Design by root locus:

What we will focus on:

A) Loci versus other parameters

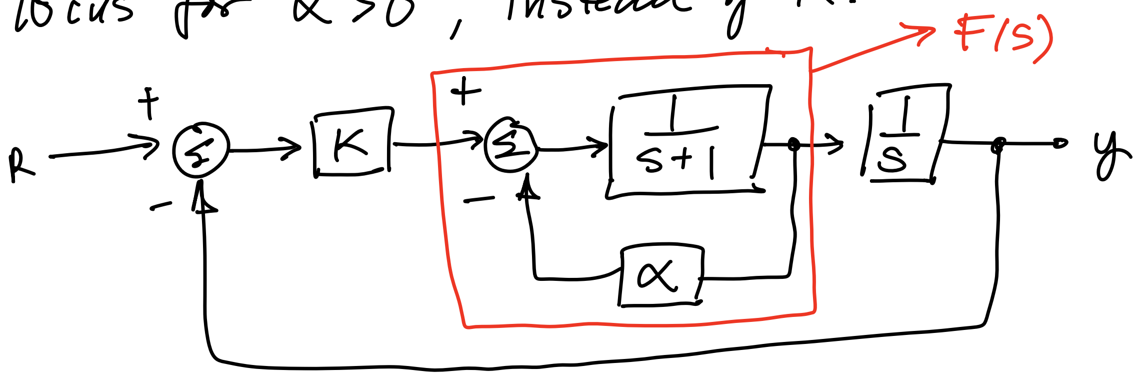
B) Improving Steady-state and transients

- Lead/Lag controllers

C) PID controllers

A) Root locus for other parameters

Consider this system - we want to draw root locus for $\alpha > 0$, instead of K.



Approach: we need to put the equation for the roots of closed-loop system into this form: $1 + \alpha \hat{G}(s) = 0$

Find the closed-loop T.F.

$$\frac{Y(s)}{R(s)} = \frac{K\,F(s)\left(\frac{1}{s}\right)}{1 + K\,F(s)\left(\frac{1}{s}\right)}$$

Poles are: $1 + K F(s)\left(\frac{1}{s}\right) = 0 = 1 + \alpha \hat{G}(s)$

Expand:

$$\Rightarrow\ 1 + K\left[\frac{\frac{1}{s+1}}{1+\alpha\left(\frac{1}{s+1}\right)}\right]\left[\frac{1}{s}\right] = 1 + \frac{K}{s(s+1+\alpha)}$$

$$\Rightarrow\ s(s+1+\alpha) + K = s^2 + s + \alpha s + K$$

Rearrange into $1 + \alpha \hat{G}(s) = 0$

$$\Rightarrow\ s^2 + s + \alpha s + K = s^2 + s + K + \alpha s$$

Divide by $s^2 + s + K$ gives:

$$\frac{s^2 + s + K}{s^2 + s + K} + \frac{\alpha s}{s^2 + s + K} = 0$$

$$\Rightarrow\ 1 + \alpha \boxed{\left[\frac{s}{s^2 + s + K}\right]} = 0$$

$\hat{G}(s)$

Therefore, to draw root locus for α,

we treat $\hat{G}(s) = \frac{s}{s^2+s+K}$ as the

"open-loop" system.

$$\hat{G}(s) = \frac{s}{s^2+s+K} \Rightarrow$$

$s=0$ open-loop zero

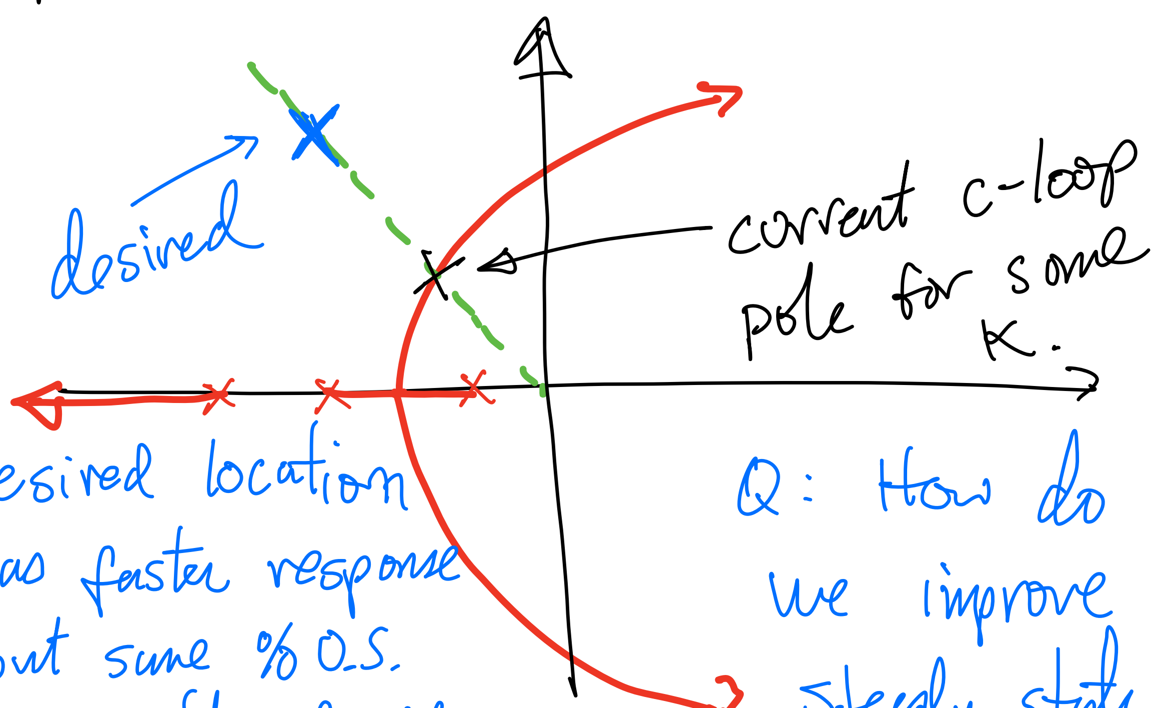
$s^2+s+K=0$ open-loop poles

we need to select nominal K value.

From here, we apply the steps described to draw root locus, but now it's for $\alpha > 0$!

B) Improving steady-state + transient response.

Suppose our closed-loop system has a root locus for some parameter K that looks like:



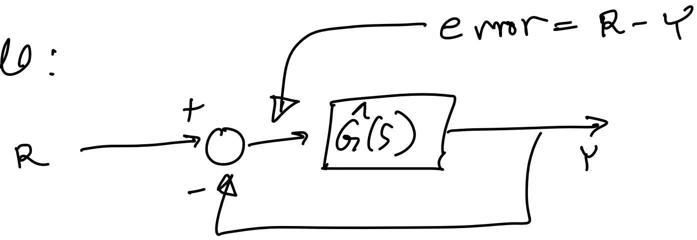
Desired location has faster response but same % O.S. we can't get there with current controller.

Q: How do we improve steady-state + transient response?

Lag Compensators

Lag controller is used to reduce steady-state error.

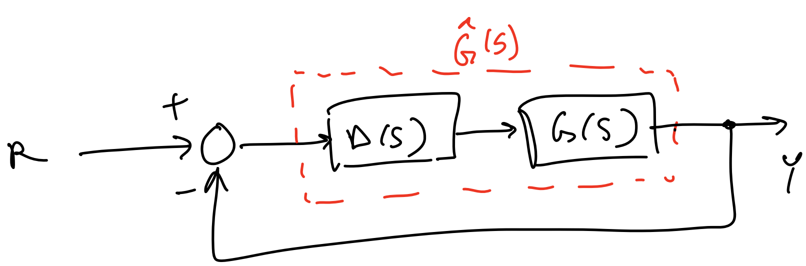
Recall:



s.s. error:

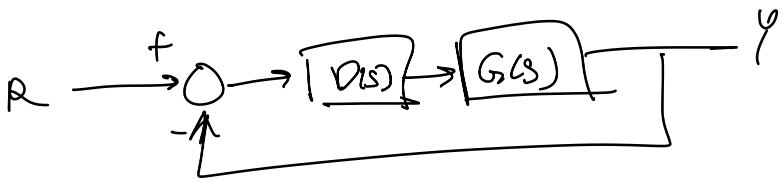
- step: $e_{ss} = \dfrac{1}{1+K_p}$ ← position constant $\qquad K_p = \lim\limits_{s \to 0} \hat{G}(s)$
- ramp: $e_{ss} = \dfrac{1}{K_v}$ $\qquad K_v = \lim\limits_{s \to 0} s\hat{G}(s)$
- parabolic: $e_{ss} = \dfrac{1}{K_a}$ $\qquad K_a = \lim\limits_{s \to 0} s^2 \hat{G}(s)$

To achieve small s.s error, we like the constants K_p, K_v, K_a to be as large as possible.



$$D(s) = \frac{s+z}{s+p} \qquad z > p \quad \Rightarrow \quad \frac{z}{p} \gg 1$$

pick $\frac{z}{p} \approx 3$ to 10 ⟵ Design guidelines



$G(s) = \frac{1}{s(s+1)}$ and $D(s) = \frac{s+0.1}{s+0.01}$

s.s. error

w/o lag controller: Assume we have a ramp input.

$$e_{ss} = \frac{1}{K_v} \qquad K_v = \lim_{s\to 0} sG(s) = \lim_{s\to 0} \frac{1}{s+1} = 1$$

$$\Rightarrow e_{ss} = \frac{1}{1} = \boxed{1}$$

w/ Lag:

$$e_{ss} = \frac{1}{K_v} \quad K_v = \lim_{s\to 0} s\hat{G}(s) = \lim_{s\to 0} s\left(\frac{s+.1}{s+.01}\right)\left(\frac{1}{s(s+1)}\right) = 10$$

$$e_{ss} = \frac{1}{10} = 0.1$$

Lead Controller Design Guidelines:

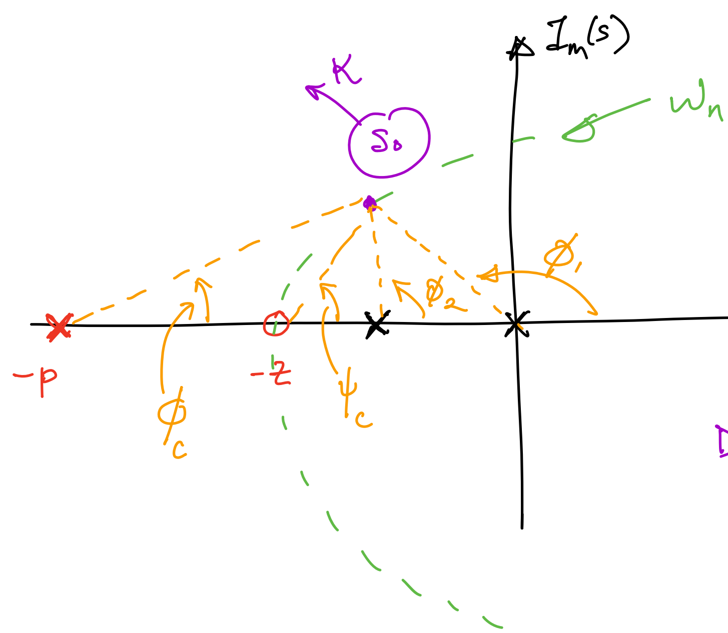
$D(s) = K \frac{s+z}{s+p}$

lead: $z < p$

* Place zero in the neighborhood of the desired closed-loop $\underline{\underline{\omega_n}}$, as determined by rise-time or settling time requirements.

* Place the pole at least 3 to 20 times the value of the zero.

* Lead controller must meet the angle condition:



s_0 = desired closed-loop pole location

$D(s) = \frac{K(s+z)}{s+p}$

s_0

$\angle D(s)G(s) = 180^\circ + 360^\circ l \qquad l = 0, \pm 1, \pm 2 \ldots$

$\sum \psi_i - \sum \phi_i = 180 + 360^\circ l$

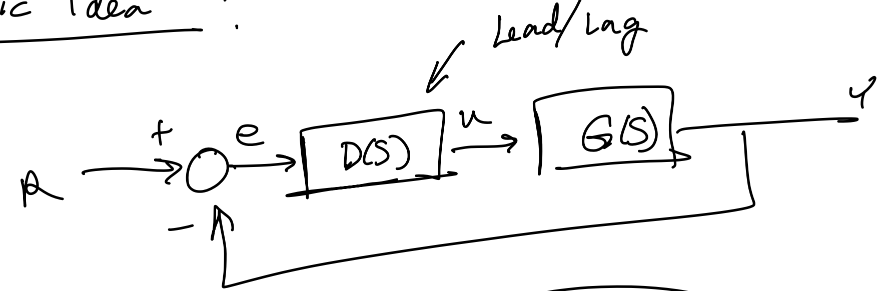
$$\psi_c - [\phi_1 + \phi_2 + \phi_c] = 180° + 360° l$$

$$\underbrace{\psi_c - \phi_c}_{\text{lead controller}} - (\phi_1 + \phi_2) = 180° + 360° l$$

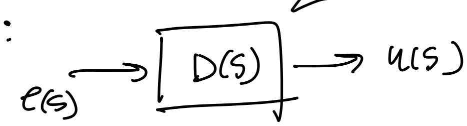
* Can chose ψ_c based on ω_n and t_s conditions, then find ϕ_c for location of pole of lead controller.

Implementing Lead / Lag Controllers (Analog circuit)

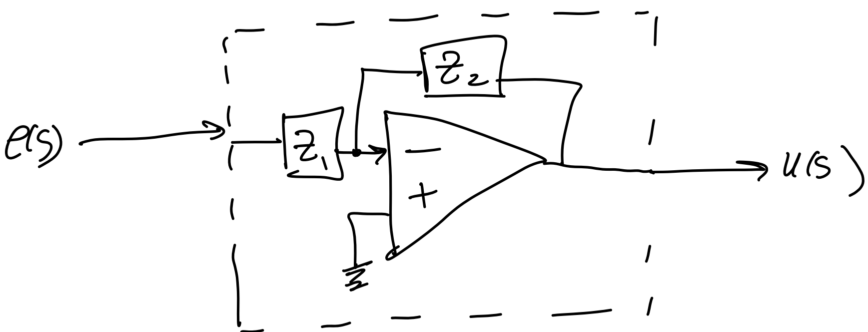
Basic idea:



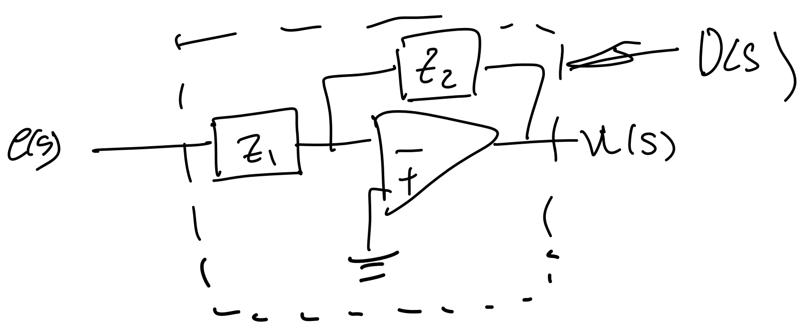
isolate:

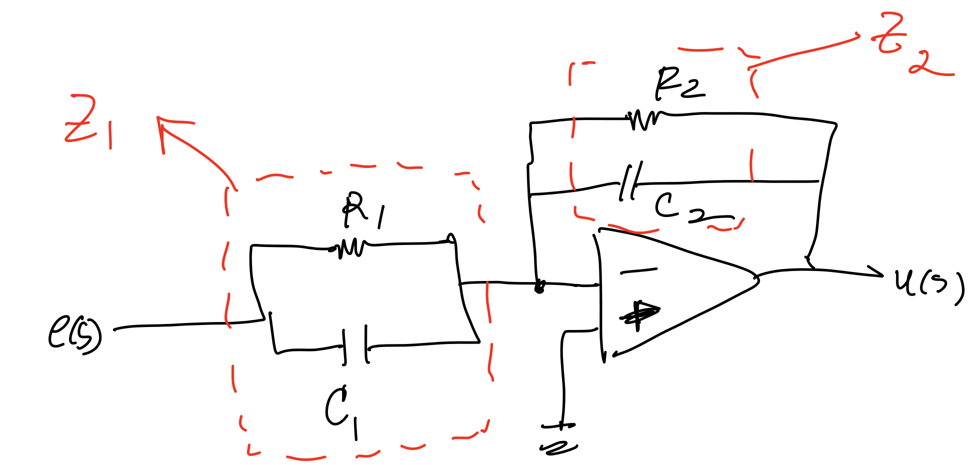


$$D(s) = \frac{s+z}{s+p}$$



$$\frac{u(s)}{e(s)} = -\frac{Z_2}{Z_1} = \frac{s+z}{s+p} = D(s)$$





$$\frac{u(s)}{e(s)} = -\frac{Z_2}{Z_1}$$

$$Z_1 = \left[\frac{1}{Z_{R_1}} + \frac{1}{Z_{C_1}}\right]^{-1} = \left[\frac{1}{R_1} + \frac{1}{1/sC_1}\right]^{-1}$$

$$Z_2 = \left[\frac{1}{Z_{R_2}} + \frac{1}{Z_{C_2}}\right]^{-1} = \left[\frac{1}{R_2} + \frac{1}{1/sC_2}\right]^{-1}$$

$$\Rightarrow \quad Z_1 = \left[\frac{1}{R_1} + sC_1\right]^{-1} = \frac{R_1}{C_1R_1s+1}$$

$$Z_2 = \left[\frac{1}{R_2} + sC_2\right]^{-1} = \frac{R_2}{C_2R_2s+1}$$

$$\frac{u(s)}{e(s)} = -\frac{Z_2}{Z_1} = -\frac{R_2(C_1R_1s+1)}{(C_2R_2s+1)(R_1)}$$

$$\frac{U(s)}{E(s)} = -\frac{\cancel{R_2}}{\cancel{R_1}}\left[\frac{C_1\cancel{R_1}\left(s+\frac{1}{C_1R_1}\right)}{C_2\cancel{R_2}\left(s+\frac{1}{C_2R_2}\right)}\right]$$

$$\boxed{\begin{gathered}\frac{U(s)}{E(s)} = -\frac{C_1}{C_2}\left[\frac{s+\frac{1}{C_1R_1}}{s+\frac{1}{C_2R_2}}\right] \\[2em] \frac{U(s)}{E(s)} = D(s) = K\frac{s+z}{s+p} \\[1em] z = \frac{1}{C_1R_1} \qquad p = \frac{1}{C_2R_2} \\ K = -\frac{C_1}{C_2}\end{gathered}}$$