Solutions Manual for

Advanced Mechanics of Materials and Applied Elasticity

Fifth Edition

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ISBN-10: 0-13-269049-7
ISBN-13: 978-0-13-269049-2

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NOTES TO THE INSTRUCTOR

The *Solutions Manual for Advanced Mechanics of Materials and Applied Elasticity, Fifth Edition* supplements the study of stress and deformation analyses developed in the book. The main objective of the manual is to provide efficient solutions for problems dealing with variously loaded members. This manual can also serve to guide the instructor in the assignments of problems, in grading these problems, and in preparing lecture materials as well as examination questions. Every effort has been made to have a solutions manual that can cut through the clutter and is as self-explanatory as possible, thus reducing the work on the instructor. It is written and class-tested by the author, Ansel Ugural.

As indicated in the book's *Preface*, the text is designed for the senior and/or first year graduate level courses in stress analysis. In order to accommodate courses of varying emphasis, considerably more material has been presented in the book than can be covered effectively in a single three-credit course. The instructor has the choice of assigning a variety of problems in each chapter. Answers to selected problems are given at the end of the text. A description of the topics covered is given in the introduction of each chapter throughout the text. It is hoped that the foregoing materials will help instructor in organizing his or her course to best fit the needs of his or her students.

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CHAPTER 1

SOLUTION (1.1)

We have

$$A = 50 \times 75 = 3.75(10^{-3})\ m^2,\ \theta = 90^o - 40^o = 50^o, \text{ and } \sigma_x = P/A.$$

Equations (1.8), with $\theta = 50^o$:

$$\sigma_{x'} = 700(10^3) = \sigma_x \cos^2 50^o = 0.413\sigma_x = 110.18P$$

or

$$P = 6.35\ kN$$

$$|\tau_{x'y'}| = 560(10^3)\sigma_x \sin 50^o \cos 50^o = 0.492\sigma_x = 131.2P$$

Solving

$$P = 4.27\ kN = P_{all}$$ ◀

SOLUTION (1.2)

Normal stress is

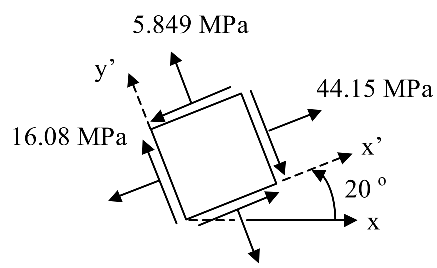
$$\sigma_x = \tfrac{P}{A} = \tfrac{125(10^3)}{0.05\times 0.05} = 50\ MPa$$

(a) Equations (1.11), with $\theta = 90^o - 70^o = 20^o$:

$$\sigma_{x'} = 50\cos^2 20^o = 44.15\ MPa$$

$$\tau_{x'y'} = -50\sin 20^o \cos 20^o = -16.08\ MPa$$ ◀

$$\sigma_{y'} = 50\cos^2(20^o + 90^o) = 5.849\ MPa$$

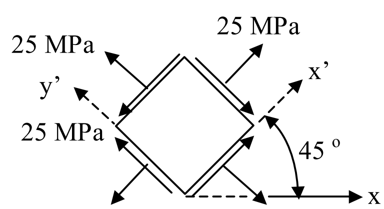


(b) Equations (1.11), with $\theta = 45^o$:

$$\sigma_{x'} = 50\cos^2 45^o = 25\ MPa$$

$$\tau_{x'y'} = -50\sin 45^o \cos 45^o = -25\ MPa$$ ◀

$$\sigma_{y'} = 50\cos^2(45^o + 90^o) = 25\ MPa$$



SOLUTION (1.3)

From Eq. (1.11a),

$$\sigma_x = \frac{\sigma_{x'}}{\cos^2\theta} = \frac{-75}{\cos^2 30^o} = -100\ MPa$$

For $\theta = 50^o$, Eqs. (1.11) give then

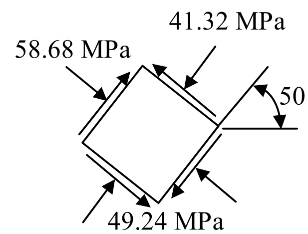
$$\sigma_{x'} = -100\cos^2 50^o = -41.32\ MPa$$

$$\tau_{x'y'} = -(-100)\sin 50^o \cos 50^o$$
$$= 49.24\ MPa$$

Similarly, for $\theta = 140^o$:

$$\sigma_{x'} = -100\cos^2 140^o = -58.68\ MPa$$

$$\tau_{x'y'} = -49.24\ MPa$$



SOLUTION (1.4)

Refer to Fig. 1.6c. Equations (1.11) by substituting the double angle-trigonometric relations, or Eqs. (1.18) with $\sigma_y = 0$ and $\tau_{xy} = 0$, become

$$\sigma_{x'} = \tfrac{1}{2}\sigma_x + \tfrac{1}{2}\sigma_x \cos 2\theta \qquad \text{and} \qquad |\tau_{x'y'}| = \tfrac{1}{2}\sigma_x \sin 2\theta$$

or

$$20 = \tfrac{P}{2A}(1+\cos 2\theta) \qquad \text{and} \qquad 10 = \tfrac{P}{2A}\sin 2\theta$$

The foregoing lead to

$$2\sin 2\theta - \cos 2\theta = 1 \tag{a}$$

By introducing trigonometric identities, Eq. (a) becomes

$4\sin\theta\cos\theta - 2\cos^2\theta = 0$ or $\tan\theta = 1/2$. Hence

$$\theta = 26.56^o$$

Thus,

$$20 = \tfrac{P}{2(1300)} = (1+0.6)$$

gives

$$P = 32.5\ kN$$

It can be shown that use of Mohr's circle yields readily the same result.

SOLUTION (1.5)

Equations (1.12):

$$\sigma_1 = \frac{P}{A} = \frac{-150(10^3)}{\frac{\pi}{4}(50)^2} = -76.4\ MPa$$

$$\tau_{max} = \frac{P}{2A} = 38.2\ MPa$$

SOLUTION (1.6)

Shaded transverse area:

$$A = 2at = 2(10)(75) = 1.5(10^3)\ mm^2$$

Metal is capable of supporting the load

$$P = \sigma A = 90(10^6)(1.5\times10^{-3}) = 135\ kN$$

Apply Eqs. (1.11):

$$\sigma_{x'} = 25(10^6) = \frac{P}{1.5(10^{-3})}(\cos^2 55^o), \quad P = 114\ kN$$

$$\tau_{x'y'} = 12(10^6) = -\frac{P}{1.5(10^{-3})}\sin 55^o \cos 55^o, \quad P = 38.3\ kN$$

Thus,

$$P_{all} = 38.3\ kN$$ ◄

SOLUTION (1.7)

Use Eqs. (1.11):

$$\sigma_{x'} = 20(10^6) = \frac{P}{1.5(10^3)}(\cos^2 40^o), \quad P = 51.1\ kN$$

$$\tau_{x'y'} = 8(10^6) = -\frac{P}{1.5(10^3)}\sin 40^o \cos 40^o, \quad P = 24.4\ kN$$

Thus,

$$P_{all} = 24.4\ kN$$ ◄

SOLUTION (1.8)

$$A = 15\times30 = 450\ mm^2$$

Apply Eqs. (1.11):

$$\sigma_{x'} = \frac{120(10^3)}{450\times10^{-6}}(\cos^2 40^o) = 156\ MPa$$

$$\tau_{x'y'} = -\frac{120(10^3)}{450\times10^{-6}}\sin 40^o \cos 40^o = -131\ MPa$$ ◄

SOLUTION (1.9)

We have $A = 450(10^{-6})\ m^2$. Use Eqs. (1.11):

$$\sigma_{x'} = \frac{-100(10^3)}{450\times10^{-6}}(\cos^2 60^o) = -55.6\ MPa$$

$$\tau_{x'y'} = -\frac{-100(10^3)}{450\times10^{-6}}\sin 60^o \cos 60^o = 96.2\ MPa$$ ◄

SOLUTION (1.10)

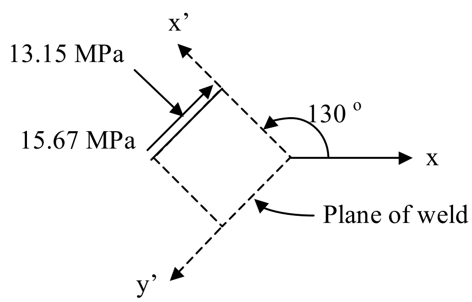
$$\theta = 40^o + 90^o = 130^o$$

$$\sigma_x = \frac{P}{A} = -\frac{150(10^3)}{\pi(0.08^2 - 0.07^2)} = -31.83\ MPa$$

Equations (1.11):

$$\sigma_{x'} = -31.83\cos^2 130^o = -13.15\ MPa \quad ◄$$

$$\tau_{x'y'} = 31.83\sin 130^o \cos 130^o = -15.67\ MPa \quad ◄$$



SOLUTION (1.11)

Use Eqs. (1.14),

$$(2x) + (-2xy) + (x) + F_x = 0$$

$$(-y^2) + (-2yz + x) + (0) + F_y = 0$$

$$(z - 4xy) + (0) + (-2z) + F_z = 0$$

Solving, we have (in MN/m^3):

$$F_x = -3x + 2xy \qquad F_y = -x + y^2 + 2xz \qquad F_z = 4xy + z \qquad \text{(a)} \quad ◄$$

Substituting x=-0.01 m, y=0.03 m, and z=0.06 m, Eqs. (a) yield the following values

$$F_x = 29.4\ kN/m^3 \qquad F_y = 14.5\ kN/m^3 \qquad F_z = 58.8\ kN/m^3$$

Resultant body force is thus

$$F = \sqrt{F_x^2 + F_y^2 + F_z^2} = 67.32\ \ kN/m^3 \quad ◄$$

SOLUTION (1.12)

Equations (1.14):

$$-2c_1 y - 2c_1 y + 0 + 0 = 0, \qquad 4c_1 y \neq 0$$

$$0 + c_3 z + 0 + 0 = 0, \qquad c_3 z \neq 0$$

$$0 + 0 + 0 + 0 = 0$$

No. Eqs. (1.14) are not satisfied. ◄

SOLUTION (1.13)

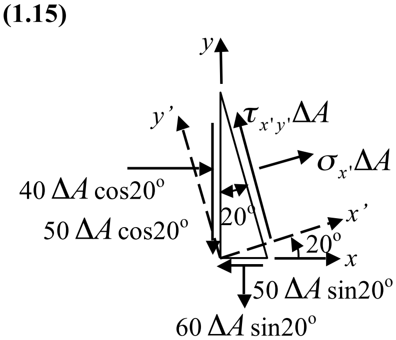
(a) No. Eqs. (1.14) are not satisfied. ◄
(b) Yes. Eqs. (1.14) are satisfied. ◄

SOLUTION (1.14)

Eqs. (1.14) for the given stress field yield: ◄

$$F_x = F_y = F_z = 0$$

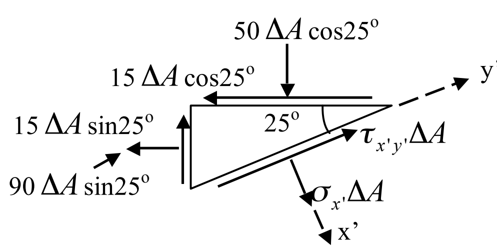
SOLUTION (1.15)



$$\sum F_{x'} = 0: \quad \sigma_{x'}\Delta A + 40\cos^2 20^o - 60\Delta A\sin^2 20^o$$
$$-2(50\Delta A\sin 20^o \cos 20^o) = 0$$
$$\sigma_{x'} = -35.32 + 7.02 + 32.14 = 3.84 \ MPa$$ ◄

$$\sum F_{y'} = 0: \quad \tau_{x'y'}\Delta A - 40\Delta A\sin 20^o \cos 20^o$$
$$-60\Delta A\sin 20^o \cos 20^o - 50\Delta A\cos^2 20^o$$
$$+50\Delta A\sin^2 20^o = 0$$
$$\tau_{x'y'} = 12.86 + 19.28 + 44.15 - 5.85 = 70.4 \ MPa$$ ◄

SOLUTION (1.16)



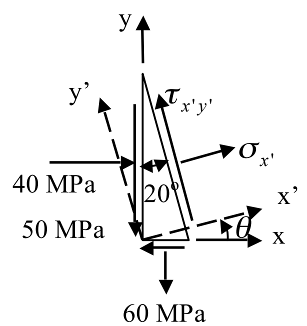
$$\sum F_{x'} = 0: \quad \sigma_{x'}\Delta A + 50\Delta A\cos^2 25^o$$
$$-90\Delta A\sin^2 25^o - 2(15\Delta A\sin 25^o \cos 25^o) = 0$$
$$\sigma_{x'} = -41.7 + 16.07 + 11.49 = -12.9 \ MPa$$ ◄

(CONT.)

1.16 (CONT.)

$$\sum F_{y'} = 0: \quad \tau_{x'y'}\Delta A - 50\Delta A \sin 25^o \cos 25^o$$
$$-90\Delta A \sin 25^o \cos 25^o - 15\Delta A \cos^2 25^o + 15\Delta A \sin^2 25^o = 0$$
$$\tau_{x'y'} = 19.15 + 34.47 + 12.32 - 2.68 = 63.3 \ MPa \quad ◀$$

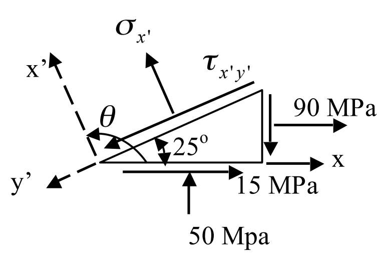
SOLUTION (1.17)



θ =20°

$$\sigma_{x'} = \frac{1}{2}(-40+60) + \frac{1}{2}(-40-60)\cos 40^o + 50\sin 40^o$$
$$= 10 - 38.3 + 32.1 = -3.8 \ MPa \quad ◀$$
$$\tau_{x'y'} = -\frac{1}{2}(-40-60)\sin 40^o + 50\cos 40^o$$
$$= 32.14 + 38.3 = 70.4 \ MPa \quad ◀$$

SOLUTION (1.18)



θ =115°

$$\sigma_{x'} = \frac{1}{2}(90-50) + \frac{1}{2}(90+50)\cos 230^o - 15\sin 230^o$$
$$= 20 - 45 + 11.5 = -13.5 \ MPa \quad ◀$$
$$\tau_{x'y'} = -\frac{1}{2}(90+50)\sin 230^o - 15\cos 230^o$$
$$= 53.62 + 9.64 = 63.3 \ MPa \quad ◀$$

SOLUTION (1.19)

Transform from $\theta = 40^o$ to $\theta = 0$. For convenience in computations, Let

$$\sigma_x = -160\ MPa, \qquad \sigma_y = -80\ MPa, \qquad \tau_{xy} = 40\ MPa \text{ and } \theta = -40^o$$

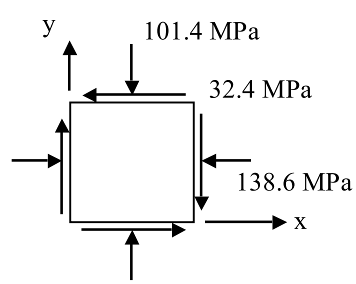
Then

$$\sigma_{x'} = \frac{1}{2}(\sigma_x + \sigma_y) + \frac{1}{2}(\sigma_x - \sigma_y)\cos 2\theta + \tau_{xy}\sin 2\theta$$
$$= \frac{1}{2}(-160-80) + \frac{1}{2}(-160+80)\cos(-80^o) + 40\sin(-80^o)$$
$$= -138.6\ MPa \quad ◄$$

$$\tau_{x'y'} = -\frac{1}{2}(\sigma_x - \sigma_y)\sin 2\theta + \tau_{xy}\cos 2\theta$$
$$= -\frac{1}{2}(-160+80)\sin(-80^o) + 40\cos(-80^o)$$
$$= -32.4\ MPa \quad ◄$$

So $\quad \sigma_{y'} = \sigma_x + \sigma_y - \sigma_{x'} = -160 - 80 + 138.6 = -101.4\ MPa$ ◄

For $\theta = 0^o$:

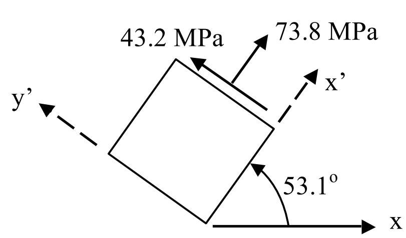


SOLUTION (1.20)

$$\theta = \tan^{-1}\frac{4}{3} = 53.1^o$$

$$\sigma_{x'} = \frac{45+90}{2} + \frac{45-90}{2}\cos 106.2^o$$
$$= 67.5 + 6.28 = 73.8\ MPa \quad ◄$$

$$\tau_{x'y'} = -\frac{45-90}{2}\sin 106.2^o = 43.2\ MPa$$



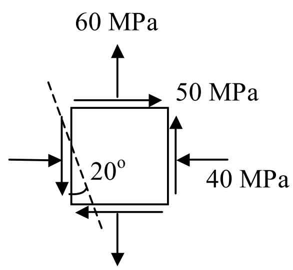
SOLUTION (1.21)

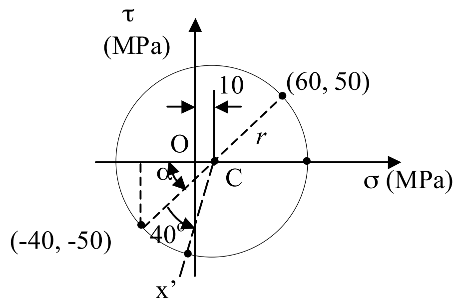
$$\tau_{xy} = 0 \qquad \theta = 70^o$$

(**a**) $\tau_{x'y'} = -30 = -\dfrac{\sigma - 60}{2}\sin 140^o \qquad \sigma = 153.3\ MPa$ ◀

(**b**) $\sigma_{x'} = 80 = \dfrac{\sigma + 60}{2} + \dfrac{\sigma - 60}{2}\cos 140^o \qquad \sigma = 231\ MPa$ ◀

SOLUTION (1.22)





$$\alpha = \tan^{-1}\frac{50}{60} = 39.8^o$$

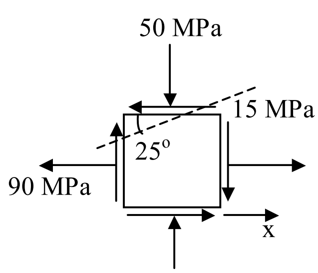
$$r = (60^2 + 50^2)^{\frac{1}{2}} = 78.1$$

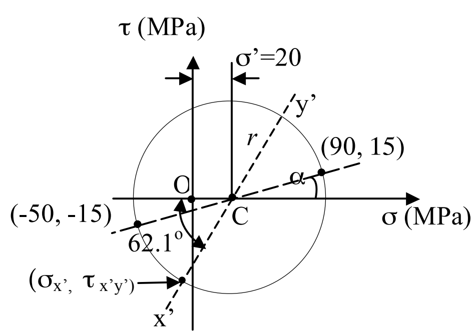
$$\tau_{x'y'} = \sin 79.8^o(78.1) = 76.9\ MPa$$ ◀

$$\sigma_{x'} = \cos 79.8^o(78.1) = -13.83\ MPa$$

Sketch of results is as shown in solution of Prob. 1.15.

SOLUTION (1.23)





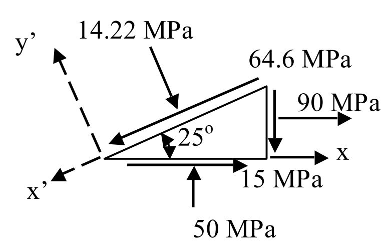
$$\alpha = \tan^{-1}\frac{15}{70} = 12.1^o$$

$$r = (15^2 + 70^2)^{\frac{1}{2}} = 73.14$$

$$\tau_{x'y'} = 73.14\sin 62.1^o = 64.6\ MPa$$

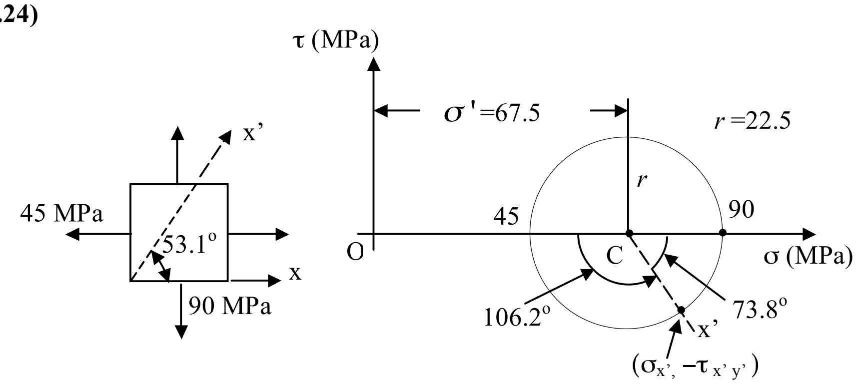
$$\sigma_{x'} = -73.14\cos 62.1^o + 20$$

$$= -14.22\ MPa$$



◀

SOLUTION (1.24)



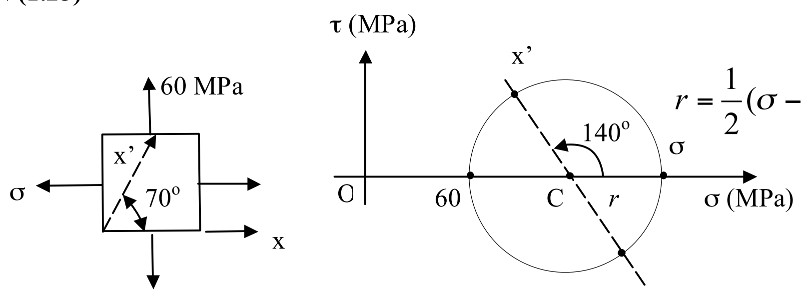
$$\tau_{x'y'} = 22.5\sin 73.8^o = 21.6\ MPa$$ ◄

$$\sigma_{x'} = 67.5 + 22.5\cos 73.8^o = 73.8\ MPa$$

Sketch of results is as shown in solution of Prob. 1.20.

SOLUTION (1.25)

(a)



$$r = \frac{1}{2}(\sigma - 60)$$

$$\tau_{x'y'} = -30 = \frac{\sigma - 60}{2}\sin 40^o; \qquad \sigma = 153.3\ MPa$$ ◄

(b) $\sigma_{x'} = 80 = 60 + \dfrac{\sigma - 60}{2}(1 - \cos 40^o)$

$\sigma = 231\ MPa$ ◄

SOLUTION (1.26)

(a) From Mohr's circle, Fig. (a):

$$\sigma_1 = 121\ MPa \qquad \sigma_2 = -71\ MPa \qquad \tau_{max} = 96\ MPa$$

$$\theta_p' = -19.3^o \qquad \theta_s' = 25.7^o$$ ◄

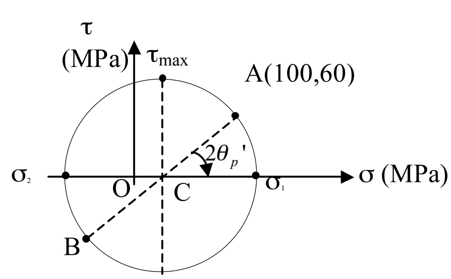


Figure (a)

(CONT.)

1.26 (CONT.)

By applying Eq. (1.20):

$$\sigma_{1,2} = \tfrac{50}{2} \pm \left[\tfrac{22,500}{4} + 36000\right]^{1/2} = 25 \pm 96$$

or

$$\sigma_1 = 121\ MPa \qquad \sigma_2 = -71\ MPa$$

Using Eq. (1.19):

$$\tan 2\theta_p = -\tfrac{12}{15} = -0.8$$

$$\theta_p' = -19.3^o \qquad \theta_s' = 25.7^o$$

(b) From Mohr's circle, Fig. (b):

$$\sigma_1 = 200\ MPa \qquad \sigma_2 = -50\ MPa \qquad \tau_{\max} = 125\ MPa$$

$$\theta_p' = 26.55^o \qquad \theta_s' = 71.55^o$$

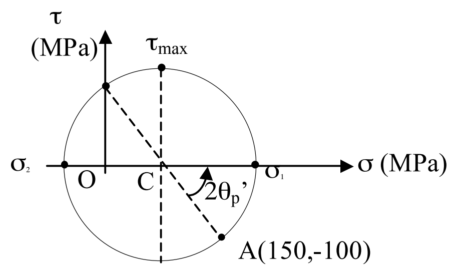


Figure (b)

Through the use of Eq. (1.20),

$$\sigma_{1,2} = 75 \pm \left[\tfrac{22,500}{4} + 10,000\right]^{1/2} = 75 \pm 125$$

or

$$\sigma_1 = 200\ MPa \qquad \sigma_2 = -50\ MPa$$

Using Eq. (1.19), $\tan 2\theta_p = 4/3$:

$$\theta_p' = 26.55^o \qquad \theta_s' = 71.55^o$$

SOLUTION (1.27)

Referring to Mohr's circle, Fig. 1.15:

$$\sigma_{x'} = \tfrac{\sigma_1+\sigma_2}{2} + \tfrac{\sigma_1-\sigma_2}{2}\cos 2\theta \qquad \text{(a)}$$

$$\sigma_{y'} = \tfrac{\sigma_1+\sigma_2}{2} - \tfrac{\sigma_1-\sigma_2}{2}\cos 2\theta$$

$$\tau_{x'y'} = \tfrac{\sigma_1-\sigma_2}{2}\sin 2\theta \qquad \text{(b)}$$

From Eqs. (a),

$$\sigma_{x'} + \sigma_{y'} = \sigma_1 + \sigma_2$$

By using $\cos^2 2\theta + \sin^2 2\theta = 1$, and Eqs. (a) and (b), we have

$$\sigma_{x'} \cdot \sigma_{y'} - \tau_{x'y'}^2 = \sigma_1 \cdot \sigma_2 = const.$$

SOLUTION (1.28)

We have

$$\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} = \frac{2(-70)}{50-(190)} = -0.583$$

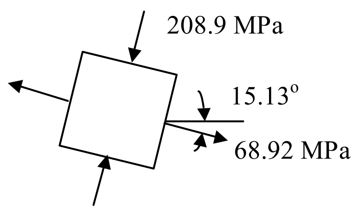
$$2\theta_p = -30.26^o \qquad \text{and } \theta_p = -15.13^o$$

Equations (1.18):

$$\sigma_{x'} = \tfrac{50-190}{2} + \tfrac{50+190}{2}\cos(-30.26^o) - 70\sin(-30.26^o)$$

$$= -70 + 103.65 + 35.275 = 68.92 \ MPa = \sigma_1$$ ◄

$$\sigma_{y'} = \sigma_x + \sigma_y - \sigma_{x'} = -208.9 \ MPa = \sigma_2$$



SOLUTION (1.29)

$$\tau_{\max} = \sqrt{(\tfrac{\sigma_x - \sigma_y}{2})^2 + \tau_{xy}^2}$$

Substituting the given values

$$140^2 = \left(\tfrac{60+100}{2}\right)^2 + \tau_{xy}^2$$

or

$$\tau_{xy,\max} = 114.19 \ MPa$$ ◄

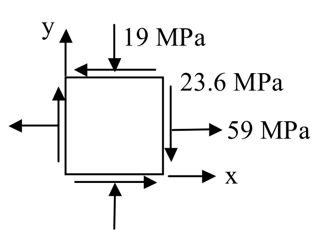
SOLUTION (1.30)

Transform from $\theta = 60^o$ to $\theta = 0^o$ with $\sigma_{x'} = -20 \ MPa$, $\sigma_{y'} = 60 \ MPa$, $\tau_{x'y'} = -22 \ MPa$, and $\theta = -60^o$. Use Eqs. (1.18):

$$\sigma_x = \tfrac{-20+60}{2} + \tfrac{-20-60}{2}\cos 2(-60^o) - 22\sin 2(-60^o) = 59 \ MPa$$

$$\sigma_y = \sigma_{x'} + \sigma_{y'} - \sigma_x = -19 \ MPa$$ ◄

$$\tau_{xy} = -23.6 \ MPa$$



SOLUTION (1.31)

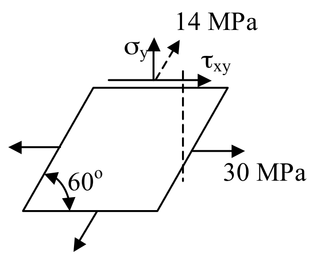


Figure (a)

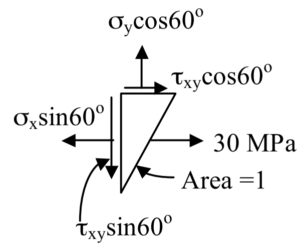


Figure (b)

(a) Figure (a):

$$\sigma_y = 14\sin 60^o = 12.12\ MPa$$

$$\tau_{xy} = 14\cos 60^o = 7\ MPa$$

Figure (b):

$$\sum F_y = 12.12\cos 60^o - \tau_{xy}\sin 60^o = 0$$

or

$$\tau_{xy} = 7\ MPa \text{ (as before)}$$

$$\sum F_x = -\sigma_x \sin 60^o + 30 + 7\cos 60^o = 0$$

or

$$\sigma_x = 38.68\ MPa$$

(b) Equation (1.20) is therefore:

$$\sigma_{1,2} = \tfrac{38.68+12.12}{2} \pm \left[\left(\tfrac{38.68-12.12}{2}\right)^2 + 7^2\right]^{\frac{1}{2}}$$

or $\sigma_1 = 40.41\ MPa,\qquad \sigma_2 = 10.39\ MPa$

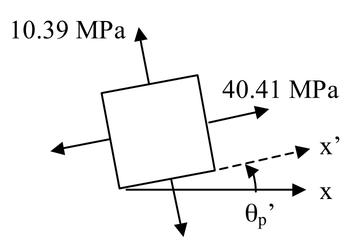
Also,

$$\theta_p = \tfrac{1}{2}\tan^{-1}\tfrac{2(7)}{38.68-12.12} = 13.9^o$$

Note: Eq. (1.18a) gives, $\sigma_{x'} = 40.41\ MPa$.

Thus,

$$\theta_p' = 13.9^o$$



SOLUTION (1.32)

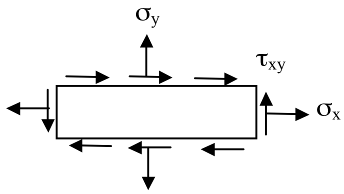


Figure (a)

Figure (a):

$$\sigma_x = 100\cos 45^o = 70.7 \ MPa$$
$$\sigma_y = 100\sin 45^o = 70.7 \ MPa$$
$$\tau_{xy} = 100\cos 45^o = 70.7 \ MPa$$

Now, Eqs. (1.18) give (Fig. b):

$$\sigma_{x'} = 70.7 + 0 + 70.7\sin 240^o = 9.47 \ MPa$$
$$\tau_{x'y'} = -0 + 70.7\cos 240^o = -35.35 \ MPa \quad ◄$$
$$\sigma_{y'} = 70.7 - 0 - 70.7\sin 240^o = 131.9 \ MPa$$

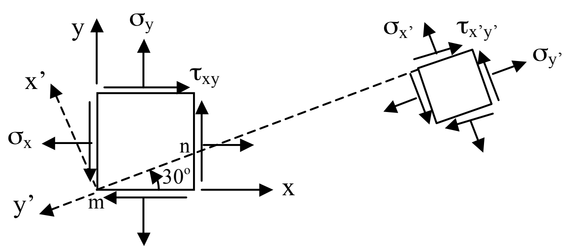


Figure (b)

SOLUTION (1.33)

$$\sigma_y = -70\sin 30^o = -35 \ MPa \quad ◄$$
$$\tau_{xy} = 70\cos 30^o = 60.6 \ MPa$$

(a) Figure (a):

$$\sum F_x = -150 + 0.5\sigma_x + 60.6(0.866) = 0$$

or $\sigma_x = 195 \ MPa$ ◄

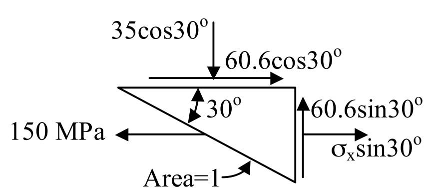


Figure (a)

(CONT.)

1.33 (CONT.)

(b) Equation (1.20):

$$\sigma_{1,2} = \tfrac{195-30}{2} \pm \left[\left(\tfrac{195+35}{2}\right)^2 + 60.6^2\right]^{\frac{1}{2}}$$

or $\quad \sigma_1 = 210\ MPa \qquad \sigma_2 = -50\ MPa$ ◀

Also,

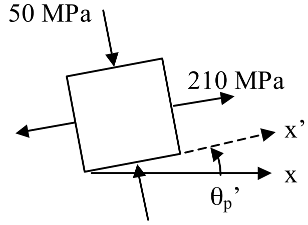
$$\theta_p = \tfrac{1}{2}\tan^{-1}\tfrac{2(60.6)}{195+35} = 13.89^o$$

Equation (1.18a):

$$\sigma_{x'} = 80 + 115\cos 2(13.89^o) + 60.6\sin 2(13.89^o) = 210\ MPa$$

Thus,

$$\theta_p' = 13.89^o$$ ◀



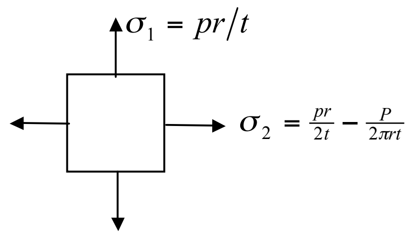
SOLUTION (1.34)

For pure shear, $\sigma_1 = -\sigma_2$:

$$\tfrac{pr}{t} = -\tfrac{pr}{2t} + \tfrac{P}{2\pi rt}$$

from which

$$P = 3\pi pr^2$$ ◀



SOLUTION (1.35)

Table C.1:

$$A = 2\pi rt$$
$$J = 2\pi r^3 t$$

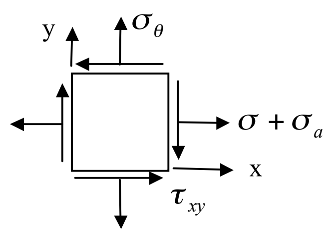


Figure (a)

Stresses are (Fig. a):

$$\sigma = \tfrac{-P}{A} = \tfrac{-30(10^3)\pi}{2\pi(0.12)(0.005)} - 25\ MPa$$

$$\sigma_a = \tfrac{pr}{2t} = \tfrac{4(10^6)120}{2(5)} = 48\ MPa$$

$$\sigma_\theta = 2\sigma_a = 96\ MPa$$

$$\tau_{xy} = \tfrac{-Tr}{J} = \tfrac{-10\pi(10^3)}{2\pi(0.12^2)(0.005)} = -69.4\ MPa$$

(CONT.)

1.35 (CONT.)

Hence,

$$\sigma_x = 48 - 25 = 23\ MPa \qquad \sigma_y = 96\ MPa$$

Therefore, we have

$$\tau_{\max} = \pm\left[\left(\tfrac{23-96}{2}\right)^2 + 69.4^2\right] = \pm 78.4\ MPa$$ ◄

Also

$$\sigma' = \tfrac{1}{2}(23+96) = 59.5\ MPa$$

and

$$\theta_s = \tfrac{1}{2}\tan^{-1}\tfrac{23-96}{2(-69.4)} = -13.87^o$$

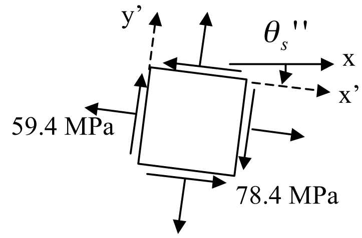


Figure (b)

Equation (1.18b) with $\theta_s = -13.87^o$:

$$\tau_{x'y'} = -16.99 - 61.42 = -78.4\ MPa$$

Thus,

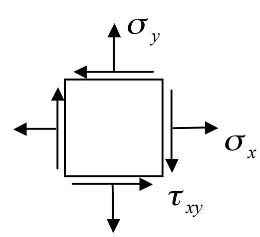
$$\theta_s'' = 13.87^o$$ ◄

SOLUTION (1.36)

$$A = 2\pi rt = 2\pi(60)(4) = 1508\ mm^2$$
$$J = 2\pi r^3 t = 2\pi(60)^3(4) = 5.429\times 10^6\ mm^4$$

$$\sigma_y = \tfrac{pr}{t} = \tfrac{5(60)}{4} = 75\ MPa$$
$$\tau_{xy} = -\tfrac{Tr}{J} = -\tfrac{600(0.05)}{5.429(10^{-6})}$$
$$= -5.526\ MPa$$
$$\sigma_x = \tfrac{pr}{2t} + \tfrac{P}{A} = 37.5 + \tfrac{P}{1508}$$

(P in newtons)



Thus

$$\sigma_1 = \tfrac{\sigma_x+\sigma_y}{2} + \sqrt{\left(\tfrac{\sigma_x-\sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

Substituting the numerical values gives

$$80 = 56.3 + 331.6\times 10^{-6}P + \left[(-18.75 + 331.6\times 10^{-6}P)^2 + (-5.526)^2\right]$$

Solving,

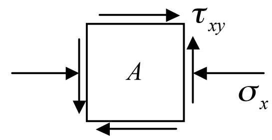
$$P = 64.01\ kN$$ ◄

SOLUTION (1.37)

At point A, $T = 8 kN$ and $P = 400\ kN$

$$\sigma_x = -\frac{P}{A} = -\frac{4(400\times10^3)}{\pi(0.1)^2} = -50.9\ MPa$$

$$\tau_{xy} = \frac{16T}{\pi d^3} = \frac{16(8\times10^3)}{\pi(0.1)^3} = 40.7\ MPa$$

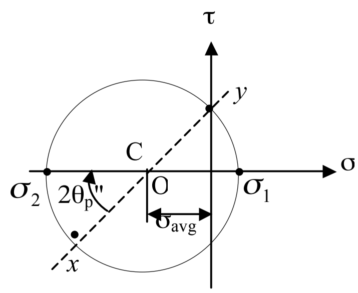


Hence

$$\tau_{\max} = \sqrt{(\frac{\sigma_x}{2})^2 + \tau_{xy}^2}$$

$$= \sqrt{(-25.45)^2 + (40.7)^2}$$

$$= 48\ MPa = R$$ ◀

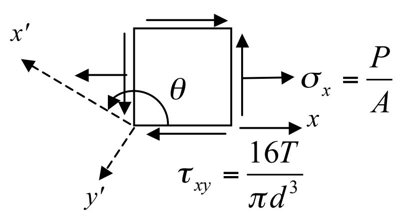


$$\sigma_{avg} = \frac{\sigma_x}{2} = -25.45\ MPa$$

$$\tan 2\theta_p'' = \frac{40.7}{25.45}, \quad \theta_p'' = 29^o$$

SOLUTION (1.38)

$$\theta = \alpha + 90 = 50 + 90 = 140^o$$



$$\sigma_x = \frac{120\times10^3}{\frac{\pi}{4}(0.04)^2} = 95.5\ MPa$$

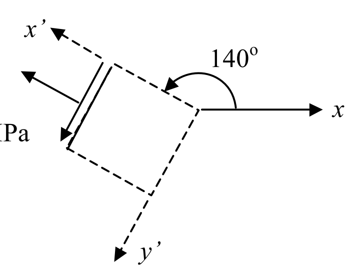
$$\tau = \frac{16(1.5\times10^{-3})}{\pi(0.04)^3} = 119.4\ MPa$$

Equations(1.18):

$$\sigma_w = \sigma_{x'} = \frac{95.5}{2} + \frac{95.5}{2}\cos 280^o + 119.4\sin 280^o = 61.5\ MPa$$ ◀

and

$$\tau_w = \tau_{x'y'} = -\frac{95.5}{2}\sin 280^o - 119.4\cos 280^o = 47.02 - 20.73 = 26.3\ MPa$$ ◀



SOLUTION (1.39)

$$A = \tfrac{\pi}{4}(180^2 - 120^2) = 11.31 \times 10^3 \ mm^2$$
$$J = \tfrac{\pi}{32}(180^4 - 120^4) = 82.70 \times 10^6 \ mm^4$$

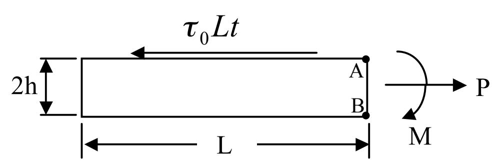
$$\sigma_x = -\tfrac{P}{A} = -\tfrac{700}{11.31} = -61.89 \ MPa, \qquad \sigma_y = 0$$
$$\tau_{xy} = \tfrac{Tr}{J} = \tfrac{20(90)}{82.70 \times 10^{-6}} = 21.76 \ MPa$$

Equation (1.20) is therefore

$$\sigma_{\max} = \sigma_1 = -\tfrac{61.89}{2} + \sqrt{(\tfrac{61.89}{2})^2 + (21.76)^2}$$
$$= 6.885 \ MPa \qquad ◀$$

SOLUTION (**1.40**)

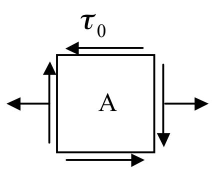
$$P = \tau_0 Lt$$
$$M = \tau_0 Lth$$
$$A = 2ht$$
$$I = \tfrac{1}{12} t(2h)^3 = \tfrac{2}{3} th^3$$



Axial stress: $\sigma_a = \tfrac{P}{A} = \tfrac{\tau_0 L}{2h}$

Bending stress: $\sigma_b = \tfrac{Mc}{I} = \tfrac{3\tau_0 L}{2h}$

Point A

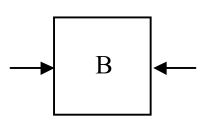


$$\sigma_a + \sigma_b = \tfrac{\tau_0 L}{2h}$$

From Eqs. (1.20) and (1.22), we obtain

$$\sigma_{1,2} = \tfrac{\tau_0 L}{h} \pm \tau_0 \sqrt{(\tfrac{L}{h})^2 + 1} \qquad ◀$$
$$\tau_{\max} = \tau_0 \sqrt{(\tfrac{L}{h})^2 + 1}$$

Point B

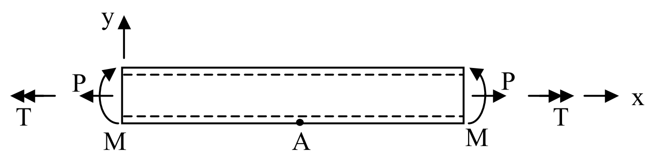


$$\sigma_b - \sigma_a = \tfrac{\tau_0 L}{h}$$

Hence

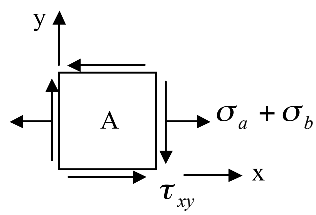
$$\sigma_1 = \tfrac{\tau_0 L}{h} \qquad \sigma_2 = 0 \qquad \tau_{\max} = \tfrac{\tau_0}{2h} \qquad ◀$$

SOLUTION (1.41)



$$A = \pi(30^2 - 15^2) = 2.121(10^{-3})\ m^2$$

$$I = \tfrac{\pi}{4}(30^4 - 15^4) = 0.596(10^{-6})\ m^4 \qquad J = 2I$$



We have

$$\sigma_a = \tfrac{P}{A} = \frac{50(10^3)}{2.12(10^{-3})} = 23.58\ MPa$$

$$\sigma_b = \tfrac{Mr}{I} = \frac{200(0.03)}{0.596(10^{-6})} = 10.07\ MPa$$

$$\tau_{xy} = \tfrac{-Tr}{J} = \frac{-500(0.03)}{1.192(10^{-6})} = -12.58\ MPa$$

Thus,

$$\sigma_x = 23.58 + 10.07 = 33.65\ MPa \qquad \sigma' = 16.83\ MPa$$

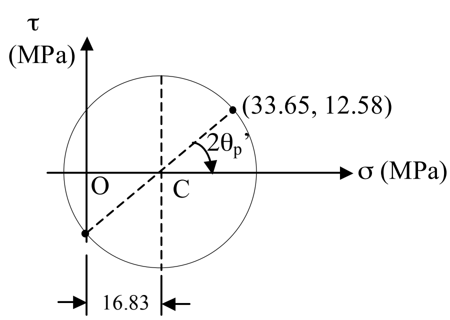


Figure (a)

From Mohr's circle (Fig. a):

$$r = \sqrt{12.58^2 + 16.83^2} = 21.01\ MPa \qquad \theta_p' = \tfrac{1}{2}\tan^{-1}\tfrac{12.58}{16.83} = 18.39^o$$

$$\sigma_1 = 16.83 + 21.01 = 37.84\ MPa$$

$$\sigma_2 = -4.18\ MPa$$ ◄

Results are shown in Fig. (b).

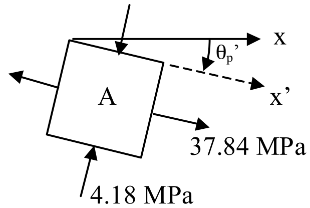


Figure (b)

SOLUTION (1.42)

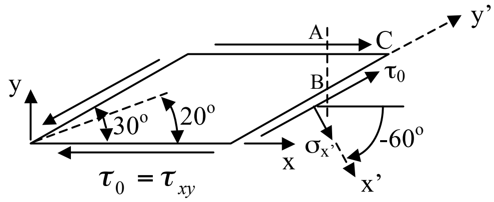


Figure (a)

(a) At $\theta = -60^o$ (Fig. a):

$$0 = \tfrac{\sigma_x+\sigma_y}{2} + \tfrac{\sigma_x-\sigma_y}{2}\cos 2(-60^o) + \tau_0 \sin 2(-60^o)$$

or

$$0 = 0.5\sigma_x + 1.5\sigma_y - 1.732\tau_0 \tag{a}$$

We also have

$$\tau_0 = -\tfrac{\sigma_x-\sigma_y}{2}\sin 2(-60^o) + \cos 2(-60^o)$$

or

$$\sigma_x = 3.464\tau_0 + \sigma_y \tag{b}$$

Substituting Eq. (b) into (a), we obtain $\sigma_y = 0$. Results are shown in Fig. b.

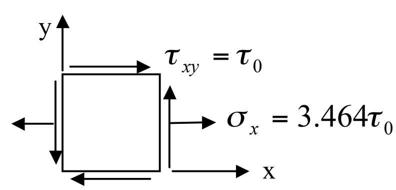


Figure (b)

Alternatively, using an element ABC (Fig. c):

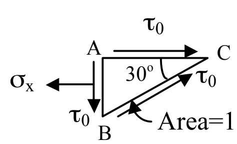


Figure (c)

$$\sum F_x = 0.5\sigma_x - 0.866\tau_0 - 0.866\tau_0 = 0$$

or $\sigma_x = 3.464\tau_0$, as before.

Stresses on planes at 20^o, taking $\theta = -70^o$ (Fig. b):

$$\sigma_{20^o} = [\tfrac{3.464}{2} + \tfrac{3.464}{2}\cos(-140^o) + \sin(-140^o)]\tau_0 = -0.237\tau_0 \quad ◄$$

$$\tau_{20^o} = [-\tfrac{3.464}{2}\sin(-140^o) + \cos(-140^o)]\tau_0 = 0.347\tau_0 \quad ◄$$

(CONT.)

1.42 (CONT.)

(b) Principal stresses:

$$\sigma_{1,2} = \tfrac{3.464}{2} \pm \left[\left(\tfrac{3.464}{2}\right)^2 + \tau_0^2\right]^{\frac{1}{2}}$$

$$\sigma_1 = 3.732\tau_0 \qquad \sigma_2 = -0.268\tau_0 \qquad \blacktriangleleft$$

The maximum principal stress is on plane inclined at

$$\theta_p' = \tfrac{1}{2}\tan^{-1}\tfrac{\tau_0}{1.732\tau_0} = 15^o \qquad \blacktriangleleft$$

SOLUTION (1.43)

At a critical point on the shaft surface, the state of stress of stress is as shown in Fig. (a).

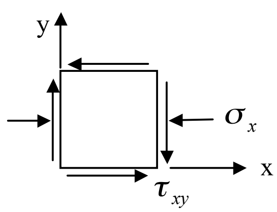


Figure (a)

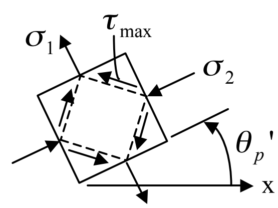


Figure (b)

We have

$$\sigma_x = -\tfrac{P}{A} - \tfrac{Mr}{I}$$

$$= -\frac{81(10^3)}{\pi(0.075)^2} - \frac{13(10^3)(0.075)}{\pi(0.075)^4/4} = -43.818\ MPa$$

$$\tau_{xy} = -\tfrac{Tr}{J} = -\frac{(15.6\times10^3)0.075}{\pi(0.075)^4/2} = -23.54\ MPa$$

Therefore,

$$\sigma_{1,2} = \tfrac{-43.818}{2} \pm \left[\left(\tfrac{43.818}{2}\right)^2 + (-23.54)^2\right]^{\frac{1}{2}}$$

or

$$\sigma_1 = 10.248\ MPa, \qquad \sigma_2 = -54.066\ MPa \qquad \blacktriangleleft$$

$$\tau_{max} = \tfrac{1}{2}(\sigma_1 - \sigma_2) = 32.157\ MPa$$

and

$$\theta_p'' = \tfrac{1}{2}\tan^{-1}\tfrac{2(23.54)}{43.818} = 23.53^o \qquad \blacktriangleleft$$

Results are shown in Fig. (b).

SOLUTION (1.44)

Apply Eqs. (1.20) to Fig. P1.44b, for $\theta = -30^o$:

$$\sigma_{xb} = -40\sin 2(-30^o) = 20\sqrt{3}\ MPa$$

$$\sigma_{yb} = -20\sqrt{3}\ MPa \qquad \text{(b)}$$

$$\tau_{xyb} = -40\cos 2(-30^o) = -20\ MPa$$

(CONT.)

Now apply Eqs. (1.18) to Fig. P1.44c, for $\theta = -60^o$:

$$\sigma_{xc} = 10\sin 2(-60^o) = -5\sqrt{3}\ MPa$$
$$\sigma_{yc} = 5\sqrt{3}\ MPa \qquad \text{(c)}$$
$$\tau_{xyc} = 10\cos 2(-60^o) = -5\ MPa$$

Superposing stresses in Eqs. (b) and (c) and those in Fig. P1.44a, we obtain Fig. (a).

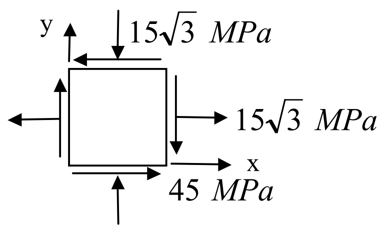


Figure (a)

Referring to Fig. (a):

$$\sigma_{1,2} = 0 \pm \left[(15\sqrt{3})^2 + (-45)^2\right]^{\frac{1}{2}}$$

or

$$\sigma_1 = 51.96\ MPa \qquad \sigma_2 = -51.96\ MPa \qquad \blacktriangleleft$$

When

$$\theta_p' = \tfrac{1}{2}\tan^{-1}\tfrac{2(-45)}{2(15\sqrt{3})} = -30^o \qquad \blacktriangleleft$$

is substituted into Eq. (1.18a), we have 51.96 MPa (Fig. b).

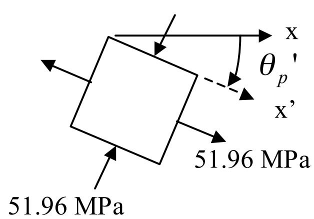


Figure (b)

SOLUTION (1.45)

Apply Eqs. (1.18) to Fig. P1.45a, for $\theta = -15^o$, to obtain stresses in Fig. (a):

$$\sigma_{xa} = -\tfrac{30}{2} - \tfrac{30}{2}\cos 2(-15^o) = -27.99\ MPa$$
$$\sigma_{ya} = -15 + 15\cos 2(-15^o) = -2.01\ MPa$$
$$\tau_{xya} = 15\sin 2(-15^o) = -7.5\ MPa$$

Superposition of stresses in Figs. (a) and P1.45b gives Fig. (b).

(CONT.)

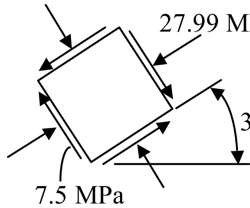


Figure (a)

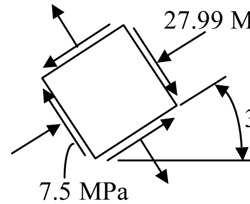


Figure (b)

Apply Eq. (1.20) to Fig. (b):

$$\sigma_{1,2} = \tfrac{-27.99+47.99}{2} \pm \left[\tfrac{1}{4}(-27.99-47.99)^2 + (-7.5)^2\right]$$

or

$$\sigma_1 = 48.72\ MPa, \qquad \sigma_2 = -28.72\ MPa \qquad \blacktriangleleft$$

When

$$\theta_p = \tfrac{1}{2}\tan^{-1}\tfrac{2(-7.5)}{-(27.99+47.99)} = 5.58^o$$

is substituted into Eq. (1.18a), we obtain –28.72 MPa (Fig. c).

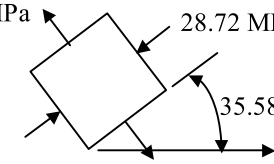


Figure (c)

SOLUTION (1.46)

Equations (1.18) are applied to Fig. P1.46a, for $\theta = -30^o$:

$$\sigma_{xa} = \tfrac{20+30}{2} + \tfrac{20-30}{2}\cos 2(-30^o) = 22.5\ MPa$$

$$\sigma_{ya} = 25 - (-5)\cos 2(-30^o) = 27.5\ MPa$$

$$\tau_{xya} = -(-5)\sin 2(-30^o) = -4.33\ MPa$$

These stresses and that of Fig. P1.46b are superimposed to yield Fig. (a).

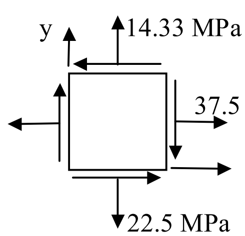


Figure (a)

(CONT.)

1.46 (CONT.)

Principal stresses are thus

$$\sigma_{1,2} = \tfrac{37.5+22.5}{2} \pm \left[\left(\tfrac{37.5-22.5}{2}\right)^2 + 14.33^2\right]$$

or

$$\sigma_1 = 46.17\ MPa \qquad \sigma_2 = 13.83\ MPa \qquad \blacktriangleleft$$

Hence

$$\tau_{\max} = \tfrac{1}{2}(\sigma_1 - \sigma_2) = 16.17\ MPa \qquad \blacktriangleleft$$

We have

$$\theta_p = \tfrac{1}{2}\tan^{-1}\tfrac{2(-14.33)}{37.5-22.5} = -31.2^o$$

Equation (1.18a) results in

$$\sigma_{x'} = \tfrac{37.5+22.5}{2} + \tfrac{37.5-22.5}{2}\cos(-62.4^o) - 14.33\sin(-62.4^o) = 46.17\ MPa$$

Therefore

$$\theta_p' = 31.2^o \qquad \blacktriangleleft$$

Results are shown in a properly oriented element in Fig. (b).

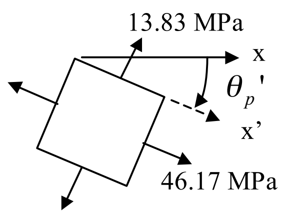


Figure (b)

SOLUTION (1.47)

State of stress is represented by Mohr's circle in Fig. (a).

From this circle, we determine

$$\sigma_x = -40\ MPa$$
$$\sigma_y = 20\ MPa \qquad \blacktriangleleft$$
$$\theta_p' = \tfrac{1}{2}\tan^{-1}\tfrac{4}{3} = 26.57^o$$

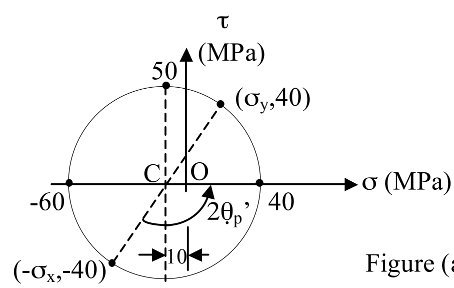


Figure (a)

Results are shown in Fig. (b).

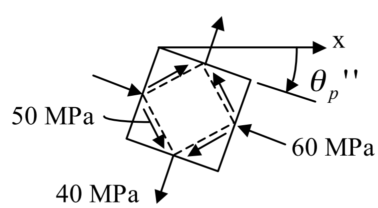
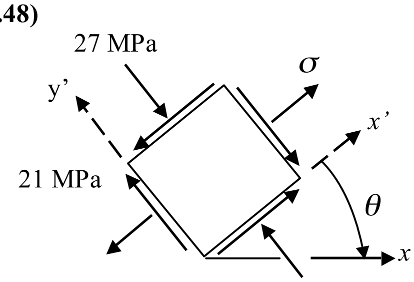


Figure (b)

SOLUTION (1.48)



$$\sigma_x + \sigma_y = \sigma_{x'} + \sigma_{y'}$$

$$45 - 30 = -27 + \sigma; \qquad \sigma = 42\ MPa$$ ◄

$$\sigma_{x'} = 42 = \frac{45-30}{2} + \frac{45+30}{2}\cos 2\theta + 15\sin 2\theta$$

or

$$49.5 = 37.5\cos 2\theta + 15\sin 2\theta \qquad (1)$$

$$\tau_{x'y'} = -21 = -37.5\sin 2\theta + 15\cos 2\theta$$

Multiply this by -2.5:

$$52.5 = 93.75\sin 2\theta - 37.5\cos 2\theta \qquad (2)$$

Add Eqs. (1) and (2),

$$102 = 108.75\sin 2\theta, \quad 2\theta = 69.71^o$$

or

$$\theta = 34.9^o$$ ◄

SOLUTION (1.49)

State of stress is represented by Mohr's circle in Fig. (a).

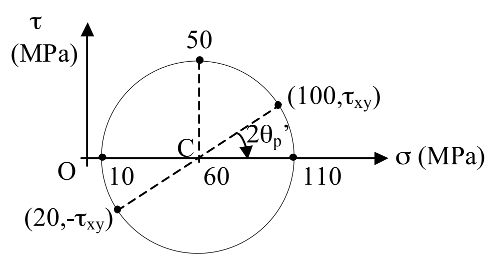


Figure (a)

Referring to this circle, we obtain the results (Fig. b).

$$\theta_p' = \tfrac{1}{2}\tan^{-1}\tfrac{3}{4} = 18.43^o$$ ◄

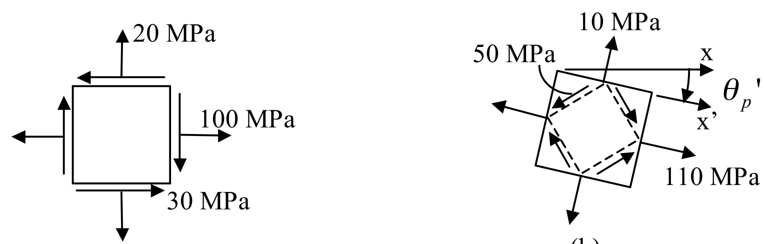


Figure (b)

SOLUTION (1.50)

$$\sigma_x = 60\ MPa \qquad \sigma_y = -18\ MPa \qquad \tau_{xy} = -15\ MPa \qquad \sigma_{x'} = 30\ MPa$$

From Equation (1.18a):

$$30 = \tfrac{60-18}{2} + \tfrac{60+18}{2}\cos 2\theta_1 - 15\sin 2\theta_1$$

or

$$13\cos 2\theta_1 - 5\sin 2\theta_1 - 3 = 0$$

Solving

$$2\theta_1 = 56.52^o \qquad \theta_1 = 28.26^o \qquad ◄$$

We have

$$\sigma_{y'} = \sigma_x + \sigma_y - \sigma_{x'} = 12\ MPa \qquad ◄$$

Equation (1.14b) gives

$$\tau_{x'y'} = -\tfrac{60+18}{2}\sin 56.52^o - 15\cos 56.52^o = -40.8\ MPa \qquad ◄$$

SOLUTION (1.51)

We have

$$\sigma = \tfrac{4M}{\pi r^3} = \tfrac{4(21\pi)10^3}{\pi(0.1)^3} = 84\ MPa$$

State of stress is represented by Mohr's circle in Fig. (a).

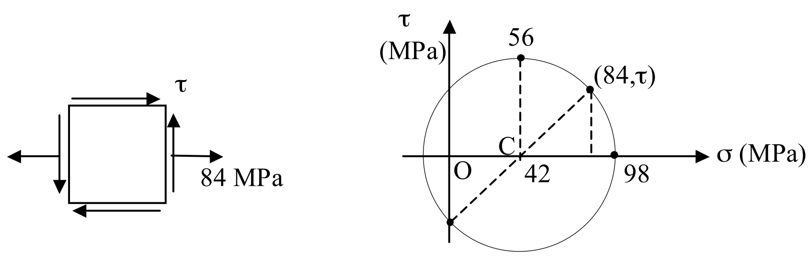


Figure (a)

$$\tau = (56^2 - 42^2)^{\frac{1}{2}} = 37.04\ MPa$$

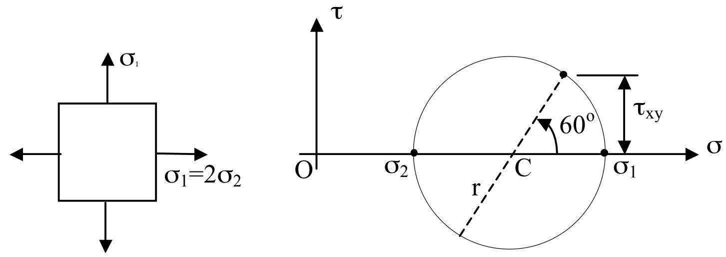
Thus

$$T = \tfrac{\tau J}{r} = \tfrac{(37.04)\pi(0.1^3)}{2} = 58.18\ kN \cdot m \qquad ◄$$

Hence

$$P = 2\pi f T = 2\pi(20)58.18 = 7311\ kW \qquad ◄$$

SOLUTION (1.52)



$$r = \frac{2\sigma_2 - \sigma_2}{2} = \frac{\sigma_2}{2}$$

From Mohr's circle,

$$\tau_{x'y'} = (\tfrac{\sigma_2}{2})\sin 60^o = 0.433\sigma_2$$

Therefore

$$30 = 0.433\sigma_2 \qquad \sigma_2 = 69.28 \ MPa$$

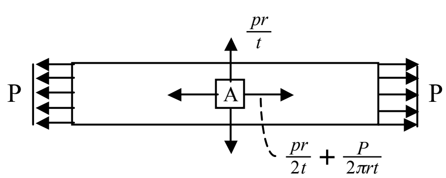
We have

$$\sigma_2 = \tfrac{pr}{2t} = 69.28 \qquad p(\tfrac{250}{2}) = 69.28$$

Solving

$$p = 2.771 \ MPa$$ ◄

SOLUTION (1.53)



Mohr's circle representing stress at point A is shown in Fig. (a).

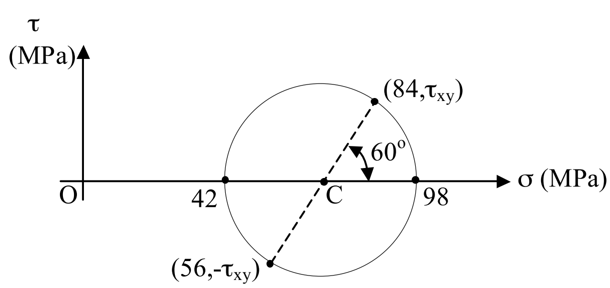


Figure (a)

From this circle:

$$42 = \tfrac{p(0.45)}{0.005} = 90p \qquad \text{or} \qquad p = 467 \ kPa$$ ◄

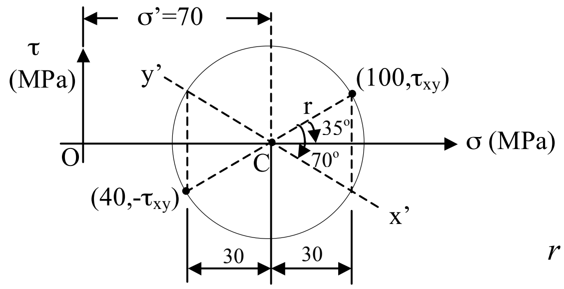
Then

$$98(10^3) = \tfrac{90P}{2} + \tfrac{P}{2\pi(0.45)(0.005)}$$

gives

$$P = 1069 \ kN$$ ◄

SOLUTION (1.54)



$$r = \frac{30}{\cos 35^o} = 36.62 \ MPa$$

(a) $\tau_{xy} = -36.62 \sin 35^o = -21 \ MPa$ ◀

(b) Because of symmetry:

$$\tau_{x'y'} = -\tau_{xy} = 21 \ MPa$$ ◀

and

$$\sigma_x + \sigma_y = \sigma_{x'} + \sigma_{y'} = 140 \ MPa$$

gives

$$\sigma_{y'} = 40 \ MPa$$ ◀

SOLUTION (1.55)

State of stress is represented by Mohr's circle in Fig. (a).

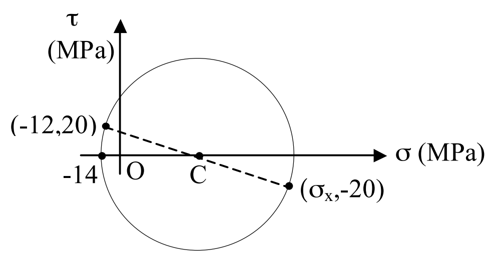


Figure (a)

(a) Using this circle, we write

$$\tau_{\max} = \left[\left(\frac{\sigma_x + 12}{2}\right)^2 + 20^2\right]^{\frac{1}{2}}$$

and

$$\tau_{\max} = 14 + OC = 14 + \tfrac{1}{2}(\sigma_x - 12)$$

Solving,

$$\sigma_x = 186 \ MPa$$ ◀

Note that, alternately,

(CONT.)

1.55 (CONT.)

$$-14 = \tfrac{\sigma_x - 12}{2} - \left[\left(\tfrac{\sigma_x + 12}{2} \right)^2 + 20^2 \right]$$

yields $\sigma_x = 186\ MPa$, as before.

(b) We have

$$\sigma_{1,2} = \tfrac{186-12}{2} \pm \left[\left(\tfrac{186+12}{2} \right)^2 + 20^2 \right]$$

or

$$\sigma_1 = 188\ MPa \qquad \sigma_2 = -14\ MPa$$ ◀

and

$$\tau_{\max} = \tfrac{1}{2}(\sigma_1 - \sigma_2) = 101\ MPa$$ ◀

Also

$$\theta_p' = \tfrac{1}{2}\tan^{-1}\tfrac{2(20)}{186+12} = 5.71^o$$ ◀

Results are shown in Fig. (b).

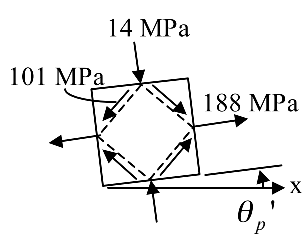


Figure (b)

SOLUTION (1.56)

(a) $\sigma_1 = 96.05\ MPa \qquad \sigma_2 = 23.05\ MPa \qquad \sigma_3 = 0$

(b) $(\tau_{12})_{\max} = \tfrac{1}{2}(\sigma_1 - \sigma_2) = 36.05\ MPa$

$(\tau_{13})_{\max} = \tfrac{1}{2}(\sigma_1 - \sigma_3) = 48.03\ MPa$ ◀

$(\tau_{23})_{\max} = \tfrac{1}{2}(\sigma_2 - \sigma_3) = 11.98\ MPa$

Plane of $(\tau_{12})_{\max}$ is shown in Fig. (a). Other maximum shear planes are sketched similarly.

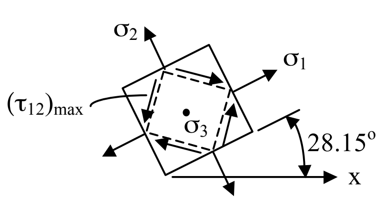
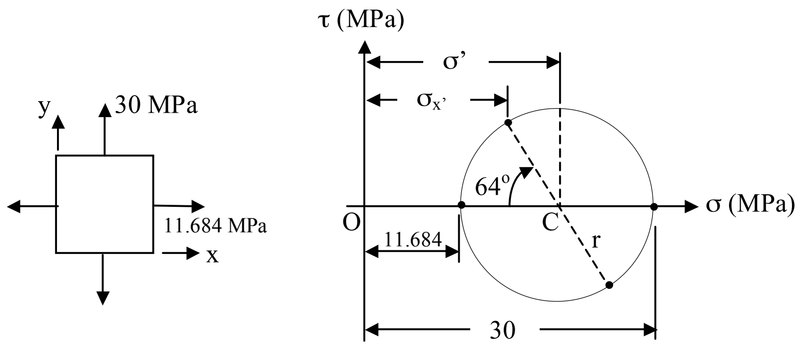


Figure (a)

SOLUTION (1.57)

$$A = 2\pi rt = 2\pi(60)(4) = 1508\ mm^2$$
$$\sigma_y = \tfrac{pr}{t} = \tfrac{2(60)}{4} = 30\ MPa$$
$$\sigma_x = -\tfrac{P}{A} + \tfrac{pr}{2t} = -\tfrac{50(10^3)}{1508(10^{-6})} + 15 = 11{,}684\ MPa$$



$$\sigma' = \tfrac{1}{2}(30 + 11.684) = 20.84\ MPa$$
$$r = \tfrac{1}{2}(30 - 11.684) = 9.158\ MPa$$

(a) $\sigma_{x'} = \sigma' - r\cos 64^o = 16.82\ MPa$ ◄

(b) $\tau_{x'y'} = r\sin 64^o = 8.231\ MPa$ ◄

SOLUTION (1.58)

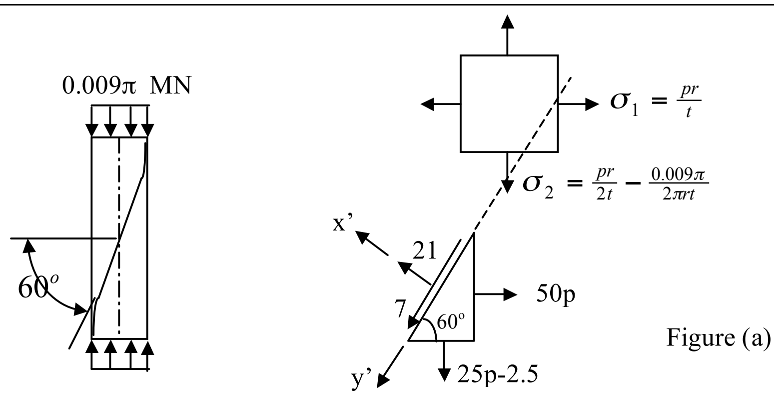


Figure (a)

Equilibrium of x' and y' directed forces results in (Fig. a):

$$21 - 50p(\tfrac{\sqrt{3}}{2})^2 - (25p - 2.5)(\tfrac{1}{2})^2 = 0$$

or

$$p_{all} = 494\ kPa$$ ◄

and

$$7 + (25p - 2.5)(\tfrac{\sqrt{3}}{4}) - 50p(\tfrac{\sqrt{3}}{4}) = 0$$

from which

$$p = 547\ kPa$$

SOLUTION (1.59)

Direction cosines are:

$$\begin{array}{lll} l_1 = \sqrt{3}/2 & m_1 = 1/2 & n_1 = 0 \\ l_2 = -1/2 & m_2 = \sqrt{3}/2 & n_2 = 0 \\ l_3 = 0 & m_3 = 0 & n_3 = 1 \end{array}$$

Equation (1.28a) is thus

$$\sigma_{x'} = 20(\tfrac{3}{4}) + 0 + 0 + 2(12)(\tfrac{\sqrt{3}}{2})(\tfrac{1}{2}) + 0 + 0 = 25.392 \ MPa$$

Similarly, applying Eqs. (1.28b) through (1.28e), we obtain $[\tau_{i'j'}]$:

$$\begin{bmatrix} 25.342 & -2.66 & -7.99 \\ -2.66 & -5.392 & 16.16 \\ -7.99 & 16.16 & 6 \end{bmatrix} MPa \quad ◄$$

Then, Eqs. (1.34) result in

$$I_1 = I_1' = 26 \ MPa \qquad I_2 = I_2' = -349 \ (MPa)^2$$
$$I_3 = I_3' = -6464 \ (MPa)^3 \quad ◄$$

SOLUTION (1.60)

Direction cosines are:

$$\begin{array}{lll} l_1 = \sqrt{3}/2 & m_1 = 1/2 & n_1 = 0 \\ l_2 = -1/2 & m_2 = \sqrt{3}/2 & n_2 = 0 \\ l_3 = 0 & m_3 = 0 & n_3 = 1 \end{array}$$

Equation (1.28a) is therefore

$$\sigma_{x'} = 60(\tfrac{3}{4}) + 0 + 0 + 2(40)(\tfrac{\sqrt{3}}{2})(\tfrac{1}{2}) + 0 + 0$$
$$= 20[(\tfrac{9}{4}) + \sqrt{3}] = 79.64 \ MPa$$

Similarly, applying Eqs. (1.28b) through (1.28e), we obtain $[\tau_{i'j'}]$:

$$\begin{bmatrix} 79.64 & -5.98 & -44.64 \\ -5.98 & -19.64 & -2.68 \\ -44.64 & -2.68 & 20 \end{bmatrix} MPa \quad ◄$$

Then, Eqs. (1.34) lead to

$$I_1 = I_1' = 80 \ MPa \qquad I_2 = I_2' = -2400 \ (MPa)^2$$
$$I_3 = I_3' = 8000 \ (MPa)^3 \quad ◄$$

SOLUTION (1.61)

Referring to Appendix B:

$$\sigma_1 = 13.212\ MPa \qquad \sigma_2 = 5.684\ MPa \qquad \sigma_3 = -8.896\ MPa$$

and

$$l_1 = 0.9556 \qquad m_1 = 0.1688 \qquad n_1 = 0.2416$$

Thus,

$$\tau_{\max} = \tfrac{1}{2}(\sigma_1 - \sigma_3) = 11.054\ MPa \qquad ◄$$

SOLUTION (1.62)

Referring to Appendix B:

$$\sigma_1 = 66.016\ MPa \qquad \sigma_2 = 28.418\ MPa \qquad \sigma_3 = -44.479\ MPa \qquad ◄$$

and

$$l_1 = 0.9556 \qquad m_1 = 0.1688 \qquad n_1 = 0.2416$$

SOLUTION (1.63)

Referring to Appendix B:

$$\sigma_1 = 30.493\ MPa \qquad \sigma_2 = 12.485\ MPa \qquad \sigma_3 = -16.979\ MPa$$

Thus,

$$\tau_{\max} = \tfrac{1}{2}(\sigma_1 - \sigma_3) = 23.736\ MPa \qquad ◄$$

SOLUTION (1.64)

Referring to Appendix B:

$$\sigma_1 = 24.747\ MPa \qquad \sigma_2 = 8.480\ MPa \qquad \sigma_3 = 2.773\ MPa \qquad ◄$$

and

$$l_1 = 0.6467 \qquad m_1 = 0.3958 \qquad n_1 = 0.6421$$

SOLUTION (1.65)

(a) Equation (1.32) becomes

$$\begin{vmatrix} (30 - \sigma_p) & 0 & 20 \\ 0 & -\sigma_p & 0 \\ 20 & 0 & -\sigma_p \end{vmatrix} = 0$$

Expanding,

$$-\sigma_p[\sigma_p(\sigma_p - 30) - 400] = 0$$

(CONT.)

or

$$\sigma_p = 0, \qquad \sigma_p = -10, \qquad \sigma_p = 40$$

Thus

$$\sigma_1 = 40\ MPa, \qquad \sigma_2 = 0, \qquad \sigma_3 = -10\ MPa \qquad ◄$$

(b) For $\sigma_1 = 40\ MPa$:

$$(30-40)l_1 + (0)m_1 + 20n_1 = 0, \qquad l_1 = 2n_1$$
$$(0)m_1 = 0, \qquad m_1 = 0$$

The condition $l_1^2 + 0 + n_1^2 = 1$ gives

$$(2n_1)^2 + n_1^2 = 1, \qquad n_1 = \tfrac{1}{\sqrt{5}}$$

Thus

$$l_1 = \tfrac{2}{\sqrt{5}}, \qquad m_1 = 0, \qquad n_1 = \tfrac{1}{\sqrt{5}} \qquad ◄$$

SOLUTION (1.66)

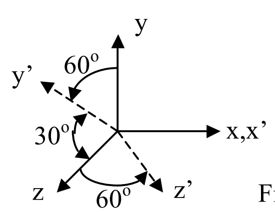


Figure (a)

(a) At point (3,1,5) with respect to xyz axis, we have $[\tau_{ij}]$:

$$\begin{bmatrix} 10 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 8 \end{bmatrix} MPa \qquad \text{(a)}$$

Then, Eqs. (1.34) result in

$$I_1 = 14\ MPa \qquad I_2 = 8\ (MPa)^2 \qquad I_3 = -320\ (MPa)^3 \qquad ◄$$

Direction cosines of x' y' z', referring to Fig. (a) are

$$l_1 = 1 \quad m_1 = 0 \qquad n_1 = 0$$
$$l_2 = 0 \quad m_2 = 1/2 \qquad n_2 = \sqrt{3}/2$$
$$l_3 = 0 \quad m_3 = -\sqrt{3}/2 \quad n_3 = 1/2$$

Now Eqs. (1.28) and (a) give $[\tau_{i'j'}]$:

$$\begin{bmatrix} 10 & 0 & 0 \\ 0 & 5 & 3\sqrt{3} \\ 0 & 3\sqrt{3} & -1 \end{bmatrix} MPa \qquad ◄$$

(CONT.)

1.66 (CONT.)

Thus, Eqs. (1.34) yield

$$I_1' = 14\ MPa \qquad I_2' = 8\ (MPa)^2 \qquad I_3' = -320\ (MPa)^3 \qquad ◀$$

as before.

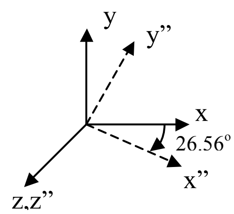


Figure (b)

(b) Direction cosines are (Fig. b):

$$\begin{aligned} l_1 &= 2/\sqrt{5} & m_1 &= -1/\sqrt{5} & n_1 &= 0 \\ l_2 &= 1/\sqrt{5} & m_2 &= 2/\sqrt{5} & n_2 &= 0 \\ l_3 &= 0 & m_3 &= 0 & n_3 &= 1 \end{aligned}$$

With these and Eq. (a), Eqs. (1.28) yield $[\tau_{i'j'}]$:

$$\begin{bmatrix} 7.2 & 5.6 & 0 \\ 5.6 & -1.2 & 0 \\ 0 & 0 & 8 \end{bmatrix} MPa \qquad ◀$$

Thus, Eqs. (1.34) result in

$$I_1'' = 14\ MPa \qquad I_2'' = 8\ (MPa)^2 \qquad I_3'' = -320\ (MPa)^3 \qquad ◀$$

The I’s are thus invariants.

SOLUTION (1.67)

Introducing the given data into Eq. (1.28a), we obtain

$$\begin{aligned} \sigma_{x'} &= 12(\tfrac{1}{2})^2 + 10(\tfrac{\sqrt{3}}{2})^2 + 14(0) + 2[6(\tfrac{1}{2})(\tfrac{\sqrt{3}}{2})] + 0 + 0 \\ &= 15.696\ MPa \end{aligned}$$

Remaining stress components are determined in a like manner. The result, $[\tau_{i'j'}]$, is

$$\begin{bmatrix} 15.696 & -3.866 & 7.089 \\ -3.866 & 6.304 & -6.294 \\ 7.089 & -6.294 & 14. \end{bmatrix} MPa \qquad ◀$$

SOLUTION (1.68)

Equations (1.34) become

$$I_1 = \sigma_x + \sigma_y \qquad I_2 = \sigma_x \cdot \sigma_y - \tau_{xy}^2 \qquad I_3 = 0$$

Equation (1.33) is then

$$\sigma_p^3 - (\sigma_x + \sigma_y)\sigma_p^2 + (\sigma_x\sigma_y - \tau_{xy}^2)\sigma_p = 0$$

or

$$\sigma_p^2 - (\sigma_x + \sigma_y)\sigma_p + (\sigma_x\sigma_y - \tau_{xy}^2) = 0$$

Solution of this quadratic equation is

$$\sigma_{1,2} = \tfrac{\sigma_x+\sigma_y}{2} \pm \tfrac{1}{2}[\sigma_x^2 + 2\sigma_x\sigma_y + \sigma_y^2 - 4(\sigma_x + \sigma_y - \tau_{xy}^2)]^{\frac{1}{2}}$$

$$= \tfrac{\sigma_x+\sigma_y}{2} \pm [(\tfrac{\sigma_x-\sigma_y}{2})^2 + \tau_{xy}^2]^{\frac{1}{2}} \qquad ◄$$

SOLUTION (1.69)

Referring to Appendix B, we obtain the following values.

(a) $\sigma_1 = 12.049\ MPa \qquad \sigma_2 = -1.521\ MPa \qquad \sigma_3 = -4.528\ MPa$ ◄

and

$$l_1 = 0.6184 \qquad m_1 = 0.5333 \qquad n_1 = 0.5772$$

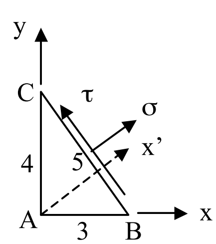
(b) $\sigma_1 = 19.238\ MPa \qquad \sigma_2 = 13.704\ MPa \qquad \sigma_3 = 4.648\ MPa$ ◄

and

$$l_1 = 0.3339 \qquad m_1 = 0.3862 \qquad n_1 = 0.8599$$

SOLUTION (1.70)

(a) Direction cosines are:



$$l = 4/5 = 0.8$$
$$m = 3/5 = 0.6$$
$$n = 0$$

Equation (1.40) is thus

$$\sigma = 100(0.8)^2 + 60(0.6)^2 + 2(40)(0.8)(0.6) = 124\ MPa \qquad ◄$$

Equations (1.26) yield

$$p_x = 100(0.8) + 40(0.6) = 104\ MPa$$
$$p_y = 40(0.8) + 60(0.6) = 68\ MPa$$
$$p_z = 80(0.6) = 48\ MPa$$

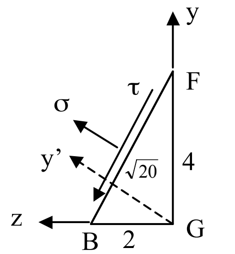
Equation (1.41) is then

$$\tau = [104^2 + 68^2 + 48^2 - 124^2]^{\frac{1}{2}} = 48.66\ MPa \qquad ◄$$

(CONT.)

1.70 (CONT.)

(b) Direction cosines are:



$$l = 0$$
$$m = 2/\sqrt{20} = 0.447$$
$$n = 4/\sqrt{20} = 0.894$$

Equation (1.40) results in

$$\sigma = 60(0.447)^2 + 20(0.894)^2 + 2(80)(0.447)(0.894) = 91.913\ MPa \quad \blacktriangleleft$$

Equations (1.26) yield

$$p_x = 40(0.447) = 17.88\ MPa$$
$$p_y = 60(0.447) + 80(0.894) = 98.34\ MPa$$
$$p_z = 80(0.447) + 20(0.894) = 53.64\ MPa$$

Equation (1.41) leads to

$$\tau = [17.88^2 + 98.34^2 + 53.66^2 - 91.913^2]^{\frac{1}{2}} = 66.481\ MPa \quad \blacktriangleleft$$

(c) Direction cosines are:

$$l = 0.512 \qquad m = 0.384 \qquad n = 0.768$$

Equation (1.40) is therefore

$$\sigma = 100(0.512)^2 + 60(0.384)^2 + 20(0.768)^2 + 2[40(0.512)(0.384) + 80(0.384)(0.768)] = 109.77\ MPa \quad \blacktriangleleft$$

Equations (1.26) yield

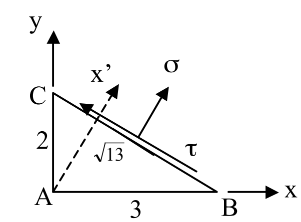
$$p_x = 100(0.512) + 40(0.384) = 66.56\ MPa$$
$$p_y = 40(0.512) + 60(0.384) + 80(0.768) = 104.96\ MPa$$
$$p_z = 80(0.384) + 20(0.768) = 46.08\ MPa$$

Equation (1.41) gives

$$\tau = [66.56^2 + 104.96^2 + 46.08^2 - 109.77^2]^{\frac{1}{2}} = 74.3\ MPa \quad \blacktriangleleft$$

SOLUTION (1.71)

(a) Direction cosines are:



$$l = 2/\sqrt{13} = 0.555$$
$$m = 3/\sqrt{13} = 0.882$$
$$n = 0$$

(CONT.)

1.71 (CONT.)

Equation (1.40) is then

$$\sigma = 100(0.555)^2 + 60(0.832)^2 + 2(40)(0.555)(0.832) = 109.277\ MPa \quad \blacktriangleleft$$

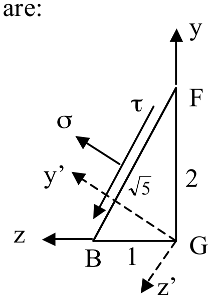
Equations (1.26) lead to

$$p_x = 100(0.555) + 40(0.832) = 88.78\ MPa$$
$$p_y = 40(0.555) + 60(0.832) = 72.12\ MPa$$
$$p_z = 80(0.832) = 66.56\ MPa$$

Equation (1.41) gives then

$$\tau = [88.78^2 + 72.12^2 + 66.56^2 - 109.277^2]^{\frac{1}{2}} = 74.67\ MPa \quad \blacktriangleleft$$

(b) Direction cosines are:



$$l = 0$$
$$m = 1/\sqrt{5} = 0.447$$
$$n = 2/\sqrt{5} = 0.894$$

Thus, the results are the same as those obtained in Solution of Prob. 1.70b.

(c) We have $\vec{r}_g = 3\vec{i}$, $\vec{r}_e = 2\vec{j}$, $\vec{r}_a = \vec{k}$.

Equation (P1.70) is therefore

$$\begin{bmatrix} x-3 & y & z \\ -3 & 2 & 0 \\ -3 & 0 & 1 \end{bmatrix} = 0$$

or

$$2x + 3y + 6z = 6$$

Direction cosines are:

$$l = \tfrac{2}{\sqrt{2^2+3^2+6^2}} = \tfrac{2}{7} \qquad m = \tfrac{3}{7} \qquad n = \tfrac{6}{7}$$

With these and given stresses, Eqs. (1.40) and (1.26) yield

$$\sigma = 102.449\ MPa \quad \blacktriangleleft$$

and

$$p_x = 45.714\ MPa \qquad p_y = 105.714\ MPa \qquad p_z = 51.429\ MPa$$

Substituting the above values into Eq. (1.41), we obtain

$$\tau = 73.582\ MPa \quad \blacktriangleleft$$

SOLUTION (1.72)

See: Hint, Prob. 1.70:

$$l = \frac{2}{\sqrt{3^2+1^2+2^2}} = \frac{2}{\sqrt{14}} \qquad m = \frac{1}{\sqrt{14}} \qquad n = -\frac{3}{\sqrt{14}}$$

Equation (1.40) gives

$$\sigma = 20(\tfrac{2}{\sqrt{14}})^2 + 30(\tfrac{1}{\sqrt{14}})^2 + 50(-\tfrac{3}{\sqrt{14}})^2$$
$$+ 2[10(\tfrac{2}{\sqrt{14}})(\tfrac{1}{\sqrt{14}}) - 10(\tfrac{2}{\sqrt{14}})(-\tfrac{3}{\sqrt{14}})] = 51.43\ MPa \quad ◄$$

Equation (1.41):

$$\tau = \{[20(\tfrac{2}{\sqrt{14}}) + 10(\tfrac{1}{\sqrt{14}}) - 10(-\tfrac{3}{\sqrt{14}})]^2$$
$$+ [10(\tfrac{2}{\sqrt{14}}) + 30(\tfrac{1}{\sqrt{14}}) + 0]^2 + [-10(\tfrac{2}{\sqrt{14}}) + 0 + 50(-\tfrac{3}{\sqrt{14}})]^2 - 51.34^2\}^{\frac{1}{2}}$$
$$= 7.413\ MPa \quad ◄$$

SOLUTION (1.73)

Direction cosines are

$$l = \cos 35^o = 0.8192 \qquad m = \cos 60^o = 0.5 \qquad n = \cos 73.6^o = 0.2823$$

Equation (1.40) results in

$$\sigma = 60(0.8192)^2 - 40(0.5)^2 + 30(0.2823)^2$$
$$+ 2[20(0.8192)(0.5) - 5(0.5)(0.2823) + 10(0.8192)(0.2823)]$$
$$= 52.25\ MPa \quad ◄$$

Equations (1.26):

$$p_x = 60(0.8192) + 20(0.5) + 10(0.2823) = 61.9725\ MPa$$
$$p_y = 20(0.8192) - 40(0.5) - 5(0.2823) = -5.0287\ MPa$$
$$p_z = 10(0.8192) - 5(0.5) + 30(0.2823) = 14.1618\ MPa$$

Equation (1.41) is thus

$$\tau = [(61.9725)^2 + (-5.0287)^2 + (14.1618)^2 - (52.25)^2]^{\frac{1}{2}}$$
$$= 36.56\ MPa \quad ◄$$

SOLUTION (1.74)

Direction cosines are

$$l = \cos 40^o = 0.766 \qquad m = \cos 75^o = 0.259 \qquad n = \cos 54^o = 0.588$$

(CONT.)

1.74 (CONT.)

Equation (1.40):

$$\sigma = 40(0.766)^2 + 20(0.259)^2 + 20(0.588)^2$$
$$+ 2[40(0.766)(0.259) + 0 + 30(0.766)(0.588)]$$
$$= 23.47 + 1.34 + 6.91 + 42.9$$
$$= 74.62 \ MPa \quad ◄$$

Equation (1.41) gives

$$\tau = \{[40(0.766) + 40(0.259) - 30(0.588)]^2$$
$$+ [40(0.766) + 20(0.259) + 0]^2 + [30(0.766) + 0 + 20(0.588)]^2 - 74.62^2\}^{\frac{1}{2}}$$
$$= [3436.3 + 1282.9 + 1206.7 - 5568.1]^{\frac{1}{2}}$$
$$= 18.93 \ MPa \quad ◄$$

SOLUTION (1.75)

Note: Planes of maximum shear stresses can be determined upon following a procedure similar to that used in Solution of Prob. 1.56.

(a) From Problem 1.69a:

$$\sigma_1 = 12.049 \ MPa \qquad \sigma_2 = -1.521 \ MPa \qquad \sigma_3 = -4.528 \ MPa$$

Thus,
$$(\tau_{13})_{\max} = \tfrac{1}{2}(\sigma_1 - \sigma_3) = 8.288 \ MPa$$
$$(\tau_{12})_{\max} = \tfrac{1}{2}(\sigma_1 - \sigma_2) = 6.785 \ MPa \quad ◄$$
$$(\tau_{23})_{\max} = \tfrac{1}{2}(\sigma_2 - \sigma_3) = 1.503 \ MPa$$

(b) From Problem 1.69b:

$$\sigma_1 = 19.237 \ MPa \qquad \sigma_2 = 13.704 \ MPa \qquad \sigma_3 = 4.648 \ MPa$$

Thus,
$$(\tau_{13})_{\max} = \tfrac{1}{2}(\sigma_1 - \sigma_3) = 7.294 \ MPa$$
$$(\tau_{12})_{\max} = \tfrac{1}{2}(\sigma_1 - \sigma_2) = 2.766 \ MPa \quad ◄$$
$$(\tau_{23})_{\max} = \tfrac{1}{2}(\sigma_2 - \sigma_3) = 4.528 \ MPa$$

SOLUTION (1.76)

We have

$$\sigma_1 = 48 \ MPa \qquad \sigma_2 = 36 \ MPa \qquad \sigma_3 = -72 \ MPa$$

(a) From Eqs. (1.43) and (1.44):

$$\tau_{oct} = \tfrac{1}{3}[(48 - 36)^2 + (36 + 72)^2 + (-72 - 48)^2]^{\frac{1}{2}} = 53.96 \ MPa \quad ◄$$
$$\sigma_{oct} = \tfrac{1}{3}(48 + 36 - 72) = 4 \ MPa$$

(b) Using Eq. (1.45),

$$\tau_{\max} = \tfrac{1}{2}(48 - 36) = 6 \ MPa \quad ◄$$

SOLUTION (1.77)

(a) $$\begin{vmatrix} (-100-\sigma_p) & 0 & -80 \\ 0 & (20-\sigma_p) & 0 \\ -80 & 0 & (20-\sigma_p) \end{vmatrix} = 0$$

Expanding,

$$(20-\sigma)[(\sigma+100)(\sigma-20)-6400]=0$$

and

$$\sigma_1 = 60 \ MPa, \qquad \sigma_2 = 20 \ MPa, \qquad \sigma_3 = -140 \ MPa \qquad ◀$$

(b) Apply Eqs. (1.43), (1.44), and (1.45):

$$\tau_{oct} = \tfrac{1}{3}[(60-20)^2 + (20+140)^2 + (-140-60)^2]^{\frac{1}{2}} = 86.41 \ MPa$$
$$\sigma_{oct} = \tfrac{1}{3}(60+20-140) = -20 \ MPa \qquad ◀$$
$$\tau_{\max} = \tfrac{1}{2}(60+140) = 100 \ MPa$$

SOLUTION (1.78)

Octahedral and shearing stresses are given by

$$\tau_{oct}^2 = \tfrac{1}{9}[(\sigma_1-\sigma_2)^2 + (\sigma_2-\sigma_3)^2 + (\sigma_3-\sigma_1)^2]$$
$$\tau_{\max}^2 = \tfrac{1}{4}(\sigma_1-\sigma_3)^2$$

Let us say, $\tau_{\max}^2 > \tau_{oct}^2$. Then

$$(\tfrac{\sigma_1-\sigma_3}{2})^2 > \tfrac{1}{9}[(\sigma_1-\sigma_2)^2 + (\sigma_2-\sigma_3)^2 + (\sigma_3-\sigma_1)^2]$$

or

$$\tfrac{9}{4}(\sigma_1-\sigma_3) > [(\sigma_1-\sigma_2)^2 + (\sigma_2-\sigma_3)^2 + (\sigma_3-\sigma_1)^2]$$

Subtracting $(\sigma_1-\sigma_3)^2$ from both sides and noting that $(\sigma_1-\sigma_3)^2 = (\sigma_3-\sigma_1)^2$,

we have

$$\tfrac{5}{4}(\sigma_1-\sigma_3)^2 > (\sigma_1-\sigma_3)^2 + (\sigma_2-\sigma_3)^2$$

But

$$(\sigma_1-\sigma_3)^2 > (\sigma_1-\sigma_2)^2 + (\sigma_2-\sigma_3)^2$$

Thus,

$$\tfrac{5}{4}(\sigma_1-\sigma_3)^2 > (\sigma_1-\sigma_2)^2 + (\sigma_2-\sigma_3)^2 \qquad \text{(a)}$$

The squares of the difference between σ_1 and σ_3 will always be greater then the sum of the squares of the difference between σ_1 and σ_2, σ_2 and σ_3, since $\sigma_1 > \sigma_2 > \sigma_3$. Hence, Eq. (a) is true and our assumption is correct. That is

$$\tau_{\max} > \tau_{oct} \qquad ◀$$

SOLUTION (1.79)

From Solution of Problem 1.64:

$$\sigma_1 = 24.747\ MPa \qquad \sigma_2 = 8.48\ MPa \qquad \sigma_3 = 2.773\ MPa$$

Applying Eqs. (1.43) and (1.44):

$$\tau_{oct} = \tfrac{1}{3}[(24.747-8.48)^2 + (8.48-2.773)^2 + (2.773-24.747)^2]^{\frac{1}{2}}$$
$$= 9.31\ MPa \qquad ◀$$

and $\sigma_{oct} = \tfrac{1}{3}(24.747+8.48+2.773) = 12\ MPa$ ◀

Therefore $p_{oct} = (9.31^2 + 12^2)^{\frac{1}{2}} = 15.19\ MPa$

SOLUTION (1.80)

Shearing stress, in terms of principal stresses, is given by

$$\tau^2 = \sigma_1^2 l^2 + \sigma_2 m^2 + \sigma_3^2 n^2 - (\sigma_1 l^2 + \sigma_2 m^2 + \sigma_3 n^2)^2 \qquad \text{(a)}$$

We substitute $n^2 = 1 - m^2 - l^2$ into Eq. (a), calculate its derivatives with respect to l and m, and equate these derivatives to zero:

$$\tfrac{\partial\tau}{\partial l} = l[(\sigma_1-\sigma_3)l^2 + (\sigma_2-\sigma_3)m^2 - \tfrac{1}{2}(\sigma_3-\sigma_1)] = 0 \qquad \text{(b)}$$

$$\tfrac{\partial\tau}{\partial m} = m[(\sigma_1-\sigma_3)l^2 + (\sigma_2-\sigma_3)m^2 - \tfrac{1}{2}(\sigma_2-\sigma_3)] = 0 \qquad \text{(c)}$$

One solution is $l = m = 0$. Solutions for the direction cosines of planes for which τ is a maximum of minimum can also be found as follows.

Take $l = 0$: Eq. (c) gives $m = \pm\sqrt{1/2}$

Take $m = 0$: Eq. (c) gives $l = \pm\sqrt{1/2}$

There are, in general, no solutions of Eqs. (b) and (c) in which l and m are both different from zero, for this case the expressions in brackets cannot both vanish.

By the above procedure we can form the following table.

Direction cosines for planes of $\tau_{\max}$ and $\tau_{\min}$

$l =$	0	0	± 1	0	$\pm\sqrt{1/2}$	$\pm\sqrt{1/2}$
$m =$	0	± 1	0	$\pm\sqrt{1/2}$	0	$\pm\sqrt{1/2}$
$n =$	± 1	0	0	$\pm\sqrt{1/2}$	$\pm\sqrt{1/2}$	0

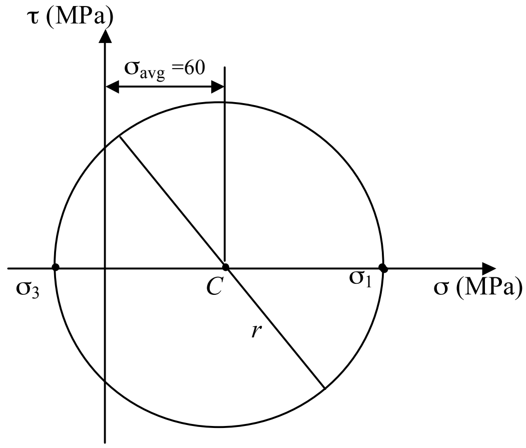
The first three columns define the planes for $\tau_{\min}$, where $\tau = 0$. The last three columns give planes through each principal axes bisecting the angles between the two other principal axes. Substituting the latter direction cosines into Eq. (a), we have

$$(\tau_{23})_{\max} = \pm\tfrac{\sigma_2-\sigma_3}{2} \qquad (\tau_{13})_{\max} = \pm\tfrac{\sigma_1-\sigma_3}{2}$$
$$(\tau_{12})_{\max} = \pm\tfrac{\sigma_1-\sigma_2}{2} \qquad ◀$$

Similarly, introducing the direction cosines given in the above table into Eq. (1.37), we obtain the normal stresses associated with the maximum shearing stresses:

$$\sigma_{12}' = \tfrac{\sigma_1+\sigma_2}{2} \qquad \sigma_{13}' = \tfrac{\sigma_1+\sigma_3}{2} \qquad \sigma_{23}' = \tfrac{\sigma_2+\sigma_3}{2} \qquad ◀$$

SOLUTION (1.81)



$$\sigma_{avg} = \frac{1}{2}(\sigma_x + \sigma_y)$$
$$= \frac{1}{2}(100 + 20) = 60\ MPa$$
$$r = \sqrt{(\frac{\sigma_x - \sigma_y}{2})^2 + \tau_{xy}^2}$$
$$= \sqrt{(\frac{100 - 20}{2})^2 + 60^2}$$
$$= 72.1\ MPa$$

$$\sigma_1 = \sigma_{avg} + R = 132.1\ MPa$$
$$\sigma_3 = -12.1\ MPa$$

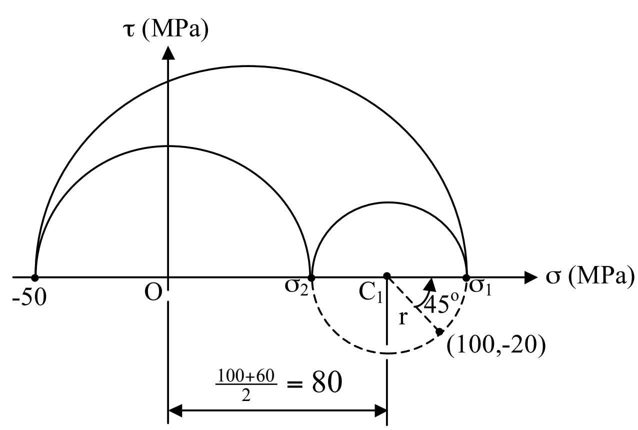
(**a**) $\sigma_2 = 30\ MPa \qquad \sigma_1 = 132.1\ MPa \qquad \sigma_3 = -12.1\ MPa$

$$(\tau_{\max})_a = \frac{1}{2}(\sigma_1 - \sigma_3)$$
$$= \frac{1}{2}[132.1 - (-12.1)] = 72.1\ MPa \quad \blacktriangleleft$$

(**b**) $\sigma_3 = -30\ MPa \qquad \sigma_1 = 132.1\ MPa \qquad \sigma_2 = -12.1\ MPa$

$$(\tau_{\max})_a = \frac{1}{2}(\sigma_1 - \sigma_3)$$
$$= \frac{1}{2}[132.1 - (-30)] = 81\ MPa \quad \blacktriangleleft$$

SOLUTION (1.82)



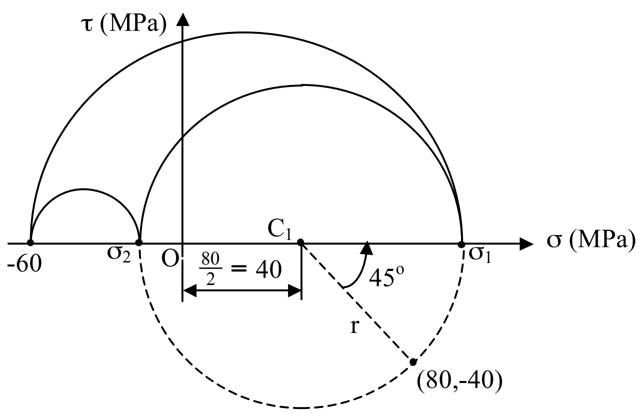
(CONT.)

1.82 (CONT.)

From Mohr's circle, we have

$$r = \sqrt{20^2 + 20^2} = 28.3\ MPa$$
$$\sigma_1 = 108\ MPa \qquad \sigma_2 = 51.7\ MPa$$
$$\sigma_3 = -50\ MPa \qquad \theta_p' = 22.5^o$$
◀

SOLUTION (1.83)

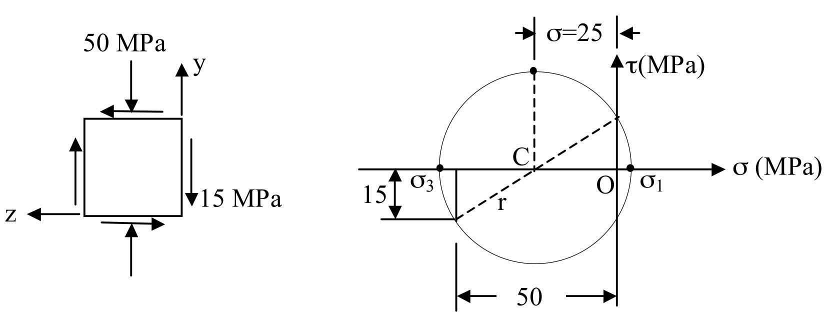


Referring to Mohr's circle:

$$r = \sqrt{40^2 + 40^2} = 56.57\ MPa$$
$$\theta_p' = 22.5^o$$
$$\sigma_1 = 96.57\ MPa \qquad \sigma_2 = 16.57\ MPa$$
◀

SOLUTION (1.84)

(a) In the yz plane:



(CONT.)

1.84 (CONT.)

We have

$$r = \sqrt{25^2 + 15^2} = 29.16 \ MPa$$

Thus

$$\sigma_1 = r - \sigma = 29.16 - 25 = 4.16 \ MPa$$
$$\sigma_3 = -r - \sigma = -29.16 - 25 = -54.16 \ MPa \quad ◄$$
$$\sigma_2 = -20 \ MPa$$

(b) Using Eqs. (1.43) and (1.44):

$$\tau_{oct} = \tfrac{1}{3}[(4.16+20)^2 + (-20+54.16)^2 + (-54.16-4.16)^2]^{\frac{1}{2}}$$
$$= 23.92 \ MPa \quad ◄$$
$$\sigma_{oct} = \tfrac{1}{3}(4.16 - 20 - 54.16) = -23.33 \ MPa \quad ◄$$

From Eq. (1.45),

$$\tau_{\max} = \tfrac{1}{2}(4.16 + 54.16) = 29.16 \ MPa \quad ◄$$

SOLUTION (1.85)

It is noted that $l^2 + m^2 + n^2 = 1$.

Applying Eq. (1.37), we have

$$\sigma = 35(\tfrac{3}{13}) - 14(\tfrac{1}{13}) - 28(\tfrac{9}{13}) = -12.39 \ MPa \quad ◄$$

Equation (1.39), substituting the given data and the direction cosines determined above, gives

$$\tau = 26.2 \ MPa \quad ◄$$

Surface tractions. Equations (1.48) give

$$p_x = \sigma_1 l = 35(\sqrt{\tfrac{3}{13}}) = 16.81 \ MPa$$
$$p_y = \sigma_2 m = -14(\sqrt{\tfrac{1}{13}}) = -3.88 \ MPa \quad ◄$$
$$p_z = \sigma_3 n = -28(\sqrt{\tfrac{9}{13}}) = -23.30 \ MPa$$

Check: $p^2 = p_x^2 + p_y^2 + p_z^2 = \sigma^2 + \tau^2 = 840 \ (MPa)^2$

Observe that Approach I is more conveniently leads to results.

SOLUTION (1.86)

$$\sigma_{oct} = \tfrac{1}{3}(40 + 25 + 15) = 26.667 \ MPa \quad ◄$$
$$\tau_{oct} = \tfrac{1}{3}[(40-25)^2 + (25-15)^2 + (15-40)^2]^{\frac{1}{2}} = 10.274 \ MPa \quad ◄$$

(CONT.)

1.86 (CONT.)

We have $l = m = n = 1/\sqrt{3}$. Note that $l^2 + m^2 + n^2 = 1$. ◄

Surface tractions. Equations (1.48) give

$$p_x = \sigma_1 l = 40(\tfrac{1}{\sqrt{3}}) = 23.09 \ MPa$$

$$p_y = \sigma_2 m = 25(\tfrac{1}{\sqrt{3}}) = 14.43 \ MPa$$

$$p_z = \sigma_3 n = 15(\tfrac{1}{\sqrt{3}}) = 8.66 \ MPa$$

Check:

$$p^2 = p_x^2 + p_y^2 + p_z^2 = \sigma^2 + \tau^2 = 816.7 \ (MPa)^2$$

End of Chapter 1

CHAPTER 2

SOLUTION (2.1)

(a) Yes. ◀
Eqs. (2.12) are satisfied.

(b) No. ◀
Eqs. (2.12) are not satisfied.

SOLUTION (2.2)

Apply Eqs. (2.4):

$$\varepsilon_x = 2c \qquad \varepsilon_y = -6cy \qquad \gamma_{xy} = 2c(x+y)$$

(a) $u_{AB} = \int_1^3 \varepsilon_x dx = 4c = 0.4\ mm$

$$v_{AD} = \int_{1/2}^{2} \varepsilon_y dy = -6c\frac{y^2}{2}\Big|_{1/2}^{2} = -11.25c = -1.125\ mm$$

Thus,

$$L_{A'B'} = 2000.4\ mm \qquad L_{A'D'} = 1498.875\ mm$$ ◀

(b) $\gamma_{xy} = 2c(1+\frac{1}{2}) = 300\ \mu$ ◀

(c) We have

$$u_A = c(2\times 1 + \tfrac{1}{4}) = 2.25c \qquad v_A = c(1^2 - 3\times\tfrac{1}{4}) = 0.25c$$

and

$$x_{A'} = 1 + 2.25c = 1000.225\ mm$$
$$y_{A'} = 0.5 + 0.25c = 500.025\ mm$$ ◀

SOLUTION (2.3)

Equations (2.4), for the given displacement field, yield $[\varepsilon_{ij}]$:

$$\begin{bmatrix} 2x & 0 & -y/2 \\ 0 & 2z & (2y-x)/2 \\ -y/2 & (2y-x)/2 & 2z \end{bmatrix} c$$

At point (0,2,1), we have $[\varepsilon_{ij}]$:

$$\begin{bmatrix} 0 & 0 & -100 \\ 0 & 200 & 200 \\ -100 & 200 & 200 \end{bmatrix} \mu$$ ◀

SOLUTION (2.4)

First two of Eqs. (2.4) give

$$\varepsilon_x = 2a_0xy^2 + a_1y^2 + 2a_2xy$$
$$\varepsilon_y = b_0x^2 + b_1x$$

Equation (2.11):

$$(4a_0 + 2a_1) + (2b_0) = 2c_0x + c_1$$

or

$$2(2a_0 - c_0)x + 2(a_1 + b_0) - c_1 = 0$$

This is satisfied if $x \neq 0$:

$$2a_0 - c_0 = 0, \qquad c_0 = 2a_0 \qquad ◀$$
$$2(a_1 + b_0) - c_1 = 0, \qquad c_1 = 2(a_1 + b_0)$$

SOLUTION (2.5)

Equation (2.11) yields

$$2a_1 + 12y^2 + 2b_1 + 12x^2 = 3c_1(x^2 + y^2) + c_1c_2$$

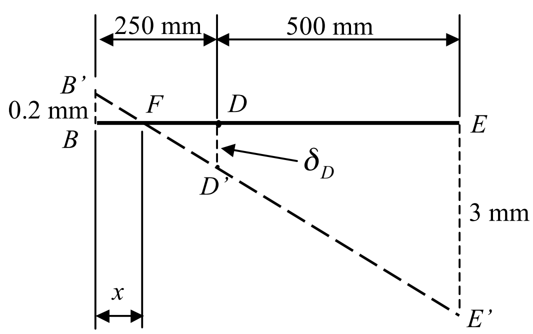
Solving,

$$c_1 = 4 \qquad c_2 = \tfrac{1}{2}(a_1 + b_1) \qquad ◀$$

SOLUTION (2.6)

The change in length of bar AB is

$$\delta_{AB} = \varepsilon_{AB}L_{AB} = (-500\times10^{-6})(400) = -0.2\ mm$$



From triangles $B'BF$ and $EE'F$:

$$\frac{0.2}{x} = \frac{3}{750 - x}, \qquad x = 117.2\ mm$$

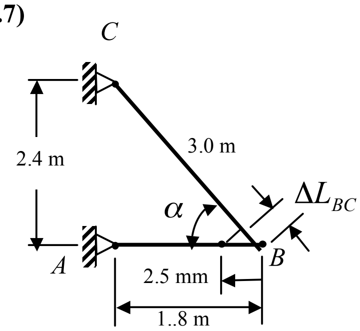
From triangles $DD'F$ and $EE'F$:

$$\frac{132.8}{\delta_D} = \frac{632.8}{3}, \qquad \delta_D = 0.63\ mm$$

Thus

$$\varepsilon_{CD} = \frac{\delta_D}{L_{CD}} = \frac{0.63}{500} = 1260\ \mu \qquad ◀$$

SOLUTION (2.7)



$$\varepsilon_{AB} = \frac{\Delta L_{AB}}{L_{AB}} = -\frac{2.5}{1800}$$
$$= -1389\ \mu$$

$$\varepsilon_{BC} = \frac{-\Delta L_{BC}}{L_{BC}} = \frac{-0.0025\cos\alpha}{3.0}$$

$$= \frac{-0.0025(1.8/3.0)}{3.0} = -500\ \mu \quad ◄$$

SOLUTION (2.8)

(**a**) $$\varepsilon_x = \frac{2.8}{2000} = 1400\ \mu$$
$$\varepsilon_y = \frac{-1.3}{1000} = -1300\ \mu \quad ◄$$
$$\gamma_{xy} = 0$$

(**b**) $$ACB = 90^o$$
$$A'C'B' = 2\tan^{-1}\frac{1.0014}{0.9987} = 90.1547^o$$
$$\gamma = 90^o - 90.1547^o = -0.1547^o = -2.7\times10^{-3}\ rad$$
$$= -2700\ \mu \quad ◄$$

SOLUTION (2.9)

(a) Equations (2.4) give
$$\varepsilon_x = \tfrac{\partial u}{\partial x} = \tfrac{0.175-0.075}{150} = 667\ \mu \quad ◄$$
$$\varepsilon_y = \tfrac{\partial v}{\partial y} = \tfrac{0.025-(-0.05)}{100} = 750\ \mu \quad ◄$$
and
$$\gamma_{xy} = \tfrac{0-0.075}{100} + \tfrac{[-0.125-(-0.05)]}{150} = -1250\ \mu \quad ◄$$

(b) Equation (2.16) is therefore
$$\varepsilon_{1,2} = \tfrac{667+750}{2} \pm [(\tfrac{667-750}{2})^2 + 625^2]^{\frac{1}{2}}$$
or
$$\varepsilon_1 = 1335\ \mu \qquad \varepsilon_2 = 82\ \mu \quad ◄$$
When $\theta_p = \frac{1}{2}\tan^{-1}\frac{-1250}{667-750} = 43.1^o$

and stresses are substituted into Eq. (2.14a), we obtain $\varepsilon_{x'} = 82\ \mu$. Thus,
$$\theta_p'' = 43.1^o \quad ◄$$

SOLUTION (2.10)

Use Eq. (2.16) with the strains obtained in Example 2.1:

$$\varepsilon_{1,2} = \tfrac{1250-2000}{2} \pm [(\tfrac{1250+2000}{2})^2 + 750^2]^{\frac{1}{2}}$$

or $\varepsilon_1 = 1415\ \mu \qquad \varepsilon_2 = -2165\ \mu$ ◄

Apply Eq. (2.15):

$$\theta_p = \tfrac{1}{2}\tan^{-1}\tfrac{1500}{1250-2000} = 12.39^o$$

Substituting this angle and the stresses, Eq. (2.14a), gives $1415\ \mu$.
Thus,

$$\theta_p' = 12.39^o$$ ◄

SOLUTION (2.11)

We have, using Eqs. (2.3):

$$\varepsilon_x = \tfrac{\partial u}{\partial x} = \tfrac{0.004}{20} = 200\ \mu$$

$$\varepsilon_y = \tfrac{\partial v}{\partial y} = \tfrac{-0.003}{12} = -250\ \mu$$

and

$$\gamma_{xy} = \tfrac{\partial v}{\partial x} + \tfrac{\partial u}{\partial y} = -1000 - 500 = -1500\ \mu$$

Then, Eq. (2.16) yields

$$\varepsilon_{1,2} = \tfrac{200-250}{2} \pm [(\tfrac{200+250}{2})^2 + (-750)^2]^{\frac{1}{2}}$$

or

$$\varepsilon_1 = 758\ \mu \qquad \varepsilon_2 = -808\ \mu$$ ◄

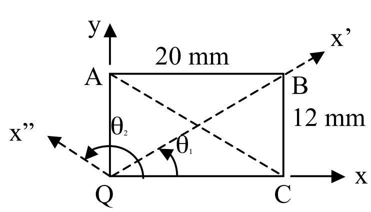
Apply Eq. (2.15):

$$\theta_p = \tfrac{1}{2}\tan^{-1}\tfrac{-1500}{200+250} = 36.65^o$$

For this angle, Eq. (2.14a) gives $\varepsilon_{x'} = 758\ \mu$. Therefore,

$$\theta_p' = -36.65^o$$ ◄

SOLUTION (2.12)



We have, from geometry:

$$AC = QB = 23.32\ mm$$

$$\theta_1 = 30.96^o \qquad \theta_2 = 149.04^o$$

Equation (2.14a) leads to

$$\varepsilon_{x'} = \tfrac{300+500}{2} + \tfrac{300-500}{2}\cos 61.92^o + 100\sin 61.92^o = 441\ \mu$$

(CONT.)

Thus,

$$\Delta_{QB} = 441\,\mu(23.32) = 0.01\ mm \qquad \blacktriangleleft$$

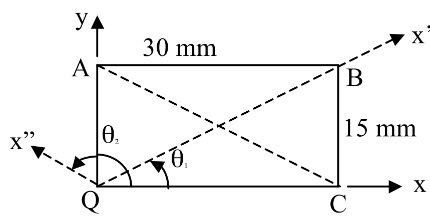
Similarly,

$$\varepsilon_{x''} = 400 - 100\cos 298.08^o + 100\sin 298.08^o = 265\ \mu$$

and

$$\Delta_{AC} = 265\,\mu(23.32) = 0.006\ mm \qquad \blacktriangleleft$$

SOLUTION (2.13)



We find, from geometry:

$$AC = QB = 33.54\ mm$$

$$\theta_1 = 26.56^o \qquad \theta_2 = 153.44^o$$

Equation (2.14a) gives

$$\varepsilon_{x'} = \tfrac{400+200}{2} + \tfrac{400-200}{2}\cos 53.12^o + 150\sin 53.12^o = 240\ \mu$$

Hence,

$$\Delta_{QB} = 240\,\mu(33.54) = 0.008\ mm \qquad \blacktriangleleft$$

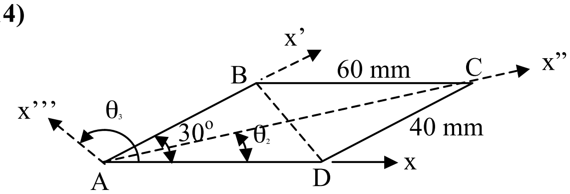
In a like manner,

$$\varepsilon_{x''} = 300 + 100\cos 306.88^o - 150\sin 306.88^o = 480\ \mu$$

and

$$\Delta_{AC} = 480\,\mu(33.54) = 0.016\ mm \qquad \blacktriangleleft$$

SOLUTION (2.14)



We obtain, from geometry:

$$AC = 96.73\ mm \qquad BD = 32.29\ mm$$

$$\theta_2 = 11.93^o \qquad \theta_3 = 141.73^o$$

From Solution of Problem 1.42:

$$\sigma_x = 3.464\tau_0 \qquad \sigma_y = 0 \qquad \tau_{xy} = \tau_0$$

Thus,

(CONT.)

2.14 (CONT.)

$$\varepsilon_x = \frac{3.464(70\times10^6)}{200(10^9)} = 1212\ \mu$$

$$\varepsilon_y = -0.3(1212) = -364\ \mu$$

$$\gamma_{xy} = \frac{2(1+0.3)(70\times10^6)}{200(10^9)} = 910\ \mu$$

(a) $$\varepsilon_{x'} = \frac{1212-364}{2} + \frac{1212+364}{2}\cos 60^o + 455\sin 60^o = 1212\ \mu$$

$$\Delta_{AB} = 1212\,\mu(40) = 0.05\ mm$$ ◀

(b) $$\varepsilon_{x''} = 424 + 788\cos 23.86^o + 455\sin 23.86^o = 1328.7\ \mu$$

$$\Delta_{AC} = 1328.7\,\mu(96.73) = 0.13\ mm$$ ◀

Similarly,

$$\varepsilon_{x'''} = 424 + 788\cos 283.46^o + 455\sin 283.46^o = 164.92\ \mu$$

$$\Delta_{BD} = 164.92(32.29) = 0.005\ mm$$ ◀

(c) $$\varepsilon_{1,2} = \frac{1212-364}{2} \pm [(\frac{1212+364}{2})^2 + 455^2]^{\frac{1}{2}}$$

or

$$\varepsilon_1 = 1334\ \mu \qquad \varepsilon_2 = 486\ \mu$$ ◀

Substitution of

$$\theta_p' = \tfrac{1}{2}\tan^{-1}\frac{910}{1212+364} = 15^o$$ ◀

into Eq. (2.14a) yields $1334\ \mu$.

SOLUTION (2.15)

(a) We have

$$\gamma_{max} = 400 - 200 = 200\ \mu$$ ◀

Maximum shearing strain occurs on a plane oriented at 45° from the plane of principal strains.

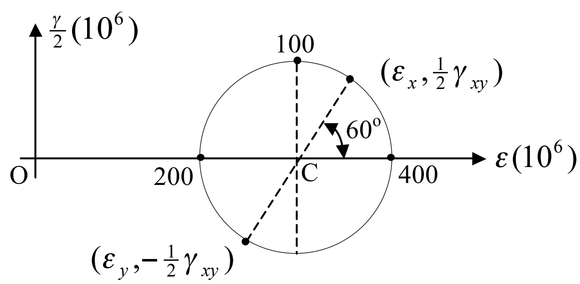


Figure (a)

(b) From Mohr's circle Fig. (a):

$$\varepsilon_x = 300 + 100\cos 60^o = 350\ \mu$$

$$\varepsilon_x = 300 - 100\cos 60^o = 250\ \mu$$ ◀

$$\gamma_{xy} = -(400-200)\sin 60^o = -173\ \mu$$

SOLUTION (2.16)

We have $u_B = 3\ mm$ and $v_B = 1.5\ mm$.

(a) Take: $u = c_1 xy \qquad v = c_2 xy$

Hence,

$$3(10^{-3}) = c_1(3\times 2), \qquad c_1 = 500(10^{-6})$$
$$1.5(10^{-3}) = c_2(3\times 2), \qquad c_2 = 250(10^{-6})$$

Thus,

$$u = 500(10^{-6})xy \qquad v = 250(10^{-6})xy \qquad ◀$$

(b) Using Eqs. (2.3),

$$\varepsilon_x = 500\,\mu y \qquad \varepsilon_y = 250\,\mu x \qquad \gamma_{xy} = 250\,\mu(2x+y)$$

which satisfy Eq. (2.11): the strain field is possible.

At point B, we thus have

$$(\varepsilon_x)_B = 1000\,\mu \qquad (\varepsilon_y)_B = 750\,\mu \qquad (\gamma_{xy})_B = 2000\,\mu \qquad ◀$$

(c) We obtain $\theta = \tan^{-1}\frac{2}{3} = 33.69^o$.

Equation (2.14a) is therefore

$$\varepsilon_{x'} = 875 + 125\cos 67.38^o + 1000\sin 67.38^o = 1846\ \mu$$

SOLUTION (2.17)

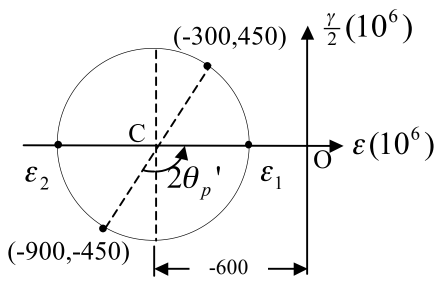


Figure (a)

Refer to Mohr's circle (Fig. a):

$$\varepsilon_{1,2} = -600 \pm [(300)^2 + (450)^2]^{\frac{1}{2}}$$

or

$$\varepsilon_1 = -59\ \mu \qquad \varepsilon_2 = -1141\ \mu \qquad ◀$$

and

$$2\theta_p'' = \tan^{-1}\frac{450(2)}{-(900-300)} = -56.31^o$$

or

$$\theta_p' = 61.85^o \qquad ◀$$

SOLUTION (2.18)

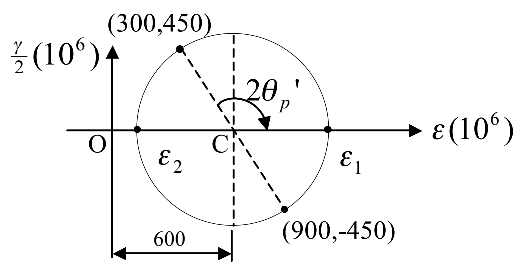


Figure (a)

Referring to Mohr's circle shown in Fig. (a), we obtain

$$\varepsilon_{1,2} = 600 \pm [(300)^2 + (450)^2]^{\frac{1}{2}}$$

from which

$$\varepsilon_1 = 1141\ \mu \qquad \varepsilon_2 = 59\ \mu$$ ◄

and

$$2\theta_p'' = \tan^{-1}\tfrac{450(2)}{900-300} = 56.31^o$$

or

$$\theta_p' = -61.85^o$$ ◄

SOLUTION (2.19)

Referring to Fig. P2.19,

$$u_A = -0.0005 = c_1(3\times1\times2), \qquad c_1 = -83.3(10^{-6})$$

$$v_A = 0.0003 = c_2(6), \qquad c_2 = 50(10^{-6})$$

$$w_A = -0.0006 = c_3(6), \qquad c_3 = -100(10^{-6})$$

Hence,

$$u = -83.3(10^{-6})xyz \qquad v = 50(10^{-6})xyz \qquad w = -100(10^{-6})xyz$$

(a) Using Eqs. (2.4), we thus have

$$\varepsilon_x = -83.3\,\mu yz \qquad \varepsilon_y = 50\,\mu xz \qquad \varepsilon_z = -100\,\mu xy$$

$$\gamma_{xy} = (-83.3xz + 50yz)\ \mu \qquad \gamma_{xz} = (-100yz - 83.4xy)\ \mu$$

$$\gamma_{yz} = (50xy - 100xz)\ \mu$$

This forgoing expressions satisfy Eqs. (2.12): the strain field is possible.

Substitute x=-3 m, y=1 m, and z=2 m into the above equations to obtain strains at A, $[\varepsilon_{ij}]$:

$$\begin{bmatrix} -167 & -200 & -225 \\ -200 & 300 & -225 \\ -225 & -225 & -300 \end{bmatrix} \mu$$ ◄

(CONT.)

(b) Let, for example, x'-axis lie along the line from A to B (Fig. P2.19). The direction of cosines of AB are

$$l_1 = -\tfrac{3}{\sqrt{13}} \qquad m_1 = 0 \qquad n_1 = -\tfrac{2}{\sqrt{13}}$$

Thus, Eq. (2.18a):

$$\varepsilon_{x'} = \varepsilon_x l_1^2 + \varepsilon_z n_1^2 + \gamma_{xz} l_1 n_1$$
$$= -167(\tfrac{9}{13}) - 300(\tfrac{4}{13}) - 450(\tfrac{-3}{\sqrt{13}})(\tfrac{-2}{\sqrt{13}})$$
$$= -416\ \mu \qquad ◄$$

(c) Let y'-axis be placed along AC (Fig. P2.19). The direction cosines of AC are

$$m_2 = -1 \qquad l_2 = 0 \qquad n_2 = 0$$

Equation (2.18b) is therefore

$$\gamma_{x'y'} = \gamma_{xy} l_1 m_2 + \gamma_{yz} n_1 m_2$$
$$= -400(\tfrac{-3}{\sqrt{13}})(-1) - 450(\tfrac{-2}{\sqrt{13}})(-1)$$
$$= -582\ \mu \qquad ◄$$

Negative sign shows that the angle BAC has increased.

SOLUTION (2.20)

We now have (Fig. P2.19):

$$u_A = 0.0006 = c_1(3\times1\times2), \qquad c_1 = 100(10^{-6})$$
$$v_A = -0.0003 = c_2(6), \qquad c_2 = -50(10^{-6})$$
$$w_A = -0.0004 = c_3(6), \qquad c_3 = -66.7(10^{-6})$$

Hence,

$$u = 100(10^{-6})xyz \qquad v = -50(10^{-6})xyz \qquad w = -66.7(10^{-6})xyz$$

(a) Applying Eqs. (2.4), we obtain

$$\varepsilon_x = 100\,\mu yz \qquad \varepsilon_y = -50\,\mu xz \qquad \varepsilon_z = -66.7\,\mu xy$$
$$\gamma_{xy} = (100xz - 50yz)\ \mu \qquad \gamma_{xz} = (-66.7yz + 100xy)\ \mu$$
$$\gamma_{yz} = (100xy - 66.7xz)\ \mu$$

These expressions satisfy Eqs. (2.12): the strain field is possible.

Introducing x=3 m, y=1 m, and z=2 m into the above equations we find strains at A, $[\varepsilon_{ij}]$:

$$\begin{bmatrix} 200 & 250 & 83.5 \\ 250 & -300 & -275 \\ 83.5 & -275 & -200 \end{bmatrix} \mu \qquad ◄$$

(b) Let, for instance, x'-axis lie along the line from A to B (Fig. P2.19). The direction of cosines of AB are

$$l_1 = -\tfrac{3}{\sqrt{13}} \qquad m_1 = 0 \qquad n_1 = -\tfrac{2}{\sqrt{13}}$$

(CONT.)

Therefore, Eq. (2.18a):

$$\varepsilon_{x'} = \varepsilon_x l_1^2 + \varepsilon_z n_1^2 + \gamma_{xz} l_1 n_1$$
$$= 200(\tfrac{9}{13}) - 200(\tfrac{4}{13}) - 167(\tfrac{-3}{\sqrt{13}})(\tfrac{-2}{\sqrt{13}})$$
$$= -154\ \mu \quad ◄$$

(c) Let y'-axis be placed along (Fig. P2.19). The direction cosines of AC are

$$m_2 = -1 \qquad l_2 = 0 \qquad n_2 = 0$$

Equation (2.18b) is thus

$$\gamma_{x'y'} = \gamma_{xy} l_1 m_2 + \gamma_{yz} n_1 m_2$$
$$= 500(\tfrac{-3}{\sqrt{13}})(-1) - 550(\tfrac{-2}{\sqrt{13}})(-1)$$
$$= 111\ \mu \quad ◄$$

Positive sign means that the angle BAC has decreased.

SOLUTION (2.21)

(a) Applying Eqs. (2.21),

$$J_1 = 200 - 100 - 400 = -300\ \mu \quad ◄$$
$$J_2 = (-2 - 8 + 4 - 9 - 4 - 25)(10^4) = -44(10^4)\ (\mu)^2$$

and

$$J_3 = \begin{vmatrix} 200 & 300 & 200 \\ 300 & -100 & 500 \\ 200 & 500 & -400 \end{vmatrix} = 58(10^6)\ (\mu)^3 \quad ◄$$

(b) Table of direction cosines:

	x	y	z
x'	$\sqrt{3}/2$	$1/2$	0
y'	$-1/2$	$\sqrt{3}/2$	0
z'	0	0	1

Thus, using Eqs. (2.18a),

$$\varepsilon_{x'} = \varepsilon_x l_1^2 + \varepsilon_y m_1^2 + \gamma_{xy} l_1 m_1$$
$$= 200(\tfrac{\sqrt{3}}{2})^2 - 100(\tfrac{1}{2})^2 + 600(\tfrac{\sqrt{3}}{2})(\tfrac{1}{2}) = 385\ \mu \quad ◄$$

(c) Use Table B.1 (with $\sigma \to \varepsilon$ and $\tau \to \gamma/2$):

$$\varepsilon_1 = 598\ \mu \qquad \varepsilon_2 = -126\ \mu \qquad \varepsilon_3 = -772\ \mu \quad ◄$$

(d) $$\gamma_{\max} = 598 + 772 = 1370\ \mu \quad ◄$$

SOLUTION (2.22)

(a) Applying Eqs. (2.21),

$$J_1 = 400 + 0 + 600 = 1(10^3)\ \mu \quad ◄$$

$$J_2 = (0 + 24 + 0 - 1 - 4 - 0)(10^4) = 19(10^4)\ (\mu)^2$$

and

$$J_3 = \begin{vmatrix} 400 & 100 & 0 \\ 100 & 0 & -200 \\ 0 & -200 & 600 \end{vmatrix} = -22(10^6)\ (\mu)^3 \quad ◄$$

(b) Using Eq. (2.18a),

$$\varepsilon_{x'} = 400(\tfrac{\sqrt{3}}{2})^2 + 200(\tfrac{\sqrt{3}}{2})(\tfrac{1}{2}) = 387\ \mu \quad ◄$$

(c) Use Table B.1 (with $\sigma \rightarrow \varepsilon$ and $\tau \rightarrow \gamma/2$):

$$\varepsilon_1 = 664\ \mu \qquad \varepsilon_2 = 416\ \mu \qquad \varepsilon_3 = -80\ \mu \quad ◄$$

(d) $$\gamma_{max} = 664 + 80 = 744\ \mu \quad ◄$$

SOLUTION (2.23)

Use Table B.1 (with $\sigma \rightarrow \varepsilon$ and $\tau \rightarrow \gamma/2$):

$$\varepsilon_1 = 1807\ \mu \qquad \varepsilon_2 = -228\ \mu \qquad \varepsilon_3 = -679\ \mu \quad ◄$$

and

$$l_1 = 0.6184 \qquad m_1 = 0.5333 \qquad n_1 = 0.5772$$

SOLUTION (2.24)

$$\varepsilon = \tfrac{\delta}{L} = \tfrac{0.10}{50} = 2000\ \mu$$

$$\sigma = \tfrac{P}{A} = \frac{16(10^3)}{(\pi/4)(0.012)^2} = 141.5\ MPa$$

(a) $$E = \tfrac{\sigma}{\varepsilon} = \frac{141.5(10^{-6})}{2000(10^{-6})} = 70.8\ GPa \quad ◄$$

(b) $$\Delta\delta = \nu\varepsilon d = 0.33(2000 \times 10^{-6})(12) = 7.92(10^{-3})\ mm \quad ◄$$

$$e = \varepsilon(1 - 2\nu) = 2000(10^{-6})(1 - 2 \times 0.33) = 6.80(10^{-4})$$

SOLUTION (2.25)

Nominal strain

$$\varepsilon_0 = \tfrac{0.025}{75} = 333\ \mu$$

(CONT.)

2.25 (CONT.)

Nominal stress

$$\sigma_0 = \frac{9(10^3)}{\pi(0.012)^2/4} = 79.577\ MPa$$

Modulus of elasticity

$$E = \frac{79.577(10^6)}{333(10^{-6})} = 238.97\ GPa \quad ◀$$

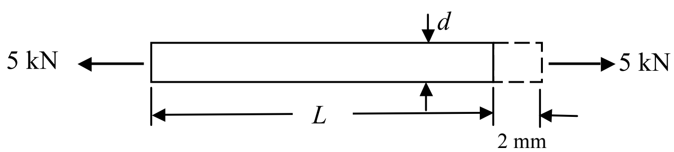
True strain

$$\varepsilon = \ln(1 + 0.000333) = 333\ \mu \quad ◀$$

True stress

$$\sigma = 79.577(1 + 0.000333) = 79.603\ MPa \quad ◀$$

SOLUTION (2.26)



$$\sigma_x = \frac{5(10^3)}{\pi d^2/4} = \frac{20(10^3)}{\pi d^2} = 160\times 10^6$$

$$d_{\min} = 0.0063\ m = 6.3\ mm \quad ◀$$

Also,

$$E = \frac{160\times 10^6}{0.002/L} = 210\times 10^9$$

or

$$L_{\min} = 2.625\ m \quad ◀$$

SOLUTION (2.27)

(**a**) Axial stress in the bar $\sigma_s = \sigma_a = \sigma$ is

$$\sigma = \varepsilon_s E_s = 600(210) = 126\ MPa$$

Hence

$$P = \sigma A = 126[\frac{\pi}{4}(40^2)] = 158.3\ kN$$

(**b**) Axial strain in aluminum equals

$$\varepsilon_a = \frac{\sigma_a}{E_a} = \frac{126(10^6)}{70(10^9)} = 1{,}800\ \mu$$

Therefore

$$\delta = \varepsilon_a L_a + \varepsilon_s L_s$$
$$= [1{,}800(0.5) + 600(1.5)]10^{-6} = 1.8\ mm$$

SOLUTION (2.28)

$$A = 2(\pi \times 40^2/4) = 2.513\pi \ mm^2$$

(**a**) $\sigma = \dfrac{P}{A} = \dfrac{600\pi(10^3)}{2513\pi(10^{-6})} = 239 \ MPa$

Since $\sigma < \sigma_{yp}$, the result is valid.

Thus, $\varepsilon = \dfrac{\sigma}{E} = \dfrac{239(10^6)}{200(10^9)} = 1195 \ \mu$

$\Delta L = L\varepsilon = 5(10^3)(1195 \times 10^{-6}) = 5.98 \ mm$ ◀

(**b**) $\varepsilon_t = -\nu\varepsilon = -0.3(1195)(10^{-6}) = -358 \ \mu$

$\Delta d = -358(10^{-6})(40) = -0.014 \ mm$ ◀

SOLUTION (2.29)

$$\varepsilon = \frac{\sigma}{E} = \frac{P}{AE} = \frac{-150(10^3)}{70(10^9)(\pi/4)(0.1^2 - 0.08^2)} = -758 \ \mu$$

(**a**) $\Delta L = \varepsilon L = -758(10^{-6})(0.4 \times 10^3) = -0.303 \ mm$ ◀

(**b**) $\Delta D = \nu(\varepsilon D) = 0.3(758)(10^{-6})100 = 0.023 \ mm$ ◀

(**c**) $\Delta t = \nu(\varepsilon t) = 0.3(758)(10^{-6})10 = 0.0023 \ mm$ ◀

SOLUTION (2.30)

(**a**) According to assumption 1, the rubber is in triaxial stress:

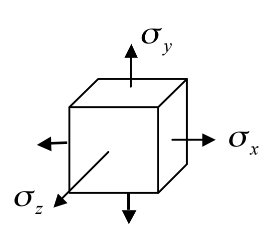
$$\sigma_x = \sigma_z = -p, \qquad \sigma_y = -\frac{Q}{\frac{\pi}{4}d^2} = -\frac{4Q}{\pi d^2}$$

Strains are: $\varepsilon_x = \varepsilon_z = 0.$ The first of Eqs. (2.34) gives

$$\varepsilon_x = 0 = \frac{1}{E}[\sigma_x - \nu(\sigma_y + \sigma_z)]$$

or

$$0 = p - \nu(-\frac{4Q}{\pi d^2} - p)$$



Solving,

$$p = \frac{4\nu Q}{\pi d^2(1-\nu)}$$ ◀

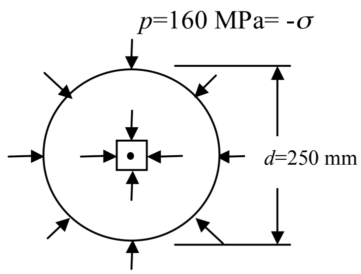
(CONT.)

2.30 (CONT.)

(**b**) Substituting the data,

$$p = \frac{4(0.3)(5\times10^3)}{\pi(0.05)^2(1-0.3)} = 1.091\ MPa\ (C) \quad ◄$$

SOLUTION (2.31)



$$\sigma_x = \sigma_y = \sigma_z = -\sigma$$

$$V_o = \frac{4}{3}\pi r^3 = \frac{4\pi}{3}(125^3) = 8.18(10^6)\ mm^3$$

(**a**) $\varepsilon_x = -\frac{1}{E}[\sigma - \nu(\sigma+\sigma)] = -\frac{\sigma}{E}(1-2\nu)$

$$= \frac{-160(10^6)}{70\times10^9}(1-0.6) = -914\ \mu$$

$$\Delta d = \varepsilon_x d = -914(10^{-6})250 = -0.229\ mm$$

Decrease in circumference:

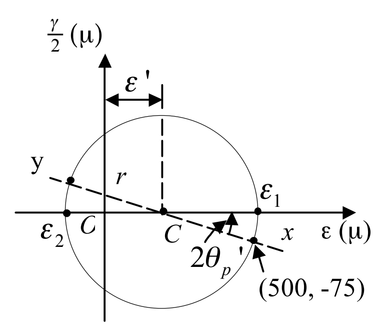
$$\pi(\Delta d) = -0.229\pi = -0.72\ mm \quad ◄$$

(**b**) $\Delta V = (1-2\nu)\varepsilon_x V_o$

$$= (0.4)(-914)(10^{-6})(8.18\times10^6) = -2291\ mm^3 \quad ◄$$

SOLUTION (2.32)

(**a**)



$$\varepsilon' = \frac{500-100}{2} = 200\ \mu$$

$$\theta_p' = \frac{1}{2}\tan^{-1}\frac{75}{300} = 7^o$$

$$\theta_s'' = 45 + 7 = 52^o$$

$$r = \sqrt{75^2 + 300^2} = 309\ \mu$$

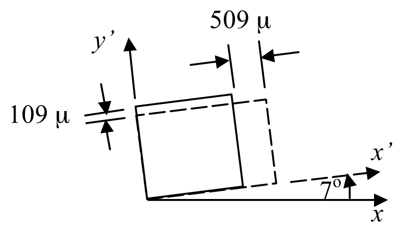
$$\gamma_{\max} = 2R = 618\ \mu$$

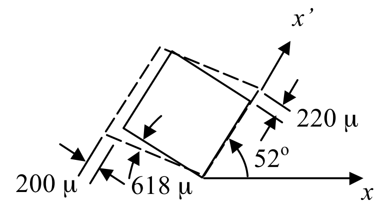
$$\varepsilon_1 = 200 + 309 = 509\ \mu \quad ◄$$

$$\varepsilon_2 = -390 + 200 = -109\ \mu$$

(CONT.)

2.32 (CONT.)



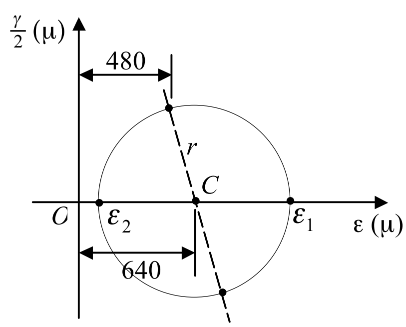


(**b**) $\varepsilon_3 = \varepsilon_z = -\dfrac{0.3}{0.7}(500-100) = -172\ \mu$

Thus,

$$(\gamma_{\max})_t = 509 + 172 = 681\ \mu \qquad ◄$$

SOLUTION (2.33)



$$\varepsilon_{avg} = \frac{1}{2}(\varepsilon_x + \varepsilon_y)$$

$$= \frac{1}{2}(480 + 800) = 640\ \mu$$

$$r = \sqrt{(640-800)^2 + 560^2}$$

$$= 582\ \mu$$

(**a**) $\varepsilon_{1,2} = \varepsilon_{avg} \pm r$

$\varepsilon_1 = 640 + 582 = 1222\ \mu$ ◄

$\varepsilon_2 = 640 - 582 = 58\ \mu$

(**b**) $\gamma_{\max} = 2r = 2(582) = 1164\ \mu$ ◄

(**c**) $(\gamma_{\max})_t = \varepsilon_1 - \varepsilon_3 = \varepsilon_1 - 0 = 1222\ \mu$ ◄

SOLUTION (2.34)

(**a**) $\sigma_x = \dfrac{P}{A} = \dfrac{25(10^3)}{(0.06 \times 0.006)} = 69.4\ MPa$

$\varepsilon_x = \dfrac{\sigma_x}{E} = \dfrac{69.4(10^6)}{200(10^9)} = 347\ \mu$ ◄

$\varepsilon_y = -\nu\varepsilon_x = -(0.3)(347 \times 10^{-6}) = -104\ \mu$

(CONT.)

(**b**) $\varepsilon_{x'} = \frac{1}{2}(\varepsilon_x + \varepsilon_y) + \frac{1}{2}(\varepsilon_x - \varepsilon_y)\cos 2\theta$

For $\theta = -30^o$:

$$\varepsilon_{x'} = \frac{1}{2}(347 - 104) + \frac{1}{2}(347 + 104)\cos(-60^o) = 182\ \mu$$

$$\varepsilon_{y'} = \frac{1}{2}(347 - 104) - \frac{1}{2}(347 + 104)\cos(-60^o) = 8.8\ \mu$$

(**c**) For $\theta = -30^o$:

$$\gamma_{x'y'} = -(\varepsilon_x - \varepsilon_y)\sin 2\theta$$

$$= -(347 + 104)\sin(-60^o) = 391\ \mu$$

SOLUTION (2.35)

Substituting $\theta_a = 0^o$, $\theta_b = 45^o$, and $\theta_c = 90^o$, Eqs (2.44) give

$$\varepsilon_a = \varepsilon_x, \quad \varepsilon_c = \varepsilon_y, \quad \varepsilon_b = \frac{1}{2}(\varepsilon_x + \varepsilon_x + \gamma_{xy})$$

$$\varepsilon_x = \varepsilon_a, \quad \varepsilon_y = \varepsilon_c, \quad \gamma_{xy} = 2\varepsilon_b - (\varepsilon_a + \varepsilon_c) \qquad \text{(P2.35)}$$

Thus, we have

$$\varepsilon_x = -800\ \mu, \qquad \varepsilon_y = 400\ \mu, \qquad \gamma_{xy} = 2(-1000) - (-800 + 400) = -1600\ \mu$$

Thus,

$$\varepsilon_{1,2} = \frac{-800 + 400}{2} \pm \sqrt{(-\frac{1200}{2})^2 + (-\frac{1600}{2})^2} = -200 \pm 1000$$

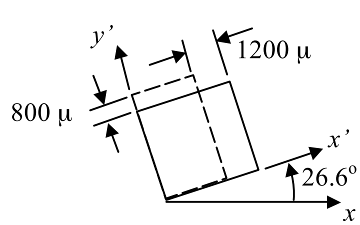
$$\varepsilon_1 = 800\ \mu, \qquad \varepsilon_2 = -1200\ \mu$$

$$\theta_p = \tfrac{1}{2}\tan^{-1}[\frac{-1600}{-800 - 400}] = 26.6^o$$

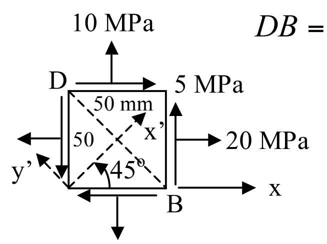
$$\varepsilon_{x'} = -200 - 600\cos 53.2^o - 800\sin 53.2^o = -1200\ \mu = \varepsilon_2$$

Thus,

$$\theta_p'' = 26.6^o$$



SOLUTION (2.36)



$DB = 0.005\sqrt{2}\ m$

(a) Hooke's law gives

$$\varepsilon_x = \tfrac{(20-0.3\times 10)}{E} = \tfrac{17}{E} \qquad \varepsilon_y = \tfrac{(10-0.3\times 20)}{E} = \tfrac{4}{E}$$

$$\gamma_{xy} = \tfrac{5(2.6)}{E} = \tfrac{13}{E}$$

Then, using Eq. (2.14a) with $\theta = \theta + \pi/2$,

$$\varepsilon_{y'} = \tfrac{1}{2E}(17+4) - 0 - \tfrac{13}{2E}\sin 90^o = \tfrac{4}{E}$$

Thus,

$$\Delta_{BD} = (\varepsilon_{y'})(BD) = \tfrac{0.283}{E}\ m$$ ◀

(b) Applying Eqs. (1.18a,c):

$$\sigma_{x'} = \tfrac{30}{2} + \tfrac{10}{2}\cos 90^o + 5\sin 90^o = 20\ MPa$$

$$\sigma_{y'} = 15 - 5\cos 90^o - 5\sin 90^o = 10\ MPa$$

As before, Hooke's law yields

$$\varepsilon_{y'} = \tfrac{(10-0.3\times 20)}{E} = \tfrac{4}{E}$$

and

$$\Delta_{BD} = (\varepsilon_{y'})(BD) = \tfrac{0.283}{E}\ m$$ ◀

SOLUTION (2.37)

The generalized Hooke's law gives

$$\varepsilon_x = 0 = \tfrac{1}{E}[\sigma_x - \nu(\sigma_y + \sigma_z)]$$

$$\varepsilon_z = 0 = \tfrac{1}{E}[\sigma_z - \nu(\sigma_x + \sigma_y)]$$

or $\quad \sigma_x - \nu\sigma_z = \nu\sigma_y \qquad -\nu\sigma_x + \sigma_z = \nu\sigma_y$

Solving,

$$\sigma_x = \sigma_z = \tfrac{1}{1-\nu}\sigma_y$$

Thus

$$\sigma_x = \sigma_z = -\tfrac{\nu p}{1-\nu}$$ ◀

SOLUTION (2.38)

Using Eqs. (2.46):

$$\varepsilon_{1,2} = \tfrac{1}{2}\{(-100+100) \pm [(-200)^2 + 100^2]^{\frac{1}{2}}\} = \tfrac{1}{2}(0 \pm 224)$$

(CONT.)

or $\varepsilon_1 = 112\ \mu \qquad \varepsilon_2 = -112\ \mu$ ◀

$\sigma_{1,2} = \frac{200(10^3)}{2}[0 \pm (\frac{224}{1.3})]$

or

$$\sigma_1 = 17.2\ MPa \qquad \sigma_2 = -17.2\ MPa$$ ◀

and

$$\theta_p = \tfrac{1}{2}\tan^{-1}\tfrac{100}{-200} = -13.28^o$$

Equations (2.44),

$$\varepsilon_x = \varepsilon_a \qquad \varepsilon_y = \varepsilon_c \qquad \gamma_{xy} = 2\varepsilon_b - \varepsilon_a - \varepsilon_c$$

Equation (2.14a) gives

$$\varepsilon_{x'} = 0 - 100\cos(-26.56^o) + 50\sin(-26.56^o) = -112\ \mu$$

Thus,

$$\theta_p'' = -13.28^o$$ ◀

SOLUTION (2.39)

We have

$$\varepsilon_x + \varepsilon_y = \varepsilon_a + \varepsilon_c = 1200\ \mu$$

and the first two of Eqs. (2.44):

$$1000 = \varepsilon_x\cos^2(-15^o) + \varepsilon_y\sin^2(-15^o) + \gamma_{xy}\sin(-15^o)\cos(-15^o)$$

$$-250 = \varepsilon_x\cos^2 30^o + \varepsilon_y\sin^2 30^o + \gamma_{xy}\sin 30^o\cos 30^o$$

These may be written

$$\varepsilon_y = 1200 - \varepsilon_x$$

$$1000 = 0.933\varepsilon_x + 0.067\varepsilon_y - 0.25\gamma_{xy}$$

$$-250 = 0.75\varepsilon_x + 0.25\varepsilon_y + 0.433\gamma_{xy}$$

Solving,

$$\varepsilon_x = 522\ \mu \qquad \varepsilon_y = 678\ \mu \qquad \gamma_{xy} = -1873\ \mu$$ ◀

SOLUTION (2.40)

(a) We have

$$\varepsilon_c = \varepsilon_y = -50\ \mu$$ ◀

First two of Eqs. (2.44):

$$400 = \varepsilon_x(\tfrac{3}{4}) - 50(\tfrac{1}{4}) + \gamma_{xy}(\tfrac{1}{2})(\tfrac{\sqrt{3}}{2})$$

$$300 = \varepsilon_x(\tfrac{3}{4}) - 50(\tfrac{1}{4}) + \gamma_{xy}(-\tfrac{1}{2})(\tfrac{\sqrt{3}}{2})$$

Solving,

$$\varepsilon_x = 482\ \mu \qquad \gamma_{xy} = 116\ \mu$$ ◀

(CONT.)

2.40 (CONT.)

Applying Eq. (2.16),

$$\varepsilon_{1,2} = \tfrac{483-50}{2} \pm [(\tfrac{483+50}{2})^2 + 58^2]^{\frac{1}{2}}$$

or

$$\varepsilon_1 = 489\ \mu \qquad \varepsilon_2 = -56\ \mu$$

Thus,

$$\gamma_{\max} = \varepsilon_1 - \varepsilon_2 = 545\ \mu$$ ◄

(b) Using Eq. (3.11b) of Chap.3,

$$\varepsilon_z = -\tfrac{\nu}{1-\nu}(483.50) = -217\ \mu = \varepsilon_3$$

Hence,

$$(\gamma_{\max})_t = \varepsilon_1 - \varepsilon_3 = 706\ \mu$$ ◄

SOLUTION (2.41)

From Eqs. (2.35), (2.37), and (2.38), we have

$$\nu = \tfrac{200(10^9)}{2(80\times10^9)} - 1 = 0.25$$

$$\lambda = \tfrac{0.25\times200(10^9)}{1.25\times0.5} = 80(10^9)$$

and

$$e = 200 + 300 = 500\ \mu$$

Then, Eqs. (2.36) lead to the following stress components, $[\tau_{ij}]$:

$$\begin{bmatrix} 72 & 16 & 0 \\ 16 & 88 & 64 \\ 0 & 64 & 40 \end{bmatrix} MPa$$ ◄

SOLUTION (2.42)

Equation (2.35) yields

$$\nu = \tfrac{200(10^9)}{2(80\times10^9)} - 0.25$$

Then, introducing the given data into the generalized Hooke's law, Eqs. (2.34), we calculate the following strain components, $[\varepsilon_{ij}]$:

$$\begin{bmatrix} 81.25 & -25 & 31.25 \\ -25 & -43.75 & 62.5 \\ 31.25 & 62.5 & 50 \end{bmatrix} \mu$$ ◄

SOLUTION (2.43)

Using Eq. (2.35),

$$G = \frac{79(10^9)}{2(1+0.3)} = 26.92\ GPa$$

For x=1 m, y=2 m, and z=4 m, we obtain $[\tau_{ij}]$:

$$\begin{bmatrix} 34 & 48 & 8 \\ 48 & 5 & 1 \\ 8 & 1 & 5 \end{bmatrix} MPa$$

Then, Eqs. (2.34) yield $[\varepsilon_{ij}]$:

$$\begin{bmatrix} 443 & 892 & 149 \\ 892 & -96 & 19 \\ 149 & 19 & -96 \end{bmatrix} \mu \qquad \blacktriangleleft$$

SOLUTION (2.44)

Substituting x=3/4 m, y=1/4 m, and z=1/2 m into Eqs. (d) of Example 1.2, we have $[\tau_{ij}]$:

$$\begin{bmatrix} -0.359 & 2.625 & 0.234 \\ 0.234 & 0.875 & 0 \\ 0.234 & 0 & 0.125 \end{bmatrix} MPa$$

Equation (2.35) gives,

$$G = \frac{200(10^9)}{2(1+0.25)} = 80\ GPa$$

Applying Hooke's law, we compute the strain components $[\varepsilon_{ij}]$:

$$\begin{bmatrix} -3 & 33 & 3 \\ 33 & 5 & 0 \\ 3 & 0 & 0 \end{bmatrix} \mu \qquad \blacktriangleleft$$

SOLUTION (2.45)

We have $\sigma_z = 0$. Using Hooke's law:

$$\varepsilon_x = \tfrac{1}{E}(\sigma_x - \nu\sigma_y) = \tfrac{1}{70\times10^3}(60 - \tfrac{90}{3}) = 1286\ \mu$$

$$\varepsilon_y = \tfrac{1}{E}(\sigma_y - \nu\sigma_z) = \tfrac{1}{70\times10^3}(90 - \tfrac{60}{3}) = 1571\ \mu$$

$$\varepsilon_z = -\tfrac{\nu}{E}(\sigma_x + \sigma_y) = -\tfrac{1}{210\times10^3}(60 + 90) = -714\ \mu$$

(a) $\Delta_{AB} = \varepsilon_y b = (1571\times10^{-6})(400) = 0.628\ mm$ ◀

(b) $e = \varepsilon_x + \varepsilon_y + \varepsilon_z = 1286 + 1571 - 714 = 2143\ \mu$

$V_0 = 300\times400\times10 = 1.2\times10^6\ mm^3$

$\Delta V = eV_0 = (2143\times10^{-6})(1.2\times10^6) = 2{,}571.6\ mm^3$ ◀

SOLUTION (2.46)

(a) Using generalized Hooke's law,

$$\varepsilon_x = \tfrac{10^6}{200(10^9)}[-60 - 0.3(-50 - 40)] = -165\ \mu$$

$$\varepsilon_y = \tfrac{1}{200(10^3)}[-50 - 0.3(-60 - 40)] = -100\ \mu$$

$$\varepsilon_z = \tfrac{1}{200(10^3)}[-40 - 0.3(-60 - 50)] = -35\ \mu$$

Thus,

$$\Delta a = a\varepsilon_x = -0.04125\ mm$$

$$\Delta b = b\varepsilon_y = -0.02\ mm \qquad \blacktriangleleft$$

$$\Delta c = c\varepsilon_z = -0.00525\ mm$$

(b)

$$e = \varepsilon_x + \varepsilon_y + \varepsilon_z = -300\ \mu$$

and

$$\Delta V = e(abc) = -2250\ mm^3 \qquad \blacktriangleleft$$

SOLUTION (2.47)

Applying generalized Hooke's law,

$$\varepsilon_x = \tfrac{10^6}{70(10^9)}[70 - \tfrac{1}{3}(-30 - 15)] = 1214\ \mu$$

$$\varepsilon_y = \tfrac{1}{70(10^3)}[-30 - \tfrac{1}{3}(70 - 15)] = -691\ \mu$$

$$\varepsilon_z = \tfrac{1}{70(10^3)}[-15 - \tfrac{1}{3}(70 - 30)] = -405\ \mu$$

Thus,

$$\Delta a = a\varepsilon_x = 0.1821\ mm$$

$$\Delta b = b\varepsilon_y = -0.0691\ mm \qquad \blacktriangleleft$$

$$\Delta c = c\varepsilon_z = -0.0304\ mm$$

(b)

$$e = \varepsilon_x + \varepsilon_y + \varepsilon_z = 118\ \mu$$

and

$$\Delta V = e(abc) = 132.75\ mm^3 \qquad \blacktriangleleft$$

SOLUTION (2.48)

We have

$$G = \tfrac{200(10^9)}{2(1+0.3)} = 76.96\ GPa$$

$$\lambda = \tfrac{0.3\times 200(10^9)}{1.3(0.4)} = 111.11(10^9)$$

Let $\varepsilon_1 = 5c \qquad \varepsilon_2 = 4c \qquad \varepsilon_3 = 3c$

We then obtain e=12c.

(CONT.)

Using the first of Eqs. (2.36),

$$\sigma_1 = 2G\varepsilon_1 + \lambda e$$

or

$$140(10^6) = 2(76.96\times 10^9)(5c) + 111.11(10^9)(12c)$$

This yields

$$c = 66.587(10^{-6})$$

Hence,

$$\varepsilon_1 = 332.935\ \mu \qquad \varepsilon_2 = 266.348\ \mu \qquad \varepsilon_3 = 199.761\ \mu$$

Now applying Eqs. (2.36), we calculate the principal stresses as follows:

$$\sigma_1 = 51.165 + 88.782 = 139.947\ MPa$$

$$\sigma_2 = 40.975 + 88.782 = 129.757\ MPa$$

$$\sigma_3 = 30.731 + 88.782 = 119.513\ MPa$$

Therefore

$$\sigma_3 : \sigma_2 : \sigma_1 = 119.513 : 129.757 : 139.947$$

or

$$\sigma_3 : \sigma_2 : \sigma_1 = 1 : 1.086 : 1.171$$ ◀

SOLUTION (2.49)

(a) $\varepsilon_x = \frac{\sigma}{E} = \frac{\partial u}{\partial x} \qquad \varepsilon_y = -\frac{\nu\sigma}{E} = \frac{\partial v}{\partial y}$

Integrating;

$$u = \tfrac{\sigma}{E}x + f(y) \qquad v = -\tfrac{\nu\sigma}{E}y + g(x) \tag{a}$$

Then

$$\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 0$$

gives

$$\frac{\partial f(y)}{\partial y} + \frac{\partial g(x)}{\partial x} = 0 \text{ or } \qquad \frac{\partial g(x)}{\partial x} = -\frac{\partial f(y)}{\partial y} = c$$

Integration leads to

$$g(x) = cx + d \qquad f(y) = -cy + e$$

Equations (a) are thus

$$u = \tfrac{\sigma}{E}x - cy + e \qquad v = -\tfrac{\nu\sigma}{E}y + cx + d$$

Boundary conditions u(0,0)=0 and v(0,0)=0 result in c=d=e=0. Thus

$$u = \tfrac{\sigma}{E}x \qquad v = -\tfrac{\nu\sigma}{E}y$$ ◀

(b) $\varepsilon_x = \frac{\sigma}{E} = const. \qquad \varepsilon_y = -\frac{\nu\sigma}{E} = const.$

Hence $\varepsilon_x = u/x$ and $\varepsilon_y = -v/y$. Therefore

$$u = \tfrac{\sigma}{E}x \text{ and } v = -\tfrac{\nu\sigma}{E}y$$ ◀

SOLUTION (2.50)

Equations (2.34) become

$$\varepsilon_x = \frac{cy^2+vcx^2}{E} \qquad \varepsilon_y = \frac{-cx^2+vcy^2}{E}$$

Integrating,

$$u(x,y) = \int \varepsilon_x dx = \frac{c}{3E}(3y^2x + vx^3) + g_1(y) \qquad (1)$$

$$v(x,y) = \int \varepsilon_y dy = \frac{c}{3E}(3y^2x + vy^2) + g_2(x) \qquad (2)$$

Given: $\tau_{xy} = 0 \qquad \tau_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 0 \qquad (3)$

From Eq. (1),

$$\frac{\partial u}{\partial y} = \frac{dg_1}{dy} + 2\frac{c}{E}xy \qquad (a)$$

Equation (2) leads to

$$\frac{\partial v}{\partial x} = \frac{dg_2}{dx} - 2\frac{c}{E}xy \qquad (b)$$

Substituting Eqs. (a) and (b) into Eq. (3):

$$\frac{dg_1}{dy} = 2\frac{c}{E}xy + a_1 \qquad \frac{dg_2}{dx} = -2\frac{c}{E}xy + a_1$$

from which, after integration,

$$g_1(y) = \frac{c}{E}xy^2 + a_1 y + a_2$$
$$g_2(x) = -\frac{c}{E}yx^2 - a_1 x - a_2 \qquad (4)$$

Constants a_1 and a_2 are obtained upon satisfying the prescribed boundary conditions. Substitution of Eqs. (4) into Eqs. (1) and (2) yields the solution for displacement field.

SOLUTION (2.51)

Equations (2.39) and (2.35) give

$$K = \frac{E}{3(1-2v)} = \frac{2G(1+v)}{3(1-2v)}$$

Also, $$K = \frac{E}{3(1-2v)} = \frac{2E(1+v)}{3(1+v)(1-2v)}$$

$$= \frac{E(3v+1-2v)}{3(1+v)(1-2v)} = \frac{vE}{(1+v)(1-2v)} + \frac{E}{3(1+v)}$$

$$= \lambda + (\tfrac{2G}{3}) \qquad \blacktriangleleft$$

Equations (2.39) and (2.35) yield

$$G = \frac{3K(1-2v)}{2(1+v)} = \frac{E/(1-2v)}{(3-1+2v)/(1-2v)} = \frac{E^2/(1-2v)}{E\{[3/(1-2v)]-1\}}$$

$$= \frac{\frac{3E^2}{3E(1-2v)}}{\frac{9E}{3}\frac{1}{1-2v} - E} = \frac{3KE}{9K-E} \qquad \blacktriangleleft$$

Equations (2.35) and (2.39) give,

$$E = 2G(1+v) \qquad E = 3K(1-2E)$$

and

$$\frac{G(3+2G)}{\lambda+G} = \frac{\frac{E}{2(1+v)}[3\lambda + \frac{E}{1+v}]}{\lambda + \frac{E}{2(1+v)}}$$

The above expression, after substituting λ from Eq. (2.38) and simplifying, reduce to E.

(CONT.)

From formula (2.39),

$$(1-2\nu) = \tfrac{E}{3K}$$

or $$\nu = \tfrac{1}{2} - \tfrac{E}{6K} = \tfrac{3K-E}{6K}$$

We also have

$$G = \tfrac{E}{2G} - 1 \qquad (1+\nu) = \tfrac{E}{2G}$$

or

$$\nu = \tfrac{E}{2G} - 1 \qquad \blacktriangleleft$$

This expression is written as

$$\nu = \tfrac{3K(1-2\nu)}{2G} - 1 = \tfrac{3K}{2G} - \tfrac{6K\nu}{2G} - 1$$

from which

$$\nu + \tfrac{6K\nu}{2G} = \nu(1 + \tfrac{6\nu}{2G}) = \tfrac{3K}{2G} - 1$$

or

$$\nu = \tfrac{3K-2G}{2(3K+G)}$$

Finally, we write

$$\lambda = \tfrac{2\nu E}{2(1+\nu)(1-2\nu)} = \tfrac{2\nu G}{1-2\nu}$$

or

$$(1-2\nu)\lambda - 2\nu G = 0$$

This yields,

$$\nu = \tfrac{\lambda}{2(\lambda+G)} \qquad \blacktriangleleft$$

SOLUTION (2.52)

Equations (2.4) yield,

$$\varepsilon_x = \tfrac{\gamma(a-x)}{E} \qquad \varepsilon_y = -\tfrac{\nu\gamma(a-x)}{E} \qquad \gamma_{xy} = -\tfrac{\nu\gamma y}{E} + \tfrac{\nu\gamma y}{E} = 0$$

The stresses are therefore,

$$\sigma_x = \tfrac{E}{1-\nu^2}(\varepsilon_x + \nu\varepsilon_y) = \gamma(a-x)$$

$$\sigma_y = \tfrac{E}{1-\nu^2}(\varepsilon_y + \nu\varepsilon_x) = 0 \qquad \blacktriangleleft$$

$$\tau_{xy} = G\gamma_{xy} = 0$$

At x=0 and x=a, we have

$$\sigma_x = \gamma a \qquad \sigma_x = 0 \qquad \blacktriangleleft$$

Applying Eqs. (1.48) at x=a, we obtain

$$\begin{Bmatrix} p_x \\ p_y \\ p_z \end{Bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

This is, boundary conditions at x=a are satisfied. ◀

(CONT.)

At $y = \pm b$, stresses are

$$\sigma_x = \gamma(a - x) \qquad \sigma_y = \tau_{xy} = 0$$

and

$$\begin{Bmatrix} p_x \\ p_y \\ p_z \end{Bmatrix} = \begin{bmatrix} (x-a)\gamma & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} 0 \\ \pm 1 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

We see that boundary conditions at $y = \pm b$ are also satisfied. ◀

SOLUTION (2.53)

Equations (2.4) give,

$$\varepsilon_x = -\tfrac{\nu\gamma z}{E}, \qquad \varepsilon_y = -\tfrac{\nu\gamma z}{E} \qquad \text{(a)}$$

$$\varepsilon_z = \tfrac{\gamma z}{E}, \qquad \gamma_{xy} = \gamma_{yz} = \gamma_{xz} = 0$$

Equations (2.12) are satisfied by the above strains.

Equations (2.36) and (a) yield,

$$\sigma_x = 2G\varepsilon_x + \lambda(\varepsilon_x + \varepsilon_y + \varepsilon_z) = 0$$

and

$$\sigma_y = 0 \qquad \sigma_z = \gamma z \qquad \gamma_{xy} = \tau_{yx} = \tau_{xz} = 0$$

Note that $F_x = F_y = 0 \quad and \quad F_z = -\gamma$

Thus, the first of Eqs. (1.14) are satisfied, and the third leads to

$$0 + 0 + \tfrac{\partial \sigma_z}{\partial z} + F_z = 0 \text{ or } \gamma - \gamma = 0$$

At ends, since x' and z are parallel, n=cos(x',z)= ± 1.
Thus, boundary conditions (1.48) at z=L:

$$p_x = 0 + 0 + 0 = 0 \qquad p_y = 0 + 0 + 0 = 0$$

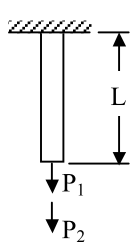
$$p_z = 0 + 0 + \gamma L(1) = \gamma L$$

as required. Similarly, at z=0:

$$p_x = 0 \qquad p_y = 0 \qquad p_z = \gamma(0)(-1) = 0$$

Therefore, all equations of elasticity are satisfied. ◀

SOLUTION (2.54)



Using Eq. (2.59), with $P_1 = P_1 + P_2$:

$$U = \tfrac{(P_1+P_2)^2 L}{2AE}$$ ◀

(CONT.)

2.54 (CONT.)

From Example 2.8, with $P_1 = P_2 = P$:

$$U_v = \tfrac{\sigma^2}{12E}(AL) = \tfrac{P^2L}{3AE} \qquad U_d = \tfrac{5\sigma^2}{12E}(AL) = \tfrac{5P^2L}{3AE}$$ ◀

Hence,

$$U = U_v + U_d = 2\tfrac{P^2L}{AE}$$ ◀

SOLUTION (2.55)

We have

$$U_1 = \tfrac{P^2L}{2AE} \qquad U_2 = \tfrac{P^2(L/4)}{2AE} + \tfrac{P^2(3L/4)}{2E(2A)} = \tfrac{5}{8}U_1$$

And

$$U_3 = \tfrac{P^2(L/8)}{2AE} + \tfrac{P^2(7L/8)}{2E(3A)} = \tfrac{5}{12}U_1$$

Comparison of these results show that strain energy decreases as the volume of the bar is increased, although all three bars have the same maximum stress. ◀

SOLUTION (2.56)

Stress field is described by

$$\sigma_x = \sigma_y = \sigma_z = -p, \qquad \tau_{xy} = \tau_{xz} = \tau_{yz} = 0$$

(a) Equation (2.37) reduces to

$$e = -\tfrac{3(1-2\nu)p}{E}; \qquad -0.005 = -\tfrac{3[1-(2/3)]p}{110(10^9)}$$

or

$$p = 550\ MPa$$ ◀

(b) Equation (2.52) becomes

$$U_0 = \tfrac{1}{2E}(p^2 + p^2 + p^2) - \tfrac{\nu}{E}(p^2 + P^2 + p^2) = \tfrac{3p^2(1-2\nu)}{2E}$$

$$= \tfrac{3(550\times10^6)^2}{2(110\times10^9)}(1 - \tfrac{2}{3}) = 1.375\ \ MPa$$

Thus,

$$U = U_0V_0 = 1.375(10^6)(\tfrac{4\pi}{3}\times 0.15^3)$$

$$= 19.439\ kN\cdot m$$ ◀

SOLUTION (2.57)

Substituting the given data into Eq. (2.52), we have

$$U_0 = \tfrac{10^3}{2(200)}(60^2 + 50^2 + 40^2) - \tfrac{0.3(10^3)}{200}(3000 + 2400 + 2000)$$

$$= 8.15\ kPa$$

Thus,

$$U = U_0(abc)$$

$$= 8.15(10^3)(0.25\times0.2\times0.15)$$

$$= 61.125\ N\cdot m$$ ◀

SOLUTION (2.58)

Equation (2.59) leads to

$$U_n = \frac{P^2(3L/4)}{2AE} + \frac{P^2(L/4)}{2(n^2A)E} = \frac{P^2L}{8AE}(3 + \frac{1}{n^2})$$

$$= \frac{1+3n^2}{4n^2}\frac{P^2L}{2AE} \qquad ◄$$

We have,

for $n = 1: \quad U_1 = \frac{P^2L}{2AE}$

Thus, for $n = \frac{1}{2}: \quad U_{1/2} = \frac{7U_1}{4}$

for $n = 2: \quad U_2 = \frac{13U_1}{16}$

Hence,

$$U_{1/2} > U_1 \quad \text{and} \quad U_2 < U_1 \qquad ◄$$

SOLUTION (2.59)

(a) $U = \frac{1}{2E}\int dx \int \sigma^2 dA = \frac{1}{2E}\int dx \int (\frac{P}{A} + \frac{My}{I})^2 dA$

Since $\int ydA = 0$, this becomes:

$$U = \int \frac{P^2dx}{2AE} + \int \frac{M^2dx}{2EI} = U_a + U_b \qquad ◄$$

Here

$$U_b = \frac{1}{2EI}\left[\int_0^a M_{AD}^2 dx + \int_0^b M_{DB}^2 dx\right]$$

$$= \frac{1}{2EI}\left[\int_0^a (-\frac{M_0x}{L})^2 dx + \int_0^b (\frac{M_0x'}{L})^2 dx'\right]$$

$$= \frac{M_0^2}{6EIL^2}(a^3 + b^3)$$

Thus, $U = \frac{P^2L}{2AE} + \frac{M_0^2}{6EIL^2}(a^3 + b^3) \qquad ◄$

(b) Substituting the given data:

$$U = \frac{(8\times10^3)^2(1.2)}{2(7.5\times10^{-3})(70\times10^9)} + \frac{(2\times10^3)^2(0.027+0.729)}{6(70\times10^9)(0.075\times0.1^3/12)(1.2)^2}$$

$$= 0.0731 + 0.8 = 0.8731 \ N\cdot m \qquad ◄$$

SOLUTION (2.60)

$$U = \sum \frac{T^2L}{2JG} = \frac{T^2(L/2)}{2(\frac{\pi}{32}d_1^4)G} + \frac{T^2(L/2)}{2(\frac{\pi}{32}d_2^4)G}$$

$$= \frac{8T^2L}{\pi G}(\frac{1}{d_1^4} + \frac{1}{d_2^4}) \qquad \text{(a)}$$

$$\phi = \sum \frac{TL}{JG} = \frac{T(L/2)}{\frac{\pi}{32}d_1^4G} + \frac{T(L/2)}{\frac{\pi}{32}d_2^4G}$$

$$= \frac{16TL}{\pi G}(\frac{1}{d_1^4} + \frac{1}{d_2^4}) \qquad \text{(b)}$$

Solving T from Eq. (b) and substituting into Eq. (a) result in Eq. (P2.60).

SOLUTION (2.61)

We have for segment AB, $T_{AB} = 3T$ and for segment BC, $T_{BC} = T$

(a) $U_{AB} = (\frac{T^2L}{2JG})_{AB} = \frac{9T^2(1.2a)}{\pi(1.4d)^4 G/16} = 44.977\frac{T^2a}{\pi d^4G}$

and $U_{BC} = (\frac{T^2L}{2JG})_{BC} = \frac{16T^2a}{\pi d^4G}$

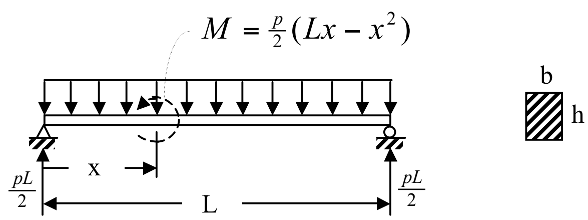
The total energy is thus

$$U = 60.981\frac{T^2a}{\pi d^4G} \quad ◀$$

(b) Substituting the given data into the above equation,

$$U = 60.981\frac{(1.4\times10^3)^2(0.5)}{\pi(0.02)^4(42\times10^9)} = 2.831\ kN\cdot m \quad ◀$$

SOLUTION (2.62)



The maximum bending moment occurs at the midspan:

$$\sigma_{\max} = \frac{M_{\max}c}{I}\frac{(pL^2/8)(h/2)}{bh^3/12} = \frac{3pL^2}{4bh^2}$$

Maximum strain energy density,

$$U_{0,\max} = \frac{\sigma_{\max}^2}{2E} = \frac{9p^2L^4}{32Eb^2h^4}$$

Using Eq. (2.63), we obtain

$$U = \frac{1}{2EI}\int_0^L (\frac{p}{2})^2(Lx - x^2)^2\,dx$$

$$= \frac{p^2L^5}{240EI} = \frac{p^2L^5}{20Ebh^3}$$

It is required to find c:

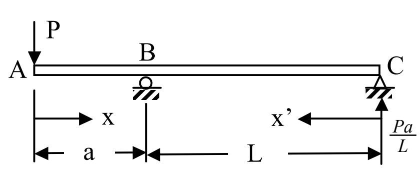
$$U_{0,\max} = \frac{cU}{V} \quad \text{or} \quad c = U_{0,\max}\frac{V}{U}$$

Thus, $c = \frac{9p^2L^4}{32Eb^2h^4}\frac{bhl}{p^2L^5/20Ebh^3} = \frac{45}{8}$

and

$$U_{0,\max} = \frac{45U}{8V} \quad ◀$$

SOLUTION (2.63)



$$M_{AB} = Px$$

$$M_{BC} = \frac{Pax}{L}$$

(CONT.)

2.63 (CONT.)

We have

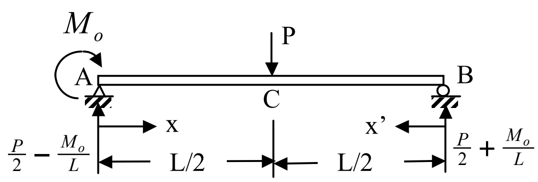
$$U_{AB} = \tfrac{1}{2EI}\int_0^a (Px)dx = \tfrac{P^2a^3}{6EI}$$

$$U_{BC} = \tfrac{1}{2EI}\int_0^a (\tfrac{Pax}{L})^2 dx' = \tfrac{P^2a^2L}{6EI}$$

Therefore

$$U = U_{AB} + U_{BC} = \tfrac{P^2a^2}{6EI}(L+a) \quad ◀$$

SOLUTION (2.64)



$$M_{AC} = (\tfrac{P}{2} - \tfrac{M_o}{L})x$$

$$M_{BC} = (\tfrac{P}{2} + \tfrac{M_o}{L})x'$$

We have

$$U_{AB} = \tfrac{1}{2EI}\int_0^{L/2} M_{AB}^2 dx = (\tfrac{P}{2} - \tfrac{M_0}{L})^2(\tfrac{L^3}{48EI}) + M_0(\tfrac{P}{2} - \tfrac{M_0}{L})(\tfrac{L^2}{8EI}) + \tfrac{M_0^2L}{4EI}$$

$$U_{BC} = \tfrac{1}{2EI}\int_0^{L/2} M_{BC}dx' = (\tfrac{P}{2} + \tfrac{M_0}{L})^2(\tfrac{L^3}{48EI})$$

Thus

$$U = U_{AB} + U_{BC}$$

$$= \tfrac{P^2L^3}{96EI} + \tfrac{PM_0L^2}{16EI} + \tfrac{M_0^2L}{6EI} \quad ◀$$

SOLUTION (2.65)

Equation (2.66) yields

$$\tau_{oct} = \tfrac{1}{3}[(-19-4.6)^2 + (4.6+8.3)^2 + (-8.3+19)^2$$

$$+ 6.45^2 + 11.8^2]^{\frac{1}{2}} = 15.11 \ MPa$$

Applying Eqs. (2.65) and (2.64), respectively,

$$U_{0d} = \tfrac{3}{4(76.9\times 10^9)}(15.11)^2 = 2226.71 \ Pa \quad ◀$$

and

$$U_{0v} = \tfrac{(-19+4.6-8.3)^2(10^{12})}{18(166.67\times 10^9)} = 171.76 \ Pa \quad ◀$$

It follows that

$$\tfrac{U_{0d}}{U_{0v}} = 12.96 \approx 13 \quad ◀$$

SOLUTION (2.66)

Dilatational stress is

$$\sigma_m = \tfrac{(200-50+40)}{3} = 63.33\ MPa$$

Distortional stresses are then

$$\sigma_x - \sigma_m = 200 - 63.33 = 136.6\ MPa$$
$$\sigma_y - \sigma_m = -50 - 63.33 = -113.3\ MPa$$
$$\sigma_z - \sigma_m = 40 - 63.33 = -23.33\ MPa$$

Dilatational and deviator stress matrices are, respectively,

$$\begin{bmatrix} 63.33 & 0 & 0 \\ 0 & 63.33 & 0 \\ 0 & 0 & 63.33 \end{bmatrix} MPa \qquad \blacktriangleleft$$

and

$$\begin{bmatrix} 136.6 & 20 & 10 \\ 20 & -113.3 & 0 \\ 10 & 0 & -23.33 \end{bmatrix} MPa \qquad \blacktriangleleft$$

Next, substitute the above deviator stresses into Eq. (1.33). Then, solve the resulting equation for principal deviator stresses, using Table B.1:

$$\sigma_{1d} = 138.8\ MPa \qquad \sigma_{2d} = -23.9\ MPa \qquad \sigma_{3d} = -114.9\ MPa \qquad \blacktriangleleft$$

SOLUTION (2.67)

Only existing stresses are: $\sigma_x = \sigma \qquad \tau_{xy} = \tau$

Here,

$$\tau = \tfrac{2T}{\pi r^3} = \tfrac{2(20\times10^3)}{\pi(0.06)^3} = 58.95\ MPa$$
$$\sigma = \tfrac{4M}{\pi r^3} = \tfrac{4(15\times10^3)}{\pi(0.06)^3} = 88.42\ MPa$$

We have

$$G = \tfrac{2E}{5} \qquad K = \tfrac{2E}{3}$$

Equations (2.64) and (2.65):

$$U_{0v} = \tfrac{\sigma^2}{18K} = \tfrac{\sigma^2}{12E} = \tfrac{(88.42)^2(10^3)}{12(200)}$$
$$= 3.258\ kPa \qquad \blacktriangleleft$$
$$U_{0d} = \tfrac{5}{12E}(\sigma^2 + 3\tau^2) = \tfrac{5(10^2)}{12(200)}(7818.1 + 10{,}425.3)$$
$$= 38.007\ kPa \qquad \blacktriangleleft$$

Thus,

$$U_0 = U_{0v} + U_{od} = 41.265\ kPa \qquad \blacktriangleleft$$

SOLUTION (2.68)

Principal stresses are

$$\sigma_{1,2} = \tfrac{\sigma}{2} \pm [\tfrac{\sigma^2}{4} + \tau^2]^{\frac{1}{2}}, \qquad \sigma_3 = 0$$

and

$$\tau_{oct} = \tfrac{1}{3}(2\sigma^2 + 6\tau^2)^{\frac{1}{2}}$$

Equations (2.64) and (2.65) are therefore

$$U_{0v} = \tfrac{1}{18K}(\sigma_1^2 + \sigma_2^2) = \tfrac{\sigma^2}{18K} = \tfrac{1-2v}{6E}\sigma^2$$

$$U_{0d} = \tfrac{3}{4G}\tau_{oct}^2 = \tfrac{1+v}{3E}(\sigma^2 + 3\tau^2) \quad ◄$$

SOLUTION (2.69)

Only existing stress components are: $\sigma_x = \sigma \qquad \tau_{xy} = \tau$

where,

$$\sigma = \tfrac{P}{\pi r^2} \qquad \tau = \tfrac{2T}{\pi r^3}$$

The area properties are: $A = \pi r^2 \qquad J = \tfrac{\pi r^4}{2}$

Thus, Eqs. (2.64) and (2.65) become

$$U_{0v} = \tfrac{\sigma^2}{18K} = \tfrac{1-2v}{6E}\sigma^2 = \tfrac{\sigma^2}{12E}$$

$$U_{0d} = \tfrac{3}{4G}\tau_{oct}^2 = \tfrac{5\sigma^2}{12E} + \tfrac{5\tau^2}{4E}$$

The components of strain energy are

$$U_v = \int U_{0v}dV = \tfrac{1}{12E}\int \tfrac{P^2}{A}dx = \tfrac{P^2L}{12\pi r^2 E} \quad ◄$$

$$U_d = \int U_{0d}dV = \tfrac{5}{12E}\int \tfrac{P^2}{E}dx + \tfrac{5}{4E}\int \tfrac{T^2}{J}dx$$

$$= \tfrac{5P^2L}{12\pi r^2 E} + \tfrac{5T^2L}{2\pi r^4 E} \quad ◄$$

Total strain energy is therefore

$$U = \tfrac{L}{2\pi r^2 E}(P^2 + 5\tfrac{T^2}{r^2}) \quad ◄$$

End of Chapter 2

CHAPTER 3

SOLUTION (3.1)

(a) We obtain

$$\frac{\partial^4\Phi}{\partial x^4} = -12pxy \qquad \frac{\partial^4\Phi}{\partial y^4} = 0 \qquad \frac{\partial^4\Phi}{\partial x^2\partial y^2} = 6pxy$$

Thus, $\nabla^4\Phi = -12pxy + 2(6pxy) = 0$

and the given stress field represents a possible solution. ◀

(b)
$$\frac{\partial^2\Phi}{\partial x^2} = pxy^3 - 2px^3y$$

Integrating twice

$$\Phi = \frac{px^3y^3}{6} - \frac{px^5y}{10} + f_1(y)x + f_2(y)$$

The above is substituted into $\nabla^4\Phi = 0$ to obtain

$$\frac{d^4f_1(y)}{dy^4}x + \frac{d^4f_2(y)}{dy^4} = 0$$

This is possible only if

$$\frac{d^4f_1(y)}{dy^4} = 0 \qquad \frac{d^4f_2(y)}{dy^4} = 0$$

We find then

$$f_1 = c_4y^3 + c_5y^2 + c_6y + c_7$$

$$f_2 = c_8y^3 + c_9y^2 + c_{10}y + c_{11}$$

Therefore,

$$\Phi = \frac{px^3y^3}{6} - \frac{px^5y}{10} + (c_4y^3 + c_5y^2 + c_6y + c_7)x + c_8y^3 + c_9y^2 + c_{10}y + c_{11}$$ ◀

(c) Edge y=0:

$$V_x = \int_{-a}^{a}\tau_{xy}tdx = \int_{-a}^{a}(\tfrac{px^4}{2} + c_3)tdx = \frac{pa^5t}{5} + 2c_3at$$ ◀

$$P_y = \int_{-a}^{a}\sigma_y tdx = \int_{-a}^{a}(0)tdx = 0$$ ◀

Edge y=b:

$$V_x = \int_{-a}^{a}(-\tfrac{3}{2}px^2b^2 + c_1b^2 + \tfrac{px^4}{2} + c_3)tdx$$

$$= -pa^3(b^2 - \tfrac{a^2}{5})t + 2a(c_1b^2 + c_3)t$$ ◀

$$P_y = \int_{-a}^{a}(pxb^3 - 2px^3b)tdx = 0$$ ◀

SOLUTION (3.2)

Edge $x = \pm a$:

$$\tau_{xy} = 0: \qquad -\tfrac{3}{2}pa^2y^2 + c_1y^2 + \tfrac{1}{2}pa^4 + c_3 = 0$$

$$\tau_{xy} = 0: \qquad -\tfrac{3}{2}pa^2y^2 + c_1y^2 + \tfrac{1}{2}pa^4 + c_3 = 0$$

Adding, $(-3pa^2 + 2c_1)y^2 + pa^4 + 2c_3 = 0$

(CONT.)

3.2 (CONT.)

or $\quad c_1 = \frac{3}{2} pa^2 \qquad c_3 = -\frac{1}{2} pa^4$ ◄

Edge $x = a$:

$$\sigma_x = 0: \qquad pa^3 y - 2c_1 ay + c_2 y = 0$$

or

$$c_2 = 2pa^3$$ ◄

SOLUTION (3.3)

(a) Equations (3.6) become

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0 \qquad \frac{\partial \tau_{xy}}{\partial x} = 0$$

Substituting the given stresses, we have

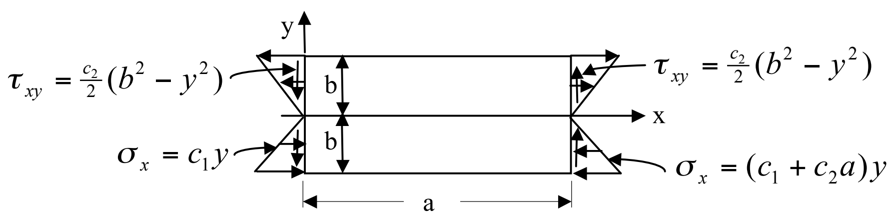
$$c_2 y - 2c_3 y = 0$$

Thus

$$c_2 = 2c_3 \qquad c_1 = \text{arbitrary}$$ ◄

(b) $\sigma_x = c_1 y + c_2 xy \qquad \tau_{xy} = \frac{c_2}{2}(b^2 - y^2)$

Assume $c_1 > 0$ and $c_2 > 0$.



SOLUTION (3.4)

Boundary conditions, Eq. (3.6):

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0 \qquad \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} = 0$$

or $\quad (2ab - 2ab)x = 0 \qquad (-2ab + 2ab)y = 0$

are fulfilled.

However, equation of compatibility:

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(\sigma_x + \sigma_y) = 0 \text{ or } 4ab \neq 0 \text{ is } \textit{not} \text{ satisfied.}$$

Thus, the stress field given does not meet requirements for solution.

SOLUTION (3.5)

It is readily shown that

$$\nabla^4 \Phi_1 = 0 \qquad \text{is satisfied}$$

$$\nabla^4 \Phi_2 = 0 \qquad \text{is satisfied}$$ ◄

(CONT.)

3.5 (CONT.)

We have

$$\sigma_x = \frac{\partial^2\Phi_1}{\partial y^2} = 2c, \qquad \sigma_y = \frac{\partial^2\Phi_1}{\partial x^2} = 2a, \qquad \tau_{xy} = -\frac{\partial^2\Phi_1}{\partial x\partial y} = -b \qquad \blacktriangleleft$$

Thus, stresses are uniform over the body.

Similarly, for Φ_2 :

$$\sigma_x = 2cx + 6dy \qquad \sigma_y = 6ax + 2by \qquad \tau_{xy} = -2bx - 2cy \qquad \blacktriangleleft$$

Thus, stresses vary linearly with respect to x and y over the body.

SOLUTION (3.6)

Note: Since $\sigma_z = 0$ and $\varepsilon_y = 0$, we have plane stress in xy plane and plane strain in xz plane, respectively.

Equations of compatibility and equilibrium are satisfied by

$$\sigma_x = -\sigma_0 \qquad \sigma_y = -c \qquad \sigma_z = 0$$
$$\tau_{xy} = \tau_{yz} = \tau_{xz} = 0 \qquad \text{(a)}$$

We have

$$\varepsilon_y = 0 \qquad \text{(b)}$$

Stress-strain relations become

$$\varepsilon_x = \frac{(\sigma_x - \nu\sigma_y)}{E}, \qquad \varepsilon_y = \frac{(\sigma_y - \nu\sigma_x)}{E} \qquad \text{(c)}$$
$$\varepsilon_z = \frac{-\nu(\sigma_x + \sigma_y)}{E}, \qquad \gamma_{xy} = \gamma_{yz} = \gamma_{xz} = 0$$

Substituting Eqs. (a,b) into Eqs. (c), and solving

$$\sigma_y = -\nu\sigma_0 \qquad \varepsilon_z = \frac{\nu(1+\nu)\sigma_0}{E}$$
$$\varepsilon_x = -\frac{(1-\nu^2)}{\sigma_0 E} \qquad \varepsilon_y = 0$$

Then, Eqs. (2.3) yield, after integrating:

$$u = -\frac{(1-\nu^2)\sigma_0 x}{E} \qquad v = 0 \qquad \blacktriangleleft$$
$$w = \frac{\nu(1+\nu)\sigma_0 z}{E}$$

SOLUTION (3.7)

Equations of equilibrium,

$$\frac{\partial\sigma_x}{\partial x} + \frac{\partial\tau_{xy}}{\partial y} = 0, \qquad 2axy + 2axy = 0$$
$$\frac{\partial\tau_{xy}}{\partial y} + \frac{\partial\sigma_y}{\partial x} = 0, \qquad ay^2 - ay^2 = 0$$

are satisfied. Equation (3.12) gives

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(\sigma_x + \sigma_y) = -4ay \neq 0$$

Compatibility is violated; solution is *not* valid.

SOLUTION (3.8)

We have

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} = 0 \qquad \frac{\partial^2 \varepsilon_y}{\partial x^2} = -2ay \qquad \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} = 2ay$$

Equation of compatibility, Eq. (3.8) is satisfied. Stresses are

$$\sigma_x = \frac{E}{1-\nu^2}(\varepsilon_x + \nu\varepsilon_y) = \frac{aE}{1-\nu^2}(x^3 + \nu x^2 y)$$

$$\sigma_y = \frac{E}{1-\nu^2}(\varepsilon_y + \nu\varepsilon_x) = \frac{aE}{1-\nu^2}(x^2 y + \nu x^3)$$

$$\tau_{xy} = G\gamma_{xy} = \frac{aE}{2(1+\nu)} xy^2$$

Equations (3.6) become

$$\frac{aE}{1-\nu^2}(3x^2 + 2\nu xy) + \frac{aE}{1+\nu} xy = 0$$

$$\frac{aE}{1-\nu^2} y^2 + \frac{aE}{1-\nu^2} x^2 = 0$$

These cannot be true for all values of x and y. Thus, Solution is *not* valid.

SOLUTION (3.9)

$$\varepsilon_x = \frac{\partial u}{\partial x} = -2\nu cx \qquad \varepsilon_y = \frac{\partial v}{\partial y} = 2ax$$

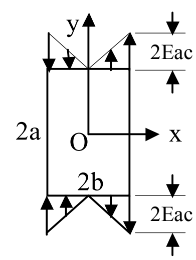
$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = -2cy + 2cy = 0$$

Thus

$$\sigma_x = \frac{E}{1-\nu^2}(\varepsilon_x + \nu\varepsilon_y) = 0 \qquad \tau_{xy} = G\gamma_{xy}$$

$$\sigma_y = \frac{E}{1-\nu^2}(\varepsilon_y + \nu\varepsilon_x) = 2Ecx$$

◄



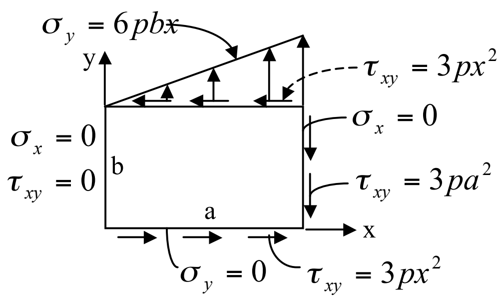
Note that this is a state of *pure bending*.

SOLUTION (3.10)

(a) $\sigma_x = \frac{\partial^2 \Phi}{\partial y^2} = 0, \qquad \sigma_y = 6pxy \qquad \tau_{xy} = -3px^2$ ◄

Note that $\nabla^4 \Phi = 0$ is satisfied.

(b)



(c) Edge $x = 0$: $\qquad V_y = P_x = 0$ ◄

Edge $x = a$: $\qquad P_x = 0$

(CONT.)

3.10 (CONT.)

$$V_y = \int_0^b \tau_{xy} t dy = 3pa^2bt \downarrow$$ ◄

Edge $y = 0$: $P_y = 0$

$$V_x = \int_0^a \tau_{xy} t dx = pa^3 t \rightarrow$$ ◄

Edge $y = b$: $V_x = pa^3 t \leftarrow$

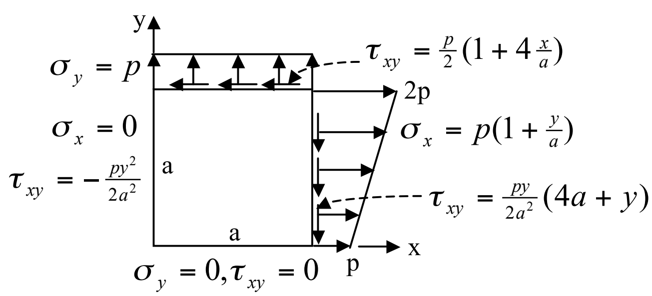
$$P_y = \int_0^a \sigma_y t dx = 3pa^2bt \uparrow$$ ◄

SOLUTION (3.11)

(a) We have $\nabla^4\Phi \neq 0$ is *not* satisfied.

$$\sigma_y = \tfrac{\partial^2\Phi}{\partial x^2} = \tfrac{py^2}{a^2}, \qquad \sigma_x = \tfrac{p(x^2+xy)}{a^2}, \qquad \tau_{xy} = -\tfrac{p(4xy+y^2)}{2a^2}$$ ◄

(b)



(c) Edge $x = 0$: $V_y = \int_0^a \frac{py^2}{2a^2} dy = \frac{1}{6} pat \qquad P_x = 0$ ◄

Edge $x = a$:

$$V_y = \int_0^a \tau_{xy} t dy = \tfrac{7}{6} pat \downarrow$$ ◄

$$P_x = \int_0^a \sigma_x t dy = \tfrac{3}{2} pat \rightarrow$$

Edge $y = 0$: $V_x = 0 \qquad P_y = 0$ ◄

Edge $y = a$:

$$V_x = \int_0^a \tau_{xy} t dx = \tfrac{3}{2} pat \leftarrow$$ ◄

$$P_y = \int_0^a \sigma_y p t dx = pat \uparrow$$

SOLUTION (3.12)

(a) We have $\nabla^4\Phi = 0$ is satisfied. The stresses are

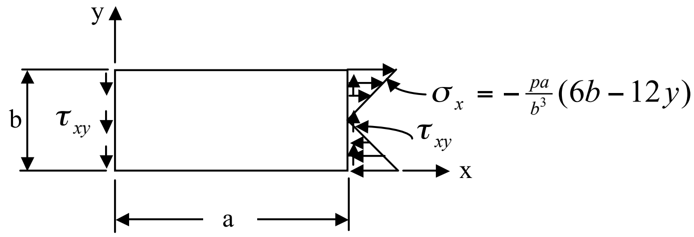
$$\sigma_x = \tfrac{\partial^2\Phi}{\partial y^2} = -\tfrac{px}{b^3}(6b - 12y) \qquad \sigma_y = \tfrac{\partial^2\Phi}{\partial x^2} = 0$$

$$\tau_{xy} = -\tfrac{\partial^2\Phi}{\partial x \partial y} = \tfrac{6py}{b^3}(b - y)$$

(CONT.)

3.12 (CONT.)

(b)



SOLUTION (3.13)

We have

$$\frac{\partial\Phi}{\partial y} = -\frac{P}{\pi}[\tan^{-1}\frac{y}{x} + \frac{xy}{x^2+y^2}], \qquad \frac{\partial\Phi}{\partial x} = -\frac{Py}{\pi}\frac{-y}{x^2+y^2}$$

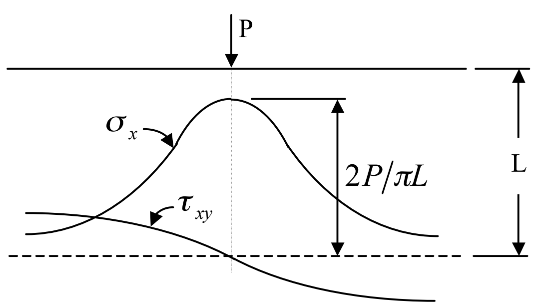
$$\frac{\partial^2\Phi}{\partial y^2} = -\frac{P}{\pi}[\frac{x}{x^2+y^2} + \frac{(x^2+y^2)x-2y^2x}{(x^2+y^2)^2}]$$

The stresses are thus,

$$\sigma_x = \frac{\partial^2\Phi}{\partial y^2} = -\frac{2P}{\pi}\frac{x^3}{(x^2+y^2)^2}$$

$$\sigma_y = \frac{\partial^2\Phi}{\partial x^2} = -\frac{2P}{\pi}\frac{xy^2}{(x^2+y^2)^2} \qquad ◀$$

$$\tau_{xy} = -\frac{\partial^2\Phi}{\partial x\partial y} = -\frac{2P}{\pi}\frac{x^2y}{(x^2+y^2)^2}$$



SOLUTION (3.14)

Various derivatives of Φ are:

$$\frac{\partial\Phi}{\partial x} = \frac{\tau_0}{4}(y - \frac{y^2}{h} - \frac{y^3}{h^2}), \qquad \frac{\partial^2\Phi}{\partial x^2} = 0$$

$$\frac{\partial^4\Phi}{\partial x^2\partial y^2} = 0, \qquad \frac{\partial^2\Phi}{\partial x\partial y} = \frac{\tau_0}{4}(1 - \frac{2y}{h} - \frac{3y^2}{h^2}) \qquad \text{(a)}$$

$$\frac{\partial^2\Phi}{\partial y^2} = \frac{\tau_0}{4h}(-2x - \frac{6xy}{h} + 2L + \frac{6Ly}{h^2})$$

$$\frac{\partial^4\Phi}{\partial x^4} = 0, \qquad \frac{\partial^4\Phi}{\partial y^4} = 0$$

It is clear that Eqs. (a) satisfy Eq. (3.17). On the basis of Eq. (a) and (3.16), we obtain

(CONT.)

$$\sigma_x = \tfrac{\tau_0}{4h}(-2x - \tfrac{6xy}{h} + 2L + \tfrac{6Ly}{h}), \qquad \sigma_y = 0$$
$$\tau_{xy} = -\tfrac{\tau_0}{4}(1 - \tfrac{2y}{h} - \tfrac{3y^2}{h^2}) \qquad \text{(b)}$$

From Eqs. (b), we determine

Edge $y = h$: $\quad \sigma_y = 0 \qquad \tau_{xy} = \tau_0$

Edge $y = -h$: $\quad \sigma_y = 0 \qquad \tau_{xy} = 0$

Edge $x = L$: $\quad \sigma_x = 0, \qquad \tau_{xy} = -\tfrac{\tau_0}{4}(1 - \tfrac{2y}{h} - \tfrac{3y^2}{h^2})$

It is observed from the above that boundary conditions are satisfied at $y = \pm h$, but *not* at $x = L$.

◀

SOLUTION (3.15)

(a) For $\nabla^4\Phi = 0$, $e = -5d$ and a, b, c are arbitrary.

Thus
$$\Phi = ax^2 + bx^2y + cy^3 + d(y^5 - 5x^2y^3) \qquad (1)$$ ◀

(b) The stresses:

$$\sigma_x = \tfrac{\partial^2\Phi}{\partial y^2} = 6cy + 10d(2y^3 - 3x^2y) \qquad (2)$$
$$\sigma_y = \tfrac{\partial^2\Phi}{\partial x^2} = 2a + 2by - 10dy^3 \qquad (3)$$
$$\tau_{xy} = -\tfrac{\partial^2\Phi}{\partial x\partial y} = -2bx - 30dxy^2 \qquad (4)$$

Boundary conditions:

$$\sigma_y = -p \qquad \tau_{xy} = 0 \qquad \text{(at y=h)} \qquad (5)$$

Equations (3), (4), and (5) give

$$b = -15dh^2 \qquad 2a - 40dh^3 = -p \qquad (6)$$
$$\int_{-h}^{h}\sigma_x dy = 0 \qquad \int_{-h}^{h} y\sigma_x dy = 0 \qquad \int_{-h}^{h}\tau_{xy}dy = 0 \qquad \text{(at x=0)} \qquad (7)$$

Equations (2), (4), and (7) yield

$$c = -2dh^2 \qquad (8)$$

Similarly

$$\sigma_y = 0 \qquad \tau_{xy} = 0 \qquad \text{(at y=-h)}$$

give
$$a = 20dh^3 \qquad (9)$$

Solution of Eqs. (6), (8), and (9) results in

$$a = -\tfrac{p}{4} \qquad b = -\tfrac{3p}{16h} \qquad c = -\tfrac{p}{40h} \qquad d = \tfrac{p}{80h^3} \qquad e = -\tfrac{p}{16h^3} \qquad (10)$$

The stresses are therefore

$$\sigma_x = -\tfrac{3py}{20h} + \tfrac{p}{8h^3}(2y^3 - 3x^2y)$$
$$\sigma_y = -\tfrac{p}{2} - \tfrac{3py}{8h} - \tfrac{py^3}{8h^3}$$
$$\tau_{xy} = \tfrac{3px}{8h}(1 - \tfrac{y^2}{h^2})$$

◀

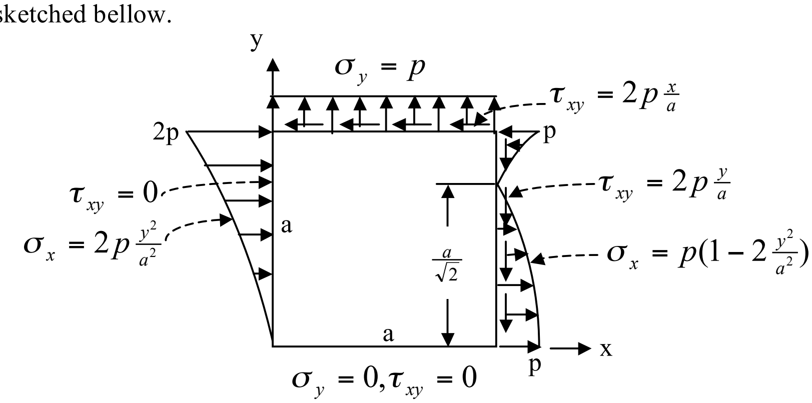
SOLUTION (3.16)

We obtain

$$\sigma_x = \frac{\partial^2\Phi}{\partial y^2} = \frac{p(x^2-2y^2)}{a^2}$$
$$\sigma_y = \frac{\partial^2\Phi}{\partial x^2} = \frac{py^2}{a^2} \qquad \text{(a)} \blacktriangleleft$$
$$\tau_{xy} = -\frac{\partial^2\Phi}{\partial x\partial y} = -\frac{2pxy}{a^2}$$

Taking higher derivatives of Φ, it is seen that Eq. (3.17) is *not* satisfied.

Stress field along the edges of the plate, as determined from Eqs. (a), is sketched bellow.



SOLUTION (3.17)

The first of Eqs. (3.6) with $F_x = 0$

$$\frac{\partial\tau_{xy}}{\partial y} = \frac{pxy}{I}$$

Integrating,

$$\tau_{xy} = \frac{pxy^2}{2I} + f_1(x) \qquad \text{(a)}$$

The boundary condition,

$$(\tau_{xy})_{y=h} = 0 = \frac{pxh^2}{2I} + f_1(x)$$

gives $f_1(x) = -pxh^2/2I$. Equation (a) becomes

$$\tau_{xy} = -\frac{px}{2I}(h^2 - y^2) \qquad \text{(b)} \blacktriangleleft$$

Clearly, $(\tau_{xy})_{y=-h} = 0$ is satisfied by Eq. (b).

Then, the second of Eqs. (3.6) with $F_y = 0$ results in

$$\frac{\partial\sigma_y}{\partial y} = \frac{p(h^2-y^2)}{2I}$$

Integrating,

$$\sigma_y = \frac{p}{2I}y(h^2 - \frac{y^2}{3}) + f_2(x) \qquad \text{(c)}$$

Boundary condition, with $t = 3I/2h^2$,

(CONT.)

$$(\sigma_y)_{y=-h} = -\frac{p}{t} = -\frac{ph}{2I}(h^2 - \frac{h^3}{3}) + f_2(x)$$

gives $f_2(x) = -ph^3/3I$. Equation (c) is thus

$$\sigma_y = \frac{p}{6I}(3h^2 y - y^3 - 2h^3) \qquad \text{(d)} \blacktriangleleft$$

This satisfies the condition that $(\sigma_y)_{y=h} = 0$.

Note that Eq. (3.12) is not satisfied: the solution obtained does *not* provide a compatible displacement field.

SOLUTION (3.18)

Substituting the stresses from Eqs. (3.10) into Eqs. (3.6) and taking $F_x = F_y = 0$:

$$\frac{E}{1-\nu^2}(\frac{\partial \varepsilon_x}{\partial x} + \nu\frac{\partial \varepsilon_y}{\partial x}) + G\frac{\partial \gamma_{xy}}{\partial y} = 0$$

$$\frac{E}{1-\nu^2}(\frac{\partial \varepsilon_y}{\partial y} + \nu\frac{\partial \varepsilon_x}{\partial y}) + G\frac{\partial \gamma_{xy}}{\partial x} = 0$$

or

$$\frac{E}{2(1+\nu)}[\frac{2}{1-\nu}\frac{\partial^2 v}{\partial x^2} + \frac{2\nu}{1-\nu}\frac{\partial^2 v}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 v}{\partial x \partial y}] = 0$$

$$\frac{E}{2(1+\nu)}[\frac{2}{1-\nu}\frac{\partial^2 v}{\partial y^2} + \frac{2\nu}{1-\nu}\frac{\partial^2 u}{\partial x \partial y} - \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 v}{\partial x^2}] = 0$$

The foregoing become

$$\frac{\partial^2 u}{\partial x^2} + \frac{1+\nu}{1-\nu}\frac{\partial^2 u}{\partial x^2} - \frac{1+\nu}{1-\nu}\frac{\partial^2 v}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$\frac{\partial^2 v}{\partial y^2} + \frac{1+\nu}{1-\nu}\frac{\partial^2 v}{\partial y^2} + \frac{1+\nu}{1-\nu}\frac{\partial^2 v}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} = 0$$

or

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{1+\nu}{1-\nu}\frac{\partial}{\partial x}(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}) = 0$$

$$\frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial x^2} + \frac{1+\nu}{1-\nu}\frac{\partial}{\partial y}(\frac{\partial v}{\partial y} + \frac{\partial u}{\partial x}) = 0$$

SOLUTION (3.19)

It is readily found that

$$\frac{\partial \sigma_x}{\partial x} = -\frac{pxy}{I} \qquad \frac{\partial \tau_{xy}}{\partial y} = \frac{pxy}{I}$$

And $\frac{\partial \sigma_y}{\partial y} = -\frac{p}{I}(-h^2 + y^2) \qquad \frac{\partial \tau_{xy}}{\partial x} = -\frac{p}{2I}(h^2 - y^2)$

Thus, Eqs. (3.6) are satisfied: stress field is possible.

We have $x_Q = 1.5\,m, y_Q = 0.05\,m$, and

$$I = \frac{2th^3}{3} = \frac{2(0.04)(0.1)^3}{3} = 2.67(10^{-5})\ m^4$$

Substituting the given data, Eqs. (P3.19) yield at Q:

$$\sigma_x = \frac{-10\times10^3}{10(2.67\times10^{-5})}(5\times10^5 + 2\times0.1^2)0.05 + \frac{-10\times10^3}{3(2.67\times10^{-5})}(1.25\times10^{-4})$$
$$= -21.09\ MPa$$

(CONT.)

$$\tau_{xy} = \tfrac{-10\times10^3}{2(2.67\times10^{-5})}(0.1^2 - 0.05^2) = -2.106\ MPa$$
$$\sigma_y = \tfrac{-10\times10^3}{6(2.67\times10^{-5})}(2\times0.1^3 - 3\times0.0005 + 0.05^3) = -0.039\ MPa$$

Applying Eq. (2.29), we have

$$G = \tfrac{200(10^9)}{2(1+0.3)} = 76.9\ MPa$$

Hooke's law is therefore

$$\varepsilon_x = \tfrac{1}{200(10^9)}(-21.09 + 0.3\times0.039)10^6 = -105\ \mu$$
$$\varepsilon_y = \tfrac{1}{200(10^9)}(-0.039 + 0.3\times21.09)10^6 = 31.4\ \mu$$
$$\gamma_{xy} = \tfrac{-2.106(10^6)}{76.9(10^9)} = -27.4\ \mu$$

Principal strains are

$$\varepsilon_{1,2} = \tfrac{-105+31.4}{2} \pm [(\tfrac{136.4}{2})^2 + (\tfrac{-27.4}{2})^2]^{\frac{1}{2}}$$

or $\varepsilon_1 = 32.8\ \mu \qquad \varepsilon_2 = -106\ \mu$ ◄

We have $\theta_p = \frac{1}{2}\tan^{-1}\frac{-27.4}{-105-31.4} = 5.65^o$

For this angle, Eq. (2.14a) yield $\varepsilon_{x'} = 32.8\,\mu$. Thus, ◄

$$\theta_p' = 5.65^o$$

SOLUTION (3.20)

Assume $\varepsilon_x = \varepsilon_y = 0, \qquad \sigma_z = 0, \qquad \sigma_x = \sigma_y =$ constant

which satisfy Eqs. (3.6) and (3.27). Hooke's law becomes

$$\varepsilon_x = \tfrac{1}{E}(\sigma_x - \nu\sigma_y) + \alpha T_1 = 0 \qquad \text{(a)}$$
$$\varepsilon_y = \tfrac{1}{E}(\sigma_y - \nu\sigma_x) + \alpha T_1 = 0 \qquad \text{(b)}$$
$$\varepsilon_z = \tfrac{\nu}{E}(-\sigma_x - \sigma_y) + \alpha T_1 = 0 \qquad \text{(c)}$$

From Eqs. (a) and (b), we obtain $\sigma_x = \sigma_y$. Therefore,

$$\varepsilon_x = \tfrac{\sigma_x}{E}(1-\nu) + \alpha T_1 = 0 \quad ◄$$

This yields

$$\sigma_x = \sigma_y = \tfrac{E\alpha T_1}{\nu - 1} \quad ◄$$

Then, Eq. (c) becomes

$$\varepsilon_z = \tfrac{2\nu\alpha T_1}{1-\nu} + \alpha T_1 \quad ◄$$

We also have: $\tau_{xy} = \tau_{yz} = \tau_{xz} = 0$ and $\gamma_{xy} = \gamma_{yz} = \gamma_{xz}$.

SOLUTION (3.21)

The nonzero strain components are

$$\varepsilon_x = \varepsilon_y = \varepsilon_z = \alpha T$$

(CONT.)

Compatibility equations reduce to

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial y \partial x}$$

$$\frac{\partial^2 \varepsilon_y}{\partial z^2} + \frac{\partial^2 \varepsilon_z}{\partial y^2} = \frac{\partial^2 \gamma_{yz}}{\partial y \partial z}$$

$$\frac{\partial^2 \varepsilon_z}{\partial x^2} + \frac{\partial^2 \varepsilon_x}{\partial z^2} = \frac{\partial^2 \gamma_{xz}}{\partial x \partial z}$$

Adding the above equations and substituting the given data:

$$2\alpha\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right) = 0$$

or

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = 0$$

This equation, for a time independent temperature field, has the solution

$$T = c_1' x + c_2' y + c_3' z + c_4'$$

which may be written as

$$\alpha T = c_1 x + c_2 y + c_3 z + c_4$$ ◀

SOLUTION (3.22)

Stress is $\sigma_x = -\sigma_0$ regardless of T_1 and still we have $\varepsilon_y = 0$. The second of Eqs. (3.26a) is thus

$$\varepsilon_y = \tfrac{1}{E}(\sigma_y + \nu\sigma_0) + \alpha T_1 = 0$$

or $$\sigma_y = -\nu\sigma_0 - E\alpha T_1 \qquad \text{(a)}$$

The first of Eqs. (3.26a) and (a) result in

$$\varepsilon_x = -\tfrac{1-\nu^2}{E}\sigma_0^2 + (1+\nu)\alpha T_1 \qquad \text{(b)}$$

Now, Hooke's law

$$\varepsilon_z = -\tfrac{\nu}{E}(\sigma_x - \sigma_y) + \alpha T_1$$

leads to

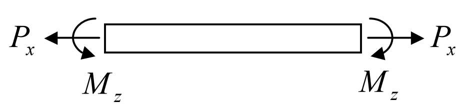
$$\varepsilon_z = \tfrac{\nu(1+\nu)}{E}\sigma_0 + (1+\nu)\alpha T_1 \qquad \text{(c)}$$

Then Equations (2.4) yield, after integration,

$$u = \varepsilon_x x \qquad v = 0 \qquad w = \varepsilon_z z$$ ◀

Here, ε_x and ε_z are given by Eqs. (b) and (c).

SOLUTION (3.23)



Equation (3.27) reduces to

$$\frac{d^2}{dy^2}(\sigma_x + \alpha E T) = 0$$

(CONT.)

from which

$$\sigma_x = -\alpha ET + c_1 y + c_2$$
$$= -\alpha ET(a_1 y + a_2) + c_1 y + c_2$$

Referring to Part (b) of Example 3.2: $c_1 = c_2 = 0$ and

$$\sigma_x = -\alpha E(a_1 y + a_2)$$

We have

$$P_x = \int_{-h}^{h} \sigma_x t dy = -2E\alpha h t a_2 \qquad \blacktriangleleft$$

$$M_z = \int_{-h}^{h} \sigma_x t y dy = -E\alpha t \left| \frac{a_1 y^3}{3} + \frac{a_2 y^2}{2} \right|_{-h}^{h}$$
$$= -\tfrac{2}{3} E\alpha t h^3 a_1 \qquad \blacktriangleleft$$

The P_x and M_z are opposite to that shown above.

SOLUTION (3.24)

Assume a stress distribution:

$$\tau_{xy} = \tau_{yz} = \tau_{xz} = \sigma_y = \sigma_z = 0 \qquad \sigma_x = \text{constant} \qquad \text{(a)}$$

which satisfy Eqs. (3.6) and (3.27). Then, Hooke's law becomes

$$\varepsilon_x = \frac{\sigma_x}{E} + \alpha T, \qquad \varepsilon_y = \varepsilon_z = -\frac{\nu \sigma_x}{E} + \alpha T \qquad \text{(b,c)}$$

$$\gamma_{xy} = \gamma_{yz} = \gamma_{xz} = 0 \qquad \text{(d)}$$

Due to constraint imposed by the walls and because of the uniformity of the temperature distribution: we take $\varepsilon_x = 0$. Equations (a) and (b) give

$$\sigma_x = -\alpha ET$$

Equation (c) is then

$$\varepsilon_y = \varepsilon_z = \alpha T(1+\nu) = \text{constant}$$

The compressive force *P* on the tube is

$$P_x = \sigma_x A = -E\alpha A t \qquad \blacktriangleleft$$

Substituting the data given

$$P_x = -120(10^9)(16.8\times10^{-6})\times(800\times10^{-6})(100)$$
$$= -161.3\ kN \qquad \blacktriangleleft$$

SOLUTION (3.25)

Derivatives of the given stress function are

$$\frac{\partial^2 \Phi}{\partial r^2} = 0, \qquad \frac{1}{r}\frac{\partial \Phi}{\partial r} = -\frac{P\theta}{\pi r}\sin\theta, \qquad \frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \theta^2} = \frac{P\theta}{\pi r}\sin\theta - \frac{2P}{\pi r}\cos\theta$$

Equation (3.40) becomes

$$\nabla^4 \Phi = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)\left(-\frac{2P}{\pi r}\cos\theta\right)$$

(CONT.)

3.25 (CONT.)

After performing the derivatives, we obtain $\nabla^4\Phi = 0$.

Substituting the derivatives obtained above, into Eqs. (3.32):

$$\sigma_r = -\tfrac{P\theta}{\pi r}\sin\theta + \tfrac{P\theta}{\pi r}\sin\theta - \tfrac{2P}{\pi r}\cos\theta$$

$$= -\tfrac{2P}{\pi r}\cos\theta \qquad \blacktriangleleft$$

Similarly, we find $\sigma_\theta = 0$ and $\tau_{r\theta} = 0$.

SOLUTION (3.26)

Refer to Fig. P3.26a. Let $A_{BC} = 1$ and hence $A_{AB} = \cos\theta,\ A_{AC} = \sin\theta$.

$$\sum F_x = 0:$$

$$\sigma_x = \sigma_r \cos\theta\cos\theta + \sigma_\theta \sin\theta\sin\theta - 2\tau_{r\theta}\sin\theta\cos\theta$$

$$\sum F_y = 0:$$

$$\tau_{xy} = \sigma_r \cos\theta\sin\theta - \sigma_\theta \sin\theta\cos\theta + \tau_{r\theta}\cos\theta\cos\theta$$
$$-\tau_{r\theta}\sin\theta\sin\theta$$

Similarly, from Fig. P3.26b:

$$\sum F_y = 0:$$

$$\sigma_y = \sigma_r \sin^2\theta + \sigma_\theta\cos^2\theta + 2\tau_{r\theta}\sin\theta\cos\theta$$

Check: $\sum F_x = 0:$

$$\tau_{xy} = \sigma_r \sin\theta\cos\theta - \sigma_\theta\sin\theta\cos\theta + \tau_{r\theta}(\cos^2\theta - \sin^2\theta)$$

Thus, quoted equations are derived.

SOLUTION (3.27)

Apply the chain rule (Sec. 3.9):

$$\tfrac{\partial\Phi}{\partial x} = \tfrac{\partial\Phi}{\partial r}\cos\theta - \tfrac{1}{r}\tfrac{\partial\Phi}{\partial\theta}\sin\theta$$

and

$$\tfrac{\partial^2\Phi}{\partial x^2} = \tfrac{\partial^2\Phi}{\partial r^2}\cos^2\theta - 2\tfrac{\partial^2\Phi}{\partial\theta\partial r}\tfrac{\sin\theta\cos\theta}{r} + \tfrac{\partial\Phi}{\partial r}\tfrac{\sin^2\theta}{r} - 2\tfrac{\partial\Phi}{\partial r}\tfrac{\sin\theta\cos\theta}{r^2} + \tfrac{\partial^2\Phi}{\partial\theta^2}\tfrac{\sin^2\theta}{r^2} \qquad \text{(a)}$$

Similarly,

$$\tfrac{\partial^2\Phi}{\partial y^2} = \tfrac{\partial^2\Phi}{\partial r^2}\sin^2\theta + 2\tfrac{\partial^2\Phi}{\partial\theta\partial r}\tfrac{\sin\theta\cos\theta}{r} + \tfrac{\partial\Phi}{\partial r}\tfrac{\cos^2\theta}{r} + 2\tfrac{\partial\Phi}{\partial\theta}\tfrac{\sin\theta\cos\theta}{r^2} + \tfrac{\partial^2\Phi}{\partial\theta^2}\tfrac{\cos^2\theta}{r^2} \qquad \text{(b)}$$

Adding Eqs. (a) and (b), we have

$$\tfrac{\partial^2\Phi}{\partial x^2} + \tfrac{\partial^2\Phi}{\partial y^2} = \tfrac{\partial^2\Phi}{\partial r^2} + \tfrac{1}{r}\tfrac{\partial\Phi}{\partial r} + \tfrac{1}{r^2}\tfrac{\partial^2\Phi}{\partial\theta^2} \qquad \text{(c)}$$

By referring to the identity

$$\tfrac{\partial^4\Phi}{\partial x^4} + 2\tfrac{\partial^4\Phi}{\partial x^2\partial y^2} + \tfrac{\partial^4\Phi}{\partial y^4} = (\tfrac{\partial^2}{\partial x^2} + \tfrac{\partial^2}{\partial y^2})(\tfrac{\partial^2\Phi}{\partial x^2} + \tfrac{\partial^2\Phi}{\partial y^2})$$

and Eq. (c), we can readily write the equation quoted, Eq. (3.40).

SOLUTION (3.28)

Equation (3.28) is written as

$$(\tfrac{d^2}{dr^2} + \tfrac{1}{r}\tfrac{d}{dr})(\tfrac{d^2\Phi}{dr^2} + \tfrac{1}{r}\tfrac{d\Phi}{dr}) + \alpha ET(\tfrac{d^2}{dr^2} + \tfrac{1}{r}\tfrac{d}{dr}) = 0$$

(CONT.)

or $$\frac{d^2\Phi}{dr^2} + \frac{1}{r}\frac{d\Phi}{dr} + \alpha ET = 0$$
or
$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d\Phi}{dr}\right) + \alpha ET = 0 \qquad \blacktriangleleft$$

SOLUTION (3.29)

(a) Let $C = \frac{-M}{2(\sin 2\alpha - 2\alpha\cos 2\alpha)}$, and
$$\Phi = C(\sin 2\theta - 2\theta\cos 2\alpha)$$

Various derivatives of Φ are:
$$\frac{\partial\Phi}{\partial r} = 0 \qquad \frac{\partial^2\Phi}{\partial r^2} = 0$$
$$\frac{\partial\Phi}{\partial\theta} = 2C\cos 2\theta - 2C\cos 2\alpha$$
$$\frac{\partial^2\Phi}{\partial\theta^2} = -4C\sin 2\theta$$
and
$$\nabla^2\Phi = \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2} = -\frac{4C\sin 2\theta}{r^2}$$
We thus obtain
$$\nabla^4\Phi = C\left[\frac{8\sin 2\theta}{r^4} - \frac{24\sin 2\theta}{r^4} + \frac{16\sin 2\theta}{r^4}\right] = 0 \qquad \blacktriangleleft$$

(b) $$\sigma_r = \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2} = -\frac{4C\sin 2\theta}{r^2} \qquad \sigma_\theta = 0$$
$$\tau_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\Phi}{\partial\theta}\right) = \frac{2C}{r^2}(\cos 2\theta - \cos 2\alpha) \qquad \blacktriangleleft$$

(c)

Letting $\alpha = \pi/2$: $C = -M/2\pi$. It follows that
$$\sigma_r = \frac{2M\sin 2\theta}{\pi r^2} \qquad \sigma_\theta = 0$$
$$\tau_{r\theta} = \frac{2C}{r^2}(\cos 2\theta + 1) = -\frac{2M\cos^2\theta}{\pi r^2} \qquad \blacktriangleleft$$
where $\cos^2\theta = (1 + \cos 2\theta)/2$

SOLUTION (3.30)

Using Eq. (3.48) and Fig. P3.30:
$$F_x = \int_0^{\pi/2}(\sigma_r r d\theta)\sin\theta = \int_0^{\pi/2}\left(\frac{2P}{\pi}\cos\theta\sin\theta\right)d\theta$$
$$= \frac{2P}{\pi}\left|\frac{1}{2}\sin^2\theta\right|_0^{\pi/2} = \frac{P}{\pi} \quad \rightleftarrows \qquad \blacktriangleleft$$

Similarly,
$$F_y = \int_{-\pi/2}^{\pi/2}(\sigma_r r d\theta)\cos\theta = \int_{-\pi/2}^{\pi/2}\left(\frac{2P}{\pi}\cos^2\theta\right)d\theta$$
$$= \frac{2P}{\pi}\left|\frac{\theta}{2} + \frac{1}{4}\sin 2\theta\right|_{-\pi/2}^{\pi/2} = P \downarrow \qquad \blacktriangleleft$$

SOLUTION (3.31)

NOTE: (In Probs. 3.31, 3.32, and 3.33), the P, L, and α are constants. It can readily be verified that, the maximum values of the functions in parentheses occur:

$$\tfrac{d}{d\theta}(\sin\theta\cos^3\theta) = 0\text{, or }\tan^2\theta = \tfrac{1}{3} \qquad \text{when } \theta = \pm 30^o$$
$$\tfrac{d}{d\theta}(\sin^2\theta\cos^2\theta) = 0\text{, or }\tan^2\theta = 1 \qquad \text{when } \theta = \pm 45^o$$

Maximum stresses, using Eqs. (3.37) and (3.43):

$$(\sigma_x)_{elast.} = \frac{P}{L(\alpha+\frac{1}{2}\sin 2\alpha)}$$
$$(\tau_{xy})_{elast.} = \frac{P\sin\theta\cos^3\theta}{L(\alpha+\frac{1}{2}\sin 2\alpha)} \qquad \text{(a)}$$

Elementary solution of maximum stresses are

$$(\sigma_x)_{elem.} = \frac{P}{2L\tan\alpha}, \qquad (\tau_{xy})_{elem.} = 0 \qquad \text{(b)}$$

(a) For $\alpha = 15^o$, $(\alpha + \frac{1}{2}\sin 2\alpha = 0.512)$:

$(\sigma_x)_{elast.} = P/0.512L$ at $\theta = 0^o$

$(\tau_{xy})_{elast.} = P/2.195L$ at $\theta = 15^o$

$(\sigma_x)_{elem.} = P/0.536L$ at any θ

Thus,

$(\sigma_x)_{elast.} = 1.047(\sigma_x)_{elem.}$ ◄

(b) For $\alpha = 60^o$, $(\alpha + \frac{1}{2}\sin 2\alpha = 1.48)$:

$(\sigma_x)_{elast.} = P/1.48L$ at $\theta = 0^o$

$(\tau_{xy})_{elast.} = P/4.557L$ at $\theta = 30^o$

$(\sigma_x)_{elem.} = P/3.464L$ at any θ

Thus,

$(\sigma_x)_{elast.} = 2.341(\sigma_x)_{elem.}$ ◄

SOLUTION (3.32)

See: **NOTE**, solution of Prob. 3.31.

Substitute $\alpha = 30^o$ into Eqs. (a) and (b) of Solution of Prob. 3.31.

$(\sigma_x)_{elast.} = P/0.957L$ at $\theta = 0^o$

$(\tau_{xy})_{elast.} = P/2.946L$ at $\theta = 30^o$

$(\sigma_x)_{elem.} = P/1.155L$ at any θ

Thus,

$(\sigma_x)_{elast.} = 1.207(\sigma_x)_{elem.}$ ◄

SOLUTION (3.33)

See: NOTE, solution of Prob. 3.31.

We have $r = L/\cos\theta$, $h_{mn}/2 = c = L\cdot\tan\alpha$, and $I = 2c^3/3$.

Equations (3.43) give

$$(\sigma_x)_{elast.} = \frac{F\sin\theta\cos^3\theta}{L(\alpha-\frac{1}{2}\sin 2\alpha)}$$

$$(\tau_{xy})_{elast.} = \frac{F\sin^2\theta\cos^2\theta}{L(\alpha-\frac{1}{2}\sin 2\alpha)} \quad \blacktriangleleft$$

Elementary solution:

$$(\sigma_x)_{elem.} = 3FL/2c^2, \qquad (\tau_{xy})_{elem.} = 3F/4c \quad \blacktriangleleft$$

(a) For $\alpha = 15^o$, $(\alpha + \frac{1}{2}\sin 2\alpha = 0.512)$:

$(\sigma_x)_{elast.} = 19.43F/L$ at $\theta = 15^o$

$(\tau_{xy})_{elast.} = 5.21F/L$ at $\theta = 15^o$

$(\sigma_x)_{elem.} = 20.89F/L$ at $\theta = 15^o$

$(\tau_{xy})_{elem.} = 2.8F/L$ at $\theta = 0^o$

Thus, $(\sigma_x)_{elast.} = 0.93(\sigma_x)_{elem.}$

$(\tau_{xy})_{elast.} = 1.86(\tau_{xy})_{elem.}$ ◀

(b) For $\alpha = 60^o$, $(\alpha - \frac{1}{2}\sin 2\alpha = 0.614)$:

$(\sigma_x)_{elast.} = 0.529F/L$ at $\theta = 30^o$

$(\tau_{xy})_{elast.} = 0.407F/L$ at $\theta = 45^o$

$(\sigma_x)_{elem.} = 0.5F/L$ at $\theta = 60^o$

$(\tau_{xy})_{elem.} = 0.433F/L$ at $\theta = 0^o$

Thus,

$(\sigma_x)_{elast.} = 1.058(\sigma_x)_{elem.}$

$(\tau_{xy})_{elast.} = 0.94(\tau_{xy})_{elem.}$ ◀

SOLUTION (3.34)

With $x = r\cdot\cos\theta$, Eqs. (3.50) become

$$\sigma_x = -(\tfrac{2P}{\pi r})\cos^3\theta$$

$$\sigma_y = -(\tfrac{2P}{\pi r})\sin^2\theta\cos\theta$$

$$\tau_{xy} = -(\tfrac{2P}{\pi r})\sin\theta\cos^2\theta$$

Substituting,

$$d\sigma_x = -\tfrac{2}{\pi r}(\tfrac{prd\theta}{\cos\theta})\cos^3\theta = -\tfrac{2p}{\pi}\cos^2\theta d\theta$$

$$d\sigma_y = -\tfrac{2p}{\pi}\sin^2\theta, \qquad d\tau_{xy} = -\tfrac{p}{\pi}\sin 2\theta d\theta$$

(CONT.)

Integrating,

$$\sigma_x = -\tfrac{2p}{\pi}\int_{\theta_1}^{\theta_2}\cos^2\theta d\theta = -\tfrac{p}{2\pi}[2(\theta_2-\theta_1)+(\sin 2\theta_2-\sin 2\theta_1)]$$

$$\sigma_y = -\tfrac{p}{2\pi}[2(\theta_2-\theta_1)-(\sin 2\theta_2-\sin 2\theta_1)] \qquad \blacktriangleleft$$

$$\tau_{xy} = \tfrac{p}{2\pi}[\cos 2\theta_2-\cos 2\theta_1]$$

SOLUTION (3.35)

APPROACH (a):

$$(\tfrac{d^2}{dr^2}+\tfrac{1}{r}\tfrac{d}{dr})(\tfrac{d^2f_1}{dr^2}+\tfrac{1}{r}\tfrac{df_1}{dr})=0 \tag{d}$$

We have

$$\tfrac{d}{dr}(\tfrac{d^2f_1}{dr^2})=\tfrac{d^3f_1}{dr^3}, \qquad \tfrac{d^2}{dr^2}(\tfrac{d^2f_1}{dr^2})=\tfrac{d^4f_1}{dr^4}$$

$$\tfrac{d}{dr}(\tfrac{1}{r}\tfrac{df_1}{dr})=-r^{-2}\tfrac{df_1}{dr}+r^{-1}\tfrac{d^2f_1}{dr^2}$$

$$\tfrac{d^2}{dr^2}(\tfrac{1}{r}\tfrac{df_1}{dr})=\tfrac{2}{r^3}\tfrac{df_1}{dr}-\tfrac{2}{r}\tfrac{d^2f_1}{dr^2}+\tfrac{1}{r}\tfrac{d^3f_1}{dr^3}$$

$$\tfrac{1}{r}\tfrac{d}{dr}(\tfrac{d^2f_1}{dr^2})=\tfrac{1}{r}\tfrac{d^3f_1}{dr^3}$$

$$\tfrac{1}{r}\tfrac{d}{dr}(\tfrac{1}{r}\tfrac{df_1}{dr})=\tfrac{1}{r}(-\tfrac{1}{r^2}\tfrac{df_1}{dr}+\tfrac{1}{r}\tfrac{d^2f_1}{dr^2})$$

Then, Eq. (d) becomes

$$\tfrac{d^4f_1}{dr^4}+\tfrac{2}{r}\tfrac{d^3f_1}{dr^3}-\tfrac{1}{r^2}\tfrac{d^2f_1}{dr^2}+\tfrac{1}{r^3}\tfrac{df_1}{dr}=0 \tag{d'}$$

The first equation of Problem 3.35 may be written as:

$$\tfrac{1}{r}\tfrac{d}{dr}\{r\tfrac{d}{dr}[r^{-1}\tfrac{d}{dr}(r\tfrac{df_1}{dr})]\}=0$$

$$\tfrac{1}{r}\tfrac{d}{dr}\{r\tfrac{d}{dr}[r^{-1}\tfrac{df_1}{dr}+\tfrac{d^2f_1}{dr^2}]\}=0$$

$$\tfrac{1}{r}\tfrac{d}{dr}\{r[-r^{-2}\tfrac{df_1}{dr}+r^{-1}\tfrac{d^2f_1}{dr^2}+\tfrac{d^3f_1}{dr^3}]\}=0$$

$$\tfrac{1}{r}\tfrac{d}{dr}\{-r^{-1}\tfrac{df_1}{dr}+\tfrac{d^2f_1}{dr^2}+r\tfrac{d^3f_1}{dr^3}\}=0$$

$$\tfrac{1}{r}\{r^{-2}\tfrac{df_1}{dr}-r^{-1}\tfrac{d^2f_1}{dr^2}+2\tfrac{d^3f_1}{dr^3}+r\tfrac{d^4f_1}{dr^4}\}=0$$

or

$$\tfrac{1}{r^3}\tfrac{df_1}{dr}-\tfrac{1}{r^2}\tfrac{d^2f_1}{dr^2}+\tfrac{2}{r}\tfrac{d^3f_1}{dr^3}+\tfrac{d^4f_1}{dr^4}\}=0$$

which is the same as Eq. (d').

Now let us integrate the expression:

$$\tfrac{1}{r}\tfrac{d}{dr}\{r\tfrac{d}{dr}[\tfrac{1}{r}\tfrac{d}{dr}(r\tfrac{df_1}{dr})]\}=0$$

$$r\tfrac{d}{dr}[\tfrac{1}{r}\tfrac{d}{dr}(r\tfrac{df_1}{dr})]=c_1$$

$$\tfrac{1}{r}\tfrac{d}{dr}(r\tfrac{df_1}{dr})=c_1\ln r+c_2$$

$$r\tfrac{df_1}{dr}=c_1\int r\ln r dr+c_2\int r dr$$

$$r\tfrac{df_1}{dr}=c_1[\tfrac{r^2}{2}\ln r-\tfrac{r^4}{4}]+c_2\tfrac{r^2}{2}+c_3$$

$$\tfrac{df_1}{dr}=c_1r\ln r+c_2r^2+c_3\ln r+c_4$$

(CONT.)

or $$f_1 = c_1 r^2 \ln r + c_2 r^2 + c_3 \ln r + c_4$$

Expression (e) may be treated in a like manner.

APPROACH (b):

Letting $t = \ln r$, we have

$$\frac{df_1}{dr} = \frac{df_1}{dt}\frac{dt}{dr} = \frac{1}{r}\frac{df_1}{dr}$$
$$\frac{d^2 f_1}{dr^2} = \frac{1}{r^2}\left(\frac{d^2 f_1}{dt^2} - \frac{df_1}{dt}\right)$$
$$\frac{d^3 f_1}{dr^3} = \frac{1}{r^3}\left(\frac{d^3 f_1}{dt^3} - 3\frac{d^2 f_1}{dt^2} + 2\frac{df_1}{dt}\right)$$
$$\frac{d^4 f_1}{dr^4} = \frac{1}{r}\left(\frac{d^4 f_1}{dt^4} - 6\frac{d^3 f_1}{dt^3} + 11\frac{d^2 f_1}{dt^2} - 6\frac{df_1}{dt}\right)$$

Substituting these derivatives into Eq. (d’), we obtain:

$$\frac{d^4 f_1}{dt^4} - 4\frac{d^3 f_1}{dt^3} + 4\frac{d^2 f_1}{dt^2} = 0$$

This is an ordinary differential equation with constant coefficients. It has a solution

$$f_1 = c_1 r^2 \ln r + c_2 r^2 + c_3 \ln r + c_4$$

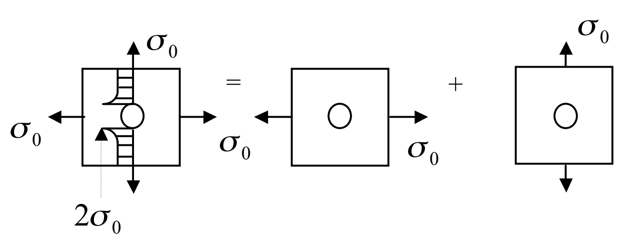
In a like manner, it can be shown that,

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{4}{r^2}\right)\left(\frac{d^2 f_2}{dr^2} + \frac{1}{r}\frac{df_2}{dr^2} - \frac{4f_2}{r^2}\right) = 0$$

is solved to yield Eq. (g) of Section 3.12.

SOLUTION (3.36)

(a)



$$\sigma_{r1} = \frac{\sigma_0}{2}\left[\left(1 - \frac{a^2}{r^2}\right) + \left(1 + \frac{3a^4}{r^4} - \frac{4a^2}{r^2}\right)\cos 2\theta\right]$$
$$\sigma_{\theta 1} = \frac{\sigma_0}{2}\left[\left(1 + \frac{a^2}{r^2}\right) - \left(1 + \frac{3a^4}{r^4}\right)\cos 2\theta\right]$$
$$\tau_{r\theta 1} = -\frac{\sigma_0}{2}\left(1 - \frac{3a^4}{r^4} + \frac{2a^2}{r^2}\right)\sin 2\theta$$

and

$$\sigma_{r2} = \frac{\sigma_0}{2}\left[\left(1 - \frac{a^2}{r^2}\right) + \left(1 + \frac{3a^4}{r^4} - \frac{4a^2}{r^2}\right)\cos 2(\theta + 90^o)\right]$$
$$\sigma_{\theta 2} = \frac{\sigma_0}{2}\left[\left(1 + \frac{a^2}{r^2}\right) - \left(1 + \frac{3a^4}{r^4}\right)\cos 2(\theta + 90^o)\right]$$
$$\tau_{r\theta 2} = -\frac{\sigma_0}{2}\left(1 - \frac{3a^4}{r^4} + \frac{2a^2}{r^2}\right)\cos 2(\theta + 90^o)$$

(CONT.)

We have, by superposition:

$$\sigma_r = \sigma_{r1} + \sigma_{r2} \qquad \sigma_\theta = \sigma_{\theta 1} + \sigma_{\theta 2} \qquad \tau_{r\theta} = \tau_{r\theta 1} + \tau_{r\theta 2}$$

Hence, at r=a and $\theta = \pi/2$,

$$\sigma_{r1} = 0 \qquad \sigma_{r2} = 0$$

$$\sigma_{\theta 1} = 3\sigma_0 \qquad \sigma_{\theta 2} = -\sigma_0$$

$$\tau_{r\theta 1} = 0 \qquad \tau_{r\theta 2} = 0$$

lead to the solution:

$$\sigma_r = 0 \qquad \sigma_\theta = 2\sigma_0 \qquad \tau_{r\theta} = 0 \qquad \blacktriangleleft$$

(b) Referring to the results of part (a), we write

$$\sigma_{r1} = 0 \qquad \sigma_{r2} = 0$$

$$\sigma_{\theta 1} = 3\sigma_0 \qquad \sigma_{\theta 2} = \sigma_0$$

$$\tau_{r\theta 1} = 0 \qquad \tau_{r\theta 2} = 0$$

Thus,

$$\sigma_r = 0 \qquad \sigma_\theta = 4\sigma_0 \qquad \tau_{r\theta} = 0 \qquad \blacktriangleleft$$

SOLUTION (3.37)

We have

$$\frac{d}{D} = \frac{1}{3}$$

Then, from Fig. D.8: $K \approx 2.3$. Hence

$$\sigma_{\max} = K\frac{P}{A} = 2.3\frac{180(10^3)}{(150-50)20} = 207 \ MPa \qquad \blacktriangleleft$$

SOLUTION (3.38)

$$K = \frac{\sigma_{\max}}{\sigma_{nom}} = \frac{130}{80} = 1.625$$

From Fig. D.1: $r/d = 0.25$. Then

$$D = 2r + d$$

gives

$$40 = 2(0.25d) + d = 1.5d$$

or

$$d = 26.7 \ mm$$

$$r = 6.67 \ mm \qquad \blacktriangleleft$$

SOLUTION (3.39)

For $\dfrac{r}{d} = 0.15$:

$$D = 2r + d; \quad 40 = 2(0.15d) + d = 1.3d$$

(CONT.)

or $d = 30.76$ We thus have

$$\frac{D}{d} = 1.3$$

Figure D.1 gives $K \approx 1.7$. Hence,

$$\sigma_{\max} = 250(10^6) = 1.7\frac{P_{all}}{(20)(30.76)}$$

or $$P_{all} = 90.5\ kN$$ ◀

SOLUTION (3.40)

$T_{BC} = 1\ kN \cdot m \longrightarrow\!\!\!\!\!\rightarrow \qquad T_{AB} = 2\ kN \cdot m \longrightarrow\!\!\!\!\!\rightarrow$

Then, $\tau_{\max} = 2T/\pi c^3$ yields

$$\tau_{BC} = \frac{2(1\times 10^3)}{\pi(0.02)^3} = 79.6\ MPa$$

$$\tau_{AB} = \frac{2(2\times 10^3)}{\pi(0.025)^3} = 81.5\ MPa$$

Thus

$$\tau_{\max} = K\tau_{AB} = (1.6)(81.5) = 130.4\ MPa$$ ◀

SOLUTION (3.41)

(**a**) We have

$$\frac{D}{d} = \frac{40}{35} = 1.14 \qquad \frac{r}{d} = \frac{2}{35} = 0.057$$

Hence, by Fig. D.4, $K \approx 1.6$.

We have

$$J = \frac{\pi}{32}(d)^4 = \frac{\pi}{32}(35)^4 = 147.3(10^3)\ mm^4$$

So $$\tau_{yp} = K\frac{Tc}{J} = 1.6[\frac{100(0.0175)}{147.3(10^{-9})}] = 19\ MPa$$ ◀

SOLUTION (3.42)

We have

$$\frac{D}{d} = \frac{40}{16} = 2.5 \qquad \frac{r}{d} = \frac{8}{16} = 0.2$$

From Fig. D.6 , $K \approx 1.31$

$$\tau_{\max} = \tau_{all} = K\frac{Tc}{J}; \qquad 250(10^6) = 1.31[\frac{16T}{\pi(0.016)^3}]$$

or

$$T = 153.5\ N \cdot m$$ ◀

SOLUTION (3.43)

$\frac{D}{d} = \frac{37.5}{25} = 1.5, \qquad \frac{r}{d} = \frac{5}{25} = 0.2$

Figure D.1, $K = 1.72$.

Hence $\sigma_{nom} = \frac{\sigma_{\max}}{1.72} = \frac{210/1.4}{1.72} = 87.2\ MPa$

and $P_{all} = A\sigma_{nom} = (25 \times 10)(87.2) = 21.8\ kN$ ◄

SOLUTION (3.44)

<u>At a section through *B*</u>

$$M_B = 400(0.3) = 120\ N \cdot m$$

$$\sigma_{nom} = \frac{M_B c}{I} = \frac{120(0.02)}{\frac{1}{12}(0.012)(0.04)^3} = 37.5\ MPa$$

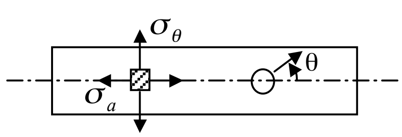
(**a**) $\frac{r}{d} = \frac{5}{40} = 0.125 \qquad \frac{D}{d} = \frac{60}{40} = 1.5: \quad K \approx 1.65$ (Fig. D.2)

$$\sigma_{\max} = 1.65(37.5) = 61.9\ MPa$$ ◄

(**b**) $\frac{r}{d} = \frac{10}{40} = 0.25 \qquad \frac{D}{d} = \frac{60}{40} = 1.5: \quad K \approx 1.41$ (Fig. D.2)

$$\sigma_{\max} = 1.41(37.5) = 52.8\ MPa$$ ◄

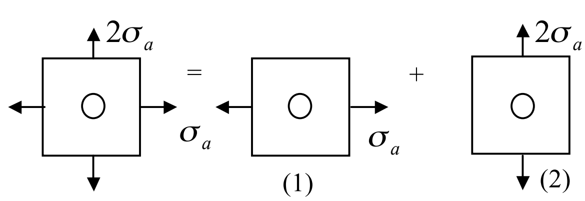
SOLUTION (3.45)



Without hole:

$$\sigma_\theta = \frac{pd}{2t} \qquad \sigma_a = \frac{pd}{4t}$$

With hole:



(CONT.)

We use Eq. (3.55b), with r=a.

$$\sigma_{\theta 1} = \tfrac{\sigma_a}{2}[(1+\tfrac{a^2}{a^2})-(1+\tfrac{3a^4}{a^4})\cos 2\theta]$$

$$= \sigma_a[1-2\cos 2\theta] \quad ◀$$

$$\sigma_{\theta 2} = 2\sigma_a[1-2\cos 2(\theta+90^o)] \quad ◀$$

For $\theta = 0^o$:

$$\sigma_{\theta 1} = -\sigma_a \qquad \sigma_{\theta 2} = 6\sigma_a$$

For $\theta = \pi/2$:

$$\sigma_{\theta 1} = 3\sigma_a \qquad \sigma_{\theta 2} = -2\sigma_a$$

Therefore, superposing the results at $\theta = 0^o$:

$$\sigma_\theta = 5\sigma_a = 5pd/4t \quad ◀$$

at $\theta = \pi/2$:

$$\sigma_\theta = \sigma_a = pd/4t \quad ◀$$

SOLUTION (3.46)

(a) We have D/d=1.1 and r/d=0.05. Then, we find from Figs. D.6, D.7, and D.5 that

$$K_t = 1.64 \qquad K_b = 2.2 \qquad K_a = 2.3$$

Then, Eqs. (b) of Example 3.5 yield

$$\sigma_x = 2.3\frac{50(10^3)}{\pi(0.2)^2} + 2.2\frac{4(10\times 10^3)}{\pi(0.2)^3} = 4.42\ MPa$$

$$\tau_{xy} = 1.64\frac{2(20\times 10^3)}{\pi(0.2)^3} = 2.61\ MPa$$

Equation (a) of Example 3.4 is therefore

$$\sigma_{1,2} = \tfrac{4.42}{2} \pm [(\tfrac{4.42}{2})^2 + (2.61)^2]^{\frac{1}{2}}$$

or $\sigma_1 = 5.63\ MPa \qquad \sigma_2 = -1.21\ MPa$ ◀

(b) $\tau_{max} = \tfrac{1}{2}(5.63+1.21) = 3.42\ MPa$ ◀

(c) $\sigma_{oct} = \tfrac{1}{3}(5.63-1.21) = 1.47\ MPa$ ◀

$$\tau_{oct} = \tfrac{1}{3}[(5.63+1.21)^2 + (-1.21)^2 + (-5.63)^2]^{\frac{1}{2}} = 2.98\ MPa \quad ◀$$

SOLUTION (3.47)

(a) We have D/d=2 and r/d=0.04. Then, we find from Figs. D.7 and D.6 that

$$K_b = 2.6 \qquad K_t = 1.9$$

Equation (b) of Example 3.5 is therefore

$$\sigma_x = 2.6\frac{4(20\times 10^3)}{\pi(0.125)^3} = 33.9\ MPa$$

$$\tau_{xy} = 1.9\frac{2(5\times 10^3)}{\pi(0.125)^3} = 9.73\ MPa$$

(CONT.)

3.47 (CONT.)

Equation (a) of Example 3.5:

$$\sigma_{1,2} = \tfrac{33.9}{2} \pm [(\tfrac{33.9}{2})^2 + (9.73)^2]^{\frac{1}{2}}$$

or $\quad \sigma_1 = 36.5\ MPa \qquad \sigma_2 = -2.59\ MPa$ ◀

(b) $\tau_{\max} = \tfrac{1}{2}(36.5 + 2.59) = 16.96\ MPa$

(c) $\sigma_{oct} = \tfrac{1}{3}(36.5 - 2.59) = 11.3\ MPa$ ◀

$\tau_{oct} = \tfrac{1}{3}[(36.5 + 2.59)^2 + (-2.59)^2 + (-36.5)^2]^{\frac{1}{2}} = 17.85\ MPa$ ◀

SOLUTION (3.48)

We apply Eqs. (3.63).

(a) $a = 0.88[\frac{2(500)(0.025\times 0.0375)}{(200\times 10^9)(0.0125)}]^{\frac{1}{3}} = 0.635\ mm$ ◀

(b) $\sigma_c = 0.62[500(200\times 10^9)^2(\frac{0.0125}{2\times 0.025\times 0.0375})^2]^{\frac{1}{3}} = 596.1\ MPa$ ◀

(c) $\delta = 1.54[(500)^2(\frac{0.0125}{2(200\times 10^9)^2(0.025\times 0.0375)})]^{\frac{1}{3}} = 5.339(10^{-3})\ mm$ ◀

SOLUTION (3.49)

(a) Use Eq. (3.62):

$$\sigma_c = 0.62[\frac{500(200\times 10^9)^2}{4(0.025)^2}]^{\frac{1}{3}} = 1233\ MPa$$ ◀

(b) Apply Eqs. (3.60) and (3.59) for $r_1 = r_2 = r$ and $E_1 = E_2 = E$ to obtain the formula

$$\sigma_c = 0.617[\frac{PE^2}{r^2}]^{\frac{1}{3}}$$

Thus, $\quad \sigma_c = 0.617[\frac{500(200\times 10^9)^2}{(0.025)^2}]^{\frac{1}{3}} = 1959\ MPa$ ◀

SOLUTION (3.50)

Using Eqs. (3.73), (3.74), and the second of (3.70), we have

$$m = \frac{4}{(1/0.4)+(1/0.25)} = 0.6154$$

$$n = \frac{4(200\times 10^9)}{3(1-0.3^2)} = 2.9304(10^{11})$$

$$\cos\alpha = \pm\frac{(1/0.4)-(1/0.25)}{(1/0.4)+(1/0.25)} = 0.2308 \quad \text{or} \quad \alpha = 76.66^o$$

Interpolating Table 3.3:

$$c_a = 1.1774 \qquad c_b = 0.8616$$

Apply Eqs. (3.69):

$$a = 1.774[\frac{4(10^3)(0.6154)}{2.9304\times 10^{11}}]^{\frac{1}{3}} = 2.393\ mm$$ ◀

$$b = 0.8616[\frac{4(10^3)(0.6154)}{2.9304\times 10^{11}}]^{\frac{1}{3}} = 1.752\ mm$$ ◀

Thus,

$$\sigma_c = 1.5\frac{4(10^3)}{\pi(2.393\times 1.752)10^{-6}} = 455.5\ MPa$$ ◀

SOLUTION (3.51)

Use Eqs. (3.67):

$$\sigma_c = 0.418[\tfrac{2.5(10^3)(200\times10^9)}{0.1(0.005)}]^{\frac{1}{2}} = 418\ MPa \quad ◀$$

$$2b = 2\{1.52[\tfrac{2.5(10^3)(0.005)}{200\times10^9(0.1)}]^{\frac{1}{2}}\} = 2(0.038) = 0.076\ mm \quad ◀$$

SOLUTION (3.52)

Equations (3.65), for $\nu_1 = \nu_2 = 0.25,\ E_1 = E_2 = E,\ r_1 = r_2 = r$:

$$\sigma_c = 0.412[\tfrac{2(10^6)(200\times10^9)(2)}{0.2}]^{\frac{1}{2}} = 824\ MPa \quad ◀$$

$$2b = 2\{1.545[\tfrac{2(10^6)(0.2)}{200\times10^9(2)}]^{\frac{1}{2}}\} = 2(1.545) = 3.09\ mm \quad ◀$$

SOLUTION (3.53)

Refer to Example 3.6.

$$1/r_1' = 0 \qquad 1/r_2' = 0 \qquad \theta = \pi/2$$

$$m = \tfrac{4}{(1/0.5)+(1/0.2)} = 0.5714$$

$$n = \tfrac{4(210\times10^9)}{3(1-0.25^2)} = 2.9867(10^{11})$$

$$\cos\alpha = \pm\tfrac{(1/0.5)-(1/0.2)}{(1/0.5)+(1/0.2)} = 0.4286 \quad \text{or} \quad \alpha = 64.62^o$$

Table 3.3: $c_a = 1.3862 \qquad c_b = 0.7758$

$$a = 1.3862[\tfrac{5\times10^3(0.5714)}{2.9867\times10^{11}}]^{\frac{1}{3}} = 2.994\ mm \quad ◀$$

$$b = 0.7558[\tfrac{5\times10^3(0.5714)}{2.9867\times10^{11}}]^{\frac{1}{3}} = 1.605\ mm \quad ◀$$

Thus, $\sigma_c = 1.5\tfrac{5(10^3)}{\pi(2.994\times1.605)10^{-6}} = 505.2\ MPa$ ◀

SOLUTION (3.54)

Refer to Example 3.6. We now have $r_1 = r_2 = r$. Thus, Equations (3.75) and (3.76) become

$$m = \tfrac{4}{(1/r)+(1/r)} = 2r = 2(0.2) = 0.4$$

$$\cos\alpha = \tfrac{(1/r)-(1/r)}{(1/r)+(1/r)} = 0, \qquad \alpha = 90^o$$

From Table 3.3 it can be concluded that surface of contact has a circular boundary: $c_a = c_b = 1$.

$$n = \tfrac{4(210\times10^9)}{3(1-0.25^2)} = 2.98667(10^{11})$$

$$a = b = 1[\tfrac{5(10^3)(0.4)}{2.98667\times10^{11}}]^{\frac{1}{3}} = 1.885\ mm \quad ◀$$

Thus,

$$\sigma_c = 1.5\tfrac{5(10^3)}{\pi(1.885)^2 10^{-6}} = 671.9\ MPa \quad ◀$$

SOLUTION (3.55)

Case B (1st column), Table 3.2 with $r_1 = r_2, \quad E_1 = E_2$.

(a) $\Delta = \frac{1}{E} + \frac{1}{E} = \frac{2}{E} \qquad m = \frac{1}{r} + \frac{1}{r} = \frac{2}{r}$

$$a = 0.88\sqrt[3]{F\frac{r}{E}} = 0.88\sqrt[3]{400\frac{0.17}{210(10^9)}} = 0.604\ mm \qquad ◀$$

(b) $p_0 = 1.5\frac{P}{\pi a^2} = 1.5\frac{400}{\pi(0.604)^2(10^{-6})} = 349\ MPa \qquad ◀$

(c) Equations (3.68) at z=0:

$$\sigma_x = \sigma_y = -p_0[(1+\nu) - \tfrac{1}{2}] = -\tfrac{1+2\nu}{2}p_0 = -0.8(349) = -279\ MPa$$

$$\sigma_z = -p_0 = -349\ MPa$$

$$\tau_{yz} = \tau_{xz} = \tfrac{1}{2}(\sigma_x - \sigma_z) = -\tfrac{p_0}{2}(0.8-1) = 34.9\ MPa \qquad ◀$$

SOLUTION (3.56)

Refer 2nd column of C, Table 3.2.

$$E_1 = E_2 = E = 210\ GPa \qquad r_1 = 8\ mm \qquad r_2 = 50\ mm \qquad F_1 = 240\ kN/m$$

Hence

$$\Delta = \frac{2}{E} = \frac{2}{210(10^9)} = \frac{1}{105(10^9)}, \qquad n = \frac{1}{0.008} - \frac{1}{0.05} = 105$$

(a) $a = 1.076[\frac{240(10^3)}{105(10^9)(105)}]^{\frac{1}{2}} = 0.159\ mm \qquad ◀$

(b) $p_0 = \frac{2}{\pi}\frac{F}{aL} = \frac{2}{\pi}\frac{240(10^3)}{0.159(10^{-3})} = 961\ MPa \qquad ◀$

SOLUTION (3.57)

We have $r_1 = 0.48\ m,\ r_2 = 0.34\ m,\ r_1' = \infty,\ r_2' = \infty$, and $\theta = 90^o$. Thus, using Eqs.(3.72) through (3.74):

$$m = \frac{4}{\frac{1}{0.48} + \frac{1}{0.34}} = 0.796 \qquad n = \frac{4(210\times10^9)}{3(1-0.09)} = 3.0769(10^{11})$$

$$A = \frac{2}{m}, \qquad B = \pm\frac{1}{2}(\frac{1}{r_1} - \frac{1}{r_2})$$

$$\cos\alpha = \frac{B}{A} = \pm\frac{1/0.48 - 1/0.34}{1/0.48 + 1/0.34} = 0.143, \qquad \alpha = 81.78^o$$

Interpolating from Table 3.3: $c_a = 1.104, \qquad c_b = 0.911$. Hence

$$a = 1.104[\frac{4(10^3)(0.796)}{3.0769(10^{11})}]^{\frac{1}{3}} = 2.4058\ mm$$

$$b = 0.911[\frac{4(10^3)(0.796)}{3.0769(10^{11})}]^{\frac{1}{3}} = 1.9852\ mm$$

Thus $\quad p_0 = 1.5\frac{F}{\pi ab} = 1.5\frac{4(10^3)}{\pi(2.4058\times1.9852\times10^{-6})} = 400\ MPa \qquad ◀$

SOLUTION (3.58)

Given quantities are:

$$r_1 = r_1' = 0.025\ m \qquad r_2 = -0.03\ m$$

$$r_2' = -0.125\ m \qquad \nu = 0.3 \qquad E = 200\ GPa$$

Thus,

$$m = \frac{4}{\frac{1}{0.025}+\frac{1}{0.025}-\frac{1}{0.3}-\frac{1}{0.125}} = 0.10345$$

$$n = \frac{4(200\times10^9)}{3(1-0.09)} = 293.04029(10^9)$$

Also

$$A = \tfrac{2}{m} = \tfrac{2}{0.10345} = 19.33301$$

$$B = \tfrac{1}{2}[(0)^2 + (\tfrac{1}{r_2} - \tfrac{1}{r_2'}) + 2(0)]^{\frac{1}{2}} = 12.66667$$

$$\alpha = \cos^{-1}\tfrac{12.66667}{19.33301} = 49.06645^o$$

From Table 3.3, we find

$$c_a = 1.78611 \qquad c_b = 0.63409$$

The semiaxes are then

$$a = 1.78611[\tfrac{1800(0.10345)}{293.04029(10^9)}]^{\frac{1}{3}} = 0.00154\ m = 1.54\ mm$$

$$b = 0.67409[\tfrac{1800(0.10345)}{293.04029(10^9)}]^{\frac{1}{3}} = 0.00055\ m = 0.55\ mm$$

Maximum contact pressure is therefore

$$\sigma_c = 1.5\frac{1800}{\pi(1.54\times0.55)(10^{-6})} = 1014.7\ MPa$$ ◄

SOLUTION (3.59)

Now given data is as follows:

$$r_1 = r_1' = 0.02\ m \qquad r_2 = -0.022\ m$$

$$r_2' = -0.125\ m \qquad \nu = 0.3 \qquad E = 200\ GPa$$

Therefore,

$$m = \frac{4}{\frac{1}{0.02}+\frac{1}{0.02}-\frac{1}{0.022}-\frac{1}{0.125}} = 0.08594$$

$$n = \frac{4(200\times10^9)}{3(1-0.09)} = 293.0403(10^9)$$

We have

$$A = \tfrac{2}{m} = 23.2721$$

$$B = \pm\tfrac{1}{2}[-\tfrac{1}{0.022} + \tfrac{1}{0.125}] = 18.7273$$

$$\alpha = \cos^{-1}\tfrac{18.7273}{23.2721} = 36.42^o$$

Using Table 3.3

$$c_a = 2.323 \qquad c_b = 0.541$$

(CONT.)

3.59 (CONT.)

Then, the semiaxes are:

$$a = 2.323\left[\frac{1800(0.08594)}{293.0403\times10^{9}}\right]^{\frac{1}{3}} = 0.00188\ m = 1.88\ mm$$

$$b = 0.541\left[\frac{1800(0.08594)}{293.0403\times10^{9}}\right]^{\frac{1}{3}} = 0.00044\ m = 0.44\ mm$$

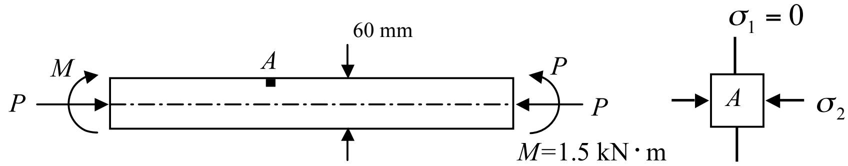
Maximum contact stress is now obtained as

$$\sigma_c = 1.5\frac{1800}{\pi(1.88\times0.44)10^{-6}} = 1039\ MPa$$ ◄

End of Chapter 3

CHAPTER 4

SOLUTION (4.1)



$$\sigma_2 = -\frac{P}{A} - \frac{Mc}{I} = -\frac{4P}{\pi(0.06)^2} - \frac{32(1.5\times10^3)}{\pi(0.06)^3}$$
$$= -353.7 - 70.74\times10^6$$

We have

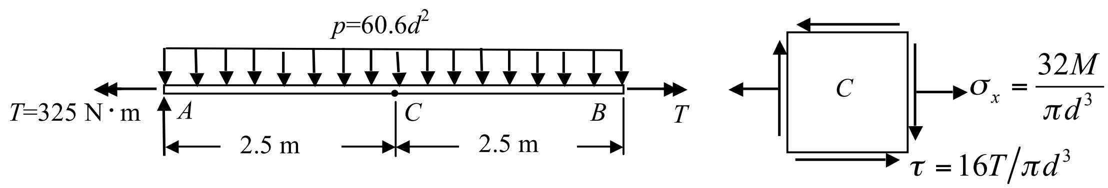
$$\cancel{\sigma_1}^2 - \cancel{\sigma_1}\sigma_2 + \sigma_2^2 = \sigma_{yp}^2 \quad \text{or} \quad |\sigma_2| = |\sigma_{yp}|$$

So,

$$353.7 + 70.74\times10^6 = 250\times10^6, \qquad P_{all} = 707\ kN$$ ◄

SOLUTION (4.2)

$$p = 7.86(9.81)(\pi d^2/4) = 60.6d^2\ kN/m$$



$$\sigma_x = \frac{32(pL^2/8)}{\pi d^3} = \frac{32(60.6d^2)(25)10^3}{8\pi d^3} = \frac{1.93\times10^6}{d}$$
$$\tau = -\frac{16(325)}{\pi d^3} = -\frac{1.66\times10^3}{d^3}$$

Equation (4.9a), $\sigma_x^2 + 3\tau^2 = \sigma_{all}^2$

$$\left(\frac{1.93\times10^6}{d}\right)^2 + 3\left(\frac{-1.66\times10^3}{d^3}\right)^2 = (100\times10^6)^2$$

or

$$\frac{3.725\times10^6}{d^2} + \frac{8.267}{d^6} = 100\times10^8$$

Solving by trial & error: $d = 32.8\ mm.$ ◄

Use a $33-mm$ diameter shaft. ◄

SOLUTION (4.3)

$$\sigma_y = 100\ MPa \qquad \tau_{xy} = -80\ MPa \qquad \sigma_x = 0$$

$\sigma_{yp} = 250\ MPa$ (Table D.1)

Hence

$$\sigma_{1,2} = \frac{\sigma_y}{2} \pm \sqrt{(\frac{-\sigma_y}{2})^2 + \tau_{xy}^2}$$

$$= 50 \pm \sqrt{(-50)^2 + (-80)^2} = 50 \pm 94.3$$

or

$$\sigma_1 = 144.3\ MPa \qquad \sigma_2 = -43.3\ MPa$$

(**a**) Maximum shear stress memory:

$$|\sigma_1 - \sigma_2| \le \frac{\sigma_{yp}}{n}$$

$$|144.3 - (-43.3)| \le \frac{250}{1.5}$$

$$187.6 > 166.7$$

Failure will occur. ◄

(**b**) Maximum energy of distortion theory:

$$\sigma_1^2 - \sigma_1\sigma_2 + \sigma_2^2 \le (\frac{\sigma_{yp}}{n})$$

$$|(144.3)^2 - (144.3)(-43.3) - (-43.3)^2|^{1/2} \le 166.7$$

$$160.1 < 166.7$$

Failure will not occur. ◄

SOLUTION (4.4)

State of stress is given by

$$\sigma_1 = \sigma = \frac{32M}{\pi(0.1)^3} + \frac{4P}{\pi(0.1)^2}, \qquad \sigma_2 = \sigma_3 = 0$$

Refer to Table 4.1 and Eq. (2.66):

$$0.47\sigma_{yp} = 0.47\sigma \quad \text{or} \qquad \sigma_{yp} = \sigma$$

Thus, $221(10^3) = \frac{32(17)}{\pi(0.1)^3} + \frac{4P}{\pi(0.1)^2}$

Solving,

$$P = 375.7\ kN$$ ◄

SOLUTION (4.5)

From Table D.1: $\sigma_{yp} = 250\ MPa$

We have

$$\sigma_{1,2} = \frac{-30+90}{2} \pm [(\frac{-30-90}{2})^2 + (-40)^2]^{\frac{1}{2}}$$

(CONT.)

or $\sigma_1 = 102.1\ MPa \qquad \sigma_2 = -42.1\ MPa$

(a) Equation (4.2a) gives

$$|102.1 + 42.1| = \tfrac{250}{n}$$

or $n = 1.73$ ◄

(b) Equation (4.5a) leads to

$$(102.1)^2 - (102.1)(-42.1) + (-42.1)^2 = (\tfrac{250}{n})^2$$

from which

$$n = 1.95$$ ◄

SOLUTION (4.6)

Referring to Appendix B, we obtain

$$\sigma_1 = 101.3\ MPa \qquad \sigma_2 = 0 \qquad \sigma_3 = -51.32\ MPa$$

(a) $|\sigma_1 - \sigma_3| = \sigma_{yp}$

or $\sigma_{yp} = |101.3 + 51.32| = 152.6\ MPa$ ◄

(b) $\tau_{oct} = \frac{1}{3}[(101.3)^2 + (101.3 + 51.32)^2 + (51.32)^2]^{\frac{1}{2}} = 63.41\ MPa$

Thus,

$$\sigma_{yp} = 63.41/0.47 = 134.9\ MPa$$ ◄

SOLUTION (4.7)

Using Eq. (4.6), we have

$$0.47\tfrac{\sigma_{yp}}{n} = \tfrac{1}{3}[(\sigma_1 - \sigma_2)^2 + \sigma_1^2 + \sigma_2^2]^{\frac{1}{2}} \qquad \text{(a)}$$

Here

$$\sigma_{1,2} = \tfrac{\sigma}{2} \pm \tfrac{1}{2}[\sigma^2 + 4\tau^2]^{\frac{1}{2}}$$

and $\sigma = \frac{32M}{\pi d^3} + \frac{4P}{\pi d^2}, \qquad \tau = \frac{16T}{\pi d^3}$

Substituting the given data:

$$\sigma_{1,2} = \tfrac{1}{2}\left[\tfrac{32(4\times10^3)}{\pi(0.12)^3} + \tfrac{4(45\times10^3)}{\pi(0.12)^2}\right]$$

$$\pm \tfrac{1}{2}\left\{\left[\tfrac{32(4\times10^3)}{\pi(0.12)^3} + \tfrac{4(45\times10^3)}{\pi(0.12)^2}\right]^2 + 4\left[\tfrac{16(11.2\times10^3)}{\pi(0.12)^3}\right]^2\right\}^{\frac{1}{2}}$$

Or $\sigma_1 = 49.55\ MPa \qquad \sigma_2 = -21.99\ MPa$

Equation (a) is then

$$1.41\tfrac{280}{n} = [(71.54)^2 + (49.55)^2 + (-21.99)^2]^{\frac{1}{2}}$$

Solving,

$$n = 4.4$$ ◄

SOLUTION (4.8)

Maximum stresses, occurring at the fixed end are:

$$\sigma = \frac{Mc}{I} = \frac{450(0.25t)}{2t^4/3} = \frac{168.75}{t^3}$$

$$\tau = \frac{3}{2}\frac{450}{2t^2} = \frac{337.5}{t^2}$$

From Eq. (4.9a), we have

$$\sigma_{yp}^2 = \sigma^2 + 3\tau^2 = (280\times 10^6)^2$$

Therefore, at neutral axis $\sigma = 0$:

$$\sigma_{yp} = \sqrt{3}\tau \quad \text{gives} \qquad t = 1.45\ mm$$

At the extreme fibers, $\tau = 0$:

$$\sigma_{yp} = \sigma \quad \text{gives} \qquad t = 8.45\ mm$$

Allowable width is thus

$$t_{all} = 8.45\ mm$$ ◀

SOLUTION (4.9)

We have

$$\tau_{yp} = \frac{\sigma_{yp}}{2} = 175\ MPa, \quad \sigma_1 = -\sigma_2 = \tau$$

(a) $$\frac{175}{1.5} = \frac{16(500)}{\pi d^3}$$

or

$$d = 27.95\ mm$$ ◀

(b) $$\sigma_{1,2} = \frac{\sigma}{2} \pm \frac{1}{2}\sqrt{\sigma^2 + 4\tau^2}$$

Here

$$\sigma = \frac{32M}{\pi d^3} = \frac{32(pL^2/8)}{\pi d^3}$$

$$= \frac{32}{8\pi d^3}[\tfrac{1}{4}\pi d^2(77\times 10^3)10^2]$$

$$= 77(10^5)/d$$

$$\tau = \frac{16T}{\pi d^3} = \frac{16(500)}{\pi d^3} = 2547.77/d^3$$

Using Eq. (4.8a):

$$\frac{\sigma_{yp}}{n} = \sqrt{\sigma^2 + 4\tau^2}$$

$$\frac{350(10^6)}{1.5} = [(\frac{77\times 10^5}{d})^2 + 4(\frac{2547.77}{d^3})^2]^{\frac{1}{2}}$$

or

$$5.444(10^{16}) = \frac{5.929(10^{13})}{d^2} + \frac{2.594(10^7)}{d^6}$$

Solving, by trial and error:

$$d = 0.0368\ m = 36.8\ mm$$ ◀

SOLUTION (4.10)

We have $\sigma_{all} = 90/1.2 = 75\ MPa$

(a)

$$\sigma_{all} = |\sigma_1 - \sigma_3| = 63.4 + 12.2 = 75.6 > 75 \qquad \text{Failure occurs} \qquad \blacktriangleleft$$

(b)

$$2\sigma_{all}^2 = (63.4 - 0.53)^2 + (0.53 + 12.2)^2 + (-12.2 - 63.4)^2$$

or

$$\sigma_{all} = 70 < 75 \qquad \text{No failure} \qquad \blacktriangleleft$$

SOLUTION (4.11)

(a) Using the torsion formula,

$$\tau = \frac{T(0.05)}{\pi(0.05)^4/2} = 5093T$$

and $\sigma_1 = -\sigma_2 = 5093T$

Equation (4.5a) yields then

$$[280(10^6)]^2 = (5093T)^2 + (5093T)(5093T) + (-5093T)^2$$

Solving,

$$T = 31.74\ kN \cdot m \qquad \blacktriangleleft$$

(b) We now have

$$\sigma = \frac{400(10^3)\pi}{\pi(0.05)^2} = 160\ MPa$$

$$\tau = 5093T \qquad \text{(as before)}$$

Principal stresses are:

$$\sigma_{1,2} = \underbrace{\frac{160(10^6)}{2}}_{a} \pm \underbrace{\tfrac{1}{2}[(160\times10^6)^2 + 4(5093T)^2]^{\frac{1}{2}}}_{b}$$

$$= a \pm b$$

With this notation, Eq. (4.5a) becomes

$$\sigma_{yp}^2 = (a+b)^2 - (a^2 - b^2) + (a-b)^2 = a^2 + 3b^2$$

Thus,

$$(280\times10^6)^2 = (\tfrac{160\times10^6}{2})^2 + \tfrac{3}{4}[(160\times10^6)^2 + 4(5093T)^2]$$

Solving,

$$T = 26.05\ kN \cdot m \qquad \blacktriangleleft$$

SOLUTION (4.12)

Maximum moment is

$$M = \frac{PL^2}{8} = \frac{6(1.5)^2}{8} = 1.688\ kN \cdot m$$

and hence

$$\sigma = \sigma_1 = \frac{My}{I} = \frac{1.688(0.125)10^3}{0.1(0.25)^3/12} = 1.62\ MPa$$

(CONT.)

4.12 (CONT.)

(a) $\frac{\sigma_{yp}}{n} = \sqrt{\sigma_1^2 - 0} = \sigma_1$

$\frac{28}{n} = 1.62$

from which

$n = 17.3$ ◀

(b) $|\sigma_1 - 0| = \frac{\sigma_{yp}}{n}$

or $n = 17.3$ ◀

SOLUTION (4.13)

Referring to Appendix B, we compute

$$\sigma_1 = 12.05\ MPa \qquad \sigma_2 = -1.521\ MPa \qquad \sigma_3 = -4.528\ MPa$$

Using Eq. (4.5a),

$$2\sigma_{yp}^2 = (12.05 + 1.521)^2 + (-1.521 + 4.528)^2 + (-4.528 - 12.05)^2$$
$$= 468$$

Solving,

$$\sigma_{yp} = 15.3\ MPa$$

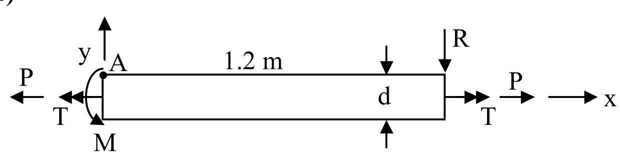
Hence,

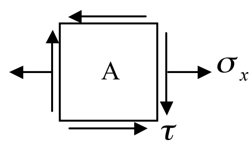
$$\tau_{yp} = 15.3(0.577) = 8.828\ MPa$$

Therefore

$$\sigma_y = 2(\tfrac{9}{8.828}) = 2.039\ MPa$$ ◀
$$\sigma_x = 3(\tfrac{9}{8.828}) = 3.058\ MPa$$

SOLUTION (4.14)





We have $P = 50R$, $T = 0.8R$, and $M = 1.2R$.

Stresses are

$$\sigma_b = \frac{32M}{\pi d^3} = \frac{32(1.2R)}{\pi(0.05)^3} = 97{,}784.8R$$
$$\tau = -\frac{16T}{\pi d^3} = \frac{-16(0.8R)}{\pi(0.05)^3} = -32{,}595R$$
$$\sigma_a = \frac{50R}{\pi(0.05)^2/4} = 25{,}464.8R$$
$$\sigma_x = \sigma_a + \sigma_b = 123{,}249.6R$$

(CONT.)

4.14 (CONT.)

(a) Equation (4.8a):

$$\frac{260(10^6)}{2} = R[(123{,}249.6)^2 + 4(-32{,}595)^2]^{\frac{1}{2}}$$

or $R = 932\ N$ ◀

(b) Equation (4.9a):

$$130 \times 10^6 = R[(123{,}249.6)^2 + 3(-32{,}595)^2]^{\frac{1}{2}}$$

or $R = 959\ N$ ◀

SOLUTION (4.15)

Referring to Appendix B,

$$\sigma_1 = 197.4\ MPa \qquad \sigma_2 = -14.44\ MPa \qquad \sigma_3 = -72.96\ MPa$$

Applying Eq. (4.4b):

$$2\sigma_{yp}^2 = (197.4 + 14.44)^2 + (-14.44 + 72.96)^2 + (-72.96 - 197.4)^2$$

or $\sigma_{yp} = 246.4\ MPa$

Hence,

$$\tau_{yp} = 346.4(0.577) = 142.2\ MPa$$

Thus,

$$\sigma_y = 40\tfrac{140}{142.2} = 39.38\ MPa$$

$$\sigma_x = 50\tfrac{140}{142.2} = 49.23\ MPa$$ ◀

SOLUTION (4.16)

Stresses are

$$\sigma_1 = \tfrac{pr}{t} = \tfrac{p(0.25)}{0.005} = 50p$$

$$\sigma_2 = \tfrac{pr}{2t} = 25p$$

(a) Applying Eq. (4.5a),

$$(50p)^2 - 1250p^2 + (25p)^2 = (280)^2$$

or $p = 6.466\ MPa$ ◀

(b) Using Eq. (4.2a),

$$|50p - 0| = 280$$

or

$$p = 5.6\ MPa$$ ◀

SOLUTION (4.17)

We have $(\sigma_{yp})_{all} = \frac{82}{1.2} = 68.33\ MPa$.

Referring to Appendix B:

$$\sigma_1 = 63.44\ MPa \qquad \sigma_2 = 0.533\ MPa \qquad \sigma_3 = -12.17\ MPa$$

(CONT.)

4.17 (CONT.)

(a) Using Eq. (4.1),

$$\sigma_{yp} = |63.44 + 12.17| = 75.6 > 68.33: \quad \text{Failure occurs} \quad ◀$$

(b) Applying Eq. (4.4b)

$$2\sigma_{yp}^2 = (63.44 - 0.533)^2 + (0.533 + 12.17)^2 + (-12.17 - 63.44)^2$$

or $$\sigma_{yp} = 70.13 > 68.33: \quad \text{Failure occurs} \quad ◀$$

SOLUTION (4.18)

Referring to Appendix B:

$$\sigma_1 = 162.4 \ MPa \qquad \sigma_2 = 46.15 \ MPa \qquad \sigma_3 = 1.468 \ MPa$$

(a) Equation (4.1) yields

$$n = \frac{300}{162.4 - 1.468} = 1.86 \quad ◀$$

(b) Equation (4.4b) gives

$$n^2 = \frac{2\sigma_{yp}^2}{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2} = \frac{2(300)^2}{(116.25)^2 + (44.682)^2 + (160.932)^2}$$

or $$n = 2.09 \quad ◀$$

SOLUTION (4.19)

Referring to Appendix B:

$$\sigma_1 = 156.2 \ MPa \qquad \sigma_2 = 42.13 \ MPa \qquad \sigma_3 = 11.7 \ MPa$$

(a) Equation (4.1) gives

$$n = \frac{220}{156.2 - 11.7} = 1.52 \quad ◀$$

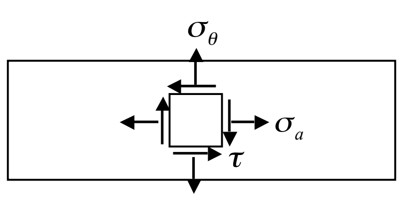
(b) Equation (4.4b) yields

$$n^2 = \frac{2\sigma_{yp}^2}{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2} = \frac{2(220)^2}{(114.07)^2 + (30.43)^2 + (-144.5)^2}$$

or

$$n = 1.67 \quad ◀$$

SOLUTION (4.20)



$$J \approx 2\pi r^3 t$$

$$\sigma_\theta = \frac{pr}{t} = \frac{5(10^6)(105)}{10}$$

$$= 52.5 \ MPa$$

$$\sigma_a = \frac{pr}{2t} = 26.25 \ MPa$$

$$\tau = -\frac{Tc}{J} = -\frac{50 \times 10^3 (0.21)}{2\pi (0.105)^3 (0.01)} = -144.4 \ MPa$$

$$\sigma_{1,2} = \frac{26.25 + 52.5}{2} \pm \sqrt{\left(\frac{26.25 - 52.5}{2}\right)^2 + (-144.4)^2} = 39.4 \pm 145$$

(CONT.)

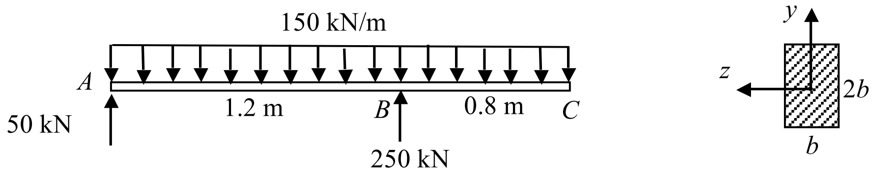
$$\sigma_1 = 184.4\ MPa \qquad \sigma_2 = -105.6\ MPa$$

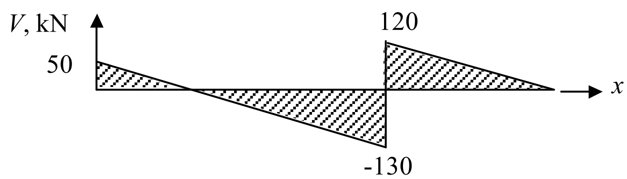
Thus, $\sigma_1 = \dfrac{\sigma_u}{n}; \qquad 184.4 = \dfrac{250}{n}, \qquad n = 1.4$

and $\sigma_2 = \dfrac{\sigma_u'}{n}; \qquad -105.6 = \dfrac{-520}{n}, \qquad n = 4.9$ ◀

SOLUTION (4.21)

$$\sum M_A = 0: \quad 150(2)(1) - R_B(1.2) = 0, \qquad R_B = 250\ kN$$
$$\sum F_y = 0: \quad R_A = 50\ kN$$





$$\tau_{\max} = \frac{3V}{2A} = \frac{3}{2}\frac{130\times10^3}{2b^2} = \frac{97.5\times10^3}{b^2}$$

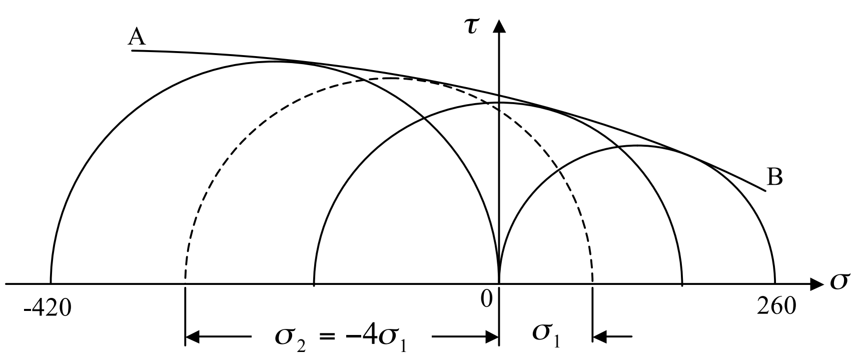
And $\tau_{\max} = \sigma_1 = -\sigma_2$

Thus,

$$\tau_{\max} = \sigma_{all}; \quad \frac{97.5\times10^3}{b^2} = 120\times10^6, \qquad b = 28.5\ mm$$

Use a 30 mm by 60 mm rectangular beam. ◀

SOLUTION (4.22)



(CONT.)

4.22 (CONT.)

(a) Upon following the procedure described in Sec. 4.11, Mohr's circle is constructed as shown in the sketch above.

The circle representing the given loading is then drawn by a trial and error procedure, as is indicated by the dashed lines. From the diagram, we measure the following values:

$$\sigma_1 = 77\ MPa \qquad \sigma_2 = -308\ MPa$$ ◀

(b) Applying Eq. (4.12a),

$$\frac{\sigma_1}{\sigma_u} - \frac{\sigma_2}{\sigma_u'} = 1$$

or

$$\frac{\sigma_1}{260} - \frac{-4\sigma_1}{420} = 1$$

Solving,

$$\sigma_1 = 75\ MPa \qquad \sigma_2 = -300\ MPa$$ ◀

SOLUTION (4.23)

Principal stresses are

$$\sigma_{1,2} = \frac{-180}{2} \pm [(\tfrac{180}{2})^2 + 200^2]^{\frac{1}{2}}$$

Or $\sigma_1 = 129.3\ MPa, \qquad \sigma_2 = -309.3\ MPa$

(a) Equation (4.11a):

$$|\sigma_1| < 290\ MPa$$

But since

$$|\sigma_2| > 290\ MPa:$$ failure occurs ◀

(b) Equation (4.12a):

$$\frac{129.3}{290} - \frac{-309.3}{650} = 1$$

gives $0.446 + 0.476 = 0.922 < 1$

Thus, no fracture ◀

Note that Coulomb-Mohr theory is the most reliable when $\sigma_u' >> \sigma_u$, as in this example.

SOLUTION (4.24)

$$\sigma_x = \frac{pr}{2t} + \frac{P}{2\pi rt}$$

$$= \frac{2.8(10^6)125}{2(5)} + \frac{45(10^3)}{2\pi(0.125)(0.005)} = 46.46\ MPa$$

$$\sigma_y = \frac{pr}{t} = \frac{2.8(10^6)125}{5} = 70\ MPa$$

$$\tau = \frac{Tr}{2\pi r^3 t} = \frac{31.36(10^3)}{2\pi(0.125^2)(0.005)} = 63.89\ MPa$$

Thus,

$$\sigma_{1,2} = \tfrac{1}{2}(46.46 + 70) \pm [\tfrac{1}{4}(46.46 - 70)^2 + (63.89)^2]^{\frac{1}{2}}$$

(CONT.)

4.24 (CONT.)

or $\sigma_1 = 123.2\ MPa \qquad \sigma_2 = -6.74\ MPa$

(a) Equation (4.12a),

$$\tfrac{123.2}{210} - \tfrac{-6.74}{500} = 1$$

gives $0.587 + 0.013 = 0.6 < 1$

Thus, no fracture ◀

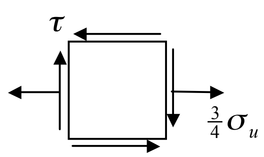
(b) Equations (4.11a) shows

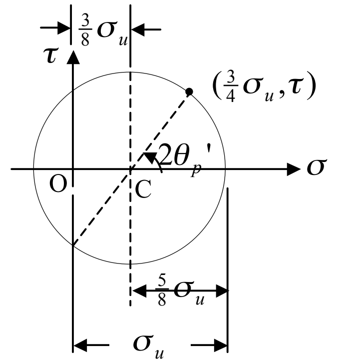
$123.2 < 210,$ no fracture

$6.14 < 210,$ no fracture ◀

SOLUTION (4.25)

State of stress is represented by Mohr's circle shown below.



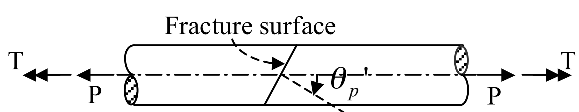


From the circle, we obtain

$$\tau = \sigma_u \sqrt{\left(\tfrac{5}{8}\right)^2 - \left(\tfrac{3}{8}\right)^2} = \tfrac{1}{2}\sigma_u$$

and $\theta_p' = \tfrac{1}{2}\tan^{-1}\tfrac{1/2}{3/8} = 26.57^o$ ◀

Orientation of the fracture plane is shown below.



SOLUTION (4.26)

Principal stresses are

$$\sigma_{1,2} = \tfrac{1}{2}(200 + 20) \pm [(90)^2 + (150^2)]^{\frac{1}{2}}$$

or

$$\sigma_1 = 284.9\ MPa \qquad \sigma_2 = -64.9\ MPa$$

(a) $284.9 = \tfrac{420}{n}, \qquad n = 1.47$ ◀

$64.9 = \tfrac{420}{n}, \qquad n = 6.47$

(CONT.)

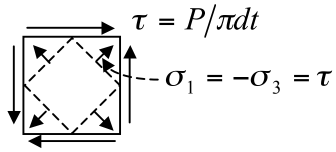
4.26 (CONT.)

(b) Equation (4.12a): $\frac{284.9}{420} - \frac{-64.9}{900} = \frac{1}{n}$

Solving, $n = 1.33$ ◄

SOLUTION (4.27)

Uniform shear stress τ acts on a typical element as shown.



(a) $|\sigma_1| = \sigma_u$ or $|\sigma_3| = \sigma_u$

$\frac{P}{\pi td} = \sigma_u, \qquad P = \pi td\sigma_u$ ◄

(b)

$\frac{\sigma_1}{\sigma_u} - \frac{\sigma_3}{\sigma_u'} = 1; \qquad \tau(1 + \frac{\sigma_u}{\sigma_u'}) = \sigma_u$

or $P = \frac{td\sigma_u}{(1+\sigma_u/\sigma_u')}$ ◄

SOLUTION (4.28)

Table 4.3: $K_c = 23\sqrt{1000}\ MPa\sqrt{mm} \qquad \sigma_{yp} = 444\ MPa$

Note that the values of crack length *a* and plate thickness *t* satisfy Table 4.3.

Table 4.2: $\frac{a}{w} = \frac{25}{125} = 0.2 \qquad \lambda = 1.37$

Equation (4.18), with n=1:

$$\sigma_{all} = \frac{K_c}{\lambda\sqrt{\pi a}} = \frac{23\sqrt{1000}}{(1.37)\sqrt{\pi(25)}} = 59.91\ MPa$$

Therefore

$$P_{all} = \sigma_{all}(wt) = 59.91(125 \times 25) = 187.2\ kN$$ ◄

Note, the nominal stress at fracture:

$$\sigma = \frac{P}{t(w-a)} = \frac{187.2(10^3)}{20(125-25)} = 93.6\ MPa < 444\ MPa$$

SOLUTION (4.29)

We have $a/w = 0.005/5 = 0.001$ and $\lambda = 1$ by Table 4.2. From Table 4.3:

$$K_c = 59\ MPa\sqrt{m} \qquad \sigma_{yp} = 1503\ ksi$$

(a) $K = \lambda\sigma\sqrt{\pi a} = (1)(100)\sqrt{\pi(0.03)} = 30.7\ MPa\sqrt{m}$

$n = \frac{K_c}{K} = \frac{59}{30.7} = 1.92$ ◄

(b) Using Eq. (4.18) with $n = 1$:

$$\sigma = \frac{K_c}{\lambda\sqrt{\pi a}} = \frac{59}{(1)\sqrt{\pi(0.03)}} = 192.2\ MPa$$

This is well below the yield strength.

SOLUTION (4.30)

By Table 4.3: $K_c = 66\sqrt{1000}\ MPa\sqrt{mm}$ and $\sigma_{yp} = 1149\ MPa.$ Table 4.2:

$$\tfrac{a}{w} = \tfrac{10}{65} = 0.15 \qquad \lambda = 1.02$$

We have

$$\sigma = \tfrac{K_c}{\lambda n\sqrt{\pi a}} = \tfrac{66\sqrt{1000}}{(1.02)(2.2)\sqrt{\pi(10)}} = 165.9\ MPa$$

Thus

$$t_{req} = \tfrac{P}{2w\sigma} = \tfrac{200(10^3)}{2(65)(165.9)} = 9.27\ mm \qquad \blacktriangleleft$$

A thickness of 9.3 mm should be used. Note that both values of *a* and *t* satisfy Table 4.3.

SOLUTION (4.31)

Refer to Example 4.5. We now have $\tfrac{a}{w} = \tfrac{8}{40} = 0.2$.

Table 4.2: $\lambda_a = 1.37 \qquad \lambda_b = 1.06$

Table 4.3: $K_c = 59\sqrt{1000}\ MPa\sqrt{mm} \qquad \sigma_{yp} = 1503\ MPa$

Equation (a) of Example 4.5, with $M = 0.17P$:

$$\lambda\sigma = (1.37)\tfrac{P}{(0.04)(0.01)} + 1.06\tfrac{6(0.17P)}{(0.01)(0.04)^2} = 71{,}000P$$

By Eq. (4.18):

$$\lambda\sigma = \tfrac{K_c}{n\sqrt{\pi a}}; \qquad 71{,}000P = \tfrac{59\sqrt{1000}(10^6)}{1.8\sqrt{\pi(0.008)}}$$

from which

$$P = 92.09\ kN \qquad \blacktriangleleft$$

The nominal stress at fracture:

$$\sigma = \tfrac{P}{t(w-a)} = \tfrac{92.09(10^3)}{(0.01)(0.04-0.008)}$$
$$= 287.8\ MPa < 1503\ MPa$$

SOLUTION (4.32)

From Table 4.3: $K_c = 23\sqrt{1000}\ MPa\sqrt{mm}$ and $\sigma_{yp} = 444\ MPa$

Case B of Table 4.2: $a/w = 0.16, \quad \lambda = 1.12$

By Eq.(4.18), with $n = 1$:

$$\sigma = \tfrac{K_c}{\lambda\sqrt{\pi a}} = \tfrac{23\sqrt{1000}}{1.12\sqrt{\pi(24)}} = 74.79\ MPa$$

It follows that

$$P = \sigma(wt) = 74.79(150\times 30) = 337\ kN$$

Then

$$\sigma = \tfrac{P}{(w-a)t} = \tfrac{337(10^3)}{(0.15-0.024)(0.03)} = 89.15\ MPa$$
$$= 89.15\ MPa < \sigma_{yp} \qquad \blacktriangleleft$$

SOLUTION (4.33)

Case A of Table 4.2 and Table 4.3:

$$K_c = 59\sqrt{1000}\ MPa\sqrt{mm} \qquad \sigma_{yp} = 1503\ MPa$$

$$\lambda = 1.01 \qquad \text{(assumed)}$$

By Eq.(4.18):

$$\sigma = \frac{K_c}{n\lambda\sqrt{\pi a}} = \frac{59\sqrt{1000}}{(2)(1.01)\sqrt{\pi(6)}} = 213\ MPa < \sigma_{yp}$$

(a) $\sigma = \frac{p_f r}{t}, \qquad p_f = \frac{\sigma t}{r} = \frac{213(10)}{30} = 71\ MPa$ ◀

(b) $\sigma = \frac{p_f r}{2t}, \qquad p_f = \frac{1}{2}(71) = 35.5\ MPa$ ◀

SOLUTION (4.34)

Table 4.3: $K_c = 31\ MPa\sqrt{m} \qquad \sigma_{yp} = 392\ MPa$

Case D of Table 4.2: $\frac{a}{w} = 0.4 \qquad \therefore \lambda = 1.32$

Using Eq.(4.18) with $n = 1$ and $\sigma = 6M/tw^2$:

$$K_c = \lambda \frac{6M}{tw^2}\sqrt{\pi a}; \qquad 31(10^6) = 1.32\frac{6M}{0.03(0.12)^2}\sqrt{\pi(0.048)}$$

Solving

$$M = 4.35\ kN \cdot m$$ ◀

SOLUTION (4.35)

$$\sigma_m = \frac{120(10^3)}{25(10^{-4})} = 48\ MPa$$

and $\frac{\sigma_a}{\sigma_{cr}} + \frac{\sigma_m}{\sigma_f} = 1; \qquad \frac{1200F_A}{240(10^6)} + \frac{48(10^6)}{700(10^6)} = 1$

or $F_A = 186.3\ kN$ ◀

SOLUTION (4.36)

$$\sigma_{1a} - \sigma_{3a} = 15p - 0 = \sigma_{ea}$$

$$\sigma_{1m} - \sigma_{3m} = 9p - 0 = \sigma_{em}$$

Then,

$$\frac{15p}{250(10^6)} + \frac{9p}{300(10^6)} = 1$$

or

$$p = 11.11\ MPa$$ ◀

SOLUTION (4.37)

We have

$$\sigma_{cr} = \frac{\sigma_a}{1-(\sigma_m/\sigma_u)} \tag{a}$$

where,

$$\sigma_m = \frac{F_{\max}+F_{\min}}{2A}, \qquad \sigma_a = \frac{F_{\max}-F_{\min}}{2A} \tag{b}$$

Substituting Eqs. (b) into (a):

$$\sigma_{cr} = \frac{(F_{\max}-F_{\min})/2A}{1-[(F_{\max}+F_{\min})/2A\sigma_u]}$$

Solving,

$$A = \frac{1}{\sigma_{cr}}[F_{\max} - \tfrac{1}{2}(F_{\max} + F_{\min})(1 - \tfrac{\sigma_{cr}}{\sigma_u})]$$ ◄

SOLUTION (4.38)

We have $\sigma_m = \sigma_a$. Using Table 4.4:

$$\frac{\sigma_m}{510/1.5} + \frac{\sigma_m}{1050/1.5} = 1$$

from which $\sigma_m = 228.8\ MPa.$

At the fixed end:

$$M_{\max} = PL = 10(0.05) = 0.5\ N\cdot m$$

Hence,

$$M_{a,m} = \frac{(M_{\max} \pm M_{\min})}{2} = 0.25\ N\cdot m$$

and

$$\sigma_a = \sigma_m = \frac{6M_m}{bt^2} = \frac{6(0.25)}{0.005t^2} = \frac{300}{t^2} = 228.8(10^6)$$

Solving,

$$t = 1.145\ mm$$ ◄

SOLUTION (4.39)

We have $\sigma_a = \sigma_m$. From Table 4.4:

$$\frac{\sigma_m}{740/2.5} + \frac{\sigma_m}{1500/2.5} = 1$$

or

$$\sigma_m = 198.2\ MPa.$$

At the center of the beam:

$$M_{\max} = PL/4 = (0.25)(20)(0.125) = 0.625\ N\cdot m$$

Hence,

$$M_{a,m} = \frac{(M_{\max} \pm M_{\min})}{2} = 0.3125\ N\cdot m$$

and

$$\sigma_a = \sigma_m = \frac{6M_m}{bt^2} = \frac{6(0.3125)}{0.01t^2} = \frac{187.5}{t^2} = 198.2(10^6)$$

Solving,

$$t = 0.973\ mm$$ ◄

SOLUTION (4.40)

$$\sigma_{\max} = \sigma_{\min} = \tfrac{Mc}{I}, \qquad \sigma_m = \tfrac{4M}{\pi r^3}, \qquad \sigma_a = 0$$
$$\tau_{\max} = \tfrac{Tr}{J}, \qquad \tau_{\min} = 0, \qquad \tau_m = \tau_a = \tfrac{Tr}{\pi r^3}$$

and

$$\sigma_x = \sigma, \qquad \tau_{xy} = \tau, \qquad \sigma_y = \sigma_z = \tau_{xz} = \tau_{yz} = 0$$

Equations (4.21) yield

$$\sqrt{0 + 3\tau_a^2} = \sigma_{ea}, \qquad \sqrt{\sigma_m^2 + 3\tau_m^2} = \sigma_{em}$$

Then, Soderberg relation becomes

$$\sigma_{cr} = \frac{\sqrt{3}\tau_a}{1 - \frac{\sqrt{\sigma_m^2 + 3\tau_m^2}}{\sigma_{yp}}}$$

or $$\sigma_{cr} = \tfrac{\sigma_{cr}}{\sigma_{yp}} \sqrt{\sigma_m^2 + 3\tau_m^2} + \sqrt{3}\tau_a$$

or

$$\sigma_{cr} = \tfrac{\sigma_{cr}}{\sigma_{yp}} [(\tfrac{4M}{\pi r^3})^2 + 3(\tfrac{T}{\pi r^3})^2]^{\frac{1}{2}} + \tfrac{\sqrt{3}T}{\pi r^3}$$

Solving this expression for r, we obtain Eq. (P4.40).

SOLUTION (4.41)

$$\sigma_{1a,2a} = \tfrac{\sigma_{xa} + \sigma_{ya}}{2} \pm [(\tfrac{\sigma_{xa} - \sigma_{ya}}{2})^2 + \tau_{xya}^2]^{\frac{1}{2}}$$
$$= \tfrac{\sigma_{xa}}{2} \pm [\tfrac{\sigma_{xa}^2}{4} + \tau_{xya}^2]$$
$$= \tfrac{900}{2} \pm [\tfrac{81(10^4)}{4} + 4(10^4)]^{\frac{1}{2}}$$

or

$$\sigma_{1a} = 942.44\ MPa, \qquad \sigma_{2a} = -42.44\ MPa$$

Similarly,

$$\sigma_{1m} = 161.8\ MPa, \qquad \sigma_{2m} = -61.8\ MPa$$

Thus,

$$\sigma_{ea} = \sigma_{1a} - \sigma_{2a} = 984.88\ MPa$$
$$\sigma_{em} = \sigma_{1m} + \sigma_{2m} = 233.6\ MPa$$

(a) Modified Goodman relation:

$$\sigma_{cr} = \tfrac{984.88}{1-(223.6/2400)} = 1086\ MPa$$
$$b = \tfrac{\ln(0.9\times 2400/800)}{\ln(10^3/10^8)} = -0.0863$$

$$N_{cr} = 10^3 (\tfrac{1086}{0.9\times 2400})^{-11.587} = 2.89(10^6) \text{ cycles}$$ ◀

(b) Soderberg criterion:

$$\sigma_{cr} = \tfrac{948.88}{1-(223.6/1600)} = 1145\ MPa$$
$$N_{cr} = 10^3 (\tfrac{1145}{0.9\times 2400})^{-11.587} = 1.57(10^6) \text{ cycles}$$ ◀

(CONT.)

4.41 (CONT.)

(c) The SAE criterion:

$$\sigma_{cr} = 1086\ MPa, \qquad b = -0.0596$$

$$N_{cr} = 1(\tfrac{1086}{2400})^{-16.778} = 0.59(10^6)\ \text{cycles} \quad \blacktriangleleft$$

(d) Gerber criterion:

$$\sigma_{cr} = \frac{948.88}{1-(223.6/2400)^2} = 993.51\ MPa$$

$$N_{cr} = 10^3(\tfrac{993.51}{0.9\times 2400})^{-11.587} = 8.12(10^6)\ \text{cycles} \quad \blacktriangleleft$$

SOLUTION (4.42)

$$\sigma_{xa} = (800+600)/2 = 700\ MPa$$
$$\sigma_{xm} = (800-600)/2 = 100\ MPa$$
$$\sigma_{ya} = (500+300)/2 = 400\ MPa$$
$$\sigma_{ym} = (500-300)/2 = 100\ MPa$$
$$\tau_{xya} = (200+150)/2 = 175\ MPa$$
$$\tau_{xym} = (200-150)/2 = 25\ MPa$$

Equations (4.21) give then

$$2\sigma_{ea}^2 = (700-400)^2 + 400^2 + 700^2 + 6(175)^2$$
$$2\sigma_{em}^2 = (200-100)^2 + 100^2 + 100^2 + 6(25)^2$$

or $\sigma_{ea} = 679.61\ MPa, \qquad \sigma_{em} = 108.97\ MPa$

(a) Modified Goodman criterion:

$$\sigma_{cr} = \frac{679.61}{1-(108.97/1600)} = 729.27\ MPa$$

$$b = \frac{\ln(0.9\times 1600/0.5\times 1600)}{\ln(10^3/10^8)} = -0.08509$$

$$N_{cr} = 10^3(\tfrac{729.27}{0.9\times 1600})^{-11.752} = 2.97(10^6)\ \text{cycles} \quad \blacktriangleleft$$

(b) Soderberg criterion:

$$\sigma_{cr} = \frac{679.61}{1-(108.97/1000)} = 762.72\ MPa$$

$$N_{cr} = 10^3(\tfrac{762.72}{0.9\times 1600})^{-11.752} = 1.75(10^6)\ \text{cycles} \quad \blacktriangleleft$$

SOLUTION (4.43)

$$\sigma_a = \frac{(M_{\max}-M_{\min})c}{2I}$$

$$\sigma_m = \frac{(M_{\max}+M_{\min})c}{2I}, \qquad M = PL$$

Substituting these into Soderberg relation, we obtain

$$M_{\max} = \frac{2\sigma_{cr} I}{(1+\lambda)c} + \frac{1-\lambda}{1+\lambda} M_{\min}$$

(CONT.)

4.43 (CONT.)

where $\lambda = \frac{\sigma_{cr}}{\sigma_{yp}} = \frac{2}{3}$

Thus, $P_{\max} = \frac{2\sigma_{cr} I}{c(1+\lambda)L} + \frac{1-\lambda}{1+\lambda} P_{\min} = \frac{2(200\times10^6)(0.05\times0.1^3)}{12(5/3)(0.05)(1.2)} + \frac{1/3}{5/3}(10{,}000)$

$= 16{,}667 + 2000 = 18.7\ kN$ ◀

SOLUTION (4.44)

We have $W = mg = 80 \times 9.81 = 784.8\ N$

From Eq. (4.29);

$$\sigma_{\max} = (1 + \sqrt{1 + \tfrac{2h}{\delta_{st}}})\tfrac{W}{A}$$

Solving, with $\delta_{st} = WL/AE$, we obtain

$$h = \tfrac{L\sigma_{\max}}{2EW}(A\sigma_{\max} - 2W) \qquad \text{(a)}$$

Substituting the given data:

$$h = \tfrac{(2)(350\times10^6)}{2(105\times10^9)(784.8)}(250 \times 350 - 1569.6) = 0.365\ m$$ ◀

SOLUTION (4.45)

Through the use of Eq. (4.29):

$$\sigma_{\max} = \tfrac{W}{A}[1 + \sqrt{1 + \tfrac{2h}{\delta_{st}}}], \qquad [\tfrac{\sigma_{\max} A}{W} - 1]^2 = 1 + 2h\delta_{st}$$

from which

$$\tfrac{\sigma_{\max}^2 A^2}{W^2} - \tfrac{2\sigma_{\max} A}{W} + 1 = 1 + \tfrac{2hAE}{WL}$$

Solving for W, we obtain Eq. (P4.45).

SOLUTION (4.46)

The kinetic energy is

$$E_k = \tfrac{W\omega^2 r^2}{2g} = \tfrac{1090(240\times2\pi/60)^2(0.35)^2}{2(9.81)} = 4300\ N \cdot m$$

But $E_k = \frac{1}{2}T\phi$ where $T = \frac{GJ\phi}{L}$

Thus,

$$\phi = \sqrt{2LE_k/JG} \qquad \text{(a)}$$

Substituting the data given:

$$\phi = [\tfrac{2\times1.5(4300)2}{80.5(10^9)\pi(0.0625)^4}]^{\frac{1}{2}} = 0.08309\ rad = 4.76^o$$ ◀

Introducing

$$\phi = \tfrac{TL}{GJ} = \tfrac{\tau J}{r}\tfrac{L}{GJ} = \tfrac{\tau L}{rG}$$

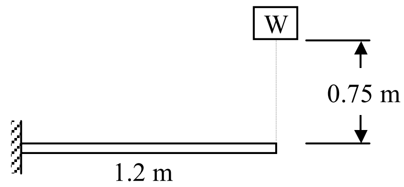
into Eq. (a), we obtain

$$\tau = \sqrt{4GE_k/AL} \qquad \text{where } A = \pi c^2$$

Substituting the numerical values,

$$\tau = [\tfrac{4(80.5\times10^9)4300}{\pi(0.0625)^4(1.5)}]^{\frac{1}{2}} = 274.3\ MPa$$ ◀

SOLUTION (4.47)



We have $\sigma_{yp} = Mc/I = PLc/I$

from which

$$P = \frac{\sigma_{yp} I}{Lc} = \frac{280(10^6)(0.05^4/12)}{1.2(0.025)} = 4.86\ kN$$

End deflection is

$$\delta_{\max} = \frac{4860(1.2)^3(12)}{3(200\times10^9)(0.05)^4} = 26.86\ mm$$

But

$$W(0.75 + 0.02686) = \tfrac{1}{2}(4860)(0.02686)$$

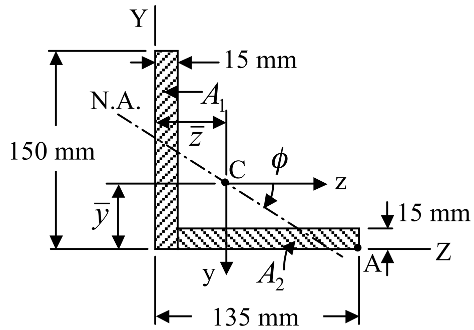
Solving,

$$W = 84.02\ N$$ ◄

End of Chapter 4

CHAPTER 5

SOLUTION (5.1)



$$\overline{z} = \frac{A_1 z_1 + A_2 z_2}{A_1 + A_2} = \frac{(150\times15)7.5+(135\times15)[15+(135/2)]}{150\times15+135\times15}$$

or $\quad \overline{z} = \overline{y} = 43\ mm$

Then,

$$I_y = \tfrac{1}{12}(150)(15)^3 + (150\times15)(35.5)^2 + \tfrac{1}{12}(15)(135)^3$$
$$+ (135\times15)(39.5)^2$$

or $\quad I_y = I_z = 9.11(10^6)\ mm^4$

$$I_{yz} = (150\times15)(-32)(-35.5) + (135\times15)(35.5)(39.5)$$
$$= 5.4(10^6)\ mm^4$$

We have the moment components:

$$M_y = 0, \qquad M_z = -11.25(0.9) = -10.125\ kN\cdot m$$

Thus,

$$(\sigma_x)_A = \frac{-10125(5.4)(0.107)+10125(9.11)(0.043)}{[(9.11)^2-(5.4)^2](10^{-6})} = -35\ MPa \quad ◀$$

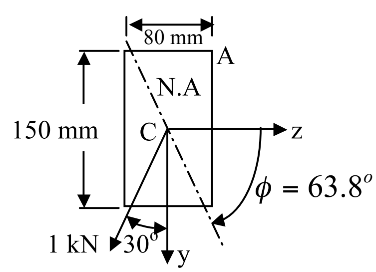
Equation (5.15) gives

$$\tan\phi = \tfrac{5.4}{9.11} = 0.593$$

or

$$\phi = 30.66^o \quad ◀$$

SOLUTION (5.2)



$$I_y = \tfrac{1}{12}hb^3 = \tfrac{1}{12}(150)(80)^3 = 6.4\times10^6\ mm^4$$

(CONT.)

5.2 (CONT.)

$$I_z = \tfrac{1}{12}bh^3 = \tfrac{1}{12}(80)(150)^3 = 22.5\times10^6 \ mm^4$$
$$M_y = (P\sin\alpha)L = 600 \ N\cdot m$$
$$M_z = (P\cos\alpha)L = 1{,}039.2 \ N\cdot m$$
$$y_d = -75 \ mm \qquad z_d = 40 \ mm$$

(a) Equation (5.15) with $I_{yz} = 0$:

$$\tan\phi = \tfrac{I_z}{I_y}\tfrac{M_y}{M_z} = \tfrac{I_z}{I_y}\tan\alpha \qquad \therefore \phi = 63.8^o$$ ◀

(b) Thus, maximum tensile stress is at point A.
Equation (5.16) gives

$$\sigma_{\max} = \tfrac{600(0.04)}{22.5\times10^{-6}} - \tfrac{1{,}039.2(-0.075)}{6.4\times10^{-6}} = 13.25 \ MPa$$ ◀

SOLUTION (5.3)

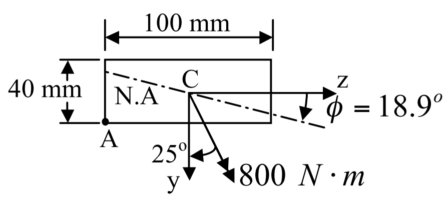
$$I_y = \tfrac{1}{12}hb^3 = \tfrac{1}{12}(40)(100)^3 = 3.33\times10^6 \ mm^4$$
$$I_z = \tfrac{1}{12}bh^3 = \tfrac{1}{12}(100)(40)^3 = 0.533\times10^6 \ mm^4$$

$$M_y = M_o\cos\alpha = 800(\cos25^o) = 725 \ N\cdot m$$
$$M_z = M_o\sin\alpha = 800(\sin25^o) = 338.1 \ N\cdot m$$

(a) Equation (5.15) with $I_{yz} = 0$:

$$\tan\phi = \tfrac{I_z}{I_y}\tfrac{M_y}{M_z} = \tfrac{I_z}{I_y}\cot\alpha \qquad \therefore \phi = 18.9^o$$ ◀

(b) There, maximum compressive stress is at A.



Equation (5.16) results in

$$\sigma_{\max} = \tfrac{725(-0.05)}{3.33(10^{-6})} - \tfrac{338.1(0.02)}{0.533(10^{-6})} = -23.57 \ MPa$$ ◀

SOLUTION (5.4)

$$I_y = 2[\tfrac{30\times8^3}{12} + 240\times34^2] + \tfrac{8\times76^3}{12} = 85(10^4) \ mm^4$$
$$I_z = 2[\tfrac{30\times8^3}{12} + 240\times19^2] + \tfrac{76\times8^3}{12} = 21.2(10^4) \ mm^4$$
$$I_{yz} = [0 + 240(19)(-34)] + [0 + 240(-19)(34) + 0 = -31(10^4) \ mm^4$$

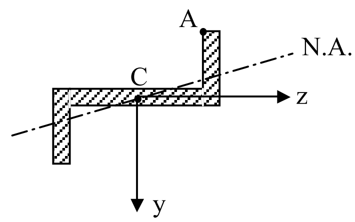
Using Eq. (5.14):

$$M_o[21.2 + 1.5(-31)]z = M_o[-31 + 1.5\times85]y$$

(CONT.)

or $z = -3.81y$

Thus, point A is the farthest from the N.A., as shown.



Hence, Eq. (5.13) gives

$$(\sigma_x)_A = 80(10^6) = \frac{[21.2+1.5(-31)]0.03-[-31+1.5(85)](-0.034)}{[85\times 21.2-(-31)^2]10^{-8}} M_o$$

from which

$$M_o = 266.8\ N\cdot m \qquad ◄$$

SOLUTION (5.5)

Bending moment at the midspan is

$$M_z = -24(4) - 24(2) = -48\ kN\cdot m$$

From Fig. 5.4 and Example 5.1:

$$z_E = 0.105\ m \qquad z_D = 0$$

$$y_E = -0.045\ m \qquad y_D = -0.045\ m$$

$$I_y = I_z = 11.596(10^{-6})\ m^4$$

$$I_{yz} = -6.79(10^{-6})\ m^4$$

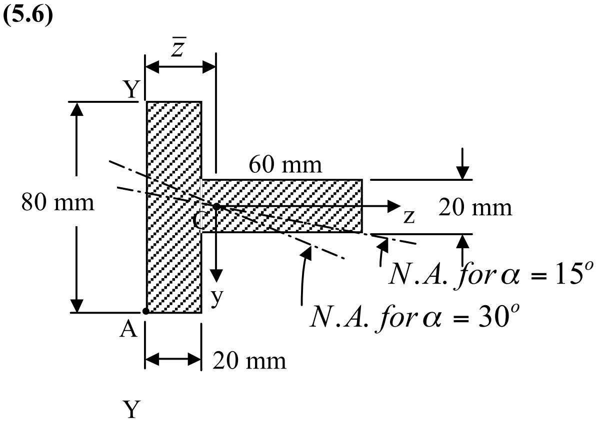
Then, Eq. (5.13) with $M = 0$:

$$(\sigma_x)_D = \frac{0-(-48000)(11.596)(-0.045)}{[(11.596)^2-(6.79)^2]10^{-6}} = -283.4\ MPa \qquad ◄$$

Similarly,

$$(\sigma_x)_E = \frac{-48000[-6.79(0.105)-11.596(-0.045)]}{[(11.596)^2-(6.79)^2]10^{-6}} = 103.8\ MPa \qquad ◄$$

SOLUTION (5.6)



(CONT.)

5.6 (CONT.)

$$\bar{z} = \frac{80\times20(10)+60\times20(50)}{80\times20+60\times20} = 27.14\ mm$$
$$I_y = \frac{80\times20^3}{12} + 20\times80(17.14)^2 + \frac{20\times60^3}{12} + 20\times60(22.86)^2$$
$$= 15.1048(10^5)\ mm^4$$
$$I_z = \frac{80\times20^3}{12} + 2[\frac{20\times30^3}{12} + 20\times30(25)^2] = 8.933(10^5)\ mm^4$$

Due to the symmetry $I_{yz} = 0$.

(a) We have

$$\alpha = 0^o \qquad M_y = 0 \qquad M_z = 1.5P$$

Equation (5.13) is thus,

$$\frac{290(10^6)}{1.2} = \frac{1.5Py}{I_z} = \frac{1.5P(0.04)}{8.933(10^{-7})}$$

or $\quad P = 3.6\ kN$ ◄

(b) Now we have $\alpha = 15^o$ and

$$M_z = 1.5P\cos 15^o = 1.4489P$$
$$M_y = 1.5P\sin 15^o = 0.3882P$$

Equation (5.14):

$$0.3882P(8.933)z = 1.4489P(15.1048)y$$

from which

$$z = 6.3105y$$

The farthest point from the N.A. is A. Equation (5.13):

$$(\sigma_x)_A = \frac{290(10^6)}{1.2} = \frac{0.3882P(8.933)(-0.02714)-1.4889P(15.1048)0.04}{8.933(15.1048)10^{-7}}$$

Solving, $P = 3.36\ kN$ ◄

SOLUTION (5.7)

Given $\alpha = 30^o$ and

$$M_z = 1.5P\cos 30^o = 1.299P$$
$$M_y = 1.5P\sin 30^o = 0.75P$$

The area properties are already found in Solution of Prob. 5.6.
Equation (5.14) gives

$$0.75P(8.933)z = 1.299P(15.1048)y$$

or $\quad z = 2.9287y$

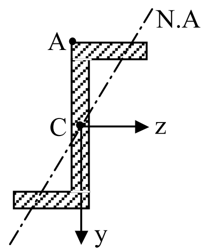
As before, the maximum stress occurs at A. Equation (5.13):

$$(\sigma_x)_A = \frac{290(10^6)}{1.2} = \frac{0.75P(8.933)(-0.02714)-1.299P(15.1048)0.04}{8.933(15.1048)10^{-7}}$$

Solving,

$$P = 3.37\ kN$$ ◄

SOLUTION (5.8)



We have $M_y = 0$ and $M_z = PL$

Equation (5.14) becomes

$$I_{yz} z = I_y y \quad \text{or} \quad -th^3 z = \tfrac{2}{3} th^3 y$$

or

$$y = -\tfrac{3z}{2}$$

Point A is the farthest from the N.A. Thus, with

$$y_A = -h - \tfrac{t}{2} \quad \text{and} \quad z_A = -\tfrac{t}{2}$$

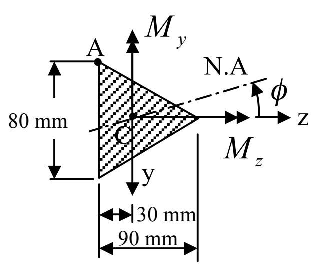
Equation (5.13) yields

$$(\sigma_x)_A = \frac{PL[-th^3(-t/2) - (2th^3/3)h^3(-h - t/2)}{(2th^3/3)(8th^3/3) - (-th^3)^2}$$

or

$$(\sigma_x)_A = \frac{3PL(2.5t + 2h)}{7th^3} = \sigma_{max}$$ ◄

SOLUTION (5.9)



$$I_y = \tfrac{bh^3}{36} = \tfrac{80(90^3)}{36} = 1.62(10^6)\ mm^4$$

$$I_z = \tfrac{hb^3}{48} = \tfrac{90(80^3)}{48} = 0.96(10^6)\ mm^4$$

$$M_y = -3(10^3)\sin 20^o = -1.026\ kN \cdot m$$

$$M_z = 3(10^3)\cos 20^o = 2.819\ kN \cdot m$$

(a) Equation (5.15):

$$\phi = \tan^{-1}[\tfrac{0.96}{1.62}\tan(-20^o)] = -12.17^o$$

(b) Point A is the farthest from the N.A. Equation (5.16):

$$\sigma_A = \frac{-1026(-0.03)}{1.62(10^{-6})} - \frac{2819(-0.04)}{0.96(10^{-6})} = 136.5\ MPa$$ ◄

SOLUTION (5.10)

(a) $$\sigma_x = \tfrac{\partial^2\Phi}{\partial y^2} = -c_4 xy - \tfrac{2}{3}c_5 x^3 y - c_6 xy^3$$

$$\sigma_y = \tfrac{\partial^2\Phi}{\partial x^2} = c_2 x - c_3 xy - \tfrac{2}{3}c_5 xy^3$$

$$\tau_{xy} = -\tfrac{\partial^2\Phi}{\partial x \partial y} = c_1 + \tfrac{c_3}{2}x^2 + \tfrac{c_4}{2}y^2 + c_5 x^2 y^2 + \tfrac{c_6}{4}y^4$$

(CONT.)

and $\quad \frac{\partial^4\Phi}{\partial x^4} = 0, \qquad \frac{\partial^4\Phi}{\partial y^4} = -6c_6xy, \qquad \frac{\partial^4\Phi}{\partial x^2\partial y^2} = -4c_5xy$

Thus, $\quad \nabla^4\Phi = 6c_6xy + 8c_5xy = 0$

or

$$6c_6 + 8c_5 = 0 \tag{a}$$

At $y = h/2: \quad \sigma_y = 0:$

$$c_2 - \frac{c_3h}{2} - \frac{c_5h^3}{12} = 0 \tag{b}$$

At $y = -h/2: \quad \sigma_y = -px/Lt:$

$$(c_2 + c_3y - \frac{2c_5y^3}{3})x = -\frac{px}{Lt}$$

or

$$c_2 + \frac{c_3h}{2} + \frac{c_5h^3}{12} = -\frac{p}{Lt} \tag{c}$$

Adding Eqs. (b) and (c),

$$c_2 = -\frac{p}{2Lt}$$

Substituting this into Eq. (b):

$$\frac{c_3h}{2} + \frac{c_5h^3}{12} = -\frac{p}{2Lt} \tag{d}$$

On $y = \pm\frac{h}{2}, \quad \tau_{xy} = 0:$

$$-c_1 - \frac{c_3}{2}x^2 - \frac{c_4}{8}h^2 - \frac{c_5}{4}h^2x^2 - \frac{c_6}{64}h^4 = 0$$

or

$$(\frac{c_3}{2} + \frac{c_5h^2}{4})x^2 + (c_1 + \frac{c_4h^2}{8} + \frac{c_6h^4}{64}) = 0 \tag{e}$$

This is of form $A \cdot B + C = 0,$ where A, B, C are independent. Thus,

$$\frac{c_3}{2} + \frac{c_5h^2}{4} = 0 \tag{f}$$

$$c_1 + \frac{c_4h^2}{8} + \frac{c_6h^2}{64} = 0 \tag{g}$$

Multiply Eq. (f) by h and subtract it from Eq. (d) to find

$$c_5 = -\frac{3p}{Lt}$$

Then, Eqs. (g) and (a) give

$$c_3 = \frac{3p}{2htL}, \qquad c_6 = \frac{4p}{th^3L}$$

On $x = 0: \quad V = 0:$

$$\int_{-h/2}^{h/2} \tau_{xy} t dy = \int_{-h/2}^{h/2} (-c_1 - \frac{c_4y^2}{2} - \frac{c_6y^4}{4}) t dy = 0$$

or $\quad c_1 + c_4(\frac{h^2}{24}) + c_6(\frac{h^4}{320}) = 0 \qquad$ (h)

Subtracting Eq. (h) from (g), together with the value of c_6 already obtained, we have

$$c_4 = -\frac{3p}{5htL}$$

Finally, substitute c_4 and c_6 into Eq. (g) to determine

$$c_1 = \frac{ph}{80tL}$$

(CONT.)

5.10 (CONT.)

The stress function is thus,

$$\Phi = \tfrac{p}{Lt}[\tfrac{h}{80}xy - \tfrac{x^3}{12} - \tfrac{x^3 y}{4h} + \tfrac{xy^3}{10h} + \tfrac{x^3 y^3}{h^3} - \tfrac{xy^5}{5h^3}]$$

and the stresses are

$$\sigma_x = \tfrac{p}{Lt}[\tfrac{3}{5h}xy + \tfrac{2}{h^3}x^3 y - \tfrac{4}{h^3}xy^3]$$

$$\sigma_y = \tfrac{p}{Lt}[-\tfrac{x}{2} - \tfrac{3}{2h}xy + \tfrac{2}{h^3}xy^3] \quad ◀$$

$$\tau_{xy} = \tfrac{p}{Lt}[\tfrac{h}{80} - \tfrac{3x^2}{4h} + \tfrac{3y^2}{10h} + \tfrac{3x^2 y^3}{h^3} - \tfrac{y^4}{h^3}]$$

(b) $$\sigma_x = \tfrac{Mc}{I} = \tfrac{(px^2/6L)(h/2)}{th^3/12} = \tfrac{px^2}{Lth^2} \quad ◀$$

(c) The maximum stresses are

$$(\sigma_x)_{elast.} = \tfrac{p}{10th}[6y + 2000y - \tfrac{40y^3}{h^2}]$$

$$(\sigma_x)_{elem.} = \tfrac{p(10h)^3}{10h^3 t} = 100(\tfrac{p}{t}) \quad ◀$$

$$(\sigma_x)_{elast.} = 99.8(\tfrac{p}{t}) \qquad \text{at } y = \pm\tfrac{h}{2}$$

Thus,

$$(\sigma_x)_{elast.} = 0.998(\sigma_x)_{elem.} \quad ◀$$

SOLUTION (5.11)

(a) We can show that given Φ satisfies $\nabla^4\Phi = 0.$ From Eqs. (3.13):

$$\sigma_x = \tfrac{p}{0.43}\{2[0.78 - \tan^{-1}\tfrac{y}{x}] - \tfrac{2xy}{x^2+y^2}\} \qquad (a)$$

$$\sigma_y = \tfrac{p}{0.43}\{2[0.78 - \tan^{-1}\tfrac{y}{x}] - 2 + \tfrac{2xy}{x^2+y^2}\} \qquad (b)$$

$$\tau_{xy} = -\tfrac{p}{0.43}\tfrac{2y^2}{x^2+y^2} \qquad (c)$$

To test which stress is maximum we rewrite σ_y in the form:

$$\sigma_y = \tfrac{p}{0.43}\{2[0.78 - \tan^{-1}\tfrac{y}{x}] - \tfrac{2(x^2+y^2)}{x^2+y^2} + \tfrac{2xy}{x^2+y^2}\}$$

$$= \tfrac{p}{0.43}\{2[0.78 - \tan^{-1}\tfrac{y}{x}] - \tfrac{2xy}{x^2+y^2} - \tfrac{2(x-y)^2}{x^2+y^2}\}$$

Comparing this with Eq. (a), noting $2(x-y)^2/(x^2+y^2) > 0,$ we conclude that $\sigma_x > \sigma_y$.
When $y = 0,$ Eqs. (a) to (c) yield $\sigma_x = 3.63p,\ \sigma_y = -p/0.43,\ \tau_{xy} = -p/0.43$

Maximum stress occurs at $y = 0$:

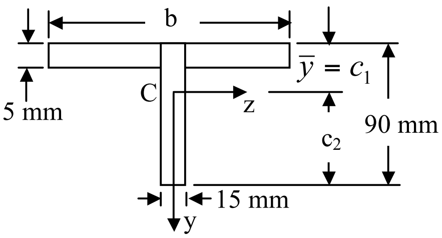
$$\sigma_{x,\max} = 3.63(2) = 7.26\ MPa \quad ◀$$

(b) $$(\sigma_x)_{elem.} = \tfrac{Mc}{I} = \tfrac{\frac{1}{2}px^2(\frac{x}{2})}{x^3/3} = 3p = 6\ MPa$$

Thus,

$$(\sigma_x)_{elast.} = 1.21(\sigma_x)_{elem.} \quad ◀$$

SOLUTION (5.12)



$$\frac{\sigma_{bottom}}{\sigma_{top}} = \frac{c_1}{c_2} = 3$$

and $c_1 = 22.5\ mm$

$c_2 = 67.5\ mm$

Thus

$$\bar{y} = c_1 = \frac{\sum A_i y_i}{\sum A_i}$$

$$= \frac{(15)(90)(45)+(b-15)(15)(7.5)}{(15)(90)+(b-15)(15)} = 22.5\ mm$$

$$= \frac{60{,}750+(b-15)(112.5)}{1{,}350+(b-15)(15)} = 22.5\ mm$$

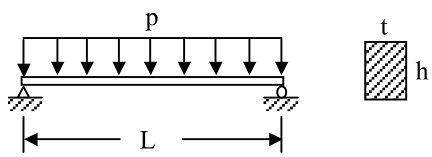
Solving,

$$b = 150\ mm \quad \blacktriangleleft$$

SOLUTION (5.13)

$$\tau_{\max} = \frac{VQ}{Ib} = \frac{pL}{2}\frac{(th/2)(h/4)}{(th^3/12)t} = \frac{3}{4}\frac{pL}{ht}$$

$$\sigma_{\max} = \frac{Mc}{I} = \frac{pL^2}{8}\frac{h/2}{th^3/12} = \frac{3}{4}\frac{pL^2}{th^2}$$



Thus,

$$\frac{\sigma_{\max}}{\tau_{\max}} = \frac{L}{h}$$

from which

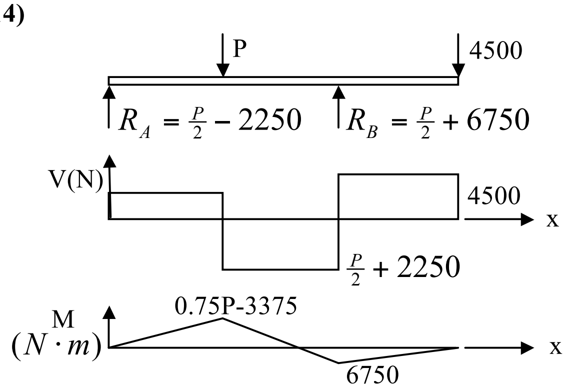
$$L = \frac{\sigma_{\max} h}{\tau_{\max}} = \frac{8.4(0.15)}{0.7} = 1.8\ m \quad \blacktriangleleft$$

Then,

$$p = \frac{4}{3}\frac{\tau_{\max}(th)}{L} = \frac{4}{3}\frac{700(0.05 \times 0.15)}{1.8}$$

$$= 3.88\ kN \cdot m \quad \blacktriangleleft$$

SOLUTION (5.14)



(CONT.)

We have

$$I = \tfrac{0.2(0.25)^3}{12} - \tfrac{0.15(0.2)^3}{12} = 160.417(10^{-6})\ m^4$$

Then, $\tau = \tfrac{VQ}{Ib} = 0.7(10^6)$

$$= \tfrac{(P/2)+2250}{160.417\times10^{-6}(0.05)}(0.2\times0.025\times0.1125 + 0.1\times0.05\times0.05)(2)$$

Solving,

$$P = 9.32\ kN$$

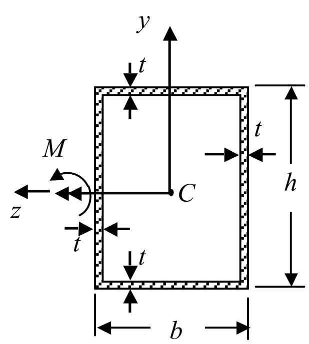
Similarly,

$$\sigma = \tfrac{Mc}{I} = 7(10^6) = \tfrac{(0.75P-3375)(0.125)}{160.417\times10^{-6}}$$

or $P = 16.478\ kN$

Thus, $P_{all} = 9.32\ kN$ ◄

SOLUTION (5.15)



$$\sigma_{all} = 150\ MPa$$

and

$$I_z = \frac{bh^3}{12} - \frac{1}{12}(b-2t)(h-2t)^3$$

$$= \frac{1}{12}(120)(170)^3 - \frac{1}{12}(100)(150)^3$$

$$= 21.005(10^6)\ mm^4$$

(**a**) Therefore

$$\sigma_{all} = \frac{Mc}{I_z},$$

or $$M = \frac{I}{c}\sigma_{all} = \frac{21.005(10^{-6})}{0.085}150(10^6) = 37.1\ kN\cdot m$$ ◄

(**b**) $$\frac{1}{r_x} = \frac{M}{EI} = \frac{37{,}100}{(70\times10^9)(21.005)(10^{-6})} = 0.0252$$

$$r_x = 39.68\ m$$ ◄

SOLUTION (5.16)

$$I = \frac{1}{64}(D^4 - d^4) = \frac{\pi}{64}(60^4 - 40^4) = 510.509(10^3)\ mm^4$$

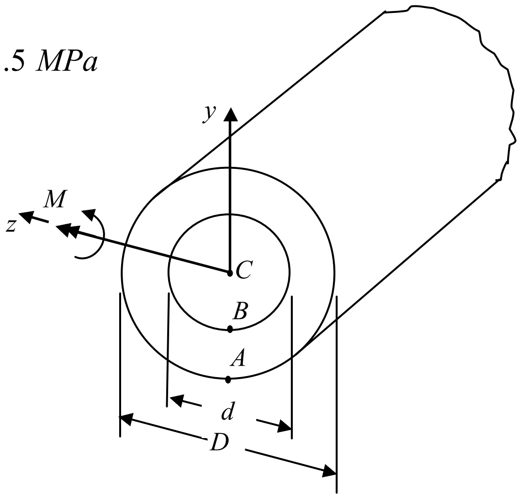
(**a**) $y_A = 30\ mm = 0.03\ m$

$$\sigma_A = \frac{My_A}{I} = \frac{(600)(0.03)}{510.509(10^{-9})} = 35.3\ MPa$$

(CONT.)

(b) $y_B = 20\ mm = 0.02\ m$

$$\sigma_B = \frac{My_B}{I} = \frac{(600)(0.02)}{510.509(10^{-9})} = 23.5\ MPa$$



(c) $$\frac{1}{r_x} = \frac{M}{EI}$$

$$= \frac{600}{70(10^9)(510.509\times10^{-9})}$$
$$= 0.0168$$

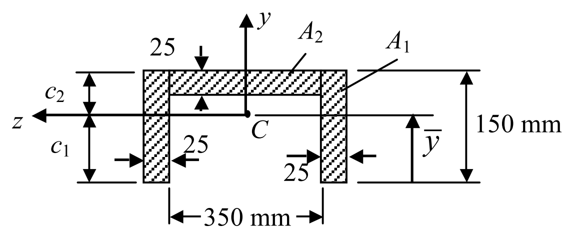
$$r_x = 59.56\ m$$

Hence,

$$r_z = -\frac{r_x}{\nu} = -\frac{59.56}{0.29} = -205.48\ m$$

SOLUTION (5.17)

$$\overline{y} = \frac{\sum A_i\overline{y}_i}{\sum A_i} = \frac{2A_1\overline{y} + A_2\overline{y}_2}{2A_1 + A_2}$$
$$= \frac{2(150\times25)(75)+(350\times25)(137.5)}{2(150\times25)+(350\times25)}$$
$$= 108.65\ mm$$



So $c_1 = 108.65\ mm$ and $c_2 = 41.35\ mm$

$$I_z = 2[\frac{1}{12}(25)(150)^3 + (25\times150)(33.65)^2] +$$
$$\frac{1}{12}(350)(25)^3 + (350\times25)(28.85)^2 = 30.29(10^6)\ mm^4$$

$$M_z = M = \frac{1}{2}PL = \frac{1}{2}P(2.4) = 1.2P$$

Therefore

$$\sigma_t = \frac{Mc_1}{I_z};\quad 60(10^6) = \frac{1.2P(0.10865)}{30.29(10^{-6})},\quad P = 14.4\ MPa$$

$$\sigma_c = \frac{Mc_2}{I_z}\quad 60(10^6) = \frac{1.2P(0.04153)}{30.29(10^{-6})},\quad P = 36.6\ kN$$

Hence $P_{all} = 14.4\ kN$ ◄

SOLUTION (5.18)

$$M_{\max} = \frac{1}{8}pL^2 = \frac{1}{8}(12)(3)^2 = 13.5\ kN \cdot m$$

$$I = \frac{1}{12}(80)(120)^3 = 11.52 \times 10^6\ mm^4$$

$$\sigma_{nom} = \frac{Mc}{I} = \frac{13.5 \times 10^3 (45 \times 10^{-3})}{11.52 \times 10^{-6}} = 52.7\ MPa$$

$$K = \frac{\sigma_{all}}{\sigma_{nom}} = \frac{95}{52.7} = 1.8$$

Use Fig. D.3. For $K = 1.8,\quad \frac{D}{d} = \frac{120}{90} = 1.33$:

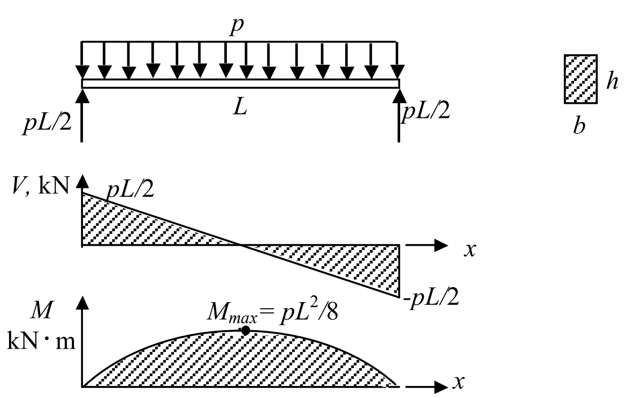
$$\frac{r}{d} = 0.14$$

Thus,

$$r_{\min} = 0.14(90) = 12.6\ mm$$ ◄

SOLUTION (5.19)

(a)



$$\tau_{\max} = \frac{3}{2}\frac{V}{A} = \frac{3}{2}\frac{pL/2}{bh} = \frac{3}{4}\frac{pL}{bh} \tag{1}$$

$$\sigma_{\max} = \frac{Mc}{I} = \frac{pL^2/8(h/2)}{bh^3/12} = \frac{3}{4}\frac{pL^2}{bh^2} \tag{2}$$

Thus,

$$\tau_{\max}/\sigma_{\max} = h/L \tag{3}$$ ◄

For example, if $L = 10h$, the above ratio is $1/10$.

(b) From Eq. (3), we have:

$$L = h\frac{\sigma_{all}}{\tau_{all}} = 0.16(\frac{9}{1.4}) = 1.029\ m$$ ◄

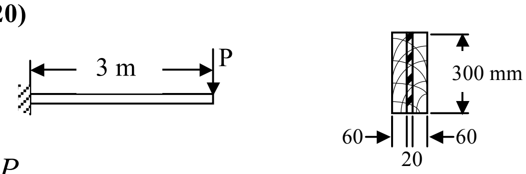
(CONT.)

5.19 (CONT.)

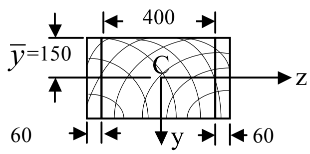
Equation (1) gives then

$$p_{all} = \frac{4}{3}\frac{bh}{L}\tau_{all} = \frac{4}{3}\frac{0.05\times 0.16}{1.029}(1.4\times 10^6) = 10.34\ kN/m \qquad \blacktriangleleft$$

SOLUTION (5.20)



$$M = 3P$$
$$n = E_s/E_t = 20$$
$$I_t = \tfrac{(520)(300)^3}{12} = 1170(10^6)\ mm^4$$



The allowable stress in the transformed section is

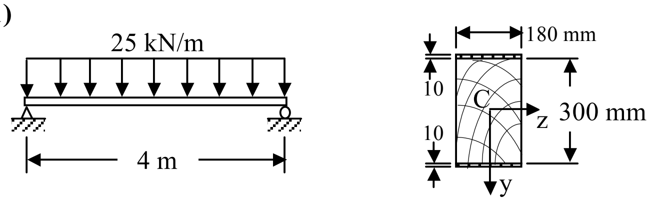
$$\tfrac{12}{20} = 6\ MPa < 7\ MPa$$

Thus, the stress in the steel is the controlling stress. Hence,

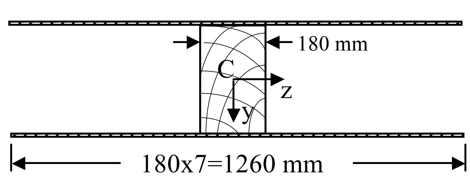
$$M_{\max} = 3P_{\max} = \tfrac{\sigma_{\max} I_t}{c}$$
$$= \tfrac{6(10^6)1170(10^{-6})}{(0.15)}$$

or $\quad P_{\max} = 15.6\ kN \qquad \blacktriangleleft$

SOLUTION (5.21)



$$M_{\max} = \tfrac{pL^2}{8} = \tfrac{25\times 10^3(4)^2}{8} = 50\ kN\cdot m$$
$$I_t = \tfrac{1}{12}(180)(300)^3 + 2[\tfrac{1260(10^3)}{12} + (1260)(10)(155)^2] = 1010.64(10^6)\ mm^4$$



$$\sigma_{w,\max} = \tfrac{Mc}{I_t} = \tfrac{50\times 10^3(0.15)}{1010.64(10^{-6})} = 7.4\ MPa \qquad \blacktriangleleft$$

$$\sigma_{a,\max} = \tfrac{7Mc}{I_t} = \tfrac{7\times 50\times 10^3(0.16)}{1010.64(10^{-6})} = 55.4\ MPa \qquad \blacktriangleleft$$

SOLUTION (5.22)

Equation (5.54) becomes

$$(kd)^2 + (kd)(\tfrac{20}{300})(1200) - (\tfrac{20}{300})(500)(1200) = 0$$

or $$(kd)^2 + 180kd - 40(10^3) = 0$$

Solving,

$$kd = 164\ mm$$

Hence,

$$500 - kd = 336\ mm$$

From Eqs. (e) of Example 5.5:

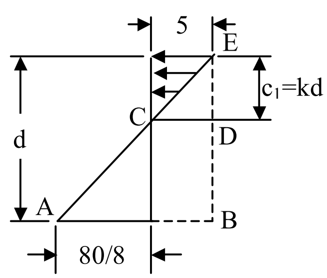
$$M_c = \tfrac{1}{2}\sigma_c(bkd)(d - \tfrac{kd}{3})$$
$$= \tfrac{1}{2}(12\times10^6)(0.3\times0.164)(0.5 - \tfrac{0.164}{3})$$
$$= 89.5\ kN\cdot m$$

and $$M_s = \sigma_s A_s(d - \tfrac{kd}{3})$$
$$= 150(10^6)(1200\times10^{-6})(0.445)$$
$$= 81.9\ kN\cdot m$$

Thus,

$$M_{all} = 81.9\ MPa$$ ◄

SOLUTION (5.23)



The stresses in concrete and the equivalent of the steel (Fig. 514b) have the values shown in the preceding figure. From the similarity of △ECD and △EAB we find

$$\tfrac{c_1}{d} = \tfrac{5}{(80/8)+5} = \tfrac{5}{15}$$

or

$$c_1 = \tfrac{5d}{15} = \tfrac{5}{15}$$

Equation (d) of Example 5.5:

$$\tfrac{1}{2}\sigma_c(c_1 b) = \sigma_s A_s$$
$$\tfrac{1}{2}(5)(\tfrac{500}{3})300 = 80A_s$$

or $$A_s = 1563\ mm^2$$ ◄

Then, we have from Eq. (e) of Example 5.5,

$$M = \sigma_s A_s(d - \tfrac{c_1}{3})$$
$$= 80(10^6)(1563\times10^{-6})(0.5 - \tfrac{0.5}{9})$$
$$= 55.6\ kN\cdot m$$ ◄

SOLUTION (5.24)

The shearing stress at a distance s is given by

$$\tau = \frac{V_y Q_z}{I_z b} = \frac{V_y}{I_z}\int_0^{\alpha} R\cos\theta(tRd\theta)$$
$$= \frac{2V_y}{\pi Rt}\sin\alpha$$

This shows that $\tau = 0$ at the free ends and $\tau_{\max}$ at the neutral axis, same as for a rectangular section. The shearing stress produces the following twisting moment about O:

$$T = \int \tau R dA = \int_0^{\pi} \frac{2V_y \sin\alpha}{\pi Rt} R(Rtd\alpha)$$
$$= \frac{4RV_y}{\pi}$$

By applying the principle of moment at O: $V_y e = M.$ Thus,

$$e = \frac{4R}{\pi} \quad ◄$$

SOLUTION (5.25)

Shearing stress in the web is neglected. Moment of the forces about S:

$$V_1 e_1 = V_2 e_2; \qquad \frac{e_1}{e_2} = \frac{V_2}{V_1} \tag{a}$$

Let M_1 and M_2 be bending moments on flanges 1 and 2, respectively.

Curvature-moment are related by

$$\frac{1}{r_1} = \frac{M_1}{EI_1}, \qquad \frac{1}{r_2} = \frac{M_2}{EI_2} \tag{b}$$

By assuming $r_1 = r_2$, we have

$$\frac{M_1}{EI_1} = \frac{M_2}{EI_2}; \qquad \frac{dM_1}{I_1} = \frac{dM_2}{I_2}$$

Introducing $dM/dx = -V$, Eq. (c) gives

$$\frac{V_1}{V_2} = \frac{I_1}{I_2}$$

Equation (a) now becomes

$$\frac{e_1}{e_2} = \frac{I_2}{I_1}$$

Since $\quad e_1 + e_2 = h; \qquad [\frac{e_1}{(h-e_1)}] = \frac{I_2}{I_1}$

Thus,

$$e_1 = \frac{I_2 h}{(I_1 + I_2)} \quad ◄$$

where

$$I_1 = \frac{b_1^3 t_1}{12} \qquad I_2 = \frac{b_2^3 t_2}{12}$$

SOLUTION (5.26)

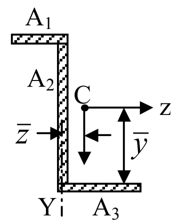


Fig. (a)

Location of centroid C (Fig. a):

$$\bar{z} = \frac{15.5\times 75(-37.5)+0+9.5\times 125(62.5)}{15.5\times 75+9.5\times 250+9.5\times 125} = 6.481\ mm$$

$$\bar{y} = \frac{15.5\times 75(-250)+0+9.5\times 125(-125)}{15.5\times 75+9.5\times 250+9.5\times 125} = -124.339\ mm$$

Moment of inertia:

$$\begin{aligned} I_y &= \tfrac{1}{12}(15.5)(75)^3 + (15.5\times 75)(43.98)^2 \\ &\quad + \tfrac{1}{12}(250)(9.5)^3 + 250(9.5)(6.48)^2 \\ &\quad + \tfrac{1}{12}(9.5)(125)^3 + 9.5(125)(56.01)^2 \\ &= 8.18(10^6)\ mm^4 \end{aligned}$$

$$\begin{aligned} I_z &= \tfrac{1}{12}(75)(15.5)^3 + (75\times 15.5)(125.6)^2 \\ &\quad + \tfrac{1}{12}(9.5)(250)^3 + (9.5\times 250)(0.65)^2 \\ &\quad + \tfrac{1}{12}(125)(9.5)^3 + (125\times 9.5)(124.3)^2 \\ &= 49.10(10^6)\ mm^4 \end{aligned}$$

$$\begin{aligned} I_{yz} &= (15.5\times 75)(-125)(-43.98) \\ &\quad + (9.5\times 250)(-0.65)(-6.48) \\ &\quad + (9.5\times 125)(124.3)(56) \\ &= 14.70(10^6)\ mm^4 \end{aligned}$$

$$\theta_p = \tfrac{1}{2}\tan^{-1}[-\tfrac{2(14.7)}{8.18-49.1}] = 17.85^o$$

Then,

$$I_{y'} = 10^6[\tfrac{8.18+49.1}{2} + \tfrac{8.18-49.1}{2}\cos 35.7^o - 14.7\sin 35.7^o] = 3.45(10^6)\ mm^4$$

Similarly, we compute

$$I_1 = I_{z'} = 53.85(10^6)\ mm^4$$

$$I_2 = I_{y'} = 3.45(10^6)\ mm^4$$

From geometry of section (Fig. b):

$$HB = 144.57\ mm \qquad BC = 39.05\ mm$$

Thus,

$$\tau_{xz} = \tfrac{V_{y'}}{I_{z'}t} = [st(0.1445 - \tfrac{s}{2}\sin 17.85^o]$$

(CONT.)

Shear force due to $V_{y'}$ (Fig. b):

$$F_1 = \int_0^s \tau_{xz} t ds = \frac{V_{y'}(0.0155)}{54.1\times10^{-6}} \int_0^{0.075} [0.1445s - \tfrac{s^2}{2}\sin 17.85^o]ds = 0.1108V_{y'}$$

We write

$$V_{y'} \cdot e_{z'} = 0.25F_1 = 0.25(0.1108V_{y'})$$

or $\quad e_{z'} = 0.0277\ m = 27.7\ mm$ ◀

Shear force due to $V_{z'}$. Now assume that the direction of F_1 shown in the figure is reversed. Then,

$$F_1 = \tfrac{V_{z'}t}{I_{y'}} \int_0^{0.075} [0.03905s - \tfrac{s^2}{2}\cos 17.85^o]ds$$

$$= 0.1928V_{z'}$$

We write

$$V_{z'} \cdot e_{y'} = 0.25(0.1928V_{z'})$$

or

$$e_{y'} = 0.0482\ m = 48.2\ mm$$ ◀

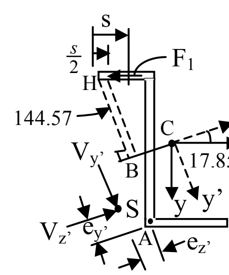
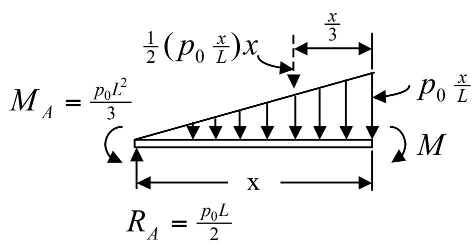


Figure (b)

SOLUTION (5.27)



We have $EIv' = M = \frac{1}{6}p_0x^3 - \frac{1}{2}p_0Lx + \frac{1}{3}p_0L^2$

Integrating

$$EIv' = \tfrac{p_0}{24L}x^4 - \tfrac{p_0L}{4}x^2 + \tfrac{p_oL^2}{3}x + c_1$$

$$v'(0) = 0; \qquad c_1 = 0.$$

$$EIv' = \tfrac{p_0}{24L}x^4 - \tfrac{p_0L}{4}x^2 + \tfrac{p_oL^2}{3}x \qquad \text{(a)}$$

Integrating

$$EIv = \tfrac{p_0}{120EIL}x^5 - \tfrac{p_0L}{12}x^3 + \tfrac{p_oL^2}{6}x^2 + c_2$$

$$v(0) = 0; \qquad c_2 = 0.$$

(a) $v = \frac{p_0}{120EIL}(x^5 - 10L^2x^3 + 20L^3x^2)$ ◀

(b) Let x=L in this equation: $v_B = \frac{11p_0L^4}{120EI}$ ◀

(c) Let x=L in Eq. (a): $\theta_B = \frac{p_0L^3}{8EI}$ ◀

SOLUTION (5.28)

$$EIv'''' = p \qquad EIv''' = px + c_1$$
$$EIv'' = \tfrac{1}{2}px^2 + c_1 x + c_2$$
$$EIv' = \tfrac{1}{6}px^3 + \tfrac{1}{2}c_1 x^2 + c_2 x + c_3$$
$$EIv = \tfrac{1}{24}px^4 + \tfrac{1}{6}c_1 x^3 + \tfrac{1}{2}c_2 x^2 + c_3 x + c_4$$

Boundary conditions:

$$EIv(0) = 0; \qquad c_4 = 0$$
$$EIv''(0) = 0; \qquad c_2 = 0$$
$$EIv(L) = 0; \qquad \tfrac{pL^3}{24} + \tfrac{c_1 L^2}{6} + c_3 = 0 \tag{a}$$
$$EIv'(L) = -EI\tfrac{pL^3}{96EI} = \tfrac{pL^3}{6} + \tfrac{c_1 L^2}{2} + c_3$$

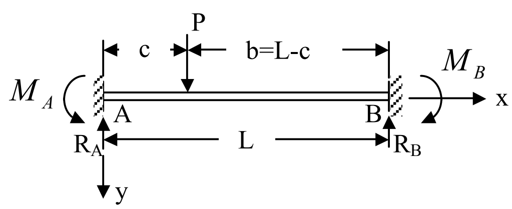
or

$$c_3 = -\tfrac{17pL^3}{96} - \tfrac{c_1 L^2}{2} \tag{b}$$

Substituting Eq. (b) into (a), we obtain reaction at right end:

$$c_1 = -\tfrac{13pL}{32} = R$$ ◀

SOLUTION (5.29)



Segment AD:

$$EIv_1'' = M_A - R_A x \qquad EIv_1' = M_A x - \tfrac{1}{2}R_A x^2 + c_1$$
$$EIv_1 = \tfrac{1}{2}M_A x^2 - \tfrac{1}{6}R_A x^3 + c_1 x + c_2 \tag{a}$$

Segment BD:

$$EIv_2'' = M_A - R_A x + P(x-c)$$
$$EIv_2' = M_A x - \tfrac{1}{2}R_A x^2 + \tfrac{1}{2}P(x-c)^2 + c_3$$
$$EIv_2 = \tfrac{1}{2}M_A x^2 - \tfrac{1}{6}R_A x^3 + \tfrac{1}{6}P(x-c)^3 + c_3 x + c_4 \tag{b}$$

Boundary conditions:

$$v_1(0) = 0; \qquad c_2 = 0$$
$$v_1'(0) = 0; \qquad c_1 = 0$$

(CONT.)

$$v_1(c) = v_2(c); \qquad c_3 c + c_4 = 0$$
$$v_1'(c) = v_2'(c); \qquad c_3 = 0, \qquad c_4 = 0$$

and

$$v_2'(L) = 0 = M_A L - \tfrac{1}{2} R_A L^2 + \tfrac{1}{2} P b^2 \tag{c}$$
$$v_2(L) = 0 = \tfrac{1}{2} M_A L^2 - \tfrac{1}{6} R_A L^3 + \tfrac{1}{6} P b^3 \tag{d}$$

Solving Eqs.(c) and (d),

$$R_A = \tfrac{Pb^3}{L^3}(3c + b) = \tfrac{P(L-c)^2}{L^3}(2c + L) \qquad ◀$$

Substituting of this into Eq. (c) gives

$$M_A = \tfrac{Pcb^2}{L^2} = \tfrac{Pc(L-c)^2}{L^2} \qquad ◀$$

Thus, introducing the values of $M_A, R_A,$ and $c_1 = c_2 = c_3 = c_4 = 0$ into Eqs. (a) and (b) we obtain the deflections.

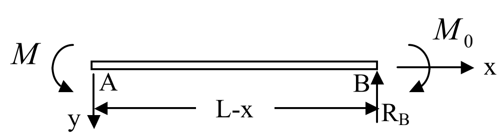
For $0 \le x \le c$:

$$v_1 = \tfrac{P(L-c)^2 x^2}{6EIL^3}(3cL - 2cx - Lx)$$

For $c \le x \le L$:

$$v_2 = \tfrac{P(L-c)^2 x^2}{6EIL^3}(3cL - 2cx - Lx) + \tfrac{P}{6EI}(x - c)^3$$

SOLUTION (5.30)



The reactions are statically indeterminate. We have

$$EIv'' = M = R_B x - R_B L + M_0$$
$$EIv' = \tfrac{1}{2} R_B x^2 - R_B L x + M_0 x + c_1$$
$$v'(0) = 0; \qquad c_1 = 0.$$

$$EIv = \tfrac{1}{6} R_B x^3 - \tfrac{1}{2} R_B L x^2 + \tfrac{1}{2} M_0 x^2 + c_2$$
$$v(0) = 0; \qquad c_2 = 0$$
$$v(L) = 0; \qquad R_B = \tfrac{3M_0}{2L} \qquad ◀$$

The preceding equation gives

$$v = \tfrac{M_0 x^2 (x - L)}{4EIL} \qquad ◀$$

SOLUTION (5.31)

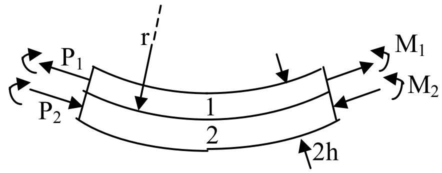


Fig. (a) Bent beam

(a) Static equilibrium gives

$$P_1 = P_2 = P \qquad M_1 + M_2 = Ph$$

Equation (5.9):

$$M_1 = \tfrac{E_1 I_1}{r} \qquad M_2 = \tfrac{E_2 I_2}{r}$$

Interface strains must be the same:

$$\underbrace{\alpha_1 \Delta T}_{\substack{due\,to\\ temp.\\ increase}} + \underbrace{\tfrac{P_1}{E_1 h}}_{\substack{due\,to\\ axial\\ force}} + \underbrace{\tfrac{h/2}{r}}_{\substack{due\,to\\ bending\\ (from\,Eq.5.9)}} = \alpha_2 \Delta T - \tfrac{P_2}{E_2 h} - \tfrac{h/2}{r}$$

This yields an expression for the interface curvature:

$$\tfrac{1}{r} = \tfrac{12(\alpha_2 - \alpha_1)\Delta T}{h(14+n+1/n)} \qquad \text{where } n = E_1/E_2 .$$ ◀

(b) At the interface

$$\sigma_1 = \tfrac{P}{h} + \tfrac{E_1 h}{2r} = (\sigma_{yp})_1$$ ◀

$$\sigma_2 = -\tfrac{P}{h} - \tfrac{E_2 h}{2r} = -(\sigma_{yp})_2$$

(c) Summing these equations.

$$(\sigma_{yp})_1 - (\sigma_{yp})_2 = -\tfrac{h(E_1 - E_2)}{2r}$$

It follows that

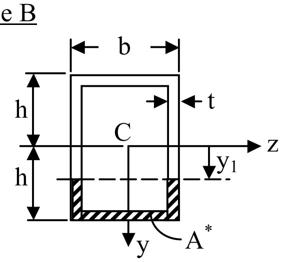
$$\tfrac{(\sigma_{yp})_1 - (\sigma_{yp})_2}{E_1 - E_2} = \tfrac{h}{2}\,\tfrac{24(\alpha_2 - \alpha_1)\Delta T}{h(14+n+1/n)}$$

from which

$$\Delta T = \tfrac{14+n+1/n}{6(\alpha_2 - \alpha_1)}\,\tfrac{(\sigma_{yp})_1 - (\sigma_{yp})_2}{E_1 - E_2}$$ ◀

SOLUTION (5.32)

Case B



$$Q = Q_{outside} - Q_{inside}$$

$$= \tfrac{b}{2}(h^2 - y_1^2) - \tfrac{b-2t}{2}[(h-t)^2 - y_1^2]$$

$$= bht - \tfrac{1}{2}bt^2 + th^2 + t^3 - ty_1^2$$

(CONT.)

5.32 (CONT.)

$$\alpha = \frac{A}{I^2}\int \frac{Q}{(width)^2}\,dA$$

$$= \frac{A}{I^2}\left\{2\left[\int_0^{h-t} \frac{(bht+th^2-\frac{bt^2}{2}-2ht^2+t^3-ty_1^2)^2}{b^2} 2t\,dy_1 + \int_{h-t}^{h} \frac{(bht-\frac{bt^2}{2}+th^2-2ht^2+t^3-ty_1^2)^2}{b^2} b\,dy_1\right]\right\}$$

or $\qquad \alpha = \frac{A}{A_{web}} = 1 + \frac{b}{2(h-t)}$

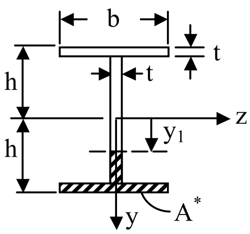
where

$$A = 2bh - (b-2t)(2h-2t) = 2bt + 4th - 4t^2$$

$$A_{web} = 2(2h-2t)t = 4ht - 4t^2$$

$$I = \tfrac{2}{3}bh^3 - \tfrac{2}{3}(b-2t)(h-t)^3$$

Case B



$$Q = (h-t-y_1)t[y_1 + \tfrac{1}{2}(h-t-y_1)]$$

$$= \tfrac{th^2}{2} - ht^2 + \tfrac{t^3}{2} + bht - \tfrac{bt^2}{2} - \tfrac{t}{2}y_1^2$$

$$\alpha = \frac{A}{I^2}\left\{2\left[\int_0^{h-t} \frac{(bht+ht^2-\frac{bh^2}{2}-\frac{bt^2}{2}-\frac{by_1^2}{2}+\frac{t^3}{2})^2}{t^2} t\,dy_1 + \int_{h-t}^{h} \frac{(bht+ht^2+\frac{bh^2}{2}-\frac{bt^2}{2}-\frac{ty_1^2}{2}+\frac{t^3}{2})^2}{b^2} b\,dy_1\right]\right\}$$

or $\qquad \alpha = \frac{A}{A_{web}} = 1 + \frac{b}{h-t}$ ◄

where

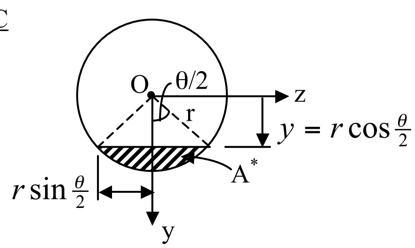
$$A = 2bt + (2h-2t)t = 2bt + 2th - 2t^2$$

$$A_{web} = (2h-2t)t = 2ht - 2t^2$$

$$I = \tfrac{1}{12}t(2h-2t)^3 + 2[\tfrac{1}{12}bt^3 + bt(h-\tfrac{t}{2})^2]$$

Note: For thin-walled sections flange and web thicknesses are small with compared with to unity and their products are neglected.

Case C



(CONT.)

We write $A = \frac{r^2}{2}(\theta - \sin\theta)$ =area of segment

$$dA = \tfrac{r^2}{2}(1-\cos\theta)d\theta = r^2(1-\cos^2\tfrac{\theta}{2})d\theta$$

$$Q = \int_{A^*} y dA = \int_{A^*}(r\cos\tfrac{\theta}{2})r^2(1-\cos^2\tfrac{\theta}{2})d\theta$$

$$= r^3\int_0^\theta(\cos\tfrac{\theta}{2}-\cos^3\tfrac{\theta}{2})d\theta = r^3[2\sin\tfrac{\theta}{2}-\tfrac{2}{3}\sin\tfrac{\theta}{2}(\cos^2\tfrac{\theta}{2}+2)]$$

$$= \tfrac{2}{3}r^3[\sin\tfrac{\theta}{2}-\sin\tfrac{\theta}{2}\cos^2\tfrac{\theta}{2}]$$

$$\alpha = \tfrac{A}{I^2}\int\frac{Q^2}{(width)^2}dA$$

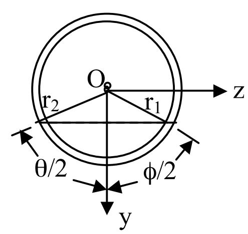
$$= \tfrac{A}{I^2}\int_0^{2\pi}\frac{[\frac{2\pi r^3}{3}(\sin\frac{\theta}{2}-\sin\frac{\theta}{2}\cos^2\frac{\theta}{2})]}{(2r\sin\frac{\theta}{2})^2}r^2(1-\cos^2\tfrac{\theta}{2})d\theta$$

Since, $\quad A = \pi r^2 \qquad I^2 = \frac{\pi^2 r^8}{16}$

We have, after simplification:

$$\alpha = \tfrac{16}{\pi r^6}\int_0^{2\pi}(\tfrac{r^6}{9})(1-\cos^2\tfrac{\theta}{2})^3 d\theta = \tfrac{10}{9} \qquad ◄$$

Case D



$$Q = Q_2 - Q_1$$

$$= \tfrac{2}{3}r_2^2[\sin\tfrac{\theta}{2}-\sin\tfrac{\theta}{2}\cos^2\tfrac{\theta}{2}] - \tfrac{2}{3}r_1^2[\sin\tfrac{\phi}{2}-\sin\tfrac{\phi}{2}\cos^2\tfrac{\phi}{2}]$$

$$\alpha = \tfrac{A}{I^2}[\int_0^{2\pi}\frac{Q_2^2}{(2r_2\sin\frac{\theta}{2})^2}r_2^2(1-\cos^2\tfrac{\theta}{2})d\theta - \int_0^{2\pi}\frac{Q_1^2}{(2r_1\sin\frac{\phi}{2})^2}r_1^2(1-\cos^2\tfrac{\phi}{2})d\phi]$$

Letting

$$A = \pi(r_2^2 - r_1^2), \qquad I^2 = \tfrac{\pi^2}{16}(r_2^4 - r_1^4)$$

after integration, we obtain

$$\alpha = 2 \qquad ◄$$

SOLUTION (5.33)

Since stress is symmetrical, Eq. (3.40) reduces to

$$(\tfrac{\partial^2}{\partial r^2}+\tfrac{1}{r}\tfrac{\partial}{\partial r})(\tfrac{\partial^2\Phi}{\partial r^2}+\tfrac{1}{r}\tfrac{\partial\Phi}{\partial r}) = \tfrac{\partial^4\Phi}{\partial r^4}+\tfrac{2}{r}\tfrac{\partial^3\Phi}{\partial r^3}-\tfrac{1}{r^2}\tfrac{\partial^2\Phi}{\partial r^2}+\tfrac{1}{r^3}\tfrac{\partial\Phi}{\partial r} = 0$$

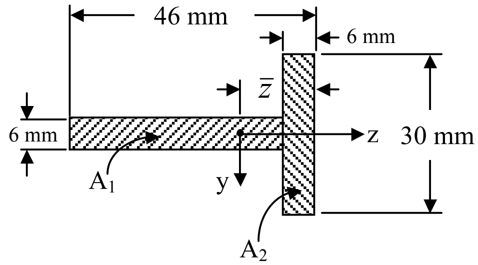
The given Φ satisfies this equation. Equations (3.32) lead to

$$\sigma_r = \tfrac{1}{r}\tfrac{\partial\Phi}{\partial r} = \tfrac{A}{r^2} + B(1+2\cdot\ln r) + 2C \qquad ◄$$

$$\sigma_\theta = \tfrac{\partial^2\Phi}{\partial r^2} = (-\tfrac{A}{r^2}) + B(3+2\cdot\ln r) + 2C$$

The constants A, B, and C are determined from the boundary conditions (b) and (d) of Sec. 5.13. In so doing, we arrive at the solution given by Eq. (5.67).

SOLUTION (5.34)



$$\overline{z} = \frac{(A_1 z_1 + A_2 z_2)}{(A_1 + A_2)} = 16.14\ mm$$

We have

$$\overline{r} = 136.14\ mm \qquad r_i = 120\ mm \qquad r_o = 166\ mm$$

$$R = \frac{A}{\sum\int \frac{dA}{r}} = \frac{420}{\int_{120}^{126} \frac{30dr}{r} + \int_{126}^{166} \frac{6dr}{r}} = 134.7045\ mm, \quad e = \overline{r} - R = 1.4355\ mm$$

We have $M = -(30 + 120 + 16.14)P = -166.14P$

Outer Edge. Applying Eq. (5.73),

$$-80 \frac{N}{mm^2} = \frac{P}{420} + \frac{166.14P(134.7045 - 166)}{420(1.4355)(166)}$$

or $\quad P = 1.614\ kN = P_{all}$ ◄

Inner Edge. Using Eq. (5.73),

$$80 \frac{N}{mm^2} = \frac{P}{420} + \frac{166.14P(134.7045 - 126)}{420(1.4355)(126)}$$

Solving,

$$P = 3.73\ kN$$ ◄

SOLUTION (5.35)

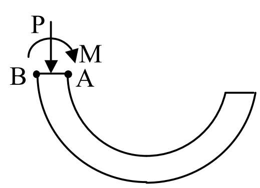


Figure (a)

(a) Equation (5.66) yields

$$N = (1 - \tfrac{0.1^2}{0.2^2})^2 - 4\tfrac{0.1^2}{0.2^2} \ln^2 \tfrac{0.1}{0.2} = 0.082$$

$$M = PR = 70(10^3) \times 0.15 = 10.5\ kN \cdot m$$

Thus, from Eq. (5.67):

$$(\sigma_\theta)_A = \frac{4(10{,}500)}{0.05(0.2)^2(0.082)}[(1 - \tfrac{0.1^2}{0.2^2})(1 + \ln 1) - (1 + 1)\ln \tfrac{0.2}{0.1}]$$
$$= 163\ MPa$$

$$(\sigma_\theta)_B = \frac{4(10{,}500)}{0.05(0.2)^2(0.082)}[(1 - \tfrac{0.1^2}{0.2^2})(1 + \ln 2) - (1 + \tfrac{0.1^2}{0.2^2})\ln 2]$$
$$= 103\ MPa$$

(CONT.)

5.35 (CONT.)

Referring to Fig. (a),

$$\sigma_{\max} = (\sigma_\theta)_A - \tfrac{P}{A} = -163 - \tfrac{70}{5} = -177\ MPa$$

$$\sigma_{\min} = (\sigma_\theta)_B - \tfrac{P}{A} = 103 - 14 = 89\ MPa \quad ◄$$

(b) We have

$$r_i = 100\ mm, \quad r_o = 200\ mm, \quad R = 100/\ln 2 = 144.2695$$

$$A = 5000\ mm^2, \quad e = 5.7305$$

Thus,

$$(\sigma_\theta)_A = -\tfrac{P}{A}[1 + \tfrac{\bar{r}(R-r_i)}{er_i}] = -176.2\ MPa$$

$$(\sigma_\theta)_B = -\tfrac{P}{A}[1 + \tfrac{\bar{r}(R-r_o)}{er_o}] = 88.1\ MPa \quad ◄$$

SOLUTION (5.36)

$$A = 20b + 2(60 \times 10) = 20b + 2400$$

We have $r_A = 60\ mm$ and $r_B = 140\ mm$. Applying Eq. (5.70):

$$-\sigma_A = \sigma_B = \frac{(-M)(R - r_A)}{Aer_A} = -\frac{(-M)(R - r_B)}{Aer_B}$$

from which

$$r_B(R - r_A) = -r_A(R - r_B), \qquad 140(R - 60) = -60(R - 140)$$

or $\quad R = 134\ mm$

Then $\quad R = \dfrac{A}{\sum\int \dfrac{dA}{r}}$

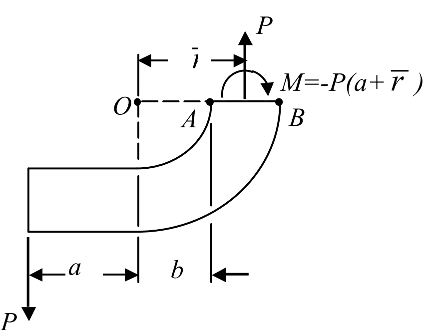
or $\quad A = 84\left[\int_{60}^{80} \dfrac{bdr}{r} + \int_{80}^{140} \dfrac{2(10)dr}{r}\right] = 84\left[b \ln\dfrac{8}{6} + 20 \ln\dfrac{14}{8}\right]$

$$= 24.1653b + 1880.3091$$

Hence $\quad 20b + 2400 = 24.1653b + 1880.3091$

or $\quad b = 124.8\ mm \quad ◄$

SOLUTION (5.37)



We have $c = \dfrac{1}{2}d = 10\ mm \qquad \bar{r} = b + c = 25\ mm$

$$R = \frac{1}{2}[\bar{r} + \sqrt{\bar{r}^2 - c^2}\,] \quad \text{(by Table 5.2)}$$

$$= \frac{1}{2}[25 + \sqrt{25^2 - 10^2}\,] = 23.9564\ mm$$

$$e = \bar{r} - R = 1.0436\ mm$$

$$A = \pi c^2 = \pi(10)^2 = 314.16\ mm^2$$

(CONT.)

5.37 (CONT.)

(a) From Eq.(5.74):

$$\sigma_A = \frac{P}{A} - \frac{M(R-r_A)}{Aer_A} = \frac{P}{A}[1+\frac{(a+\overline{r})(R-r_A)}{er_A}]$$

$$= \frac{800}{314.16}[1+\frac{(50)(23.9564-15)}{(1.0436)(15)}] = 75.4\ MPa \quad ◀$$

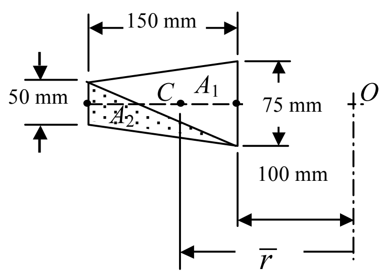
(b) Using Eq.(5.74):

$$\sigma_B = \frac{P}{A} - \frac{M(R-r_B)}{Aer_B} = \frac{P}{A}[1+\frac{(a+\overline{r})(R-r_B)}{er_B}$$

$$= \frac{800}{314.16}[1+\frac{50(29.1421-35)}{1.0436(35)}] = -34.2\ MPa \quad ◀$$

SOLUTION (5.38)

Locate centroid :



$$\overline{r} = \frac{A_1\overline{r}_1 + A_2\overline{r}_2}{A_1 + A_2}$$

$$= \frac{(5625)(150)+(3750)(200)}{5625+3750} = 170\ mm$$

$$A = \frac{1}{2}(50+75)(150) = 9375\ mm^2$$

$$R = \frac{\frac{1}{2}h^2(b_1+b_2)}{(b_1r_o - b_2r_i)\ln\frac{r_o}{r_i} - h(b_1-b_2)}$$

$$= \frac{(0.5)(150)^2(75+50)}{[(75)(250)-(50)(100)]\ln\frac{25}{10} - (150)(75-50)} = 158.9165\ mm$$

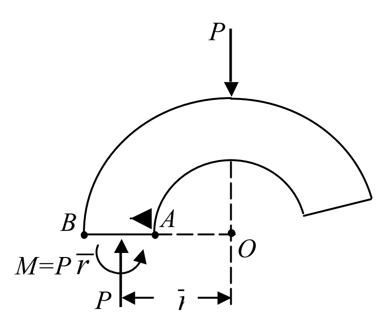
$$e = \overline{r} - R = 11.0825\ mm$$

(**a**) Use Eq.(5.75a),

$$(\sigma_\theta)_A = -\frac{P}{A} - \frac{M(R-r_A)}{Aer_A} = -\frac{P}{A}[1+\frac{\overline{r}(R-r_A)}{er_A}]$$

$$= -\frac{50(10^3)}{9375}\left[1+\frac{(170)(158.9175-100)}{(11.0825)(100)}\right]$$

$$= -53.5\ MPa$$



(CONT.)

(**b**) From Eq.(5.75b):

$$(\sigma_\theta)_B = -\frac{P}{A}[1+\frac{\overline{r}(R-r_B)}{er_B}]$$

$$= -\frac{50(10^3)}{9375}\left[1+\frac{(170)(158.9175-250)}{11.0825(250)}\right] = -24.5\ ksi \quad ◄$$

SOLUTION (5.39)

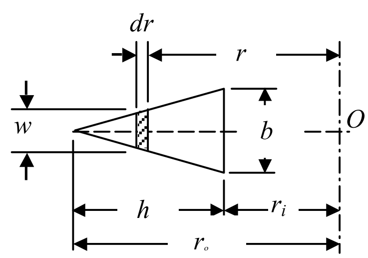
$$A = \frac{1}{2}bh$$

The section width w varies linearly with r. Thus

$$w = c_0 + c_1 r \qquad (1)$$

Since

$$w = b \quad (at\ r = r_i)$$
$$w = 0 \quad (at\ r = r_o) \qquad (2)$$



Substituting Eq.(1) into Eq.(2);

$$c_1 = -\frac{b}{h} \qquad c_0 = \frac{br_o}{h}$$

Then

$$\int_A \frac{dA}{r} = \int_{r_i}^{r_o} \frac{w}{r}dr = \frac{c_0 + c_1 r}{r}dr$$

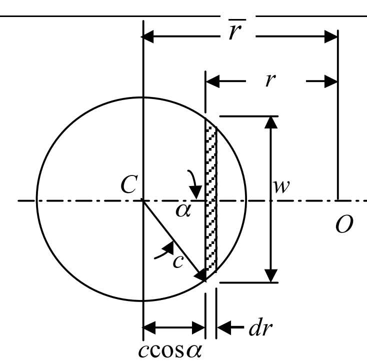
Inserting c_1 and c_0 into this, after integrating and rearranging, we have

$$\int \frac{dA}{r} = b(\frac{r_o}{h}\ln\frac{r_o}{r_i} - 1)$$

Therefore

$$R = \frac{A}{\int \frac{dA}{r}} = \frac{\frac{1}{2}h}{\frac{r_o}{h}\ln\frac{r_o}{r_i} - 1} \quad ◄$$

SOLUTION (5.40)



$$A = \pi c^2$$

Through use of the polar coordinates we write:

$$w = 2c\sin\alpha \qquad r = \overline{r} - c\cos\alpha$$
$$dr = c\sin\alpha\, d\alpha$$
$$dA = wdr = 2c^2\sin^2\alpha\, d\alpha$$

$$\int \frac{dA}{r} = \int_0^\pi \frac{2c^2\sin^2\alpha}{\overline{r} - c\cos\alpha}d\alpha$$

(CONT.)

$$= 2\int_0^\pi \frac{c^2(1-\cos^2\alpha)}{\overline{r} - c\cos\alpha} = 2\int_0^\pi \frac{\overline{r}^2 - c^2\cos^2\alpha - (r^2 - c^2)}{\overline{r} - c\cos\beta}$$

$$= 2\int_0^\pi (\overline{r} + c\cos\alpha)d\alpha - 2(\overline{r}^2 - c^2)\int_0^\pi \frac{d\alpha}{\overline{r} - c\cos\alpha}$$

$$= 2\overline{r}\alpha\Big|_0^\pi + 2c\sin\alpha\Big|_0^\pi - 2(\overline{r}^2 - c^2)\frac{2}{\sqrt{r^2 - c^2}}\tan^{-1}\frac{\sqrt{r^2 - c^2}\tan\frac{\alpha}{2}}{\overline{r} + c}\Bigg|_0^\pi$$

This gives

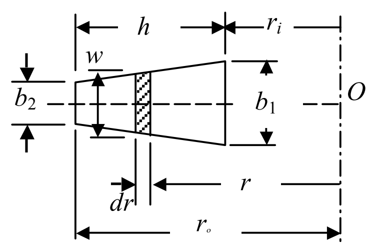
$$\int \frac{dA}{r} = 2\pi(\overline{r} - \sqrt{r^2 - c^2})$$

Hence, it can be shown that

$$R = \frac{A}{\int \frac{dA}{r}} = \frac{1}{2}(\overline{r} + \sqrt{\overline{r}^2 + c^2}) \qquad \blacktriangleleft$$

SOLUTION (5.41)

$$A = \frac{1}{2}(b_1 + b_2)h$$



The section width w varies linearly with r as

$$w = c_0 + c_1 r \qquad (1)$$

We have

$$w = b_1 \quad (at\ r = r_i)$$
$$w = b_2 \quad (at\ r = r_o) \qquad (2)$$

Introduce Eq.(1) into Eq.(2), then solve for c_0 and c_1

$$c_0 = \frac{r_o b_1 - r_i b_2}{h} \qquad c_1 = -\frac{b_1 - b_2}{h} \qquad (3)$$

Then, we write

$$\int \frac{dA}{r} = \int_{r_i}^{r_o} \frac{w}{r}dr = \int_{r_i}^{r_o} \frac{c_0 + c_1 r}{r}dr = c_0 \ln\frac{r_o}{r_i} + c_1(r_o - r_i)$$

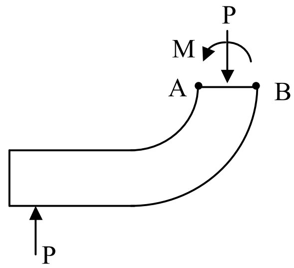
This gives, substituting Eqs.(3):

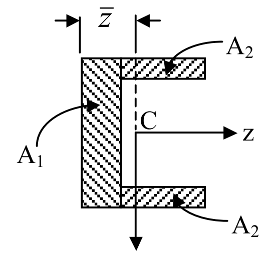
$$\int \frac{dA}{r} = \frac{r_o b_1 - r_i b_2}{h}\ln\frac{r_o}{r_i} - (b_1 - b_2)$$

Hence

$$R = \frac{A}{\int \frac{dA}{r}} = \frac{\frac{1}{2}h^2(b_1 + b_2)}{(r_o b_1 - r_i b_2)\ln\frac{r_o}{r_i} - (b_1 - b_2)} \qquad \blacktriangleleft$$

SOLUTION (5.42)





We have

$$\bar{z} = \frac{(A_1 z_1 + A_2 z_2)}{(A_1 + A_2)} = 12.5\ mm$$

$$\bar{r} = 52.5\ mm \qquad A = 0.0008\ m^2 \qquad r_i = 40\ mm \qquad r_o = 80\ mm$$

$$R = \frac{A}{\sum \int \frac{dA}{r}} = \frac{800}{\int_{40}^{50} \frac{50dr}{r} + \int_{50}^{80} \frac{2(5)dr}{r}} = 50.4502\ mm, \quad e = \bar{r} - R = 2.0498\ mm$$

$$M = (0.06 + 0.04 + 0.0125)P = 337.5\ N \cdot m$$

Inner Edge. Using Eq. (5.73),

$$(\sigma_\theta)_A = -\frac{3000}{0.0008} - \frac{337.5(50.4502-40)10^{-3}}{0.0008(2.0498)(40)10^{-6}}$$

$$= -57.5\ MPa \qquad ◀$$

Outer Edge. Applying Eq. (5.73),

$$(\sigma_\theta)_B = -\frac{3000}{0.0008} - \frac{337.5(50.4502-80)10^{-3}}{0.0008(2.0498)(80)10^{-6}}$$

$$= 74.6\ MPa \qquad ◀$$

SOLUTION (5.43)

Using Eq. (P5.44), at section A-B of Fig. P5.43:

$$M = -0.182P\bar{r} = -0.182(0.75b)P = -0.136Pb$$

(a) Equation (5.66) yields,

$$N = (1 - \tfrac{1}{4})^2 - 4(\tfrac{1}{4})(\ln 2)^2 = 0.082$$

Point A. Applying Eq. (5.67),

$$(\sigma_\theta)_A = \frac{4M}{tb^2N}[(1-\tfrac{1}{4})(1+\ln 1) - 2 \cdot \ln 2] + \frac{P}{bt}$$

$$= \frac{4(-0.136Pb)}{tb^2(0.082)}[\tfrac{3}{4} - 1.3863] + \frac{P}{bt}$$

$$= 5.22P/bt$$

Point B. Using Eq. (5.67),

$$(\sigma_\theta)_B = \frac{4(-0.136Pb)}{tb^2(0.082)}[\tfrac{3}{4}(1+\ln 2) - \tfrac{5}{4}\ln 2] + \frac{P}{bt}$$

$$= -1.68P/bt \qquad ◀$$

(CONT.)

(b) We have $\bar{r} = 0.75b,\ r_i = 0.5b,$ and $r_o = b$. Thus, $R = (b/2)/\ln 2 = 0.7213b$

Points A and B. Apply Eq. (5.74) with $P = P/2$ and $M = -0.136Pb$:

$$(\sigma_\theta)_A = \tfrac{P}{bt} + 4.2\tfrac{P}{bt} = 5.2P/bt \quad ◀$$

$$(\sigma_\theta)_B = \tfrac{P}{bt} - 2.64\tfrac{P}{bt} = -1.64P/bt \quad ◀$$

(c) $$(\sigma_\theta)_{A,B} = (\tfrac{P}{2A}) \pm (\tfrac{Mc}{I})$$

Here $c = \frac{b}{4}$ $\quad I = \frac{tb^3}{96}$

Thus, $$(\sigma_\theta)_A = \tfrac{P}{bt} + \frac{0.136Pb(b/4)}{tb^2/96} = \tfrac{P}{bt} + 3.26\tfrac{P}{bt} = 4.26P/bt \quad ◀$$

and $$(\sigma_\theta)_B = \tfrac{P}{bt} - 3.26\tfrac{P}{bt} = -2.26P/bt \quad ◀$$

Comparing the result, we observe the following differences:

At the point A:

Elasticity vs. hyperbolic 0.4 % ◀

Elasticity vs. linear 18 %

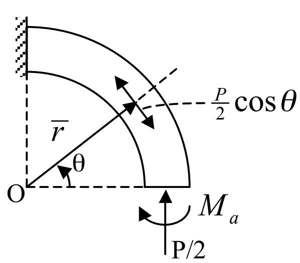
At the point B:

Elasticity vs. hyperbolic 2.4 % ◀

Elasticity vs. Linear 34.5 %

SOLUTION (5.44)

From the condition of symmetry, the distribution of stress in any quadrant is known to be the same as the other.



At any angle θ, the bending moment referring to this figure is expressed as follows

$$M_\theta = M_a - \tfrac{1}{2}P\bar{r}(1-\cos\theta) \qquad \text{(a)}$$

The problem is statically indeterminate and value of M_a is found first. Note that shear component of the load will be omitted in our solution.

Applying Castigliano's theorem, we have:

$$\frac{\partial U}{\partial M_a} = 4\int_0^{\pi/2} \frac{1}{EI} M_\theta \frac{\partial M_\theta}{\partial M_a} ds$$

$$0 = \frac{4}{EI}\int_0^{\pi/2} [M_a - \tfrac{P\bar{r}}{2}(1-\cos\theta)]\bar{r}d\theta$$

$$0 = \frac{4}{EI}[\tfrac{\pi}{2}M_a - \tfrac{P\bar{r}}{2}(\tfrac{\pi}{2}-1)]$$

(CONT.)

solving,

$$M_a = P\bar{r}(\tfrac{1}{2} - \tfrac{1}{\pi}) = 0.182P\bar{r}$$

Then, Eq. (a) becomes

$$M_\theta = 0.182P\bar{r} - \tfrac{1}{2}P\bar{r}(1-\cos\theta) \qquad \text{(b)} \quad ◀$$

Therefore stress at any point of a section, using Eq. (5.74), is expressed in the form

$$\sigma_\theta = -\tfrac{P}{2}\tfrac{\cos\theta}{A} + \tfrac{M_\theta}{A}(\tfrac{R-r}{er}) \quad ◀$$

Here, the moment M_θ is given by Eq. (b).

SOLUTION (5.45)

(a) Using Eq. (P5.44) at $\theta = \pi/4$ (Fig. P5.44):

$$M_\theta = 0.182(1.5)P - \tfrac{1}{2}P(1.5)(1-\cos\tfrac{\pi}{4}) = 0.00533P \ N\cdot m$$

We have

$$A = 0.05(0.1) = 0.005 \ m^2 \qquad R = \frac{h}{\ln\frac{r_o}{r_i}} = \frac{100}{\ln\frac{2}{1}} = 144.2695$$

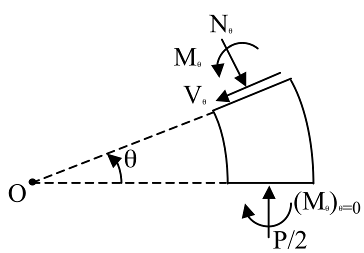
and

$$e = \bar{r} - R = 150 - 144.2695 = 5.7355 \ mm$$

At inner fiber, $r_i = -0.1 \ m$:

$$(\sigma_\theta)_{\pi/4} = \tfrac{-(P/2)\cos(\pi/4)}{0.005} - \tfrac{0.00533P}{0.005}[\tfrac{144.2695-100}{5.7355(0.1)}] = -182.3P \ Pa \quad ◀$$

(b)



At $\theta = 0^o$:

$$M_\theta = 0.182PR \qquad N_\theta = -P/2$$

At any angle:

$$M_\theta = 0.182P\bar{r} - 0.5P\bar{r}(1-\cos\theta) = -0.318P\bar{r} + 0.5P\bar{r}\cos\theta$$

$$\tfrac{\partial M_\theta}{\partial(P/2)} = -0.636\bar{r} + \bar{r}\cos\theta$$

$$N_\theta = -\tfrac{P}{2}\cos\theta, \qquad \tfrac{\partial N_\theta}{\partial(P/2)} = -\cos\theta$$

$$V_\theta = -\tfrac{P}{2}\sin\theta, \qquad \tfrac{\partial V_\theta}{\partial(P/2)} = -\sin\theta$$

(CONT.)

5.46 (CONT.)

Based on the symmetry of the ring, we have

$$\delta = 2\int_0^{\pi/2} \frac{N_\theta}{AE}\frac{\partial N_\theta}{\partial(P/2)} + \frac{M_\theta}{EI}\frac{\partial M_\theta}{\partial(P/2)} + \frac{\alpha V_\theta}{AG}\frac{\partial V_\theta}{\partial(P/2)}]\overline{r}d\theta$$

Substituting the values of $M_\theta, N_\theta,$ and $V_\theta,$ this expression becomes

$$\delta_P = 2\int_0^{\pi/2}[\tfrac{P}{AE}\cos^2\theta + \tfrac{P\overline{r}^2}{2EI}(-0.636+\cos\theta)^2 + \tfrac{\alpha}{AG}\tfrac{P}{2}\sin^2\theta]\overline{r}d\theta$$

$$= 2[\tfrac{P\overline{r}}{2AE}\Big|\tfrac{\theta}{2} + \tfrac{1}{4}\sin 2\theta\Big|_0^{\pi/2} + \tfrac{P\overline{r}^3}{2EI}\Big|0.904\theta - 1.27\sin\theta + \tfrac{\sin 2\theta}{4}\Big|_0^{\pi/2}$$

$$+ \tfrac{\alpha P\overline{r}}{2AG}\Big|\tfrac{\theta}{2} - \tfrac{1}{4}\sin 2\theta\Big|_0^{\pi/2}]$$

from which

$$\delta_P = 2[\tfrac{P\overline{r}}{2AE}(\tfrac{\pi}{4}) + \tfrac{P\overline{r}^3}{2EI}(0.15) + \tfrac{\alpha}{AG}\tfrac{P\overline{r}}{2}(\tfrac{\pi}{2})] \quad ◄$$

For the given rectangular cross section:

$$\overline{r} = 0.15\ m \qquad A = 0.005\ m^2$$

$$\alpha = \tfrac{6}{5} = 1.2 \qquad G = \tfrac{2}{5}E = 0.4E$$

$$I = \tfrac{1}{12}(0.05)(0.1)^3 = 4.17(10^{-6})\ m^4$$

The deflection is therefore

$$\delta_P = 2[\tfrac{P(0.15)\pi}{0.005E(8)} + \tfrac{P(0.15)^3(0.15)}{2E(4.17\times10^{-6})} + \tfrac{1.2}{0.4E(0.005)}\tfrac{P(0.15)\pi}{8}]$$

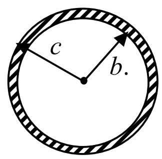
This results in

$$\delta_P = 215.65P/E\ m \quad ◄$$

End of Chapter 5

CHAPTER 6

SOLUTION (6.1)



$$\tau_{all} = \frac{Tc}{\frac{\pi}{2}(c^4 - b^4)}$$

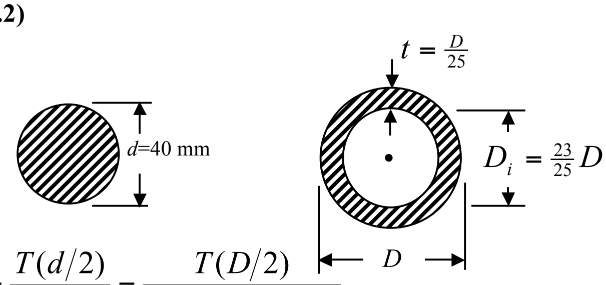
or $$100(10^6) = \frac{2(3\times10^3)(0.035)}{\pi[(0.035)^4 - b^4]}$$

or $$100(10^6)[1.5(10^{-6}) - b^4] = 66.845$$

Solving,

$$b = 0.0302\ m = 30.2\ mm$$ ◄

SOLUTION (6.2)



$$\tau_{\max} = \frac{T(d/2)}{\pi d^4/32} = \frac{T(D/2)}{\pi(D^4 - D_i^4)/32}$$

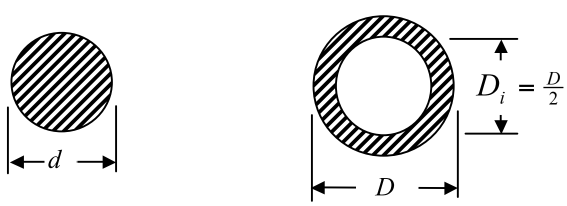
or

$$d^3 = D^3[1 - (\frac{23}{25})^4]$$

$$(40)^3 = D^3(0.2836)$$

$$D = 60.9\ mm$$ ◄

SOLUTION (6.3)



$$\tau_{\max} = \frac{16T}{\pi d^3} = \frac{T(D/2)}{\frac{\pi}{32}(D^4 - D_i^4)} = \frac{16T}{\pi D^3(1 - \frac{1}{16})}$$

or $$d^3 = D^3(\frac{15}{16}), \qquad D = 1.0217d$$

Thus

$$d = \frac{D}{1.0217} = \frac{60}{1.0217} = 58.7\ mm$$ ◄

SOLUTION (6.4)

Apply the method of sections between the change of load points:

$$T_{CD} = 0.5\ kN \cdot m \longrightarrow\!\!\!\!\!\rightarrow \qquad\qquad T_{BC} = 1.5\ kN \cdot m \longrightarrow\!\!\!\!\!\rightarrow$$

$$T_{AB} = 2\ kN \cdot m \quad \leftarrow\!\!\!\!\!\longleftarrow$$

Therefore, $\tau_{\max} = 2T/\pi c^3$:

$$\tau_{AB} = \frac{2(2\times10^3)}{\pi(0.025)^3} = 81.5\ MPa$$

$$\tau_{BC} = \frac{2(1.5\times10^3)}{\pi(0.02)^3} = 119.4\ MPa \qquad \blacktriangleleft$$

$$\tau_{CD} = \frac{2(0.5\times10^3)}{\pi(0.015)^3} = 94.3\ MPa$$

SOLUTION (6.5)

We have, applying the method of sections:

$$T_{CD} = 0.5\ kN \cdot m \longrightarrow\!\!\!\!\!\rightarrow \qquad\qquad T_{BC} = 1.5\ kN \cdot m \longrightarrow\!\!\!\!\!\rightarrow$$

$$T_{AB} = 2\ kN \cdot m \quad \leftarrow\!\!\!\!\!\longleftarrow$$

Hence,

$$\tau_{\max} = \frac{2Tc}{\pi(c^4 - b^4)}$$

gives,

$$\tau_{AB} = \frac{2(2\times10^3)(0.025)}{\pi[(0.025)^4 - (0.008)^4]} = 82.4\ MPa$$

$$\tau_{BC} = \frac{2(1.5\times10^3)(0.02)}{\pi[(0.02)^4 - (0.008)^4]} = 122.5\ MPa \qquad \blacktriangleleft$$

$$\tau_{CD} = \frac{2(0.5\times10^3)(0.015)}{\pi[(0.015)^4 - (0.008)^4]} = 102.6\ MPa$$

SOLUTION (6.6)

$$T_{BC} = 1\ kN \cdot m \longrightarrow\!\!\!\!\!\rightarrow \qquad T_{AB} = 2\ kN \cdot m \longrightarrow\!\!\!\!\!\rightarrow$$

Then, $\tau_{\max} = 2T/\pi c^3$ yields

$$\tau_{BC} = \frac{2(1\times10^3)}{\pi(0.02)^3} = 79.58\ MPa \qquad \blacktriangleleft$$

$$\tau_{AB} = \frac{2(2\times10^3)}{\pi(0.025)^3} = 81.49\ MPa$$

Therefore,

$$(\tau_{\max})_B = K\tau_{AB} = 1.6(81.49)$$
$$= 130.4\ MPa$$

SOLUTION (6.7)

We have

$$T_{BC} = 4\ kN \cdot m \longrightarrow\!\!\!\!\!\!\rightarrow \qquad T_{AB} = 4\ kN \cdot m \leftarrow\!\!\!\!\!\!\longleftarrow$$

Then

$$\phi_C = \sum (TL/GJ) \text{ yields}$$

$$\phi_C = \frac{4(10^3)}{(28\times10^9)\frac{\pi}{2}}\left[\frac{0.35}{(0.05)^4} - \frac{0.7}{\frac{3}{2}(0.1)^4}\right] = \frac{1}{3.5\pi}\left(\frac{0.35}{6.25} - \frac{0.7}{150}\right)$$

$$= 4.67\times10^{-3}\ rad = 0.27^o$$ ◄

SOLUTION (6.8)

$$\theta = 50 + 90 = 140^o$$

$$J = \frac{\pi}{2}[(0.05)^4 - (0.045)^4] = 3.376\times10^{-6}\ m^4$$

From Eq. (6.5a) at $\theta = 140^o$:

$$\sigma_{x'} = \tau \sin 280^o = -0.985\tau$$

$$\left|120\times10^6\right| = 0.985\frac{Tc}{J} = 0.985\frac{T(0.05)}{3.376\times10^{-6}}$$

or $\quad T = 8.23\ kN \cdot m$

Similarly, Eq. (6.5b):

$$\tau_{x'y'} = \tau \cos 280^o = 0.174\tau$$

gives

$$\left|50\times10^6\right| = 0.174\frac{T(0.05)}{3.376\times10^{-6}}, \qquad T = 19.4\ kN \cdot m$$

Thus,

$$T_{all} = 8.23\ kN \cdot m$$ ◄

SOLUTION (6.9)

From solution of Prob. 6.8:

$$J = 3.376\times10^{-6}\ m^4$$

From Equations (6.5) at $\theta = 35 + 90 = 125^o$,

$$\sigma_{x'} = \tau \sin 250^o = -0.94\tau$$

$$\left|120\times(10^6)\right| = 0.94\frac{Tc}{J} = 0.94\frac{T(0.05)}{3.376\times10^{-6}}$$

or $\quad T = 8.62\ kN \cdot m$

and

$$\tau_{x'y'} = \tau \cos 250^o = -0.342\tau$$

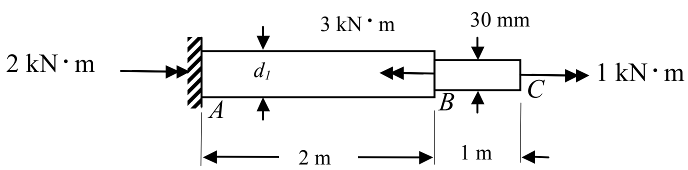
(CONT.)

$$\left|50\times10^{6}\right| = 0.342\frac{T(0.05)}{3.376\times10^{-6}}, \qquad T = 9.87\ kN\cdot m$$

Thus,

$$T_{all} = 8.62\ kN\cdot m \qquad \blacktriangleleft$$

SOLUTION (6.10)



Apply the method of sections:

$$T_{AB} = 2\ kN\cdot m \qquad\qquad T_{BC} = 1\ kN\cdot m$$

Then,

$$\frac{J}{c} = \frac{\pi d_1^3}{16} = \frac{T}{\tau_{all}} = \frac{2\times10^3}{50\times10^6}$$

$$d_1 = 0.05884\ m = 58.84\ mm$$

Also,

$$\phi = \sum (TL/GJ):$$

or

$$0.02 = \frac{10^3}{(39\times10^9)\frac{\pi}{32}}\left[\frac{1(1)}{(0.04)^4} - \frac{2(2)}{d_1^4}\right]$$

$$76576.32 = 390625 - \frac{4}{d_1^4}$$

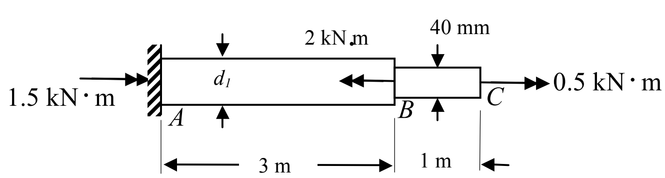
Solving

$$d_1 = 59.74\times10^{-3}\ m = 59.74\ mm$$

We therefore use

$$d_1 = 60\ mm \qquad \blacktriangleleft$$

SOLUTION (6.11)



Apply the method of sections:

$$T_{AB} = 1.5\ kN\cdot m \qquad\qquad T_{BC} = 0.5\ kN\cdot m$$

(CONT.)

6.11 (CONT.)

We have

$$\frac{J}{c} = \frac{\pi d_1^3}{16} = \frac{T}{\tau_{all}} = \frac{1.5\times10^3}{50\times10^6}$$

$$d_1 = 53.46\times10^{-3}\ m = 53.5\ mm$$

Also,

$$\phi = \sum (TL/GJ):$$

or $$0.02 = \frac{10^3}{(40\times10^9)\frac{\pi}{32}}[\frac{0.5(1)}{(0.04)^4} - \frac{1.5(3)}{d_1^4}]$$

$$78539.82 = 195,312.5 - \frac{4.5}{d_1^4}$$

Solving

$$d_1 = 78.79\times10^{-3}\ m = 78.8\ mm$$

Use $d_1 = 78.8\ mm$ ◀

SOLUTION (6.12)

(**a**) $\frac{D}{d} = \frac{60}{50} = 1.2 \qquad \frac{r}{d} = \frac{1}{50} = 0.02$

Figure D.4: $K = 2.1$.

$$\tau_{nom} = \frac{2T}{\pi c^3} = \frac{2(2\times10^3)}{\pi(0.025)^3} = 81.5\ MPa$$

$$\tau_{\max} = 2.1(81.5) = 171.2\ MPa$$ ◀

(**b**) $\frac{r}{d} = \frac{5}{50} = 0.1, \qquad \frac{D}{d} = 1.2; \quad K = 1.33$ (Fig. D.4)

$$\tau_{\max} = 1.33(81.5) = 108.4\ MPa$$ ◀

SOLUTION (6.13)

$\frac{r}{d} = \frac{10}{50} = 0.2 \qquad \frac{D}{d} = \frac{100}{50} = 2.0; \quad K = 1.25$ (from Fig. D.4)

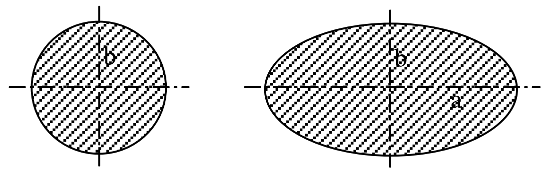
For smaller part of the shaft, $c = d/2 = 25\ mm$:

$$\tau_{\max} = \tau_{all} = K(\frac{16T}{\pi d^3}); \quad 80(10^6) = 1.25[\frac{16T}{\pi(0.05)^3}]$$

Solving

$$T = 1.571\ kN\cdot m$$ ◀

SOLUTION (6.14)



(a) For circular bar:

$$\theta_c = \frac{2T}{G\pi b^4} \qquad \tau_c = \frac{2T}{\pi b^3} = G\theta_c b$$

For elliptical bar:

$$\theta_e = \frac{T(a^2+b^2)}{\pi a^3 b^3 G}, \qquad T = \frac{\pi a^3 b^3 G}{a^2+b^2}\theta_e$$

$$\tau_e = \frac{2T}{\pi a b^2} = \frac{20\theta_e b a^3 G}{a^2+b^2}$$

We have

$$\frac{\tau_e}{\tau_c} = \frac{2\theta_e b a^2 G}{(a^2+b^2)\theta_c b G}$$

Setting $\theta_e = \theta_c$:

$$\frac{\tau_e}{\tau_c} = \frac{2a^2}{(a^2+b^2)}$$

Since $a > b$, $(a^2 + b^2) < 2a^2$, and $\tau_e/\tau_c > 1$, or

$$\tau_e > \tau_c \qquad ◀$$

(b)

$$T_c = \frac{\theta_c \pi b^4 G}{2}; \qquad G\theta_c b = \tau_c$$

$$T_c = \frac{\tau_c}{Gb}\frac{\pi b^4 G}{2} = \frac{\tau_c \pi b^3}{2}$$

$$T_e = \frac{\theta_e \pi a^3 b^3 G}{a^2+b^2}, \qquad \tau_e = \frac{2\theta_e b a^2 G}{a^2+b^2}$$

Rearranging,

$$\theta_e = \frac{\tau_e(a^2+b^2)}{2ba^2 G}$$

Thus, $T_e = \frac{\tau_e \pi a b^2}{2}$

We obtain

$$\frac{T_c}{T_e} = \frac{(\tau_c \pi b^3/2)}{(\tau_e \pi a b^2/2)}$$

Setting $\tau_e = \tau_c$: $\qquad T_c/T_e = b/a$, or

$$T_e > T_c \qquad ◀$$

SOLUTION (6.15)

We have $\pi(c^2 - b^2) = \pi a^2$; $\qquad a^2 = c^2 - b^2$.

(a)

$$T_h = \frac{J\tau_a}{c} = \frac{\pi\tau_a(c^4-b^4)}{2c}$$

$$T_s = \frac{J\tau_a}{a} = \frac{\pi\tau_a a^3}{2}$$

(CONT.)

6.15 (CONT.)

Hence, $\frac{T_h}{T_s} = \frac{c^4-b^4}{ca^3} = \frac{c^4-b^4}{c(c^2-b^2)^{3/2}}$

For $c = 1.4a$:

$$\frac{T_h}{T_s} = \frac{2.8416}{1.3168} = 2.16 \quad ◄$$

(b) $T_h = \frac{JG\phi_a}{L} = (\frac{G\phi_a}{L})[\frac{\pi(c^4-b^4)}{2}]$

$T_s = (\frac{G\phi_a}{L})(\frac{\pi a^4}{2})$

Hence, $\frac{T_h}{T_s} = \frac{c^4-b^4}{a^2} = \frac{c^4-b^4}{(c^2-b^2)^2} = \frac{c^2+b^2}{c^2-b^2}$

For $c = 1.4b$:

$$\frac{T_h}{T_s} = \frac{2.96}{0.96} = 3.08 \quad ◄$$

SOLUTION (6.16)

Use Eqs. (f) of Example 6.2:

$$T_A = \frac{T}{1+(aJ_b/bJ_a)} = \frac{T}{1+\frac{0.4(15)^4(\pi/32)}{0.2(20)^4(\pi/32)}} = \frac{T}{(1+0.6328)} = 0.6124T$$

$$T_B = T - T_A = 0.3876T$$

Based on shear in segment AC:

$$\tau_a = \frac{16T_A}{\pi d_a^3} = \frac{16(0.6124T)}{\pi(0.02)^3} = 150(10^6) \qquad \text{or} \qquad T = 384.7\ N\cdot m$$

Based on shear in segment CB:

$$\frac{16T_B}{\pi d_b^3} = \frac{16(0.3876T)}{\pi(0.015)^3} = 150(10^6)$$

or

$$T = 256.5\ N\cdot m = T_{all} \quad ◄$$

SOLUTION (6.17)

Use Eqs. (f) of Example 6.2:

$$T_A = \frac{T}{1+\frac{0.8(15)^4(\pi/32)}{0.5(25)^4(\pi/32)}} = \frac{T}{1+0.2074} = 0.8283T$$

$$T_B = T - T_A = 0.1717T$$

Based on shear in segment AC:

$$\frac{16T_A}{\pi d_a^3} = \frac{16(0.8283T)}{\pi(0.025)^3} = 70(10^6)$$

or $\quad T = 259.3\ N\cdot m = T_{all} \quad ◄$

Based on shear in segment CB:

$$\frac{16T_B}{\pi d_b^3} = \frac{16(0.1717T)}{\pi(0.015)^3} = 70(10^6)$$

or

$$T = 270.2\ N\cdot m$$

SOLUTION (6.18)

$$\frac{\partial^2\Phi}{\partial x^2} = k[2(a^2b - a^2) + 2y^2(b^2 + 1) - 12bx^2]$$
$$\frac{\partial^2\Phi}{\partial y^2} = k[2(a^2b - a^2) + 2x^2(b^2 + 1) - 12by^2]$$

Substituting these in Eq. (6.9):

$$k = \frac{G\theta}{2a^2(b-1)+(b^2-6b+1)(x^2+y^2)}$$

In order k be a constant: $b^2 - 6b + 1 = 0$. Thus,

$$k = \frac{G\theta}{2a^2(b-1)} \quad ◄$$

SOLUTION (6.19)

$$u = -\theta z(y-b), \qquad v = \theta z(x-z), \qquad w = w(x,y)$$
$$\varepsilon_x = \varepsilon_y = \varepsilon_z = 0, \qquad \gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = 0$$
$$\gamma_{xz} = \frac{\partial w}{\partial x} - \theta(y-b), \qquad \gamma_{yz} = \frac{\partial w}{\partial y} + \theta(x-a)$$

Thus
$$\tau_{xz} = G[\tfrac{\partial w}{\partial x} - \theta(y-b)] = \frac{\partial\Phi}{\partial y}$$
$$\tau_{yz} = G[\tfrac{\partial w}{\partial y} - \theta(x-a)] = -\frac{\partial\Phi}{\partial x} \quad ◄$$

Substituting these in Eqs. (6.6), (6.7), and (6.11), we obtain

$$\frac{\partial\tau_{xz}}{\partial y} - \frac{\partial\tau_{yz}}{\partial x} = -2G\theta, \qquad \frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} = -2G\theta \quad ◄$$

and
$$T = \iint[(x-a)\tau_{yz} - (y-b)\tau_{xz}]dxdy$$
$$= [-\iint(x-a)\frac{\partial\Phi}{\partial x} - \iint(y-b)\frac{\partial\Phi}{\partial y}]dxdy$$
$$= 2\iint\Phi dxdy \quad ◄$$

We observe that characteristic equations remain unchanged.

SOLUTION (6.20)

$$T = \iint(x\tau_{yz} - y\tau_{xz})dxdy = -\iint(x\frac{\partial\Phi}{\partial x} - y\frac{\partial\Phi}{\partial y})dxdy$$

Here

$$-\int dy\int x\frac{\partial\Phi}{\partial x}dx = -\int xd\Phi\Big|_{x_1}^{x_2}dy - [-\iint\Phi dxdy]$$
$$= -c\int(x_2 - x_1)dy + \iint\Phi dxdy$$
$$= -c\iint dxdy + \iint\Phi dxdy$$

Note that $(x_2 - x_1)dy$ is area and equals $\iint dxdy$. Similarly,

$$-\int dx\int y\frac{\partial\Phi}{\partial y}dy = -c\iint dxdy + \iint\Phi dxdy$$

Thus,

$$T = 2\iint(\Phi - c)dxdy \quad ◄$$

SOLUTION (6.21)

$$T_\theta = T\cos^2\theta$$

Hence,

$$\theta = \int \tfrac{T_\theta dz}{GJ} = \tfrac{1}{GJ}\int T\cos^2\theta a d\theta$$

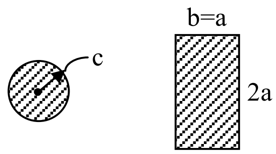
This gives, at sections A and B:

$$\theta_A = \tfrac{Ta}{GJ}\int_0^{\pi/2}(\tfrac{1}{2} + \tfrac{\cos 2\theta}{2})d\theta = \tfrac{Ta}{2r^4G}$$

◀

$$\theta_B = \tfrac{Ta}{GJ}\int_0^{\pi}\cos^2\theta d\theta = \tfrac{Ta}{r^4G}$$

SOLUTION (6.22)



From Table 6.2: $\alpha = 0.246,\ \beta = 0.229$

$$\tau_r = \tfrac{T}{\alpha ab^2} = \tfrac{T}{0.492a^3}$$

$$\tau_c = \tfrac{Tc}{J} = \tfrac{2T}{\pi c^3}$$

Thus, $\tfrac{T}{0.492a^3} = \tfrac{2T}{\pi c^3};\qquad c = 0.684a$

Similarly,

$$\theta_r = \tfrac{T}{0.229(2a)a^3G} = \tfrac{T}{0.458a^4G}$$

$$\theta_c = \tfrac{T}{JG} = \tfrac{2T}{\pi c^4 G}$$

Then,

$$\tfrac{T}{0.458a^4G} = \tfrac{2T}{\pi c^4G};\qquad c = 0.736a = c_{all}$$

◀

SOLUTION (6.23)

$$\tfrac{\partial^2\Phi}{\partial y^2} = k[(x+\tfrac{h}{3}) + (x+\sqrt{3}y-\tfrac{2}{3}h) + (x+\tfrac{h}{3}) + (x-\sqrt{3}y-\tfrac{2}{3}h) + 2(x-\tfrac{2}{3}h)]$$

$$\tfrac{\partial^2\Phi}{\partial y^2} = k[-3(x+\tfrac{h}{3}) - 3(x+\tfrac{h}{3})]$$

Substituting these in Eq. (6.9),

$$-4kh = -2G\theta;\qquad k = \tfrac{G\theta}{2h}$$

Thus,

$$\Phi = -G\theta[\tfrac{1}{2}(x^2+y^2) - \tfrac{1}{2h}(x^3-3xy^2) - \tfrac{2}{27}h^2]$$

Along the x axis $\tau_{xz} = 0$, due the symmetry. Equation (6.8) is therefore,

for $y = 0$:

$$\tau_{yz} = -\tfrac{\partial\Phi}{\partial x} = \tfrac{3G\theta}{2h}(\tfrac{2hx}{3} - x^2)$$

(CONT.)

6.23 (CONT.)

When $x = 0:\qquad \tau_{yz} = 0$

$x = \frac{2h}{3}:\qquad \tau_{yz} = 0$

$x = -\frac{h}{3}:\qquad \tau_{yz} = \tau_{\max} = \frac{G\theta h}{2}$ ◀

Next, substitute the preceding value of Φ into Eq. (6.11) to obtain

$$T = 2\iint \Phi dxdy$$
$$= -4G\theta \int_0^{h/\sqrt{3}} \int_0^{-\sqrt{3}y+\frac{2}{3}h} [\tfrac{1}{2}(x^2+y^2) - \tfrac{1}{2h}(x^3 - 3xy^2) - \tfrac{2}{27}h^2]dxdy$$
$$= \frac{G\theta h^4}{15\sqrt{3}}$$

This gives $\qquad \theta = \frac{15\sqrt{3}T}{h^4 G}$

The shear stress is thus

$$\tau_{\max} = \frac{20T}{(2h/\sqrt{3})^3} = \frac{15\sqrt{3}T}{2h^3}$$ ◀

SOLUTION (6.24)

For a circular bar:

$$T = G\theta J = C_c\theta$$

where

$$C_c = \frac{\pi r^4 G}{2} \qquad \text{(a)}$$

For an elliptical bar (from Example 6.3):

$$H = -\frac{2T(a^2+b^2)}{\pi a^3 b^3} = -2G\theta$$
$$T = \frac{\pi a^3 b^3}{a^2+b^2} G\theta = C_e\theta$$

or $\qquad C_e = \frac{\pi a^3 b^3}{a^2+b^2} G \qquad$ (b)

For an equilateral bar (from Prob. 6.23):

$$T = \frac{G\theta h^4}{15\sqrt{3}} = \theta C_t$$

or $\qquad C_t = \frac{h^4 G}{15\sqrt{3}} \qquad$ (c)

Bars have equal areas:

$$A_c = \pi r^2, \qquad A_e = \pi ab, \qquad A_t = \frac{h^2}{\sqrt{3}}$$

Setting $A_c = A_e = A_t$:

$$r^4 = a^2 b^2 = \frac{h^4}{3\pi^2}$$

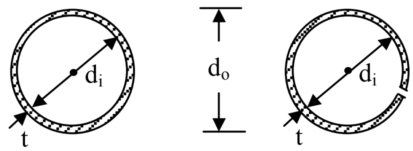
Then, Eqs. (a), (b), and (c) give

$$\frac{C_e}{C_c} \frac{2a^3b^3}{r^4(a^2+b^2)} = \frac{2ab}{a^2+b^2}$$ ◀

and

$$\frac{C_t}{C_c} = \frac{h^4 G}{15\sqrt{3}} \frac{2}{\pi r^4 G} = \frac{2\pi\sqrt{3}}{15}$$ ◀

SOLUTION (6.25)



For the seamless tube, from $\theta_1 = T/GJ$

$$\theta_1 = \frac{32T}{\pi d_o^4(1-d_i^4/d_o^4)G}$$

For the split tube, referring to Eq. (6.17):

$$\theta_2 = \frac{3T}{G}\frac{1}{\pi(\frac{d_o+d_i}{2})(\frac{d_o-d_i}{2})^3}$$

We have

$$\frac{\theta_1}{\theta_2} = \frac{2(d_o-d_i)}{3(d_o^2+d_i^2)} \qquad \text{(a)} \blacktriangleleft$$

For <u>very thin</u> tubes $d_o^2 + d_i^2 \approx 2d_o^2$, and Eq. (a) becomes

$$\frac{\theta_1}{\theta_2} = \frac{4}{3}\left(\frac{t}{d_o}\right) \qquad \blacktriangleleft$$

SOLUTION (6.26)

Apply Eqs. (6.17) and (6.19):

$$\tau_{\max} = \frac{3T}{bt^2} = \frac{3(80)}{0.125(0.005)^2} = 76.8\ MPa \qquad \blacktriangleleft$$

$$\theta = \frac{3T}{bt^3G} = \frac{3(80)}{0.125(0.005)^3(80\times10^9)} = 0.192\ rad/m \qquad \blacktriangleleft$$

SOLUTION (6.27)

Referring to Table 6.1, we have

$$\phi = \theta L = \frac{TL}{\beta ab^2 G} \qquad \tau_{\max} = \frac{T}{\alpha ab^2}$$

Thus

$$\frac{\phi}{\tau} = \frac{L}{bG}\frac{\alpha}{\beta} \qquad (1)$$

For $a/b = 24/16 = 1.5$: $\alpha = 0.231$ and $\beta = 0.196$

Introducing given data into Eq. (1):

$$\frac{1.5(\pi/180)}{\tau_{\max}} = \frac{0.4(0.231)}{(0.024)(80\times10^9)(0.196)}$$

Solving,

$$\tau_{\max} = 106.6\ MPa \qquad \blacktriangleleft$$

SOLUTION (6.28)

From Eq. (1) of solution of Prob. 6.27:

$$b = \frac{\tau_{\max}}{G}\frac{L}{\phi}\frac{\alpha}{\beta} \qquad \text{(a)}$$

By Table 6.1: $\alpha = 0.208$ and $\beta = 0.141$.

Substituting the numerical values into Eq. (a), we obtain

$$b = \frac{120(10^6)}{79(10^9)}\frac{2}{25(\pi/180)}\frac{0.204}{0.141} = 10.1\ mm \qquad \blacktriangleleft$$

SOLUTION (6.29)

Equation (6.20) yields

$$J_e = \sum \tfrac{1}{3} bt^3 = \tfrac{1}{3}(100)(10)^3 + \tfrac{1}{3}(115)(4)^3$$

$$= 3.5787(10^4)\ mm^4 \quad ◄$$

Maximum shear stress occurs on the lower leg:

$$\tau_{\max} = \tfrac{Tt_1}{J_e} = \tfrac{500(0.1)}{3.5787(10^{-8})} = 139.7\ MPa \quad ◄$$

Angle of twist per unit length is

$$\theta = \tfrac{T}{J_e G} = \tfrac{500}{3.5787(10^{-8})(200\times10^9)}$$

$$= 69.86(10^{-3})\ rad/m = 4.00^o \text{ per meter} \quad ◄$$

SOLUTION (6.30)

Refer to Table 6.2.

(a) $T = \frac{a_2^3 \tau_{\max}}{20} = \frac{(0.045)^3(50\times10^6)}{20} = 228\ N\cdot m$

Also

$$T = \tfrac{a_2^4 G\theta_{all}}{46.2} = \tfrac{(0.045)^4(80\times10^9)(1.5\pi/180)}{46.2}$$

$$= 186\ N\cdot m = T_{all} \quad ◄$$

(b) $\phi = \frac{46.2T_{\max}}{G}[\frac{L_1}{a_1^4} + \frac{L_2}{a_2^4}] = \frac{46.2(186)}{80\times10^9}[\frac{2.5}{(0.06)^4} + \frac{1.5}{(0.045)^4}]$

$$= 1.0742(10^{-7})[192{,}901.235 + 364{,}797.897]$$

$$= 0.06\ rad = 3.44^o \quad ◄$$

SOLUTION (6.31)

Referring to Fig. P6.31, an expression for t is written as

$$t = t_0(1 - \tfrac{y}{b})$$

Substitute this into the given stress function to obtain,

$$\Phi = G\theta[\tfrac{t_0^2}{4}(1-\tfrac{y}{b})^2 - x^2]$$

We have

$$T = 2\iint \Phi\, dxdy = 4\int_0^{t_0/2}\int_0^{b(1-\frac{2x}{t_0})}[\tfrac{t_0^2}{4}(1-\tfrac{y}{b})^2 - x^2]dxdy$$

$$= 4G\theta\int_0^{t_0/2}\{\tfrac{t_0^2}{4}[b(1-\tfrac{2x}{t_0}) - b(1-\tfrac{2x}{t_o})^2 + \tfrac{b}{3}(1-\tfrac{2x}{t_0})^3] - bx^3(1-\tfrac{2x}{t_0})\}dx$$

Let $u = \frac{(1-2x)}{t_0}$, $\quad x = \frac{(1-u)t_o}{2}$

$$du = -\tfrac{2dx}{t_0}, \qquad dx = -\tfrac{t_0 du}{2}$$

Then, the preceding expression for the torque becomes

$$T = 4G\tfrac{t_0^3 b}{8}\int_0^1 (u^2 - \tfrac{2}{3}u^3)du$$

Integrating,

$$T = G\theta t_0^3/12 \quad ◄$$

SOLUTION (6.32)

(a) From Table 6.2: $T = (2\pi r^3 tG)\theta = C\theta$

or $C = 2\pi r^3 tG = 2\pi(0.02375)^3(0.0025)G = 2.1(10^{-7})G$

$$\tau_{\max} = \tfrac{T}{2\pi r^2 t} = \tfrac{T}{2\pi(0.02375)^2(0.0025)} = 112.860T \quad ◄$$

(b) Equation (6.16):

$$C = \tfrac{bt^3 G}{3} = \tfrac{(0.02375)(0.0025)^3 G}{3} = 1.24(10^{-10})G$$

Equation (6.18):

$$\tau_{\max} = \tfrac{3T}{bt^2} = \tfrac{3T}{(0.02375)(0.0025)^2} = 20{,}210{,}526T \quad ◄$$

(c) From Table 6.2: for $a = b,\ t = t_1$:

$$T = (0.02375)^3 tG\theta = 1.34(10^{-5})0.0025G\theta$$

$$= 3.35(10^{-8})G\theta = C\theta$$

$$\tau_{\max} = \tfrac{T}{2a^2 t} = \tfrac{T}{2(0.02375)^2(0.0025)} = 354{,}571T \quad ◄$$

SOLUTION (6.33)

Referring to Table 6.2:

$$\tau_A = \tfrac{20T}{a^3}; \qquad \tfrac{420(2)}{3} = \tfrac{20T}{(0.05)^3}$$

Solving, $T = 1.75\ kN \cdot m$ ◄

Also

$$\theta = \tfrac{46.2T}{a^4 G} = \tfrac{46.2(1.75\times10^3)}{(6.25\times10^{-6})(80\times10^9)} = 0.1617\ rad/m \quad ◄$$

SOLUTION (6.34)

For a thin-walled tube, letting

$$r_o \approx r_i \approx r_{avg.} = r$$

$$J = \tfrac{\pi(r_o^4 - r_i^4)}{2} = \tfrac{\pi}{2}(r_o^2 + r_i^2)(r_o + r_i)(r_o - r_i) = 2\pi r t$$

Equation (6.2):

$$\tau = \tfrac{T\rho}{J} = \tfrac{T\rho}{2\pi r^3 t} = \tfrac{T\rho}{2Art}$$

Since $r = \rho$, we obtain Eq. (6.22): $\tau = T/2At$. ◄

SOLUTION (6.35)

For a regular hexagon, we can write

$$A = \tfrac{3\sqrt{3}}{2}a^3 \qquad ds = 6a$$

Equation (6.22), substituting the value of A, results in the shear stress

$$\tau = \tfrac{T}{2At} = \tfrac{T\sqrt{3}}{9a^2 t} \quad ◄$$

Angle of twist per unit length, using Eq. (6.23):

(CONT.)

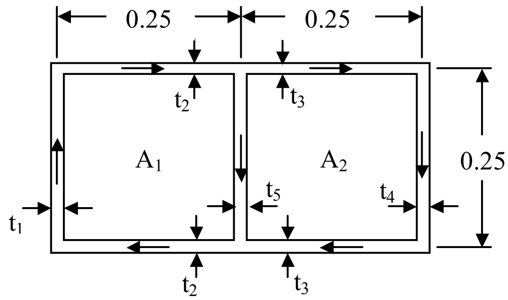
6.35 (CONT.)

$$\theta = \frac{\tau(6a)}{2GA} = \frac{3Ta}{2A^2Gt}$$

or

$$\theta = \frac{2T}{9Ga^3t} \quad ◀$$

SOLUTION (6.36)



Given:

$$t_1 = 0.012\ m, \qquad t_2 = t_3 = 0.006\ m,$$
$$t_4 = t_5 = 0.01\ m, \qquad G = 28\ GPa$$
$$T = 56.5\ kN \cdot m, \qquad A_1 = A_2 = A = 0.0625\ m^2$$
$$s_1 = s_2 = s_3 = s_4 = s_5 = 0.25\ m.$$

We write

$$T = 2A_1h_1 + 2A_2h_2 = 2A_1t_1\tau_1 + 2A_2t_3\tau_3$$

or

$$2\tau_1 + \tau_3 = 75.3(10^6) \tag{1}$$

The shear flow yields

$$\tau_1 t_1 = \tau_2 t_2 \tag{2}$$
$$\tau_2 t_2 = \tau_3 t_3 + \tau_5 t_5 \tag{3}$$
$$\tau_3 t_3 = \tau_4 t_4 \tag{4}$$

Also,

$$\tau_1 s_1 + 2\tau_2 s_2 + \tau_5 s_5 = 2G\theta A_1 \tag{5}$$
$$-\tau_5 s_5 + 2\tau_3 s_3 + \tau_4 s_4 = 2G\theta A_2 \tag{6}$$

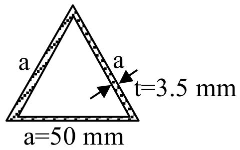
Simultaneous solution of Eqs. (1) through (6), after substitution of the given numerical values yields:

$$\tau_1 = 19.1\ MPa \qquad \tau_2 = 38.2\ MPa$$
$$\tau_3 = 37.2\ MPa \qquad \tau_4 = 22.3\ MPa \qquad \tau_5 = 0.62\ MPa \quad ◀$$

and

$$\theta = 6.86(10^{-3})\ rad/m \quad ◀$$

SOLUTION (6.37)



We have

$$A = \frac{(0.05)(0.05\sin 60^o)}{2} = 1.0825(10^{-3})\ m^2.$$

Then,

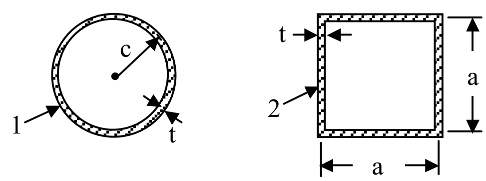
$$h = \frac{T}{2A} = \frac{40}{2(0.0010825)} = 18{,}475.209\ N/m$$

$$\tau = \frac{h}{t} = \frac{18{,}475{,}209}{0.0035} = 5.279\ MPa \qquad ◄$$

$$\theta = \frac{\tau}{2GA}\int ds = \frac{5.279(10^6)}{2(28\times10^9)(0.0010825)}[0.05 + 0.05 + 0.05]$$

$$= 13.063(10^{-3})\ rad/m \qquad ◄$$

SOLUTION (6.38)



For circular tube:

Area enclosed by c, $A_{m1} = \pi c^2$

Area of the section, $A_1 = 2\pi ct$

Polar moment of inertia, $J_1 = 2\pi c^3 t$

For square tube:

Enclosed area by a, $A_{m2} = a^2$

Area of the section, $A_2 = 4at$

Polar moment of inertia (from Table 6.2 with $t = t_1$ and $a = b$):

$$J_2 = \frac{2tt_1a^2b^2}{(at+bt_1)} = a^3t$$

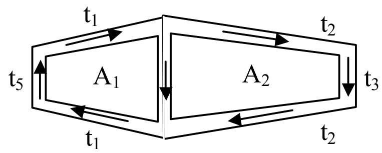
Then, $A_1 = A_2$ gives: $a = \pi c/2$ and hence, from Eq. (6.22) we obtain

$$\frac{\tau_1}{\tau_2} = \frac{A_{m2}}{A_{m1}} = \frac{a^2}{\pi c^2} = 0.785 \qquad ◄$$

Similarly, from $\phi = T/GJ$:

$$\frac{\phi_1}{\phi_2} = \frac{J_2}{J_1} = \frac{a^3}{2\pi c^3} = 0.617 \qquad ◄$$

SOLUTION (6.39)



We have

$$s_1 = [0.25^2 + 0.0375^2]^{\frac{1}{2}} = 0.2528\ m$$
$$s_2 = [0.5^2 + 0.05^2]^{\frac{1}{2}} = 0.5025\ m$$
$$s_3 = 0.15\ m, \qquad s_4 = 0.05\ m, \qquad s_5 = 0.075\ m$$

and

$$\tau_1 t_1 = \tau_3 t_3 + \tau_2 t_2 \tag{1}$$
$$\tau_5 t_5 = \tau_1 t_1, \qquad \tau_2 t_2 = \tau_4 t_4 \tag{2,3}$$
$$T = 2A_1 t_1 \tau_1 + 2A_2 t_2 \tau_2 \tag{4}$$

Using Eq. (6.23),

$$\tau_5 s_5 + 2\tau_1 s_1 + \tau_3 s_3 = 2G\theta A_1 \tag{5}$$
$$2\tau_2 s_2 + \tau_4 s_4 - \tau_3 s_3 = 2G\theta A_2 \tag{6}$$

Given: $t_1 = t_2 = t_4 = t_5 = 0.0005\ m$, $t_3 = 0.00075\ m$, $G = 28\ GPa$, and $T = 4\ kN \cdot m$.

The cell areas are calculated as

$$A_1 = 0.028125\ m^2 \qquad A_2 = 0.05\ m^2$$

Substituting the numerical values and solving Eqs. (1) through (6):

$$\tau_1 = \tau_5 = 50.6\ MPa, \qquad \tau_3 = 0.593\ MPa$$
$$\tau_2 = \tau_4 = 50.88\ MPa, \qquad \theta = 0.0194\ rad/m$$

◀

SOLUTION (6.40)

(a) We have $\delta_s = \delta_c$ and Eq. (6.37) becomes

$$\frac{P_s R_s^3}{G_s} = \frac{P_c R_c^3}{G_c}; \qquad \frac{P_s(0.062)^3}{79(10^9)} = \frac{P_c(0.05)^3}{41(10^9)}$$

Solving,

$$P_s = 1.011 P_c$$

Assume that copper controls

$$\tau_c = \frac{16 P_c R_c}{\pi d^3}; \qquad 300(10^6) = \frac{16 P_c(0.05)}{\pi(0.01)^3}$$

from which

$$P_c = 1178.1\ N$$

Then,

$$P_s = 1188.4\ N$$

(CONT.)

Check the assumption:

$$\tau_s = \frac{16(1188.4)(0.062)}{\pi(0.01)^3} = 375.25\ MPa$$

Since $\tau_s < 500\ MPa$, the assumption is correct.

Thus, the total force is

$$P_t = P_c + P_s = 2366.5\ N \quad ◄$$

(b) For the condition specified,

$$\frac{P_s}{k_s} = \frac{P_c}{k_c}$$

from which

$$\frac{k_s}{k_c} = \frac{P_s}{P_c}1.011 \quad ◄$$

End of Chapter 6

CHAPTER 7

SOLUTION (7.1)

Refer to Fig. 7.2 and Eq. (7.10):

$$\frac{\partial^4 w}{\partial x^4} = \frac{1}{h^4}(w_9 - 4w_1 + 6w_0 + w_{11})$$

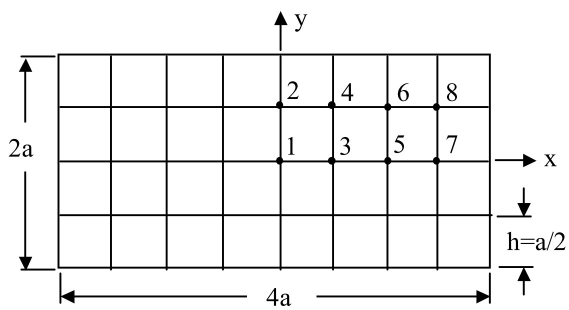
$$\frac{\partial^4 w}{\partial y^4} = \frac{1}{h^4}(w_{10} - 4w_2 + 6w_0 - 4w_4 + w_{12})$$

$$\frac{\partial^4 w}{\partial x^2 \partial y^2} = \frac{1}{h^4}[w_5 - w_6 + w_7 + w_8 - 2(w_1 - w_2 - w_3 - w_4 + 2w_0)]$$

Substituting these into $\nabla^4 w$:

$$h^4 \nabla^4 w = 20w_0 - 8(w_1 + w_2 + w_3 + w_4) - 2(w_5 + w_6 + w_7 + w_8) + w_9 - w_{10} - w_{11} - w_{12}$$

SOLUTION (7.2)



Only a quarter of the section need be considered. Using the symmetry the nodal points are labeled as shown in the preceding figure, and $\Phi = 0$ at the boundary. The finite difference equations for the nodes 1 through 8 are, respectively:

$$-4\Phi_1 + 2\Phi_2 + 2\Phi_3 = -2G\theta h^2 \quad (1)$$

$$\Phi_1 - 4\Phi_2 + 2\Phi_4 = -2G\theta h^2 \quad (2)$$

$$\Phi_1 - 4\Phi_3 + 2\Phi_4 + \Phi_5 = -2G\theta h^2 \quad (3)$$

$$\Phi_2 + \Phi_3 - 4\Phi_4 + \Phi_6 = -2G\theta h^2 \quad (4)$$

$$\Phi_3 - 4\Phi_5 + 2\Phi_6 + \Phi_7 = -2G\theta h^2 \quad (5)$$

$$\Phi_4 + \Phi_5 - 4\Phi_6 + \Phi_8 = -2G\theta h^2 \quad (6)$$

$$\Phi_5 - 4\Phi_7 + 2\Phi_8 = -2G\theta h^2 \quad (7)$$

$$\Phi_6 + \Phi_7 - 4\Phi_8 = -2G\theta h^2 \quad (8)$$

Solving,

$$\Phi_1 = 3.587G\theta h^2 \qquad \Phi_2 = 2.708G\theta h^2$$

$$\Phi_3 = 3.467G\theta h^2 \qquad \Phi_4 = 2.622G\theta h^2$$

$$\Phi_5 = 3.037G\theta h^2 \qquad \Phi_6 = 2.313G\theta h^2$$

$$\Phi_7 = 2.055G\theta h^2 \qquad \Phi_8 = 1.592G\theta h^2$$

(CONT.)

Tables of differences are in P7.2a and P7.2b.

Table P7.2a
(when y=b, $\Phi = \Phi_1$; y=h, $\Phi = \Phi_2$, etc.)

y	$\Phi/G\theta h^2$	$\Delta/G\theta h^2$	$\Delta^2/G\theta h^2$
0	3.587	−0.380	−1.828
h	2.708	−2.708	
$2h$	0		

Table P7.2b
(when x=0, $\Phi = \Phi_1$; x=h, $\Phi = \Phi_3$, etc.)

x	$\Phi/G\theta h^2$	$\Delta/G\theta h^2$	$\Delta^2/G\theta h^2$	$\Delta^3/G\theta h^2$	$\Delta^4/G\theta h^2$
0	3.587	−0.120	−0.310	−0.242	−0.28
h	3.467	−0.430	−0.552	−0.522	
$2h$	3.037	−0.982	−1.074		
$3h$	2.055	−2.055			
$4h$	0				

We have

$$\frac{\partial \Phi}{\partial x} = \frac{\Delta\Phi_0}{h} + \frac{\Delta^2\Phi_0}{2h^2}[2x - h] + \frac{\Delta^3\Phi_0}{6h^3}[3x - 6xh + 2h^2]$$

$$+ \frac{\Delta^4\Phi_0}{24h^4}[4x^3 - 18x^2h + 22xh^2 - 6h^3]$$

$$\left(\frac{\partial \Phi}{\partial x}\right)_{x=4h} = \frac{\Delta\Phi_0}{h} + \frac{\Delta^2\Phi_0}{2h^2}(7h) + \frac{\Delta^3\Phi_0}{6h^3}(26h^2) + \frac{\Delta^4\Phi_0}{24h^4}(50h^3)$$

Using Table P7.2b, we obtain $\left(\frac{\partial \Phi}{\partial x}\right)_{x=4h} = -1.418G\theta a.$ That is

$$(\tau)_{x=\pm 2a} = -1.418G\theta a \quad \blacktriangleleft$$

Similarly, using Table P7.2a:

$$\left(\frac{\partial \Phi}{\partial y}\right)_{y=2h} = \frac{\Delta\Phi_0}{h} + \frac{\Delta^2\Phi_0}{2h^2}(2y - h) = -1.811G\theta a$$

$$(\tau)_{y=\pm 2h} = -1.811G\theta a \quad \blacktriangleleft$$

Hence, the error using this method is $[(1.86 - 1.811)/1.86]100 = 2.63\%$

SOLUTION (7.3)

Equation (7.19) is applied at points b, c, d, respectively:

$$2\Phi_c + 1.178\Phi_g - 4.857\Phi_b = -2G\theta h^2$$

$$\Phi_b + 1.178\Phi_f + \Phi_d - 4.857\Phi_c = -2G\theta h^2$$

$$1.178\Phi_c + 1.178\Phi_e - 5.714\Phi_d = -2G\theta h^2$$

(CONT.)

7.3 (CONT.)

Similarly, Eq. (7.16) is applied at points e, f, and g, respectively:

$$2\Phi_d + \Phi_f - 4\Phi_e = -2G\theta h^2$$
$$2\Phi_c + \Phi_g + \Phi_e - 4\Phi_f = -2G\theta h^2$$
$$2\Phi_b + 2\Phi_f - 4\Phi_g = -2G\theta h^2$$

In matrix form these equations are written as:

$$\begin{bmatrix} 2 & 0 & 0 & 0 & 2 & -4 \\ 0 & 2 & 0 & 1 & -4 & 1 \\ 0 & 0 & 2 & -4 & 1 & 0 \\ 0 & 1.178 & -5.714 & 1.178 & 0 & 0 \\ 1 & -4.857 & 1 & 0 & 1.178 & 0 \\ -4.857 & 2 & 0 & 0 & 0 & 1.178 \end{bmatrix} \begin{Bmatrix} \Phi_b \\ \Phi_c \\ \Phi_d \\ \Phi_e \\ \Phi_f \\ \Phi_g \end{Bmatrix}$$
$$= -2G\theta h^2 \{1, \quad 1, \quad 1, \quad 1, \quad 1, \quad 1\}$$

Solution is

$$\Phi_b = 1.612G\theta h^2 \qquad \Phi_c = 1.487G\theta h^2$$
$$\Phi_d = 0.975G\theta h^2 \qquad \Phi_e = 1.547G\theta h^2$$
$$\Phi_f = 2.236G\theta h^2 \qquad \Phi_g = 2.424G\theta h^2$$

The finite differences are then

$$\Delta = \Phi_e - \Phi_B^o = 1.547G\theta h^2$$
$$\Delta^2 = \Phi_f - 2\Phi_e + \Phi_B^o = -0.858G\theta h^2$$
$$\Delta^3 = \Phi_g - 3\Phi_f + 3\Phi_e - \Phi_B^o = 0.357G\theta h^2$$
$$\Delta^4 = \Phi_f - 4\Phi_g + 6\Phi_f - 4\Phi_B^o + \Phi_B^o = -0.232G\theta h^2$$
$$\Delta^5 = \Phi_e - 5\Phi_f + 10\Phi_g - 10\Phi_f + 5\Phi_e - \Phi_B^o = -0.018G\theta h^2$$
$$\Delta^6 = \Phi_B^o - 6\Phi_e + 15\Phi_f - 20\Phi_g + 15\Phi_f - 6\Phi_e + \Phi_B^o = 0.036G\theta h^2$$

Hence,

$$\tau_B = \left(\tfrac{\partial \Phi}{\partial x}\right)_B = \tfrac{1}{h}\left(\Delta - \tfrac{\Delta^2}{2} + \tfrac{\Delta^3}{3} - \tfrac{\Delta^4}{4} + \tfrac{\Delta^5}{5} - \tfrac{\Delta^6}{6}\right)G\theta h^2$$
$$= \left(1.547 + \tfrac{0.858}{2} + \tfrac{0.357}{3} + \tfrac{0.232}{4} - \tfrac{0.018}{5} - \tfrac{0.036}{6}\right)G\theta h$$

or

$$\tau_B = 2.144G\theta h = 0.0107G\theta \qquad \blacktriangleleft$$

SOLUTION (7.4)

Applying Eq. (7.16) at points b, c, e, f, and g, we obtain

$$\Phi_g + 2\Phi_c - 4\Phi_b = -2G\theta h^2 \qquad \Phi_b + \Phi_d + \Phi_f - 4\Phi_c = -2G\theta h^2$$

$$2\Phi_d + \Phi_f - 4\Phi_e = -2G\theta h^2 \qquad \Phi_g + \Phi_e + 2\Phi_c - 4\Phi_f = -2G\theta h^2$$

$$2\Phi_f + 2\Phi_b - 4\Phi_g = -2G\theta h^2$$

At point d, we apply Eq. (7.19):

$$1.308\Phi_c + \Phi_e - 5.77\Phi_d = -2G\theta h^2$$

Solving these equations, we have

$$\Phi_b = 2.238G\theta h^2 \qquad \Phi_c = 2.000G\theta h^2 \qquad \Phi_d = 1.096G\theta h^2$$

$$\Phi_e = 1.715G\theta h^2 \qquad \Phi_f = 2.667G\theta h^2 \qquad \Phi_g = 2.953G\theta h^2$$

The finite differences are the computed as shown in Table P7.4a.
Thus,

$$\tau_A = (\tfrac{\partial\Phi}{\partial x})_A = \tfrac{1}{h}(\Delta - \tfrac{\Delta^2}{2} + \tfrac{\Delta^3}{3} - \tfrac{\Delta^4}{4})G\theta h^2$$

$$= (2.238 + \tfrac{1.523}{2} + \tfrac{0.093}{3} + \tfrac{0.0186}{4})G\theta h$$

$$= 3.035G\theta h = 0.0129G\theta$$ ◀

Table P7.4a

y	$\Phi/G\theta h^2$	$\Delta/G\theta h^2$	$\Delta^2/G\theta h^2$	$\Delta^3/G\theta h^2$	$\Delta^4/G\theta h^2$
0	0	2.238	−1.523	0.093	−0.019
h	2.238	0.715	−1.430	−0.003	
$2h$	2.953	−0.715	−1.523		
$3h$	2.238	−2.238			
$4h$	0				

Alternatively, we construct the table as shown in Fig. P7.4b:

Table P7.4b

y	$\Phi/G\theta h^2$	$\Delta/G\theta h^2$	$\Delta^2/G\theta h^2$
0	2.953	−0.715	−1.523
h	2.238	−2.238	
$2h$	0		

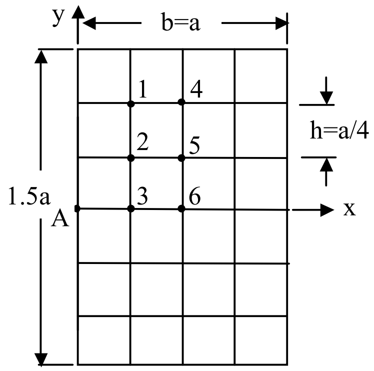
Then we obtain,

$$\tau_A = (\tfrac{\partial\Phi}{\partial y})_{y=2h} = \tfrac{\Delta\Phi_0}{h} + \tfrac{3}{2}\tfrac{\Delta^2\Phi_0}{h} = [-0.715 - \tfrac{3}{2}(1.523)]G\theta h$$

$$= 3.0G\theta h$$ ◀

This result is approximately the same as calculated before. Clearly now computation involved are reduced considerably.

SOLUTION (7.5)



Applying Eq. (7.16) at nodes in to 6, we obtain

$$\begin{bmatrix} -4 & 1 & 0 & 1 & 0 & 0 \\ 1 & -4 & 1 & 0 & 1 & 0 \\ 0 & 2 & -4 & 0 & 0 & 1 \\ 2 & 0 & 0 & -4 & 1 & 0 \\ 0 & 2 & 0 & 1 & -4 & 1 \\ 0 & 0 & 2 & 0 & 2 & -4 \end{bmatrix} \begin{Bmatrix} \Phi_1 \\ \Phi_2 \\ \Phi_3 \\ \Phi_4 \\ \Phi_5 \\ \Phi_6 \end{Bmatrix} = -2G\theta h^2 \begin{Bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{Bmatrix}$$

Solving,

$$\Phi_1 = 1.551G\theta h^2 \qquad \Phi_2 = 2.206G\theta h^2 \qquad \Phi_3 = 2.387G\theta h^2$$
$$\Phi_4 = 1.997G\theta h^2 \qquad \Phi_5 = 2.887G\theta h^2 \qquad \Phi_6 = 3.137G\theta h^2$$

Then, the differences are as shown in Table P7.5.

Table P7.5

x	$\Phi/G\theta h^2$	$\Delta/G\theta h^2$	$\Delta^2/G\theta h^2$	$\Delta^3/G\theta h^2$	$\Delta^4/G\theta h^2$
0	0	2.387	−1.637	0.137	−0.274
h	2.387	0.75	−1.50	−0.137	
$2h$	3.137	−0.75	−1.637		
$3h$	2.387	−2.387			
$4h$	0				

At point A, we have

$$\left(\tfrac{\partial\Phi}{\partial x}\right)_{x=0} = \tfrac{1}{h}\left(\Delta\Phi_0 - \tfrac{\Delta^2\Phi_0}{2} + \tfrac{\Delta^3\Phi_0}{3} - \tfrac{\Delta^4\Phi_0}{4}\right)G\theta h^2$$

Substituting the first row of Table P7.5:

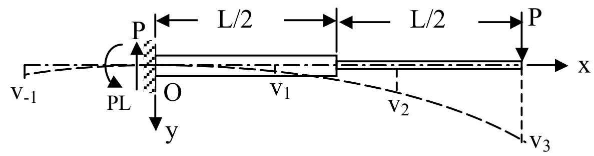
$$\left(\tfrac{\partial\Phi}{\partial x}\right)_A = \left(2.387 + \tfrac{1.637}{2} + \tfrac{0.137}{3} + \tfrac{0.274}{4}\right)G\theta h$$

or

$$\left(\tfrac{\partial\Phi}{\partial x}\right)_A = 0.83G\theta a = \tau_{\max} \qquad \blacktriangleleft$$

This differs 2.12 % from the exact solution $0.848G\theta a$ (Table 6.2)

SOLUTION (7.6)



Boundary conditions yield,

$$v(0) = 0; \quad v_0 = 0 \quad \text{and} \quad v'(0) = 0; \quad v_1 = v_{-1}$$

Apply Eq. (7.22) at nodes 0, 1, and 2, respectively:

$$v_{-1} - 2v_0 + v_0 = \tfrac{h^2(PL)}{2EI}; \qquad v_1 = \tfrac{h^2}{4EI}PL$$

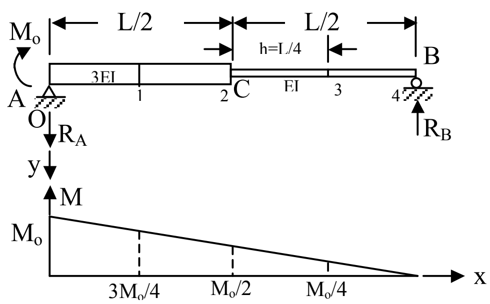
$$v_2 - 2v_1 + v_0 = \tfrac{h^2(2PL/3)}{2EI}; \qquad v_2 = \tfrac{5h^2}{6EI}PL$$

$$v_3 - 2v_2 + v_1 = \tfrac{h^2(PL/3)}{2EI} \tag{a}$$

Substituting v_1 and v_2 into Eq. (a) we obtain

$$v_3 = \tfrac{7}{32}\tfrac{PL^3}{EI}$$ ◀

SOLUTION (7.7)



Let $C = -\tfrac{M_0}{EI}h^2 \qquad v_0 = v_4 = 0$

Applying Eq. (7.22) at 1, 2, 3:

$$v_2 - 2v_1 + \cancel{v_0}^{\,o} = \tfrac{C}{4} \qquad v_3 - 2v_2 + v_1 = \tfrac{C}{4} \qquad \cancel{v_4}^{\,o} - 2v_3 + v_2 = \tfrac{C}{4}$$

Solving,

$$v_1 = -\tfrac{3}{8}C \qquad v_2 = -\tfrac{1}{2}C \qquad v_3 = -\tfrac{3}{8}C$$

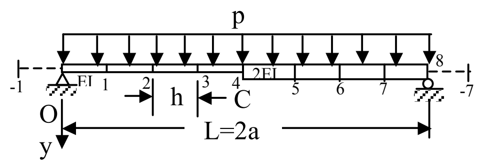
(a) $\theta_C = \theta_3 = \tfrac{1}{2h}(v_3 - v_1) = 0$ ◀

(b) $v_C = v_2 = \tfrac{1}{36}\tfrac{M_0L^2}{EI}$ ◀

Results are the same as the exact solutions.

SOLUTION (7.8)

Let $C = (p/EI)h^4$. We have $h = a/4 = L/8$ and $v_0 = v_8 = 0$.



Apply Eq. (7.23) at 1 through 7:

$$v_3 - 4v_2 + 5v_1 = C \qquad (1)$$

$$v_4 - 4v_3 + 6v_2 - 4v_1 = C \qquad (2)$$

$$v_5 - 4v_4 + 6v_3 - 4v_2 + v_1 = C \qquad (3)$$

$$v_6 - 4v_5 + 6v_4 - 4v_3 + v_2 = \tfrac{C}{1.5} \qquad (4)$$

$$v_7 - 4v_6 + 6v_5 - 4v_4 + v_3 = \tfrac{C}{2} \qquad (5)$$

$$-4v_7 + 6v_6 - 4v_5 + v_4 = \tfrac{C}{2} \qquad (6)$$

$$5v_7 - 4v_6 + v_5 = \tfrac{C}{2} \qquad (7)$$

Solving,

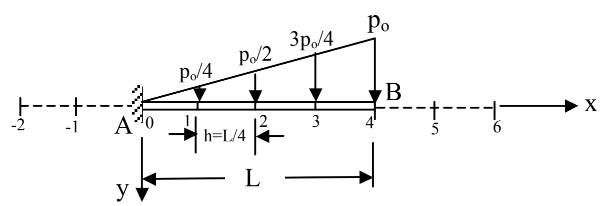
$$v_C = v_4 = 0.00966\tfrac{pL^4}{EI} = 0.1546\tfrac{pa^4}{EI}$$

Note:

$$(v_C)_{exact} = 5\,pa^4/32EI \qquad ◀$$

SOLUTION (7.9)

Let $N = p_0 L^4/EI$



Boundary conditions at A:

$$v'(0) = 0; \qquad v_1 = v_{-1}$$

$$v(0) = 0; \qquad v_0 = 0$$

$$v''''(0) = 0; \qquad v_2 - 8v_1 + v_{-2} = 0$$

Apply Eq. (7.23) at 1 through 4:

$$7v_1 - 4v_2 + v_3 = (0.25)^5\ N$$

$$-4v_1 + 6v_2 - 4v_3 + v_4 = (0.5)^9\ N$$

$$v_1 - 4v_2 + 6v_3 - 4v_4 + v_5 = 3(0.5)^{10}\ N$$

$$v_2 - 4v_3 + 6v_4 - 4v_5 + v_6 = (0.25)^4\ N$$

(CONT.)

Boundary conditions at B:

$$v'''(L) = 0; \qquad -v_2 + 2v_3 - 2v_5 + v_6 = 0$$

$$v''(L) = 0; \qquad v_3 - 2v_4 + v_5 = 0$$

The foregoing six equations are solved to yield

$$v_1 = 0.010742\ N \qquad v_2 = 0.035156\ N$$

$$v_3 = 0.066406\ N \qquad v_4 = 0.099609\ N \qquad \blacktriangleleft$$

$$v_5 = 0.132812\ N \qquad v_2 = 0.167969\ N$$

The error is therefore

$$\tfrac{0.099609-0.091667}{0.091667} \times 100 = 8.7\% .$$

SOLUTION (7.10)

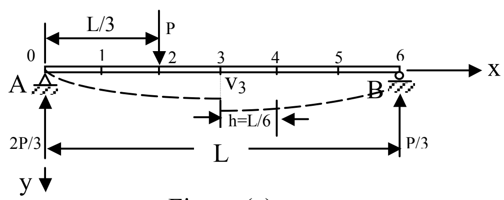


Figure (a)

Application of Eq. (7.22) at points 1 through 5 gives, respectively:

$$v_2 - 2v_1 + v_0 = -\tfrac{PL}{9EI} h^2$$

$$v_3 - 2v_2 + v_1 = -\tfrac{2PL}{9EI} h^2$$

$$v_4 - 2v_3 + v_2 = -\tfrac{PL}{EI} h^2 \qquad \text{(a)}$$

$$v_5 - 2v_4 + v_3 = -\tfrac{PL}{EI} h^2$$

$$v_6 - 2v_5 + v_4 = -\tfrac{PL}{EI} h^2$$

The boundary conditions are $v_0 = v_6 = 0.$ Then Eqs. (a) may be represented in matrix form as

$$\begin{bmatrix} -2 & 1 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 1 & -2 \end{bmatrix} \begin{Bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{Bmatrix} = \begin{Bmatrix} 2 \\ 4 \\ 3 \\ 2 \\ 1 \end{Bmatrix} C$$

where $C = -PL^3/648EI$. Solving the foregoing; $v_1 = -6.67C,\ v_2 = -11.33,$ $v_4 = -9.67C,\ v_5 = -5.33C,$ and $v_3 = -12C,$ or

$$v_3 = 0.01852 \tfrac{PL^3}{EI} \qquad \blacktriangleleft$$

Note: The exact value of the deflection at the center (see Table D.4) is $0.01775PL^3/EI$.

SOLUTION (7.11)

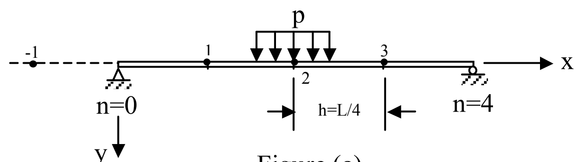


Figure (a)

We observe that because of symmetry only half of the beam span need be considered as shown in Fig. (a). From the boundary conditions $v(0) = v''(0) = 0$ (Fig. 7.6) and symmetry:

$$v_0 = v_4 = 0 \qquad v_1 = -v_{-1} \qquad v_1 = v_3 \tag{a}$$

By Eq. (7.23) at points 1 and 2,

$$\begin{aligned} &v_3 - 4v_2 + 6v_1 - 4v_0 + v_{-1} = 0 \\ &v_4 - 4v_3 + 6v_2 - 4v_1 + v_0 = \tfrac{ph^4}{EI} \end{aligned} \tag{b}$$

Substituting Eqs. (a) into Eqs. (b), we obtain

$$6v_1 - 4v_2 = 0$$

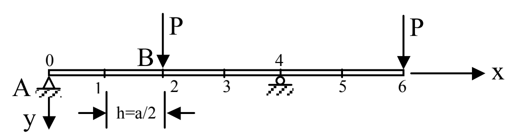
$$-4v_1 + 3v_2 = \tfrac{ph^4}{EI}$$

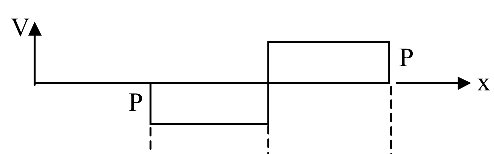
Solving the above and setting $h = L/4$ yields $v_1 = 0.0039\, pL^4/EI$ and

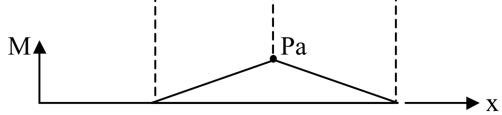
$$v_2 = v_{\max} = 0.0059 \tfrac{pL^4}{EI}$$

◀

SOLUTION (7.12)







Equation (7.22) is applied at nodes 1, 2, 3, respectively:

$$0 - 2v_1 + v_2 = 0; \qquad v_1 = v_2/2$$

$$v_1 - 2v_2 + v_3 = 0; \qquad v_2 = 2v_3/3$$

$$v_2 - 2v_3 + 0 = (h^2/EI)(Pa/2)$$

or $\qquad v_3 = -3Pah^2/8EI$

(CONT.)

Thus,

$$v_B = v_2 = -\tfrac{1}{16}\tfrac{Pa^3}{EI} = -0.0625\tfrac{Pa^3}{EI}$$ ◄

and

$$v_1 = -Pa^3/32EI$$

Slop at A is then

$$\theta_A = \tfrac{v_1 - v_{-1}}{2h} = \tfrac{v_1}{h} = -\tfrac{Pa^2}{16EI}$$ ◄

We have $v_B = 0.08333\,Pa^3/EI$ as exact solution. Error is thus

$$\tfrac{0.08333-0.0625}{0.08333} \times 100 = 25\%$$

SOLUTION (7.13)

Boundary conditions yield

$$v(0) = 0; \qquad v_0 = 0$$
$$v'(0) = 0; \qquad v_1 = v_{-1}$$

and

$$v''(L) = 0 = v_4 - 2v_3 + v_2$$

or $\quad v_4 = 2v_3 - v_2 \qquad (1)$

$$v'''(L) = 0 = v_5 - 2v_4 + 2v_2 - v_1$$

or

$$v_5 = 4v_3 - 4v_2 + v_1 \qquad (2)$$

Apply Eq. (7.23) at points 1, 2, and 3, with $v_0 = 0$ and $v_1 = v_{-1}$:

$$v_3 - 4v_2 + 7v_1 = \tfrac{pL^4}{162EI} \qquad (3)$$
$$v_4 - 4v_3 - 6v_2 - 4v_1 = \tfrac{pL^4}{81EI} \qquad (4)$$
$$v_5 - 4v_4 + 6v_3 - 4v_2 + v_1 = \tfrac{pL^4}{81EI} \qquad (5)$$

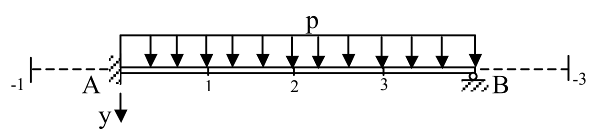
Solving Eqs. (1) to (6),

$$v_3 = 7pL^4/54EI$$ ◄

and $\quad v_1 = 4v_3/21 \qquad v_2 = 4v_3/7$

SOLUTION (7.14)

$$h = L/4 \qquad C = ph^4/EI \qquad v_1 = v_{-1}$$



(CONT.)

7.14 (CONT.)

Apply Eq. (7.22) at 1, 2, 3:

$$v_3 - 4v_2 + 7v_1 = C \qquad -4v_3 + v_2 - 4v_1 = C$$

$$5v_3 - 4v_2 + v_1 = C$$

Solving, $v_1 = 0.90909 \frac{ph^4}{EI} \qquad v_3 = 1.36364 \frac{ph^4}{EI}$

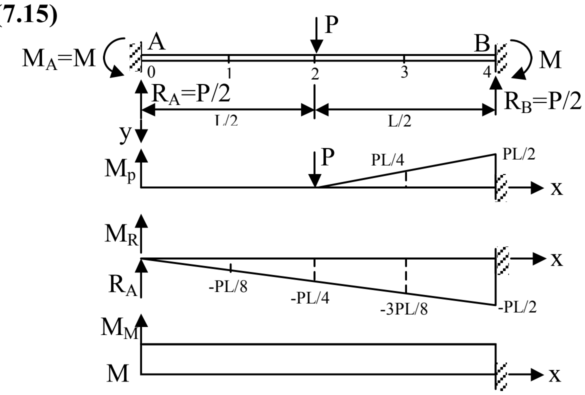
$$v_2 = v_{\max} = 1.68182 \frac{ph^4}{EI}$$

$$= 0.00657 \frac{pL^4}{EI} \downarrow \qquad \blacktriangleleft$$

$$\theta_B = \theta_{\max} = \frac{1}{2h}(v_5 - v_3) = -\frac{v_3}{h}$$

$$= -0.02131 \frac{pL^3}{EI} \qquad \blacktriangleleft$$

SOLUTION (7.15)



$v_1 = v_{-1}, \quad v_1 = v_3$ (due to symmetry), $v_3 = v_5$

$v_0 = v_4 = 0, \quad C = L^3/16EI$.

Apply Eq.(7.22) at 1through 4:

$$v_2 - 2v_1 + \cancel{v_0}^{\,o} = (-\tfrac{PL}{8} + M)C \qquad (1)$$

$$v_3 - 2v_2 + v_1 = (-\tfrac{PL}{4} + M)C \qquad (2)$$

$$\cancel{v_4}^{\,o} - 2v_3 + v_2 = (\tfrac{PL}{4} - \tfrac{3PL}{8} + M)C \qquad (3)$$

$$v_5 - 2\cancel{v_4}^{\,o} + v_3 = (\tfrac{PL}{4} - \tfrac{PL}{2} + M)C \qquad (4)$$

Equations (4), (3), (2) lead to

$$2v_3 = MC$$

$$v_2 = (2M - \tfrac{PL}{8})C$$

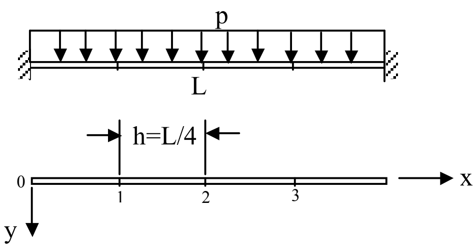
$$v_1 = (\tfrac{9M}{2} - \tfrac{PL}{2})C$$

Then, Eq. (1) gives

$$M = \tfrac{1}{8}PL \qquad \blacktriangleleft$$

Note: $M_{exact} = \frac{1}{8}PL$

SOLUTION (7.16)



Boundary conditions are $v_0 = v_4 = 0$ and from symmetry $v_1 = v_3$.
Then, using Eq. (7.23) at nodes 1 and 2:

$$v_1 + 6v_1 - 4v_2 + v_3 = \tfrac{ph^4}{EI}$$

$$-4v_1 + 6v_2 - 4v_3 = \tfrac{ph^4}{EI}$$

Solving,

$$v_2 = \tfrac{ph^4}{EI} = \tfrac{p}{EI}(\tfrac{L}{4})^4 = 0.00391\tfrac{pL^4}{EI}$$ ◀

The exact solution is

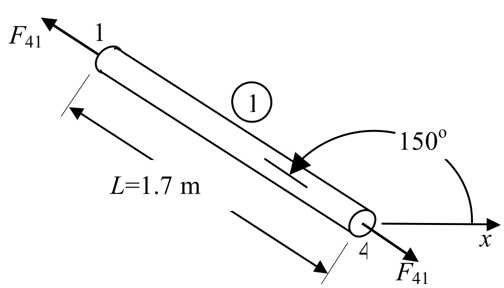
$$v_2 = 0.002604\tfrac{pL^4}{EI}$$

The slope is zero at nodes 0, 2, and 4. Thus,

$$\theta_1 = \theta_3 = \tfrac{v_2 - v_0}{2h} = \tfrac{(0.0039)4}{2L}\tfrac{pL^4}{EI}$$

$$= 0.078\tfrac{pL^3}{EI}$$ ◀

SOLUTION (7.17)



$$\frac{AE}{L} = \frac{13.5(10^{-4})(69\times10^9)}{1.7}$$
$$= 54.79\times10^6 \ N/m$$

$$c = \cos150^o = -\sqrt{3}/2 \qquad s = \sin150^o = 1/2$$

(**a**) Equation (7.38):

$$[k]_e = 54.79\times10^6\begin{bmatrix} 3/4 & -\sqrt{3}/4 & -3/4 & \sqrt{3}/4 \\ -\sqrt{3}/4 & 1/4 & \sqrt{3}/4 & -1/4 \\ -3/4 & -\sqrt{3}/4 & 3/4 & -\sqrt{3}/4 \\ \sqrt{3}/4 & -1/4 & -\sqrt{3}/4 & 1/4 \end{bmatrix}$$

(CONT.)

or

$$[k]_e = 54.79\begin{bmatrix} 0.75 & -0.433 & -0.75 & 0.433 \\ -0.433 & 0.25 & 0.433 & -0.25 \\ -0.75 & -0.433 & 0.75 & -0.433 \\ 0.433 & -0.25 & -0.433 & 0.75 \end{bmatrix} MN/m$$

(**b**) Equation (7.39) with i=4 and j=1:

$$F_{41} = 54.79(10^6)\begin{bmatrix} -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}\begin{Bmatrix} 2+1.1 \\ -1.5+1.2 \end{Bmatrix}(10^{-3})$$

or $$F_1 = F_{41} = -73.14\ kN$$ ◀

(**c**) Then $\{\delta\}_e = [T]\{\delta\}_e$, give

$$\begin{Bmatrix} \bar{u}_4 \\ \bar{v}_4 \\ \bar{u}_1 \\ \bar{v}_1 \end{Bmatrix} = \begin{bmatrix} -0.866 & 0.5 & 0 & 0 \\ -0.5 & -0.866 & 0 & 0 \\ 0 & 0 & -0.866 & 0.5 \\ 0 & 0 & -0.5 & -0.866 \end{bmatrix}\begin{Bmatrix} -1.1 \\ -1.2 \\ 2 \\ 1.5 \end{Bmatrix} = \begin{Bmatrix} 0.35 \\ 1.59 \\ -0.98 \\ -0.3 \end{Bmatrix} mm$$ ◀

SOLUTION (7.18)

We have

$$(\frac{AE}{L})_1 = \frac{AE}{L} \qquad (\frac{AE}{L})_2 = \frac{2AE}{L} \qquad (\frac{AE}{L})_3 = \frac{3AE}{L}$$

Equation (7.25a):

$$[k]_1 = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \qquad [k]_2 = \frac{AE}{L}\begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} \qquad [k]_3 = \frac{AE}{L}\begin{bmatrix} 3 & -3 \\ -3 & 3 \end{bmatrix}$$

(**a**) System stiffness matrix, $[k] = [k]_1 + [k]_2 + [k]_3$ is order of 4×4. Thus

$$[K] = \frac{AE}{L}\begin{array}{c} \begin{matrix} u_1 & u_2 & u_3 & u_4 \end{matrix} \\ \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 3 & -2 & 0 \\ 0 & -2 & 5 & -3 \\ 0 & 0 & -3 & 3 \end{bmatrix} \end{array}$$ ◀

(CONT.)

(**b**)

$$\begin{Bmatrix} F_{1x} \\ F_{2x} \\ F_{3x} \\ F_{4x} \end{Bmatrix} = \frac{AE}{L} \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 3 & -2 & 0 \\ 0 & -2 & 5 & -3 \\ 0 & 0 & -3 & 3 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} \tag{1}$$

Boundary conditions are $u_1 = u_4 = 0$. We have $F_{3x} = -P$. Thus

$$\begin{Bmatrix} 0 \\ -P \end{Bmatrix} = \frac{AE}{L} \begin{bmatrix} 3 & -2 \\ -2 & 5 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix}$$

Solving

$$u_2 = -\frac{2PL}{11AE} \qquad u_3 = -\frac{3PL}{11AE}$$ ◀

(**c**) Equations (1) :

$$\begin{Bmatrix} F_{1x} \\ F_{2x} \\ F_{3x} \\ F_{4x} \end{Bmatrix} = \frac{AE}{L} \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 3 & -2 & 0 \\ 0 & -2 & 5 & -3 \\ 0 & 0 & -3 & 3 \end{bmatrix} \begin{Bmatrix} 0 \\ -2P/11 \\ -3P/11 \\ 0 \end{Bmatrix} \frac{L}{AE} = \begin{Bmatrix} 2P/11 \\ 0 \\ -P \\ 9P/11 \end{Bmatrix}$$ ◀

The reactions are

$$R_1 = \frac{2}{11}P \rightarrow \qquad R_4 = \frac{9}{11}P \rightarrow$$ ◀

SOLUTION (7.19)

We have

$$(\frac{AE}{L})_{1,2} = \frac{A(2E)}{L/3} = \frac{6AE}{L}$$

$$(\frac{AE}{L})_3 = \frac{2A(E)}{L/3} = \frac{6AE}{L}$$

There are four displacement components (u_1, u_2, u_3, u_4) and so the order of the system matrix is 4x4. Using Eq. (7.25a):

$$[k]_1 = [k]_2 = [k]_3 = \frac{6AE}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

(CONT.)

(**a**)

$$\begin{array}{cc} & \begin{array}{cccc} u_1 & u_2 & u_3 & u_4 \end{array} \\ [K]=\dfrac{6AE}{L} & \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & (1+1) & -1 & 0 \\ 0 & -1 & (1+1) & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \end{array} \begin{array}{cc} & \begin{array}{cccc} u_1 & u_2 & u_3 & u_4 \end{array} \\ =\dfrac{6AE}{L} & \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \end{array} \quad ◀$$

(**b**)

$$\begin{Bmatrix} F_{1x} \\ F_{2x} \\ F_{3x} \\ F_{4x} \end{Bmatrix} = \frac{AE}{L}\begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} \qquad (1) \quad ◀$$

Boundary conditions are $u_1 = u_4 = 0$ and $F_{x3} = P$. Equation (1) is then

$$\begin{Bmatrix} 0 \\ P \end{Bmatrix} = \frac{AE}{L}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix}$$

Solving

$$u_2 = \frac{PL}{9AE} \qquad u_3 = \frac{PL}{18AE} \quad ◀$$

(**c**) Equations (1) result in

$$\begin{Bmatrix} F_{1x} \\ F_{2x} \\ F_{3x} \\ F_{4x} \end{Bmatrix} = \frac{6AE}{L}\begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}\begin{Bmatrix} 0 \\ PL/9AE \\ PL/18AE \\ 0 \end{Bmatrix}\frac{L}{AE} = \begin{Bmatrix} -2P/3 \\ P \\ 0 \\ -P/3 \end{Bmatrix} \quad ◀$$

The reactions are

$$R_1 = \frac{2}{3}P \leftarrow \qquad R_4 = \frac{1}{3}P \leftarrow \quad ◀$$

SOLUTION (7.20)

$$\left(\frac{AE}{L}\right)_1 = \frac{AE}{L} \qquad \left(\frac{AE}{L}\right)_2 = \frac{AE}{L} \qquad \left(\frac{AE}{L}\right)_3 = \frac{4AE}{5L} = 0.8\frac{AE}{L}$$

Equation (7.25a):

$$[k]_1 = [k]_2 = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \qquad [k]_3 = \frac{AE}{L}\begin{bmatrix} 0.8 & -0.8 \\ -0.8 & 0.8 \end{bmatrix}$$

(CONT.)

(**a**) System matrix, $[k]=[k]_1+[k]_2+[k]_3$, is then

$$[K]=\frac{AE}{L}\begin{bmatrix}1 & -1 & 0 & 0\\ -1 & (1+1) & -1 & 0\\ 0 & -1 & (1+0.8) & -0.8\\ 0 & 0 & -0.8 & 0.8\end{bmatrix}$$ ◄

(**b**)

$$\begin{Bmatrix}F_{1x}\\F_{2x}\\F_{3x}\\F_{4x}\end{Bmatrix}=\frac{AE}{L}\begin{array}{c}\begin{array}{cccc}u_1 & u_2 & u_3 & u_4\end{array}\\ \begin{bmatrix}1 & -1 & 0 & 0\\ -1 & 2 & -1 & 0\\ 0 & -1 & 1.8 & -0.8\\ 0 & 0 & -0.8 & 0.8\end{bmatrix}\end{array}\begin{Bmatrix}u_1\\u_2\\u_3\\u_4\end{Bmatrix} \qquad (1)$$

Boundary conditions are $u_1=u_4=0$, We have $F_{x3}=P$. Thus

$$\begin{Bmatrix}0\\P\end{Bmatrix}=\frac{AE}{L}\begin{bmatrix}2 & -1\\ -1 & 1.8\end{bmatrix}\begin{Bmatrix}u_2\\u_3\end{Bmatrix}$$

Solving

$$u_2=-\frac{PL}{2.6AE} \qquad u_3=\frac{PL}{1.3AE}$$ ◄

(c) Equations (1) yield

$$\begin{Bmatrix}F_{1x}\\F_{2x}\\F_{3x}\\F_{4x}\end{Bmatrix}=\frac{AE}{L}\begin{bmatrix}1 & -1 & 0 & 0\\ -1 & 2 & -1 & 0\\ 0 & -1 & 1.8 & -0.8\\ 0 & 0 & -0.8 & 0.8\end{bmatrix}\begin{Bmatrix}0\\P/2.6\\P/1.3\\0\end{Bmatrix}\frac{L}{AE}=\begin{Bmatrix}-P/2.6\\0\\P\\-1.6P/2.6\end{Bmatrix}$$ ◄

The reactions are

$$R_1=\frac{1}{2.6}P\leftarrow \qquad R_4=\frac{0.8}{1.3}P\leftarrow$$ ◄

SOLUTION (7.21)

Table P7.21 Data for the truss of Fig P7.21

Element	*Length*(m)	θ	c	s	c^2	cs	s^2
1	7.5	36.9^o	0.8	0.6	0.639	0.48	0.36
2	6	0^o	1	0	1	0	0
3	4.5	90^o	0	1	0	0	1
4	4.5	0^o	1	0	1	0	0
5	$4.5\sqrt{2}$	135^o	−0.707	0.707	0.5	−0.5	0.5

(CONT.)

7.21 (CONT.)

$$
\begin{array}{cc}
 & \begin{array}{cccc} u_1 & v_1 & u_2 & v_2 \end{array} \\
[k]_1 = \dfrac{AE}{7.5} & \begin{bmatrix} 0.639 & 0.48 & -0.639 & -0.48 \\ 0.48 & 0.361 & -0.48 & -0.361 \\ -0.639 & -0.48 & 0.639 & 0.48 \\ -0.48 & -0.361 & 0.48 & 0.361 \end{bmatrix}
\end{array}
$$ ◄

$$
\begin{array}{cc}
 & \begin{array}{cccc} u_1 & v_1 & u_3 & v_3 \end{array} \\
[k]_2 = \dfrac{AE}{6.0} & \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}
\end{array}
$$ ◄

$$
\begin{array}{cc}
 & \begin{array}{cccc} u_2 & v_2 & u_3 & v_3 \end{array} \\
[k]_3 = \dfrac{AE}{4.5} & \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}
\end{array}
$$ ◄

$$
\begin{array}{cc}
 & \begin{array}{cccc} u_3 & v_3 & u_4 & v_4 \end{array} \\
[k]_4 = \dfrac{AE}{4.5} & \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}
\end{array}
$$ ◄

$$
\begin{array}{cc}
 & \begin{array}{cccc} u_2 & v_2 & u_4 & v_4 \end{array} \\
[k]_5 = \dfrac{AE}{4.5\sqrt{2}} & \begin{bmatrix} 0.5 & -0.5 & -0.5 & 0.5 \\ -0.5 & 0.5 & 0.5 & -0.5 \\ -0.5 & 0.5 & 0.5 & -0.5 \\ 0.5 & -0.5 & -0.5 & 0.5 \end{bmatrix}
\end{array}
$$ ◄

SOLUTION (7.22)

Table P7.22 Data for the truss of Fig.P7.22

Element	*Length* (m)	θ	c	s	c^2	cs	s^2
1	2.4	0^o	1	0	1	0	0
2	2.6	22.62^o	0.923	0.385	0.852	0.355	0.148
3	1.0	-90^o	0	1	0	0	1
4	2.4	0^o	1	0	1	0	0
5	2.6	22.62^o	0.923	0.385	0.852	0.355	0.148

(CONT.)

(**a**) Use Eq.(7.38);

$$[k]_1 = \frac{AE}{2.4}\begin{array}{c} \begin{array}{cccc} u_A & v_A & u_C & v_C \end{array} \\ \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

$$[k]_2 = \frac{AE}{2.6}\begin{array}{c} \begin{array}{cccc} u_A & v_A & u_B & v_B \end{array} \\ \begin{bmatrix} 0.852 & 0.355 & -0.852 & -0.355 \\ 0.355 & 0.148 & -0.355 & -0.148 \\ -0.852 & -0.355 & 0.852 & 0.355 \\ -0.355 & -0.148 & 0.355 & 0.148 \end{bmatrix} \end{array} \blacktriangleleft$$

$$[k]_3 = \frac{AE}{1.0}\begin{array}{c} \begin{array}{cccc} u_B & v_B & u_C & v_C \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \end{array} \blacktriangleleft$$

$$[k]_4 = \frac{AE}{2.4}\begin{array}{c} \begin{array}{cccc} u_B & v_B & u_D & v_D \end{array} \\ \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \blacktriangleleft$$

$$[k]_5 = \frac{AE}{2.6}\begin{array}{c} \begin{array}{cccc} u_C & v_C & u_D & v_D \end{array} \\ \begin{bmatrix} 0.852 & 0.355 & -0.852 & -0.355 \\ 0.355 & 0.148 & -0.355 & -0.148 \\ -0.852 & -0.355 & 0.852 & 0.355 \\ -0.355 & -0.148 & 0.355 & 0.148 \end{bmatrix} \end{array} \blacktriangleleft$$

(CONT.)

7.22 (CONT.)

(**b**) Global Stiffness Matrix $[K] =$

$$
\begin{array}{c}
\begin{array}{cccccccc} u_A & v_A & u_B & v_B & u_C & v_C & u_D & v_D \end{array} \\
AE\begin{bmatrix}
0.745 & 0.137 & -0.328 & -0.137 & -0.417 & 0 & 0 & 0 \\
0.137 & -0.057 & -0.137 & -0.057 & 0 & 0 & 0 & 0 \\
-0.328 & -0.137 & 0.745 & 0.137 & 0 & 0 & -0.833 & 0 \\
-0.137 & -0.057 & 0.137 & 1.057 & 0 & -2 & 0 & 0 \\
-0.417 & 0 & 0 & 0 & 0.745 & 0.137 & -0.328 & -0.137 \\
0 & 0 & 0 & -2 & 0.137 & 1.057 & -0.137 & -0.057 \\
0 & 0 & -0.833 & 0 & -0.655 & -0.273 & 0.745 & 0.137 \\
0 & 0 & 0 & 0 & -0.137 & -0.057 & 0.137 & 0.057
\end{bmatrix}
\end{array}
\blacktriangleleft
$$

$$
\{F\} = [K]\{\delta\}; \qquad \begin{Bmatrix} R_{Ax} \\ R_{Ay} \\ 0 \\ 0 \\ 0 \\ 0 \\ P \\ R_{Dy} \end{Bmatrix} = [K]\begin{Bmatrix} 0 \\ 0 \\ u_B \\ v_B \\ u_C \\ v_C \\ u_D \\ 0 \end{Bmatrix} \qquad \blacktriangleleft
$$

SOLUTION (7.23)

(**a**) Element 1 $\theta = 135^o$ $c^2 = 0.5$ $cs = -0.5$ $s^2 = 0.5$

Equation (7.38):

$$
\begin{array}{c}
\begin{array}{cccc} u_1 & v_1 & u_2 & v_2 \end{array} \\
[k]_1 = \dfrac{AE}{L}\begin{bmatrix}
0.5 & -0.5 & -0.5 & 0.5 \\
-0.5 & 0.5 & 0.5 & -0.5 \\
-0.5 & 0.5 & 0.5 & -0.5 \\
0.5 & -0.5 & -0.5 & 0.5
\end{bmatrix}
\end{array}
\blacktriangleleft
$$

Element 2 $\theta = 180^o$ $c^2 = 1.0$ $cs = 0$ $s^2 = 0$

$$
\begin{array}{c}
\begin{array}{cccc} u_1 & v_1 & u_3 & v_3 \end{array} \\
[k]_2 = \dfrac{AE}{L}\begin{bmatrix}
1 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 \\
-1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\end{array}
\blacktriangleleft
$$

(CONT.)

Element 3 $\theta = 270^o$ $c^2 = 0$ $cs = 0$ $s^2 = 1.0$

$$[k]_3 = k' \begin{array}{c} \begin{array}{cccc} u_1 & v_1 & u_4 & v_4 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1.0 & 0 & -1.0 \\ 0 & 0 & 0 & 0 \\ 0 & -1.0 & 0 & 1.0 \end{bmatrix} \end{array} \quad \blacktriangleleft$$

where $k' = kL/AE$

(**b**) System matrix is 8x8. Superposition. $[K] = \sum [k]$ results in

$$[K] = \frac{AE}{L} \begin{array}{c} \begin{array}{cccccccc} u_1 & v_1 & u_2 & v_2 & u_3 & v_3 & u_4 & v_4 \end{array} \\ \begin{bmatrix} 1.5 & -0.5 & -0.5 & 0.5 & -1 & 0 & 0 & 0 \\ -0.5 & 0.5 & 0.5 & -0.5 & 0 & 0 & 0 & -k' \\ -0.5 & 0.5 & 0.5 & -0.5 & 0 & 0 & 0 & 0 \\ 0.5 & -0.5 & -0.5 & 0.5 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -k' & 0 & 0 & 0 & 0 & 0 & k' \end{bmatrix} \end{array} \quad \blacktriangleleft$$

where $k' = kL/AE$.

(**c**) Boundary conditions are

$$u_2 = v_2 = u_3 = v_3 = u_4 = v_4 = 0$$

Therefore

$$\begin{Bmatrix} 0 \\ -P \\ F_{2x} \\ F_{2y} \\ F_{3x} \\ F_{3y} \\ F_{4x} \\ F_{4y} \end{Bmatrix} = [K] \begin{Bmatrix} u_1 \\ u_2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \quad \blacktriangleleft$$

The total 7×7 stiffness matrix can be reduced to a 7×4 matrix by using the following antisymmetrical conditions:

$$u_1 = -u_5, \quad u_2 = -u_4, \text{ and } v_2 = v_1$$

SOLUTION (7.24)

Table P7.24 Data for the truss of Fig.P7.24

Element	*Length*(m)	θ	c	s	c^2	cs	s^2
1	5	53.13^o	0.6	0.8	0.36	0.48	0.64
2	4	180^o	−1	0	1	0	0

(**a**) Use Eq.(7.38):

$$[k]_1 = \frac{AE}{5}\begin{array}{c} \begin{array}{cccc} u_1 & v_1 & u_2 & v_2 \end{array} \\ \begin{bmatrix} 0.36 & 0.48 & -0.36 & -0.48 \\ 0.48 & 0.64 & -0.48 & -0.64 \\ -0.36 & -0.48 & 0.36 & 0.48 \\ -0.48 & -0.64 & 0.48 & 0.64 \end{bmatrix} \end{array} \quad ◄$$

$$[k]_2 = \frac{AE}{4}\begin{array}{c} \begin{array}{cccc} u_2 & v_2 & u_3 & v_3 \end{array} \\ \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

(**b**) Global Stiffness Matrix

$$[K] = AE\begin{array}{c} \begin{array}{cccccc} u_1 & v_1 & u_2 & v_2 & u_3 & v_3 \end{array} \\ \begin{bmatrix} 0.072 & 0.096 & -0.072 & -0.096 & 0 & 0 \\ 0.096 & 0.128 & -0.096 & -0.128 & 0 & 0 \\ -0.072 & -0.096 & 0.322 & 0.096 & -0.25 & 0 \\ -0.096 & -0.128 & 0.096 & 0.128 & 0 & 0 \\ 0 & 0 & -0.25 & 0 & 0.25 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

(**c**)

$$\begin{Bmatrix} 0 \\ -6000 \end{Bmatrix} = AE\begin{bmatrix} 0.322 & 0.096 \\ 0.093 & 0.128 \end{bmatrix}\begin{Bmatrix} u_2 \\ v_2 \end{Bmatrix}$$

$$\begin{Bmatrix} u_2 \\ v_2 \end{Bmatrix} = \frac{1}{10(10^6)}\begin{bmatrix} 4 & -3 \\ -3 & 10.063 \end{bmatrix}\begin{Bmatrix} 0 \\ -6000 \end{Bmatrix} = \begin{Bmatrix} 0.9 \\ -3.02 \end{Bmatrix}(10^{-3}) \quad m \quad ◄$$

(**d**)

$$\begin{Bmatrix} R_{1x} \\ R_{1y} \\ R_{3x} \end{Bmatrix} = AE\begin{bmatrix} -0.072 & -0.096 \\ -0.096 & -0.128 \\ -0.25 & 0 \end{bmatrix}\begin{Bmatrix} u_2 \\ v_2 \end{Bmatrix} = \begin{Bmatrix} 4.5024 \\ 6.0032 \\ -4.5 \end{Bmatrix} \quad kN$$

(CONT.)

(**e**) Use Eq.(7.39)

$$F_{12} = AE\begin{bmatrix}0.6 & 0.8\end{bmatrix}\begin{Bmatrix}u_2\\ v_2\end{Bmatrix} = 20(10^{6})\begin{bmatrix}0.6 & 0.8\end{bmatrix}\begin{Bmatrix}0.0018\\ -0.00604\end{Bmatrix} = -75\ kN \quad (C)$$

◄

$$F_{23} = AE\begin{bmatrix}-1 & 0\end{bmatrix}\begin{Bmatrix}-u_2\\ -v_2\end{Bmatrix} = 20(10^{6})\begin{bmatrix}-1 & 0\end{bmatrix}\begin{Bmatrix}-0.0018\\ 0.00604\end{Bmatrix} = 36\ kN \quad (T)$$

SOLUTION (7.25)

We have $E = 105\ GPa \quad A = 10\times10^{-4}\ m^2$

Table P7.25 Data for the truss of Fig.P7.25

Element	*Length*(*m*)	θ	c	s	c^2	cs	s^2
1	5	53.13^o	0.6	0.8	0.36	0.48	0.64
2	4	90^o	0	1	0	0	1

(**a**) Apply Eq.(7.38):

$$[k]_1 = \frac{AE}{5}\begin{array}{c} \begin{array}{cccc} u_1 & v_1 & u_2 & v_2 \end{array} \\ \begin{bmatrix} 0.36 & 0.48 & -0.36 & -0.48\\ 0.48 & 0.64 & -0.48 & -0.64\\ -0.36 & -0.48 & 0.36 & 0.48\\ -0.48 & -0.64 & 0.48 & 0.64 \end{bmatrix}\end{array}$$ ◄

$$[k]_2 = \frac{AE}{4}\begin{array}{c} \begin{array}{cccc} u_1 & v_1 & u_3 & v_3 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0\\ 0 & 1 & 0 & -1\\ 0 & 0 & 0 & 0\\ 0 & -1 & 0 & 1 \end{bmatrix}\end{array}$$ ◄

(**b**) Global Stiffness Matrix

$$[K] = 10^7\begin{array}{c} \begin{array}{cccccc} u_1 & v_1 & u_2 & v_2 & u_3 & v_3 \end{array} \\ \begin{bmatrix} 0.756 & 1.008 & -0.756 & -1.008 & 0 & 0\\ 1.008 & 3.969 & -1.008 & -1.344 & 0 & -2.625\\ -0.756 & -1.008 & 0.756 & 1.008 & 0 & 0\\ -1.008 & -1.344 & 1.008 & 1.344 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & -2.625 & 0 & 0 & 0 & 2.625 \end{bmatrix}\end{array}$$

(CONT.)

7.25 (CONT.)

(**c**)

$$\begin{Bmatrix} F_{1x} \\ 10,000 \end{Bmatrix} = 10^7 \begin{bmatrix} 0.756 & 1.008 \\ 1.008 & 3.969 \end{bmatrix} \begin{Bmatrix} -0.015 \\ v_1 \end{Bmatrix}$$

$$10\times 10^3 = (1.008\times 10^7)(-0.015) + 3.696\times 10^7 v_1 \qquad \text{or} \qquad v_1 = 0.0044\ m$$

$$F_{1x} = (0.756\times 10^7)(-0.015) + 1.008\times 10^7 v_1 \qquad \text{or} \qquad F_{1x} = -69.4\ kN$$ ◄

(**d**) Support reactions:

$$\begin{Bmatrix} F_{2x} \\ F_{2y} \\ F_{3x} \\ F_{3y} \end{Bmatrix} = 10^7 \begin{bmatrix} -0.756 & -1.008 \\ -1.008 & -1.344 \\ 0 & 0 \\ 0 & -2.625 \end{bmatrix} \begin{Bmatrix} -0.015 \\ 0.0044 \end{Bmatrix} = \begin{Bmatrix} 69 \\ 92.1 \\ 0 \\ -115.5 \end{Bmatrix} kN$$ ◄

SOLUTION (7.26)

Equation (7.38):

$$[k] = [K] = \frac{EI}{L^3} \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}$$ ◄

(**a**) Boundary conditions are $v_2 = \theta_2 = 0$. Equation (7.45a) becomes

$$\begin{Bmatrix} -P \\ 0 \end{Bmatrix} = \frac{EI}{L^2} \begin{bmatrix} 12 & 6L \\ 6L & 4L^2 \end{bmatrix} \begin{Bmatrix} v_1 \\ \theta_1 \end{Bmatrix}$$

Solving, $\theta_1 = \dfrac{PL^2}{2EI} \qquad v_1 = -\dfrac{PL^3}{3EI}$

(**b**) Equation (7.45a):

$$\begin{Bmatrix} F_{1y} \\ M_1 \\ F_{2y} \\ M_2 \end{Bmatrix} = \frac{EI}{L^3} \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \begin{Bmatrix} -PL^3/3EI \\ PL^2/2EI \\ 0 \\ 0 \end{Bmatrix}$$

Solving

$$\begin{Bmatrix} F_{1y} \\ M_1 \\ F_{2y} \\ M_2 \end{Bmatrix} = \begin{Bmatrix} -P \\ 0 \\ P \\ -PL \end{Bmatrix}$$ ◄

where $F_{2y} = R_2 = P$. ◄

SOLUTION (7.27)

Due to the symmetry only one-half of the beam need be considered. Referring to Case 3 of Table D.5, equivalent nodal forces obtained (see Fig. a).

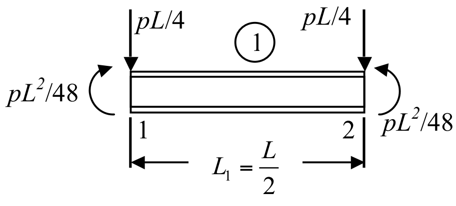


Figure (a) Equivalent nodal forces

Boundary conditions are $v_1 = \theta_2 = 0$

We have $M_1 = -pL^2/48$ and

$$F_{2y} = -pL/4$$

Equation (7.45a), with element of length $L_1 = L/2$.

$$\begin{Bmatrix} pL^2/48 \\ -pL/4 \end{Bmatrix} = \frac{EI}{L_1^3}\begin{bmatrix} 4L_1^2 & -6L_1 \\ -6L_1 & 12 \end{bmatrix}\begin{Bmatrix} \theta_1 \\ v_2 \end{Bmatrix} = \frac{8EI}{L^3}\begin{bmatrix} L_1^2 & -3L \\ -3L_1 & 12 \end{bmatrix}\begin{Bmatrix} \theta_1 \\ v_2 \end{Bmatrix}$$

Inverting

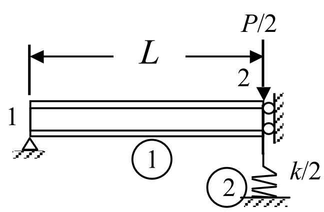
$$\begin{Bmatrix} \theta_1 \\ v_2 \end{Bmatrix} = \frac{L^3}{24EI}\begin{bmatrix} 12/L^2 & 3/L \\ 3/L & 1 \end{bmatrix}\begin{Bmatrix} pL^2/48 \\ -pL/4 \end{Bmatrix}$$

or
$$\begin{Bmatrix} \theta_1 \\ v_2 \end{Bmatrix} = \begin{Bmatrix} -pL^3/24EI \\ -5pL^4/384EI \end{Bmatrix} \quad \blacktriangleleft$$

This is the exact solution (See Table D.4)

SOLUTION (7.28)

Due to symmetry, only one-half of the beam need be considered.



$$[k]_1 = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}$$

$$[k]_2 = \frac{EI}{L^3}\begin{bmatrix} 0 & 0 & 0 & 0 \\ kL^3/EI & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Therefore

$$[K] = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 + \dfrac{kL^3}{EI} & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}$$

(CONT.)

(**a**) Boundary conditions are $v_1 = 0$ and $\theta_2 = 0$. Equation (7.45a) with $F_{2y} = -P/2$ and $M_1 = 0$:

$$\begin{Bmatrix} 0 \\ -P/2 \end{Bmatrix} = \frac{EI}{L^3}\begin{bmatrix} 4L^2 & -6L \\ -6L & 12+\dfrac{kL^3}{EI} \end{bmatrix}\begin{Bmatrix} \theta_1 \\ v_2 \end{Bmatrix}$$

Introduce the data and solve:

$$v_2 = -7.9338\ mm \qquad \theta_1 = -2.9752\ rad$$ ◄

(**b**)

$$\begin{Bmatrix} F_{1y} \\ 0 \\ F_{2y} \\ M_2 \end{Bmatrix} = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12+\dfrac{kL^3}{EI} & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}\begin{Bmatrix} 0 \\ -2.9752 \\ -7.9338 \\ 0 \end{Bmatrix}(10^{-3})$$

or

$$\begin{Bmatrix} F_{1y} \\ F_{2y} \\ M_2 \end{Bmatrix} = \begin{Bmatrix} 5.20\ kN \\ -3.80\ kN \\ -20.8\ kN\cdot m \end{Bmatrix}$$

$$F_{spring} = 200(7.9338) = 1.587\ kN \quad (C)$$ ◄

From symmetry: $F_{1y} = F_{3y} = 5.20\ kN \uparrow$

SOLUTION (7.29)

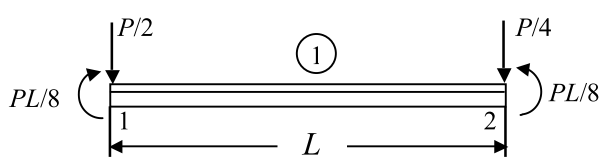


Figure (a). Equivalent Nodal forces.

Boundary conditions are

$$v_1 = 0 \qquad \theta_1 = 0$$

We have

$$M_2 = -M_1 = \frac{PL}{8}$$

$$F_{1y} = F_{2y} = -\frac{P}{2}$$

Equation (7.45a) reduces to

$$\begin{Bmatrix} -P/2 \\ PL/8 \end{Bmatrix} = \frac{EI}{L^3}\begin{bmatrix} 12 & -6L \\ -6L & 4L^2 \end{bmatrix}\begin{Bmatrix} v_1 \\ \theta_2 \end{Bmatrix}$$

Inverting

$$\begin{Bmatrix} v_2 \\ \theta_2 \end{Bmatrix} = \frac{L}{6EI}\begin{bmatrix} L^2 & 3L \\ 3L & 6 \end{bmatrix}\begin{Bmatrix} -P/2 \\ PL/8 \end{Bmatrix} = \begin{Bmatrix} -5PL^3/48EI \\ -PL^2/8EI \end{Bmatrix}$$ ◄

SOLUTION (7.30)

We have $(\frac{EI}{L})_1 = \frac{EI}{L^3}$ and $(\frac{EI}{L})_2 = \frac{EI/4}{L^3} = \frac{EI}{4L^3}$

So, the stiffness materices $[k]_1 + [k]_2$ and $[K]$ are the same as obtained in Example 7.10.

Boundary conditions are $v_1 = 0,\ \theta_1 = 0,$ and $v_3 = 0.$ Equation (f) of Example 7.10 becomes

$$\begin{Bmatrix} v_2 \\ \theta_2 \\ \theta_3 \end{Bmatrix} = \frac{L^2}{276EI}\begin{bmatrix} 28L & 18 & -30 \\ 18 & 51/L & -39/L \\ -30 & -39/L & 111/L \end{bmatrix}\begin{Bmatrix} -P \\ 3PL \\ 0 \end{Bmatrix}$$

$$= \frac{PL^3}{276EI}\begin{Bmatrix} 8 \\ 84/L \\ -48/L \end{Bmatrix} = \frac{PL^3}{69EI}\begin{Bmatrix} -2 \\ 21/L \\ -12/L \end{Bmatrix}$$

Substituting the given data:

$$\begin{Bmatrix} v_2 \\ \theta_2 \\ \theta_3 \end{Bmatrix} = \frac{(30\times10^3)(1.2)^3}{69(207\times10^3)(15)}\begin{Bmatrix} -8 \\ 168 \\ -96 \end{Bmatrix} = \begin{Bmatrix} -1.936\ mm \\ 0.041\ rad \\ -0.023\ rad \end{Bmatrix}$$ ◄

SOLUTION (7.31)

(**a**) The three element stiffness are identical. Equation (7.46):

$$[k]_1 = [k]_2 = [k]_3 = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}$$ ◄

(**b**) The beam stiffness matrix, $[K] = [k]_1 + [k]_2 + [k]_3$, is assembled as

$$[K] = \frac{EI}{L^3}\begin{array}{c} \begin{matrix} v_1 & \theta_1 & v_2 & \theta_2 & v_3 & \theta_3 & v_4 & \theta_4 \end{matrix} \\ \begin{bmatrix} 12 & 6L & -12 & 6L & 0 & 0 & 0 & 0 \\ 6L & 4L^2 & -6L & 2L^2 & 0 & 0 & 0 & 0 \\ -12 & -6L & 24 & 0 & -12L & 6L & 0 & 0 \\ 6L & 2L^2 & 0 & 8L^2 & -6L & 2L^2 & 0 & 0 \\ 0 & 0 & -12L & -6L & 24 & 0 & -12 & 6L \\ 0 & 0 & 6L & 2L^2 & 0 & 8L^2 & -6L & 2L^2 \\ 0 & 0 & 0 & 0 & -12 & -6L & 12 & -6L \\ 0 & 0 & 0 & 0 & 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \end{array}$$ ◄

(CONT.)

7.31 (CONT.)

Boundary conditions are :

$$v_2 = v_3 = v_4 = \theta_4 = 0$$

System governing relations, from . (7.40):

$$\begin{Bmatrix} -P \\ 0 \\ F_{y2} \\ M_2 \\ F_{y3} \\ M_3 \\ F_{y4} \\ M_4 \end{Bmatrix} = [K] \begin{Bmatrix} v_1 \\ \theta_1 \\ 0 \\ \theta_2 \\ 0 \\ \theta_3 \\ 0 \\ 0 \end{Bmatrix}$$

SOLUTION (7.32)

(**a**) Use Eq.(7.45a):

$$[k]_1 = \frac{EI}{L^3} \begin{array}{c} \begin{matrix} v_1 & \theta_1 & v_2 & \theta_2 \end{matrix} \\ \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \end{array}$$

$$[k]_2 = \frac{EI}{L^3} \begin{array}{c} \begin{matrix} v_2 & \theta_2 & v_3 & \theta_3 \end{matrix} \\ \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \end{array}$$

(**b**) Assemble the global stiffness matrix of the beam: $[K] = [k]_1 + [k]_2$. Then,

$$\begin{Bmatrix} R_1 \\ M_1 \\ R_2 \\ M_2 \\ F_{3y} \\ M_3 \end{Bmatrix} = \frac{EI}{L^3} \begin{bmatrix} 12 & 6L & -12 & 6L & 0 & 0 \\ 6L & 4L^2 & -6L & 2L^2 & 0 & 0 \\ -12 & -6L & 24 & 0 & -12 & 6L \\ 6L & 2L^2 & 0 & 8L^2 & -6L & 2L^2 \\ 0 & 0 & -12 & -6L & 12 & -6L \\ 0 & 0 & 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \\ v_2 \\ \theta_3 \end{Bmatrix} \tag{1}$$

(CONT.)

7.32 (CONT.)

(c) The boundary conditions are $v_1 = 0,\ \theta_1 = 0,$ and $v_2 = 0$, Hence

$$\begin{Bmatrix} -P \\ 0 \\ 0 \end{Bmatrix} = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & 6L \\ 6L & 4L^2 & 2L^2 \\ 6L & 2L^2 & 8L^2 \end{bmatrix}\begin{Bmatrix} v_3 \\ \theta_3 \\ \theta_2 \end{Bmatrix}$$

Solving

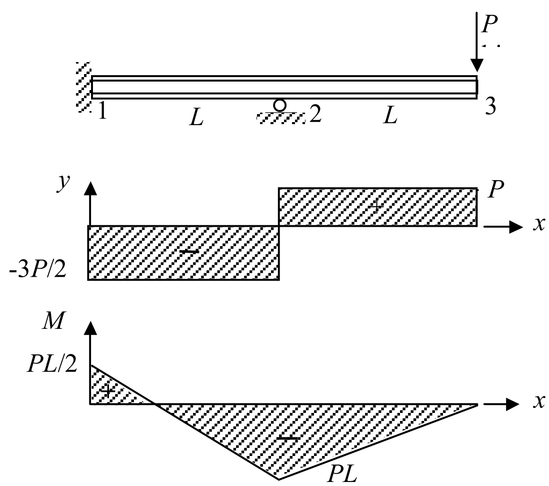
$$v_3 = -\frac{7PL^3}{12EI}, \qquad \theta_3 = \frac{3PL^2}{4EI}, \qquad \theta_2 = \frac{PL^2}{4EI} \qquad ◄$$

(d) Introducing these equations into Eq. (1), after multiplying :

$$F_{3y} = -P, \qquad M_3 = 0, \qquad R_2 = \frac{5}{2}P$$

$$M_2 = -PL, \qquad R_1 = -\frac{3}{2}P, \qquad M_1 = \frac{1}{2}PL \qquad ◄$$

(e)



◄

SOLUTION (**7.33**)

Table P7.33 Data for the truss of Fig P7.33

Element	*Length*(m)	θ	c	s	c^2	cs	s^2
1	7.5	36.9^o	0.8	0.6	0.639	0.48	0.36
2	6	0^o	1	0	1	0	0
3	4.5	90^o	0	1	0	0	1
4	4.5	0^o	1	0	1	0	0
5	$4.5\sqrt{2}$	135^o	– 0.707	0.707	0.5	– 0.5	0.5

(CONT.)

Apply Eq.(7.38):

$$[k]_1 = \frac{AE}{7.5}\begin{array}{c} \begin{array}{cccc} u_1 & v_1 & u_2 & v_2 \end{array} \\ \begin{bmatrix} 0.639 & 0.48 & -0.639 & -0.48 \\ 0.48 & 0.361 & -0.48 & -0.361 \\ -0.639 & -0.48 & 0.639 & 0.48 \\ -0.48 & -0.361 & 0.48 & 0.361 \end{bmatrix} \end{array} \begin{array}{c} \\ u_1 \\ v_1 \\ u_2 \\ v_2 \end{array}$$

$$[k]_2 = \frac{AE}{6.0}\begin{array}{c} \begin{array}{cccc} u_1 & v_1 & u_3 & v_3 \end{array} \\ \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \begin{array}{c} \\ u_1 \\ v_1 \\ u_3 \\ v_3 \end{array}$$

$$[k]_3 = \frac{AE}{4.5}\begin{array}{c} \begin{array}{cccc} u_2 & v_2 & u_3 & v_3 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \end{array} \begin{array}{c} \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{array}$$

$$[k]_4 = \frac{AE}{4.5}\begin{array}{c} \begin{array}{cccc} u_3 & v_3 & u_4 & v_4 \end{array} \\ \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \begin{array}{c} \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{array}$$

$$[k]_5 = \frac{AE}{4.5\sqrt{2}}\begin{array}{c} \begin{array}{cccc} u_2 & v_2 & u_4 & v_4 \end{array} \\ \begin{bmatrix} 0.5 & -0.5 & -0.5 & 0.5 \\ -0.5 & 0.5 & 0.5 & -0.5 \\ -0.5 & 0.5 & 0.5 & -0.5 \\ 0.5 & -0.5 & -0.5 & 0.5 \end{bmatrix} \end{array} \begin{array}{c} \\ u_2 \\ v_2 \\ u_4 \\ v_4 \end{array}$$

SOLUTION (7.34)

Table P7.34 Data for the truss of Fig.P7.34

Element	θ	c	s	c^2	cs	s^2
1	0^o	1	0	1	0	0
2	60^o	0.5	0.866	0.25	0.433	0.75
3	120^o	– 0.5	0.866	0.25	– 0.433	0.75
4	0^o	1	0	1	0	0
5	60^o	0.5	0.866	0.25	0.433	0.75

(a) Use Eq.(7.38):

$$[k]_1 = \frac{AE}{L}\begin{array}{c} \begin{array}{cccc} u_1 & v_1 & u_4 & v_4 \end{array} \\ \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \begin{array}{c} \\ u_1 \\ v_1 \\ u_4 \\ v_4 \end{array}$$

$$[k]_2 = \frac{AE}{L}\begin{array}{c} \begin{array}{cccc} u_1 & v_1 & u_2 & v_2 \end{array} \\ \begin{bmatrix} 0.25 & 0.433 & -0.25 & -0.433 \\ 0.433 & 0.75 & -0.433 & -0.75 \\ -0.25 & -0.433 & 0.25 & 0.433 \\ -0.433 & -0.75 & 0.433 & 0.75 \end{bmatrix} \end{array} \begin{array}{c} \\ u_1 \\ v_1 \\ u_2 \\ v_2 \end{array}$$

$$[k]_3 = \frac{AE}{L}\begin{array}{c} \begin{array}{cccc} u_2 & v_2 & u_4 & v_4 \end{array} \\ \begin{bmatrix} 0.25 & -0.433 & -0.25 & 0.433 \\ -0.433 & 0.75 & 0.433 & -0.75 \\ -0.25 & 0.433 & 0.25 & -0.433 \\ 0.433 & -0.75 & -0.433 & 0.75 \end{bmatrix} \end{array} \begin{array}{c} \\ u_2 \\ v_2 \\ u_4 \\ v_4 \end{array}$$

$$[k]_4 = \frac{AE}{L}\begin{array}{c} \begin{array}{cccc} u_2 & v_2 & u_3 & v_3 \end{array} \\ \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \begin{array}{c} \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{array}$$

(CONT.)

$$
\begin{array}{cccccc}
 & & u_3 & v_3 & u_4 & v_4 & \\
[k]_5 = & \dfrac{AE}{L} & \multicolumn{4}{c}{\left[\begin{array}{cccc} 0.25 & 0.433 & -0.25 & -0.433 \\ 0.433 & 0.75 & -0.433 & -0.75 \\ -0.25 & -0.433 & 0.25 & 0.433 \\ -0.433 & -0.75 & 0.433 & 0.75 \end{array}\right]} & \begin{array}{c} u_3 \\ v_3 \\ u_4 \\ v_4 \end{array}
\end{array}
$$

(b) Global Stiffness Matrix $[K] =$

$$
\begin{array}{cc}
 & \begin{array}{cccccccc} u_1 & v_1 & u_2 & v_2 & u_3 & v_3 & u_4 & v_4 \end{array} \\
\dfrac{AE}{L} & \left[\begin{array}{cccccccc}
1.25 & 0.433 & -0.25 & -0.433 & 0 & 0 & -1 & 0 \\
0.433 & 0.75 & -0.433 & -0.75 & 0 & 0 & 0 & 0 \\
-0.25 & -0.433 & 1.5 & 0 & -1 & 0 & -0.25 & 0.433 \\
-0.433 & -0.75 & 0 & 1.5 & 0 & 0 & 0.433 & -0.75 \\
0 & 0 & -1 & 0 & 1.25 & 0.433 & -0.25 & -0.433 \\
0 & 0 & 0 & 0 & 0.433 & 0.75 & -0.433 & -0.75 \\
-1 & 0 & -0.25 & 0.433 & -0.25 & -0.433 & 1.5 & 0 \\
0 & 0 & 0.433 & -0.75 & -0.433 & -0.75 & 0 & 1.5
\end{array}\right] \begin{array}{c} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{array}
\end{array}
$$

$$
\{F\} = [K]\{\delta\}; \qquad \begin{Bmatrix} R_{1x} \\ R_{1y} \\ 0 \\ P \\ Q \\ 0 \\ R_{4x} \\ R_{4y} \end{Bmatrix} = [K] \begin{Bmatrix} 0 \\ 0 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ 0 \\ 0 \end{Bmatrix}
$$

SOLUTION (7.35)

We have AE=30 MN.

Table P7.35 Data for the truss of Fig.P7.35

Element	*Length*	θ	c	s	c^2	cs	s^2
1	5	53.13^o	0.6	0.8	0.36	0.48	0.64
2	4	0^o	1	0	1	0	0

(CONT.)

(a) Use Eq.(7.38):

$$
\begin{array}{c} \\ [k]_1 = \dfrac{AE}{5} \end{array}
\begin{array}{c}
\begin{array}{cccc} u_1 & v_1 & u_2 & v_2 \end{array} \\
\begin{bmatrix} 0.36 & 0.48 & -0.36 & -0.48 \\ 0.48 & 0.64 & -0.48 & -0.64 \\ -0.36 & -0.48 & 0.36 & 0.48 \\ -0.48 & -0.64 & 0.48 & 0.64 \end{bmatrix}
\end{array}
\begin{array}{c} \\ u_1 \\ v_1 \\ u_2 \\ v_2 \end{array}
$$

$$
\begin{array}{c} \\ [k]_2 = \dfrac{AE}{4} \end{array}
\begin{array}{c}
\begin{array}{cccc} u_2 & v_2 & u_3 & v_3 \end{array} \\
\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}
\end{array}
\begin{array}{c} \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{array}
$$

(b) Global Stiffness Matrix

$$
\begin{array}{c} \\ [K] = AE \end{array}
\begin{array}{c}
\begin{array}{cccccc} u_1 & v_1 & u_2 & v_2 & u_3 & v_3 \end{array} \\
\begin{bmatrix} 0.072 & 0.096 & -0.072 & -0.096 & 0 & 0 \\ 0.096 & 0.128 & -0.096 & -0.128 & 0 & 0 \\ -0.072 & -0.096 & 0.322 & 0.096 & -0.25 & 0 \\ -0.096 & -0.128 & 0.096 & 0.128 & 0 & 0 \\ 0 & 0 & -0.250 & 0 & 0.25 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}
\end{array}
\begin{array}{c} \\ u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{array}
$$

(c) Boundary Conditions: $u_1 = v_1 = u_3 = v_3 = 0.$

$$
\begin{Bmatrix} F_{2x} \\ F_{2y} \end{Bmatrix} = AE \begin{bmatrix} 0.322 & 0.096 \\ 0.096 & 0.128 \end{bmatrix} \begin{Bmatrix} u_2 \\ v_2 \end{Bmatrix}
$$

$$
\begin{Bmatrix} u_2 \\ v_2 \end{Bmatrix} = \frac{1}{AE} \begin{bmatrix} 4 & -3 \\ -3 & 10.063 \end{bmatrix} \begin{Bmatrix} 0 \\ -10{,}000 \end{Bmatrix} = \begin{Bmatrix} 0.0010 \\ -0.0039 \end{Bmatrix} m
$$

(d)

$$
\begin{Bmatrix} F_{1x} \\ F_{1y} \\ F_{3x} \\ F_{3y} \end{Bmatrix} = AE \begin{bmatrix} -0.072 & -0.096 \\ -0.096 & -0.128 \\ -0.250 & 0 \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} 0.0010 \\ -0.0039 \end{Bmatrix} = \begin{Bmatrix} 7632 \\ 10176 \\ -7500 \\ 0 \end{Bmatrix} N
$$

(CONT.)

7.35 (CONT.)

(e) Use Eq.(7.39)

$$F_{12} = \frac{AE}{5}\begin{bmatrix}0.6 & 0.8\end{bmatrix}\begin{Bmatrix}0.001 \\ -0.0039\end{Bmatrix} = -15.12\ kN \quad (C)$$

$$F_{23} = \frac{AE}{4}\begin{bmatrix}1 & 0\end{bmatrix}\begin{Bmatrix}-0.001 \\ 0.0039\end{Bmatrix} = -7.5\ kN \quad (C)$$

SOLUTIONS (7.36 through 7.39)

It is important to take into account any condition of symmetry which may exist. Use a two-dimensional finite element program or ANSYS.

SOLUTION (7.40)

(a) Refer to Fig. 7.25a:

$$Q_j = \tfrac{1}{2} p_j h_1 t + \tfrac{1}{3}\tfrac{p_m - p_j}{2} h_1 t = h_1 t \tfrac{(2p_j + p_m)}{6}$$

$$Q_m = \tfrac{1}{2} p_j h_1 t + \tfrac{2}{3}\tfrac{p_m - p_j}{2} h_1 t = h_1 t \tfrac{(2p_m + p_j)}{6}$$

(b) Refer to Fig. 7.25b:

$$\tau_{xy} = \tfrac{PQ}{It} = \tfrac{3P}{4th^3}(h^2 - y^2) \tag{a}$$

Substituting Eq. (a) into the given expression for Q_m :

$$Q_m = \tfrac{1}{y_m - y_j}\int_{y_j}^{y_m} \tfrac{3P}{4h^3}(h^2 - y^2)(y - y_j)dy$$

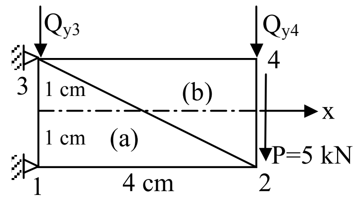
$$= \tfrac{1}{y_m - y_j}\tfrac{3P}{4h}\left[-y_j\left(y - \tfrac{y^3}{3h^2}\right)\Big|_{y_j}^{y_m} + \left(\tfrac{y^2}{2} - \tfrac{y^4}{4h^2}\right)\Big|_{y_j}^{y_m}\right]$$

This leads to the first of Eqs. (7.79). Similarly,

$$Q_j = \int_{y_j}^{y_m} \tau_{xy} t dy - Q_m \tag{b}$$

Substituting Eq. (a) and Q_m (from the first of Eqs. 7.79), Eq. (b) leads to the second of Eqs. (7.79).

SOLUTION (7.41)



Body force effects:

$$\{Q\}_a^b = \tfrac{1}{3}At\{Q_{x1}, Q_{x2}, Q_{x3}, Q_{y1}, Q_{y2}, Q_{y3}\}$$

$$= \tfrac{1}{3}(4\times 0.3)\{0,\ 0,\ 0,\ -0.077,\ -0.077,\ -0.077\}\ N$$

or

$$\{Q\}_a^b = \{0,\ 0,\ 0,\ 0,\ -0.0308,\ -0.0308,\ -0.0308,\ -0.0308,\ 0\}\ N$$

Similarly,

$$\{Q\}_b^b = \{0,\ 0,\ 0,\ 0,\ 0,\ -0.0308,\ -0.0308,\ -0.0308\}\ N$$

Hence,

$$\{Q\}^b = \{Q\}_a^b + \{Q\}_b^b$$

or

$$\{Q\}^b = \{0,\ 0,\ 0,\ 0,\ -0.0308,\ -0.0616,\ -0.0616,\ -0.0308\}\ N$$

Effect of shear force, *P*:
From Case Study 7.1,

$$\{Q\}^P = \{0,\ 0,\ 0,\ 0,\ -2500,\ -2500\}\ N$$

Surface traction effects, *p*:
Total load $(4\times 0.3)700 = 840\,N$ is equally divided between nodes 3 and 4. Thus,

$$\{Q\}^P = \{0,\ 0,\ 0,\ 0,\ 0,\ 0,\ -420,\ -420\}\ N$$

Thermal strain effects:
We have $\varepsilon_0 = 12(10^{-6})50 = 600\ \mu$ and

$$[B]_a = \frac{1}{2A}\begin{bmatrix} b_1 & b_2 & b_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_1 & a_2 & a_3 \\ a_1 & a_2 & a_3 & b_1 & b_2 & b_3 \end{bmatrix}$$

$$= \frac{1}{8}\begin{bmatrix} -2 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -4 & 0 & 4 \\ -4 & 0 & 4 & -2 & 2 & 0 \end{bmatrix}$$

Hence,

$$\{Q\}_a^t = [B]_a^T[D]\{\varepsilon_0\}(At)$$

(CONT.)

$$= \frac{1}{8}\begin{bmatrix} -2 & 0 & -4 \\ 2 & 0 & 0 \\ 0 & 0 & 4 \\ 0 & -4 & -2 \\ 0 & 0 & 2 \\ 0 & 4 & 0 \end{bmatrix} \frac{2(10^7)}{0.91}\begin{bmatrix} 1 & 0.3 & 0 \\ 0.3 & 1 & 0 \\ 0 & 0 & 0.35 \end{bmatrix}\begin{Bmatrix} 600\,\mu \\ 600\,\mu \\ 0 \end{Bmatrix}(1.2)$$

After multiplication, this gives

$$\{Q\}_a^t = \{-5142.85,\ 5142.85,\ 0,\ -10285.7,\ 0,\ 10285.7\}\ N$$

or

$$\{Q\}_a^t = \{-5142.85,\ 5142.85,\ 0,\ 0,\ -10285.7,\ 0,\ 10285.7,\ 0\}\ N$$

Similarly,

$$[B]_b = \frac{1}{2A}\begin{bmatrix} b_2 & b_4 & b_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_2 & a_4 & a_3 \\ a_2 & a_4 & a_3 & b_2 & b_4 & b_3 \end{bmatrix}$$

$$= \frac{1}{8}\begin{bmatrix} 0 & 2 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -4 & 4 & 0 \\ -4 & 4 & 0 & 0 & 2 & -2 \end{bmatrix}$$

Thus,

$$\{Q\}^t = [B]_a^T[D]\{\varepsilon_0\}(At)$$
$$= \{0,\ 5142.85,\ -5142.85,\ -10285.7,\ 10285.7,\ 0\}\ N$$

or $$\{Q\}_b^t = \{0,\ 0,\ -5142.85,\ 5142.85,\ 0,\ -10285.7,\ 0,\ 10285.7\}\ N$$

Hence,

$$\{Q\}^t = \{Q\}_a^t + \{Q\}_b^t$$

or $$\{Q\}^t = \{-5142.85,\ 5142.85,\ -5142.85,\ 5142.85,\ -10285.7,\ -10285.7,\ 10285.7,\ 10285.7\}\ N$$

The system nodal force matrix:

$$\{Q\} = \{Q\}^b + \{Q\}^P + \{Q\}^p + \{Q\}^t$$
$$= \{-5142.85,\ 5142.85,\ -5142.85,\ 5142.85,\ -10285.7,\ -12785.76,\ 986.64,\ 7365.67\}\ N$$

System equation:

$$\{Q\} = [K]\{0,\ u_2,\ 0,\ u_4,\ 0,\ v_2,\ 0,\ v_4\}$$

where

$[K]$ is given by Eq. (n) of Case Study 7.1.

(CONT.)

7.41 (CONT.)

Since we have only 4 unknown quantities u_2, u_4, v_2, v_4 (and 8 equations are available), there are redundant equations. Examination of these system of equations shows that:

$[K]$ is reduced by crossing out; row 1 and column 1 for $u_1 = 0,$ row 3 and column 3 for $u_3 = 0,$ row 5 and column 5 for $v_1 = 0,$ row 7 and column 7 for $v_3 = 0.$

$\{Q\}$ is reduced by crossing out $Q_{x1}, Q_{x3}, Q_{y1}, Q_{y3}.$

for

$$u_1 = u_3 = v_1 = v_3 = 0$$

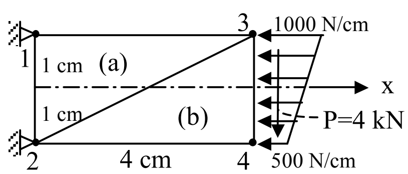
Thus, from the reduced equations, we obtain

$$\begin{Bmatrix} u_2 \\ u_4 \\ v_2 \\ v_4 \end{Bmatrix} = 10^{-6} \begin{bmatrix} 0.429 & 0.180 & 0.252 & 0.247 \\ 0.180 & 0.483 & -0.256 & -0.351 \\ 0.252 & -0.256 & 1.366 & 1.373 \\ 0.247 & -0.351 & 1.373 & 1.546 \end{bmatrix} \begin{Bmatrix} 5142.85 \\ 5142.85 \\ -12785.76 \\ 7365.67 \end{Bmatrix}$$

or

$$\begin{Bmatrix} u_2 \\ u_4 \\ v_2 \\ v_4 \end{Bmatrix} = \begin{Bmatrix} 1728 \\ 4098 \\ -7302 \\ -6702 \end{Bmatrix} 10^{-6} \ cm$$

SOLUTION (7.42)



We have

$$n = \tfrac{2.1(10^7)}{7(10^6)} = 3$$

$$m = \tfrac{2.8(10^7)}{7(10^6)} = 0.4$$

$$[D] = \frac{7(10^6)}{1-3(0.1)^2} \begin{bmatrix} 3 & 0.3 & 0 \\ 0.3 & 1 & 0 \\ 0 & 0 & 0.39 \end{bmatrix}$$

Then

$$[D^*] = \tfrac{t[D]}{4A} = \tfrac{(2/7)[D]}{16}$$

$$= 0.13(10^6) \begin{bmatrix} 3 & 0.3 & 0 \\ 0.3 & 1 & 0 \\ 0 & 0 & 0.39 \end{bmatrix} \qquad \text{(a)}$$

(CONT.)

Element *a*:

Equations (7.68), with reference to the preceding figure yield

$$\begin{aligned} a_i &= a_2 = -4 \qquad & b_i &= b_2 = 0 \\ a_j &= a_3 = 0 \qquad & b_j &= b_3 = 2 \\ a_m &= a_1 = 4 \qquad & b_m &= b_1 = -2 \end{aligned} \tag{b}$$

Substituting Eqs. (a) and (b) into Eqs. (7.54), we obtain (in 10^6):

$$\begin{aligned}
k_{uu,11} &= 0.13[3(4) + 0.39(16) + 0] = 2.37 \\
k_{uu,12} &= 0.13[0 + 0.39(-16) + 0] = -0.81 \\
k_{uu,13} &= 0.13[3(-4) + 0 + 0] = -1.56 \\
k_{uu,21} &= 0.13[0 + 0.39(-16) + 0] = -0.81 \\
k_{uu,22} &= 0.13[0 + 0.39(-16) + 0] = 0.81 \\
k_{uu,23} &= 0.13[0 + 0 + 0] = 0 \\
k_{uu,31} &= 0.13[3(-4) + 0 + 0] = -1.56 \\
k_{uu,32} &= 0.13[0 + 0 + 0] = 0 \\
k_{uu,33} &= 0.13[3(4) + 0 + 0] = 1.56
\end{aligned}$$

Similarly, remaining matrices are determined. Stiffness matrix for the element *a* (in 10^6) is:

$$[k]_a = \left[\begin{array}{ccc:ccc}
2.37 & -0.81 & -1.56 & -0.72 & 0.31 & 0.41 \\
-0.81 & 0.81 & 0 & 0.41 & 0 & -0.41 \\
-1.56 & 0 & 1.56 & 0.31 & -0.31 & 0 \\ \hdashline
-0.72 & 0.41 & 0.31 & 2.28 & -2.08 & -0.20 \\
0.31 & 0 & -0.31 & -2.08 & 2.08 & 0 \\
0.41 & -0.41 & 0 & -0.20 & 0 & 0.20
\end{array}\right]$$

or

$$[k]_a = \left[\begin{array}{cccc:cccc}
2.37 & -0.81 & -1.56 & 0 & 0.72 & 0.31 & 0.41 & 0 \\
-0.81 & 0.81 & 0 & 0 & 0.41 & 0 & -0.41 & 0 \\
-1.56 & 0 & 1.56 & 0 & 0.31 & -0.31 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hdashline
-0.72 & 0.41 & 0.31 & 0 & 2.28 & -2.08 & -0.20 & 0 \\
0.31 & 0 & -0.31 & 0 & -2.08 & 2.08 & 0 & 0 \\
0.41 & -0.41 & 0 & 0 & -0.20 & 0 & 0.20 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right]$$

(CONT.)

Element *b*:
Equations (7.68), referring to the preceding figure, now give

$$\begin{aligned} a_j &= a_2 = 0 \qquad & b_i &= b_2 = -2 \\ a_j &= a_4 = -4 \qquad & b_j &= b_4 = 2 \\ a_m &= a_3 = 4 \qquad & b_m &= b_3 = 0 \end{aligned} \tag{c}$$

Introducing Eqs. (a) and (c) into Eqs. (7.54), we obtain (in 10^6):

$$\begin{aligned} k_{uu,22} &= 0.13[3(4)+0+0] = 1.56 \\ k_{uu,23} &= 0.13[0+0+0] = 0 \\ k_{uu,24} &= 0.13[3(-4)+0+0] = -1.56 \end{aligned}$$

The remaining terms are obtained in a like manner. The stiffness matrix for the element *b* (in 10^6):

$$[k]_b = \left[\begin{array}{ccc|ccc} 1.56 & 0 & -1.56 & 0 & -0.31 & 0.31 \\ 0 & 0.81 & -0.81 & -0.41 & 0 & 0.41 \\ -1.56 & -0.81 & 2.37 & 0.41 & 0.31 & -0.72 \\ \hline 0 & -0.41 & 0.41 & 0.2 & 0 & -0.2 \\ -0.31 & 0 & 0.31 & 0 & 2.08 & -2.08 \\ 0.31 & 0.41 & -0.72 & -0.2 & -2.08 & 2.28 \end{array}\right]$$

or

$$[k]_b = \left[\begin{array}{cccc|cccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1.56 & 0 & -1.56 & 0 & 0 & -0.31 & 0.31 \\ 0 & 0 & 0.81 & -0.81 & 0 & -0.41 & 0 & 0.41 \\ 0 & -1.56 & -0.81 & 2.37 & 0 & 0.41 & 0.31 & -0.72 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -0.41 & 0.41 & 0 & 0.20 & 0 & -0.20 \\ 0 & -0.31 & 0 & 0.31 & 0 & 0 & 2.08 & -2.08 \\ 0 & 0.31 & 0.41 & -0.72 & 0 & -0.20 & -2.08 & 2.28 \end{array}\right]$$

The system matrix is found by addition of the matrices of the elements *a* and *b*:

$$[K] = [k]_a + [k]_b$$

That is (in 10^6),

(CONT.)

$$[K]=\left[\begin{array}{cccc|cccc}
2.37 & -0.81 & -1.56 & 0 & -0.72 & 0.31 & 0.41 & 0\\
-0.81 & 2.37 & 0 & -1.56 & 0.41 & 0 & 0.72 & 0.31\\
-1.56 & 0 & 2.37 & -0.81 & 0.31 & -0.72 & 0 & 0.41\\
0 & -1.56 & -0.81 & 2.37 & 0 & 0.41 & 0.31 & -0.72\\
\hline
-0.72 & 0.41 & 0.31 & 0 & 2.28 & -2.08 & -0.20 & 0\\
0.31 & 0 & -0.72 & 0.41 & -2.08 & 2.28 & 0 & -0.20\\
0.41 & -0.72 & 0 & 0.31 & -0.20 & 0 & 2.28 & -2.08\\
0 & 0.31 & 0.41 & -0.72 & 0 & -0.20 & -2.08 & 2.28
\end{array}\right]$$

Nodal force matrix:
Equations (7.78) lead to

$$Q_j = Q_{x4} = -\tfrac{2[2(500)+1000]}{6} = -667\ N$$
$$Q_m = Q_{x3} = -\tfrac{2[2(1000)+500]}{6} = -833\ N$$

Due to the shear, we also have
$$Q_{y3} = -2000\ N, \qquad Q_{y4} = -2000\ N$$
Thus,
$$\{Q\} = \{0,\ 0,\ -833,\ -667,\ 0,\ 0,\ -2000,\ -2000\}\ N$$

Nodal displacement equation:
$$\{\delta\} = \{0,\ 0,\ u_3,\ u_4,\ 0,\ 0,\ v_3,\ v_4\}$$

The system equation:
The redundant equations are eliminated by crossing out the 1st, 2nd, 5th, and 6th rows and columns. This leaves

$$\begin{Bmatrix} -833\\ -667\\ -2000\\ -2000 \end{Bmatrix} = 10^6 \begin{bmatrix} 2.37 & -0.81 & 0 & 0.41\\ -0.81 & 2.37 & 0.31 & -0.72\\ 0 & 0.31 & 2.28 & -2.08\\ 0.41 & -0.72 & -2.08 & 2.28 \end{bmatrix} \begin{Bmatrix} u_3\\ u_4\\ v_3\\ v_4 \end{Bmatrix}$$

Solving,
$$u_3 = 0.001226\ cm \qquad u_4 = -0.002324\ cm$$
$$v_3 = -0.013092\ cm \qquad v_4 = -0.013742\ cm$$

Stress in element *a*:
The strain matrix is

(CONT.)

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \frac{1}{2A}\begin{bmatrix} b_i & b_j & b_m & 0 & 0 & 0 \\ 0 & 0 & 0 & a_i & a_j & a_m \\ a_i & a_j & a_m & b_i & b_j & b_m \end{bmatrix}\begin{Bmatrix} u_i \\ u_j \\ u_m \\ v_i \\ v_j \\ v_m \end{Bmatrix}$$

$$= \frac{10^{-6}}{8}\begin{bmatrix} 0 & 2 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -4 & 0 & 4 \\ -4 & 0 & 4 & 0 & 2 & -2 \end{bmatrix}\begin{Bmatrix} 0 \\ 1226 \\ 0 \\ 0 \\ -13{,}092 \\ 0 \end{Bmatrix}$$

$$= \begin{Bmatrix} 307 \\ 0 \\ -3273 \end{Bmatrix}\mu$$

Then,

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{Bmatrix}_a = [D]\{\varepsilon\}_a$$

$$= \frac{70(10^9)}{0.97}\begin{bmatrix} 3 & 0.3 & 0 \\ 0.3 & 1 & 0 \\ 0 & 0 & 0.39 \end{bmatrix}\begin{Bmatrix} 307 \\ 0 \\ -3273 \end{Bmatrix}10^{-6}$$

$$= \{66.46,\ 6.65,\ -92.12\}\,MPa$$

Stress in element *b*:
The strain matrix is

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}_b = \frac{10^{-6}}{8}\begin{bmatrix} -2 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -4 & 4 \\ 0 & -4 & 4 & -2 & 2 & 0 \end{bmatrix}\begin{Bmatrix} 0 \\ 2324 \\ -1226 \\ 0 \\ -13{,}742 \\ 13{,}092 \end{Bmatrix}$$

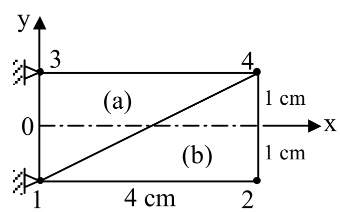
$$= \{-581,\ 325,\ -1661\}\,\mu$$

(CONT.)

7.42 (CONT.)

Thus, $$\begin{Bmatrix}\sigma_x\\ \sigma_y\\ \tau_{xy}\end{Bmatrix}_b = \frac{70(10^3)}{0.97}\begin{bmatrix}3 & 0.3 & -581\\ 0.3 & 1 & 325\\ 0 & 0 & -1661\end{bmatrix}\begin{Bmatrix}-581\\ 325\\ -1661\end{Bmatrix}$$
$$= \{-118.75,\ 10.88,\ -46.75\}\ MPa$$

SOLUTION (**7.43**)



Stiffness matrix of element *a*:

Let $i = 1,\ j = 4,\ m = 3.$ Then,

$$x_1 = 0 \qquad x_4 = 4 \qquad x_3 = 0$$
$$y_1 = -1 \qquad y_4 = 4 \qquad y_3 = 1$$

Equations (7.68) give

$$a_1 = 0 - 4 = -4 \qquad b_1 = 1 - 1 = 0$$
$$a_4 = 0 - 0 = 0 \qquad b_4 = 1 + 1 = 2$$
$$a_3 = 4 - 0 = 4 \qquad b_3 = -1 - 1 = -2$$

We have

$$[D^*] = \frac{10^6}{8}\begin{bmatrix}3.33 & 0.99 & 0\\ 0.99 & 3.3 & 0\\ 0 & 0 & 1.16\end{bmatrix}$$

Equations (7.76) in 10^6 are thus,

$$k_{uu,11} = [0 + 1.16(16)]/8 = 2.32 \qquad k_{uu,14} = [0 + 0 + 0]/8 = 0$$
$$k_{uu,13} = [0 + 1.16(-16) + 0]/8 = -2.32 \qquad k_{uu,44} = [3.3(4) + 0 + 0]/8 = 1.65$$
$$k_{uu,43} = [3.3(-4) + 0 + 0]/8 = -1.65 \qquad k_{uu,33} = [3.3(4) + 0 + 0]/8 = 3.97$$

These can be written as follows:

$$k_{uu} = \begin{bmatrix}2.32 & -2.32 & 0\\ -2.32 & 3.97 & -1.65\\ 0 & -1.65 & 1.65\end{bmatrix}(10^6)$$

(CONT.)

Similarly, we find submatrices k_{vv} and k_{uv}. In so doing and after assembling these matrices, we obtain the stiffness matrix for the element *a* (in 10^6):

$$[k]_a = \left[\begin{array}{ccc|ccc} 2.32 & -2.32 & 0 & 0 & 1.16 & -1.16 \\ -2.32 & 3.97 & -1.65 & 0.99 & -2.15 & 1.16 \\ 0 & -1.65 & 1.65 & -0.99 & 0.99 & 0 \\ \hline 0 & 0.99 & -0.99 & 6.6 & -6.6 & 0 \\ 1.16 & -2.15 & 0.99 & -6.6 & 7.18 & -0.58 \\ -1.16 & 1.16 & 0 & 0 & -0.58 & 0.58 \end{array}\right]$$

Stiffness matrix for element *b*:

Let $i = 1,\ j = 2,\ m = 4.$ Then,

$$a_1 = 4 - 4 = 0 \qquad b_1 = -1 - 1 = -2$$
$$a_2 = 0 - 4 = -4 \qquad b_2 = 1 + 1 = 2$$
$$a_4 = 4 - 0 = 4 \qquad b_4 = -1 + 1 = 0$$

Then, Eqs. (7.54) yield (in 10^6):

$$k_{uu,11} = [3.3(4) + 0 + 0]/8 = 1.65$$
$$k_{uu,12} = [3.3(-4) + 0 + 0]/8 = -1.65$$
$$k_{uu,14} = [0 + 0 + 0]/8 = 0$$
$$k_{uu,22} = [3.3(4) + 1.16(16) + 0]/8 = 3.97$$
$$k_{uu,24} = [0 + 1.16(-16) + 0]/8 = -2.32$$
$$k_{uu,44} = [0 + 1.16(16) + 0]/8 = 2.32$$

or
$$k_{uu} = \begin{bmatrix} 1.65 & -1.65 & 0 \\ -1.65 & 3.97 & -2.32 \\ 0 & -2.32 & 2.32 \end{bmatrix}(10^6)$$

Similarly, we obtain submatrices k_{vv} and k_{uv}. In so doing and after assembling these matrices, we determine the stiffness matrix for the element *b* (in 10^6):

$$[k]_b = \left[\begin{array}{ccc|ccc} 1.65 & -1.65 & 0 & 0 & 0.99 & -0.99 \\ -1.65 & 3.97 & -2.32 & 1.16 & -2.15 & 0.99 \\ 0 & -2.32 & 2.32 & -1.16 & 1.16 & 0 \\ \hline 0 & 1.16 & -1.16 & 0.58 & -0.58 & 0 \\ 0.99 & -2.15 & 1.16 & -0.58 & 7.18 & -6.6 \\ -0.99 & 0.99 & 0 & 0 & -6.6 & 6.6 \end{array}\right]$$

(CONT.)

Prior to addition: the 2nd and 6th rows and columns of zeros are added to the matrix $[k]_a$; the 3rd and 7th rows and columns of zeros are added to the matrix $[k]_b$.

The system matrix:

$$[K] = [k]_a + [k]_b$$

is then (in 10^6):

$$[K] = \left[\begin{array}{cccc:cccc} 3.97 & -1.65 & -2.32 & 0 & 0 & 0.99 & 1.16 & -2.15 \\ -1.65 & 3.97 & 0 & -2.32 & 1.16 & -2.15 & 0 & 0.99 \\ -2.32 & 0 & 3.97 & -1.65 & 0.99 & 0 & -2.15 & 1.16 \\ 0 & -2.32 & -1.65 & 3.97 & -2.15 & 1.16 & 0.99 & 0 \\ \hdashline 0 & 1.16 & 0.99 & -2.15 & 7.18 & -0.58 & -6.6 & 0 \\ 0.99 & -2.15 & 0 & 1.16 & -0.58 & 7.18 & 0 & -6.6 \\ 1.66 & 0 & -2.15 & 0.99 & -6.6 & 0 & 7.18 & -0.58 \\ -2.15 & 0.99 & 1.16 & 0 & 0 & -6.6 & -0.58 & 7.18 \end{array}\right]$$

The force-displacement relation, Eq. (q) of Case Study 7.1 becomes

$$\begin{Bmatrix} 0 \\ 0 \\ -1.2 \\ -2.5 \end{Bmatrix} = \begin{bmatrix} 3.97 & -2.32 & -2.15 & 0.99 \\ -2.32 & 3.97 & 1.16 & 0 \\ -2.15 & 1.16 & 7.18 & -6.6 \\ 0.99 & 0 & -6.6 & 7.18 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_4 \\ v_2 \\ v_4 \end{Bmatrix}$$

This yields

$$\begin{Bmatrix} u_2 \\ u_4 \\ v_2 \\ v_4 \end{Bmatrix} = 10^{-6} \begin{bmatrix} 483.2 & 179.9 & 357.9 & 225.8 \\ 178.6 & 429 & -246.8 & 251.7 \\ 357.9 & -247 & 1546 & -1373 \\ 255.8 & -251.7 & -1373 & -1366 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ -2.5 \\ -2.5 \end{Bmatrix}$$

$$= \begin{Bmatrix} -1534.35 \\ 1246.325 \\ -433 \\ 17.125 \end{Bmatrix} (10^{-6})\ cm$$

(CONT.)

Stresses in element *b*:

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}_b = \frac{1}{2A}\begin{bmatrix} b_1 & b_2 & b_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_1 & a_2 & a_4 \\ a_1 & a_2 & a_4 & b_1 & b_2 & b_4 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \\ u_4 \\ v_1 \\ v_2 \\ v_4 \end{Bmatrix}$$

$$= \frac{10^{-6}}{8}\begin{bmatrix} -2 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -4 & 4 \\ 0 & -4 & 4 & -2 & 2 & 0 \end{bmatrix}\begin{Bmatrix} 0 \\ -1534.35 \\ 1246.325 \\ 0 \\ -433 \\ 17.125 \end{Bmatrix}$$

$$= \{-383.6,\ 225,\ 1282\}\ \mu$$

Thus,

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}_b = \frac{200(10^3)}{0.91}\begin{bmatrix} 1 & 0.3 & 0 \\ 0.3 & 1 & 0 \\ 0 & 0 & 0.35 \end{bmatrix}\begin{Bmatrix} -581 \\ 325 \\ -1661 \end{Bmatrix}$$

$$= \{-118.75,\ 10.88,\ -46.75\}\ MPa$$

Stresses in element *a*:

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}_a = \frac{10^{-6}}{8}\begin{bmatrix} 0 & -2 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -4 & 4 & 0 \\ -4 & 4 & 0 & 0 & -2 & 2 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ 1246.325 \\ 0 \\ 0 \\ 17.125 \end{Bmatrix}$$

$$= \{311.5,\ 0,\ 4.28\}\ \mu$$

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}_a = \frac{200(10^3)}{0.91}\begin{bmatrix} 1 & 0.3 & 0 \\ 0.3 & 1 & 0 \\ 0 & 0 & 0.35 \end{bmatrix}\begin{Bmatrix} 311.5 \\ 0 \\ 4.28 \end{Bmatrix}$$

$$= \{68.46,\ 20.54,\ 0.33\}\ MPa$$

End of Chapter 7

CHAPTER 8

SOLUTION (8.1)

(a) From Eq. (8.13), we have

$$\sigma_{\theta,\min} = \frac{a^2 p_i}{b^2-a^2}\left(1+\frac{b^2}{b^2}\right) = p_i\,\frac{2a^2}{b^2-a^2}$$

$$\sigma_{\theta,\max} = \frac{a^2 p_i}{b^2-a^2}\left(1+\frac{a^2}{b^2}\right) = p_i\,\frac{a^2+b^2}{b^2-a^2}$$

Hence,

$$\frac{\sigma_{\theta,\max}}{\sigma_{\theta,\min}} = \frac{a^2+b^2}{2a^2} = \frac{a^2+(1.1a)^2}{2a^2} = 1.105 \qquad \blacktriangleleft$$

(b) Using Eq. (8.16),

$$\sigma_{\theta,\max} = -2p_0\,\frac{b^2}{b^2-a^2}$$

$$\sigma_{\theta,\min} = -p_0\,\frac{a^2+b^2}{b^2-a^2}$$

Thus,

$$\frac{\sigma_{\theta,\max}}{\sigma_{\theta,\min}} = \frac{2b^2}{b^2-a^2} = \frac{2(1.21a)^2}{2.21a^2} = 1.1 \qquad \blacktriangleleft$$

SOLUTION (8.2)

Equation (8.20):

$$\sigma_z = \frac{p_i a^2}{b^2-a^2} = \frac{0.6^2\,p_i}{1^2-0.6^2} = 0.5625p_i = 140$$

or

$$p_i = 248.9\ MPa$$

Equation (8.13):

$$\sigma_{\theta,\max} = \frac{b^2+a^2}{b^2-a^2}\,p_i = \frac{1^2+0.6^2}{1^2-0.6^2}\,p_i = 2.12p_i = 140$$

or

$$p_i = 65.9\ MPa$$

Equation (8.10):

$$\tau_{\max} = \frac{p_i b^2}{b^2-a^2} = \frac{1^2}{1^2-0.6^2}\,p_i = 1.5625p_i = 80$$

or

$$p_i = 51.2\ MPa = p_{all} \qquad \blacktriangleleft$$

SOLUTION (8.3)

(a) Initial maximum tangential stress, from Eq. (8.13),

$$\sigma_\theta = p_i\,\frac{b^2+a^2}{b^2-a^2} = \frac{n^2+1}{n^2-1}\,p_i$$

or $$p_i = \sigma_\theta\,\frac{n^2-1}{n^2+1}$$

After boring, denoting the inner radius by r_x, we have

$$\Delta\sigma_\theta + \sigma_\theta = p_i\,\frac{n^2a^2+r_x^2}{n^2a^2-r_x^2} = \frac{n^2-1}{n^2+1}\,\sigma_\theta\,\frac{n^2a^2+r_x^2}{n^2a^2-r_x^2}$$

or $$(\Delta\sigma_\theta + \sigma_\theta)(n^2+1)(n^2a^2-r_x^2) = (n^2-1)\sigma_\theta(n^2a^2+r_x^2)$$

(CONT.)

8.3 (CONT.)

or $$(\Delta\sigma_\theta + \sigma_\theta)(n^2+1)n^2a^2 - (n^2-1)\sigma_\theta n^2 a^2$$
$$= r_x^2[(\Delta\sigma_\theta + \sigma_\theta)(n^2+1) + (n^2-1)\sigma_\theta]$$

This gives the new inner radius in the form

$$r_x = [\tfrac{2n^2a^2\sigma_\theta + \Delta\sigma_\theta(n^2+1)n^2a^2}{\Delta\sigma_\theta(n^2+1)+2\sigma_\theta n^2}]^{\frac{1}{2}} \quad ◄$$

(b)

$$r_x = [\tfrac{2(4)(0.025)^2\sigma_\theta + 0.1\sigma_\theta(5)4(0.025)^2}{0.1\sigma_\theta(5)+2\sigma_\theta(4)}]^{\frac{1}{2}}$$
$$= 0.02712\ m = 27.12\ mm \quad ◄$$

SOLUTION (8.4)

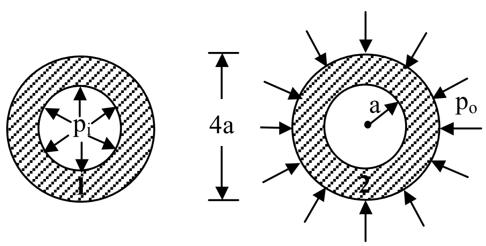
Using Eq. (8.13)

$$\sigma_{\theta,\max} = p_i \tfrac{a^2+b^2}{b^2-a^2}$$
$$\tfrac{280}{2} = 7\tfrac{(0.6)^2+b^2}{b^2-(0.6)^2}$$

Solving, $b = 0.6308\ m$. Therefore,

$$t = b - a = 630.8 - 600 = 30.8\ mm \quad ◄$$

SOLUTION (8.5)



(a) Equation (8.13) and (8.16) give at r=a:

$$\sigma_{\theta 1} = p_i \tfrac{b^2+a^2}{b^2-a^2}, \qquad \sigma_{\theta 2} = -2p_o \tfrac{b^2}{b^2-a^2}$$

Then, $|\sigma_{\theta 1}| = |\sigma_{\theta 2}|; \quad p_i \tfrac{b^2+a^2}{b^2-a^2} = 2p_o \tfrac{b^2}{b^2-a^2}$

or

$$\tfrac{p_i}{p_o} = \tfrac{2b^2}{a^2+b^2} = \tfrac{2(4a^2)}{a^2+4a^2} = 1.6 \quad ◄$$

(b) By neglecting the strain ε_L in the longitudinal direction,

$$\varepsilon_{\theta 1} = \tfrac{1}{E}(\sigma_{\theta 1} + \nu p_i), \qquad \varepsilon_{\theta 2} = \tfrac{1}{E}(\sigma_{\theta 2} + \nu p_o)$$

Then,

$$|\varepsilon_{\theta 1}| = |\varepsilon_{\theta 2}|$$

gives

$$\sigma_{\theta 1} + \nu p_i = \sigma_{\theta 2} - \nu p_o \qquad \text{(a)}$$

But

(CONT.)

8.5(CONT.)

$$\frac{\sigma_{\theta 1}}{p_i} = \frac{a^2+b^2}{b^2-a^2} = \frac{a^2+4a^2}{4a^2-a^2} = 1.66$$

Hence, $\sigma_{\theta 1} = 1.66 p_i$ (b)

We also have

$$\frac{\sigma_{\theta 2}}{p_o} = \frac{2b^2}{b^2-a^2} = \frac{2(4a^2)}{4a^2-a^2} = 2.66$$

or

$$\sigma_{\theta 2} = 2.66 p_o \qquad \text{(c)}$$

Substituting Eqs. (b) and (c) into (a) and letting $\nu = 1/3$:

$$\frac{p_i}{p_o} = 1.16 \qquad ◄$$

SOLUTION (8.6)

Equation (8.14), substituting the given data yields

$$u = \frac{ap_i}{E}\left(\frac{a^2+b^2}{b^2-a^2} + \nu\right)$$

$$= \frac{0.6(7\times10^6)}{200\times10^9}\left(\frac{0.6^2+0.6308^2}{0.6308^2-0.6^2} + 0.3\right)$$

$$= 0.426\ mm \qquad ◄$$

SOLUTION (8.7)

(a) We have $\varepsilon_\theta = u/r$, where u is defined by Eq. (8.14). Thus, at r=a:

$$\varepsilon_{\theta,\max} = \frac{p_i}{E}\left[\frac{b^2+a^2}{b^2-a^2} + \nu\right] \qquad \text{(a)}$$

(b) Introducing σ_θ, σ_r, and σ_z from Eqs. (8.12), (8.13) and (8.20) into Hooke's law

$$\varepsilon_\theta = \frac{1}{E}[\sigma_\theta - \nu(\sigma_r + \sigma_z)]$$

we have

$$\varepsilon_{\theta,\max} = \frac{p_i}{E}\left[\frac{b^2+(1-\nu)a^2}{b^2-a^2} + \nu\right] \qquad \text{(b)}$$

Substituting the data, Eq. (a):

$$0.001 = \frac{60(10^6)}{200(10^9)}\left[\frac{4+a^2}{4-a^2} + \frac{1}{3}\right]$$

Solving, $a = 1.41\ m.$ Then,

$$t = 2 - 1.41 = 0.59\ m = t_{req.} \qquad ◄$$

Similarly, Eq. (b) yields

$$0.001 = \frac{60(10^6)}{200(10^9)}\left[\frac{4+2a^2/3}{4-a^2} + \frac{1}{3}\right]$$

or $a = 1.48\ m;\qquad t = 0.52\ m$

SOLUTION (8.8)

Equation (8.13) at r=a:

$$\sigma_{\theta,\max} = \frac{0.25^2+0.05^2}{0.25^2-0.05^2}(60) = 65\ MPa$$

(CONT.)

Equation (8.12) at r=a:

$$\sigma_{r,\max} = -p_i = -60 \ MPa$$

Equation (8.10):

$$\tau_{\max} = \tfrac{60(0.25)^2}{0.25^2 - 0.05^2} = 62.5 \ MPa$$

Equation (8.20) with $p_o = 0$:

$$\sigma_z = \tfrac{0.05(60)}{0.25 - 0.05^2} = 2.5 \ MPa$$

Stress-strain relationship is given by

$$\varepsilon_\theta = \tfrac{1}{E}[\sigma_\theta - \nu(\sigma_r + \sigma_z)] = \tfrac{u}{r} \qquad \text{(a)}$$

Substituting Eqs. (8.12), (8.13), and (8.20), this expression results in at r=a:

$$u = \tfrac{ap_i}{E(b^2 - a^2)}[(1 - 2\nu)a^2 + (1 + \nu)b^2] \qquad \text{(P8.8)}$$

Introducing the given numerical values, we obtain

$$u = \tfrac{0.05(60\times10^6)}{72\times10^6(0.25^2 - 0.05^2)}[0.4(0.05)^2 + 1.3(0.25)^2]$$

$$= 0.0571 \ mm$$

The change in the internal diameter is therefore,

$$\Delta d = 2u = 0.1142 \ mm \qquad ◄$$

SOLUTION (8.9)

Equation (8.19):

$$\sigma_{\theta,\max} = -\tfrac{2(0.25)^2(60)}{0.25^2 - 0.05} = -125 \ MPa$$

Equation (8.15) at r=b:

$$\sigma_{r,\max} = -p_o = -60 \ MPa$$

Equation (8.20) for $p_i = 0$:

$$\sigma_z = -\tfrac{0.25^2(60)}{0.25^2 - 0.05^2} = -62.5 \ MPa$$

Equation (8.9) at r=a and $p_i = 0$:

$$\tau_{\max} = -\tfrac{p_o b^2}{b^2 - a^2} = -\tfrac{60(0.25)^2}{0.25^2 - 0.05^2} = -62.5 \ MPa$$

Substitution of Eqs. (8.15), (8.16), and (8.20), into Eq. (a) of Solution of Prob. 8.8 leads to for r=a:

$$u = -\tfrac{ap_o b^2}{E(b^2 - a^2)}(2 - \nu) \qquad \text{(P8.9)}$$

$$= -\tfrac{0.05(60\times10)(0.25^2)}{72\times10^9(0.25^2 - 0.05^2)}(2 - 0.3) = -0.0738 \ mm$$

Thus, $\Delta d = 2u = -0.1476 \ mm$ ◄

SOLUTION (8.10)

Given: $b = 0.1 \ m \qquad c = 0.3 \ m \qquad a = 0$

$$\delta = 0.001(0.1) = 0.0001 \ m$$

Using Eq. (8.23):

(CONT.)

$$p = \frac{E\delta}{b}\frac{c^2-b^2}{2c^2} = \frac{200\times10^9(0.0001)}{0.1}\frac{0.09-0.01}{2(0.09)}$$ ◄

$$= 88.89\ MPa$$

Then, letting $p = p_i = 88.89\ MPa$, Eq. (8.8) gives at r=b:

$$\sigma_\theta = 0 = \frac{b^2p-c^2p_o}{c^2-b^2} + \frac{(p-p_o)}{c^2-b^2}$$

After substituting the numerical values, this equation results in

$$p_o = 49.38\ MPa$$

SOLUTION (8.11)

(a) Using Eq. (8.18),

$$\frac{\sigma_{\theta,\max}}{p_i} = \frac{a^2+b^2}{b^2-a^2} = \frac{4}{3}$$

or $b = 2.65a$

Then, $\frac{b}{a} = \frac{a+t}{a} 2.65;\qquad t = 1.65a$

Hence,

$$\frac{t}{2a} = \frac{1.6a}{2a} = 0.825$$ ◄

(b) Neglect longitudinal strain and consider

$$\sigma_r = -p_i = 6.3\ MPa$$

$$\sigma_\theta = \frac{4(6.3)}{3} = 8.4\ MPa$$

Therefore, for $a = 0.075\ m$ and $b = 2.65a$:

$$\Delta d = \varepsilon_d(2a) = \frac{2p_ia}{E}\left[\frac{a^2+b^2}{b^2-a^2} + \nu\right] = \frac{2(6.3\times10^6)(0.075)}{210(10^9)}\left[\frac{4}{3} + \frac{1}{3}\right]$$

$$= 7.4(10^{-3})\ mm$$ ◄

Alternatively, use Eq. (8.14) and let $\Delta d = 2u$.

SOLUTION (8.12)

Introducing various values of P, as given in Fig. 8.4, into Eqs. (a) and (8.21) of Sec. 8.3, it is seen that σ_θ and S values as shown in the figure are found.

For example, let $P = 1$ or $p_i = p_o$.

Then,

$$\sigma_\theta = p_i\frac{1-R^2}{R^2-1} + p_ib^2\frac{1-1}{(R^2-1)r};\qquad \sigma_\theta = -p_i$$ ◄

And $S = \frac{\sigma_{\theta i}}{\sigma_{\theta o}} = \frac{1+0}{1+0} = 1$

or

$$\sigma_{\theta i} = \sigma_{\theta o}\quad \text{or}\quad S = 1$$ ◄

SOLUTION (8.13)

(a) For $\varepsilon_z = 0; [\sigma_z - \nu(\sigma_r + \sigma_\theta)/E = 0,$ using Eqs. (8.8),

(CONT.)

8.13 (CONT.)

$$\sigma_z = \nu(\sigma_r + \sigma_\theta) = \frac{2\nu(a^2 p_i - b^2 p_o)}{b^2 - a^2} \quad ◀$$

(b) Similarly, for $\sigma_z = 0$:

$$\varepsilon_z = -\frac{\nu}{E}(\sigma_r + \sigma_\theta) = -\frac{2\nu(a^2 p_i - b^2 p_o)}{E(b^2 - a^2)} \quad ◀$$

SOLUTION (8.14)

At r=a: from Eq. (8.18),

$$\sigma_{\theta,\max} = \frac{4a^2 + a^2}{4a^2 - a^2} p_i = \frac{5}{3} p_i$$

and from Eq. (8.12),

$$\sigma_{r,\max} = -p_i.$$

Energy of distortion theory:

$$p_i[(\tfrac{5}{3})^2 - (\tfrac{5}{3})(-1) + (-1)^2]^{\frac{1}{2}} = \sigma_{yp}$$

or $\quad p_i = 0.429\sigma_{yp}$ ◀

Maximum shearing stress theory:

$$\tfrac{5}{3} p_i - (-p_i) = \sigma_{yp}$$

or $\quad p_i = 0.375\sigma_{yp}$ ◀

SOLUTION (8.15)

We have, at r=a:

$$\sigma_{\theta,\max} = \frac{b^2 + a^2}{b^2 - a^2} p_i = \frac{9a^2 + a^2}{9a^2 - a^2} p_i = \frac{5}{4} p_i$$

$$\sigma_{r,\max} = -p_i$$

(a) $\quad |\sigma_u| = |\tfrac{5}{4} p_i|$

or

$$p_i = 0.8(350) = 280 \ MPa \quad ◀$$

And $\quad |\sigma_u| = |p_i|, \qquad p_i = 350 \ MPa$

(b) Using Eq. (4.12a),

$$\frac{5p_i}{4(350)} - \frac{-p_i}{630} = 1$$

Solving,

$$p_i = 193.8 \ MPa \quad ◀$$

SOLUTION (8.16)

From Eq. (8.18) for r=a:

$$p_i = 35 \frac{(0.25)^2 - (0.05)^2}{(0.25)^2 + (0.05)^2} = 32.31 \ MPa$$

Total radial force on the contact surface is $2\pi a L p$ and total frictional force equals $2\pi a p L \mu$.

(CONT.)

8.16 (CONT.)

Hence, Torque $= 2\pi a p L \mu(a)$

$$= 2\pi(0.05^3)(32.31\times10^6)(0.2)$$

$$= 5.073\ kN\cdot m \quad ◄$$

SOLUTION (8.17)

Let ε_{s1}, ε_{c1}, and ε_{s2}, ε_{c2} be initial and final compressive and tensile strains in the shaft and in the cylinder, respectively.

Also, let $\sigma_{\theta1}$, p_1 and $\sigma_{\theta2}$, p_2 denote the initial and final maximum stresses and contact pressures, respectively. Then,

$$\varepsilon_{s1}+\varepsilon_{c1}=\varepsilon_{s2}+\varepsilon_{c2} \tag{a}$$

Here,

$$\varepsilon_{s1}=\tfrac{1}{E}(p_1-\nu p_1)=\tfrac{2}{3E}p_1$$

$$\varepsilon_{c1}=\tfrac{1}{E}(\sigma_{\theta1}+\nu p_1)=\tfrac{p_1}{E}(2+\tfrac{1}{3})=\tfrac{7}{3E}p_1$$

$$\varepsilon_{s2}=\tfrac{1}{E}[(p_2-\nu p_2)+\tfrac{\nu p_L}{\pi a^2}]$$

$$=\tfrac{1}{E}[p_2(1-\tfrac{1}{3})+\tfrac{1}{3}\tfrac{4.5}{\pi(0.05)^2}]$$

$$=\tfrac{1}{E}[\tfrac{2}{3}p_2+\tfrac{1909.86}{3}]$$

From the condition of linearity,

$$\tfrac{\sigma_{\theta2}}{\sigma_{\theta1}}=\tfrac{p_2}{p_1};\qquad \sigma_{\theta2}=p_2\tfrac{\sigma_{\theta1}}{p_1}=p_2\tfrac{2p_1}{p_1}=2p_2$$

and $\varepsilon_{c2}=\tfrac{1}{E}(\sigma_{\theta2}+\nu p_2)=\tfrac{1}{E}(2p_2+\nu p_2)=\tfrac{7p_2}{3E}$

Equation (a) is thus,

$$\tfrac{2p_1}{3E}+\tfrac{7p_1}{3E}=\tfrac{1}{E}(\tfrac{2}{3}p_2+\tfrac{1909.86}{3})+\tfrac{7p_2}{3E}$$

from which

$$p_1=p_2+636.62$$

Hence,

$$\Delta p=p_2-p_1=-636.62\ kPa \quad ◄$$

SOLUTION (8.18)

From Eq. (8.22),

$$\delta=\tfrac{bp}{E_b}(\tfrac{b^2+c^2}{c^2-b^2}+\nu_b)+\tfrac{bp}{E_s}(\tfrac{a^2+b^2}{b^2-a^2}-\nu_s)$$

or
$$p=\frac{\delta}{b[\frac{1}{E_b}(\frac{c^2+b^2}{c^2-b^2}+\nu_b)+\frac{1}{E_s}(\frac{a^2+b^2}{b^2-a^2}-\nu_s)]}$$

Substituting the given data, this gives $p=9.01\ MPa$

Stresses in the steel cylinder, using Eq. (8.16),

$$(\sigma_\theta)_{r=a}=-\tfrac{2(9.01\times10^6)(0.08)^2}{(0.08)^2-(0.04)^2}=-24.03\ MPa$$

(CONT.)

8.18 (CONT.)

$$(\sigma_\theta)_{r=0.08} = -9.01(10^6)\tfrac{(0.08)^2+(0.04)^2}{(0.08)^2-(0.04)^2}$$

$$= -15.02\ MPa \quad ◀$$

Stresses in the brass cylinder, from Eq. (8.13):

$$(\sigma_\theta)_{r=0.08} = 9.01(10^6)\tfrac{(0.14)^2+(0.08)^2}{(0.14)^2-(0.08)^2}$$

$$= 17.75\ MPa \quad ◀$$

$$(\sigma_\theta)_{r=0.14} = \tfrac{2(9.01\times10^6)(0.08)^2}{(0.14)^2-(0.08)^2} = 8.74\ MPa \quad ◀$$

SOLUTION (8.19)

(a) Using Eq. (P8.18),

$$p = \frac{(0.5/2)(10^{-3})}{0.1\left[\frac{1}{72(10^9)}\left(\frac{0.15^2+0.1^2}{0.15^2-0.1^2}+0.33\right)+\frac{1-0.29}{200(10^9)}\right]}$$

$$= 56.49\ MPa \quad ◀$$

(b) From Eq. (8.14), for r=b:

$$u = \tfrac{2a^2bp_i}{E(b^2-a^2)} = \tfrac{2(0.1)^2(0.15)(56.49\times10^6)}{72\times10^9(0.15^2-0.1^2)}$$

$$= 0.1883\ mm$$

$$\delta = 2u = 0.3766\ mm \quad ◀$$

SOLUTION (8.20)

(a) Using Eq. (8.14), we have at r=a:

$$\tfrac{\delta_0}{a} = \tfrac{pa}{E(b^2-a^2)}[(1-\nu)a^2 + (1+\nu)b^2]$$

from which

$$p = \tfrac{\delta_0 E(b^2-a^2)}{a^2[(1-\nu)a^2+(1+\nu)b^2]} \qquad \text{(P8.20)} \quad ◀$$

(b) Substitution of Eq. (P8.20) into Eqs. (8.12) result in

$$\sigma_r = \tfrac{\delta_0 E}{(1-\nu)a^2+(1+\nu)b^2}(1-\tfrac{b^2}{r^2})$$

$$\sigma_\theta = \tfrac{\delta_0 E}{(1-\nu)a^2+(1+\nu)b^2}(1+\tfrac{b^2}{r^2}) \quad ◀$$

SOLUTION (8.21)

Equation (8.14) at r=b:

$$u = \tfrac{\delta_0}{2} = \tfrac{bp}{E}\left(\tfrac{b^2+c^2}{c^2-b^2}+\nu\right) = \tfrac{bp}{E}\left(\tfrac{b^2+2.25b^2}{2.25b^2-b^2}+\tfrac{1}{3}\right)$$

Solving, $p = \tfrac{E\delta_0}{5.86b}$

Then, Eq. (8.17) yields at r=b:

$$u = \tfrac{bp}{E}(1-\nu) = \tfrac{2\delta_0}{3(5.86)} = 0.114\delta_0$$

Hence, $\Delta d = 2u = 0.23\delta_0$ ◀

SOLUTION (8.22)

(a) Intial difference in diameter:

$$\Delta = 2b(\varepsilon_1 + \varepsilon_2)$$

where,

ε_1 =tangential (comp.) strain in the shaft

ε_2 =tangential (tens.) strain in the cylinder

Thus,

$$\Delta = \tfrac{2b}{E}[(p - \nu p) + (\sigma_{\theta,\max} + \nu p)]$$
$$= \tfrac{2b}{E}[(\sigma_{\theta,\max} + p) = \tfrac{2b}{E}(2p + p)$$
$$= \tfrac{6pb}{E}$$

(b) Compressive (uniform) strain ε_L, due to axial load P, is

$$\varepsilon_L = \tfrac{P}{\pi b^2 E}$$

We now have

$$\varepsilon_1 = \tfrac{1}{E}(p_1 - \nu p_1 - \tfrac{\nu P}{\pi b^2}) \qquad \text{(comp.)}$$
$$\varepsilon_2 = \tfrac{1}{E}(\sigma_{\theta 1,\max} + \nu p_1) \qquad \text{(tens.)}$$

where $\sigma_{\theta 1,\max}$ is the increased tangential stress. Thus,

$$\Delta_1 = 2b(\varepsilon_1 + \varepsilon_2) = \tfrac{2b}{E}(\sigma_{\theta 1,\max} + p_1 - \tfrac{\nu P}{\pi b^2})$$

Based on the linearity condition:

$$\tfrac{\sigma_{\theta 1,\max}}{p_1} = \tfrac{\sigma_{\theta,\max}}{P} = 2$$

or

$$\sigma_{\theta 1,\max} = 2p_1$$

Setting $\Delta = \Delta_1$

$$\tfrac{6pb}{E} = \tfrac{2b}{E}(2p_1 + p_1 - \tfrac{P}{3\pi b^2})$$

or

$$P = 9\pi b^2(p_1 - p) \qquad ◄$$

SOLUTION (8.23)

Radial strain is

$$\varepsilon = \varepsilon_{b,comp} + \varepsilon_{s,tens.}$$
$$= \tfrac{1}{E_b}(p - \nu p) + \tfrac{1}{E_s}(\sigma_{\theta,\max} + \nu p)$$

We also have

$$\varepsilon = (T_2 - T_1)(\alpha_b - \alpha_s)$$
$$= \Delta T(19.5 - 11.7)10^{-6} = 7.8(10^{-6})\Delta T$$

Note that from Eq. (8.18) at r=a:

$$\tfrac{\sigma_{\theta,\max}}{p} = \tfrac{4b^2 + b^2}{4b^2 - b^2} = \tfrac{5}{3}$$

(CONT.)

8.23 (CONT.)

Thus, $7.8(10^{-6})\Delta T = \frac{1}{E_b}(p - \frac{p}{3}) + \frac{1}{E_s}(\frac{5p}{3} + \frac{p}{3})$

or

$$p = \frac{11.7(\Delta T)E_b E_s}{(E_s + 3E_b)10^6}$$

Hence,

$$\sigma_{\theta,\max} = \frac{1.95 E_b E_s (T_2 - T_1)}{(E_s + 3E_b)10^5} \quad ◄$$

SOLUTION (8.24)

Axial Stress, Eq. (8.20):

$$\sigma_z = \frac{p_i a^2}{b^2 - a^2} = \frac{0.8^2 p_i}{1.2^2 - 0.8^2} = 0.8 p_i = 80, \qquad p_i = 100\ MPa$$

Tangential stress, Eq. (8.13):

$$(\sigma_\theta)_{\max} = \frac{b^2 + a^2}{b^2 - a^2} p_i = \frac{1.2^2 + 0.8^2}{1.2^2 - 0.8^2} p_i = 2.6 p_i = 80, \qquad p_i = 30.8\ MPa$$

Shear Stress, Eq. (8.10):

$$\tau_{\max} = \frac{p_i b^2}{b^2 - a^2} = \frac{1.2^2}{1.2^2 - 0.8^2} p_i = 1.8 p_i = 50$$

Solving

$$p_i = 27.8\ MPa = p_{all} \quad ◄$$

SOLUTION (8.25)

The maximum radial displacement $u_{\max}$ occurs at the inner edge of the tank.

So, Eq. (8.14) for $r=a$, results in

$$u_{\max} = \frac{ap_i}{E} = (\frac{a^2 + b^2}{b^2 - a^2} + \nu)$$

Introducing the given data

$$u_{\max} = \frac{0.5(60 \times 10^6)}{70(10^9)}(\frac{0.5^2 + 0.8^2}{0.8^2 - 0.5^2} + 0.3) = 1.107(10^{-3})\ m = 1.12\ mm \quad ◄$$

SOLUTION (8.26)

Equation (8.18) and Eq.(8.12) at r=a:

$$\sigma_1 = \sigma_{\theta,\max} = p_i \frac{a^2 + b^2}{b^2 - a^2} = \frac{5}{4} p_i \qquad \sigma_2 = \sigma_{r,\max} = -p_i$$

(a) $\sigma_1 - \sigma_2 = \sigma_{yp}$; $\frac{5}{4} p_i - (-p_i) = 350, \qquad p_i = 155.6\ MPa$ ◄

(b) $\sigma_1^2 - \sigma_1 \sigma_2 + \sigma_2^2 = \sigma_{yp}^2$

$$p_i[(\tfrac{5}{4})^2 - (\tfrac{5}{4})(-1) + (-1)^2]^{\frac{1}{2}} = 350, \qquad p_i = 179.3\ MPa \quad ◄$$

SOLUTION (8.27)

$$\sigma_{\theta,\max} = p_i \tfrac{b^2+a^2}{b^2-a^2} = \tfrac{5}{3} p_i = \sigma_1, \quad \sigma_{r,\max} = -p_i = \sigma_2$$

(a) $\sigma_u = \left|\tfrac{5}{3} p_i\right|, \qquad p_i = \tfrac{3}{5}(320) = 192\ MPa$ (governs) ◀

and $\sigma_u = |p_i|, \qquad p_i = 320\ MPa$

(b) $\frac{\sigma_1}{\sigma_u} - \frac{\sigma_2}{\sigma_{u'}} = 1; \qquad \frac{5p_i}{3(320)} - \frac{-p_i}{620} = 1$

or $p_i = 146.8\ MPa$ ◀

SOLUTION (8.28)

We have a=20 mm, b=30 mm, c=60 mm.
Equation (8.22):

$$0.05 = \tfrac{30p}{210\times10^9}[\tfrac{60^2+30^2}{60^2-30^2} + 0.3] + \tfrac{30p}{105\times10^9}[\tfrac{30^2+20^2}{30^2-20^2} - 0.3], \qquad p = 53.3\ MPa$$

Steel, Eq.(8.18):

$$\sigma_{\theta,\max} = p\tfrac{c^2+b^2}{c^2-b^2} = 53.3\tfrac{60^2+30^2}{60^2-30^2} = 88.8\ MPa$$ ◀

Bronze, Eq.(8.19):

$$\sigma_{\theta,\max} = -2p\tfrac{b^2}{b^2-a^2} = -2(53.3)\tfrac{30^2}{30^2-20^2} = -191.9\ MPa$$ ◀

SOLUTION (8.29)

(a) Use Eq.(8.18) with $p_i = p, \quad a = b, \quad b = c$: (Fig. 8.6)

$$\sigma_{\theta,\max} = p\tfrac{c^2+b^2}{c^2-b^2} = 80\ MPa, \qquad \tfrac{c^2+b^2}{c^2-b^2} = \tfrac{80(10^6)}{p} \qquad \text{(a)}$$

By Eq.(8.22), with $a = 0, \quad b = 50, \quad \delta = 0.06\ mm$:

$$0.06 = \tfrac{50p}{100\times10^9}[\tfrac{80\times10^6}{p} + 0.3] + \tfrac{50p}{200\times10^9}[1-0.3]$$

or

$$60\times10^6 = 40\times10^6 + 0.15p + 0.175p, \qquad p = 61.5\ MPa$$ ◀

(b) Equation (a) becomes

$$\tfrac{80}{61.5} = \tfrac{c^2+0.005^2}{c^2-0.005^2}, \qquad c = 138\ mm \qquad 2c = 276\ mm$$ ◀

SOLUTION (8.30)

Equation (8.22) with a=0, b=50 mm, c=140 mm.

$$\delta = \tfrac{bp}{E_c}[\tfrac{b^2+c^2}{c^2-b^2} + \nu_c] + \tfrac{bp}{E_s}[1-\nu_s]$$

(CONT.)

$$0.04 = \frac{50p}{120\times10^9}\left[\frac{140^2+50^2}{140^2-50^2}+0.25\right]+\frac{50p}{210\times10^9}(0.7)$$

Solving $p = 49.4\ MPa$

Shaft: $\sigma_\theta = \sigma_r = -p = -49.4\ MPa$ ◄

Cylinder:

$$\sigma_{\theta,\max} = p\frac{c^2+b^2}{c^2-b^2} = 49.4\left[\frac{140^2+50^2}{140^2-50^2}\right] = 63.8\ MPa$$

$$\sigma_{r,\max} = -p = -49.4\ MPa$$ ◄

SOLUTION (8.31)

δ_o from Equation (8.22):

$$\tfrac{\lambda}{2} = (u_d)_{r=b} = \tfrac{bp}{E}\left[\tfrac{b^2+c^2}{c^2-b^2}+\nu\right] = \tfrac{bp}{E}\left(\tfrac{17}{15}+0.3\right) = 1.433\tfrac{bp}{E}$$

or $\quad p = \frac{E\lambda}{2.866b}$

Then, δ_i from Eq.(8.22) with a=0,

$$u_s = \tfrac{bp}{E}(1-\nu) = \tfrac{0.7\lambda}{2.866} = 0.244\lambda$$

Therefore,

$$\Delta d_s = 0.489\lambda$$ ◄

SOLUTION (8.32)

We have (Fig. 8.6):

$$a = 0.05\ m \qquad b = 0.1\ m \qquad c = 0.15\ m$$

Maximum tangential stress occurs at r=0.1 m in gear wheel:

$$\sigma_{\max} = p\frac{c^2+b^2}{c^2-b^2}$$

where, p is the internal pressure exerted by the contact surfaces. Substituting the given data into this equation, we have

$$0.21 = p\frac{0.15^2+0.1^2}{0.15^2-0.1^2}$$

Solving,

$$p = 0.081\ MPa = 81\ kPa$$

This interface pressure produces the torque at the contact surface.
Area of contact is

$$2\pi bL = 2\pi(0.1)(0.1) = 0.02\pi$$

The torque transmitted is thus,

$$T = [(0.2)(81\times10^3)(0.02\pi)](0.1)$$
$$= 101.79\ N\cdot m$$ ◄

SOLUTION (8.33)

From the torsion, formula, $\tau = Tb/(\pi b^3/2)$:

$$T = \frac{\pi\tau b^3}{2} = \frac{\pi(120)b^3}{2} = 188.5b^3$$

Also, Eq.(8.35b):

$$T = 2\pi b^2 fpl = 2\pi b^2 fp(3b) = 6\pi f b^3 p$$

And

$$6\pi f b^3 p = 188.5b^3, \qquad p = \frac{188.5}{6\pi(0.2)} = 50\ MPa$$

We have $E_h = E_s = E,\ \ \nu_h = \nu_s = \nu$, and a=0. Equation (8.22) becomes

$$\delta = \frac{2bpc^2}{E(c^2-b^2)} = \frac{2bp(4b^2)}{E(4b^2-b^2)} = \frac{8pb}{3E}$$

Hence

$$\delta = \frac{8(50)b}{3(200\times10^3)} = 0.6667(10^{-3})b \qquad ◄$$

SOLUTION (8.34)

(a) a=10 mm, b=40 mm

$$F_\theta = \frac{T}{a} = \frac{120}{0.01} = 12\ kN$$

Thus $\quad 12(10^3) = 2\pi afpt = 2\pi(0.01)(0.16)(0.04)p \qquad$ (Eq.8.35a)

or $\quad p = 29.84\ MPa$

Equation (8.22) gives then

$$\delta = \frac{10(20.84)}{100(10^3)}\left[\frac{50^2+10^2}{50^2-10^2}+0.3\right] + \frac{10(29.84)}{200\times10^3}(1-0.3) = (4.128+1.044)10^{-3} = 0.005\ mm \qquad ◄$$

(b) $\sigma_{\theta,\max} = p\frac{b^2+a^2}{b^2-a^2} = 29.84\left[\frac{50^2+10^2}{50^2-10^2}\right] = 32.3\ MPa \qquad ◄$

SOLUTION (8.35)

From Eq. (8.28a),

$$\frac{\partial\sigma_r}{\partial r} = \frac{3+\nu}{8}\rho\omega^2\left[b^2 + a^2 - \frac{\partial}{\partial r}(r^2) - a^2b^2\frac{\partial}{\partial r}\left(\frac{1}{r^2}\right)\right] = 0$$

or

$$\frac{2a^2b^2}{r^3} - 2r = 0; \qquad r = \sqrt{ab} \qquad ◄$$

This value of r is substituted into Eq. (8.28a) to yield

$$\sigma_{r,\max} = \rho\omega^2(b-a)^2$$

We also find from Eq. (8.28b), for r=a:

$$\sigma_{\theta,\max} = 2\frac{3+\nu}{8}\rho\omega^2\left(b^2 + \frac{1-\nu}{3+\nu}\right)$$

Thus,

$$\frac{\sigma_{\theta,\max}}{\sigma_{r,\max}} = \frac{2\left(b^2+\frac{1-\nu}{3+\nu}a^2\right)}{(b-a)^2} \qquad ◄$$

SOLUTION (8.36)

At r=0: $\sigma_\theta = \sigma_r = (3+\nu)b^2 \rho\omega^2/8$. Thus,

$$\sigma_\theta^2 - \cancel{\sigma_r}^{\,0}\sigma_\theta + \cancel{\sigma_r^2}^{\,0} = \sigma_{yp}^2$$

or $$\omega_{all} = \tfrac{1}{b}\sqrt{\tfrac{8\sigma_{yp}}{(3+\nu)\rho}} \qquad \text{(P8.36)}$$

$$= \tfrac{1}{0.125}\left[\tfrac{8(260\times10^6)}{10(2.7\times10^3)/3}\right]^{\frac{1}{2}} = 3845.9\ rad/\text{sec}$$

$$= 36{,}726\ rpm$$ ◄

SOLUTION (8.37)

We have at inner edge:

$$\tau_{\max} = \tfrac{\sigma_\theta - 0}{2} \quad \text{or} \quad \sigma_\theta = 2\sigma_\theta = 2(90) = 180\ MPa$$

Equation (8.28b) with $r = 0.03\ m$ is thus

$$180(10^6) = \tfrac{10/3}{8}[(0.03)^2 + (0.1) - \tfrac{1+1}{10/3}(0.03)^2 + (0.1)^2](7800)\omega^2$$

from which

$$\omega = 525.8\ rad/s = 5021\ rpm$$ ◄

SOLUTION (8.38)

(a) When $p = 0$, at interface,

$$u_d - u_s = 0.05(10^{-3}) \qquad \text{(a)}$$

Use Eq. (8.28b) with r=a:

$$(\sigma_\theta)_d = \tfrac{\rho\omega^2}{4}[(3+\nu)b^2 + (1-\nu)a^2]$$

$$= \tfrac{7.8(10^3)\omega^2}{4}[3.3(0.125^2) + 0.7(0.025^2)] = 101.473\omega^2$$

Radial displacement of disk (with $\sigma_r = 0$):

$$(u_d)_{r=a} = \tfrac{(\sigma_\theta)_d\, a}{E} = \tfrac{101.473\omega^2 a}{200(10^9)}$$

For shaft, using Eq. (8.30b) with r=a:

$$(\sigma_\theta)_s = \tfrac{3.3(0.025)^2}{8}(1 - \tfrac{1.9}{3.3})7.8(10^3)\omega^2 = 0.8531\omega^2$$

Radial displacement of shaft (with $\sigma_r = 0$):

$$(u_s)_{r=a} = \tfrac{(\sigma_\theta)_s\, a}{E} = \tfrac{0.8531\omega^2(0.025)}{200(10^6)}$$

Thus, Eq. (a) gives

$$\tfrac{101.473\omega^2(0.025)}{200(10^6)} - \tfrac{0.8531\omega^2(0.025)}{200(10^6)} = 0.05$$

or

$$\omega = 1{,}933.884\ rad/\text{sec} = 19{,}040\ rpm$$ ◄

(b) Maximum stress occurs in disk

$$(\sigma_\theta)_{\max} = 101.473(1{,}993.884)^2 = 403\ MPa$$ ◄

SOLUTION (8.39)

We have

$$a = 0.5c \qquad b = 2c \qquad r = \sqrt{ab} = c$$

(a) Equation (8.28a):

$$50(10^6) = \tfrac{3.3c^2}{8}(0.25 + 4 - 1 - \tfrac{1}{2})7.8(10^3)(5000\tfrac{2\pi}{60})^2$$

or

$$c = 0.1587\ m$$

Thus,

$$t_r = 1.5c = 238.1\ mm \qquad \blacktriangleleft$$

(b) Equation (8.28b) at r=a:

$$\sigma_{\theta,\max} = \tfrac{3.3c^2}{8}[0.25 + 4 - \tfrac{1.9}{3.3}(0.25) + 4]7.8(10^3)(5000\tfrac{2\pi}{60})^2$$
$$= 180.1\ MPa \qquad \blacktriangleleft$$

SOLUTION (8.40)

(a) Equation (8.23), with a=0:

$$p = \tfrac{210(10^9)(0.000075)}{2(0.075)}\tfrac{0.16-0.005625}{0.16} = 101.31\ MPa \qquad \blacktriangleleft$$

Then, Eq. (8.18) gives

$$\sigma_{\theta,\max} = 101.31(10^6)\tfrac{0.16+0.005625}{0.16-0.005625} = 108.68\ MPa \qquad \blacktriangleleft$$

(b) Applying Eq. (8.28c),

$$0.000075 = \tfrac{3.3(0.7)}{8(210\times10^9)}[0.005625 + 0.16 - \tfrac{1.3}{3.3}(0.005625)$$
$$+ \tfrac{1.3}{0.7}(0.16)](7.8\times10^3)(0.075)\omega^2$$

Solving,

$$\omega = 450\ rad/\sec = 4300\ rpm \qquad \blacktriangleleft$$

SOLUTION (8.41)

From Eqs. (8.30a) and (8.30b), for r=0 and $\nu = 1/3$:

$$\sigma_{\max} = \tfrac{3+\nu}{8}\rho(\omega b)^2 = \tfrac{5}{12}\rho v^2 \qquad \blacktriangleleft$$

SOLUTION (8.42)

Apply Eq. (8.28c) with $u = 0.04\ mm$ at $r = 25\ mm$.

Substituting the given data:

$$0.04(10^{-3}) = \tfrac{3.3(0.7)}{8(210\times10^9)}[(0.025)^2 + (0.25)^2 - \tfrac{1.3}{3.3}(0.025)^2$$
$$\tfrac{1.3}{0.7}(0.25)^2](7800)\omega^2(0.025)$$

Solving,

$$\omega = 913.1\ rad/s = 8719\ rpm \qquad \blacktriangleleft$$

SOLUTION (8.43)

We have

$$\frac{t_i}{t_o} = (\frac{b}{a})^s; \qquad \frac{0.125}{0.0625} = (\frac{0.625}{0.125})^s$$

Solving, s=0.431. Then,

$$m_{1,2} = -\frac{0.431}{2} \pm [(\frac{0.431}{2})^2 + (1 + 0.3 \times 0.431)]^{\frac{1}{2}}$$

or

$$m_1 = 0.869 \qquad m_2 = -1.3$$

Equation (8.39) gives

$$\sigma_r = \frac{c_1}{t_1} r^{0.3} + \frac{c_2}{t_1} r^{-1.869} - 0.50169\rho(\omega r)^2 \qquad \text{(a)}$$

Given conditions are:

$$(\sigma_r)_{r=0.625} = 0, \qquad (\sigma_r)_{r=0.125} = 0$$

Thus, substituting the given data into Eq. (a) and solving:

$$\frac{c_1}{t_1} = 0.2322\rho\omega^2, \qquad \frac{c_2}{t_1} = -0.0024\rho\omega^2$$

The second of Eqs. (8.39), at the bore r=0.125 m, leads to

$$140(10^6) = 0.2322(0.125^{0.3})(0.869)\rho\omega^2$$
$$+ (-0.0024)(-1.3)(0.125^{-1.869})\rho\omega^2 - \frac{(1+0.9)(0.125)^2}{8-(3.3)(0.431)}\rho\omega^2$$

or

$$\rho\omega^2 = 549(10^6)$$

It follows, from the second of Eqs. (8.39), that

$$(\sigma_\theta)_{r=0.625} = [0.2322(0.625^{0.3})(0.869)$$
$$- 0.0024(-1.3)(0.625^{-1.869}) - \frac{1.9(0.625)^2}{8-3.3(0.431)}](549 \times 10^6)$$
$$= 37.5 \ MPa$$

Circumferential force is thus

$$37{,}500(0.0625) = 2344 \ kN/m$$ ◄

SOLUTION (8.44)

(a) Uniform thickness

Centrifugal force due to the blades, $m\omega^2 r$, is

$$\frac{540}{9.81}(\frac{10{,}000\pi}{60})^2(0.575) = 8.677(10^6) \ N$$

The pressure at *b* is then

$$p_o = \frac{8.667(10^6)}{2\pi(0.5)(0.05)} = 55.24 \ MPa$$

We have

$$\rho\omega^2 = 7.8(10^{-3})(\frac{10{,}000\pi}{60})^2 = 2.1384(10^3)$$

The condition of zero pressure at the bore is satisfied by, using Eq. (8.27b):

$$(\sigma_r)_{r=a} = 0 = \frac{E}{1-\nu^2}[-\frac{(3+\nu)(1-\nu^2)\rho\omega^2 a^2}{8E} + (1+\nu)c_1 - (1-\nu)\frac{c_2}{a^2}]$$

(CONT.)

or $$0 = -\frac{(3+\nu)a^2\rho\omega^2}{8} + \underbrace{\left[\frac{E(1+\nu)}{1-\nu^2}c_1\right]}_{A_1} + \underbrace{\left[-\frac{E(1-\nu)}{1-\nu^2}c_2\right]}_{A_2}\frac{1}{a^2}$$

$$0 = A_1 + A_2\frac{1}{a^2} - \frac{(3+\nu)a^2\rho\omega^2}{8}$$

Substituting the data, we have

$$0 = A_1 + 256A_2 - 3.4456 \qquad \text{(a)}$$

The condition at the outer circumference is satisfied by:

$$(\sigma_r)_{r=b} = 55.24 = A_1 + \frac{A_2}{(0.5)^2} - \frac{3.3(0.5)^2(2138.4)}{8}$$

or

$$55.24 = A_1 + 4A_2 - 220.52 \qquad \text{(b)}$$

Solution of Eqs. (a) and (b) gives

$$A_1 = 280.08 \qquad A_2 = -1.08 \qquad \text{(c)}$$

Equation (8.27c) is also written as follows

$$\sigma_\theta = A_1 - A_2\frac{1}{r^2} - \frac{(1+3\nu)\rho\omega^2 r^2}{8} \qquad \text{(d)}$$

where A_1 and A_2 are given by (c). At r=a, we have from Eq. (d):

$$(\sigma_\theta)_{r=a} = 280.08 + 276.48 - 1.98$$

$$= 554.58\ MPa = \sigma_{\max} \qquad \blacktriangleleft$$

Similarly, Eq. (d) gives r=b:

$$(\sigma_\theta)_{r=b} = 280.08 + 4.32 - 126.97$$

$$= 157.43\ MPa$$

Note that $\sigma_{r,\max}$ occurs at $r = \sqrt{ab} = 0.1768\ m.$ Thus Eq. (8.27):

$$(\sigma_r)_{r=0.1768} = -27.67 + 280 - 34.55$$

$$= 217.88\ MPa$$

(b) Hyperbolic section

We have

$$\frac{0.4}{0.05} = \left(\frac{0.5}{0.0625}\right)^s; \qquad s = 1$$

Then,

$$m_{1,2} = -\frac{1}{2} \pm [0.25 + 1.3]^{\frac{1}{2}}$$

or

$$m_1 = 0.745 \qquad m_2 = 1.745$$

Letting

$$\frac{c_1}{t_1} = B_1 \qquad \frac{c_2}{t_1} = B_2$$

Eq. (8.39) becomes then

$$\sigma_r = B_1 r^{m_1+s-1} + B_2 r^{m_2+s-1} - \frac{(3+\nu)\rho\omega^2}{8-(3+\nu)s}r^2$$

(CONT.)

and $(\sigma_r)_{r=a} = 0$

$$= B_1(0.0625)^{0.745} + B_2(0.0625)^{-1.745} - 0.7021(0.0625)^2\rho\omega^2$$

or $0 = B_1 + 996.35B_2 - 0.0216\rho\omega^2$ (e)

Also

$$(\sigma_r)_{r=b} = 55.24 - B_1(0.5)^{0.745} + B_2(0.5)^{-1.745} - 0.702(0.5)^2\rho\omega^2$$

or $55.24 = 0.55967B_1 + 3.352B_2 - 0.1755\rho\omega^2$ (f)

Substituting the given data and solving Eqs. (e) and (f):

$B_1 = 725.4 \qquad B_2 = -0.6814$ (g)

The second of Eqs. (8.33),

$$\sigma_\theta = B_1 m_1 r^{m_1+s-1} + B_2 m_2 r^{m_2+s-1} - \tfrac{(1+3\nu)\rho\omega^2}{8-(3+\nu)s} r^2$$

gives at r=a:

$$(\sigma_\theta)_{r=a} = 725.14(0.745)(0.0625)^{0.745}$$
$$- 0.6814(-1.745)(0.0625)^{-1.745} - 0.4043(0.0625)^2\rho\omega^2$$

or

$$(\sigma_\theta)_{r=a} = 215.2\ MPa = \sigma_{\max}$$ ◀

Also similarly we obtain that

$$(\sigma_\theta)_{r=b} = 110.2\ MPa$$

(c) Uniform stress

Using Eq. (8.41),

$$\tfrac{t_o}{t_1} = e^{-(\rho\omega^2/2\sigma)b^2}$$

where,

$$-(\rho\omega^2/2\sigma)b^2 = \tfrac{-7.8\times10^3(523.59)^2(0.5)^2}{2(84)}$$
$$= -3.182$$

Thus,

$$t_o = 0.02425e^{-3.182} = 0.001\ m = 1\ mm$$

and

$$t = 0.02425e^{-12.728r^2}$$ ◀

SOLUTION (8.45)

Because u=0 at r=b, set $c_2 = 0$ in Eq. (8.43). Thus, Eq. (8.43) with $c_2 = 0$, T =constant, and u=0 at r=b:

$$0 = c_1 b + \tfrac{(1+\nu)}{b}\alpha T\int_0^b r dr \quad \text{or} \quad c_1 = -\tfrac{(1+\nu)\alpha T}{2}$$

(CONT.)

8.45 (CONT.)

Substituting this into Eq. (d) and (e) of Sec. 8.10, we obtain

$$\sigma_r = -\frac{\alpha ET}{2} - \frac{\alpha ET}{2(1-\nu^2)}(1+\nu)^2$$

$$\sigma_\theta = \frac{\alpha ET}{2} - \alpha ET - \frac{\alpha ET}{2(1-\nu^2)}(1+\nu)^2$$

Letting $1-\nu^2 = (1+\nu)(1-\nu)$ and simplifying we obtain

$$\sigma_r = \sigma_\theta = -\frac{\alpha ET}{1-\nu} \quad ◀$$

SOLUTION (8.46)

We substitute the given T into Eqs. (8.44) to obtain

$$\sigma_r = \frac{\alpha E(T_a - T_b)}{2\ln(b/a)}[-\ln\frac{b}{a} - \frac{a^2(r^2-b^2)}{r^2(b^2-a^2)}\ln\frac{b}{a}]$$

Then,

$$\frac{\partial\sigma_r}{\partial r} = 0 = \frac{E\alpha(T_a-T_b)}{2\ln(b/a)}\{\frac{d}{dr}(\ln\frac{b}{a}) - \frac{a^2}{b^2-a^2}[1-\frac{d}{dr}(\frac{b^2}{r^2})]\ln\frac{b}{a}\}$$

yields, after differentiation:

$$r = ab(\frac{2}{b^2-a^2}\ln\frac{b}{a})^{\frac{1}{2}} \quad ◀$$

It is noted that, Eq. (8.53) gives the same result.

SOLUTION (8.47)

Using Eq. P8.46, we have

$$r = 0.01(0.015)[\frac{2\ln(1.5)}{(0.015)^2-(0.01)^2}]^{\frac{1}{2}} = 12.1\ mm$$

Equation (8.53) are therefore

$$(\sigma_r)_{r=12.1} = \frac{10.4(10^{-6})(90\times10^9)(-8)}{2(0.7)\ln(1.5)}[-\ln\frac{15}{12.1} - \frac{100(12.1^2-15^2)}{12.1^2(15^2-10^2)}\ln(1.5)]$$

$$= 1.319(10^7)[0.041] = 0.541\ MPa$$

$$(\sigma_r)_{r=10} = 1.319(10^7)[1-\ln(1.5) - \frac{100(10^2+15^2)}{100(15^2-10^2)}\ln(1.5)]$$

$$= 6.045\ MPa$$

$$(\sigma_\theta)_{r=15} = 1.319(10^7)[1-\ln(1) - \frac{100(15^2+15^2)}{100(15^2-10^2)}\ln(1.5)]$$

$$= -4.64\ MPa$$

$$(\sigma_\theta)_{r=12.1} = 1.319(10^7)[1-\ln(\frac{15}{12.1}) - \frac{100(12.1^2+15^2)}{12.5^2(15^2-10^2)}\ln(1.5)]$$

$$= 0.488\ MPa$$

Similarly,

$$(\sigma_z)_{r=10} = -1.319(10^7)[1-2\ln(1.5) - \frac{2(10^2)}{15^2-10^2}\ln(1.5)]$$

$$= 6.055\ MPa = \sigma_{\max} \quad ◀$$

$$(\sigma_z)_{r=15} = -1.319(10^7)[1-2\ln(1) - \frac{2(10^2)}{15^2-10^2}\ln(1.5)]$$

$$= -4.64\ MPa$$

SOLUTION (8.48)

Introducing

$$T(r) = T_0\,(b - r)/b$$

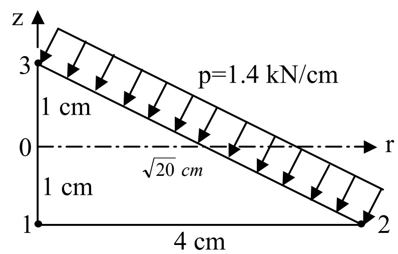
into Eqs. (8.45), after integration, we obtain the following expressions for stresses

$$\sigma_r = \tfrac{1}{3}T_0(\tfrac{r}{b} - 1)\alpha E$$
$$\sigma_\theta = \tfrac{1}{3}T_0(\tfrac{2r}{b} - 1)\alpha E$$

◄

From these we observe that, at r=b: the radial stress vanishes while tangential stress assumes its maximum value.

SOLUTION (8.49)



Equivalent nodal force matrix

We have

$$r_1 = 0 \qquad z_1 = -1 \qquad \bar{r} = 4/3$$
$$r_2 = 4 \qquad z_2 = -1 \qquad \bar{z} = -1/3$$
$$r_3 = 0 \qquad z_3 = 1$$

Weight of the element is

$$2\pi\bar{r}A(0.077) = 2\pi(4/3)(4)(0.077) = 2.58\ N$$

Body forces:

$$\{Q\}_e = \{Q_{r1},\, Q_{r2},\, Q_{r3},\, Q_{z1},\, Q_{z2},\, Q_{z3}\}$$

Therefore,

$$\{Q\}_e^b = \{0,\ 0,\ 0,\ -0.86,\ -0.86,\ -0.86\}\ N$$

Surface forces:

$$q_2 = q_3 = \tfrac{1400\sqrt{20}}{2} = 3130.4\ N/cm$$
$$q_{r2} = q_{r3} = -3130.4(\tfrac{2}{\sqrt{20}}) = -1400\ N/cm$$
$$q_{z2} = q_{z3} = -3130.4(\tfrac{4}{\sqrt{20}}) = -2800\ N/cm$$

Since

$$Q_r = 2\pi\bar{r}q_r \qquad Q_z = 2\pi\bar{r}q_z$$

Then,

$$Q_{r2} = Q_{r3} = -11.729\ kN$$
$$Q_{z2} = Q_{z3} = -23.457\ kN$$

(CONT.)

Therefore,

$$\{Q\}_e^p = \{0,\ -11.729,\ -11.729,\ 0,\ -23.457,\ -23.457\}\ kN$$

Thermal forces:

$$\{Q\}_e^t = 2\pi \bar{r} A[\overline{B}]^T [D][B]\{\varepsilon_0\}$$

Here,

$$[D] = 38.46(10^6)\begin{bmatrix} 0.3 & 0.7 & 0.3 & 0 \\ 0.7 & 0.3 & 0.3 & 0 \\ 0.3 & 0.3 & 0.7 & 0 \\ 0 & 0 & 0 & 0.2 \end{bmatrix}$$

We also have

$$a_i = a_1 = 4 \qquad b_1 = -2$$
$$a_j = a_2 = 0 \qquad b_2 = 2$$
$$a_m = a_3 = 4 \qquad b_3 = 0$$
$$c_1 = -4 \qquad d_1 = \tfrac{a_1}{\bar{r}} + b_1 + \tfrac{c_1 \bar{z}}{\bar{r}} = 2$$
$$c_2 = 0 \qquad d_2 = 2$$
$$c_3 = 4 \qquad d_3 = 2$$

Therefore,

$$[B] = \frac{1}{8}\begin{bmatrix} -2 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -4 & 0 & 4 \\ 2 & 2 & 2 & 0 & 0 & 0 \\ -4 & 0 & 4 & -2 & 2 & 0 \end{bmatrix}$$

It follows that

$$\{Q\}_e^t = 2\pi(\tfrac{4}{3})4[\overline{B}]^T [D][\overline{B}]\begin{Bmatrix} 0.0006 \\ 0.0006 \\ 0.0006 \\ 0 \end{Bmatrix}$$
$$= \{0,\ 502.3,\ 251.2,\ -502.3,\ 0,\ 502.3\}\ kN$$

Equivalent nodal force matrix is thus,

$$\{Q\}_e = \{Q\}_e^b + \{Q\}_e^p + \{Q\}_e^t$$

where $\{Q\}_e^b$, $\{Q\}_e^p$, $\{Q\}_e^t$ are already obtained.

Element stiffness matrix:

$$[k]_e = 2\pi \bar{r} A[\overline{B}]^T [D][\overline{B}] = 2\pi(\tfrac{4}{3})4[\overline{B}]^T [D][\overline{B}]$$

(CONT.)

Substituting the values of $[D]$ and $[\overline{B}]$, after carrying out the matrix matrix multiplications, we obtain:

$$[k]_e = \begin{bmatrix} 4.8 & 1.6 & -1.6 & 4.8 & -1.6 & -3.2 \\ 1.6 & 6.4 & 4 & -8 & 0 & 8 \\ -1.6 & 4 & 6 & -4 & 1.6 & 2.4 \\ 4.8 & -8 & -4 & 5.6 & -0.8 & -4.8 \\ -1.6 & 0 & 1.6 & -0.8 & 0.8 & 0 \\ -3.2 & 8 & 2.4 & -4.8 & 0 & 4.8 \end{bmatrix} (20.14 \times 10^6)$$

Element governing equation

$$\{Q\}_e = [k]_e \{\delta\}_e \tag{a}$$

Here,

$$(\delta)_e = \{u_1, u_2, u_3, w_1, w_2, w_3\}$$

When boundary conditions are given, $(\delta)_e$ is computed from Eq. (a), as illustrated in Chap. 7.

SOLUTION (8.50 through 8.53)

Owing to the symmetry, only any one-quarter of the cylinder need be analyzed.
Use a finite element computer program, such as ANSYS.

End of Chapter 8

CHAPTER 9

SOLUTION (**9.1**)

Using Eq. (9.3),

$$\beta = \left(\tfrac{k}{4EI}\right)^{\frac{1}{4}} = \left[\tfrac{1.4(10^6)}{4(200\times10^9)(5.04\times10^{-6})}\right]^{\frac{1}{4}}$$

$$= 0.7676\ m^{-1}$$

Equation (9.8) yields, for $x = 0$:

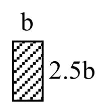
$$M_{\max} = \tfrac{-P}{4\beta} f_3(\beta x) = -\tfrac{P}{4\beta} f_3(0) = -\tfrac{P}{4\beta}$$

Thus,

$$P = \tfrac{\sigma_{\max} I(4\beta)}{c} = \tfrac{210\times10^6(5.04\times10^{-6})4(0.7676)}{0.0635}$$

$$= 51.18\ kN \qquad ◄$$

SOLUTION (9.2)



$$I = b(2.5b)^3/12 \ = 1.302b^4$$

$$M_{\max} = \tfrac{\sigma_{\max} I}{c} = \tfrac{250\times10^6(1.302b^4)}{1.25b} = 260.4(10^6)b^3 \qquad \text{(a)}$$

$$\beta = \left[\tfrac{20(10^6)}{4(200\times10^9)(1.302b^4)}\right]^{\frac{1}{4}} = 0.0662/b$$

Equation (9.8):

$$M_{\max} = \tfrac{P}{4\beta} \qquad \text{(b)}$$

From Eqs. (a) and (b),

$$260.4(10^6)b^3 = \tfrac{40(10^3)b}{4(0.0662)}$$

or

$$b = 0.0241\ m = 24.1\ mm \qquad ◄$$

SOLUTION (9.3)

Select a particular solution of the form

$$v_p = a\sin\tfrac{2\pi x}{L} \qquad a = \text{const.}$$

Introduce this into Eq. (9.1),

$$\left(\tfrac{2\pi}{L}\right)^4 a + \tfrac{k}{EI} a = \tfrac{p_1}{EI}$$

or

$$a = \frac{p_1}{k + \frac{16\pi^4 EI}{L^4}} = \frac{p_1}{k[1+4(\pi/\beta L)^4]}$$

Thus,

$$v = \frac{p_1}{k + \frac{16\pi^4 EI}{L^4}} \sin\tfrac{2\pi x}{L} \qquad \text{(a)}$$

General solution is

$$v = e^{\beta x}[A\cos\beta x + B\sin\beta x] + e^{-\beta x}[c\cos\beta x + D\sin\beta x] + v_p$$

(CONT.)

9.3 (CONT.)

Boundary conditions are:

$$v(\infty) = 0; \qquad A = B = 0$$

Loading repeats itself periodically, $p_1 \sin(2\pi x/L)$. But

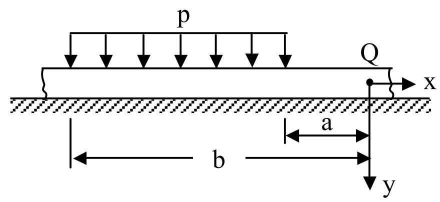
$$v = e^{-\beta x}[C\cos\beta x + D\sin\beta x]$$

cannot repeat periodically and represents a damped wave. Thus, in order deflection to repeat itself with the same wave length as loading it is required that $C = D = 0$. Accordingly, solution is

$$v = v_c + v_p = 0 + v_p = v_p$$

given by Eq. (a).

SOLUTION (9.4)



From Example 9.1:

$$v_Q = \int_0^b \tfrac{pdx}{2k}\beta e^{-\beta x}[\cos\beta x + \sin\beta x] - \int_0^a \tfrac{pdx}{2k}\beta e^{-\beta x}[\cos\beta x + \sin\beta x]$$

$$= \tfrac{p}{2k}[e^{-\beta x}\cos\beta a - e^{-\beta b}\cos\beta x]$$

Note that, if a=0 and b=L (large):

$$v_Q \approx \tfrac{p}{2k}$$

When *a* and *b* increase (large):

$$v_Q \to 0 \qquad \text{(a)}$$

While according to the result of Example 9.1, when *a*=0 and *b*=L (large):

$$v_Q \approx \tfrac{p}{2k}$$

When *a* and *b* increase (large);

$$v_Q \approx \tfrac{p}{2k} \qquad \text{(b)}$$

the answers differ, as observed by comparing Eqs. (a) and (b).

SOLUTION (9.5)

Applying Eqs. (9.3) and (9.8) we obtain

$$\beta = [\tfrac{k}{4EI}]^{\frac{1}{4}} = [\tfrac{16.8}{4(8.437)}]^{\frac{1}{4}} = 0.84\ m^{-1}$$

$$v(0) = v_{\max} = \tfrac{P\beta}{2k}$$

$$= \tfrac{0.135(0.84)}{2(16.8)} = 3.375(10^{-3})\ m \qquad \blacktriangleleft$$

and

$$M(0) = M_{\max} = \tfrac{P}{4\beta} = \tfrac{135}{4(0.84)} = 40.179\ kN \cdot m$$

Maximum stress is therefore

$$\sigma_{\max} = \tfrac{M_{\max}}{S} = \tfrac{40.179(10^3)}{3.9(10^{-4})} = 103.02\ MPa \qquad \blacktriangleleft$$

SOLUTION (9.6)

Refer to Solution of Prob. 9.1:

$$\beta = \left(\frac{k}{4EI}\right)^{1/4} = \left[\frac{12(10^6)}{4(200\times10^9)(5.04\times10^{-6})}\right]^{1/4}$$
$$= 1.3135\ m^{-1}$$

Equation (9.8) gives, for $x = 0$:

$$M_{\max} = -\frac{P}{4\beta}$$

Therefore

$$P = \frac{\sigma_{\max} I(4\beta)}{c} = \frac{(210/2.5)10^6(5.04\times10^{-6})4(1.3135)}{0.0635}$$
$$= 35\ kN$$

SOLUTION (9.7)

$$I = b^4/12$$

$$M_{\max} = \frac{\sigma_{\max} I}{c} = \frac{(260/1.8)10^6(b^4/12)}{b/2} \qquad \text{(a)}$$
$$= 24.074(10^6)b^3$$

$$\beta = \left[\frac{7(10^6)}{4(70\times10^9)(b^4/12)}\right]^{1/4} = 0.1316/b$$

Equation (9.8):

$$M_{\max} = \frac{P}{4\beta} \qquad \text{(b)}$$

From Eqs. (a) and (b),

$$24.074(10^6)b^3 = \frac{50(10)^3 b}{4(0.1316)}$$

Solving,

$$b = 0.0628\ m = 62.8\ mm$$

◄

SOLUTION (9.8)

We have

$$\beta = \left[\frac{k}{4EI}\right]^{1/4} = \left[\frac{15}{4(8.437)}\right]^{1/4} = 0.8165\ m^{-1}$$

$$v(0) = v_{\max} = \frac{P\beta}{2k}$$
$$= \frac{0.25(0.8165)}{2(15)} = 6.8(10^{-3})\ m = 6.8\ mm$$

(CONT.)

9.8 (CONT.)

And

$$M(0) = M_{\max} = \frac{P}{4\beta} = \frac{250}{4(0.8165)} = 76.55\ kN \cdot m$$

Maximum stress is thus

$$\sigma_{\max} = \frac{M_{\max}}{S} = \frac{76.55(10^3)}{3.9} = 196.3\ MPa \qquad ◄$$

SOLUTION (9.9)

Equations (9.3) and (9.8):

$$\beta = [\tfrac{k}{4EI}]^{\frac{1}{4}}, \qquad v_{\max} = \tfrac{P\beta}{2k}, \qquad M_{\max} = \tfrac{P}{4\beta}$$

(a)

k is 1.25 times the actual value.

β changes by $\sqrt[4]{1.25} = 1.057.$

$v_{\max}$ changes by $1.057/1.25 = 0.846.$

$\sigma_{\max}$ (or $M_{\max}$) changes by $1/1.057 = 0.946.$

(b)

k is 1.4 times the actual value.

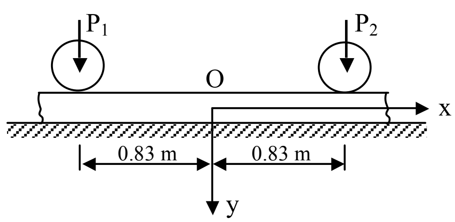
β changes by $\sqrt[4]{1.4} = 1.088.$

$v_{\max}$ changes by $1.088/1.4 = 0.777.$

$\sigma_{\max}$ (or $M_{\max}$) changes by $1/1.088 = 0.919.$

Note that stress calculation is not affected appreciably in both cases.

SOLUTION (9.10)



By using the principle of superposition, deflection of any point, say O, of rail is expressed as the algebraic sum of v_1 and v_2 caused by P_1 and P_2, respectively. Thus, from Eq. (9.8), we have

$$v_0 = \tfrac{P_1\beta}{2k} f_1(\beta x_1) + \tfrac{P_2\beta}{2k} f_1(\beta x_2) \qquad \text{(a)}$$

where, $P_1 = P_2 = P$, $\beta = 0.84\ m^{-1}$. Then, $f_1(\beta x_1)$ and $f_1(\beta x_2)$ are found from Table 9.1 for $x_1 = 0.83\ m$ and $x_2 = -0.83\ m$. We have $\beta x_1 = 0.697$ and $\beta x_2 = -0.697$. Thus

$$f_1(\beta x_1) = 0.702.$$

(CONT.)

Since, from symmetry $f_1(\beta x_1)$ has the same value for a (+) or (-) value of βx, Eq. (a) may be written as

$$v_0 = \tfrac{P\beta}{2k}(0.702 + 0.702) = 1.404\tfrac{P\beta}{2k} \qquad ◄$$

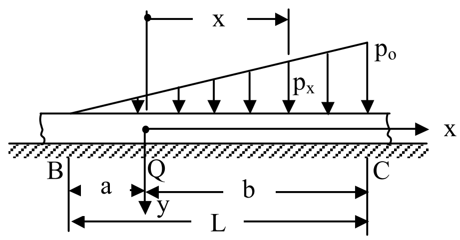
Resultant bending moment at O, from Eq. (9.8), is

$$M_0 = \tfrac{P}{4\beta} f_3(\beta x_1) + \tfrac{P}{4\beta} f_3(\beta x_2)$$

$$= \tfrac{P}{4\beta}[2 f_3(0.697)] = 0.125\tfrac{P}{4\beta} \qquad ◄$$

It may be verified by comparing the results of Problems 9.5 and 9.7 that addition of one or more load (reduces appreciably value of maximum moment) causes a large increase in the maximum deflection of the rail.

SOLUTION (9.11)



Expression for loading are:

$$p_x = \tfrac{p_0}{L}(a - x) \qquad \text{(segment BQ)}$$

$$P_x = \tfrac{p_0}{L}(a + x) \qquad \text{(segment QC)}$$

Deflection at Q is obtained by substituting ($p_x dx$) for P in Eq. (9.6). That is

$$v_Q = \tfrac{p_0\beta}{2kL}\{\int_0^a (a - x)e^{-\beta x}[\cos\beta x + \sin\beta x]dx$$

$$+\int_0^b (a + x)e^{\beta x}[\cos\beta x + \sin\beta x]dx\}$$

Integrating, we have

$$v_Q = \tfrac{p_0}{4\beta k}[f_3(\beta a) - f_3(\beta b) - 2\beta L f_4(\beta b) + 4\beta a]$$

SOLUTION (9.12)

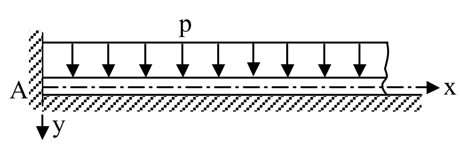


Figure (a)

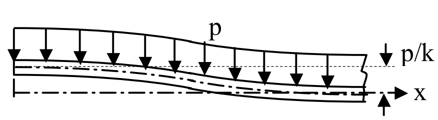


Figure (b)

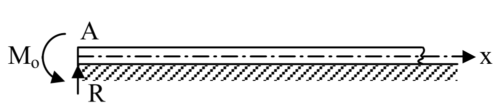


Figure (c)

(CONT.)

Referring to Fig. (b):

$$v_{A1} = -\tfrac{p}{k} \qquad \theta_{A1} = 0$$

Referring to Fig. (c) and using Eq. (9.12),

$$v_{A2} = \tfrac{2\beta}{k}[-Rf_4(0) + \beta M_0 f_3(0)]$$

$$\theta_{A2} = -\tfrac{2\beta^2}{k}[-Rf_1(0) + 2\beta M_0 f_4(0)]$$

The problem may be separated into two different cases both semi-infinite beams (as is seen in the figure above) provided that deflection at A is zero. Thus,

$$v_{A1} - v_{A2} = 0; \qquad -\tfrac{p}{k} = \tfrac{2\beta}{k}[-R + \beta M_0]$$

$$\theta_{A1} - \theta_{A2} = 0; \qquad \tfrac{2\beta^2}{k}[-R + 2\beta M_0]$$

solving,

$$M_0 = 2\beta^2 EI\tfrac{p}{k}, \qquad R = 4\beta^2 EI\tfrac{p}{k} = \tfrac{p}{\beta} \qquad ◄$$

SOLUTION (9.13)

(a) Using Eq. (9.12), $v = \tfrac{2\beta^2}{k} M_A f_3(\beta x)$

$$v_{\max} = \tfrac{2\beta^2 M_A}{k} \qquad \text{at } x = 0 \qquad \text{(down)}$$

Scan Table 9.1, $f_3(\beta x) = -0.2079$ at $\beta x = \frac{\pi}{2}$, and

$$v_{\min} = \tfrac{2\beta^2}{k} M_A(-0.2079) \qquad \text{(up)}$$

Therefore

$$\tfrac{v_{\max}}{v_{\min}} = \tfrac{1}{-0.2079} = -4.81 \qquad ◄$$

(b) Applying Eq. (9.12), $M = M_A f_1(\beta x)$

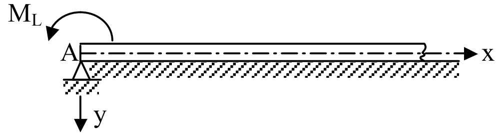
$$v_{\max} = M_A \qquad \text{at } x = 0$$

Scan Table 9.1, $f_1(\beta x) = -0.0432$ at $\beta x = \pi$, and

$$M_{\min} = -0.0432 M_A$$

Thus $\quad \tfrac{M_{\max}}{M_{\min}} = \tfrac{1}{-0.0432} = -23.15 \qquad ◄$

SOLUTION (9.14)



Equation (9.11) with $v(0) = 0$ gives

$$0 = 2\beta(-R_A + \beta M_L)/k$$

(CONT.)

from which $R_A = M_L$. When R_A is substituted for $-P$ into Eq. (9.12):

$$v = -\frac{M_L}{2\beta^2 EI} e^{-\beta} \sin \beta x = -\frac{M_L}{2\beta^2 EI} f_2(\beta x)$$

Successive differentiations of this expression give:

$$v' = -\frac{M_L}{2\beta EI} f_3(\beta x)$$

$$M = \frac{M_L}{2EI} f_4(\beta x) \qquad \blacktriangleleft$$

$$V = -\frac{M_L}{2EI} f_1(\beta x)$$

SOLUTION (9.15)

We have $k = \frac{K_s}{a} = \frac{180}{0.625} = 288 \ kPa$

$$= [\frac{288(10^3)}{4(200\times10^9)(5.04\times10^6)}]^{\frac{1}{4}} = 0.51697 \ m^{-1}$$

Using Eq. (9.8),

$$v(0) = v_{\max} = \frac{P\beta}{2k} = \frac{6.75(0.5697)}{2(288)} = 5.61 \ mm \qquad \blacktriangleleft$$

$$M(0) = M_{\max} = \frac{P}{4\beta} = \frac{6.75}{4(0.51697)} = 3.26421 \ kN \cdot m$$

Thus, $\sigma_{\max} = \frac{Mc}{I} = \frac{3264.21(0.0625)}{5.04\times10^{-6}} = 41.13 \ MPa$ ◀

SOLUTION (9.16)

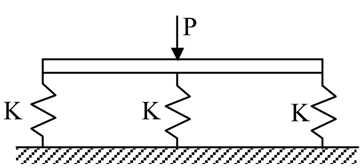
From Eq. (9.3),

$$\beta = [\frac{18(10^3)}{4(0.375\times200\times10^9)(5.2\times10^{-7})}]^{\frac{1}{4}} = 0.824 \ m^{-1}$$

Since $\beta L = 0.824(0.74) = 0.618 < \pi/4$ the beam can be considered rigid.

(a) Uniform deflection is

$$v = \frac{F}{3K} = \frac{540}{3(18)} = 10 \ mm \qquad \blacktriangleleft$$



(b) We may replace the given beam by the beams shown in Figs. (a) and (b).

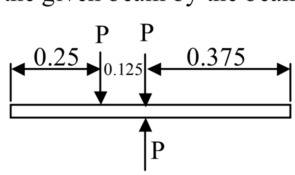


Figure (a)

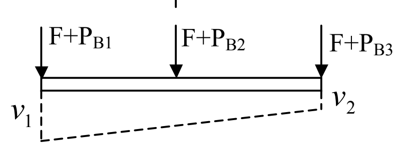


Figure (b)

(CONT.)

9.16 (CONT.)

Note that F is the direct load on each spring and P_{Bi} is the bending loads with the values:

$$F = P/3 = 540/3 = 180 \ N$$

and

$$P_{Bi} = \frac{M_B r_i}{\sum r_i^2} \qquad (i = 1, 2, 3)$$

Here

$$P_{B1} = \frac{540(0.125)(0.375)}{2(15\times 0.125)^2} = 90 \ N, \ P_{B2} = 0, \quad P_{B3} = -P_{B1}$$

End deflections are thus,

$$v_1 = (180 + 90)/18 = 15 \ mm$$
$$v_2 = (180 - 90)/18 = 5 \ mm \qquad \blacktriangleleft$$

SOLUTION (9.17)

Using Eq. (9.3),

$$\beta = [\tfrac{14}{4(8.4)}]^{\frac{1}{4}} = 0.8034 \ m^{-1}$$

$$\beta L = 0.8034(0.6) = 0.482 \ rad = 27.62^o$$

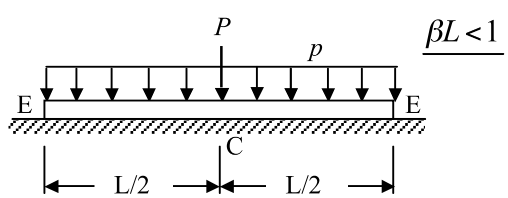
Hence, from Eq. (9.13):

$$v_c = \frac{4500(0.8034)}{2(14\times 10^6)} \frac{2+\cos 27.62^o + \cosh 27.62^o}{\sin 27.62^o + \sinh 27.62^o}$$

$$= 186(10^{-6}) \ m = 0.186 \ mm \qquad \blacktriangleleft$$

Note: Slope may be found as in Prob. 9.15, with p=0.

SOLUTION (9.18)



Referring to Sec. 9.8 and Table D.4

$$v_c = \frac{PL^3}{48EI} + \frac{5pL^4}{384EI} - \frac{5[(P+pL)/L]L^4}{384EI}$$

$$= \frac{(0.15)^3}{48(4.8\times 10^{-6})}[9000 + \frac{5(7500\times 0.15)}{8} - \frac{5(9000+1125)}{8}]$$

$$= 2.81(10^{-8}) \ m \qquad \blacktriangleleft$$

Similarly,

$$\theta_E = \frac{PL^2}{16EI} + \frac{pL^3}{24EI} - \frac{[(9000+1125)/0.15](0.15)^3}{24EI}$$

Substituting the given data,

$$\theta_E = 5(10^{-7}) \ rad \qquad \blacktriangleleft$$

SOLUTION (9.19)

Equation (9.3),

$$\beta = [\frac{k}{4EI}]^{1/4} = [\frac{8}{4(10)}]^{1/4} = 0.6687\ m^{-1}$$

$$\beta L = 0.6687(0.8) = 0.535\ rad = 30.65^o$$

From Eq. (9.13):

$$v_c = \frac{15(10^3)(0.6687)}{2(8\times 10^6)}\frac{2+\cos 30.65^o + \cosh 0.535}{\sin 30.65^o + \sinh 0.535}$$

$$= 629.9[18.4208](10^{-6})$$

$$= 5,279(10^{-6})\ m = 5.28\ mm$$

◄

SOLUTION (9.20)

Refer to Sec. 9.8 and Table D.4, we have $I = bh^3/12 = 46.8(10^{-6})\ m^4$, $EI = 3.4\ MN\cdot m^2$, and $\beta L < 1$.

$$v_c = \frac{PL^3}{48EI} + \frac{5pL^4}{384EI} - \frac{5[(P+pL)/L]L^4}{384EI}$$

$$= \frac{(0.4)^3}{48(3.4\times 10^6)}[8000 + \frac{5(7000\times 0.4)}{8} - \frac{5(8000+2800)}{8}]$$

$$= 0.00276(10^{-3})\ m = 2.8\ mm$$

Likewise,

$$\theta_E = \frac{PL^2}{16EI} + \frac{pL^3}{24EI} - \frac{[(8000+2800)/0.4](0.4)^3}{24EI}$$

$$= \frac{(0.4)^2}{16(3.4\times 10^{-6})}[8000 + \frac{7000(0.4)}{3/2} - \frac{10,800}{3/2}]$$

$$= 0.29(10^{-3})\ rad$$

SOLUTION (9.21)

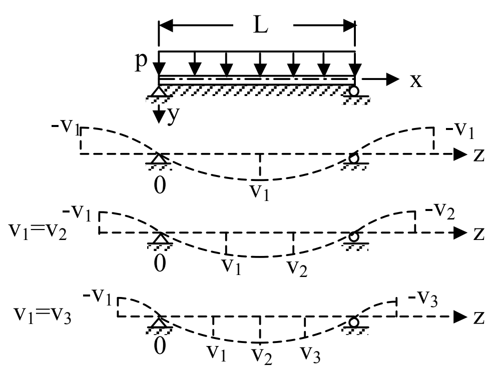


Figure (a)

Figure (b)

Figure (c)

Figure (d)

Boundary conditions are:

$$v(0) = v(L) = 0, \qquad v''(0) = v''(L) = 0 \tag{a}$$

(CONT.)

9.21 (CONT.)

These are transformed into the following central difference conditions by using Eq. (7.4) and (7.7):

$$v_0 = 0, \qquad -v_{-1} = v_1, \qquad v_n = 0, \qquad v_{n-1} = -v_{n+1} \tag{b}$$

<u>For m=2 (Fig. b)</u>:

$$v_{n-2} - 4v_{n-1} + 6(\tfrac{m^4+1}{m^4})v_n - 4v_{n+1} + v_{n+2} = \tfrac{1.6}{m^4}$$

or

$$-v_1 + 6\tfrac{2^4+1}{2^4}v_1 - v_1 = \tfrac{1.6}{2^4}; \qquad v_1 = 21.4\ mm \qquad \blacktriangleleft$$

<u>For m=3 (Fig. c)</u>:

$$-4v_1 + 6\tfrac{3^4+1}{3^4}v_2 - v_2 = \tfrac{1.6}{3^4}$$

$$-v_1 + 6\tfrac{3^4+1}{3^4}v_1 - 4v_2 = \tfrac{1.6}{3^4}$$

Solving,

$$v_1 = v_2 = 17.2\ mm \qquad \blacktriangleleft$$

<u>For m=4 (Fig. d)</u>:

$$-v_1 + 6\tfrac{4^4+1}{4^4}v_1 - 4v_1 + v_1 = \tfrac{1.6}{4^4}$$

$$-4v_1 + 6\tfrac{4^4+1}{4^4}v_2 - 4v_1 = \tfrac{1.6}{4^4}$$

Solving,

$$v_3 = v_1 = 10.2\ mm$$

$$v_2 = 13.9\ mm \qquad \blacktriangleleft$$

SOLUTION (9.22)

From Example 9.1:

$$v_p = \tfrac{p}{2k}(2 - e^{-\beta a}\cos\beta a - e^{-\beta a}\cos\beta b)$$

where $k = \frac{48EI}{aL_t^3}$

Thus, $$v_p = \tfrac{paL_t^3}{2(48)EI}[2 - e^{-\beta a}\cos\beta a - e^{-\beta b}\cos\beta b)$$

$$= \tfrac{paL_t^3}{96EI}[2 - f_4(\beta a) - f_4(\beta b)]$$

We have $\beta = 3.936/6 = 0.656\ m^{-1}$

At midspan, we have $\beta a = \beta b$ and $f_4(\beta a) \approx 0$

Thus, $v_p = \frac{paL_t^3}{48EI}$

Since, $v_m = \frac{paL_t^3}{48EI} = \frac{R_{cc}L_t^3}{48EI}$

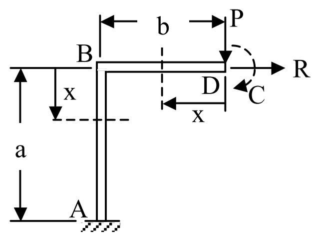
Solving,

$$R_{cc} = ap = 0.3p \qquad \blacktriangleleft$$

End of Chapter 9

CHAPTER 10

SOLUTION (10.1)



<u>Defections at point D</u>:
We write

$$M_{DB} = Px \qquad M_{BA} = Pb + Rx$$

Using Eq. (10.5),

$$U_{DB} = \int_0^b [\tfrac{(Px)^2}{2EI} + \tfrac{R^2}{2AE}]dx = \tfrac{P^2b^3}{6EI} + \tfrac{R^2b}{2AE}$$

$$U_{BA} = \int_0^a [\tfrac{(Pb+Rx)^2}{2EI} + \tfrac{P^2}{2AE}]dx$$

$$= \tfrac{1}{2EI}[p^2ab^2 + PRa^2b + \tfrac{1}{3}R^2a^3] + \tfrac{P^2a}{2AE}$$

Total strain energy $U = U_{BD} + U_{BA}$. Vertical deflection at D is thus

$$\delta_v = \tfrac{\partial U}{\partial P} = \tfrac{1}{EI}[\tfrac{Ra^2b}{2} + P(ab^2 + \tfrac{b^3}{3})] + \tfrac{Pa}{AE} \qquad \blacktriangleleft$$

Horizontal deflection at D:

$$\delta_h = \tfrac{\partial U}{\partial R} = \tfrac{1}{EI}[\tfrac{Ra^2b}{2} + \tfrac{Ra^3}{3}] + \tfrac{Rb}{AE} \qquad \blacktriangleleft$$

<u>Angular rotation of D</u>:
Introduce a couple moment C at D as shown in the figure. Then,

$$M_{DB} = Px + C \qquad M_{BA} = Pb + Rx + C$$

and

$$U_{DB} = \int_0^b [\tfrac{(Px+C)^2}{2EI} + \tfrac{R^2}{2AE}]dx$$

$$= \tfrac{1}{2EI}[\tfrac{P^2b^3}{3} + C^2b + PCb^2] + \tfrac{R^2b}{2AE}$$

$$U_{BA} = \tfrac{1}{2EI}\int_0^a [Pb + Rx + C]^2 dx + \tfrac{R^2b}{2AE}$$

$$= \tfrac{1}{2EI}[P^2ab^2 + \tfrac{R^2a^3}{3} + Ca^2 + PRa^2b + 2PCab + CRa^2] + \tfrac{P^2a}{2AE}$$

Hence,

$$\theta_D = \tfrac{\partial U}{\partial C}\Big|_{C=0} = \tfrac{1}{2EI}[Pb^2 + 2Pab + Ra^2] \qquad \blacktriangleleft$$

Note that displacements of a simple (straight) cantilever beam may be readily found by setting b=0 in the foregoing results.

SOLUTION (10.2)

Equations of statics are applied to obtain the reactions:

$$\sum F_x = 0: \qquad R_{Ax} = 0$$
$$\sum M_x = 0: \qquad R_C = \tfrac{1}{4}P_1 + \tfrac{1}{2}P_2 + \tfrac{3}{4}P_3$$
$$\sum M_C = 0: \qquad R_{Ay} = \tfrac{3}{4}P_1 + \tfrac{1}{2}P_2 + \tfrac{1}{4}P_3$$

The axial force in each member is obtained by applying the method sections, as required. The results are as follows:

$$\begin{aligned}
N_{DC} &= \tfrac{\sqrt{2}}{4}P_1 + \tfrac{\sqrt{2}}{2}P_2 + \tfrac{3\sqrt{2}}{4}P_3 \\
N_{AE} &= \tfrac{3\sqrt{2}}{4}P_1 + \tfrac{\sqrt{2}}{2}P_2 + \tfrac{\sqrt{2}}{4}P_3 \\
N_{DE} &= \tfrac{1}{2}P_1 + P_2 + \tfrac{1}{2}P_3 \\
N_{BE} &= -\tfrac{\sqrt{2}}{4}P_1 + \tfrac{\sqrt{2}}{2}P_2 + \tfrac{\sqrt{2}}{4}P_3 \\
N_{BD} &= \tfrac{\sqrt{2}}{4}P_1 + \tfrac{\sqrt{2}}{2}P_2 - \tfrac{\sqrt{2}}{4}P_3 \\
N_{AB} &= \tfrac{3}{4}P_1 + \tfrac{1}{2}P_2 + \tfrac{1}{4}P_3 \\
N_{BC} &= \tfrac{1}{4}P_1 + \tfrac{1}{2}P_2 + \tfrac{3}{4}P_3
\end{aligned}$$

Numerical results are determined for $P_1 = P_2 = P_3 = 45\ kN$ and $L = 3\ m$. These are tabulated bellow.

Bar	*Axial force* (kN)	$\partial N/\partial P_2$	*Length* (m)
DC	$N_{DC} = 99.5$	$\sqrt{2}/2$	2.125
AE	$N_{AE} = 99.5$	$\sqrt{2}/2$	2.125
DE	$N_{DE} = 90$	1	3.000
BE	$N_{BE} = 31.8$	$\sqrt{2}/2$	2.125
BD	$N_{BD} = 31.8$	$\sqrt{2}/2$	2.125
AB	$N_{AB} = 67.5$	$1/2$	3.000
BC	$N_{BC} = 67.5$	$1/2$	3.000

Thus, the vertical deflection at B:

$$\begin{aligned}
\delta_B &= \tfrac{1}{AE}\sum_{j=1}^{7} N_j \tfrac{\partial N_j}{\partial P_2} L_j \\
&= \tfrac{10^3}{AE}[2(67.5)(0.5)(3) + 2(31.8)(0.707)(2.125) + 90(1)(3) + \\
&\qquad 2(99.5)(0.707)2.125] \\
&= 686{,}773.675/AE\ \ m
\end{aligned}$$

◄

SOLUTION (10.3)

The moment is expressed by

$$M = P(2a - x); \qquad \partial M/\partial P = 2a - x$$

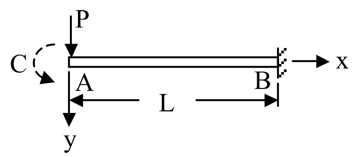
Castigliano's theorem gives then

$$v_P = \int \frac{M}{EI}\frac{\partial M}{\partial P}dx = \int_0^a \frac{P}{E}(2a - x)^2(c_1 x + c_2)dx + \int_0^{2a} \frac{P}{EI_2}(2a - x)^2 dx$$

or

$$v_P = \frac{11Pc_1a^4}{12E} + \frac{7Pc_2a^3}{3E} + \frac{Pa^3}{3EI_2} \qquad \blacktriangleleft$$

SOLUTION (10.4)



<u>Deflection at A</u> (with $M = Px$):

$$v_A = \frac{1}{EI}\int M \frac{\partial M}{\partial P}dx = \frac{1}{EI}\int_0^L Px(x)dx$$

$$v_A = \frac{PL^3}{3EI} \qquad \blacktriangleleft$$

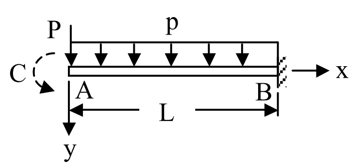
<u>Slope at A</u> (with $M = Px + C$):

$$\theta_A = \frac{1}{EI}\int M \frac{\partial M}{\partial C}dx = \frac{1}{EI}\int_0^L (Px + C)(1)dx$$

Setting $C = 0$ and integrating

$$\theta_A = \frac{PL^2}{2EI} \qquad \blacktriangleleft$$

SOLUTION (10.5)



<u>Deflection at A</u> (with $M = Px + \frac{1}{2}px^2$):

$$\delta_A = \frac{1}{EI}\int M \frac{\partial M}{\partial P}dx = \frac{1}{EI}\int_0^L (Px + \tfrac{1}{2}px^2)(x)dx$$

$$= \frac{PL^3}{3EI} + \frac{pL^4}{8EI} \qquad \blacktriangleleft$$

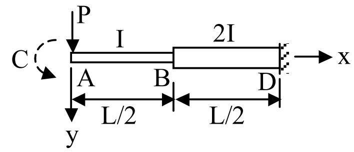
<u>Slope at A</u> (with $M = Px + \frac{1}{2}px^2 + C$):

$$\theta_A = \frac{1}{EI}\int M \frac{\partial M}{\partial C}dx = \frac{1}{EI}\int_0^L (Px + \tfrac{1}{2}px^2 + C)dx$$

Setting $C = 0$ and integrating

$$\theta_A = \frac{PL^2}{2EI} + \frac{pL^3}{6EI} \qquad \blacktriangleleft$$

SOLUTION (10.6)



Deflection at A (with $M = Px$):

$$\delta_A = \tfrac{1}{EI}\int M \tfrac{\partial M}{\partial P}\,dx = \tfrac{1}{EI}\int_0^{L/2}(Px)(x)dx + \tfrac{1}{2EI}\int_{L/2}^{L}(Px)(x)dx$$

Integrating,

$$\delta_A = \tfrac{PL^3}{24EI} + \tfrac{7PL^3}{48EI} = \tfrac{3PL^3}{16EI} \qquad \blacktriangleleft$$

Slope at A (with $M = Px + C$):

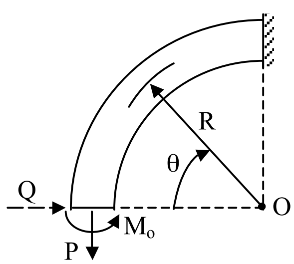
$$\theta_A = \tfrac{1}{EI}\int M \tfrac{\partial M}{\partial C}\,dx = \tfrac{1}{EI}\int_0^{L/2}(Px + C)(1)dx + \tfrac{1}{2EI}\int_{L/2}^{L}(Px + C)dx$$

Setting $C = 0$ and integrating

$$\theta_A = \tfrac{PL^2}{8EI} + \tfrac{3PL^2}{16EI}$$

or $$\theta_A = \tfrac{5PL^2}{16EI} \qquad \blacktriangleleft$$

SOLUTION (10.7)



ds=Rdθ

(a) Horizontal deflection

$$M_\theta = PR(1 - \cos\theta) + QR\sin\theta + M_0$$

and $$\delta_h = \tfrac{1}{EI}\int M_\theta \tfrac{\partial M_\theta}{\partial Q}\,ds$$

$$= \tfrac{1}{EI}\int_0^{\pi/2}[PR^3(1 - \cos\theta)\sin\theta + \cancelto{0}{Q}R^3\sin^2\theta + M_0R^2\sin\theta]d\theta$$

$$= \tfrac{R^2}{2EI}(\tfrac{1}{2}PR + M_0) \qquad \blacktriangleleft$$

Vertical deflection (Q=0)

$$\delta_v = \tfrac{1}{EI}\int M_\theta \tfrac{\partial M_\theta}{\partial P}\,ds$$

$$= \tfrac{1}{EI}\int_0^{\pi/2}[PR^3(1 - \cos\theta)^2 + MR^2(1 - \cos\theta)]d\theta$$

$$= \tfrac{R^2}{4EI}[PR(3\pi - 8) + 2M_0(\pi - 2)] \qquad \blacktriangleleft$$

(b) Rotation of the free end (Q=0)

$$\theta = \tfrac{1}{EI}\int M_\theta \tfrac{\partial M_\theta}{\partial M_0}\,ds = \tfrac{1}{EI}\int_0^{\pi/2}[PR(1 - \cos\theta) + M_0]d\theta$$

$$= \tfrac{R}{2EI}[PR(\pi - 2) + \pi M_0] \qquad \blacktriangleleft$$

SOLUTION (10.8)

We now have

$$M = -FR\sin\theta, \qquad N = F\sin\theta, \qquad V = -F\cos\theta$$

Applying Eq. (10.6), with $\alpha = 6/5$ and $T = 0$:

$$\delta_h = \tfrac{R^3}{EI}\int_0^\pi F\sin^2\theta d\theta + \tfrac{R}{AE}\int_0^\pi F\sin^2\theta d\theta + \tfrac{6R}{5AG}\int_0^\pi F\cos^2\theta d\theta$$

Integrating,

$$\delta_h = \tfrac{\pi FR^3}{2EI} + \tfrac{\pi FR}{2EA} + \tfrac{3\pi FR}{5AG} \qquad \blacktriangleleft$$

Substituting the given data:

$$\delta_h = \tfrac{\pi(4000)(0.05)^3}{400(6.67)} + \tfrac{\pi(4000)(0.05)}{400(10^5)2} + \tfrac{3\pi(4000)(0.05)}{5(80\times10^5)2}$$

$$= (0.59 + 0.008 + 0.02)10^{-3}$$

$$= 0.62\ mm \qquad \blacktriangleleft$$

The error, if N and V are omitted, is 4.5 %.

SOLUTION (10.9)

Introducing a rightward horizontal force Q at point D, we write

$$M_1 = -Qx \qquad 0 \le x \le a$$

$$M_2 = -\tfrac{1}{2}Fx - Qa \qquad 0 \le x \le \tfrac{b}{2}$$

$$M_3 = -\tfrac{1}{2}Fx - Qa + F(x - \tfrac{b}{2}) \qquad \tfrac{b}{2} \le x \le b$$

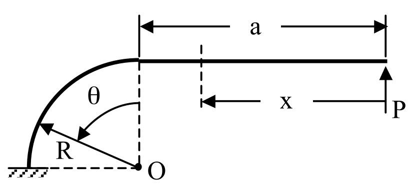
$$M_4 = -\tfrac{1}{2}Fb + \tfrac{1}{2}Fb - Q(a - x) \qquad 0 \le x \le a$$

Applying Castigliano's theorem, after setting Q=0, we have

$$\delta_h = \tfrac{1}{EI}\int_0^{b/2}\tfrac{1}{2}Faxdx + \tfrac{1}{EI}\int_{b/2}^{b}(-\tfrac{1}{2}Fax + \tfrac{1}{2}Fab)dx$$

Integrating, $\delta_h = \tfrac{Fab^2}{8EI}$ $\blacktriangleleft$

SOLUTION (10.10)



From symmetry,

$$M_1 = -Px \qquad 0 \le x \le a$$

$$M_2 = -Pa - PR\sin\theta \qquad 0 \le \theta \le \tfrac{\pi}{2}$$

Hence, $\tfrac{\delta}{2} = \tfrac{1}{EI}\int_0^a M_1 \tfrac{\partial M_1}{\partial P}dx + \tfrac{1}{EI}\int_0^{\pi/2} M_2 \tfrac{\partial M_2}{\partial P}Rd\theta$

or

$$\delta = \tfrac{2}{EI}[\int_0^a Px^2dx + \int_0^{\pi/2} P(a + R\sin\theta)^2 Rd\theta]$$

$$= \tfrac{P}{6EI}(4a^3 + 6\pi Ra^2 + 24R^2a + 3\pi R^3) \qquad \blacktriangleleft$$

SOLUTION (10.11)

Referring to Fig. P10.11, we write

$$M = PR\sin\theta \qquad T = PR(1-\cos\theta)$$
$$m = R\sin\theta \qquad t = R(1-\cos\theta)$$

Here t denotes torque caused by a unit load.

Applying Eq. (10.13), deflection at the free end (perpendicular to the plane of the ring):

$$\delta = \tfrac{1}{EI}\int_0^{\pi/2}(PR^2\sin^2\theta)Rd\theta + \tfrac{1}{JG}\int_0^{\pi/2}[PR^2(1-\cos\theta)^2]Rd\theta$$
$$= \tfrac{PR^3}{EI}\left|\tfrac{\theta}{2} - \tfrac{\sin 2\theta}{4}\right|_0^{\pi/2} + \tfrac{PR^3}{JG}\left|\theta - 2\sin\theta + \tfrac{\theta}{2} + \tfrac{\sin 2\theta}{4}\right|_0^{\pi/2}$$
$$= \tfrac{PR^3}{4}\left(\tfrac{\pi}{EI} + \tfrac{3\pi-8}{JG}\right)$$

Letting $J = 2I = \pi r^4/2$, we have

$$\delta = \tfrac{PR^3}{r^4}\left(\tfrac{1}{E} + \tfrac{0.226}{G}\right) \qquad \blacktriangleleft$$

SOLUTION (10.12)

Referring to Fig. P10.12:

$$M = -PR\sin\theta \qquad m = -R\sin\theta$$
$$T = PR(1-\cos\theta) \qquad t = R(1-\cos\theta)$$

where t denotes torque caused by a unit load. We have

$$\sin(2\pi-\theta) = -\sin\theta \quad \text{and} \quad \cos(2\pi-\theta) = \cos\theta$$

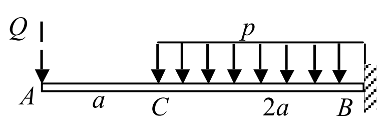
Applying Eq. (10.13), deflection at the free end (perpendicular to the plane of the ring):

$$\delta = \tfrac{PR^3}{EI}\int_0^{2\pi}\sin^2\theta d\theta + \tfrac{PR^3}{JG}\int_0^{2\pi}(1-\cos\theta)^2 d\theta$$

Integrating,

$$\delta = PR^3\pi\left[\tfrac{1}{EI} + \tfrac{3}{JG}\right] \qquad \blacktriangleleft$$

SOLUTION (10.13)



<u>Segment *AC*</u>

$$M_1 = -Qx \qquad \partial M_1/\partial Q = -x$$

<u>Segment *BC*</u>

$$M_2 = -Qx - \frac{p}{2}(x-a)^2 \qquad \frac{\partial M_2}{\partial Q} = -x$$

Thus,

$$v_A = \int_0^a \frac{M_1}{EI}\frac{\partial M_1}{\partial Q}dx + \int_a^{3a}\frac{M_2}{EI}\frac{\partial M_2}{\partial Q}dx$$

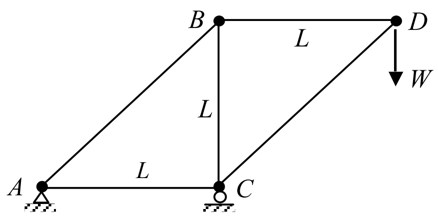
Let Q=0. Hence

(CONT.)

10.13 (CONT.)

$$v_A = \frac{p}{2EI}\int_a^{3a}(x-a)^2 x dx = \frac{p}{2EI}\int_a^{3a}(x^3 - 2ax^2 + a^2 x)dx$$
$$= \frac{p}{2EI}\left|\frac{x^4}{4} - \frac{2ax^3}{3} + \frac{a^2x^2}{2}\right|_a^{3a}$$
$$= \frac{10}{3}\frac{pa^4}{EI} \downarrow \qquad \blacktriangleleft$$

SOLUTION (10.14)



$$L_{AB} = L_{CD} = \sqrt{2}L$$
$$L_{BC} = L_{AC} = L_{BD} = L$$

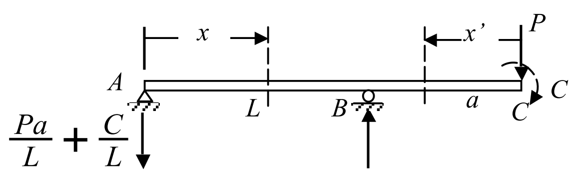
The method of joints(or sections) is applied:

$N_{AB} = \sqrt{2}W \qquad N_{BC} = -W \qquad N_{CD} = -\sqrt{2}W \qquad N_{BD} = W \qquad N_{AC} = -W$

Thus,

$$\delta_V = \frac{1}{AE}\sum N_j \frac{\partial N_j}{\partial W} L_j$$
$$= \frac{WL}{AE}[\sqrt{2}(\sqrt{2})\sqrt{2} + 1 + 1 + 1 + \sqrt{2}(\sqrt{2})\sqrt{2}] = 8.657\frac{WL}{AE} \downarrow \qquad \blacktriangleleft$$

SOLUTION (10.15)



Segment *AB*

$$M_1 = -P\frac{a}{L}x - \frac{C}{L}x$$

Segment *BC*

$$M_2 = -Px' - C$$

(**a**) For this case C=0:

$$v_C = \frac{1}{EI}\int_0^L(-\frac{Pax}{L})(-\frac{ax}{L})dx + \frac{1}{EI}\int_0^a -Px'(-x')dx'$$
$$= \frac{Pa^2}{EIL^2}\left|\frac{x^3}{3}\right|_0^L + \frac{P}{EI}\left|\frac{x'^3}{3}\right|_0^a = \frac{Pa^2}{3EI}(L+a) \downarrow \qquad \blacktriangleleft$$

(CONT.)

(**b**) $$\frac{\partial M_1}{\partial C} = -\frac{x}{L}, \qquad \frac{\partial M_2}{\partial C} = -1.$$

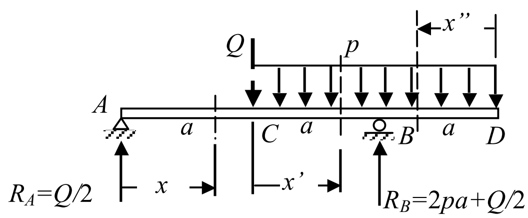
For C=0, we have :

$$\theta_C = \frac{1}{EI}\int_0^L (-\frac{Pax}{L})(-\frac{x}{L})dx + \frac{1}{EI}\int_0^a -Px'(-x')dx'$$

$$= \frac{Pa}{EIL^2}\left|\frac{x^3}{3}\right|_0^L + \frac{P}{EI}\left|\frac{x'^2}{2}\right|_0^a$$

$$= \frac{Pa}{6EI}(2L+3a)$$ ◀

SOLUTION (10.16)



<u>Segment *AC*</u>

$$M_1 = \frac{1}{2}Qx \qquad \frac{\partial M_1}{\partial Q} = \frac{x}{2}$$

Let Q=0:

$$\frac{1}{EI}\int_0^a M_1 \frac{\partial M_1}{\partial Q}dx = 0$$

<u>Segment *CB*</u>

$$M_2 = \frac{1}{2}Q(a+x') - Qx' - \frac{1}{2}px'^2$$

$$= \frac{Q}{2}(a-x') - \frac{1}{2}px'^2$$

$$\frac{\partial M_2}{\partial Q} = \frac{1}{2}(a-x')$$

Let Q=0:

$$\frac{1}{EI}\int M_2 \frac{\partial M_2}{\partial Q}dx' = \frac{1}{EI}\int_0^a (-\frac{px'^2}{2})\frac{1}{2}(a-x')dx' = -\frac{pa^4}{48EI}$$

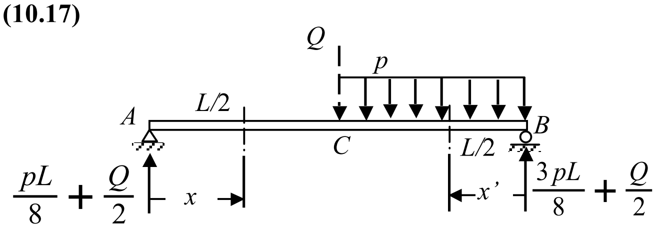
<u>Segment *BD*</u>

$$M_3 = -\frac{1}{2}px''^2 \qquad \frac{\partial M_3}{\partial Q} = 0$$

Therefore

$$v_C = \int M_i \frac{\partial M_i}{\partial Q}dx = \frac{pa^4}{48EI} \uparrow$$ ◀

SOLUTION (10.17)



Segment *AC*

$$M_1 = (\frac{pL}{8} + \frac{Q}{2})x \qquad \frac{\partial M_1}{\partial Q} = \frac{x}{2}$$

Segment *CB*

$$M_2 = (\frac{3pL}{8} + \frac{Q}{2})x' - \frac{px'^2}{2} \qquad \frac{\partial M_2}{\partial Q} = \frac{x'}{2}$$

Let Q=0:

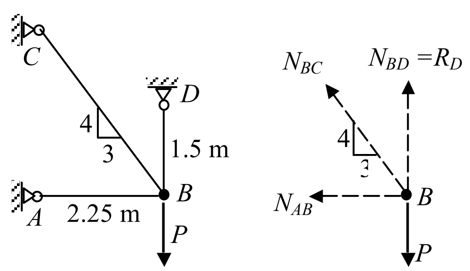
$$EIv_C = \int_0^{L/2} \frac{pLx}{8}\frac{x}{2}dx + \int_0^{L/2} (\frac{3pLx'}{8} - \frac{px'^2}{2})\frac{x'}{2}dx'$$

$$= p\int_0^{L/2} \frac{pLx^2}{16}dx + p\int_0^{L/2} (\frac{3pLx'^2}{16} - \frac{px'^3}{4})dx' = \frac{pL^4}{96}$$

or $\qquad v_C = \dfrac{pL^4}{96EI} \downarrow$ ◄

SOLUTION (10.18)

Consider R_D as redundant.



$$\frac{\partial N_{BD}}{\partial R_D} = 1$$

$$\sum F_y = 0: \quad N_{BC} = 1.25P - 1.25R_D$$

$$\sum F_x = 0: \quad N_{AB} = -0.75P + 0.75R_D$$

where $\quad \partial N_{BC}/\partial R_D = -1.25 \qquad \partial N_{AB}/\partial R_D = 0.75$

Thus,

$$AE\delta_D = \sum N_j L_j \frac{\partial N_j}{\partial R_D} = 0$$

$$= (1.25P - 1.25R_D)(7.5)(-1.25)$$

$$+(-0.75P + 0.75R_D)(4.5)(0.75) + R_D(3)1 = 0$$

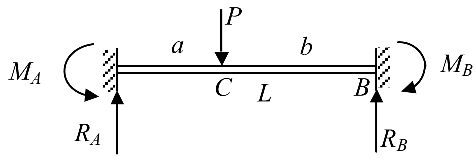
$$= -14.25P + 17.25R_D = 0$$

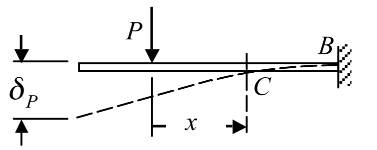
or $\qquad R_D = 0.826P$

Then $\qquad N_{BD} = 0.826P \qquad N_{AB} = -0.131P \qquad N_{BC} = 0.22P$ ◄

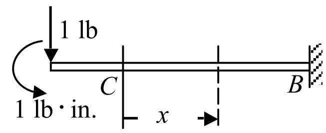
SOLUTION (10.19)

Let R_A and M_A be redundant.





$$M = -Px$$



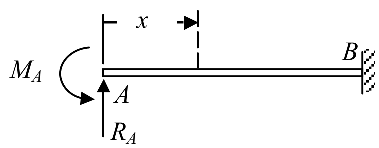
$$m = -1(a + x) \qquad m' = -1$$

$$v_A = \int \frac{Mm}{EI} dx \qquad \theta_A = \int \frac{Mm'}{EI} dx$$

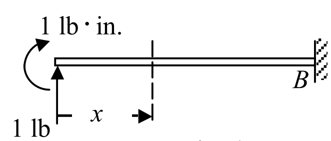
$$v_{AP} = \frac{1}{EI}\int_0^b (-Px)[-(a + x)]dx = \frac{Pb^2}{EI}(\frac{a}{2} + \frac{b}{3})$$

$$\theta_{AP} = \frac{1}{EI}\int_0^b (-Px)(-1)dx = \frac{Pb^2}{2EI}$$

Let now R_A and M_A are applied.



$$M = R_A x - M_A$$



$$m = x \qquad m' = 1$$

$$v_{AR} = \int_0^L \frac{(R_A x - M_A)x}{EI} dx = \frac{1}{EI}(\frac{R_A L^3}{3} - \frac{M_A L^2}{2})$$

$$\theta_{AR} = \int_0^L \frac{(R_A x - M_A)1}{EI} dx = \frac{1}{EI}(\frac{R_A L^2}{2} - M_A L)$$

Since $\quad v_{AP} = v_{AR} \qquad \theta_{AP} = \theta_{AR}$

or

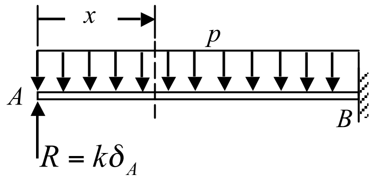
$$\frac{Pb^2}{EI}(\frac{a}{2} + \frac{b}{3}) = \frac{1}{EI}(\frac{R_A L^3}{3} - \frac{M_A L^2}{2})$$

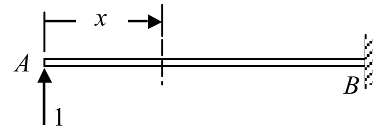
$$\frac{Pb^2}{2EI} = \frac{1}{EI}(\frac{R_A L^2}{3} - M_A L)$$

Solving,

$$R_A = \frac{Pb^2}{L^3}(3a + b) \uparrow \qquad M_A = \frac{Pab^2}{L^2} \circlearrowleft$$ ◄

SOLUTION (10.20)





Refer to the above figures, we write:

$$M = Rx - \frac{1}{2}px^2 \qquad m = -1x$$

Hence, deflection spring at A:

$$\delta_A = \int \frac{Mm}{EI}dx = \frac{R}{k}$$

$$= \frac{1}{EI}\int_0^L (Rx - \frac{1}{2}px^2)(-x)dx = \frac{R}{k}$$

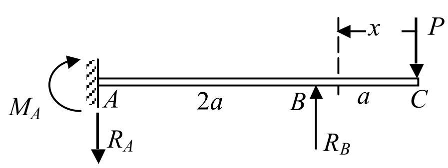
Solving

$$R = \frac{3pL/8}{1+(3EI/kL^3)} \qquad ◀$$

Reactions at B may then be found from statics.

SOLUTION (10.21)

Consider R_B as redundant.



Segment BC:

$$M_1 = -Px \qquad \partial M_1/\partial R_B = 0$$

Segment AB:

$$M_2 = -Px + R_B(x-a) \qquad \partial M_2/\partial R_B = x - a$$

Thus,

$$EIv_B = 0 = \int_0^a (-Px)(0)dx + \int_a^{3a}[-Px + R_B(x-a)](x-a)dx$$

$$= \int_a^{3a}[-Px^2 + R_Bx^2 - 2R_Bax + Pax + R_Ba^2]dx$$

$$= \left| -\frac{Px^3}{3} + \frac{R_Bx^3}{3} - R_Bax^2 + \frac{Pax^2}{2} + R_Bxa^2 \right|_a^{3a}$$

$$= -\frac{14}{3}P + \frac{3}{8}R_B$$

(CONT.)

from which

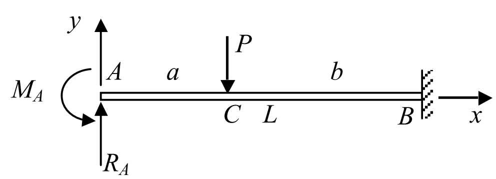
$$R_B = \frac{7}{4}P \uparrow \qquad \blacktriangleleft$$

Statics:

$$M_A = \frac{1}{2}Pa \qquad R_A = \frac{3}{4}P \downarrow \qquad \blacktriangleleft$$

SOLUTION (10.22)

Consider R_A and M_A as redundants.



<u>Segment *AC*</u>:

$$M_1 = R_A x - M_A \qquad \frac{\partial M_1}{\partial R_A} = x \qquad \frac{\partial M_1}{\partial M_A} = -1$$

<u>Segment *BC*</u>:

$$M_2 = R_A x - M_A - P(x-a)$$

$$\partial M_2 / \partial R_A = x \qquad \partial M_2 / \partial M_A = -1$$

Thus,

$$EIv_A = 0 = \int_0^a (R_A x - M_A)x dx + \int_a^L [R_A x - M_A - P(x-a)]x dx$$

$$= R_A \frac{a^3}{3} - M_A \frac{a^2}{2} + R_A \frac{L^3 - a^3}{3} - M_A \frac{L^2 - a^2}{2} - P\frac{L^3 - a^3}{3}$$

$$+ Pa\frac{L^2 - a^2}{2} = 0$$

Simplifying,

$$\frac{1}{3}R_A L^3 - \frac{1}{2}M_A L^2 - \frac{P}{6}(3a+2b)b^2 = 0 \qquad (1)$$

Similarly

$$EI\theta_A = 0 = \int_0^a (R_A x - M_A)(-1)dx + \int_a^L [(R_A x - M_A - P(x-a)](-1)dx$$

$$= -R_A \frac{a^2}{2} + M_A a - R_A \frac{L^2 - a^2}{2} + M_A(L-a) + P\frac{L^2 - a^2}{2}$$

$$-Pa(L-a) = 0$$

This reduces to

$$\frac{1}{2}R_A L^2 - M_A L - \frac{1}{2}Pb^2 = 0 \qquad (2)$$

(CONT.)

10.22 (CONT.)

Solving Eqs.(1) and (2):

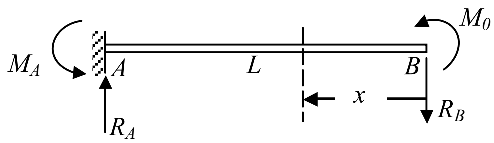
$$M_A = \frac{Pab^2}{L^2} \circlearrowleft \qquad R_A = \frac{Pb^2}{L^3}(3a+b) \uparrow \qquad \blacktriangleleft$$

Statics:

$$R_B = \frac{Pa^2}{L^3}(a+3b) \uparrow \qquad M_B = \frac{Pa^2 b}{L^2} \circlearrowright \qquad \blacktriangleleft$$

SOLUTION (10.23)

Consider R_B as redundant.



$$M = -R_B x + M_0$$
$$\partial M / \partial R_B = -x$$

Therefore

$$EIv_B = 0 = \int_0^L (R_B x^2 - M_0 x)dx = \frac{1}{3}R_B L^3 - \frac{1}{2}M_0 L^2$$

Solving

$$R_B = \frac{3}{2}\frac{M_0}{L} \downarrow \qquad \blacktriangleleft$$

Statics:

$$R_A = \frac{3}{2}\frac{M_0}{L} \uparrow \qquad M_A = \frac{1}{2}M_0 \circlearrowleft \qquad \blacktriangleleft$$

SOLUTION (10.24)

We introduce a couple moment C at point B. The expression for the moments are then,

$$M_1 = -R_D x \qquad 0 \le x \le \tfrac{L}{2}$$
$$M_2 = -R_D x + P(x - \tfrac{L}{2}) + C \qquad \tfrac{L}{2} \le x \le L$$

Applying Eq. (10.6), $\delta_B = \partial U / \partial P$:

$$\delta_B = \tfrac{1}{EI}\{\int_{L/2}^{L}[-R_D x + P(x - \tfrac{L}{2}) + C](x - \tfrac{L}{2})dx\}$$

Setting C=0 and integrating,

$$\delta_B = -\tfrac{PL^3}{48EI}[\tfrac{25/16}{1-(3EI/kL^3)} - 2] \qquad \blacktriangleleft$$

Similarly, applying Eq. (10.7), $\theta_B = \partial U / \partial C$:

$$\theta_B = \tfrac{1}{EI}\{\int_{L/2}^{L}[-R_D x + P(x - \tfrac{L}{2}) + C]dx\}$$

Setting C=0 and integrating,

$$\theta_B = -\tfrac{15PL^2}{128EI}\tfrac{1}{1-(3EI/kL^3)} \qquad \blacktriangleleft$$

SOLUTION (10.25)

Apply Eq. (10.7) to write

$$\theta_A = \frac{1}{JG}\left[\int_0^a T_A \frac{\partial T_A}{\partial T_A}dx + \int_0^b (T_A - T)\frac{\partial(T_A - T)}{\partial T_A}\right] = 0$$

Integrating and simplifying:

$$T_A a + (T_A - T)b = 0 \qquad \text{or} \qquad T_A = \frac{b}{L}T \qquad \blacktriangleleft$$

Condition of equilibrium gives

$$T_B = T - T_A = \frac{a}{L}T \qquad \blacktriangleleft$$

SOLUTION (10.26)

We write

$$M_1 = Qx \qquad 0 \le x \le a$$

$$M_2 = Q(a + R\sin\theta) + PR(1-\cos\theta) \qquad 0 \le \theta \le \pi$$

Applying Eq. (10.6), with $\delta_Q = 0$:

$$\frac{\partial U}{\partial Q} = 0 = \frac{1}{EI}\int_0^a Qx^2 dx + \int_0^\pi [Q(a + R\sin\theta) + PR(1-\cos\theta)](a + R\sin\theta)Rd\theta$$

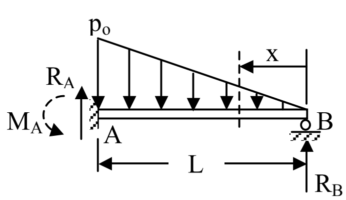
Integrating,

$$0 = \frac{QL^3}{3EI} + \frac{QR}{EI}(\pi a^2 + 4Ra + \frac{\pi}{2}R^2) + \frac{PR^2}{EI}(\pi a + 2R)$$

This may be written in the following form:

$$Q = \frac{-PR^3(\pi a + 2R)}{a^3[\frac{1}{3} + \frac{\pi R}{a} + 4(\frac{R}{a})^2 + \frac{\pi}{2}(\frac{R}{a})^3]} \qquad \blacktriangleleft$$

SOLUTION (10.27)



We have

$$M = -R_B x + \frac{p_0 x}{L}\frac{x}{2}\frac{x}{3} = -R_B x + \frac{p_0 x^3}{6}$$

Then,

$$\delta_B = 0 = \frac{1}{EI}\int M\frac{\partial M}{\partial R_B}dx = \frac{1}{EI}\int_0^L (R_B x - \frac{p_0 x^3}{6L})xdx$$

This yields, after integrating,

$$R_B = \frac{p_0 L}{10} \qquad \blacktriangleleft$$

Then, from equations of statics:

$$R_A = \frac{4p_0 L}{10} \qquad M_A = \frac{p_0 L^2}{15} \qquad \blacktriangleleft$$

SOLUTION (10.28)

Moments are expressed by

$$M_1 = -Rx \qquad 0 \le x \le \tfrac{L}{2}$$

$$M_2 = -Rx + M_0 \qquad \tfrac{L}{2} \le x \le L$$

Applying Eq. (10.6), with $\delta_R = 0$:

$$\tfrac{\partial U}{\partial R} = 0 = \tfrac{1}{EI}\int_0^{L/2} Rx^2 dx + \tfrac{1}{EI}\int_{L/2}^{L} (-Rx + M_0)(-x)dx$$

from which

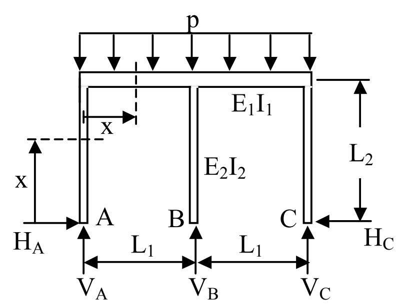
$$R = \tfrac{9M_0}{8L}$$ ◄

Using Eq. (10.7), slope at C:

$$\theta_C = \tfrac{\partial U}{\partial M} = 0 + \tfrac{1}{EI}\int_{L/2}^{L} (-Rx + M_0)dx$$

$$= \tfrac{5M_0 L}{64EI}$$ ◄

SOLUTION (10.29)



$$M_1 = H_A x \qquad 0 \le x \le L_2$$

$$M_2 = -V_A x + H_A L_2 + \tfrac{1}{2} px^2 \qquad 0 \le x \le L_1$$

Applying Eq. (10.6), with $\delta_{Av} = 0$ and $\delta_{Ah} = 0$, respectively:

$$\tfrac{\partial U}{\partial V_A} = \tfrac{1}{E_1I_1}\int_0^{L_1} (-V_A x + H_A L_2 + \tfrac{px^2}{2})(-x)dx = 0 \qquad \text{(a)}$$

$$\tfrac{\partial U}{\partial H_A} = \tfrac{1}{E_2I_2}\int_0^{L^2} H_A x(x)dx + \tfrac{1}{E_1I_1}\int_0^{L_1} (-V_A x + H_A L_2 + \tfrac{1}{2} px^2)L_2 dx = 0 \qquad \text{(b)}$$

Letting

$$\lambda = \tfrac{E_2I_2L_1}{E_1I_1L_2}$$

Equations (a) and (b) become

$$8V_A - 12H_A L_2 = 3pL_1^2$$

$$3\lambda V_A L_1 - 2(3\lambda + 1)H_A L_2 = \lambda p L_1^2$$

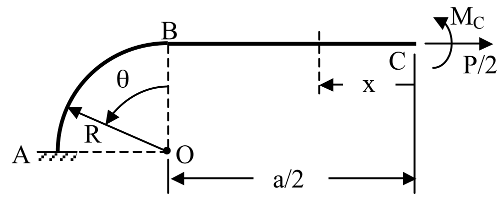
or

$$V_A = \tfrac{3(\lambda+1)}{2(3\lambda+4)} pL_1 = R_{Av}$$ ◄

$$H_A = \tfrac{\lambda}{4(3\lambda+4)} \tfrac{pL_1^2}{L_2} = R_{Ah}$$

The remaining reactions may then be found by using the equations of equilibrium.

SOLUTION (10.30)



Clearly, the problem is statically in determined. We have

$$M_x = -M_C \qquad\qquad 0 \le x \le \tfrac{a}{2}$$
$$M_x = -M_C + \tfrac{1}{2}PR(1-\cos\theta) \qquad\qquad 0 \le \theta \le \tfrac{\pi}{2}$$

Slope at C is zero:

$$\frac{\partial U}{\partial M_C} = \frac{1}{EI}\int_0^{\pi/2}(-M_C R + \tfrac{PR^2}{2} - \tfrac{PR^2}{2}\cos\theta)d\theta + \frac{1}{EI}\int_0^{a/2} - M_C dx = 0$$

Integrating,

$$M_C = \frac{PR^2(\pi-2)}{2(a+\pi R)}$$

Hence, for $0 \le \theta \le \frac{\pi}{2}$:

$$M = -\frac{PR^2(\pi-2)}{2(a+\pi R)} + \frac{PR}{2} - \frac{PR}{2}\cos\theta \qquad ◀$$

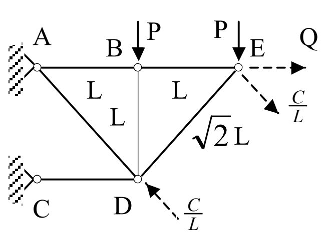
For $0 \le x \le a$:

$$M = \frac{PR^2(\pi-2)}{2(a+\pi R)} \qquad ◀$$

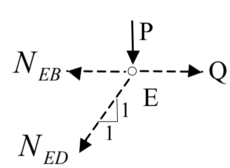
Therefore, M is maximum along *a*, for $\theta = 0$.

SOLUTION (10.31)

(a) Introduce Q at point E, as shown. (disregard C/L's).



Joint E



$$N_{ED} = -\sqrt{2}P\,(C)$$
$$N_{EB} = \sqrt{2}P + Q\,(T)$$

Similarly we obtain the remaining member forces.

Joint B

$$N_{BD} = -P\,(C)$$
$$N_{BA} = \sqrt{2}P + Q\,(T)$$

Joint D

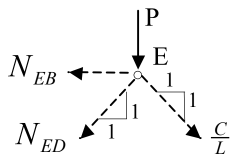
$$N_{DA} = 2P\sqrt{2}$$
$$N_{DC} = -3P\,(C)$$

$$\delta_E = \sum \frac{N_j L_j}{AE}\frac{\partial N_j}{\partial Q} = \frac{1}{AE}[(\sqrt{2}P)(5)(1) + (\sqrt{2}P)(5)(1) + 0 + 0 + 0 + 0] = \frac{10\sqrt{2}P}{AE} \qquad ◀$$

(CONT.)

(b) Introduce couple C/L applied perpendicular to line DE at points D and E as shown in the figure (now disregard Q).

Joint E



$$N_{ED} = -(P + \tfrac{C}{\sqrt{2}L})\sqrt{2}(C)$$
$$N_{EB} = P + \tfrac{2C}{\sqrt{2}L}(T)$$

Similarly, Joint D

$$N_{DA} = 2\sqrt{2}P \qquad N_{DC} = -3P - \tfrac{2C}{\sqrt{2}L}(C)$$

Thus

$$\theta_{DE} = \tfrac{1}{AE}\sum N_j \tfrac{\partial N_j}{\partial C} = \tfrac{P}{AE}(\tfrac{2}{\sqrt{2}} + \tfrac{2}{\sqrt{2}} + \sqrt{2} + \tfrac{6}{\sqrt{2}}) = \tfrac{P}{AE}(\tfrac{10}{\sqrt{2}} + \sqrt{2})$$ ◀

SOLUTION (10.32)

The complementary energy for the ith member of length L_i, from Eq. (2.49) is

$$U_{0i}^* = \int_0^{\sigma_i} \tfrac{\sigma_i^3}{K^3} d\sigma = \tfrac{\sigma_i^4}{4K^3} = \tfrac{1}{4K^3}(\tfrac{N_i}{A_i})^4$$

The complementary energy of the truss is thus

$$U^* = \sum_{j=1}^{6} \tfrac{A_j L_j}{4K^3}(\tfrac{N_j}{A_j})^4$$

Equation (10.16) is then

$$\delta_E = \sum_{j=1}^{6} \tfrac{L_j}{K^3}(\tfrac{N_j}{A_j})^3 \tfrac{\partial N_j}{\partial P} \qquad \text{(P10.32)}$$

Applying the method of joints, as needed, we obtain

$$N_1 = \tfrac{3P}{2} \qquad N_2 = -\tfrac{5P}{4} \qquad N_3 = 0$$
$$N_4 = \tfrac{5P}{4} \qquad N_5 = N_6 = -\tfrac{3P}{4}$$

We have $L_1 = L_5 = L_6 = 3\ m,\ L_3 = 4\ m,\ L_2 = L_4 = 5\ m$ and $A_i = A.$
Equation (P10.32) becomes

$$\delta_E = (\tfrac{1}{AK})^3\{L_1 N_1^3(\tfrac{\partial N_1}{\partial P}) + L_2 N_2^3(\tfrac{\partial N_2}{\partial P}) + \cdots\cdots + L_6 N_6^3(\tfrac{\partial N_6}{\partial P})\}$$

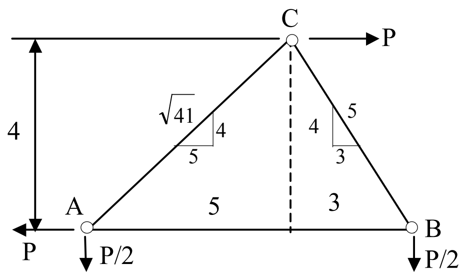
or

$$\begin{aligned}\delta_E &= (\tfrac{P}{AK})^3\{3(\tfrac{3}{2})^3(\tfrac{3}{2}) + 5(-\tfrac{5}{4})^3(\tfrac{5}{4}) + 0 \\ &\qquad + (\tfrac{5}{4})^3(\tfrac{5}{4}) + 3(\tfrac{3}{4})^3(\tfrac{3}{4}) + 3(-\tfrac{3}{4})^3(-\tfrac{3}{4})\} \\ &= (\tfrac{P}{AK})^3\ \tfrac{48\times81+5\times625\times2+3\times81\times2}{256}\end{aligned}$$

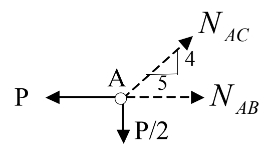
or

$$\delta_E = 41.5(\tfrac{P}{AK})^3$$ ◀

SOLUTION (10.33)



$$L_{AC} = 6.403\ m, \qquad L_{CB} = 5\ m$$
$$L_{AB} = 8\ m$$



Joint A

$$\sum N_y = 0: \quad N_{AC} = \tfrac{\sqrt{41}}{8}P$$
$$\sum N_x = 0: \quad N_{AB} = \tfrac{3}{8}P$$

Joint B $\qquad N_{BC} = -\tfrac{5}{8}P$

A horizontal unit load is applied a C:

$$n_{AB} = \tfrac{3}{8} \qquad n_{BC} = -\tfrac{5}{8} \qquad n_{AC} = \tfrac{\sqrt{41}}{8}$$

Thus

$$\delta_C = \tfrac{1}{AE}\sum n_j N_j L_j$$
$$= \tfrac{P}{AE}[(\tfrac{\sqrt{41}}{8})(\tfrac{\sqrt{41}}{8})(6.403) + (-\tfrac{5}{8})(\tfrac{5}{8})(5) + (\tfrac{3}{8})(\tfrac{3}{8})(8)]$$
$$= 7.18\tfrac{P}{AE} \rightarrow$$

SOLUTION (10.34)

(a) Refer to Fig. (a) with *no* unit load:

$$M_{AB} = 0 \qquad M_{BE} = Rx$$
$$M_{EC} = Rx - P(x-L)$$
$$M_{CD} = R(2L) - PL$$

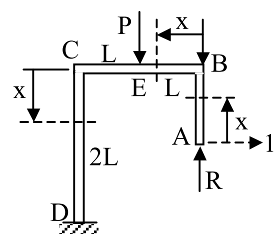


Figure (a)

The vertical deflection at A is zero.
Thus

$$\delta_v = \tfrac{1}{EI}\sum M_i \tfrac{\partial M_i}{\partial R}dx = 0$$
$$= \int_0^L 0dx + \int_0^L Rx(x)dx + \int_L^{2L}[Rx - P(x-L)]xdx + \int_0^{2L}(2LR - PL)(2L)dx = 0$$

Integrating, we have

$$R = \tfrac{29}{64}P \qquad \blacktriangleleft$$

(b) Introducing a horizontal unit load at A, Fig. (a), we write

$$m_{AB} = x \qquad m_{BE} = L \qquad m_{EC} = L \qquad m_{CD} = L - x$$

Hence

$$\delta_h = \tfrac{1}{EI}\sum M_i m_i dx$$

(CONT.)

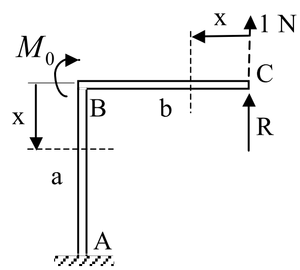
10.34 (CONT.)

$$= \frac{1}{EI}\{\int_0^L 0dx + \int_0^L Rx(L)dx + \int_L^{2L}[Rx - P(x-L)]Ldx$$
$$+ \int_0^{2L}(2LR - PL)(L-x)dx\}$$

Integrating, after substituting $R = 29P/64$, results in

$$\delta_h = \frac{13}{32}\frac{PL^3}{EI}$$ ◄

SOLUTION (10.35)



Introduce a vertical upward unit force 1 N at point C. Then

Segment CB:

$$M_1 = Rx \qquad m_1 = x$$

Segment BA:

$$M_2 = Rb = M_0 \qquad m_2 = b$$

Thus

$$EI\delta_C = 0 = \int_0^b M_1 m_1 dx + \int_0^a M_2 m_2 dx$$
$$= \int_0^b Rx^2 dx + \int_0^b (Rb - M_0)(b)dx$$
$$= \tfrac{1}{3}Rb^3 + Rab^2 - M_0 ba$$

or

$$R = \frac{3M_0 a}{b(3a+b)} \uparrow$$ ◄

SOLUTION (10.36)

We have $\alpha = 6/5$ (Table 5.1), $dx = ds = Rd\theta$

Let a downward unit load of 1 N in addition to load P is applied at the free end in Fig. P17.25. We write

$$N = P\cos\theta \qquad n = \cos\theta$$
$$V = P\sin\theta \qquad v = \sin\theta \qquad \text{(a)}$$
$$M = PR(1-\cos\theta) \qquad m = R(1-\cos\theta)$$

Thus,

$$\delta_v = \frac{1}{EI}\int_0^{3\pi/2} MmRd\theta + \frac{1}{AG}\int_0^{3\pi/2}\alpha Vvdx + \frac{1}{AE}\int_0^{3\pi/2} Nnds$$

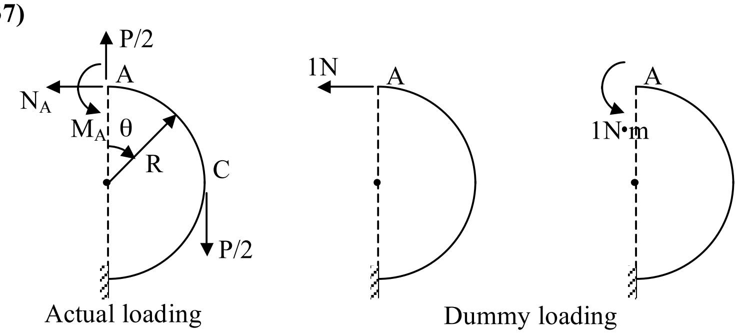
Substitution of Eqs.(a) gives

$$\delta_v = \frac{PR^3}{EI}\int_0^{3\pi/2}(1-\cos\theta)^2 d\theta + \frac{6PR}{5AG}\int_0^{3\pi/2}\sin\theta d\theta + \frac{PR}{AE}\int_0^{3\pi/2}\cos^2\theta d\theta$$

Integrating

$$\delta_v = (\tfrac{9\pi}{4} + 2)\frac{PR^3}{EI} + \frac{3.6PR}{4AG} + \frac{3\pi PR}{4AE} \downarrow$$ ◄

SOLUTION (10.37)



Because of symmetry about the vertical axis, only one half of the circle need be analyzed. Referring to the figure above, we have the following moments:

For $0 \le \theta \le \frac{\pi}{2}$:

$$M_1 = M_A + N_A R(1-\cos\theta) - \tfrac{1}{2}PR\sin\theta$$
$$m_1 = R(1-\cos\theta), \qquad m_1' = 1 \tag{a}$$

For $\frac{\pi}{2} \le \theta \le \pi$:

$$M_2 = M_A + N_A R(1-\cos\theta) - \tfrac{1}{2}PR \tag{b}$$
$$m_2 = R(1-\cos\theta), \qquad m_2' = 1$$

Horizontal deflection and slope are zero at *A*. Thus, from Eqs. (10.13) and (10.14):

$$\delta_{Ah} = \tfrac{1}{EI}\int_0^\pi MR(1-\cos\theta)Rd\theta = 0$$

or

$$\int_0^\pi M(1-\cos\theta)d\theta = 0 \tag{c}$$

$$\theta_A = \tfrac{1}{EI}\int_0^\pi Mm'dx = \tfrac{1}{EI}\int_0^\pi M(1)dx = 0$$

or

$$\int_0^\pi M = 0, \text{ where } M \text{ represents } M_1 \text{ and } M_2. \tag{d}$$

Substituting Eq. (d) into Eq. (c), the latter reduces to

$$\int_0^\pi M\cos\theta d\theta = 0 \tag{e}$$

Introducing Eqs. (a) and (b) into Eqs. (d) and (e), we have

$$M_A\int_0^\pi d\theta + N_A R\int_0^\pi (1-\cos\theta)d\theta$$
$$-\tfrac{1}{2}PR\int_0^{\pi/2}\sin\theta d\theta - \tfrac{1}{2}PR\int_{\pi/2}^\pi d\theta = 0$$

and
$$M_A\int_0^\pi \cos\theta d\theta + N_A R\int_0^\pi (1-\cos\theta)\cos\theta d\theta$$
$$-\tfrac{1}{2}PR\int_0^{\pi/2}\sin\theta\cos\theta d\theta - \tfrac{1}{2}PR\int_{\pi/2}^\pi \cos\theta d\theta = 0$$

Integrating and solving,

$$N_A = \tfrac{P}{2\pi} \qquad M_A = \tfrac{PR}{4}$$

◀

SOLUTION (10.38)

From Eq. (P10.38), we have

$$U = \frac{GJ}{2}[(\frac{\theta_C}{a})^2 a + (\frac{\theta_C}{b})^2 b]$$

Using Eq. (10.22):

$$\frac{\partial U}{\partial \theta_C} = T; \qquad GJ\theta_C(\frac{1}{a} + \frac{1}{b}) = T$$

Solving,

$$\theta_C = \frac{Tab}{GJL} \qquad \blacktriangleleft$$

Then

$$T_A = \frac{GJ}{a}\theta_C = \frac{b}{L}T \qquad \blacktriangleleft$$

and

$$T_B = \frac{GJ}{b}\theta_C = \frac{a}{L}T \qquad \blacktriangleleft$$

SOLUTION (10.39)

$$L_{AD} = L_{DC} = 4.24\ m$$

$$U = \frac{1}{2}ALE\varepsilon^2 = \frac{1}{2}ALE(\delta_v \frac{\cos\alpha}{L})^2$$

Vertical load of the joint, using Eq. (10.22),

$$\frac{\partial U}{\partial \delta_v} = P = \sum_{i=1}^{3} \frac{E_i A_i}{L_i}\delta_v \cos^2\alpha$$

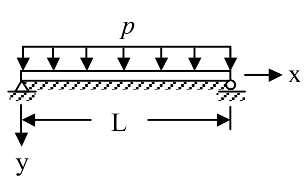
or

$$P = EA\delta_v[\frac{\cos^2 45^o}{L_{AD}} + \frac{\cos^2 45^o}{L_{DC}} + \frac{1}{L_{BD}}] = 0.5689AE\delta_V$$

Substituting the numerical values,

$$P = 373.34\ kN \qquad \blacktriangleleft$$

SOLUTION (10.40)



The deflection curve may be expressed by Eq. (10.23) and the bending strain energy is given by Eq. (10.25). The strain energy of deformation of the foundation is (Chap. 9):

$$U_2 = \frac{1}{2}k\int_0^L v^2 dx = \frac{1}{4}kL\sum_{n=1}^{\infty} a_n^2$$

Work done by the uniform load:

$$W = \frac{1}{2}\int_0^L pvdx = p(\frac{L}{\pi})\sum_{n=1}^{\infty}\frac{1}{n}a_n \cos\frac{n\pi x}{L}\Big|_0^L$$

or

$$W = \frac{2pL}{\pi}\sum_{n=1,3,\cdots}^{\infty}\frac{a_n}{n}$$

(CONT.)

10.40 (CONT.)

Then, principle of virtual work yields

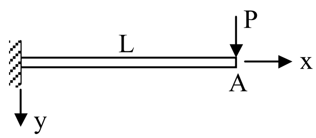
$$\tfrac{\pi^4 EI}{2L^3} n^4 a_n + \tfrac{kL}{2} a_n = \tfrac{2pL}{n\pi}$$

or

$$a_n = \frac{4pL^4}{n\pi(n^4\pi^4 EI + kL^4)} \quad ◄$$

Substituting this into Eq. (10.23) we obtain the required deflection curve.

SOLUTION (10.41)



We obtain

$$v'' = 3\tfrac{a_1}{L^3}(L-x)$$

and

$$U = \tfrac{EI}{2}\int_0^L (v'')^2 dx = \tfrac{9EIa_1^2}{2L^6}\int_0^L (L-x)^2 dx$$

$$= \tfrac{3EI}{2L^3} a_1^2 \qquad \text{(a)}$$

Also $\delta W = P \cdot \delta v_A$ (b)

Thus, $\delta W = \delta U$ gives,

$$P \cdot \delta a_1 = \tfrac{3EI}{2L^3}(2a_1 \delta a_1)$$

or

$$v_A = a_1 = \tfrac{PL^3}{3EI} \quad ◄$$

SOLUTION (10.42)

Strain energy, by Eq. (c) of Sec. 10.10:

$$U = \tfrac{\pi^4 EI}{64L^3} a^2$$

Work done by the load P is

$$W = \int_0^L pv dx = pa\int_0^L dx - pa\int_0^L \cos\tfrac{\pi x}{2L} dx$$

$$= pa(L - \tfrac{2L}{\pi})$$

Applying the Ritz method, we obtain

$$\tfrac{d\Pi}{da} = \tfrac{d}{da}(U - W) = \tfrac{\pi^4 EI}{64L^3} - pL(1 - \tfrac{2L}{\pi}) = 0$$

Solving

$$a = 0.1194\tfrac{pL^4}{EI} \quad ◄$$

Substitution of this into Eq. (P10.31) and letting x=L result in the deflection v_A at the free end of the beam.

SOLUTION (10.43)

$$W = P \cdot v_A = P(a_1 L^2 + a_2 L^3) = PL^2(a_1 + a_2 L)$$
$$U = \tfrac{EI}{2}\int_0^L (v'')^2 dx = 2EIL(a_1^2 + 3a_1 a_2 L + 3a_2^3 L^2)$$

Thus,

$$\tfrac{\partial \Pi}{\partial a_1} = 0: \qquad PL^2 = 2EIL(2a_1 + 3a_2 L)$$
$$\tfrac{\partial \Pi}{\partial a_2} = 0: \qquad PL^3 = 6EIL^2(a_1 + 2a_2 L)$$

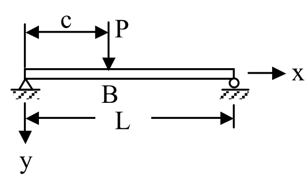
Solving,

$$a_1 = \tfrac{PL}{2EI} \qquad a_2 = -\tfrac{P}{6EI}$$

Substituting back into Eq. (P10.32):

$$v = \tfrac{Px^2}{6EI}(3L - x)$$ ◄

SOLUTION (10.44)



We have

$$\Pi = \tfrac{EI}{2}\int_0^L (v'')^2 dx - P \cdot v_B \qquad \text{(a)}$$

Here

$$v = ax(L - x) = axL - ax^2$$
$$v' = aL - 2ax, \qquad v'' = -2a$$

Substituting these into Eq. (a), after integration, we obtain

$$\Pi = 2a^2 EIL - PacL + Pac^2$$

and

$$\tfrac{d\Pi}{da} = 4aEIL - PcL + Pc^2 = 0$$

or

$$a = \tfrac{Pc(L-c)}{4EIL}$$

The deflection of the beam at point B is therefore

$$v_B = \tfrac{Pc^2(L-c)^2}{4EIL}$$ ◄

SOLUTION (10.45)

The assumed deflection is of the general cubic form:

$$v = a_1 x^3 + a_2 x^2 + a_3 x + a_4 \qquad \text{(a)}$$

For the left hand portion of the beam:

$v = 0$ (at $x = 0$); $\qquad v' = 0$ (at $x = 0$) $\qquad$ (1, 2)

$v' = 0$ (at $x = L/2$); $\qquad v = \Delta$ (at $x = L/2$); $\qquad$ (3, 4)

Here Δ is, deflection at midspan, to be determined.

Eqs. (1) and (2) give $a_3 = a_4 = 0$

(CONT.)

Eq. (3) yields $a_2 = -\frac{3a_1 L}{4}$

Eq. (4) gives $a_1 = -\frac{16\Delta}{L^3}$

Introducing these into Eq. (a):

$$v = \frac{4\Delta x^2}{L^3}(3L - 4x) \qquad 0 \le x \le \frac{L}{2} \tag{b}$$

Strain energy is

$$U = 2\left(\frac{EI/2}{2}\right)\int_0^{L/4} (v'')^2\,dx + 2\left(\frac{EI}{2}\right)\int_{L/4}^{L/2} (v'')^2\,dx \tag{c}$$

Inserting Eq. (b) into Eq. (c) and integrating, we obtain: $U = \frac{72EI\Delta^2}{L^3}$

We have

$$\Pi = \frac{72EI\Delta^2}{L^3} - P\Delta$$

The midspan deflection, from $\partial\Pi/\partial\Delta = 0$, is thus

$$\Delta = \frac{PL^3}{144EI} = v_{\max}$$ ◀

SOLUTION (10.46)

We obtain

$$v'' = \sum_{n=1}^{\infty} a_n \left(\frac{n\pi}{L}\right)^2 \cos\frac{n\pi x}{L}$$

$$U = \frac{EI}{2}\int_0^L (v'')^2\,dx = \frac{EI}{2}\sum_{n=1}^{\infty} a_n^2 \left(\frac{n\pi}{L}\right)^4 \frac{L}{2}$$

$$W = P(v)_{x=\frac{L}{2}} = P\sum_{n}^{\infty} 2a_n \qquad (n = 2, 4, 6, \cdots)$$

From the minimizing condition, $\partial\Pi/\partial a_n = 0$, we obtain

$$EIa_n\left(\frac{n\pi}{L}\right)^4 \frac{L}{2} - 2P = 0$$

or

$$a_n = \sum_{n}^{\infty} \frac{4PL^3}{n^4\pi^4 EI} \qquad (n = 2, 4, 6, \cdots)$$

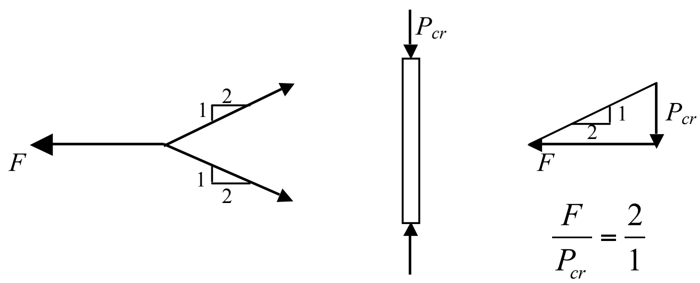
Therefore,

$$v_{\max} = v\left(\frac{L}{2}\right) = \frac{8PL^3}{\pi^4 EI}\sum_{n}^{\infty}\frac{1}{n^4} \qquad (n = 2, 4, 6, \cdots)$$ ◀

End of Chapter 10

CHAPTER 11

SOLUTION (11.1)



$$\frac{F}{P_{cr}} = \frac{2}{1}$$

$$I = \frac{\pi}{4}(c^4 - b^4) = \frac{\pi}{4}(25^4 - 20^4) = 18.11\times 10^4 \ mm^4$$

$$\frac{P_{cr}}{n} = \frac{\pi^2 EI}{nL^2} = \frac{\pi^2(200\times 10^9)(0.1811\times 10^{-6})}{2.5(1)^2} = 143 \ kN$$

$$F = 2.5(143) = 357.5 \ kN \qquad \blacktriangleleft$$

Justification of the formula used:

$$A = \pi(c^2 - b^2) = \pi(25^2 - 20^2) = 706.86 \ mm^2$$

$$r = \sqrt{\frac{I}{A}} = \sqrt{\frac{181,100}{706.86}} = 5.06 \ mm$$

and

$$\frac{L}{r} = \frac{1000}{5.06} = 197.6 \ \text{O.K.}$$

SOLUTION (11.2)

(**a**) Same area:

$$\frac{\pi}{4}(d_o^2 - d_i^2) = b_o^2 - b_i^2$$

$$b_i^2 = b_o^2 - \frac{\pi}{4}(d_o^2 - d_i^2) = 50^2 - \frac{\pi}{4}(50^2 - 40^2)$$

or

$$b_i = 42.35 \ mm \qquad t = \frac{1}{2}(b_o - b_i) = 3.83 \ mm \qquad \blacktriangleleft$$

(**b**) Circular bar

$$I = \frac{\pi}{64}(d_o^4 - d_i^4) = \frac{\pi}{64}(50^4 - 40^4) = 181\times 10^{-9} \ in.^4$$

$$P_{cr} = \frac{\pi^2 EI}{L_e^2} = \frac{\pi^2(72\times 10^9)(181\times 10^{-9})}{(3)^2} = 14.29 \ kN \qquad \blacktriangleleft$$

(CONT.)

11.2 (CONT.)

Square bar

$$I = \frac{1}{12}(b_o^4 - b_i^4) = \frac{1}{12}(50^4 - 42.35^4) = 252.8\times10^{-9}\ m^4$$

$$P_{cr} = \frac{\pi^2 EI}{L_e^2} = \frac{\pi^2(72\times10^6)(252.8\times10^{-9})}{(3)^2} = 19.96\ kN \qquad \blacktriangleleft$$

SOLUTION (11.3)

$$L_{BC} = \sqrt{2^2+1^2} = 2.236\ m \qquad r = \frac{d}{4} = \frac{50}{4} = 12.5\ mm$$

$$I = \frac{\pi}{64}d^4 = \frac{\pi}{64}(40)^4 = 125.7(10^{-9})\ m^4, \qquad L_{AB} = \sqrt{1^2+1^2} = 1.414\ m$$

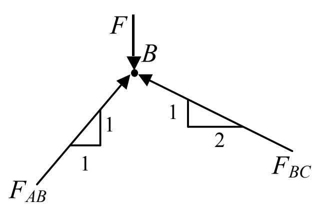
Bar BC $(L/r = 2{,}236/12.5) = 178$. Thus

$$(F_{BC})_{all} = \frac{P_{cr}}{n} = \frac{\pi^2(70)(125.7)}{1.8(2.236)^2} = 9.65\ kN$$

Bar AB

$$(F_{AB})_{all} = \frac{P_{cr}}{n} = \frac{\pi^2(70)(125.7)}{1.8(1.414)^2} = 24.1\ kN$$

Joint B



$$\sum F_x = 0: \frac{1}{\sqrt{2}}F_{AB} - \frac{2}{\sqrt{5}}F_{BC} = 0,\ F_{BC} = 0.791F_{AB}$$

$$\sum F_y = 0: \frac{1}{\sqrt{2}}F_{AB} + \frac{1}{\sqrt{5}}F_{BC} - F = 0,\ F = 1.061F_{AB}$$

Solving

$$F = 1.061F_{AB} \qquad F = 1.341F_{BC}$$

The allowable value for F:

$$F < 1.341(9.65) = 12.93\ kN$$

$$F < 1.061(24.1) = 25.6\ kN$$

Thus

$$F_{all} = 12.93\ kN \qquad \blacktriangleleft$$

SOLUTION (11.4)

Differential equation for a free-body of a segment of x is given by Eq.(11.1):

$$EI\frac{d^2v}{dx^2} + Pv = 0$$

(CONT.)

or

$$\frac{d^2v}{dx^2} + p^2 v = 0 \qquad p = \sqrt{\frac{P}{EI}}$$

General solution is

$$v = A\sin px + B\cos px \tag{1}$$

Boundary conditions :

$$v(0) = 0: \quad 0 + 0 + B = 0, \qquad B = 0$$

$$v'(L) = 0: \quad AP\cos pL = 0, \qquad Ap = 0$$

Since $A \neq 0$:

$$\cos\sqrt{\frac{P}{EI}}L = 0 \quad \text{or} \quad \sqrt{\frac{P}{EI}}L = n(\frac{\pi}{2})$$

For $n = 1$, $\quad \frac{P}{EI} = \frac{\pi^2}{4L^2}$

Therefore

$$P_{cr} = \frac{\pi^2 EI}{4L^2}$$ ◀

SOLUTION (11.5)

$$I_{\min} = \frac{1}{12}(240\times120^3 - 190\times70^3) = 29.13\times10^6 \ mm^4$$

$$A = 240\times120 - 190\times70 = 15.5\times10^3 \ mm^2$$

$$r_{\min} = \sqrt{I_{\min}/A} = 43.35 \ mm \qquad L_e = 0.5L = 4.5 \ m$$

$$L_e/r = 4500/43.35 = 103.8$$

Hence,

$$\sigma_{cr} = \frac{\pi^2 E}{(L_e/r)^2} = \frac{\pi^2(200\times10^9)}{(103.8)^2} = 183.2 \ MPa$$ ◀

SOLUTION (11.6)

$L_e = 0.7L = 6.3 \ m$. From solution of Prob.11.5: $r_{\min} = 43.35 \ mm$.

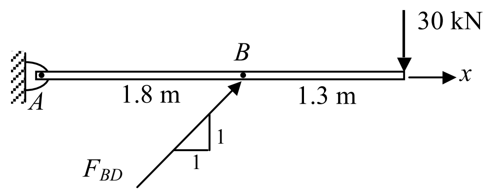
We now have

$$L_e/r = 6300/43.35 = 145.3$$

Hence,

$$\sigma_{cr} = \frac{\pi^2 E}{(L_e/r)} = \frac{\pi^2(200\times10^9)}{(145.3)^2} = 93.5 \ MPa$$ ◀

SOLUTION (11.7)



$$\sum M_A = 0: \quad -30(3.1) + \frac{1}{\sqrt{2}} F_{BD}(1.8) = 0, \qquad F_{BD} = 73.1 \ kN$$

$$I = b^4/12 = 50^4/12 = 520.8 \times 10^3 \ mm^4, \qquad A = 50 \times 50 = 2.5 \times 10^3 \ mm^2$$

$$r = \sqrt{\frac{I}{A}} = 14.4 \ mm \qquad \frac{L}{r} = \frac{1800}{14.4} = 125$$

So,

$$\sigma_{cr} = \frac{\pi^2 E}{(L/r)^2} = \frac{\pi^2 (210 \times 10^9)}{(125)^2} = 132.6 \ MPa$$

We have

$$\sigma_{cr} < \sigma_{yp} \qquad \text{(solution is valid)}$$

$$(F_{BD})_{cr} = 132.6 \times 10^6 (2.5 \times 10^{-3}) = 331.5 \ kN$$

and

$$n = \frac{(F_{BD})_{cr}}{F_{BD}} = \frac{331.5}{73.1} = 4.5$$ ◄

SOLUTION (11.8)

A solid circular section of diameter d:

$$A = \frac{\pi}{4} d^2 \qquad I = \frac{\pi}{64} d^4 \qquad r = \sqrt{\frac{I}{A}} = \frac{d}{4}$$

Using Eq. (11.13a), we have

$$\frac{4P_{cr}}{\pi d^2} = \sigma_{yp} - \frac{1}{E}\left(\frac{\sigma_{yp}}{2\pi}\frac{4L_e}{d}\right)^2$$

Solving,

$$d = 2\left(\frac{P_{cr}}{\pi \sigma_{yp}} + \frac{\sigma_{yp}^2 L_e^2}{\pi^2 E}\right)^{1/2} \qquad \text{Q.E.D.}$$ ◄

A rectangular section of height h and width *b*:

$$A = bh \qquad I = \frac{1}{12} bh^3 \qquad r^2 = \frac{h^2}{12}$$

By Eq. (11.13a), we obtain

$$\frac{P_{cr}}{bh} = \sigma_{yp} - \frac{1}{E}\left(\frac{\sigma_{yp}}{2\pi}\frac{\sqrt{12}L_e}{h}\right)^2$$

(CONT.)

11.8 (CONT.)

from which

$$b = \frac{P_{cr}}{h\sigma_{yp}(1 - \frac{3L_e^2\sigma_{yp}}{\pi^2 Eh^2})} \qquad \text{Q.E.D.} \qquad \blacktriangleleft$$

It is assumed that $h \le b$.

SOLUTION (11.9)

$$\sigma_{all} = \frac{P}{A} = \frac{90\times10^3}{3\times10^{-3}} = 30\ MPa \qquad (1)$$

$$C_c = \sqrt{\frac{2\pi^2(200\times10^9)}{250\times10^6}} = 125.7$$

Assuming $L/r \ge C_c$, use Eq. (11.8) and (1):

$$\sigma_{all} = \frac{\pi^2(200\times10^9)}{1.92(L/r)^2} = 30\times10^6, \qquad \frac{L}{r} = 185$$

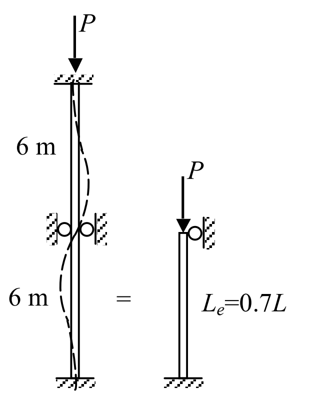
Our assumption was correct. Hence,

$$L = (185)r = (185)(25) = 4,625\ mm$$

or

$$L = 4,625\ m \qquad \blacktriangleleft$$

SOLUTION (11.10)



$$C_c = \sqrt{\frac{2\pi^2(210\times10^9)}{280\times10^6}} = 121.7$$

$$\frac{L_e}{r} = \frac{4.2}{0.1} = 42$$

Since $L/r_y < C_c$, use Eq. (11.9) and (11.8):

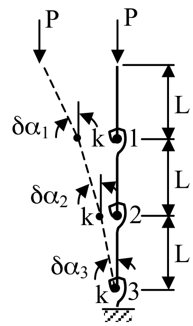
$$n = \frac{5}{3} + \frac{3}{8}(\frac{42}{121.7}) - \frac{1}{8}(\frac{42}{121.7})^3 = 1.75$$

$$\sigma_{all} = \frac{1-(42)^2/[2(121.7)^2]}{1.75}(280\times10^6) = 150.5\ MPa$$

So,

$$P_{all} = 150.5(27\times10^3) = 4063\ kN \qquad \blacktriangleleft$$

SOLUTION (11.11)



The moment expression about joints 1 and 2:

$$\sum M_1 = 0; \qquad PL\delta\alpha_1 - k(\delta\alpha_1 - \delta\alpha_2) = 0$$
$$\sum M_2 = 0; \qquad PL(\delta\alpha_1 - \delta\alpha_2) - k(\delta\alpha_2 - \delta\alpha_3) = 0$$

or, in general,

$$PL(\sum_{i=1}^{n} \delta\alpha_i) - k\delta\alpha_n = 0 \qquad \text{(P11.11)}$$

In matrix form, we have

$$\begin{bmatrix} PL-k & k & 0 & 0 & \cdot & \cdot & 0 \\ PL & PL-k & k & 0 & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ PL & PL & PL & PL & \cdot & \cdot & PL-k \end{bmatrix} \begin{Bmatrix} \delta\alpha_1 \\ \delta\alpha_2 \\ \cdot \\ \cdot \\ \delta\alpha_n \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ 0 \end{Bmatrix}$$

That is

$$[A]\{\delta\alpha\} = \{0\}$$

Determinant $|A| = 0$, yields P_{cr}. In our case, $n = 3$.

SOLUTION (11.12)

(a) P_{cr} is proportional to I. We have

$$I_s = \tfrac{\pi r^4}{4}$$
$$I_h = \tfrac{\pi}{4}[r^4 - (\tfrac{r}{2})^4] = \tfrac{15}{16}\tfrac{\pi r^4}{4} = \tfrac{15}{16} I_s$$

Thus, reduction in P_{cr} is 6.25 %

(b) Substituting the given data:

$$I_h = \tfrac{15\pi(0.015)^4}{64} = 37.276(10^{-9})\ m^4$$
$$P_{cr} = \tfrac{\pi^2 EI_h}{L_e^2} = \tfrac{\pi^2(110\times10^9)(37.276\times10^{-9})}{(1.5)^2} = 17.99\ kN \qquad \blacktriangleleft$$

SOLUTION (11.13)

(a) $P_{cr} = 2(100) = 200\ kN$

$$I = \frac{P_{cr}L^2}{\pi^2 E} = \frac{200(10^3)(2)^2}{\pi^2(11\times10^9)} = 7.369(10^6)\ mm^4$$

$$\frac{a^4}{12} = 7.369(10^6); \qquad a = 96.97\ mm$$

We have

$$\sigma = \frac{P}{A} = \frac{100(10)^3}{(0.09697)^2} = 10.63\ MPa < 15\ MPa$$

Thus, a cross section of $97\times97\ mm$ is acceptable. ◀

(b) $P_{cr} = 2(200) = 400\ kN$

$$I = \frac{400(10^3)(2)^2}{\pi^2(11\times10^9)} = 14.738(10^6)\ mm^4; \qquad a = 115.32\ mm$$

We obtain

$$\sigma = \frac{P}{A} = \frac{200(10^3)}{(0.11532)^2} = 15.04\ MPa > 15\ MPa$$

Dimension is *not* acceptable. Therefore,

$$a^2 = A = \frac{200(10^3)}{15(10^6)}; \qquad a = 115.5\ mm$$

Use a cross section of

$$116\times116\ mm$$ ◀

SOLUTION (11.14)

We have

$$I = \frac{1(50)^4}{12} = 0.521\times10^6\ mm^4.$$

Hence

$$P_{cr} = \frac{\pi^2 EI}{L^2} = \frac{\pi^2(200\times10^3)(0.521)}{(3)^2} = 114.3\ kN$$

$$P_{all} = \frac{P_{cr}}{n} = \frac{114.3}{2.2} = 52\ kN$$

Thus

$$F = \frac{P_{all}}{2.5} = \frac{52}{2.5} = 20.8\ kN$$ ◀

SOLUTION (11.15)

(a) Using Eq. (11.5),

$$P_{cr} = \frac{\pi^2(210\times10^9)(0.075\times0.05^3/12)}{0.49(3.6)^2(2)} = 127.5\ kN$$

$$\sigma_{cr} = \frac{127.5(10^3)}{3.79(10^{-3})} = 34\ MPa$$ ◀

(b) We have

$$I_{\min} = \frac{0.075(0.05)^3}{12}$$

and

$$r^2 = \frac{I_{\min}}{A} = 2.08(10^{-4})$$

(CONT.)

11.15 (CONT.)

Thus,

$$\frac{L_e}{r} = \frac{0.7L}{r} = \frac{0.7(3.6)}{0.0144} = 175$$

Also, $C_c = [\frac{2\pi^2 E}{\sigma_{yp}}]^{\frac{1}{2}} = 121.673$

As 121.673 < 175, use Eq. (11.13):

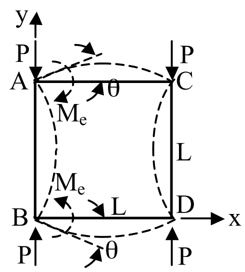
$$\sigma_{all} = \frac{\pi^2(210\times10^9)}{1.92(175)^2} = 35.25\ MPa$$

Note that

$$\sigma = \frac{P}{A} = \frac{450(10^3)}{(0.05)(0.075)} = 120\ MPa$$

and yielding does *not* occur. But member fails as a column, since $P_{cr} < P$. ◀

SOLUTION (11.16)



Symmetrical buckling shown in the figure, creates relative bending moments M_e which resist free rotation of the ends of the member AB and CD. Thus, for member AB:

$$EI\frac{d^2v}{dx^2} = -Pv + M_e$$

with the general solution

$$v = C_1 \cos\lambda x + C_2 \sin\lambda x + \frac{M_2}{P} \qquad \text{(a)}$$

where, $\lambda^2 = \frac{P}{EI}$

Boundary conditions are

$$v(0) = 0 \qquad v'(\tfrac{L}{2}) = 0$$

$$v'(0) = \theta = \frac{M_e L}{2EI}$$

Introducing v from Eq. (a) into these:

$$C_1 + \frac{M_e}{P} = 0, \qquad -C_1\lambda\sin\frac{\lambda L}{2} + C_2\lambda\cos\frac{\lambda L}{2} = 0$$

$$C_2\lambda = \frac{M_e L}{2EI}$$

The foregoing lead to the following trancendental equation

$$\tan\frac{\lambda L}{2} + \frac{PL}{2\lambda EI} = 0$$

or

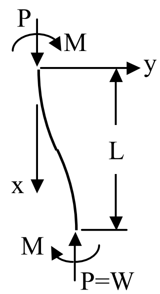
$$\tan\frac{\lambda L}{2} + \frac{\lambda L}{2} = 0$$

from which $\lambda L/2 = 2.029$. Thus,

$$P_{cr} = \frac{16.47EI}{L^2} = \frac{\pi^2 EI}{(0.774L)^2}$$

The effective length in the situation described is therefore equal to 0.774L. ◀

SOLUTION (11.17)



Let

$$\lambda_1 = \tfrac{P}{EI}$$
$$\lambda_2 = \tfrac{M}{EI}$$

The governing differential equation is

$$EIv'' = M - Pv$$

or

$$v'' + \lambda_1 v = \lambda_2^2$$

Solution is

$$v = A\cos\lambda_1 x + B\sin\lambda_2 x + (\tfrac{\lambda_2}{\lambda_1})^2$$

Boundary conditions give:

$$v'(0) = 0 = B$$
$$v(0) = A + (\tfrac{\lambda_2}{\lambda_1})^2 = 0; \qquad A_1 = -(\tfrac{\lambda_2}{\lambda_1})^2$$
$$v'(L) = (\tfrac{\lambda_2}{\lambda_1})^2 \sin\lambda_1 L = 0$$

or $\sin\lambda_1 L = 0; \qquad \lambda_1 L = 0, \pi, 2\pi, \cdots$

Choose $\lambda_1 L = \pi$. Then,

$$\lambda_1 = \tfrac{\pi}{L} = (\tfrac{P}{EI})^{\frac{1}{2}}; \qquad P = \tfrac{\pi^2 EI}{L^2}$$

Thus, $L_e = L$ ◀

SOLUTION (11.18)

For the vertical bar, from Eq. (11.5):

$$P_{cr} = \tfrac{\pi^2 EI}{4L}$$

The midspan deflection of the beam is thus

$$\delta = \tfrac{5PL^4}{384EI} - \tfrac{P_{cr}L^3}{48EI}$$

or

$$\delta = \tfrac{5PL^4}{384EI} - \tfrac{\pi^2 L}{192} = \tfrac{L}{192}(\tfrac{5PL^3}{2EI} - \pi^2)$$ ◀

SOLUTION (11.19)

(a) We have $\alpha(\Delta T)L = \delta/2$

or $\Delta T = \tfrac{\delta}{2\alpha L}$ ◀

(b) $$\alpha(\Delta T)L - \tfrac{PL}{AE} = \tfrac{\delta}{2}$$

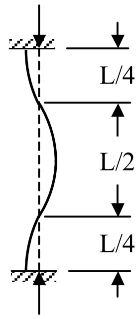
where

$$P = \tfrac{\pi^2 EI}{4L^2}$$

Thus,

$$\Delta T = \tfrac{\delta}{2\alpha L} + \tfrac{\pi^2 EI}{4L^2}\tfrac{L}{AE}\tfrac{1}{\alpha L} = \tfrac{\delta}{2\alpha L} + \tfrac{\pi^2 I}{4L^2 A\alpha}$$ ◀

SOLUTION (11.20)



For a fixed ended column L_e may be determined as follows. Inflection points and midpoint divide the bar into 4 equal portions. Each portion is the same as the fundamental case (Fig. 11.2a). Thus $L_e = L/4$.

For the dimension given, we obtain

$$\frac{L}{r} = \frac{1.932}{0.0294} = 65.99$$

Equation (11.6) gives then

$$\sigma_{cr} = \frac{4\pi^2 E}{(L/r)^2} = 0.0091E$$

Substituting $E = 174(10^9)$,

$$\sigma_{cr} = 0.0091(175 \times 10^9) = 1592.5 \ MPa$$

Since, from Fig. P11.20, E is valid only up to 175 MPa, the 1592,5 MPa cannot be critical stress.

Similarly, for inelastic range, from Eq. (11.7):

$$\sigma_{cr} = 0.0091 E_{t1}$$

$$= 0.0091(46.7 \times 10^9) = 425 \ MPa$$

Applying the same reasoning as before, this value is also not a critical stress.

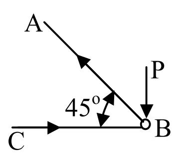
For E_{t2} :

$$\sigma_{cr} = 0.0091(28 \times 10^9) = 254.8 \ MPa$$

We now observe from the sketch that 254.8 falls in the stress range for which E_{t2} is valid. Thus, the buckling load:

$$P_t = \sigma_{cr} A = 254.8(0.0323) = 823 \ kN$$ ◄

SOLUTION (11.21)



For bar AB:

$$1.41P = 4200(5.4 \times 10^{-5}) = 2268$$

(CONT.)

or $P = 1604\ N$

For bar BC:

$$P = P_{cr} = \frac{\pi^2 EI}{L^2} = \frac{\pi^2(210\times10^9)(3.9\times10^{-9})}{6.25}$$
$$= 1297\ N$$

And $\sigma_{cr} = \frac{1297}{5.4(10^{-5})} = 24\ MPa$

Conclusion: Bar BC fails as a column, $P_{cr} = 1297\ N$. ◀

SOLUTION (11.22)

Referring to Fig. P11.22, we write

$$I_y = 2Ar_c^2 + 2Ad^2$$
$$= 2(0.0225)^2 + 2(1.719\times10^{-3})(0.0125 + 0.2325)^2$$
$$= 6.13(10^{-6})\ m^4$$

and

$$r_y = \left(\frac{I_y}{2A}\right)^{\frac{1}{2}} = (r_c^2 + d^2)^{\frac{1}{2}}$$
$$= (0.0225^2 + 0.03575^2)^{\frac{1}{2}} = 42.24\ mm$$

We see that r_y for two channels is the same as for one channel.

But $r_z = r_c = 0.0225\ m$. Thus, $r_z < r_y$. Columns tends to buckle with respect to z axis.

The slenderness ratios:

$$\frac{L_e}{r_z} = \frac{2.10}{0.0225} \approx 94, \qquad \frac{L_e}{r_z} = \frac{4.20}{0.0225} \approx 187$$

From Sec. 11.7:

$$C_c^2 = \frac{2\pi^2 E}{\sigma_{yp}} = \frac{2\pi^2(210\times10^9)}{203(10^6)} = 20{,}420; \qquad C_c = 143$$

(a) In this case $0 < (L_e/r) < C_c$. Then, substituting the given data, the first of Eqs. (11.8) yields

$$\sigma_{all} = 97.685\ MPa$$ ◀

(b) Now we have: $C_c \le (L_e/r) \le 200$ and the second of Eqs. (11.13) gives

$$\sigma_{all} = 30.87\ MPa$$ ◀

SOLUTION (11.23)

From Fig. P11.22, we observe that $I_y > I_z$.

Therefore the column buckles with respect to the z axis. Since,

$$\frac{L}{r} = \frac{4.2}{0.0225} = 186.7$$

We obtain

$$\sigma_{cr} = \frac{\pi^2 E}{(L_e/r)^2} = \frac{\pi^2(210\times10^9)}{(186.7)^2} = 59.46\ MPa$$ ◀

SOLUTION (11.24)

From Eq. (11.6),

$$(\tfrac{L}{r})_{\lim it} = [\tfrac{\pi^2(140\times10^9)}{280\times10^6}]^{\frac{1}{2}} = 70.2$$

We have

$$I = \tfrac{bh^3}{12} = Ar^2$$

or $$I = \tfrac{(0.05)(0.025)^3}{12} = 1.25\times10^{-3}r^2$$

Solving, $r = 7.216\ mm$

Hence, $(\tfrac{L}{r})_{actual} = \tfrac{1.2}{7.216(10^{-3})} = 166.3$

Thus, elastic buckling occurs, since $(L/r)_{\lim it} < (L/r)_{actual}$.

We have $\Delta = \alpha(\Delta T)L$ and $\varepsilon = \frac{\Delta}{L} = \alpha(\Delta T)$.

The condition that

$$\varepsilon = \alpha(\Delta T) - \tfrac{\pi^2}{(L/r)^2} = 0$$

results in

$$\Delta T = \tfrac{\pi^2}{\alpha(L/r)^2} = \tfrac{\pi^2}{10(10^{-6})(166.3)^2} = 35.7^o\ C$$ ◄

SOLUTION (11.25)

$$\sigma_{all} = \tfrac{P}{A} = \tfrac{125(10^3)}{3.06(10^{-3})} = 40.85\ MPa$$

Also $$C_c = [\tfrac{2\pi^2(200\times10^9)}{250(10^6)}]^{\frac{1}{2}} = 125.7$$

Assuming $(L/r) \geq C_c : \sigma_{all} = \frac{\pi^2(200\times10^9)}{1.92(L/r)^2} = \frac{1028.08(10^9)}{(L/r)^2}$

Solving, $\frac{L}{r} = 158.6$ O.K.

Choosing the smallest radii of gyration:

$$\tfrac{L}{r_y} = \tfrac{L}{0.0246} = 158.6$$

from which

$$L = 3.9\ m$$ ◄

SOLUTION (11.26)

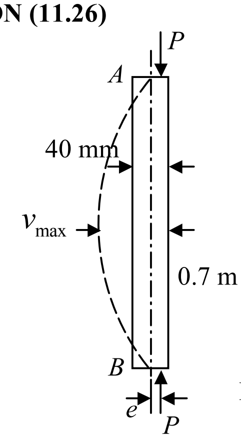


Figure (a)

$$A = \pi(20)^2 = 1256.64\ mm^2$$

$$I = \frac{\pi}{4}(20)^4 = 125.66\times10^3\ mm^4$$

$$P_{cr} = \frac{\pi^2 EI}{L^2}$$

$$= \frac{\pi^2(200\times10^9)(125.66\times10^{-9})}{(0.7)^2}$$

$$= 506.2\ kN$$

(CONT.)

11.26 (CONT.)

(**a**) Using Eq. (11.18):

$$1\times10^{-3}=e\left[\sec\left(\frac{\pi}{2}\sqrt{\frac{60}{506.2}}\right)-1\right]=e\left[\sec(31^o)-1\right],\qquad e=6\ mm \quad ◄$$

(**b**) Referring to Fig. (a):

$$M=P(v_{\max}+e)=80(1+6)=560\ N\cdot m$$

Hence, $$\sigma_{\max}=\frac{P}{A}+\frac{Mc}{I}=\frac{80\times10^3}{1256.64\times10^{-6}}+\frac{560(20)10^{-3}}{125.66\times10^{-9}}$$

$$=63.66+89.13=152.8\ MPa \quad ◄$$

SOLUTION (11.27)

$L_e=2L=1.4\ m$. Refer to solution of Prob. 11.26

$$P_{cr}=\frac{\pi^2EI}{L_e^2}=\frac{\pi^2(200\times10^9)(125.66\times10^{-9})}{(1.4)^2}=126.6\ kN$$

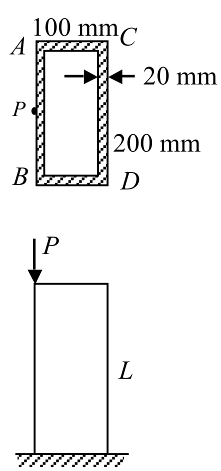
(**a**) Equation (11.18):

$$1\times10^{-3}=e\left[\sec\left(\frac{\pi}{2}\sqrt{\frac{60}{126.6}}\right)-1\right]=e[\sec(62^o)-1],\qquad e=0.88\ mm \quad ◄$$

(**b**) $M=P(v_{\max}+e)=80(1+0.88)=150.4\,N\cdot m$. Therefore,

$$\sigma_{\max}=\frac{P}{A}+\frac{Mc}{I}=63.66+\frac{150.4(20\times10^{-3})}{125.66\times10^{-9}}=63.66+23.9=87.6\ MPa \quad ◄$$

SOLUTION (11.28)



$$L_e=2L=3.6\ m$$

$$A=200\times100-160\times60$$
$$=10.4\times10^3\ mm^2$$

$$I=\frac{1}{12}(200\times100^3-160\times60^3)$$
$$=13.79\times10^6\ mm^4$$

$$r=\sqrt{I/A}=36.4\ mm$$

$$e=c=50\ mm$$

Thus,

$$\frac{ec}{r^2}=\frac{50\times50}{(36.4)^2}=1.89\qquad \frac{L_e}{2r}=49.45$$

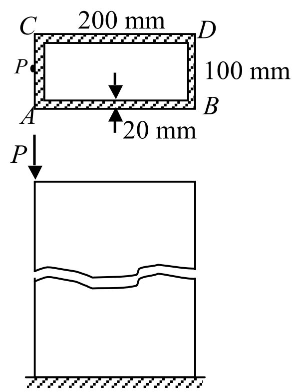
(CONT.)

11.28 (CONT.)

Use Eq.(11.19) with $L = L_e$:

$$\sigma_{\max} = \frac{250\times10^3}{10.4\times10^{-3}}\left[1+1.89\sec\left(49.45\sqrt{\frac{250\times10^3}{70\times10^9(10.4\times10^{-3})}}\right)\right] = 98.7\ MPa \quad ◄$$

SOLUTION (11.29)



$$L_e = 2L = 3.6\ m$$
$$A = 200\times100 - 160\times60$$
$$= 10.4\times10^3\ mm^2$$

$$I = \frac{1}{12}(100\times200^3 - 60\times160^3)$$
$$= 46.19\times10^6\ mm^4$$

$$e = c = 100\ mm$$
$$r = \sqrt{I/A} = 66.6\ mm$$

Therefore,

$$\frac{ec}{r^2} = \frac{100\times100}{(66.6)^2} = 2.25 \qquad \frac{L_e}{2r} = 27.03$$

Apply Eq.(11.19) with $L = L_e$:

$$\sigma_{\max} = \frac{250\times10^3}{10.4\times10^{-3}}\left[1+2.25\sec\left(27.03\sqrt{\frac{250\times10^3}{70\times10^9(10.4\times10^{-3})}}\right)\right] = 85.7\ MPa \quad ◄$$

SOLUTION (11.30)

$$I = \tfrac{1}{12}(50)(25)^3 = 65.1\times10^3\ mm^4$$
$$P_{cr} = \tfrac{\pi^2 EI}{L^2} = \tfrac{\pi^2(210)(65.1)}{(1.5)^2} = 59.97\ kN$$
$$e = \tfrac{h}{2} = \tfrac{25}{2} = 12.5\ mm, \qquad P = 10\ kN$$

Equation (11.13) gives

$$v_{\max} = 0.0125[\sec(\tfrac{\pi}{2}\sqrt{\tfrac{10}{59.97}}) - 1] = 3.10\ mm \quad ◄$$

Thus

$$M_{\max} = P(e + v_{\max}) = 10(12.5 + 3.10) = 156\ N\cdot m \quad ◄$$

SOLUTION (11.31)

$$p = 77(10^3)(0.05 \times 0.05) = 192.5\ N/m$$
$$I = \tfrac{(0.05)^4}{12} = 5.2(10^{-7})\ m^4$$

(a) $\sigma_{\max} = \frac{Mc}{I} = \frac{192.5(9)^2}{8}\frac{0.025}{5.2(10^{-7})} = 93.705\ MPa$ ◀

$$v_{\max} = \tfrac{5pL^4}{384EI}$$
$$= \frac{5(192.5)(9)^4}{384(210\times10^9)(5.2\times10^{-7})} = 0.1506\ m = 150.6\ mm$$ ◀

(b) Taking

$$a_0 = \tfrac{5pL^4}{384EI} = 150.6\ mm$$

We obtain, using Eq. (11.15):

$$v_{\max} = \frac{0.1506}{1-\frac{4500(9)^2}{(210\times10^9)(5.2\times10^{-7})}} = 227.5\ mm$$ ◀

Applying Eq. (11.16):

$$\sigma_{\max} = \tfrac{4500}{0.025}[1 + 2275\frac{0.025}{5.2(10^{-7})/0.025}]$$
$$= 51.019\ MPa$$ ◀

SOLUTION (11.32)

Given e=0.05 m, c=0.1016 m

and $r_z^2 = \frac{I_z}{A}$

$$= \tfrac{45.66}{5880} = 7.765(10^{-3})\ mm$$

Then, $P_{cr} = \frac{\pi^2 EI_y}{L^2} = \frac{\pi^2(210\times10^3)15.4}{(4.5)^2} = 1575\ kN$

Also,

$$P_{cr} = \frac{\pi^2(210\times10^3)45.66}{(4.5)^2} = 4669\ kN$$

Thus, Eq. (11.19) becomes

$$210(10^6) = \frac{P}{588(10^{-6})}[1 + \frac{0.05(0.1016)}{7.765(10^{-3})}\sec(\tfrac{\pi}{2}\sqrt{\tfrac{P}{4669\times10^3}})]$$

from which, by trial and error,

$$P \approx 700\ kN = P_{\max}$$ ◀

SOLUTION (11.33)

Governing equation is

$$EIv_1'' = -(v_0 + v_1)$$

where,

$$v_0 = a_1 \sin\tfrac{\pi x}{L} + 5a_1 \sin\tfrac{2\pi x}{L}$$

This may be written

$$v_1'' + \lambda^2 v_1 = -\lambda^2 a_1 \sin\tfrac{\pi x}{L} - 5\lambda^2 a_1 \sin\tfrac{2\pi x}{L} \qquad \text{(a)}$$

(CONT.)

We have

$$v_1 = c_1 \cos \lambda x + c_2 \sin \lambda x + v_p \tag{b}$$

Particular solution is

$$v_p = A \sin \tfrac{\pi x}{L} + B \cos \tfrac{\pi x}{L} + D \sin \tfrac{2\pi x}{L} + E \cos \tfrac{2\pi x}{L} \tag{c}$$

Substituting Eq. (c) into Eq. (a):

$$A = \frac{\lambda^2 a_1}{(\frac{\pi^2}{L^2} - \lambda^2)} \qquad B = 0$$

$$D = \frac{5\lambda^2 a_1}{(\frac{4\pi^2}{L^2} - \lambda^2)} \qquad E = 0$$

Thus, $v_p = \frac{\lambda^2 a_1}{(\pi/L)^2 - \lambda^2} \sin \frac{\pi x}{L} + \frac{5\lambda^2 a_1}{4(\pi/L)^2 - \lambda^2} \sin \frac{2\pi x}{L}$

Boundary conditions

$$v_1(0) = 0 \qquad v_1(L) = 0$$

yield $c_1 = c_2 = 0.$

General solution is

$$v = v_1 + v_p = v_0 + v_p$$

Letting $b = \frac{\pi^2}{\lambda^2 L^2} = \frac{PL^2}{\pi^2 EI}$

we obtain

$$v = \frac{a_1}{1-b} \sin \frac{\pi x}{L} + \frac{20 a_1}{4-b} \sin \frac{2\pi x}{L}$$

Solution of the b is found from

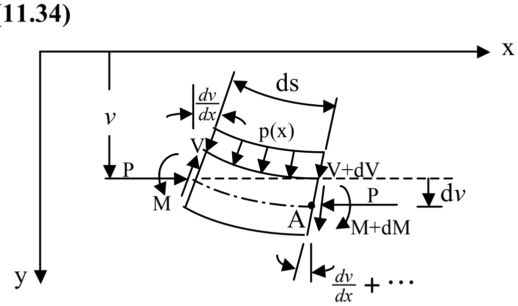
$$v(\tfrac{3L}{4}) = 0 = \frac{a_1}{1-b} \sin \frac{\pi(\frac{3L}{4})}{L} + 20 \frac{a_1}{1-b} \sin \frac{2\pi(\frac{3L}{4})}{L}$$

or $b = 0.89$

Thus, $b = \frac{PL^2}{\pi^2 EI}$, $\quad P = \frac{b\pi^2 EI}{L^2}$

and $P = 0.89 \frac{\pi^2 EI}{L^2}$ ◄

SOLUTION (11.34)



An element isolated from the beam is shown in a deformed state in the figure above. Assume $\sin\theta \approx \theta$, $\cos\theta \approx 1$, and $ds \approx dx$. Then

$$\sum F_y = 0: \qquad -V + pbx + (V + dV) = 0$$

or $p = -\frac{dV}{dx}$ (a)

(CONT.)

11.34 (CONT.)

$$\sum M_A = 0: \qquad M - Pdv - Vdx + pdx\,\tfrac{dx}{2} - (M + dM) = 0$$

Neglecting terms of second order, this becomes

$$V = -\tfrac{dM}{dx} - p\,\tfrac{dv}{dx} \qquad \text{(b)}$$

If the shear and axial deformations are neglected, the moment at any point is

$$M = EI\,\tfrac{d^2v}{dx^2} \qquad \text{(c)}$$

Substituting of Eqs. (c) and (b) into Eq. (a) gives

$$\tfrac{d^2}{dx^2}(EI\,\tfrac{d^2v}{dx^2}) + P\,\tfrac{d^2v}{dx^2} = p \qquad \text{(d)}$$

For EI = constant,

$$\tfrac{d^2v}{dx^2} + \tfrac{P}{EI}\,\tfrac{d^2v}{dx^2} = \tfrac{p}{EI} \qquad \text{(e)}$$

Homogeneous solution of this equation is

$$v = c_1 \sin\sqrt{\tfrac{P}{EI}}x + c_2 \cos\sqrt{\tfrac{P}{EI}}x + c_3 + c_4$$

where, c_1 through c_4 will require for evaluation, four boundary conditions.

SOLUTION (11.35)

Given $v = a_0[1 - (\tfrac{4x^2}{L^2})]$

and hence,

$$v' = -\tfrac{8xa_0}{L^2} \qquad v'' = -\tfrac{8a_0}{L^2}$$

Potential energy function is

$$\Pi = 2\int_0^{L/2} \tfrac{1}{2}EI(v'')^2\,dx - 2\int_0^{L/2}\tfrac{1}{2}P(v')^2\,dx = \tfrac{32EIa_0^2}{L^3} - \tfrac{32}{12}\,\tfrac{Pa_0^2}{L}$$

Thus, $\tfrac{\delta\Pi}{\delta a_0} = \tfrac{64EIa_0}{L^3}\,\tfrac{64}{12}\,\tfrac{Pa_0}{L} = 0$

or $P_{cr} = \tfrac{12EI}{L^2}$ ◀

SOLUTION (11.36)

Assume $v = v_0\sin(\tfrac{\pi x}{L})$. Then,

$$U = 2\int_0^{L/2} EI_1(1 + \tfrac{x}{L/2})(v'')^2\,dx$$

$$= 2EI_1 v_0''\,\tfrac{\pi^4}{L^4}[\int_0^{L/2}\sin^2\tfrac{\pi x}{L}\,dx + \tfrac{2}{L}\int_0^{L/2} x\sin^2\tfrac{\pi x}{L}\,dx]$$

$$= 2EI_1 v_0^2\,\tfrac{\pi^4}{L^4}[\tfrac{L}{4} + \tfrac{2L}{\pi^2}(\tfrac{\pi^2}{16} + \tfrac{1}{4})] \qquad \text{(a)}$$

We also have

$$\int_0^L (v')^2\,dx = v_0^2\,\tfrac{\pi^2}{L^2}\int_0^L \cos^2\tfrac{\pi x}{L}\,dx = \tfrac{v_0^2\pi^2}{2L}$$

Thus,

$$\int_0^L EI(v'')^2\,dx = P\int_0^L (v')^2\,dx$$

yields

$$P_{cr} = \tfrac{\pi^2 EI_1}{L^2}(\tfrac{3}{2} + \tfrac{2}{\pi^2}) = 1.7\,\tfrac{\pi^2 EI}{L^2}$$ ◀

SOLUTION (11.37)

We have $v' = 2v_1 \frac{x}{L^2}$ $\quad v'' = 2\frac{v_1}{L^2}$

Potential energy function is

$$\Pi = \frac{EI_1}{2}\int_0^L (1-\frac{x}{2L})\frac{4v_1^2}{L^4}dx - \frac{P}{2}\int_0^L \frac{4v_1^2}{L^4}x^2 dx = \frac{3}{2}\frac{EI_1v_1^2}{L^3} - \frac{2}{3}\frac{v_1^2 P}{L}$$

Hence, $\frac{\delta\Pi}{\delta v_1} = \frac{3EI_1v_1}{L^3} - \frac{4v_1P}{3L} = 0$

gives

$$P_{cr} = \frac{9EI_1}{4L^2}$$ ◀

SOLUTION (11.38)

Assume $v = \sum_1^\infty a_n \sin(\frac{n\pi x}{L})$. Then,

$$U = \frac{\pi^4 EI}{4L^3}\sum n^4 a_n^2$$

and $$W = \frac{1}{2}\int_0^L P(v')^2 dx + \int_0^L pv dx$$

$$= \frac{\pi^2 P}{4L}\sum n^2 a^2 + \frac{2PL}{\pi}\sum a_n \frac{1}{n}$$

Hence, from $\delta U = \delta W$, by letting

$$b = \frac{P}{P_{cr}} = \frac{PL^2}{\pi^2 EI}$$

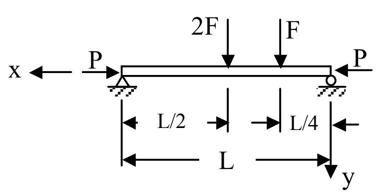
we obtain

$$a_n = \frac{4PL^4}{\pi^5 EI}\sum_{1,3,\cdots}^\infty \frac{1}{n^3(n^2-b)}$$

Thus, ◀

$$v = \frac{4PL^4}{\pi^5 EI}\sum_{1,3,\cdots}^\infty \frac{1}{n^3(n^2-b)}\sin\frac{n\pi x}{L}$$

SOLUTION (11.39)



Let $c_1 = \frac{L}{4}$ and $c_2 = \frac{L}{2}$. Deflection curve is expressed by

$$v = \sum_1^\infty a_n \sin\frac{n\pi x}{L} \qquad \text{(a)}$$

and $$U = \frac{EI}{2}\int_0^L (v'')^2 dx$$

$$= F\sum a_n \sin\frac{n\pi c_1}{L} + 2F\sum a_n \sin\frac{n\pi c_2}{L} + \frac{P}{2}\int_0^L (v')^2 dx$$

(CONT.)

11.39 (CONT.)

Therefore,

$$\frac{\partial(U-W)}{\partial a_n} = 0$$

gives

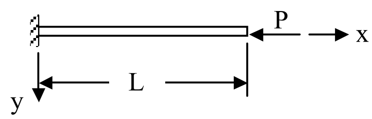
$$\frac{\pi^4 EI}{2L^3} n^4 a_n \delta a_n = F\delta a_n \sin\frac{n\pi c_1}{L} + 2F\delta a_n \sin\frac{n\pi c_2}{L} + \frac{P\pi^2}{2L} a_n n^2$$

or

$$a_n = \frac{F\sin(n\pi/4) + 2F\sin(n\pi/2)}{\frac{\pi^2 EIn^2}{2L^3}(n^2 - \frac{P}{P_{cr}})}$$ ◀

Solution is found by substituting this into Eq. (a).

SOLUTION (11.40)



Boundary conditions are: $v(0) = v(L) = 0$

We have

$$v = aLx^2 + bLx^3 - ax^3 - bx^4$$

Thus,

$$v' = 2aLx + 3bLx^2 - 3ax^2 - 4bx^3$$
$$v'' = 2aL + 6BLx - 6ax - 12bx^2$$

Hence,

$$\Pi = \frac{EI}{2}\int_0^L (v'')^2 dx - \frac{P}{2}\int_0^L (v')^2 dx$$
$$= \frac{EI}{2}[4a^2L^3 + 8abL^4 + \frac{24}{5}b^2L^5] - \frac{P}{2}[\frac{2}{15}a^2L^5 + \frac{1}{5}abL^6 + \frac{3}{35}b^2L^7]$$

It is required that

$$\frac{\partial\Pi}{\partial a} = \frac{EI}{2}(8aL^3 + 8abL^4) - \frac{P}{2}(\frac{4}{15}aL^5 + \frac{bL^6}{5}) = 0$$
$$\frac{\partial\Pi}{\partial b} = \frac{EI}{2}(8aL^4 + \frac{48}{5}bL^5) - \frac{P}{2}(\frac{1}{5}aL^6 + \frac{bL^7}{35}) = 0$$

Letting $\lambda = \frac{PL^2}{EI}$, these become

$$(4 - \frac{2\lambda}{35})a + (4 - \frac{\lambda}{10})bL = 0$$
$$(4 - \frac{\lambda}{10})a + (\frac{24}{5} - \frac{3\lambda}{35})bL = 0$$

Since $a \neq 0$ and $b \neq 0$:

$$-(4 - \frac{\lambda}{10})^2 + (4 - \frac{2\lambda}{15})(\frac{24}{5} - \frac{3\lambda}{35}) = 0$$

or

$$\lambda^2 - 128\lambda + 2240 = 0$$

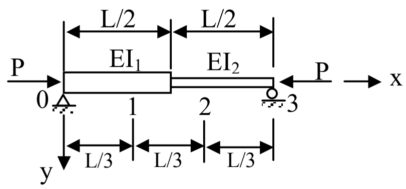
Solving,

$$\lambda_1 = 20.9 \qquad \lambda_2 = 107.1$$

Hence,

$$P_{cr} = \frac{20.9EI}{L} = 2.12\frac{\pi^2 EI}{L^2}$$ ◀

SOLUTION (11.41)



From Eq. (11.29):

$$v_{m+1} + (\lambda_1 h^2 - 2)v_m + v_{m-1} = 0 \tag{a}$$

Here

$$\lambda_1^2 = \tfrac{P}{EI_1} \qquad \lambda_2^2 = \tfrac{P}{EI_2}$$

Applying Eq. (a) at 1 and 2:

$$\begin{bmatrix} \lambda_1 h^2 - 2 & 1 \\ 1 & \lambda_2 h^2 - 2 \end{bmatrix} \begin{Bmatrix} v_1 \\ v_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Thus,

$$(\lambda_1 h^2 - 2)(\lambda_2 h^2 - 2) - 1 = 0$$

or

$$\lambda_1\lambda_2 h^4 - 2\lambda_1 h^2 - 2\lambda_2 h^2 + 3 = 0$$

This is written as

$$P^2[\tfrac{L^4}{81E^2 I_1 I_2}] - P[\tfrac{2L^2}{9EI_1} + \tfrac{2L^2}{9EI_2}] + 3 = 0$$

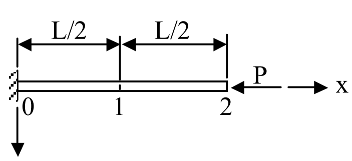
Solution is

$$(P_{1,2})_{cr} = \tfrac{9E}{L^2}(I_1 + I_2) \pm \tfrac{9E}{L^2}[I_1^2 - I_1 I_2 + I_2^2]^{\frac{1}{2}}$$

When $I_1 = I_2$:

$$P_{cr} = \tfrac{9EI}{L^2}$$ ◀

SOLUTION (11.42)



Boundary conditions yield

$$v'(0) = 0; \qquad v_0 = 0$$

$$v(0) = 0; \qquad v_1 = v_{-1}$$

$$v''(L) = 0; \qquad v_3 - 2v_2 + 2v_1 = 0 \tag{1}$$

$$v'''(L) = 0; \qquad v_4 - 2v_3 + 2v_1 = 0 \tag{2}$$

(CONT.)

Then, applying Eq. (l) of Example 11.8 at points 1, 2, 0, respectively:

$$(7-2\lambda^2h^2)v_1+(\lambda^2h^2-4)v_2+v_3=0 \tag{3}$$

$$(\lambda^2h^2-4)v_1+(6-2\lambda^2h^2)v_2+(\lambda^2h^2-4)v_3+v_4=0 \tag{4}$$

$$(\lambda^2h^2-4)v_1+v_2=0 \tag{5}$$

From Eq. (5):

$$v_2=-v_1(\lambda^2h^2-4)$$

Equation (1) becomes

$$v_3=2v_1-v_1=-2v_1(\lambda^2h^2-4)v_1-2v_1(\lambda^2h^2-4)-v_1$$

Substituting this into Eq. (3),

$$(7-2\lambda^2h^2)v_1-(\lambda^2h^2-4)(\lambda^2h^2-4)v_1-2v_1(\lambda^2h^2-4)-v_1=0$$

from which

$$\lambda^4h^4-4\lambda^2h^2+2=0$$

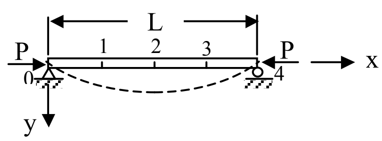
or

$$\lambda^2h^2=0.59=\tfrac{Ph^2}{EI}$$

Thus,

$$P_{cr}=\tfrac{2.36EI}{L^2}$$ ◄

SOLUTION (11.43)



From symmetry: $v_1=v_3$.

We have

$$I(x)=(1+\tfrac{2x}{L})EI_1 \qquad 0\le x\le \tfrac{L}{2}$$

$$I(x)=(3-\tfrac{2x}{L})EI_1 \qquad \tfrac{L}{2}\le x\le L$$

Equation (11.29), gives at 0, 1, 2, 3:

$$v_1+v_{-1}=0; \qquad v_1=-v_{-1}$$

$$v_2+[\tfrac{Ph^2}{(3/2)EI_1}-2]v_1=0$$

$$v_1+[\tfrac{Ph^2}{EI_1}-2]v_2+v_1=0$$

$$[\tfrac{Ph^2}{(3/2)EI_1}-2]v_1+v_2=0$$

The foregoing equations lead to

$$\begin{bmatrix}\frac{2}{3}\lambda^2h^2-2 & 1\\ 2 & \frac{1}{2}\lambda^2h^2-2\end{bmatrix}\begin{Bmatrix}v_1\\ v_2\end{Bmatrix}=\begin{Bmatrix}0\\ 0\end{Bmatrix}$$

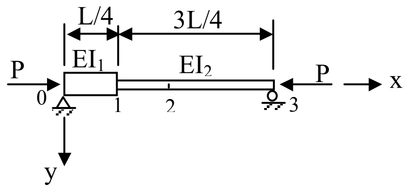
from which, since $v_1\ne 0$ and $v_2\ne 0$:

$$(\lambda^2h^2-6)(\lambda^2h^2-1)=0$$

Thus,

$$P_{cr}=16\tfrac{EI_1}{L^2}$$ ◄

SOLUTION (11.44)



We have

$$\alpha_1 = \frac{L/4}{L/4} = 1 \qquad \alpha_2 = \frac{L/2}{L/4} = 2.$$

Applying Eq. (11.30) at 1 and 2:

$$[\tfrac{PL^2}{16EI_1} - 2]v_1 + v_2 = 0 \tag{a}$$

$$\tfrac{2}{3}v_1[\tfrac{PL^2}{16EI_1} - 1]v_2 = 0 \tag{b}$$

Let

$$k_1 = \frac{PL^2}{16EI_1} - 1 \qquad k_2 = \frac{PL^2}{16EI_2}$$

Equations (a) and (b) yield then

$$\begin{vmatrix} k_1 - 2 & 1 \\ \frac{2}{3} & k_2 - 1 \end{vmatrix} = 0$$

from which

$$(k_1 - 2)(k_2 - 1) - \tfrac{2}{3} = 0$$

or

$$P^2[\tfrac{L^4}{256E^2I_1I_2}] - P[\tfrac{2L^2}{16EI_2} + \tfrac{L^2}{16EI_1}] + \tfrac{4}{3} = 0$$

Solution of this quadratic equation gives the critical load as follows:

$$P_{cr} = \frac{8E(2I_1 + I_2)}{L^2} - \frac{8E}{L^2}[4I_1^2 + I_2^2 - 1.24I_1I_2]^{\frac{1}{2}}$$

In a special case, for $I_1 = I_2 = I$, the preceding reduces to

$$P_{cr} = \frac{24EI}{L^2} - \frac{15.3EI}{L^2} = 8.7\frac{EI}{L^2}$$ ◄

End of Chapter 11

CHAPTER 12

SOLUTION (12.1)

Components of stress are

$$\sigma_x = \frac{P}{A} + \frac{Mr}{I}$$
$$= \frac{90}{\pi(0.05)^2} + \frac{3375(0.05)}{\pi(0.05)^4/4} = 45.84\ MPa$$
$$\tau_{xy} = \frac{Tr}{J} = \frac{4500(0.05)}{\pi(0.05)^4/2} = 22.92\ MPa$$

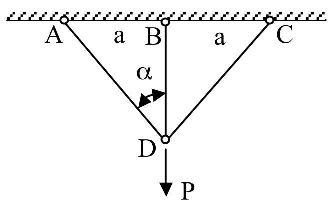
and

$$\sigma_y = \sigma_z = \tau_{xz} = \tau_{yz} = 0$$

Thus, from Sec. 2.13 (in MPa):

$$\begin{bmatrix} \frac{2\sigma_x}{3} & \tau_{xy} & 0 \\ \tau_{xy} & -\frac{\sigma_x}{3} & 0 \\ 0 & 0 & -\frac{\sigma_x}{3} \end{bmatrix} = \begin{bmatrix} 30.56 & 22.92 & 0 \\ 22.92 & -15.28 & 0 \\ 0 & 0 & -15.28 \end{bmatrix}$$

SOLUTION (12.2)



At instability, member AD (or DC) and BD become in length:

$$L'_{AD} = L'_{DC} = L_{AD} + \frac{n_1}{1-n_1} L_{AD} = \frac{L_{AD}}{1-n_1}$$
$$L'_{BD} = L_{BD} + \frac{n_2}{1-n_2} L_{BD} = \frac{L_{BD}}{1-n_2}$$

Therefore,

$$\left(\frac{L_{AD}}{1-n_1}\right)^2 = a^2 + \left(\frac{L_{BD}}{1-n_2}\right)^2 \qquad \text{(a)}$$

Initially:

$$L_{AD}^2 = a^2 + L_{BD}^2 \qquad \text{(b)}$$

Eliminating a from Eqs. (a) and (b), we obtain

$$\cos\alpha = \frac{L_{BD}}{L_{AD}} = \left(\frac{1-n_2}{1-n_1}\right)\sqrt{\frac{n_1(2-n_1)}{n_2(2-n_2)}}$$

For

$$n_1 = 0.2 \qquad \text{and} \qquad n_2 = 0.3$$

We have

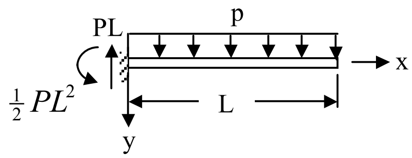
$$\cos\alpha = \frac{0.7}{0.8}\left[\frac{0.2(1.8)}{0.3(1.7)}\right]^{\frac{1}{2}} = 0.7351$$

or

$$\alpha = 42.68^o$$

◀

SOLUTION (12.3)



We write

$$M(x) = \tfrac{1}{2}p(L-x)^2$$

Equation (12.7) is then

$$v'' = \left(\tfrac{M}{KI_n}\right)^{\frac{1}{n}} = \left(\tfrac{P}{2KI_n}\right)^{\frac{1}{n}}(L-x)^{\frac{2}{n}}$$

$$v' = -\frac{\lambda(L-x)^{\frac{2}{n}+1}}{(\frac{2}{n}+1)(\frac{2}{n}+2)} + c_1$$

$$v = \frac{\lambda(L-x)^{\frac{2}{n}+2}}{(\frac{2}{n}+1)(\frac{2}{n}+2)} + c_1x + c_2 \qquad \text{(a)}$$

Boundary conditions yield

$$v'(0) = 0; \qquad c_1 = \frac{\lambda L^{\frac{2}{n}+1}}{\frac{2}{n}+1} \qquad\qquad v(0) = 0; \qquad c_2 = -\frac{\lambda L^{\frac{2}{n}+2}}{(\frac{2}{n}+1)(\frac{2}{n}+2)}$$

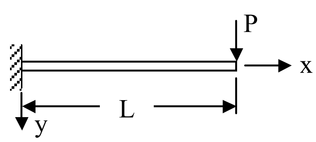
Substituting these and Eq. (g) of Sec. 12.5 into Eq. (a), we obtain

$$v = \left(\tfrac{p}{2K-n}\right)^{\frac{1}{n}}\left[\frac{(L-x)^{\frac{2}{n}+2}}{(\frac{2}{n}+1)(\frac{2}{n}+2)} - \frac{L^{\frac{2}{n}+2}}{(\frac{2}{n}+1)(\frac{2}{n}+2)} + \frac{L^{\frac{2}{n}+1}}{\frac{2}{n}+1}x\right]$$

For $n = 1$, $K = E$, and $x = L$:

$$v = \tfrac{pL^4}{6EI} - \tfrac{pL^4}{24EI} = \tfrac{pL^4}{8EI} \qquad ◄$$

SOLUTION (12.4)



We write

$$M = P(L-x)$$

Hence,

$$v'' = \left(\tfrac{P}{KI_n}\right)^{\frac{1}{n}}(L-x)^{\frac{1}{n}} = \lambda(L-x)^{\frac{1}{n}}$$

$$v' = -\frac{\lambda(L-x)^{\frac{1}{n}+1}}{\frac{1}{n}+1} + c_1$$

$$v = \frac{\lambda(L-x)^{\frac{1}{n}+2}}{(\frac{1}{n}+1)(\frac{1}{n}+2)} + c_1x + c_2$$

(a)

Boundary conditions yield

$$v'(0) = 0; \qquad c_1 = \frac{\lambda L^{\frac{1}{n}+1}}{\frac{1}{n}+1} \qquad\qquad v(0) = 0; \qquad c_2 = -\frac{\lambda L^{\frac{1}{n}+2}}{(\frac{1}{n}+1)(\frac{1}{n}+2)}$$

Substituting these onto Eq. (a), we find an expression for the deflection.

For a special case of $n = 1$ and $K = E$, the deflection at free end:

$$v(L) = \tfrac{PL^3}{2EI} - \tfrac{PL^3}{6EI} = \tfrac{PL^3}{3EI} \qquad ◄$$

SOLUTION (12.5)

We have $\sigma = K\varepsilon^{\frac{1}{4}}$ and $I = \frac{2}{3}bh^3$. Here

$$\varepsilon = \tfrac{y}{h}\varepsilon_{\max} = ay, \qquad a = \tfrac{\varepsilon_{\max}}{h}$$

Moment is thus,

$$M = \int \sigma y dA = \int_{-h}^{h} yK(ay)^{\frac{1}{4}} bdy = \tfrac{8}{9}Ka^{\frac{9}{4}}$$

But $\quad \sigma_{\max} = K\varepsilon_{\max}^{\frac{1}{4}} = Ka^{\frac{1}{4}}h^{\frac{1}{4}}$

Hence, $\quad M = (\tfrac{8}{9})bh^2\sigma_{\max}$

or

$$\sigma_{\max} = \tfrac{9M}{8bh^2} = \tfrac{3Mh}{4I} \qquad \blacktriangleleft$$

SOLUTION (12.6)

$$M = -Px \qquad\qquad 0 \le x \le a$$

Hence, $\quad v_1'' = (-\tfrac{Px}{KI_n})^{\frac{1}{n}} = -\lambda x^{\frac{1}{n}}, \qquad v_1' = -\dfrac{\lambda x^{\frac{1}{n}+1}}{\frac{1}{n}+1} + c_1$

$$v_1 = -\frac{\lambda x^{\frac{1}{n}+2}}{(\frac{1}{n}+1)(\frac{1}{n}+2)} + c_1 x + c_2 \qquad \text{(a)}$$

Similarly, $M = -Pa$, at $a \le x \le (L-a)$:

$$v_2'' = (-\tfrac{Pa}{KI_n})^{\frac{1}{n}} = -\lambda a^{\frac{1}{n}}, \qquad v_2' = -\lambda a^{\frac{1}{n}}x + c_3$$

$$v_2 = -\lambda a^{\frac{1}{n}}\tfrac{x^2}{2} + c_3 x + c_4 \qquad \text{(b)}$$

Boundary conditions yield,

$$v_1(0) = 0; \qquad c_2 = 0$$

$$v_2(\tfrac{L}{2}) = 0; \qquad c_3 = \lambda a^{\frac{1}{n}}\tfrac{L}{2}$$

$$v_1'(a) = v_2'(a); \qquad c_1 = \lambda a^{\frac{1}{n}}\tfrac{L}{2} - \frac{\lambda a^{\frac{1}{n}+1}}{n(\frac{1}{n}+1)}$$

$$v_1(a) = v_2(a); \qquad c_4 = \frac{\lambda a^{\frac{1}{n}+2}}{2} - \frac{\lambda a^{\frac{1}{n}+2}}{(\frac{1}{n}+1)(\frac{1}{n}+2)} - \frac{\lambda a^{\frac{1}{n}+2}}{n(\frac{1}{n}+1)}$$

Substitution of these of these constants into Eqs. (a) and (b) gives the required solution.

For a special case of $n = 1$ and $K = E$, the midspan deflection is

$$v(\tfrac{L}{2}) = v_{\max} = \tfrac{PaL^2}{8EI} - \tfrac{Pa^3}{6EI}$$

or $\quad v_{\max} = \tfrac{Pa}{24EI}(3L^2 - 4a^2) \qquad \blacktriangleleft$

We compute

$\bar{y} = 23.19\ mm$ (measured from top surface) and $I = 4.4(10^{-7})\ m^4$.

Therefore

$$v_{\max} = \frac{0.45(8000)}{24(4.4\times10^{-7})200\times10^9}[3(1.2)^2 - 4(0.45)^2]$$

$$= 0.00598\ m = 6\ mm \qquad \blacktriangleleft$$

SOLUTION (12.7)

$$A_{AB} = \frac{\pi}{4}(0.06)^2 = 2827.44\times10^{-6}\ m^2$$

$$A_{BC} = \frac{\pi}{4}(0.05)^2 = 1965.5\times10^{-6}\ m^2$$

(**a**) Since $A_{BC} < A_{AB}$:

$$P_{\max} = \sigma_{yp}A_{BC} = 240\times10^6(1963.5\times10^{-6})$$
$$= 471.2\ kN \qquad ◄$$

(**b**) Loading

Segment *AB* deforms elastically:

$$\delta_{AB} = \frac{PL}{AE} = \frac{471.2\times10^3(1.5)}{2827.44\times10^{-6}(210\times10^9)} = 1.19\ mm$$

Segment *BC* deforms plastically:

$$(\delta_{BC})_{\max} = \delta_{AC} - \delta_{AB} = 10 - 1.19 = 8.81\ mm$$

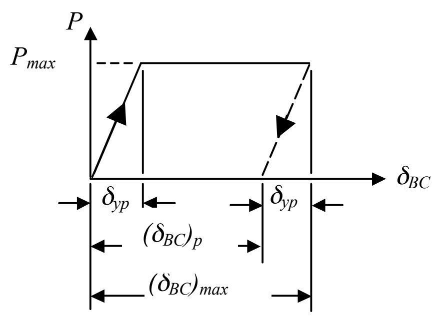
After unloading

Segment *BC* remains elastic. Thus

$$(\delta_{AB})_p = 0$$

Segment *BC* remains plastic. We have:

$$\delta_{yp} = \varepsilon_{yp}L = \frac{\sigma_{yp}}{E}L = \frac{240\times10^6}{210\times10^9}(1) = 1.15\ mm$$



Referring to the above figure:

$$(\delta_{BC})_p = (\delta_{BC})_{\max} - \delta_{yp} = 8.81 - 1.15 = 7.66\ mm \qquad ◄$$

SOLUTION (12.8)

$$A_{AB} = \frac{\pi}{4}(0.03)^2 = 706.85\times10^{-6}\ m^2$$

$$A_{BC} = \frac{\pi}{4}(0.02)^2 = 314.16\times10^{-6}\ m^2$$

(CONT.)

(**a**) Since $A_{BC} < A_{AB}$:

$$P_{\max} = \sigma_{yp} A_{BC} = 280\times10^6(314.16\times10^{-6})$$
$$= 87.96\ kN$$ ◀

(**b**) Loading

Segment AB deforms elastically:

$$\delta_{AB} = \frac{PL}{AE} = \frac{87.96\times10^3(1.5)}{706.85\times10^{-6}(210\times10^9)} = 0.89\ mm$$

Segment BC deforms plastically:

$$(\delta_{BC})_{\max} = \delta_{AC} - \delta_{AB} = 10 - 0.89 = 9.11\ mm$$

After loading

Segment AB remains elastic. Thus

$$(\delta_{AB})_p = 0$$ ◀

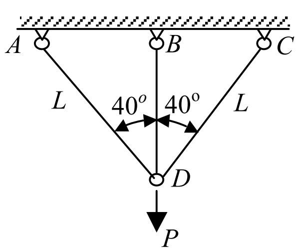
Segment BC remains plastic. We have:

$$\delta_{yp} = \frac{\sigma_{yp}}{E}L = \frac{280\times10^6}{210\times10^9}(1) = 1.33\ mm$$

Referring to the solution of prob. 12.1:

$$(\delta_{BC})_p = (\delta_{BC})_{\max} - \delta_y$$
$$= 9.11 - 1.33 = 7.78\ mm$$ ◀

SOLUTION (12.9)



Using Eqs. (P12.9):

$$N_{AD} = N_{CD} = \frac{P\cos^2 40^o}{1+2\cos^3 40^o}$$
$$= 0.309P_u$$

$$N_{BD} = \frac{P}{1+2\cos^3 40^o} = 0.527P$$

Since $N_{BD} > N_{AD}$, we have

$$0.527P_u = \sigma_{yp}A = 250\times10^6(400\times10^{-6})$$

or

$$P_u = 189.8\ kN$$ ◀

SOLUTION (12.10)

See Eqs.(i) & (j) of Example 12.3:

$$P_s' = 4.32P_a' \qquad 2P_a' + P_s' = 400 \qquad (1, 2)$$

Solving,

$$P_a' = 63.3\ kN \qquad P_s' = 189.9\ kN$$

Then

$$\sigma_s = \frac{189.9\times10^3}{750(10^{-6})} = 253\ MPa \qquad \sigma_a = \frac{63.3\times10^3}{500(10^{-6})} = 126.6\ MPa$$

Since 253 MPa > 240 MPa and 126.6 MPa < 320 MPa, steel bar yields while aluminum bars remain elastic. Thus

$$(\sigma_a)_{res} = 0 \qquad (\sigma_s)_{res} = 253 - 240 = 13\ ksi \qquad \blacktriangleleft$$

SOLUTION (12.11)

Rod begins to yield at:

$$(P_r)_{yp} = (\sigma_r)_{yp} A_r = (250)(45) = 11.25\ kN$$

$$(\delta_r) = (\varepsilon_r)_{yp} L = \frac{(\sigma_r)_{yp}}{E_r} L = \frac{250\times10^6}{200\times10^9}(1.2) = 1.5\ mm$$

The result is shown in Fig. (a). Here Y_r corresponds to the onset of yield in the rod.

Tube begins to yield at:

$$(P_t)_{yp} = (\sigma_t)_{yp} A_t = (310)(60) = 18.6\ kN$$

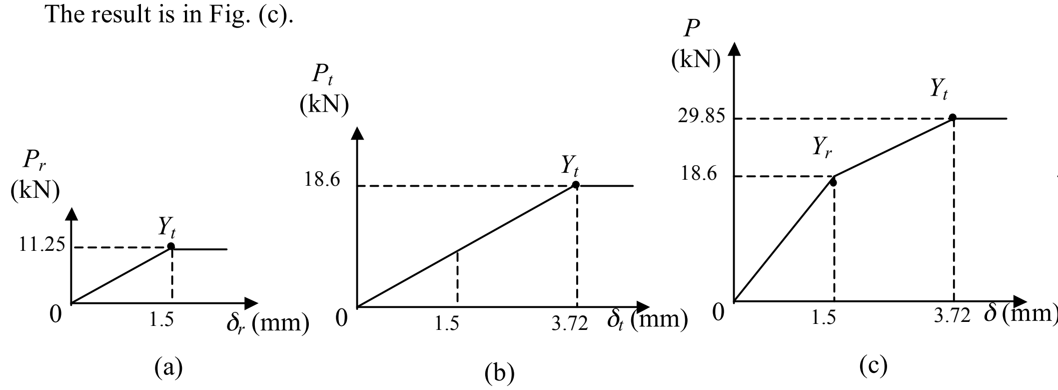
$$(\delta_t)_{yp} = \frac{(\sigma_t)_{yp}}{E_t} L = \frac{310\times10^6}{100\times10^9}(1.2) = 3.72\ mm \qquad \blacktriangleleft$$

The result is shown in Fig. (b), where Y_t represents the onset of yield in the tube.

Total P-δ of the rod-tube combination:

$$P = P_r + P_t \qquad \delta = \delta_r = \delta_t$$

The result is in Fig. (c).



$\blacktriangleleft$

SOLUTION (12.12)

We have $e = 6 - 2 = 4\ mm$, $I = (1/12)(12)^4 = 1.728 \times 10^3\ mm^4$ and $c = 6\ mm$.

(a) Applying Eq. (12.9):

$$M_{yp} = \frac{\sigma_{yp} I}{c} = \frac{350 \times 1.728}{6}$$

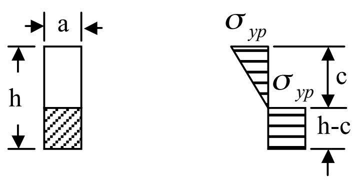
or

$$M_{yp} = 100.8\ N \cdot m \quad ◄$$

(b) Equation (12.10) results in

$$M_u = a\sigma_{yp}(a^2 - \tfrac{e^2}{3}) = (0.012)(350)(12^2 - \tfrac{4^2}{3})$$
$$= 582.4\ N \cdot m \quad ◄$$

SOLUTION (12.13)



Referring to this figure,

$$(\sigma_{yp})\tfrac{ca}{2} = (h - c)a\sigma_{yp}$$

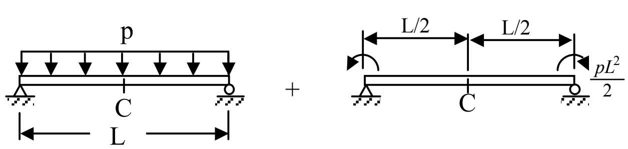
or

$$c = \tfrac{2h}{3}$$

Hence,

$$M = \tfrac{1}{2}\tfrac{2ha}{3}\sigma_{yp}\tfrac{2}{3}\tfrac{2h}{3} + \tfrac{h}{3}a\sigma_{yp}\tfrac{h}{6}$$
$$= (\tfrac{11}{54})ah^2\sigma_{yp} \quad ◄$$

SOLUTION (12.14)



Deflection at C (Case 7 of Tables D.4), in <u>elastic range</u>:

$$v_{\max} = \frac{pL^4}{384EI}$$

<u>Start of yielding</u>:

$$p = p_{yp} = \frac{12M_{yp}}{L^2}$$

Thus,

$$v_{\max} = \frac{M_{yp}L}{32EI} = \frac{(2bh^2\sigma_{yp}/3)L^2}{32E(2bh^3/3)} = \frac{\sigma_{yp}L^2}{32Eh} \quad ◄$$

SOLUTION (12.15)

Equation (12.9):

(**a**) $$M_{yp} = \frac{2}{3}bh^2\sigma_{yp}$$

$$= \frac{2\times 0.06(0.04)^2}{3}(240\times 10^6) = 15.36\ kN\cdot m \quad ◀$$

(**b**) Equation (12.10b):

$$M = \frac{3}{2}(15.36\times 10^3)[1-\frac{1}{3}(\frac{20}{40})^2] = 21.12\ kN\cdot m \quad ◀$$

SOLUTION (12.16)

(**a**) Equations (12.9) and (12.10b):

$$\frac{M}{M_{yp}} = 1.3 = \frac{3}{2}[1-\frac{1}{3}(\frac{e}{h})^2]$$

or

$$0.867 = 1-\frac{1}{3}(\frac{e^2}{h^2}), \qquad e = 0.632h \quad ◀$$

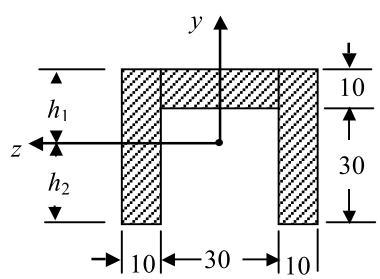
(**b**) The residual stress pattern will be as in Fig. 12.16c. ◀

SOLUTION (12.17)

$$A = 2(10)(40)+(10)(30) = 1100\ mm^2$$

Neutral axis divides section into two equal areas:

$$(30)(10)+2(10)(h_1) = \frac{A}{2} = 550\ mm^2$$



Solving

$$h_1 = 12.5\ mm \qquad h_2 = 27.5\ mm$$

Therefore,

$$Z = \frac{A(y_1+y_2)}{2}$$

where

$$y_1 = \frac{\sum A_i\bar{y}_i}{\sum A_i} = \frac{1}{A/2}[2(h_1)(10)(\frac{h_1}{2})+10(30)(h_1-5)^2] = 6.93\ mm$$

$$y_2 = \frac{1}{A/2}[2(10)(h_2)(\frac{h_2}{2})] = 13.75\ mm$$

(CONT.)

12.17 (CONT.)

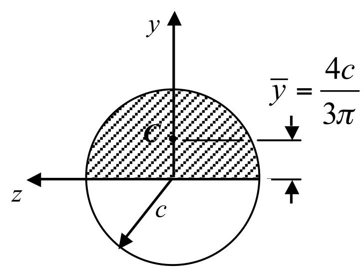
So

$$Z = \frac{1100}{2}(6.93 + 13.75) = 11{,}374\ mm^3$$

$$M_u = \sigma_{yp} Z = (260\times10^6)(11{,}374\times10^{-9})$$

$$= 2.96\ kN\cdot m \quad ◄$$

SOLUTION (12.18)



$$A = \frac{\pi c^2}{2} \qquad S = \frac{\pi c^3}{4}$$

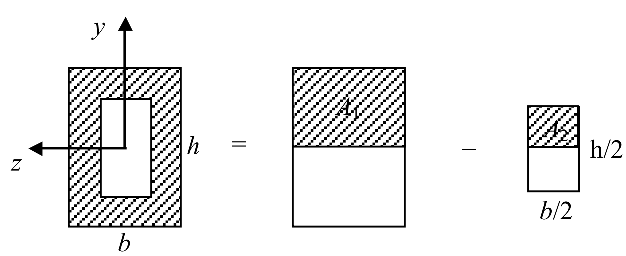
$$Z = 2A\overline{y} = 2\frac{\pi c^2}{2}\frac{4c}{3\pi}$$

$$= \frac{4c^3}{3}$$

Thus,

$$f = \frac{Z}{S} = \frac{16}{3\pi} \approx 1.7 \quad ◄$$

SOLUTION (12.19)



$$Z = 2A_1\overline{y}_1 - 2A_2\overline{y}_2 = 2[\frac{bh}{2}\frac{h}{4} - \frac{bh}{2(4)}\frac{h}{8}] = \frac{7bh^2}{32}$$

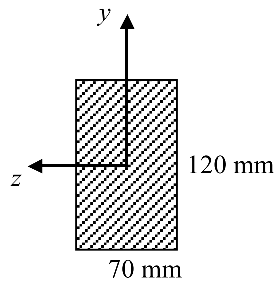
$$S = \frac{I}{h/2} = \frac{2}{h}\frac{1}{12}[bh^3 - \frac{b}{2}(\frac{h}{2})^3]$$

$$= \frac{bh^2}{6}(1 - \frac{1}{16}) = \frac{15}{96}bh^2$$

Thus,

$$f = \frac{Z}{S} = \frac{7bh^2}{32}\frac{96}{15bh^2} = 1.4 \quad ◄$$

SOLUTION (12.20)



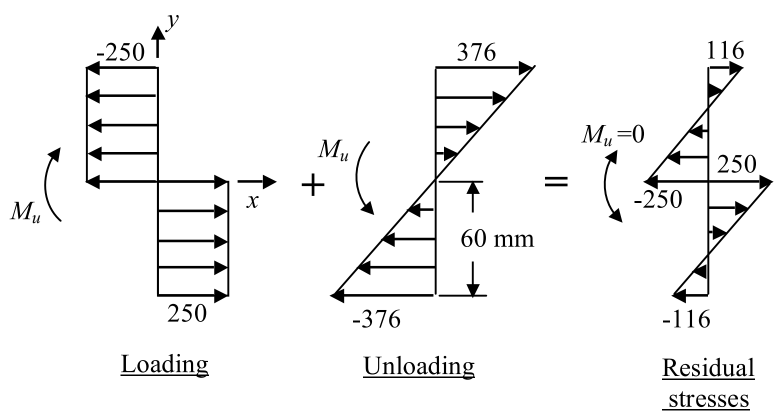
$$I = \frac{1}{12}(70)(120)^3$$
$$= 10\times10^6 \ mm^4$$
$$M_{yp} = \frac{I}{c}\sigma_{yp}$$
$$= \frac{10\times10^{-6}}{0.06}(250\times10^6)$$
$$= 41.7 \ kN\cdot m$$

$$M_u = fM_{yp} = \frac{3}{2}(41.7) = 62.6 \ kN\cdot m$$

Elastic rebound stress

$$\sigma'_{\max} = \frac{Mc}{I} = \frac{62.6\times10^3(0.06)}{10\times10^{-6}}$$
$$= 376 \ kPa$$

The results are sketched (in MPa) below.



◀

SOLUTION (12.21)

Initial Yielding:

$$\sigma_{yp} = \frac{N_1}{\pi r^2} + \frac{4M_1}{\pi r^3} \qquad \text{(a)}$$

where

$$N_{yp} = \pi r^2\sigma_{yp}, \qquad M_{yp} = \sigma_{yp}\pi r^3/4$$

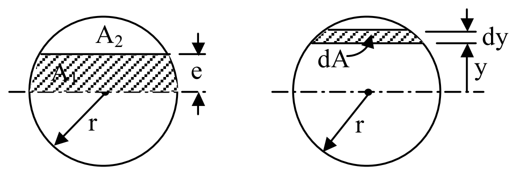
(CONT.)

12.21 (CONT.)

We express Eq. (a) in the from

$$\frac{N_1}{N_{yp}} + \frac{M_1}{M_{yp}} = 1 \qquad (1)$$

<u>Fully Plastic Deformation</u>:



From a Mathematics Handbook table:

$$A_2 = \tfrac{\pi r^2}{2} - [e(r^2 - e^2)^{\frac{1}{2}} + r^2 \sin^{-1}(\tfrac{e}{r})]$$

$$A_1 = e(r^2 - e^2)^{\frac{1}{2}} + r^2 \sin^{-1}(\tfrac{e}{r})$$

Thus,

$$N_2 = 2\sigma_{yp}[e(r^2 - e^2)^{\frac{1}{2}} + r^2 \sin^{-1}(\tfrac{e}{r})] \qquad (2)$$

We also write

$$dA = 2(r^2 - y^2)^{\frac{1}{2}} dy$$

$$Q = \int_e^r 2y(r^2 - y^2)^{\frac{1}{2}} dy = \tfrac{2}{3}(r^2 - e^2)^{\frac{3}{2}}$$

Here Q is the first moment of the area A_2. Hence,

$$M_2 = 2Q\sigma_{yp} = \tfrac{4}{3}(r^2 - e^2)^{\frac{3}{2}}\sigma_{yp} \qquad (3)$$

$$M_u = \tfrac{4}{3} r^3 \sigma_{yp} = \tfrac{16}{3\pi} M_{yp}$$

Solving Eq.(3),

$$e = [r^2 - (\tfrac{3}{4}\tfrac{M_2}{\sigma_{yp}})^{\frac{2}{3}}]^{\frac{1}{2}}$$

Substituting this into Eq. (2):

$$N_2 = 2\sigma_{yp}[r^2 - (\tfrac{3}{4}\tfrac{M_2}{\sigma_{yp}})^{\frac{2}{3}}]^{\frac{1}{2}}(\tfrac{3}{4}\tfrac{M_2}{\sigma_{yp}})^3 + 2r^2\sigma_{yp}\sin^{-1}[1 - (\tfrac{3}{4r^2}\tfrac{M_2}{\sigma_{yp}})^{\frac{2}{3}}]^{\frac{1}{2}}$$

The foregoing results in

$$\frac{\pi N_2}{2N_{yp}} = (\tfrac{3\pi}{16})^{\frac{1}{3}}(\tfrac{M_2}{M_{yp}})^{\frac{1}{3}}[1 - (\tfrac{3\pi}{16})^{\frac{2}{3}}(\tfrac{M_2}{M_{yp}})^{\frac{2}{3}}]^{\frac{1}{2}} + \sin^{-1}[1 - (\tfrac{3\pi}{16})^{\frac{2}{3}}(\tfrac{M_2}{M_{yp}})^{\frac{2}{3}}]^{\frac{1}{2}} \qquad (4)$$

The governing equations for yielding to impend and for fully plastic deformation are given by Eqs. (1) and (4). A sketch of these, <u>interaction curves</u>, are shown below.

(CONT.)

12.21 (CONT.)

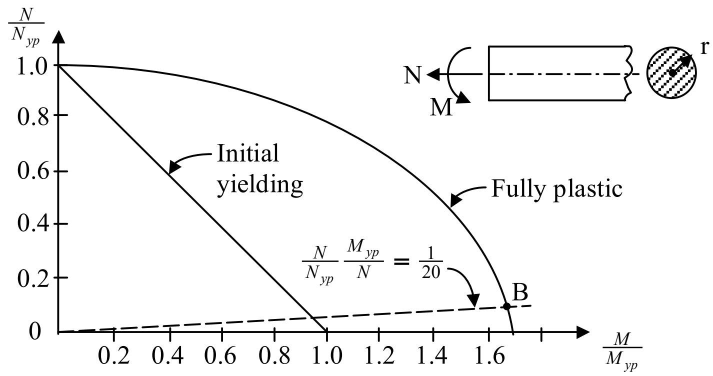


Figure P12.21

SOLUTION (12.22)

Let P=N. Then referring to Fig. P12.22, we have

$$M = Nd = N(0.05 + 0.05 + 0.025) = 0.125N$$

Thus, $N/M = 1/0.125$

Also, $\frac{M_{yp}}{N_{yp}} = \frac{(\pi r^3/4)\sigma_{yp}}{\pi r^3 \sigma_{yp}} = \frac{0.025}{4}$

and

$$M/M_{yp} = 20(N/N_{yp})$$

Now referring to Fig. P12.21, we find that $B(1.68,\ 0.084)$.

Therefore,

$$N_2 = 0.084 N_{yp} = 0.084(\pi r^3 \sigma_{yp})$$
$$= 0.084\pi(0.025)^2(280 \times 10^6)$$
$$= 46.18\ kN$$ ◄

SOLUTION (12.23)

The plastic hinges for at 1 and 2, Fig. (a).

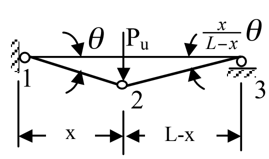


Figure (a)

Apply the principle of virtual work:

$$P_u(x\delta\theta) = M_u[2(\delta\theta + \tfrac{x\delta\theta}{L-x})]$$

from which

$$P_u = \frac{2L}{x(L-x)} M_u$$

The condition $dP_u/dx = 0$ gives $x = L/2$. Minimum magnitude of the ultimate load is thus

$$P_u = \frac{8M_u}{L}$$ ◄

SOLUTION (12.24)

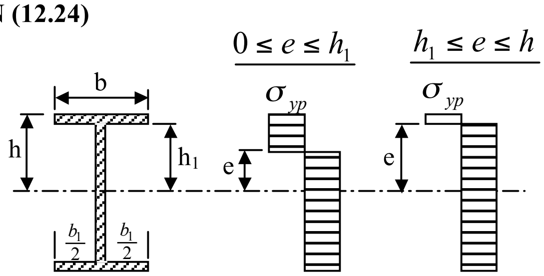


Figure (a) Figure (b) Figure(c)

Initial Yielding:

$$\sigma_{yp} = \frac{N_1}{A} + \frac{M_1 c}{I} = \frac{N_1}{2(bh - b_1 h_1)} + \frac{M_1 h}{(2/3)(bh^3 - b_1 h_1^3)} \tag{a}$$

Here,

$$N_{yp} = 2(bh - b_1 h_1)\sigma_{yp}$$

$$M_{yp} = \frac{2}{3h}(bh^3 - b_1 h_1^3)\sigma_{yp}$$

Equation (a) may now be written

$$\frac{N_1}{N_{yp}} + \frac{M_1}{M_{yp}} = 1 \tag{1}$$

Fully Plastic Deformation:

For $0 \le e \le h_1$ (Fig. b):

$$N_2 = 2A_1\sigma_{yp} = 2(b - b_1)e\sigma_{yp}$$

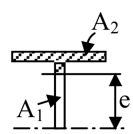


Figure (d)

from which

$$e = \frac{N_2}{2(b - b_1)\sigma_{yp}}$$

We have (Fig. d):

$$A_2 = b(h - h_1) + (h_1 - e)(b - b_1)$$

$$= bh - eb + eb_1 - b_1 h_1$$

$$\bar{y} = \frac{\sum Ay}{A_2} = \frac{b(h - h_1)(\frac{h - h_1}{2} + 1)}{A_2} = \frac{1}{2}\frac{bh^2 - b_1 h_1^2 - e^2 b + e^2 b_1}{bh - eb + eb_1 - b_1 h_1}$$

Thus,

$$M_2 = 2\sigma_{yp} A_2 \bar{y} = (bh^2 - b_1 h_1^2 - e^2 b^2 + e^2 b_1)\sigma_{yp} \tag{b}$$

For $e = 0$: $M_2 = M_u = (bh^2 - b_1 h_1^2)\sigma_{yp}$.

Substituting the given data, the preceding expressions become

$$N_{yp} = 1.68h^2\sigma_{yp}, \qquad M_{yp} = 0.922h^3\sigma_{yp}$$

$$M_u = 1.113h^3\sigma_{yp}$$

and $\frac{M_u}{M_{yp}} = 1.21, \qquad M_{yp} = 0.55hN_{yp}$

Hence, Eq. (b) leads to

$$\frac{M_2}{1.21M_{yp}} = 1 - \frac{1}{3.16}\left(\frac{N_2}{N_{yp}}\right)^2 \tag{2a}$$

This is valid for

$$0 \le 0 \le h_1 \qquad \text{or} \qquad 0 \le \frac{N_2}{N_{yp}} \le 1.67$$

(CONT.)

For $h_1 \le e \le h_2$ (Figs. *c* and *e*):

$$N_2 = 2[0.7h + 0.2h + (e - 0.7)2h]\sigma_{yp} \quad = (-2.52h^2 + 4eh)\sigma_{yp}$$

or $$e = \tfrac{N_2}{4h\sigma_{yp}} + 0.63h$$

$$M_2 = 2h(h - e)2(e + \tfrac{h-e}{2})\sigma_{yp}$$
$$= 2h^3\sigma_{yp} - 2he^2\sigma_{yp}$$

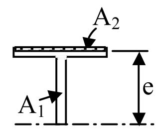


Figure (e)

Hence,

$$\tfrac{M_2}{2h^3\sigma_{yp}} = 1 - \tfrac{e^2}{h^2} = 1 - \tfrac{N_2^2}{16h^4\sigma_{yp}^2} - \tfrac{0.315N_2}{h^2\sigma_{yp}} - 0.397$$

or $$\tfrac{1}{2.17}\tfrac{M_2}{M_{yp}} = 0.603 - 0.176(\tfrac{N_2}{N_{yp}})^2 - 0.529(\tfrac{N_2}{N_{yp}}) \qquad (2b)$$

which is valid for

$$h_1 \le e \le h \qquad \text{or} \qquad 1.67 \le \tfrac{N_2}{N_{yp}} \le 1$$

Equations (1) and (2) are the governing expressions of the plastic bending. A sketch of these, interaction curves, are shown in the figure given below.

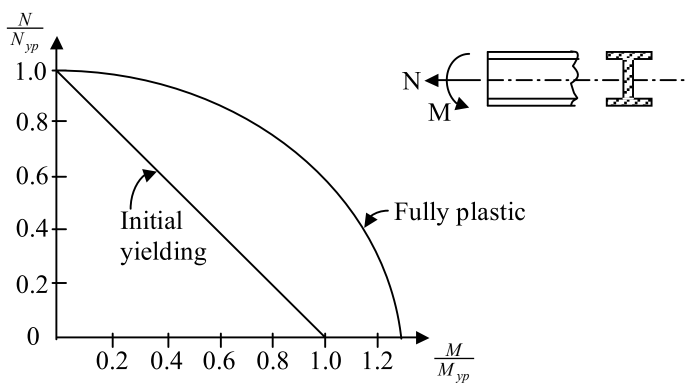
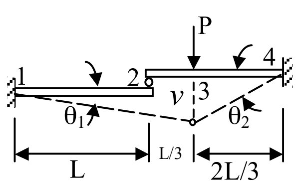


Figure P12.24

SOLUTION (12.25)



We have

$$v = \tfrac{4}{3}L\theta_1 = \tfrac{2}{3}L\theta_2$$

and $$\delta v = \tfrac{4}{3}L\delta\theta_1 = \tfrac{2}{3}L\delta\theta_2$$

from which $2\delta\theta_1 = \delta\theta_2$.

Applying the principle of virtual work:

$$P_u\delta v = M_u\delta\theta_1 + M_u(\delta\theta_1 + \delta\theta_2) + M_u\delta\theta_2$$

or $$\tfrac{4}{3}P_uL\delta\theta_1 = 6M\delta\theta_1$$

Solving, $$P_u = \tfrac{9M_u}{2L}$$ ◀

SOLUTION (12.26)

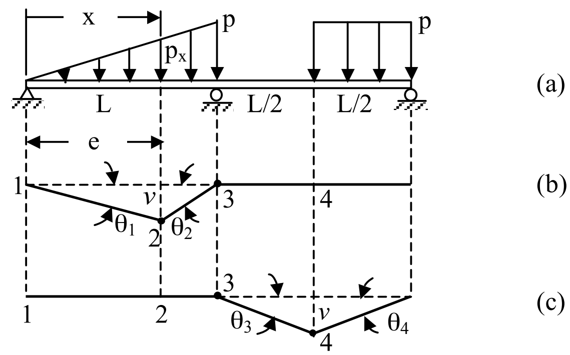


Figure P12.26

We have two different modes to be checked for collapse loading.

Mode A, Fig. P12.26b:

$$\theta_1 = \tfrac{\theta_2(L-e)}{e}; \qquad \theta_1 + \theta_2 = \tfrac{L\theta_2}{e}$$

Applying the principle of virtual work:

$$\delta\int_0^e (\theta_1 x)\tfrac{x}{L}\,p\,dx + \delta\int_e^L [\theta_1 e - \theta_2(x-e)]\tfrac{x}{L}\,p\,dx = M_u\delta\theta_1 + M_u\delta\theta_2 + M_u\delta\theta_2$$

or
$$\delta\int_0^e \theta_1 x\tfrac{x}{L}\,p\,dx + \delta\int_e^L [\theta_1 e - \theta_2(x-e)]\tfrac{x}{L}\,p\,dx = M_u\delta\theta_2(\tfrac{L}{e}+1) \qquad \text{(a)}$$

Integrating the left hand side of this equation, carrying out the algebra and simplifying:

$$\delta\theta_2 p[\tfrac{L^2-e^2}{6}] = M_u\delta\theta_2(\tfrac{L}{e}+1)$$

Solving,

$$p = \tfrac{6M_u}{e(L-e)} \qquad \text{(b)}$$

Then, $dp/de = 0$ gives,

$$0 = -\tfrac{6M_u}{e^2(L-e)} + \tfrac{6M_u}{e(L-e)}; \qquad e = \tfrac{L}{2}$$

Introducing this value of e into Eq. (b), the collapse load is

$$P_u = \tfrac{24M_u}{L^2} \qquad \blacktriangleleft$$

Mode B, Fig. P12.26c:

From symmetry $\theta_3 = \theta_4 = \theta$. Principle of virtual work gives:

$$\delta\int_{L/2}^L [\theta\tfrac{L}{2} - \theta(x-\tfrac{L}{2})]p\,dx = 3M_u\delta\theta$$

or

$$\delta\int_{L/2}^L [\theta L - \theta x]p\,dx = 3M_u\delta\theta$$

Integrating,

$$\delta\theta\tfrac{L^2 p}{8} = 3M_u\delta\theta$$

or

$$P_u = \tfrac{24M_u}{L} \qquad \blacktriangleleft$$

Note that the collapse load is the same for modes A and B.

SOLUTION (12.27)

Assume that plastic hinges force at 1 and 2, as shown in Fig. (a).
On the average, plastic limit load

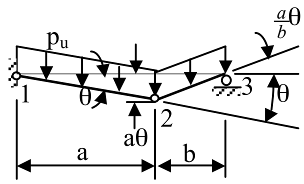


Figure (a)

Note that p_u goes through a virtual displacement of $a\delta\theta/2$. Thus, applying the principle of virtual work:

$$(p_u L)\tfrac{1}{2}a\delta\theta = M_u\delta\theta + M_u(1+\tfrac{a}{b})\delta\theta$$

Solving,

$$p_u = \tfrac{2M_u}{L}\left(\tfrac{2L-a}{La-a^2}\right)$$ ◀

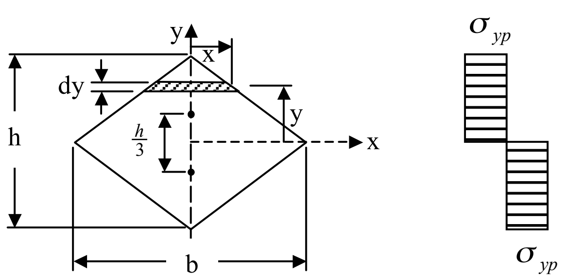
The unknown distance a is determined from $dp_u/da = 0$. In so doing and simplifying the result, we obtain

$$a = (2-\sqrt{2})L$$ ◀

and

$$b = L - a = (\sqrt{2}-1)L.$$

SOLUTION (12.28)



(a) We have $M_{yp} = \sigma_{yp} I/h$

From geometry,

$$\tfrac{x}{b/2} = \tfrac{(h/2)-y}{h/2}; \qquad x = \tfrac{b(h-2y)}{2h}$$

Then, $$I = 2\int_0^{h/2}(2x)dy(y^2) = \tfrac{bh^3}{48}$$

Hence, total yielding moment

$$M_u = 2\tfrac{bh^2}{48}\tfrac{1}{h}\sigma_{yp} = \tfrac{bh^2}{24}\sigma_{yp}$$

Also,

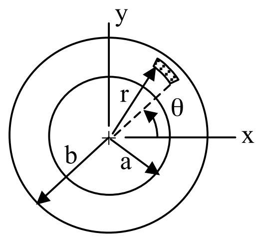
$$M_u = \tfrac{1}{2}(\text{area of rhombus})\times(\text{distance between centroids})(\sigma_{yp})$$
$$= \tfrac{bh}{4}(\tfrac{h}{3})\sigma_{yp} = \tfrac{bh^2}{12}\sigma_{yp}$$

Thus, $$\tfrac{M_u}{M_{yp}} = 2$$ ◀

(CONT.)

12.28 (CONT.)

(b)



$$I = \tfrac{\pi(b^4 - a^4)}{4}$$

$$M_{yp} = \tfrac{\pi}{4}(b^4 - a^4)\tfrac{\sigma_{yp}}{b} = \tfrac{\pi}{4b}(b^4 - a^4)\sigma_{yp}$$

Also, referring to the figure:

$$M_x = 2\int_0^{\pi}\int_0^{b} r^2 \sin\theta dr d\theta = \tfrac{4}{3}(b^3 - a^3)$$

Hence,

$$y_c = \tfrac{M_x}{A} = \tfrac{4}{3}\tfrac{b^3 - a^3}{(\pi/2)(b^2 - a^2)} = \tfrac{4}{3\pi}\tfrac{b^3 - a^3}{b^2 - a^2}$$

We therefore have,

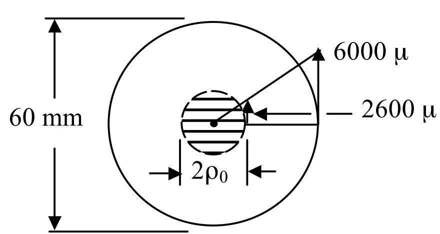
$$\tfrac{M_u}{M_{yp}} = \tfrac{(\pi/2)(b^2 - a^2)\sigma_{yp} y_c}{(\pi/4b)(b^4 - a^4)\sigma_{yp}} = \tfrac{16b}{3\pi}\tfrac{b^3 - a^3}{b^4 - a^4} \qquad ◀$$

SOLUTION (12.29)

(**a**) $$\phi = \frac{\gamma_{\max} L}{c} = \frac{6000\times10^{-6}(0.5)}{0.03} = 0.10\ rad$$

$$\gamma_y = \frac{\tau_{yp}}{G} = \frac{180\times10^6}{70\times10^9} = 2600\ \mu < 6000\ \mu \qquad ◀$$

and shaft is yielded.



From similar triangles:

$$\frac{\rho_0}{2600} = \frac{30}{6000}$$

or

$$\rho_0 = 13\ mm$$

(**b**) Elastic core. Use Eq.(b) of Sec. 12.9:

$$T_1 = \frac{\pi\rho_0^3}{2}\tau_{yp} = \frac{\pi}{2}(0.013)^3(180\times10^6) = 621\ N\cdot m$$

Outer part. Use Eq.(c) of Sec. 12.9:

$$T_2 = \frac{2\pi}{3}(c^3 - \rho_0^3)\tau_{yp} = \frac{2\pi}{3}(0.03^3 - 0.013^3)(180\times10^6)$$

$$= 7.01\ kN\cdot m$$

Thus,

$$T = T_1 + T_2 = 7.63\ kN\cdot m \qquad ◀$$

SOLUTION (12.30)

$$G = 26\times10^6\ GPa, \qquad \tau_{yp} = 140\ MPa \quad \text{(Table D.1)}$$

(**a**) For partially plastic shaft, using Eq.(12.19):

$$(\frac{\rho_0}{c})^3 = 4 - \frac{3T}{T_{yp}} = 4 - \frac{6T}{\pi c^3 \tau_{yp}}$$

Substituting the given values

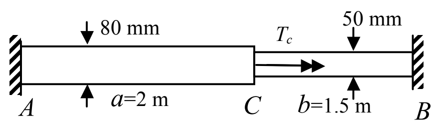
$$(\frac{\rho_0}{0.025})^3 = 4 - \frac{6(4.5\times10^3)}{\pi(0.025)^3(140\times10^6)} = 0.0711$$

$$\rho_0 = 12.4\ mm$$ ◄

(**b**) $$\gamma_y = \frac{\rho_0\phi}{L} = \frac{\tau_{yp}}{G}, \qquad \phi = \frac{\tau_{yp}}{G}\frac{L}{\rho_0}$$

$$\phi = \frac{140\times10^6(1.2)}{26\times10^9(0.0124)} = 0.5211\ rad = 29.9^o$$ ◄

SOLUTION (12.31)

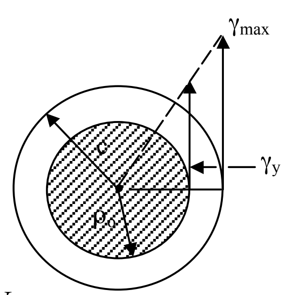


$$\gamma_{yp} = \frac{\tau_{yp}}{G} = \frac{240\times10^6}{80\times10^9} = 3000\ \mu$$

$$(\gamma_{\max})_{AC} = \frac{c\phi}{a} = \frac{0.04(0.25)}{2} = 5000\ \mu$$

$$(\gamma_{\max})_{CB} = \frac{0.025(0.25)}{1.5} = 4167\ \mu$$

Both segments are yielded and partially plastic.



Segment *AC*

$$\rho_0 = \frac{c\gamma_y}{\gamma_{\max}}$$

$$= \frac{0.04(3000\times10^{-6})}{5000\times10^{-6}} = 24\ mm$$

$$T_{yp} = \frac{\pi c^3}{2}\tau_{yp} = \frac{\pi(0.04)^3}{2}(240\times10^6) = 24.127\ kN\cdot m$$

Use Eq. (12.19):

$$T_{AC} = \frac{4}{3}(24{,}127)[1 - \frac{1}{4}(\frac{24}{40})^3] = 30.43\ kN\cdot m$$

(CONT.)

12.31 (CONT.)

Segment *BC*

$$\rho_0 = \frac{c\gamma_{yp}}{\gamma_{\max}} = \frac{0.025(3000\times10^{-6})}{4167\times10^{-6}} = 1.8\ mm$$

$$T_{yp} = \frac{\pi c^3}{2}\tau_{yp} = \frac{\pi(0.025)^3}{2}(240\times10^6) = 5.89\ kN\cdot m$$

Use Eq. (12.19):

$$T_{BC} = \frac{4}{3}(5.89)[1-\frac{1}{4}(\frac{18}{25})^3] = 7.12\ kN\cdot m$$

Total applied torque is therefore

$$T = T_{AC} + T_{BC} = 37.55\ kN\cdot m$$ ◄

SOLUTION (12.32)

For r=0, Eq. (f) of Sec. 12.11 leads to $c_1 = 0$. Then, in plastic zone

$$\sigma_r = \sigma_{yp} - \tfrac{\rho\omega^2 r^2}{3}, \qquad \sigma_\theta = \sigma_{yp}$$

If the plastic zone extends to radius c:

$$\sigma_c = \sigma_{yp} - \tfrac{\rho\omega^2 c^2}{3}$$

which may be found directly from Eq. (12.30) by setting a=0. The outer elastic zone is represented by an annular disk yielding at the inner radius c, wherein radial stress is σ_c.

We follow a procedure similar to that described in Sec. 12.11 for an annular disk. Boundary conditions:

$$(\sigma_r)_{r=c} = \sigma_c, \qquad (u)_{r=0} = 0$$

are substituted into Eqs. (8.37) to obtain c_1 and c_2. We then determine the stresses in the elastic region as follows:

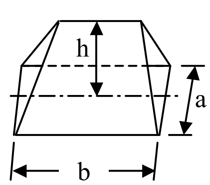
$$\sigma_r = \tfrac{\sigma_{yp}}{24N^2}[3(1+\nu) - (1+3\nu)\tfrac{c^4}{r^2b^2}](1-\tfrac{r^2}{b^2})$$ ◄

$$\sigma_\theta = \tfrac{\sigma_{yp}}{24N^2}[\tfrac{c^4}{b^4}(1+\tfrac{b^4}{r^4})(1+3\nu) + 3\nu(3+\nu) - 3(1+3\nu)\tfrac{r^2}{b^2}]$$

where

$$N^2 = \tfrac{8+(1+3\nu)[(c/b)^2-1]}{24}$$

SOLUTION (12.33)



The sand volume is

$$V = \tfrac{1}{2}(b-a)ah + 2(\tfrac{1}{3}a^2\,\tfrac{h}{2})$$

(CONT.)

12.33 (CONT.)

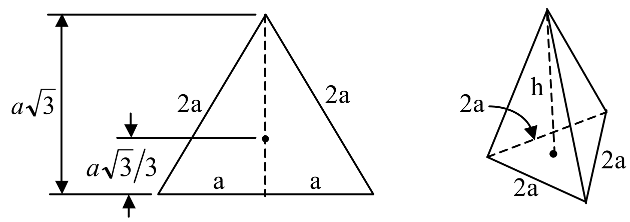
Slope $\tau_{yp} = \frac{h}{(a/2)}$

The ultimate torque is thus

$$T_u = \frac{(3b-a)a^2}{6}\tau_{yp} \quad ◀$$

The yield torque given in Table 6.2, by letting $\tau = \tau_{yp}$.

SOLUTION (12.34)



Volume $= \frac{1}{3}h(\frac{1}{2}\cdot 2a\cdot a\sqrt{3}) = \frac{\sqrt{3}}{3}ha^2$

Slope $= \frac{h}{a\sqrt{3}/3} = \frac{3h}{a\sqrt{3}} = \tau_{yp}, \qquad h = \frac{a\sqrt{3}}{3}\tau_{yp}$

(a) $T_u = 2V = \frac{2}{3}a^3\tau_{yp}$ ◀

(b) From Table 6.2: $\tau_A = 20\frac{T}{a_1^3} = \frac{20T}{8a^3}$

Thus, $T_{yp} = \frac{8a^3}{20}\tau_{yp}$

(c) Referring to the preceding results in items (a) and (b):

$$\frac{T_u}{T_{yp}} = \frac{5}{3} \quad ◀$$

SOLUTION (12.35)

Equilibrium condition, from Eq. (8.2):

$$\frac{d}{dr}(t\sigma_r) - \frac{t(\sigma_\theta - \sigma_r)}{r} = 0$$

Profile is, using Eq. (8.36) with $s = 1,\ rt = at_a$. For full plasticity

$$\sigma_\theta - \sigma_r = \sigma_{yp}$$

Thus, $t\sigma_r = \int \frac{at_a}{r}\frac{\sigma_{yp}}{r}dr = -\frac{\sigma_{yp}at_a}{r} + c_1$

Since, $(\sigma_r)_{r=b} = 0; \qquad c_1 = \frac{\sigma_{yp}at_a}{b}$

Then, noting that $\sigma_r = -p_i$ at $t = t_a$:

$$\frac{p_i}{\sigma_{yp}} = a(\frac{1}{r} - \frac{1}{b})$$

or

$$p_i = \frac{a(b-r)}{rb}\sigma_{yp} \quad ◀$$

SOLUTION (12.36)

We have $\alpha = 1/2$ and substituting the given data into Eq. (12.42):

$$t_0 = \frac{0.606\, pr_0}{(2K/\sqrt{3})(n/\sqrt{3})} = \frac{0.606(14\times10^6)(0.5)}{(2\times900\times10^6/1.73)(0.2/1.73)^{0.2}}$$
$$= 0.0063\ m = 6.3\ mm \qquad ◄$$

SOLUTION (12.37)

In this case, we have $\sigma_z > \sigma_\theta$.

The total force is

$$P = 2\pi rt\sigma_1 \qquad \text{(a)}$$

where $\sigma_1 = \sigma_z \qquad \sigma_2 = \sigma_\theta$

The values of r and t are given by Eqs. (g) and (f) of Sec. 12.12.

Substituting these into Eq. (a):

$$P = 2\pi r_0 e^{-\varepsilon_1}\sigma_1$$

At instant of stability,

$$dP = (\tfrac{\partial P}{\partial \sigma_1})d\sigma_1 + (\tfrac{\partial P}{\partial \varepsilon_1})d\varepsilon_1 = 0$$

or $$\frac{d\sigma_1}{d\varepsilon_1} = \sigma_1$$

Equations (12.28) has the form

$$\sigma_1 = f(\alpha)\varepsilon_1^n$$

from which stability condition is

$$\varepsilon_1 = n$$

Then,

$$n = (\tfrac{\sigma_1}{K})^{\frac{1}{n}}(\alpha^2 - \alpha + 1)(\tfrac{2-\alpha}{2})$$

Solving,

$$\sigma_1 = K(2n)^n(\tfrac{1}{\alpha^2-\alpha+1})^{\frac{(1-n)}{2}}(\tfrac{1}{2-\alpha})^n \qquad \text{(b)}$$

Since $\varepsilon_1 + \varepsilon_2 + \varepsilon_3 = 0,$ the maximum principal strain is

$$\varepsilon_1 = \ln\tfrac{L}{L_0} = \ln\tfrac{t}{t_0} + \ln\tfrac{r}{r_0}$$

Minimum strain is

$$\varepsilon_3 = \ln\tfrac{t}{t_0}; \qquad t = t_0 \ln^{-1}\varepsilon_3$$

Thus,

$$\sigma_1 = \frac{P}{2\pi rt} = \frac{P}{2\pi r_o \ln^{-1}\varepsilon_2 t_0 \ln^{-1}\varepsilon_3}$$

Substituting Eqs. (12.38b) and (12.38c) into this equation:

$$\sigma_1 = \frac{P}{2\pi r_0 t_0 \ln^{-1}[(\frac{\sigma_1}{K})^{1/2}(\alpha^2-\alpha+1)\frac{1-n}{2}(\frac{\alpha}{2}-1)]}$$

or $$t_0 = \frac{P}{2\pi r_0 \sigma_1 \ln^{-1}[(\frac{\sigma_1}{K})^{1/2}(\alpha^2-\alpha+1)\frac{1-n}{2}(\frac{\alpha}{2}-1)]} \qquad ◄$$

where, σ_1 is given by Eq. (b).

Results of the preceding expressions are simplified by setting $\alpha = 1/2$.

SOLUTION (12.38)

Since the mean radius does not change, we have $d\varepsilon_0 = 0$. We also take $\sigma_3 = \sigma_r = 0$.

Material is incompressible $d\varepsilon_L = d\varepsilon_3$, or $\varepsilon_L = -\varepsilon_3$.

The Levy-Mises equation is thus

$$\frac{d\varepsilon_L}{\sigma_L - \sigma_\theta} = \frac{d\varepsilon_L}{\sigma_\theta} \qquad \text{or} \qquad \sigma_L = 2\sigma_\theta$$

Hence, $\sigma_e = \frac{\sqrt{3}}{2}\sigma_L$

And $\varepsilon_e = (\frac{2}{3})d\varepsilon_L = -(\frac{2}{\sqrt{3}})d\varepsilon_3$

Radius r_0 = constant. Therefore,

$$P = 2\pi r_0 \cdot t\sigma_L$$

At instant of instability, $dP = 0$:

$$\frac{d\sigma_L}{\sigma_L} = -\frac{dt}{t} = -d\varepsilon_3 = d\varepsilon_L$$

Hence,

$$\varepsilon_e = \frac{2}{\sqrt{3}}n = \frac{2}{\sqrt{3}}\varepsilon_L$$

$$\sigma_e = K(\tfrac{2}{\sqrt{3}})^n = \frac{\sqrt{3}}{2}\sigma_L, \qquad t = t_0 e^{-n}$$

At instant of instability, the load is then given by Eq. (P12.38).

SOLUTION (12.39)

(a) Use Eq. (8.10), with $\tau_{\max} = \sigma_{yp}/2n$:

$$\frac{\sigma_{yp}}{2n} = \frac{p_i b^2}{b^2 - a^2} = \frac{p_i}{1-(\frac{a}{b})^2};$$

Substituting the given data:

$$50 = \frac{60}{1-(\frac{50}{b})^2} \qquad \text{or} \qquad \frac{5}{6} = 1 - (\frac{50}{b})^2$$

Solving, $b = 122.5\ mm$ ◄

(b) Apply Eq. (12.47) with $k = 1$ and $n = 3$:

$$p_u = \frac{\sigma_{yp}}{n}\ln(\frac{a}{b}); \qquad -60 = \frac{250}{2.5}\ln(\frac{50}{b})$$

or $\frac{50}{b} = e^{-(\frac{150}{250})} = 0.549$

Solving,

$$b = 91.11\ mm$$ ◄

SOLUTION (12.40)

Apply Eq. (12.61a): with $c = 1.4a$ and $k = \frac{2}{\sqrt{3}}$. We have $p_i = \sigma_r$ at $r = a$. Thus

$$p_i = \frac{2}{\sqrt{3}}(260)[\ln(\frac{a}{1.4a}) - \frac{(2a)^2-(1.4a)^2}{2(2a)^2}]$$
$$= 300.2[-0.336 - 0.255]$$
$$= -177.4\ MPa$$ ◄

SOLUTION (12.41)

(a) Using Eq. (8.11):

$$p_{yp} = \frac{420}{3}\frac{36-25}{2(36)} = 21.39\ MPa$$

We have $k = 1$, and thus, ◀

$$p_u = \frac{\sigma_{yp}}{n}\ln\frac{b}{a} = \frac{420}{3}\ln\frac{6}{5} = 25.53\ MPa$$

(b) $(\frac{\sigma_{yp}}{n})^2 = \sigma_\theta^2 - \sigma_\theta\sigma_r + \sigma_r^2$

or $$(140)^2 = p_{yp}^2[(\tfrac{36+25}{36-25})^2 + \tfrac{36+25}{36-25} + 1]$$

Solving, $p_{yp} = 22.92\ MPa$ ◀

Now $k = 2/\sqrt{3}$, and hence,

$$p_u = \frac{2}{\sqrt{3}}(25.53) = 29.48\ MPa$$ ◀

SOLUTION (12.42)

Using Eqs. (12.57) and (12.58) with $k = 1$ and $r = 0.25\ m$,

$$p_u = 400\ln(\tfrac{0.3}{0.2}) = 162.2\ MPa$$

$$\sigma_r = -400\ln(\tfrac{0.3}{0.25}) = -72.93\ MPa$$

$$\sigma_\theta = 400[1 - \ln\tfrac{0.3}{0.25}] = 327.1\ MPa$$

and $$\sigma_z = \tfrac{1}{2}(\sigma_r + \sigma_\theta) = 127.1\ MPa$$

Unloading from p_u. At $r = 0.25\ m$, Eqs. (8.12) and (8.20):

$$\sigma_r = \frac{0.2^2(162.2)}{0.3^2-0.2^2}(1 - \frac{0.3^2}{0.25^2}) = -57.09\ MPa$$

$$\sigma_\theta = \frac{0.2^2(162.2)}{0.3^2-0.2^2}(1 + \frac{0.3^2}{0.25^2}) = 316.6\ MPa$$ ◀

$$\sigma_z = 162.2\frac{0.2^2}{0.3^2-0.2^2} = 129.8\ MPa$$

Residual stresses at r=0.25 m:

$$(\sigma_\theta)_{res.} = 327.1 - 316.6 = 10.5\ MPa$$

$$(\sigma_r)_{res.} = -72.93 - 57.09 = -15.84\ MPa$$ ◀

$$(\sigma_z)_{res.} = 127.1 - 129.8 = -2.7\ MPa$$

SOLUTION (12.43)

We have $k = 1$. Refer to Eq. (12.57).

Inner cylinder, at $r = b$:

$$-p_b = -p_u + (\sigma_{yp})_i\ln\frac{b}{a} \qquad \text{(a)}$$

Outer cylinder, at $r = c$:

$$0 = -p_b + (\sigma_{yp})_o\ln\frac{c}{b} \qquad \text{(b)}$$

(CONT.)

From Eqs. (a) and (b), after eliminating p_b, we obtain

$$p_u = (\sigma_{yp})_i \ln\tfrac{b}{a} + (\sigma_{yp})_o \ln\tfrac{c}{b}$$
$$= 280\ln\tfrac{30}{20} + 400\ln\tfrac{50}{30}$$
$$= 317.9\ MPa \quad \blacktriangleleft$$

SOLUTION (12.44)

(a) Equation (12.60):

$$p_c = k\sigma_{yp}\tfrac{3^2-2^2}{2(3)^2} = 0.2778k\sigma_{yp} \quad \blacktriangleleft$$

(b) Equation (12.61a):

$$(\sigma_r)_{r=a} = k\sigma_{yp}[\ln\tfrac{1}{2} - 0.2778] = -0.9709k\sigma_{yp} \quad \blacktriangleleft$$

(c) Equation (12.59b):

$$(\sigma_\theta)_{r=b} = \tfrac{0.2778k\sigma_{yp}(3^2)}{3^2-2^2}(1+\tfrac{3^2}{3^2}) = 0.444k\sigma_{yp} \quad \blacktriangleleft$$

Equation (12.59b):

$$(\sigma_\theta)_{r=c} = \tfrac{0.2778k\sigma_{yp}(3^2)}{3^2-2^2}(1+\tfrac{3^2}{2^2}) = 0.7222k\sigma_{yp}$$

Equation (12.61b):

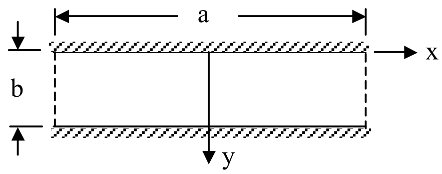
$$(\sigma_\theta)_{r=a} = k\sigma_{yp}(1+\ln\tfrac{1}{2} - 0.2778) = 0.0291k\sigma_{yp} \quad \blacktriangleleft$$

We see from these results that the maximum stress occurs at the elastic-plastic boundary.

End of Chapter 12

CHAPTER 13

SOLUTION (13.1)



(a) Boundary conditions at y=0 and y=b:

$$w = 0 \qquad \frac{dw}{dy} = 0 \tag{a}$$

We have

$$w'''' = \frac{p_0}{D} \qquad w'' = \frac{p_0 y^2}{2D} + c_1 y + c_2$$

$$w' = \frac{p_0 y^3}{6D} + \frac{1}{2} c_1 y^2 + c_2 y + c_3$$

$$w = \frac{p_0 y^4}{24D} + \frac{c_1 y^3}{6} + \frac{c_2 y^2}{2} + c_3 y + c_4$$

Conditions (a) yield $c_4 = 0,\ c_3 = 0,$

$$c_1 = -\frac{p_0 b}{2D} \qquad c_2 = \frac{p_0 b^2}{12D}$$

Thus, $$w = \frac{p_0 b^4}{24D}[(\tfrac{y}{b})^4 - 2(\tfrac{y}{b})^3 + (\tfrac{y}{b})^2] \quad ◀$$

(b) Differentiating twice the foregoing expression, we have

$$w'' = \frac{p_0 b^4}{24D}[\tfrac{12}{b^4} y^2 - \tfrac{12}{b^3} y + \tfrac{2}{b^2}]$$

For $y = \frac{b}{2}$:

$$\frac{d^2 w}{dy^2} = -\frac{p_0 b^2}{24D}$$

Hence, $M_y = \frac{p_0 b^4}{24}$

Thus, $\sigma_{y,\max} = \frac{p_0}{4}(\frac{b}{t})^2, \qquad \sigma_{x,\max} = \frac{p_0}{12}(\frac{b}{t})^2$

Similarly, for y=0:

$$\frac{d^2 w}{dy^2} = -\frac{p_0 b^2}{12D}, \qquad M_y = \frac{p_0 b^2}{12}$$

and $$\sigma_{y,\max} = \frac{1}{2} p_0 (\tfrac{b}{t})^2 = \sigma_{\max} \quad ◀$$

$$\sigma_{x,\max} = \frac{1}{6} p_0 (\tfrac{b}{t})^2$$

SOLUTION (13.2)

We have $r_x = r, \quad r_y = \infty$

and $r = 0.12\ m, \qquad t = 0.3(10^{-3})\ m$

Equation (13.3b):

$$\varepsilon_{\max} = \frac{t}{2r} = \frac{0.3}{2(120)} = 1250\ \mu$$

Equation (13.5):

$$\sigma_{\max} = -\frac{Et}{2(1-\nu^2)r} = -\frac{200\times10^9(0.3)}{2(0.91)120} = 274.7\ MPa \quad ◀$$

SOLUTION (13.3)

Using Eq. (13.8),

$$D = \frac{9(200\times10^9)(0.012)^3}{12(8)} = 32{,}400$$

From Example 13.1:

$$w = (\tfrac{b}{\pi})^4 \tfrac{p_0}{D}\sin(\tfrac{\pi y}{b})$$

$$w_{\max} = (\tfrac{0.6}{\pi})^4 \tfrac{20\times10^3}{32{,}400} = 0.82(10^{-3})\ m = 0.82\ mm \quad ◄$$

$$M_y = -D\tfrac{d^2w}{dy^2} = -(\tfrac{b}{\pi})^2 p_0 \sin\tfrac{\pi y}{b}$$

Thus,

$$\sigma_{y,\max} = \tfrac{6M_{y,\max}}{t^2} = 0.61p_0(\tfrac{b}{t})^2$$

$$= 0.61p_0(20\times10^3)(\tfrac{0.6}{0.012})^2 = 30.4\ MPa$$

$$\sigma_{x,\max} = \nu(30.4) = 10.13\ MPa$$

Then,

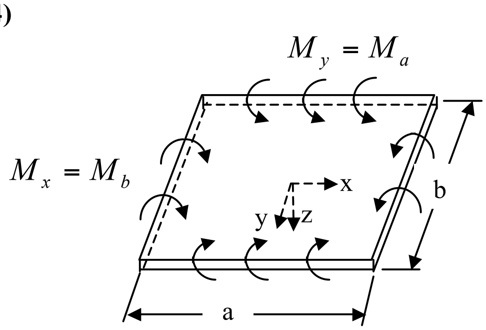
$$\varepsilon_{y,\max} = \tfrac{1}{E}(\sigma_{y,\max} - \nu\sigma_{x,\max})$$

$$= \tfrac{1}{200\times10^3}(30.4 - \tfrac{10.13}{3}) = 135\ \mu \quad ◄$$

and

$$r_y = \tfrac{t}{2\varepsilon_{y,\max}} = \tfrac{0.012\times10^6}{2(135)} = 44.41\ m \quad ◄$$

SOLUTION (13.4)



(a) Using Eq. (13.7),

$$\frac{\partial^2 w}{\partial x^2} = -\frac{M_b - \nu M_a}{D(1-\nu^2)} \qquad \text{(a)}$$

$$\frac{\partial^2 w}{\partial y^2} = -\frac{M_a - \nu M_b}{D(1-\nu^2)}, \qquad \frac{\partial^2 w}{\partial x \partial y} = 0$$

Integrating these equations,

$$w = -\frac{M_b - \nu M_a}{2D(1-\nu^2)}x^2 - \frac{M_a - \nu M_b}{2D(1-\nu^2)}y^2 + c_1 x + c_2 y + c_3$$

If the origin of *xyz* is located at the center and midplane of the plate, the c's will vanish, and

$$w = -\frac{M_b - \nu M_a}{2D(1-\nu^2)}x^2 - \frac{M_a - \nu M_b}{2D(1-\nu^2)}y^2 \qquad \text{(b)}$$

(b) By setting $M_a = -M_b$ in Eq. (a):

$$\frac{\partial^2 w}{\partial x^2} = -\frac{\partial^2 w}{\partial y^2} = \frac{M_a}{D(1-\nu)} = \frac{1}{r_x} = -\frac{1}{r_y}$$

Integrating and locating the origin of *xyz*, as in item (a):

$$w = \frac{M_a}{2D(1-\nu)}(x^2 - y^2) \quad ◄$$

SOLUTION (13.5)

Equation (13.19) becomes,

$$p_{mn} = \frac{144P}{a^4b^4}\int_0^a [\int_0^b (b-y)\sin\frac{n\pi y}{b}dy](a-x)\sin\frac{m\pi x}{a}dx$$

Integrating by parts,

$$p_{mn} = \frac{144P}{a^4b^4}\frac{b^2}{n\pi}\frac{a^2}{m\pi} = \frac{144P}{mn\pi^2a^2b^2} \qquad \text{(a)}$$

Substituting Eqs. (13.18) into $\nabla^4 w = p/D$:

$$a_{mn} = \frac{p_{mn}}{\pi^4 D(m^2/a^2+n^2/b^2)^2} \qquad \text{(b)}$$

Inserting Eqs. (a) and (b) into Eq. (13.18b), we find the required expression for the deflection.

SOLUTION (13.6)

(a) From Eq. (13.19), we obtain

$$p_{mn} = p_0$$

Then, Eq. (13.20) becomes for a square plate (a=b):

$$w = \frac{p_0a^4}{\pi^4D}\sum\sum\frac{\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}}{(m^2+n^2)} \qquad \text{(a)}$$

At $x = y = a/2$,

$$w_{\max} = \frac{p_0a^4}{\pi^4D}\sum\sum\frac{(-1)^{\frac{m+n}{2}-1}}{(m^2+n^2)^2} \qquad \blacktriangleleft$$

(b) $D = \frac{210\times10^9(0.025)^3}{12(1-0.3^2)} = 300{,}480.77$

Equation (a) is then,

$$8(10^{-3}) = \frac{p_0(3)^4}{\pi^4(300{,}480.77)}[\tfrac{1}{4}-\tfrac{1}{100}]$$

or $\quad p_0 = 12.05\ kPa$ $\qquad \blacktriangleleft$

SOLUTION (13.7)

The flexural rigidity of the plate is

$$D = \frac{Et^3}{12(1-\nu^2)} = \frac{200\times10^9t^3}{12(1-0.09)} = 18.315\times10^9t^3$$

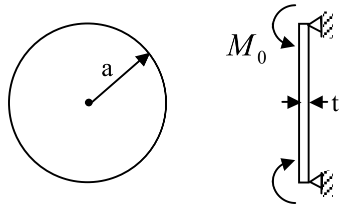
The maximum deflection occurs at the center of the plate. Equation (13.27) is thus

$$w_{\max} = \frac{p_0a^4}{64D}; \qquad 1.5\times10^{-3} = \frac{10(10^6)(0.05)^4}{64(18.315\times10^9t^3)}$$

Solving,

$$t = 3.29(10^{-3})\ m = 3.29\ mm \qquad \blacktriangleleft$$

SOLUTION (13.8)



(CONT.)

Since $Q = 0$, Eq. (13.24c) becomes

$$\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{dw}{dr}\right)\right] = 0$$

from which, after integration,

$$w = -\frac{c_1 r^2}{4} - c_2 \ln\frac{r}{a} + c_3 \qquad \text{(a)}$$

Substituting this into Eq. (13.24a):

$$M_r = D\left[\frac{c_1}{2} - \frac{c_2}{r^2} + \nu\left(\frac{c_1}{2} + \frac{c_2}{r^2}\right)\right] \qquad \text{(b)}$$

Boundary conditions

$$(M_r)_{r=a} = M_0 \qquad w(0) = 0 \qquad \left(\frac{dw}{dr}\right)_{r=0} = 0$$

yield $c_2 = 0$ and

$$c_1 = \frac{2M_0}{D(1+\nu)} \qquad c_3 = \frac{M_0 a^2}{2D(1+\nu)}$$

Equation (a) becomes then,

$$w = \frac{M_0}{2D(1+\nu)}(a^2 - r^2)$$ ◀

Introducing this into Eqs. (13.24a) and (13.24b) yields $M_r = M_\theta = M_0$.

Hence, $\sigma_{r,\max} = \sigma_{\theta,\max} = \frac{6M_0}{t^2}$ ◀

SOLUTION (13.9)

We have $Q_r = 0$ and Eq. (13.24c) after integration gives

$$w = -\frac{c_1 r^2}{4} - c_2 \ln\frac{r}{a} + c_3 \qquad \text{(a)}$$

Introducing this into Eq. (13.24a):

$$M_r = D\left[\frac{c_1}{2} - \frac{c_2}{r^2} + \nu\left(\frac{c_1}{2} + \frac{c_2}{r^2}\right)\right. \qquad \text{(b)}$$

Boundary conditions

$$w(a) = 0 \qquad M_r(b) = M_0$$

and Eqs. (a) and (b) result in

$$c_1 = \frac{2b^2 M_0}{(1+\nu)(b^2-a^2)}, \qquad c_2 = \frac{a^2 b^2 M_0}{(1-\nu)(b^2-a^2)} \qquad c_3 = \frac{a^2 b^2 M_0}{2(1+\nu)D(a^2-b^2)}$$

Carrying these into Eq. (a), we have the equation for deflection.

SOLUTION (13.10)

From Example 13.3:

$$\sigma_1 = \sigma_{r,\max} = -\frac{3}{4}p_0\left(\frac{a}{t}\right)^2, \qquad \sigma_2 = -\frac{1}{4}p_0\left(\frac{a}{t}\right)^2, \qquad \sigma_3 = 0$$

Maximum shearing stress is then

$$\tau_{\max} = \frac{1}{2}(\sigma_1 - 0) = -\frac{3}{8}p_0\left(\frac{a}{t}\right)^2$$

According to maximum shear stress theory:

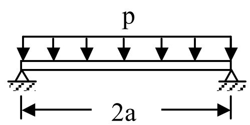
$$\frac{\sigma_{yp}}{n} = \frac{3}{4}p_0\left(\frac{a}{t}\right)^2$$

Introducing the given data,

$$\frac{100(10^6)}{n} = (0.4\times10^6)\left(\frac{0.2}{0.02}\right)^2\left(\frac{3}{4}\right)$$

or $n = 3.33$ ◀

SOLUTION (13.11)



Solution proceeds as in Example 13.3. Boundary conditions are

$$(w)_{r=a} = 0 \qquad (M_r)_{r=a} = 0 \qquad \text{(a)}$$

We have

$$\tfrac{D}{p} w = \tfrac{r^4}{64} - \tfrac{c_2 r^2}{4} + c_4 \qquad \text{(b)}$$

Carrying Eqs. (a) and (b) into Eq. (13.24a), we obtain two equations. From these c_2 and c_4 are evaluated. In so doing, and substituting the values obtained into Eq. (b):

$$w = \tfrac{p(a^2-r^2)}{64D}\left(\tfrac{5+\nu}{1+\nu} a^2 - r^2\right) \qquad ◄$$

SOLUTION (13.12)

Let $r = r_y = \partial^2 w / \partial y^2$, then $r_x = r_{xy} = \infty$, $M = M_y$, $M_x = M_{xy} = 0$

Hence, Eq.(13.5), for $z = -t/2$:

$$\sigma_{\max} = \tfrac{Et}{2(1-\nu^2)r}; \qquad 120(10^6) = \tfrac{70(10^9)(0.006)}{2(0.91)r}, \quad \text{or} \quad r\text{=1.923 m}, \; d\text{= 3.85 m} \qquad ◄$$

Equation (13.9),

$$120(10^6) = \tfrac{6M_{\max}}{t^2} = \tfrac{6M_{\max}}{(0.006)^2}; \qquad M_{\max} = 320N \qquad ◄$$

SOLUTION (13.13)

(*a*) $$Dw^{IV} = p_{0,} \qquad Dw''' = p_0 y + c_1, \qquad Dw'' = \tfrac{1}{2} p_0 y^2 + c_1 y + c_2$$

$$Dw' = \tfrac{1}{6} p_0 y^3 + \tfrac{1}{2} c_1 y^2 + c_2 y + c_3$$

and

$$Dw = \tfrac{1}{24} p_0 y^4 + \tfrac{1}{6} c_1 y^3 + \tfrac{1}{2} c_2 y^2 + c_3 y + c_4 \qquad (\boldsymbol{a})$$

Boundary conditions:

$$w(0) = 0\text{:} \;\; c_4 = 0, \qquad w''(0) = 0\text{:} \;\; c_2 = 0$$

$$w(b) = 0\text{:} \;\; \tfrac{p_0 b^4}{24} + \tfrac{b^3}{6} c_1 + c_3 b = 0$$

$$w'(b) = 0\text{:} \;\; \tfrac{p_0 b^3}{6} + \tfrac{b^2}{2} c_1 + c_3 = 0$$

Solving, $c_1 = -\tfrac{3}{8} b \quad c_3 = \tfrac{b^3}{48}$

Equation (***a***) is thus

$$w = \tfrac{p_0 b^4}{48D}\left[\left(\tfrac{y}{b}\right) - 3\left(\tfrac{y}{b}\right)^3 + 2\left(\tfrac{y}{b}\right)^4\right] \qquad (\boldsymbol{b}) \qquad ◄$$

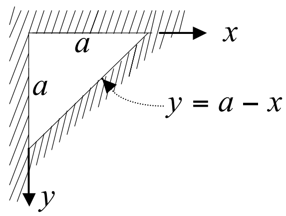
(*b*) We have

$$\tfrac{dw}{dy} = \tfrac{p_0 b^4}{24D}\left[\tfrac{4y^3}{b^4} - \tfrac{9y^2}{2b^3} + \tfrac{1}{2b}\right], \qquad \tfrac{d^2 w}{dy^2} = \tfrac{p_0}{2D}\left[y^2 - \tfrac{3yb}{4}\right]$$

At $y = b$:

$$M_{\max} = -D\tfrac{d^2 w}{dy^2} = -p_0 \tfrac{b^2}{8}, \qquad \sigma_{y,\max} = \tfrac{6M_{\max}}{t^2} = -0.75 p_0 \left(\tfrac{b}{t}\right)^2 \qquad ◄$$

SOLUTION (13.14)



Let $W = 1 - \frac{x}{a} - \frac{y}{a}$.

Then $w = cx^2y^2W$.

$$\frac{\partial w}{\partial x} = 2cxy^2W^2 + 2cx^2y^2W(-\tfrac{1}{a})$$

$$\frac{\partial^2 w}{\partial x \partial y} = 4cxyW + 4cxy^2W(-\tfrac{1}{a}) + 4cx^2y(-\tfrac{1}{a}) + 2cx^2y^2W^{-1}(-\tfrac{1}{a})$$

$$\frac{\partial w}{\partial y} = 2cx^2yW + 2cx^2y^2W(-\tfrac{1}{a})$$

$$\frac{\partial^2 w}{\partial x^2} = 2cy^2W^2 + 4cxy^2W(-\tfrac{1}{a}) + 4cxy^2(-\tfrac{1}{a}) + 2cx^2y^2W^{-1}(-\tfrac{1}{a})^2$$

$$\frac{\partial^2 w}{\partial y^2} = 2cx^2W^2 + 4cx^2yW(-\tfrac{1}{a}) + 4cx^2y(-\tfrac{1}{a}) + 4cx^2y^2W^{-1}(-\tfrac{1}{a})^2$$

(*a*) At x=0: w=0, $\frac{\partial w}{\partial x} = 0$

At y=0: w=0, $\frac{\partial w}{\partial y} = 0$

At y=a-x: w=0, $\frac{\partial w}{\partial x} = 0$, $\frac{\partial w}{\partial y} = 0$

(*b*) At x=0, y=a:

$$\sigma_y = -\frac{Et}{2(1-\nu^2)}\left[\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2}\right] = \cdots\cdots = 0, \quad \tau_{xy} = \frac{E}{2(1+\nu)}\frac{\partial^2 w}{\partial x \partial y} = \cdots\cdots = 0$$ ◄

At y=0, $x = a/2$: $\frac{\partial^2 w}{\partial y^2} = \frac{ca^2}{8}$, $\frac{\partial^2 w}{\partial x^2} = 0$, $\frac{\partial^2 w}{\partial x \partial y} = 0$

and $\sigma_{y,\max} = -\frac{Et}{2(1-\nu^2)}\left[\frac{ca^2}{8}\right] = -\frac{Ecta^2}{16(1-\nu^2)}$, $\tau_{xy} = 0$ ◄

SOLUTION (13.15)

We have

$$M_{xy} = -D(1-\nu)\frac{\partial^2 w}{\partial x \partial y} = M_0$$

Let, $\frac{\partial^2 w}{\partial x \partial y} = \frac{-M_0}{D(1-\nu)} = k$

Integrating with respect to x :

$$\frac{\partial w}{\partial y} = kx + f(y) + c_1$$

Then, integrating the above with respect to y gives

$$w = kxy + \int f_1(y)dy + c_2 \qquad \text{where } f_1(y) = f(y) + c_1$$

Due to the symmetry in deflection : $\int f_1(y)dy = 0.$

Also, owing to the symmetry, center (a/2, a/2) should be free of displacement,

$$w = 0 = \tfrac{1}{4}ka^2 + c_2 \qquad \therefore\ c_2 = -\tfrac{1}{4}ka^2$$

It follows that

$$w = -\frac{M_0}{D(1-\nu)}\left(xy - \frac{a^2}{4}\right)$$ ◄

We observe that this solution satisfies boundary conditions, $M_x = 0$ and $M_y = 0$ at plate edges.

SOLUTION (13.16)

Cylinder end can be approximated as a clamped edge plate subjected to uniform loading.

(*a*) Equation (13.29),

$$\sigma_{r,\max} = \tfrac{3p_0}{4}(\tfrac{a}{t})^2; \qquad (\tfrac{a}{t})^2 = \frac{135(10^6)\times 4}{3\times 1.5(10^6)} = 120$$

or a/t=10.954. Hence,

$$t = \frac{200}{10.954} = 18.26\ mm$$

(*b*) Then, Eq.(13.27) for r=0, gives

$$w_{\max} = \frac{p_0 a^4}{64D} = \frac{1.5\times 10^6 (0.2)^4 \times 12(1-0.3^2)}{64\times 200\times 10^9 (0.01826)^3} = 0.336\ mm \quad ◄$$

SOLUTION (13.17)

(*a*) From Eq. (13.29):

$$\sigma_{\max} = \tfrac{3}{4}p_0(\tfrac{a}{t})^2 = \tfrac{3}{4}p_{yp}(\tfrac{125}{10})^2 = 117.19\sigma_{yp}$$

Setting $\sigma_{\max} = p_{yp}$

$$p_{yp} = \frac{\sigma_{yp}}{117.19} = \frac{345}{117.19} = 2.944\ MPa$$

We have $D = \frac{Et^3}{12(1-\nu^2)} = \frac{200(10^9)(0.01)^3}{12(1-0.3^2)} = 18.315\ kPa$

Eq. (13.27) for r=0 is then

$$w_{\max} = \frac{p_{yp} a^4}{64D} = \frac{2.944(10^6)(0.125)^4}{64(18.315\times 10^3)} = 0.613\ mm \quad ◄$$

(*b*) $p_{allow} = \frac{p_{yp}}{n} = \frac{2.944}{1.2} = 2.45\ MPa$ ◄

SOLUTION (13.18)

Referring to Example 13.2:

$$\sigma_{\max} = \frac{6M_{\max}}{t^2} = 6(0.0534p_0)(\tfrac{50}{2})^2$$

Thus, $240(10^6) = 200.25p_0$

or $p_0 = 1.2\ MPa$ ◄

Similarly,

$$w_{\max} = 0.0454p_0\frac{a^4}{Et^3} = 0.0454(1.2\times 10^6)\frac{(0.5)^4}{70\times 10^9 (0.02)^3}$$

$$= 0.0608\ m = 6.1\ mm \quad ◄$$

SOLUTION (13.19)

We observe that

$$\int_0^a \sin\frac{m\pi x}{a}\sin\frac{m\pi x}{a}dx = \int_0^b \sin\frac{n\pi y}{b}\sin\frac{n\pi y}{b}dy = 0$$

if $m \neq m'$ and $n \neq n'$ Therefore, integrating, we consider only the squares of the terms in the parenthesis in Eq. (b) of Sec. 13.9. Using the formula:

(CONT.)

12.19 (CONT.)

$$\int_0^a \int_0^b \sin^2 \tfrac{m\pi x}{a} \sin^2 \tfrac{n\pi y}{b}\, dxdy = \tfrac{ab}{4}$$

calculation of the first term of the integral in Eq. (b) gives

$$\tfrac{\pi^4 abD}{8} \sum_m^\infty \sum_n^\infty a_{mn}^2 (\tfrac{m^2}{a^2} + \tfrac{n^2}{b^2})^2 \qquad \blacktriangleleft$$

Also, the second term of the integral in Eq. (b):

$$p_0 \int_0^a \int_0^b a_{mn} \sin \tfrac{m\pi x}{a} \sin \tfrac{n\pi y}{b}\, dxdy$$

$$= \tfrac{p_0 ab}{\pi^2 mn} a_{mn} (1 - \cos m\pi)(1 - \cos n\pi)$$

$$= \tfrac{4p_0 ab}{\pi^2 mn} a_{mn} \qquad (m,\ n = 1, 3, \ldots) \qquad \blacktriangleleft$$

SOLUTION (13.20)

Deflection is given by Eq. (13.18b). Loading is expressed as follows:

$$p = 2p_0 \tfrac{x}{a} \qquad 0 \le x \le \tfrac{a}{2}$$
$$p = 2p_0 - 2p_0 \tfrac{x}{a} \qquad \tfrac{a}{2} \le x \le a \qquad \text{(a)}$$

Potential energy, Eq. (13.33):

$$W = 2\sum\sum \int_0^{a/2} \int_0^a \tfrac{2p_0 x}{a} \sin \tfrac{m\pi x}{a} \sin \tfrac{n\pi y}{a}\, dxdy$$

$$= \sum\sum \tfrac{8p_0 a^2}{m^2 n \pi^3} a_{mn} \sin \tfrac{m\pi}{2} \qquad \text{(b)}$$

Strain energy, Eq. (13.32):

$$U = \tfrac{D}{2} \int_0^a \int_0^a [a_{mn} (\tfrac{m^2\pi^2}{a^2} + \tfrac{n^2\pi^2}{b^2}) \sin \tfrac{m\pi x}{a} \sin \tfrac{n\pi y}{a}]^2\, dxdy$$

$$= \tfrac{1}{8} D\pi^4 a^2 \sum\sum a_{mn}^2 (\tfrac{m^2}{a^2} + \tfrac{n^2}{a^2})^2$$

We thus have

$$\tfrac{\partial \Pi}{\partial a_{mn}} = \tfrac{D\pi^4 a^2}{4} a_{mn} (\tfrac{m^2}{a^2} + \tfrac{n^2}{a^2})^2 - \tfrac{8p_0 a^2}{m^2 n \pi^3} \sin \tfrac{m\pi}{2} = 0$$

or

$$a_{mn} = \tfrac{32 p_0 a^4 \sin(m\pi/2)}{m^2 n \pi^7 D (m^2 + n^2)^2} \qquad (m, n = 1, 3, .) \qquad \text{(c)}$$

Substitute this into Eq. (13.18b) to obtain deflection.

SOLUTION (13.21)

Let p represent the pressure differential.

Cylinder:

$$\sigma_\theta = \tfrac{ap}{t} \qquad \text{or} \qquad \sigma_\theta = \tfrac{150p}{6} = 25p$$

From the above, $p = \sigma_\theta / 25 = 15 \times 10^6 / 25 = 600\ kPa$ ◀

Sphere:

$$\sigma = \tfrac{pa}{2t} = 12.5p$$

Thus,

$$p = \sigma / 12.5 = 15 \times 10^6 / 12.5 = 1.2\ MPa$$

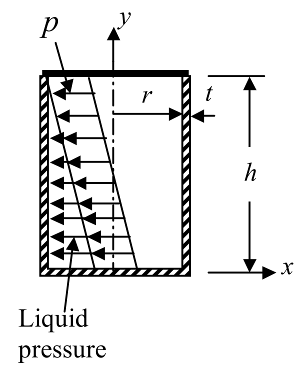
SOLUTION (13.22)

$$p = \frac{2t\sigma_{all}}{r} = \frac{2(0.05)(30\times10^6)/2}{2.5\times12} = 1.5\ MPa$$

We have

$$9.81(10^3)h = 1.5\times10^6, \qquad h = 152.9\ m$$ ◄

SOLUTION (13.23)



Total pressure at any depth:

$$p = 500(10^3) + \gamma(h - y)$$

(**a**) At $y = h$:

$$p = 500(10^3)\ Pa$$

Therefore,

$$t = \frac{pr}{\sigma_{all}} = \frac{500(10^3)4000}{180(10^6)} = 11.11\ mm$$ ◄

(**b**) At $y = h/4$:

$$p = 500(10^3) + 15(10^3)(13.5) = 702.5(10^3)\ Pa$$

$$t = \frac{702.5(10^3)4000}{180(10^6)} = 15.6\ mm$$ ◄

(**c**) At $y = 0$:

$$p = 500(10^3) + 15(10^3)(18) = 770(10^3)\ Pa$$

$$t = \frac{770(10^3)4000}{180(10^6)} = 17.1\ mm$$ ◄

SOLUTION (13.24)

Total pressure at any depth:

$$p = 200(10^3) + \gamma(h - y)$$

(**a**) At $y = h$:

$$p = 200(10^3)\ Pa$$

$$t = \frac{pr}{\sigma_{all}} = \frac{200(10^3)4000}{180\times10^6} = 4.44\ mm$$ ◄

(CONT.)

12.24 (CONT.)

(b) At $y = h/4$:

$$p = 200(10^3) + 15(10^3)(13.5) = 402.5(10^3)\ Pa$$

$$t = \frac{402.5(10^3)4000}{180\times10^6} = 8.94\ mm \quad ◄$$

(c) At $y = 0$::

$$p = 200(10^3) + 15(10^3)18 = 470(10^3)\ Pa$$

$$t = \frac{470(10^3)4000}{180\times10^6} = 10.4\ mm \quad ◄$$

SOLUTION (13.25)

The thickness for circumferential stress:

$$t = \frac{pr}{\sigma_{all}} = \frac{1.2(10^3)(0.6)}{24.0/2.4} = 12\ mm$$

The thickness for axial stress:

$$t = \frac{pr}{2\sigma_{all}} = \frac{12}{2} = 6\ mm$$

Thus, $t_{req} = 12\ mm$ ◄

SOLUTION (13.26)

$$\sigma_\theta = \frac{pr}{t} \qquad p = \gamma h$$

$$t_{req} = \frac{pr}{\sigma_{all}} = \frac{\gamma hr}{\sigma_{all}} = \frac{9.81(10^3)(150)(400)}{120(10^6)/1.8} = 8.83\ mm \quad ◄$$

SOLUTION (13.27)

$$t_{req} = \frac{pr}{\sigma_{all}} = \frac{\gamma hr}{\sigma_{all}} = \frac{9.81(10^3)(150)(400)}{100(10^6)} = 5.89\ mm \quad ◄$$

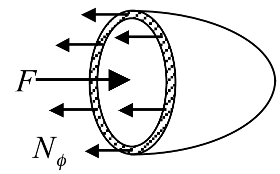
SOLUTION (13.28)

(a) $$\tau_{\max} = \frac{\sigma_1 - \sigma_2}{2} = \frac{1}{2}\left(\frac{pr}{t} - \frac{pr}{2t}\right) = \frac{pr}{4t} = \frac{pd}{8t}$$

$$d = \frac{8t\tau_{\max}}{p} = \frac{8\times12\times35\times10^6}{10\times10^6} = 336\ mm \quad ◄$$

(b) $$\sigma_1 = \frac{pr}{t} = \frac{10\times10^6\times168}{12} = 140\ MPa \quad ◄$$

SOLUTION (13.29)



The given numerical values are:

$$r_\phi = 180 - 1 = 179 \ mm$$

$$r_\theta = 80 - 1 = 79 \ mm \qquad r_i = 80 - 2 = 78$$

$$p = -0.08 \ MPa \qquad F = \pi r_i^2 p$$

Equation (13.46b) is therefore

$$N_\phi = \frac{F}{2\pi r_\theta(1)} = \frac{\pi(0.078)^2(0.08\times 10^6)}{2\pi(0.079)}$$

$$= 3.081 \ kN/m$$

and

$$\sigma_\phi = \frac{3081}{0.002} = 1.541 \ MPa \qquad ◀$$

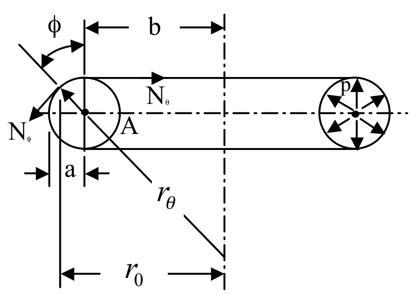
Using Eq. (13.46a),

$$\frac{\sigma_\theta}{0.079} + \frac{1.541}{0.179} = \frac{0.08}{0.02}$$

or

$$\sigma_\theta = 2.48 \ MPa = \sigma_{\max} \qquad ◀$$

SOLUTION (13.30)



Consider the portion of shell defined by ϕ. Vertical equilibrium of force yields,

$$2\pi r_0 N_\phi \sin\phi = \pi p(r_0^2 - b^2)$$

from which

$$N_\phi = \frac{p(r_0^2 - b^2)}{2r_0 \sin\phi} = \frac{pa(r_0 + b)}{2r_0}$$

or

$$N_\phi = \frac{pa}{b + a\sin\phi}\left(\frac{a}{2}\sin\phi + b\right) \qquad ◀$$

Substituting N_ϕ into Eq. (13.46a), setting $p_z = -p$, and $r_\phi = a$:

$$N_\theta = \frac{pr_\theta(r_0 - b)}{2r_0} = \frac{pa}{2} \qquad ◀$$

Since $\sin\phi = r_0/r_\theta = (r_0 - b)/a$, from symmetry.

Note that N_θ is constant throughout the shell from the condition of symmetry.

SOLUTION (13.31)

Referring to Solution of Prob. 13.30, we have

$$2a = (1-0.7)/2, \qquad a = 0.075\ m$$
$$2b = (1+0.7)/2, \qquad b = 0.425\ m$$

At point A (crotch):

$$\sigma_A = \sigma_{\phi,\max} = pa(r_0 + b)/(2r_0 t)$$

or

$$t = pa(r_0 + b)/(2r_0\sigma_{\phi,\max})$$
$$= \tfrac{2(10^6)(0.075)(0.35+0.425)}{2(0.35)(210\times 10^6)}$$
$$= 0.791\ mm = t_{req.}$$ ◀

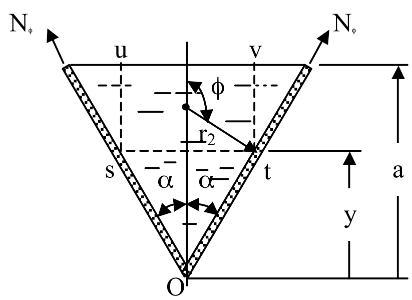
Similarly,

$$\sigma_\theta = \tfrac{pa}{2t}$$

or

$$t = \tfrac{2(10^6)75}{2(210\times 10^6)} = 0.357\ mm$$

SOLUTION (13.32)



Pressure at any level *st* is $p_r = \gamma(a-y)$. We have

$$\phi = \tfrac{\pi}{2} + \alpha \qquad r_0 = y\tan\alpha$$

Thus, the first of Eqs. (13.49) becomes

$$N_\theta = \tfrac{\gamma(a-y)y\tan\alpha}{\cos\alpha}$$

and

$$\sigma_\theta = \tfrac{\gamma(a-y)y}{t}\tfrac{\tan\alpha}{\cos\alpha}$$ ◀

The load F is equal to the weight of liquid in cylindrical portion *stuv*:

$$F = -\pi\gamma y^2(a - y + \tfrac{y}{3})\tan^2\alpha$$

Then second of Eqs. (13.49) gives

$$N_\phi = \gamma y(a - \tfrac{2}{3}y)\tan\alpha/2\cos\alpha$$

and

$$\sigma_\phi = \tfrac{\gamma y(a-2y/3)}{2t}\tfrac{\tan\alpha}{\cos\alpha}$$ ◀

SOLUTION (13.33)

Expressions for the components of pressure are:

$$p_\theta = -p\cos\theta \qquad p_r = p\sin\theta \qquad p_x = 0$$

Thus,

$$N_{x\theta} = -\int[-p\cos\theta + \tfrac{1}{a}(-pa\cos\theta)]dx + f_1(\theta)$$

$$= 2px\cos\theta + f_1(\theta) \tag{a}$$

$$N_\theta = -pa\sin\theta \tag{b}$$

$$N_x = -\int \tfrac{2}{a}(-px\sin\theta)dx + \tfrac{df_1}{d\theta} + f_2(\theta)$$

$$= \tfrac{1}{a}px^2\sin\theta + \tfrac{df_1}{d\theta} + f_2(\theta) \tag{c}$$

Boundary conditions: $N_x = 0$ (at x=0 and x=L) give $f_2 = 0$ and

$$\tfrac{pL^2}{a}\sin\theta + L\tfrac{df_1}{d\theta} = 0$$

or

$$f_1(\theta) = -\tfrac{pL}{2}\sin\theta + c$$

Note that no torque is applied to the shell; $c = 0$. Hence,

$$N_\theta = -pa\sin\theta \qquad N_x = -\tfrac{L-x}{a}px\sin\theta$$

$$N_{x\theta} = -(L-x)p\cos\theta$$

◄

SOLUTION (13.34)

Now the cylinder length does not change:

$$\int_{-L/2}^{L/2}(N_x - \nu N_\theta)dx = 0$$

Substituting Eqs. (b) of Example 13.7 into this, taking $f_1 = 0$ and integrating the resulting expression, we have

$$f_2(\theta) = \nu\gamma a^2(1-\cos\theta) - \tfrac{\gamma L^2}{24}\cos\theta$$

Referring to Example 13.7, the solution is thus,

$$N_\theta = \gamma a^2(1-\cos\theta) \qquad N_{x\theta} = \gamma ax\sin\theta$$

$$N_x = \tfrac{\gamma x^2}{2}\cos\theta + \nu\gamma a^2(1-\cos\theta) - \tfrac{\gamma L^2}{24}\cos\theta$$

◄

SOLUTION (13.35)

Referring to Fig. 13.13b:

$$p_x = p\sin\phi \qquad p_z = p\cos\phi \qquad r_0 = x\cos\phi$$

<u>Stress resultants due to weight</u>:

$$F = 2\pi r\cdot p\sin\phi\cdot rd\phi$$

(CONT.)

Since $rd\phi = dx$ and $r = x\cot\phi$. Then

$$F = 2\pi\int_0^x x\cot\phi p\sin\phi dx = 2\pi p\cos\phi(\tfrac{x^2}{2}) + c$$

For a cone supported at its edge $c = 0$, since $F = 0$ at $x = 0$. Therefore, Eq. (13.46b) gives

$$N_\phi = -\tfrac{px}{2\sin\phi} \tag{1}$$

Equation (13.46a):

$$N_\theta = \tfrac{p_z r_0}{\sin\phi} = -px\tfrac{\cos^2\phi}{\sin\phi} \tag{2}$$

<u>Stress resultants due to pressure</u>:
Equation (13.46a) yields,

$$N_\theta = -p_r\tfrac{x\cos\phi}{\sin\phi} = -p_r x\cot\phi \tag{3}$$

We now have

$$F = (2\pi p_r\sin\phi rd\phi)\cos\phi$$

Following a procedure similar to that the preceding, we obtain

$$F = 2\pi p_r\cos^2\phi(\tfrac{x^2}{2})$$

Equation (13.46a) leads to

$$N_\phi = -\tfrac{p_r x\cos\phi}{2\sin\phi} = -\tfrac{1}{2}p_r x\cot\phi \tag{4}$$

Solution is determined by the superposition of the preceding results: adding Eqs. (4) and (1), and (3) and (2). In so doing, we have

$$N_x = -\tfrac{x}{2\sin\phi}(p + p_r\cos\phi) \quad ◄$$

and

$$N_\theta = -x\cot\phi(p\cos\phi + \tfrac{1}{2}p_r) \quad ◄$$

End of Chapter 13