocity for flow past a Rankine halfbody in a channel.

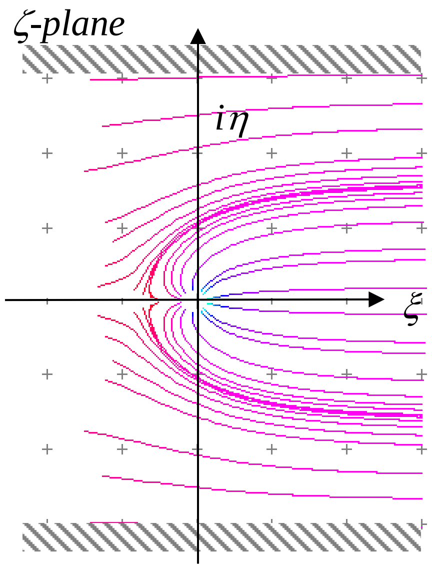


$\zeta(z)$

$\zeta = \log_e z$

$z = e^{\zeta}$

$z(\zeta)$



Mapping? Starting Flow?

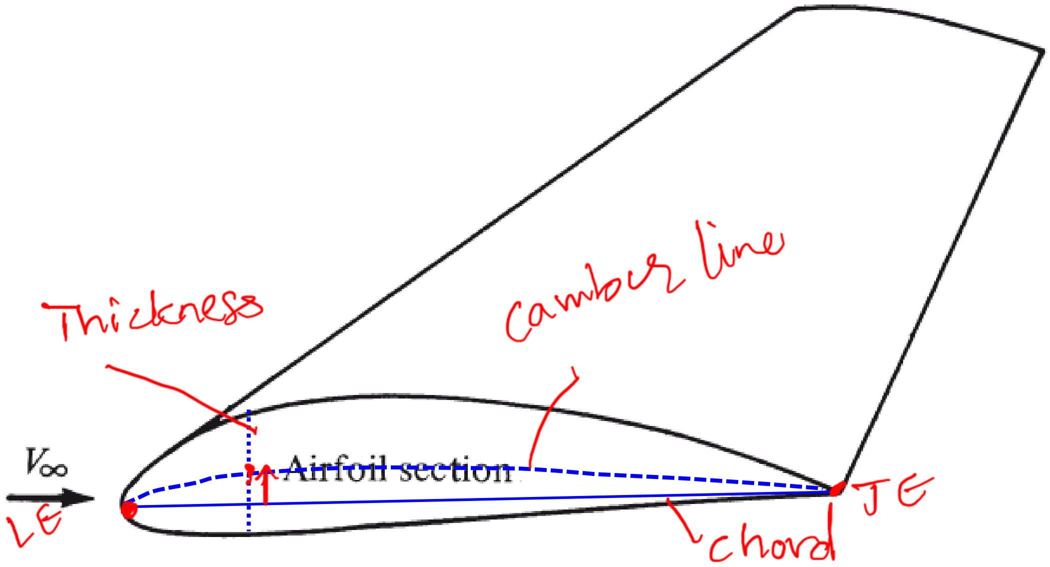
$F(z)$? $\tilde{F}(\zeta)$?

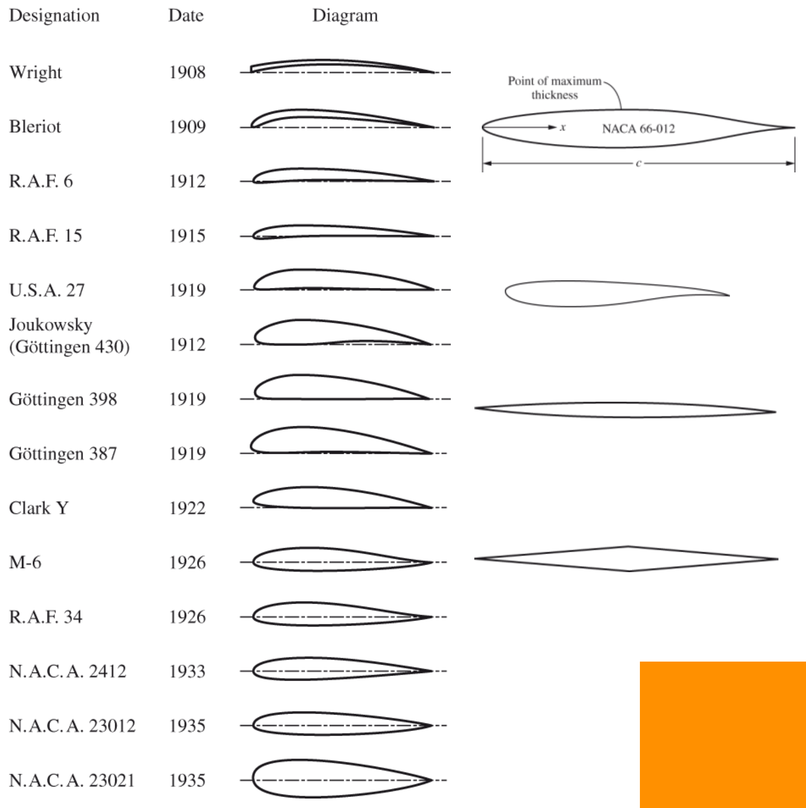
$$F(z) = \frac{q}{2\pi}\log_e(z-1) + \frac{q}{2\pi}\log_e z$$

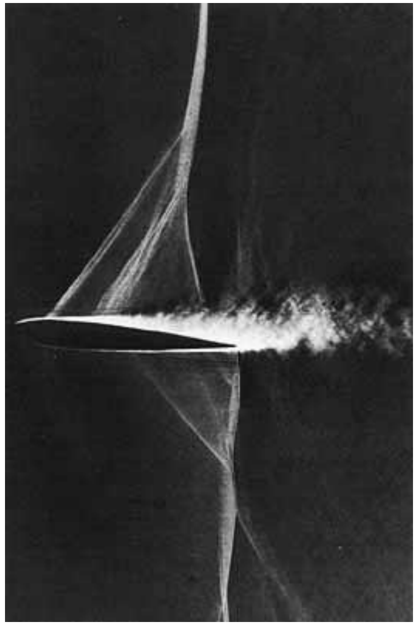
$$\tilde{F}(\zeta) = F(z(\zeta)) = \frac{q}{2\pi}\log_e\left(e^{\zeta}-1\right) + \cancel{\frac{q}{2\pi}\log_e\left(e^{\zeta}\right)} \quad \frac{q}{2\pi}\zeta$$

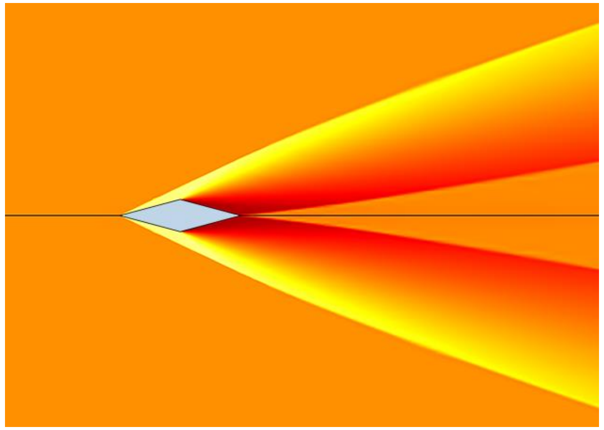
$$\tilde{W}(\zeta) = \frac{d\tilde{F}}{d\zeta} = \frac{dF}{dz}\cdot\frac{dz}{d\zeta}$$

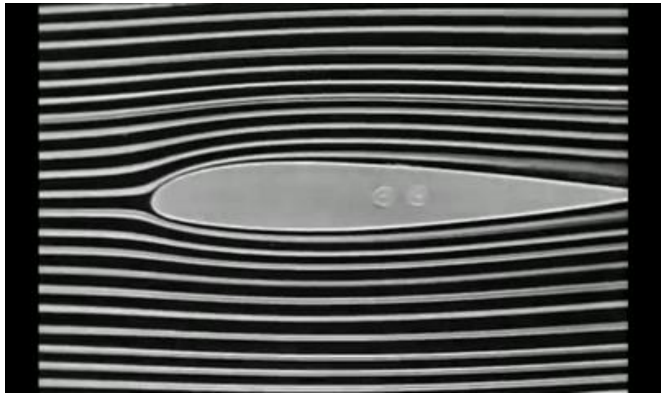
Airfoil ?











Airfoil: Any section taken perpendicular to span

Effects of Mapping on Macroscopic Geometry

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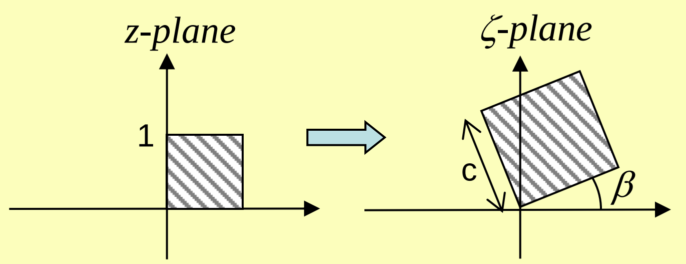
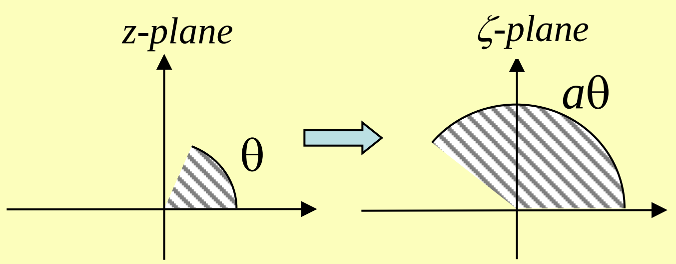
$A = const. = ce^{i\beta}$



2. Power $\zeta = z^a$ $\quad a = real > 0$



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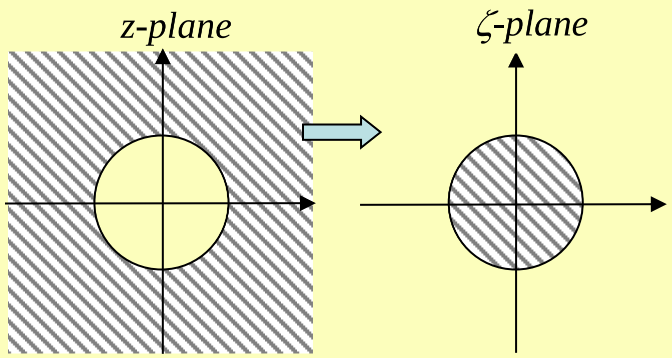

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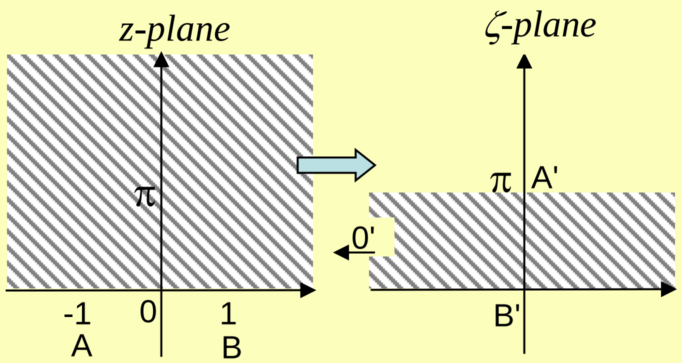

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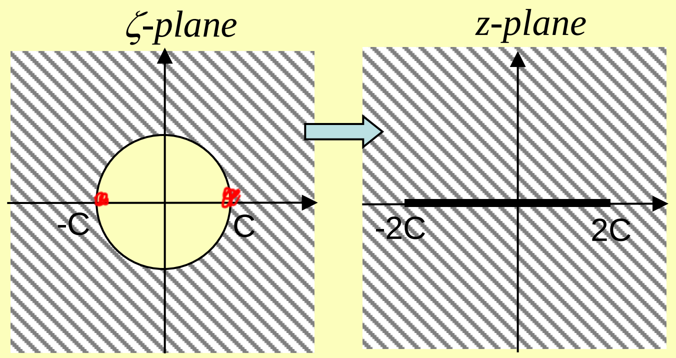
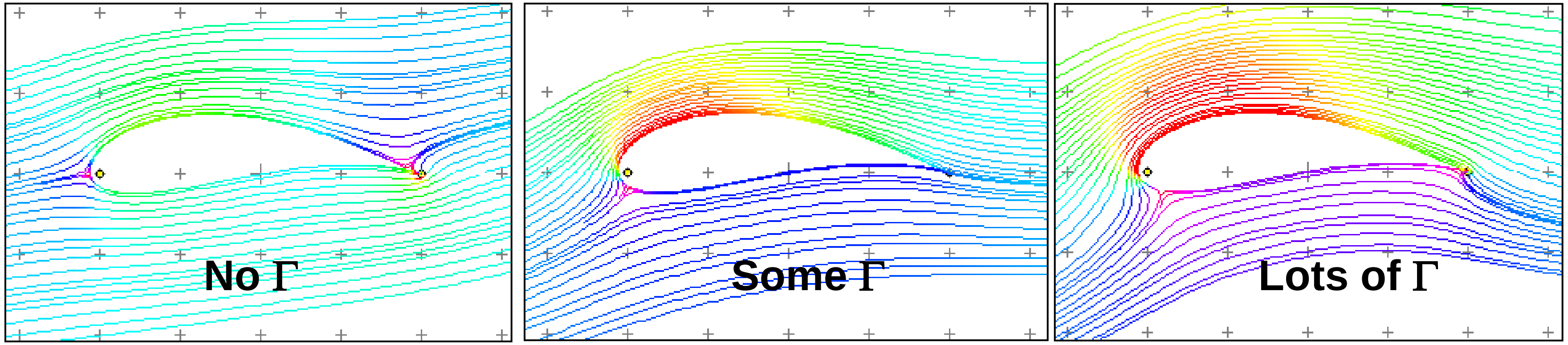


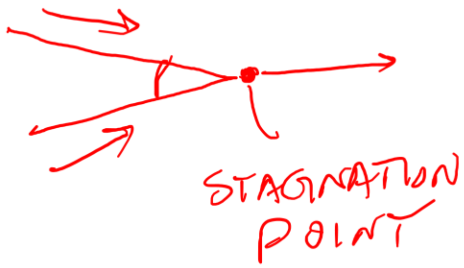
The Problem of the Airfoil



KUTTA CONDITION : FLOW LEAVES SMOOTHLY FROM THE TRAILING EDGE

⇒ EMPIRICAL OBSERVATION

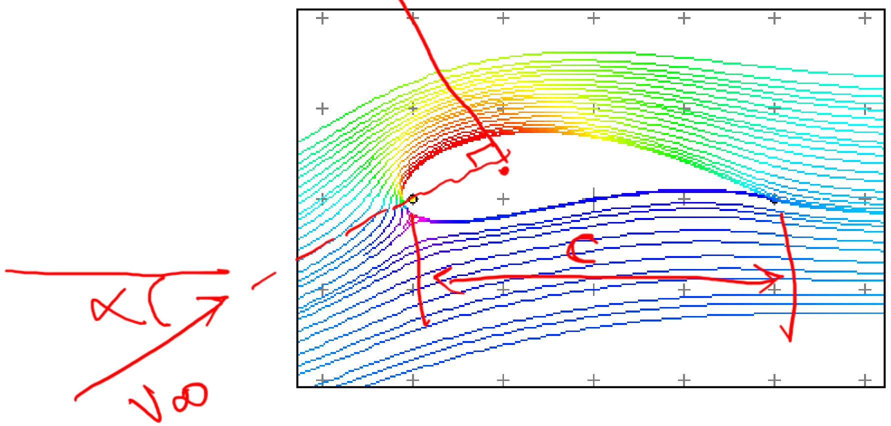
WEDGE



CUSPED

The Problem of the Airfoil

l LIFT PER UNIT SPAN



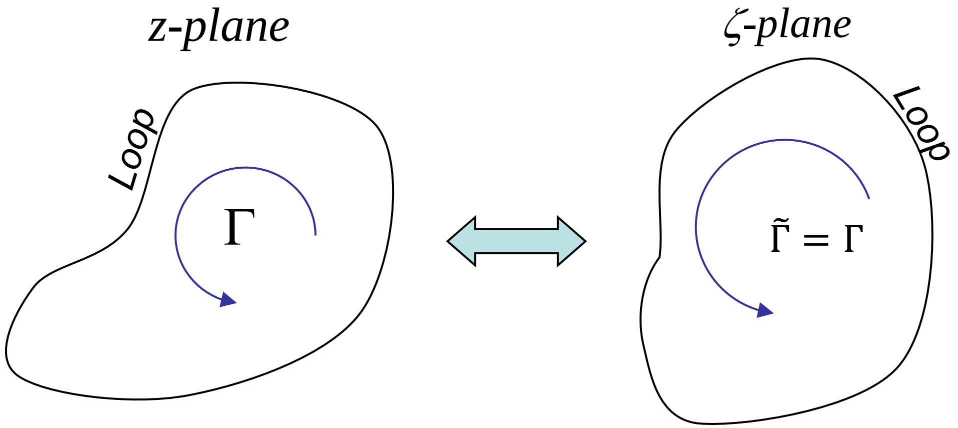
Terminology

LIFT COEFFICIENT $C_l = \dfrac{l}{\frac{1}{2}\rho V_\infty^2 \cdot c}$

AND

$$l = -\rho V_\infty \Gamma$$

Invariance of Circulation under Mapping

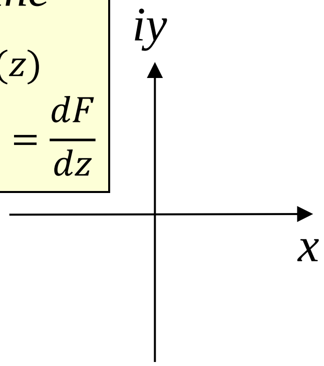


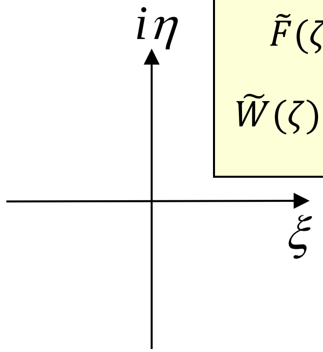
$$\widetilde{W}(\zeta) = W(z)\frac{dz}{d\zeta}$$

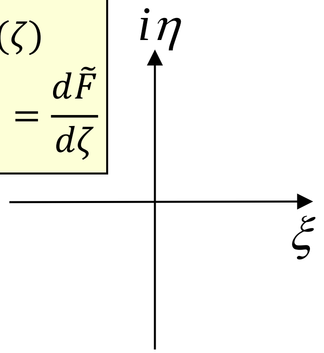
$$\begin{aligned}\Gamma + iq &= \oint_{loop} W(z)dz \\ &= \oint_{loop} \widetilde{W}(\zeta)\frac{d\zeta}{dz}dz \\ &= \oint_{loop} \widetilde{W}(\zeta)\, d\zeta \\ &= \tilde{\Gamma} + i\tilde{q}\end{aligned}$$

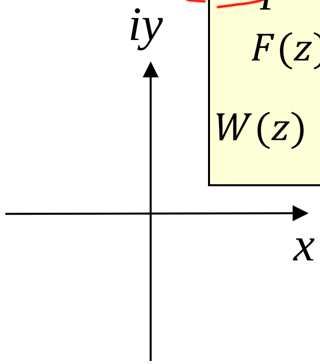
Symbol Conventions

	Initial Flow	Mapping	Mapped Flow
Our Mappings to this Point	*z-plane* $F(z)$ $W(z) = \frac{dF}{dz}$	$\zeta = \zeta(z)$ *Critical at* $\frac{d\zeta}{dz} = 0$	*ζ-plane* $\tilde{F}(\zeta) = F(z(\zeta))$ $\widetilde{W}(\zeta) = \frac{d\tilde{F}}{d\zeta} = W\frac{dz}{d\zeta}$
The Joukowski Mapping	*ζ-plane* $\tilde{F}(\zeta)$ $\widetilde{W}(\zeta) = \frac{d\tilde{F}}{d\zeta}$	$z = z(\zeta)$ *Critical at* $\frac{dz}{d\zeta} = 0$	*z-plane* $F(z) = \tilde{F}(\zeta(z))$ $W(z) = \frac{dF}{dz} = \widetilde{W}\frac{d\zeta}{dz}$



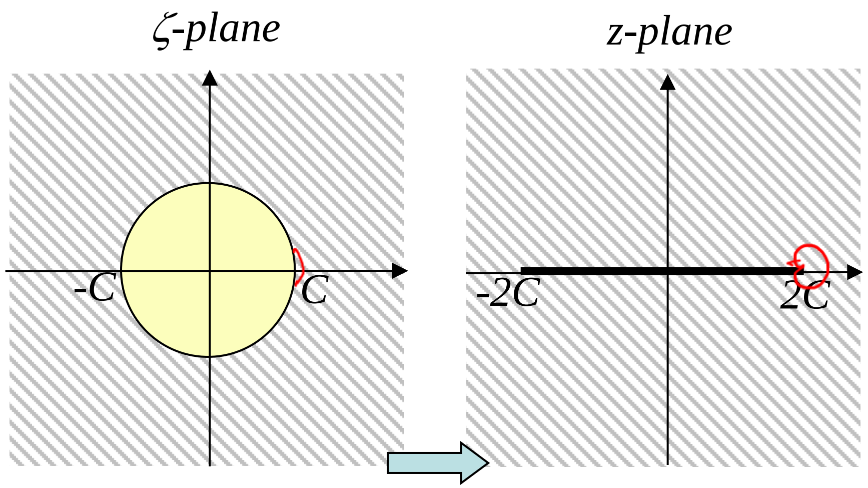


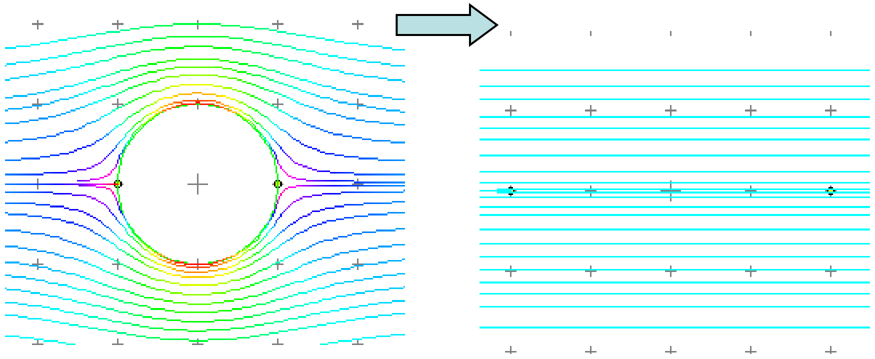




Joukowski Mapping

Effects on Space





$$z = \zeta + \frac{c^2}{\zeta}$$

At $\zeta \to \infty$, $z = \zeta$

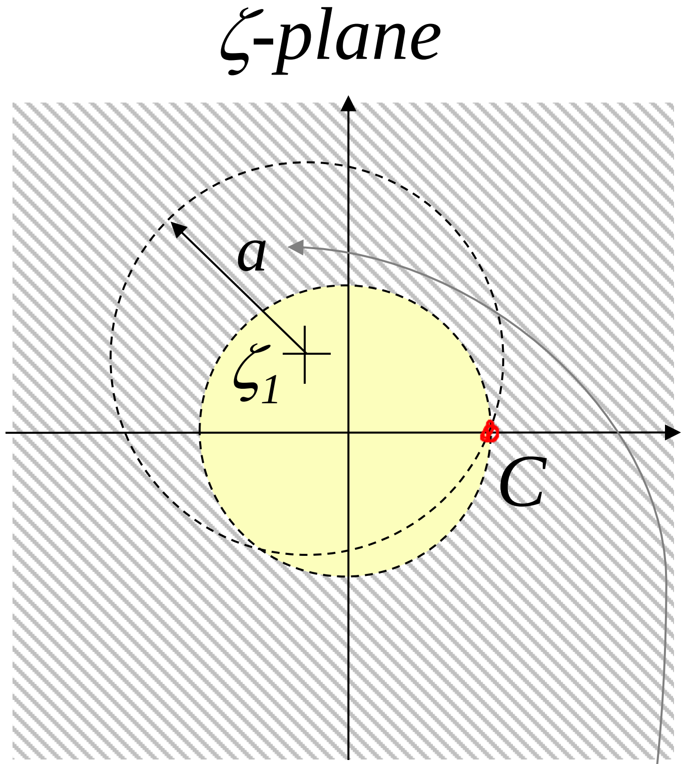
$$\frac{dz}{d\zeta} = 1 - \frac{c^2}{\zeta^2} = 0$$

CRITICAL POINTS : $\zeta = \pm c$

(or) $z = \pm 2c$

Consider a Series of Circles Cutting the Right-Hand Critical Point

1. $\zeta_1=0$
2. $Re\{\zeta_1\}=0, Im\{\zeta_1\}>0$
3. $Re\{\zeta_1\}<0, Im\{\zeta_1\}=0$
4. $Re\{\zeta_1\}<0, Im\{\zeta_1\}>0$

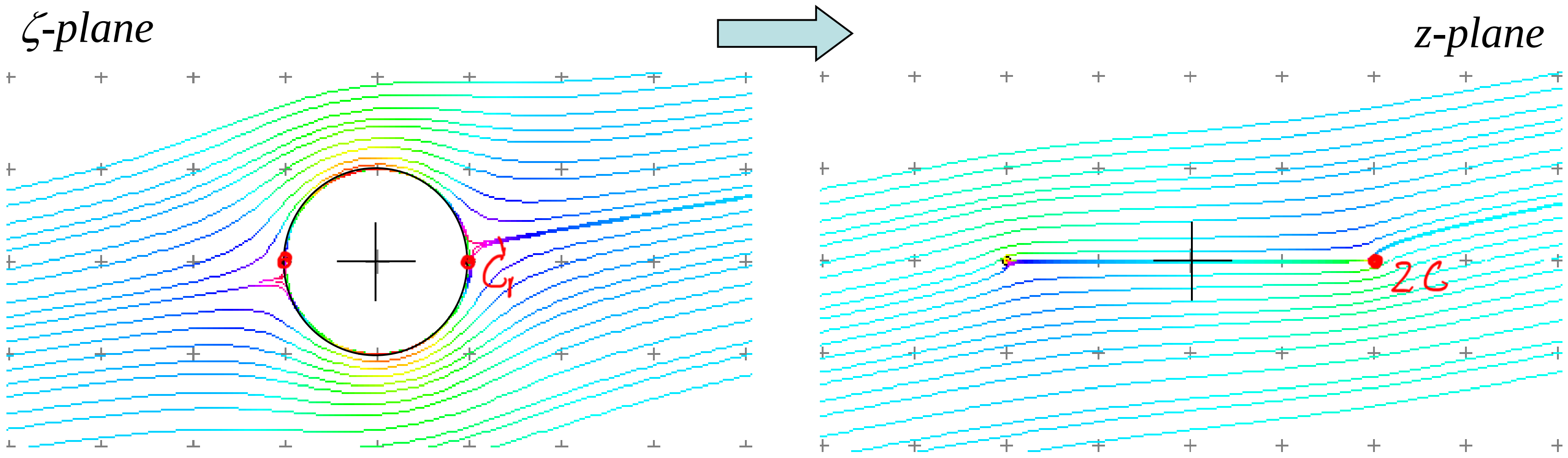


'*a*' adjusted so circle always cuts right-hand critical point

1. $\zeta_1 = 0$

Circle coincident with mapping circle

$\Rightarrow z = \zeta + C^2/\zeta$

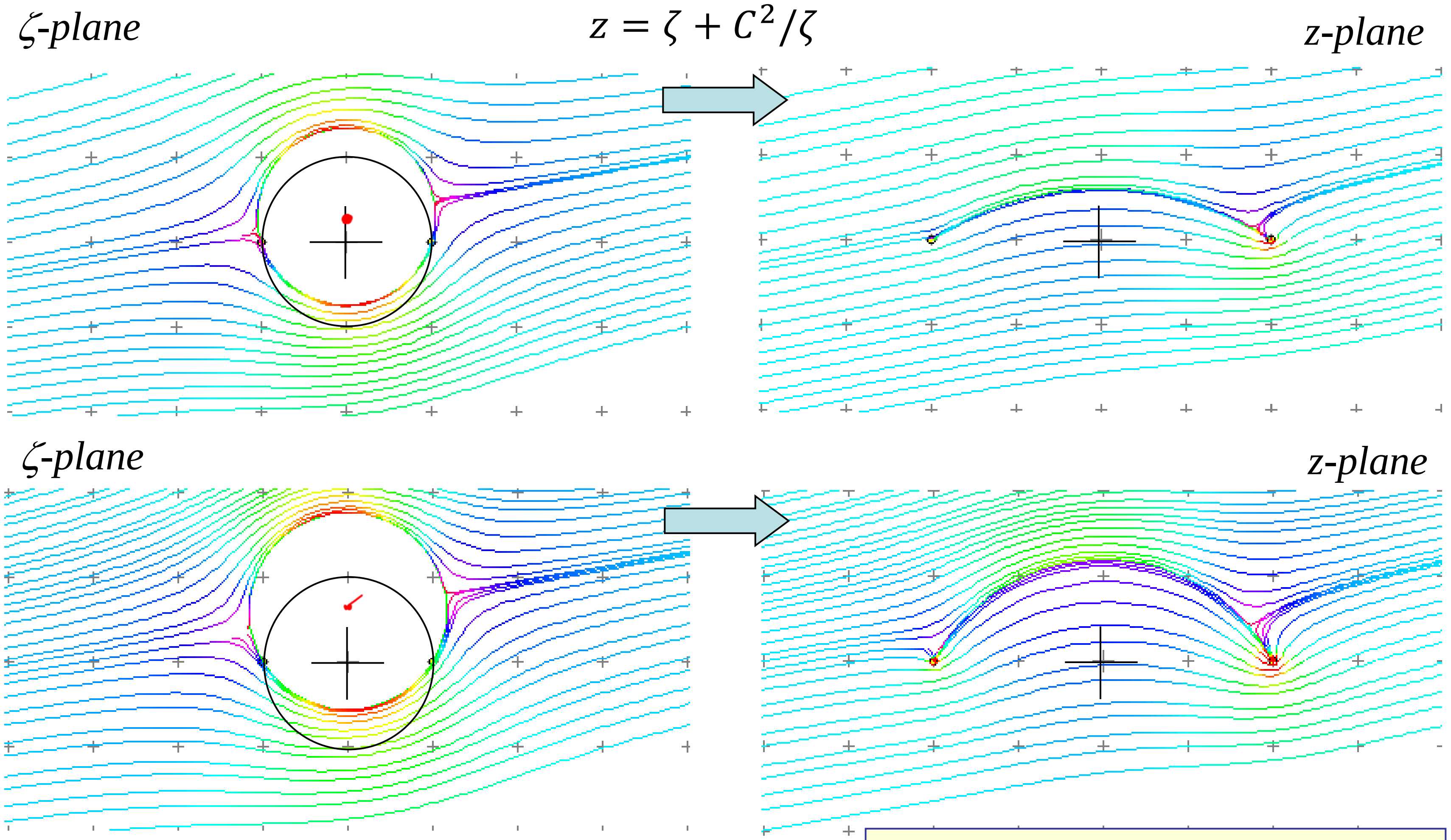


$$\tilde{F}(\zeta) = V_\infty e^{-i\alpha}\zeta + \frac{V_\infty a^2 e^{i\alpha}}{\zeta - \zeta_1}$$

The Flat Plate

2. $Re\{\zeta_1\}=0$, $Im\{\zeta_1\}>0$

Circle centered on imaginary axis

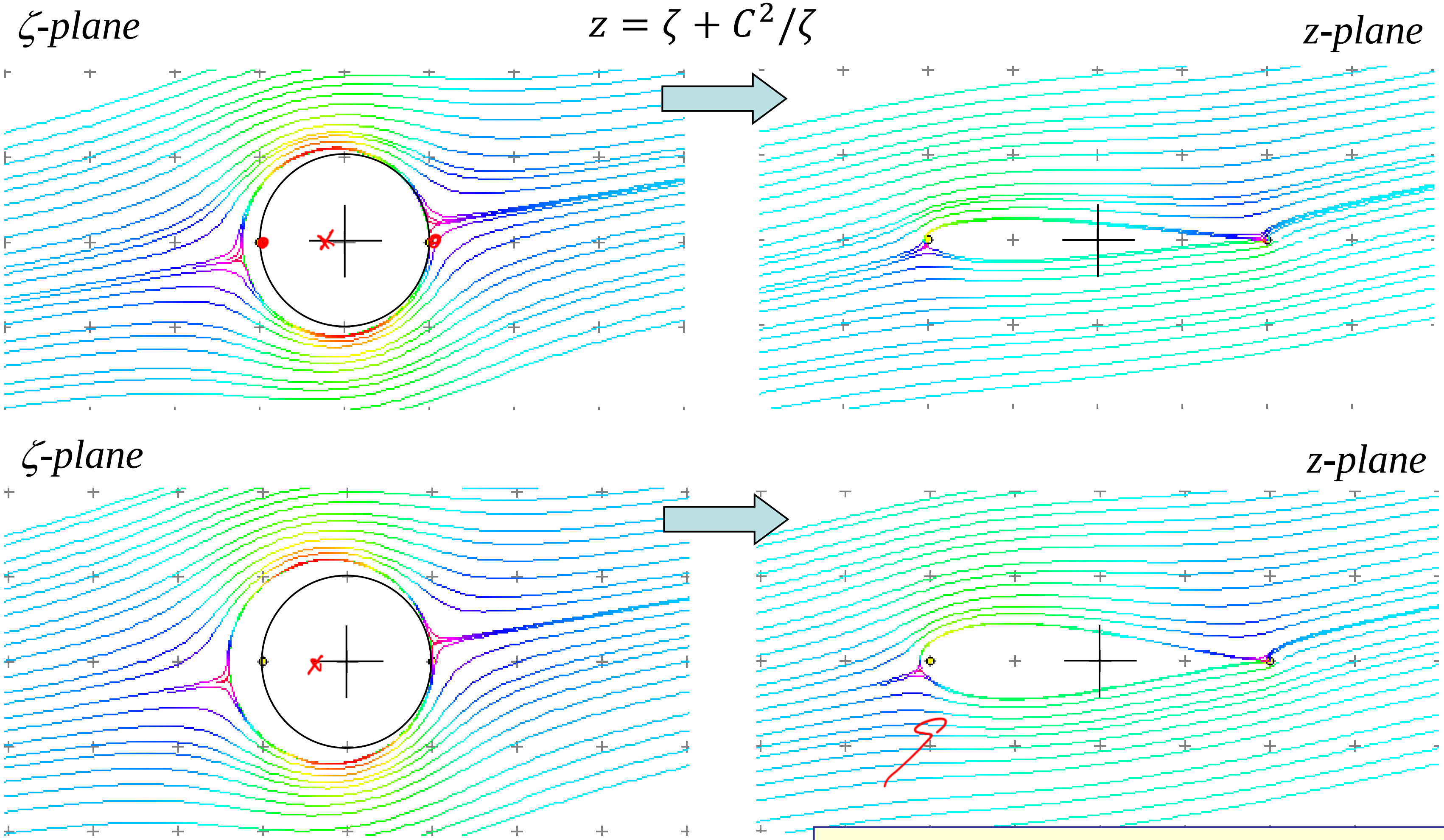


The Circular Arc

$Im\{\zeta_1\}$ controls camber

3. $Re\{\zeta_1\}<0,\ Im\{\zeta_1\}=0$

Circle centered on negative real axis

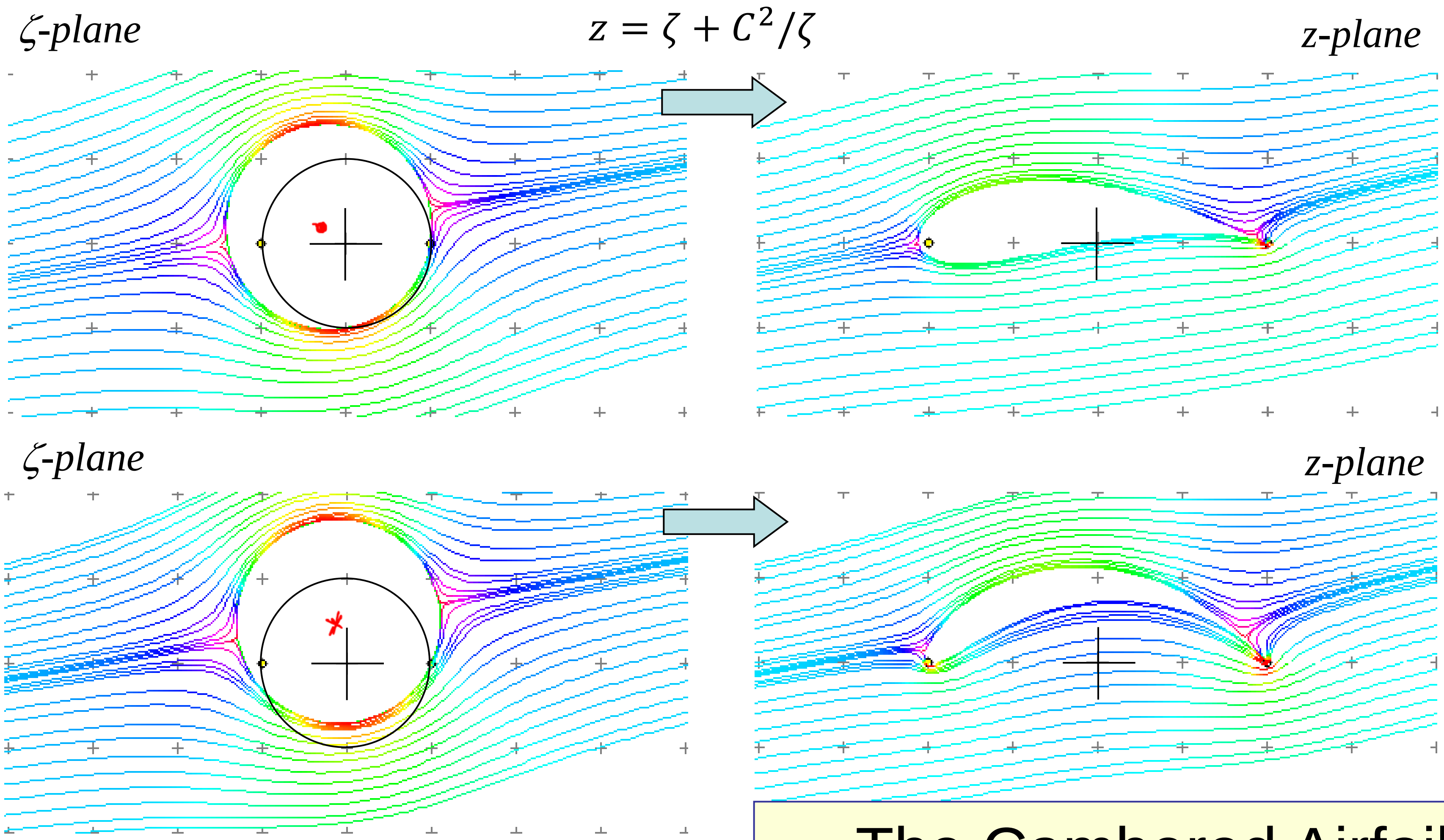


The Symmetric Airfoil

$Re\{\zeta_1\}$ controls thickness

4. $Re\{\zeta_1\}<0, Im\{\zeta_1\}>0$

Circle centered in
2nd quadrant

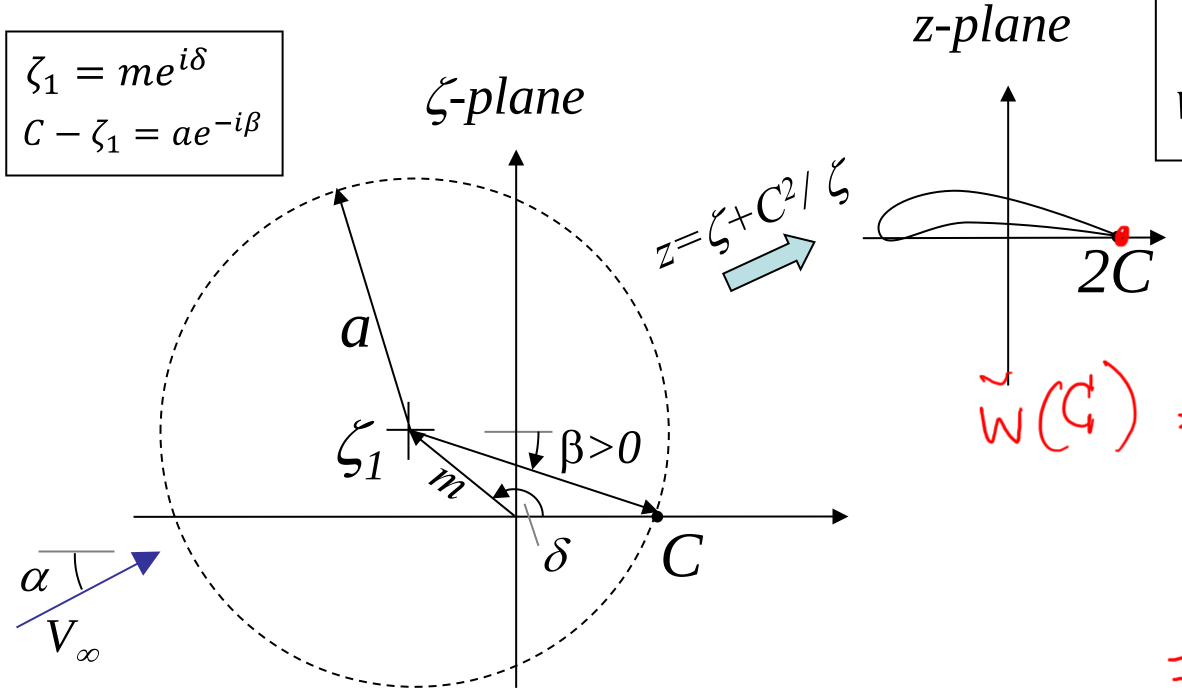


The Cambered Airfoil

$Re\{\zeta_1\}$ controls thickness.
$Im\{\zeta_1\}$ controls camber

Mapping an Airfoil Flow

$$\zeta_1 = m e^{i\delta}$$
$$C - \zeta_1 = a e^{-i\beta}$$



$$F(z) = \tilde{F}(\zeta(z))$$
$$W(z) = \frac{dF}{dz} = \tilde{W}\frac{d\zeta}{dz}$$

Γ?

$$\tilde{W}(C) = 0$$

$$\tilde{W}(C) = V_\infty e^{-i\alpha} - \frac{V_\infty a^2 e^{i\alpha}}{(C-\zeta_1)^2} - \frac{i\Gamma}{2\pi(C-\zeta_1)} = 0$$

$$= V_\infty e^{-i\alpha} - \frac{V_\infty \cancel{a^2} e^{i\alpha}}{\cancel{a^2} e^{-2i\beta}} - \frac{i\Gamma}{2\pi a e^{-i\beta}} = 0$$

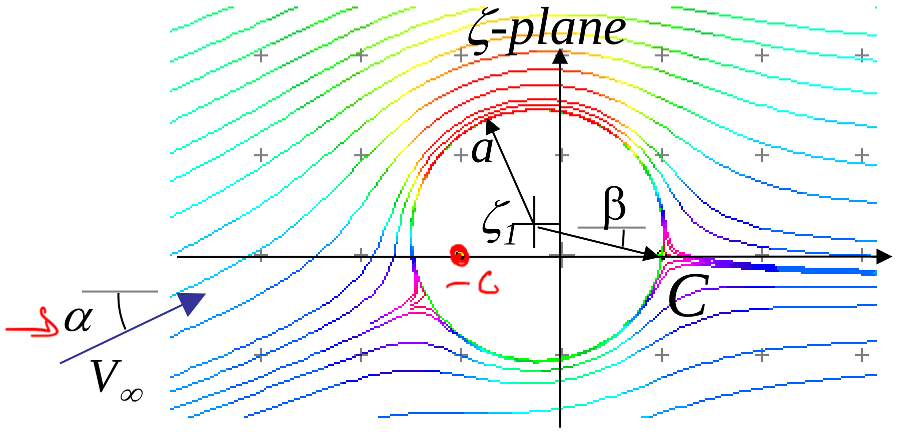
$\times e^{-i\beta}$

$$= V_\infty e^{-i(\alpha+\beta)} - V_\infty e^{i(\alpha+\beta)} - \frac{i\Gamma}{2\pi a} = 0$$

$$\Gamma = -4\pi a V_\infty \sin(\alpha+\beta)$$

$$\tilde{F}(\zeta) = V_\infty e^{-i\alpha}\zeta + \frac{V_\infty a^2 e^{i\alpha}}{\zeta - \zeta_1} - \frac{i\Gamma}{2\pi}\log_e(\zeta - \zeta_1)$$
$$\tilde{W}(\zeta) = V_\infty e^{-i\alpha} - \frac{V_\infty a^2 e^{i\alpha}}{(\zeta-\zeta_1)^2} - \frac{i\Gamma}{2\pi(\zeta-\zeta_1)}$$

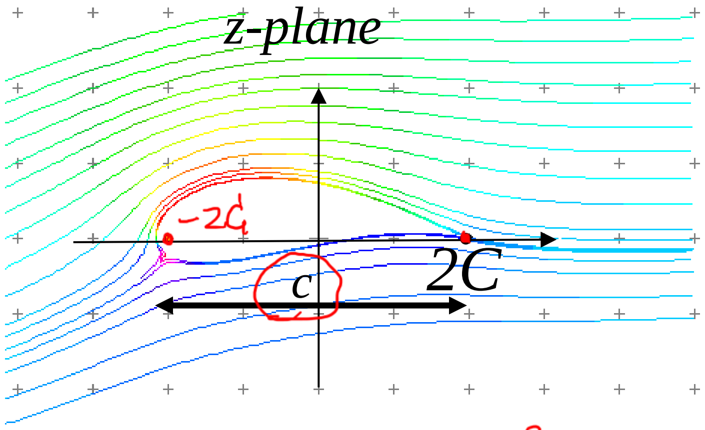
Results for Lift



$$\Gamma = 4\pi a V_\infty \sin(\alpha+\beta)$$

$z=\zeta+C^2/\zeta$

$\frac{dz}{d\zeta} = 1 - \frac{C^2}{\zeta^2}$



Lift $= -\rho V_\infty \Gamma = 4\pi \rho V_\infty^2 a \sin(\alpha+\beta)$

$C_l = \frac{\text{lift}}{\frac{1}{2}\rho V_\infty^2 c} = 8\pi \frac{a}{c} \sin(\alpha+\beta)$

$a \geq C$, $c \geq 4C$

$a/c \geq 1/4$

Note:- For thin airfoils at small 'α'

$a/c \approx 1/4$, $\sin(\alpha+\beta) \approx \alpha+\beta$

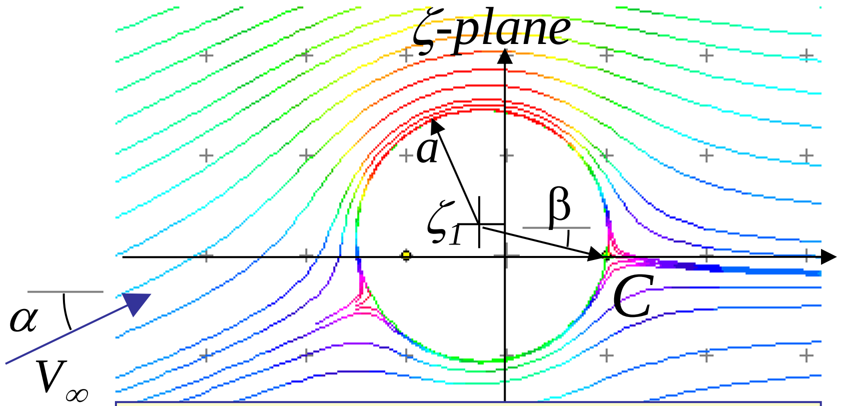
$C_l = 2\pi(\alpha+\beta)$

1. Lift varies as $\sin(\alpha)$

2. The primary effect of camber is in setting the angle of attack where the lift is zero

3. Lift curve slope is 2π for a thin airfoil but increases with thickness and camber

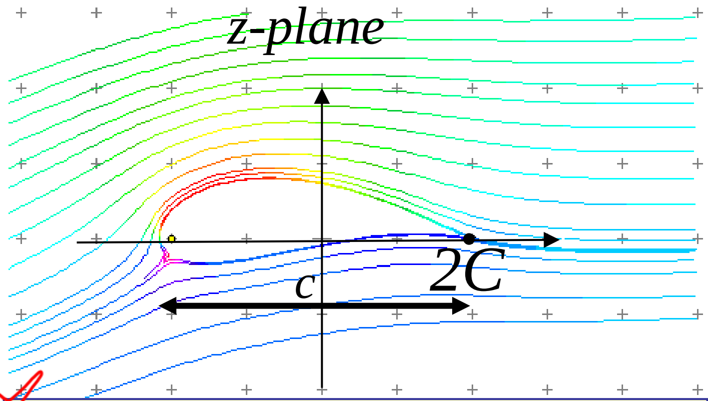
Obtaining the Pressure Distribution



$$\tilde{W}(\zeta) = V_\infty e^{-i\alpha} - \frac{V_\infty a^2 e^{i\alpha}}{(\zeta-\zeta_1)^2} - \frac{i\Gamma}{2\pi(\zeta-\zeta_1)}$$

$z = \zeta + C^2/\zeta$

$$\frac{dz}{d\zeta} = 1 - \frac{C^2}{\zeta^2}$$



$$W(z) = \underline{\tilde{W}(\zeta)}\frac{d\zeta}{dz}$$

1. Choose a set of points on the circle $\qquad \zeta|_{circle} = \zeta_1 + a e^{i\theta}$
2. For these points determine

   Velocity on the circle $\qquad \tilde{W}(\zeta|_{circle})$ with $\Gamma = -4\pi V_\infty \sin(\alpha+\beta)$

   Derivative of mapping $\qquad \left.\frac{d\zeta}{dz}\right|_{circle} = \frac{1}{1 - C^2/\zeta^2_{circle}}$

   Airfoil coordinates $\qquad z|_{airfoil} = \zeta|_{circle} + C^2/\zeta|_{circle}$

3. Evaluate C_p on the airfoil using Bernoulli $\quad \rightarrow 1 - \frac{|W(z)|^2}{V_\infty^2} = 1 - \frac{|\tilde{W}(\zeta)|^2 \left|\frac{d\zeta}{dz}\right|^2}{V_\infty^2}$
4. Plot C_p vs x, i.e $C_p|_{airfoil}$ vs $Re\{z_{airfoil}\}$ $\quad \Rightarrow -C_p$ vs x/c, 'x' measured from L.E.