Intermediate Fluid Mechanics (ME EN 5700/6700)
Final Exam, Fall 2025

Name: **UiD:**

(*Note: The exam is closed notes, book, and notebook. Students are allowed the use of a calculator and the provided equation sheet.*)

P1. [10 points] Which of the following are valid expressions in index notation? If an expression is not valid, explain why.

(a) $a = b_i c_{ij} d_j \implies$ Valid.

(b) $a = b_i c_i + d_j \implies$ Invalid; free index 'j' is only on one term.

(c) $a_i = \delta_{ij} b_i + c_i \implies$ Invalid; free index 'j' is only on one term.

(d) $a_k = b_i c_{ki} \implies$ Valid

(e) $a_k = b_k c + d_i e_{ik} \implies$ Valid

(f) $a_i = b_i + c_{ij} d_{ji} e_i \implies$ Invalid; index 'i' is repeated three times.

(g) $a_\ell = \epsilon_{ijk} b_j c_k \implies$ Invalid; free index 'l' doesn't match free index 'i'.

(h) $a_{ij} = b_{ji} \implies$ Valid

(i) $a_{ij} = b_i c_j + e_{jk} \implies$ Invalid; free index 'k' only appears once.

(j) $a_{k\ell} = b_i c_{ki} d_\ell + e_{ki}. \implies$ Invalid; free index 'l' does not appear on the last term.

P2. [10 points] Write the following vector quantities in index notation.

(a) $\vec{u} \times \left(\vec{\nabla} \times \vec{u} \right)$

$$= u_i \frac{\partial u_k}{\partial x_l} \epsilon_{klm} \epsilon_{imn}$$

(b) $\vec{\nabla} \cdot \left(\vec{\nabla} \times \vec{u} \right)$

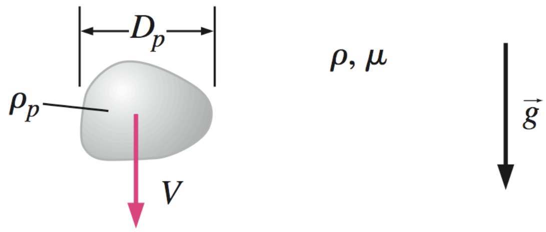
$$= \frac{\partial}{\partial x_i} \frac{\partial u_j}{\partial x_k} \epsilon_{jki}$$

(c) $\vec{\nabla} \times \vec{\nabla} \phi$

$$= \frac{\partial}{\partial x_i} \frac{\partial \phi}{\partial x_j} \epsilon_{ijk}$$

P3. [20 points] **Dimensional Analysis**

You would like to predict the terminal velocity V_t of a rain drop (diameter $D_p = 1$ mm, density $\rho_p = 1000$ kg/m^3) by conducting a dynamically similar laboratory experiment using a glass bead falling in a tank of water.



(a) Use the Buckingham Pi theorem to find the relevant non-dimensional parameters for this problem. [HINT: start with $V_t = f(D_p, \mu, \Delta\rho, g)$ where $\Delta\rho\,(\equiv \rho_p - \rho)$ denotes the density difference between the object and the fluid through which it is falling.]

8pt

Following the Buckingham-Pi theorem, $n = 5$, $(V_t, D_p, \mu, \Delta\rho, g)$ and $r = 3$, (M, L, T).

To solve this problem we choose $\Delta\rho$, g, D_p as repeating variables.

As a result, the Pi-groups obtained are:

$$\pi_1 = \frac{V_t}{\sqrt{gD_p}} \qquad\qquad \pi_2 = \frac{\mu}{\Delta\rho D_p \sqrt{gD_p}} \tag{1}$$

(b) Explain how you would go about finding an empirical relationship for the general function you found in part (a).

4pt

The approach to follow would be to make a table with two columns, one with values of π_1 and the second column with values of π_2. I would then go ahead and populate the different rows in the table, by progressively making changes to the variables that we can easily measure with our experimental setup. Note that changing certain variables in π_1 may lead to corresponding changes in π_2.

Once one has a decent amount of data points, the approach to follow is to include these in a scatter plot, so one can visualize whether any relationship appears from the data or not.

(c) In the laboratory, you measure the terminal velocity of the glass bead to be $V_t = 14$ m/s. Use this measurement along with the results from part a to calculate (i) the diameter of the glass bead required to guarantee dynamic similarity, and (ii) the terminal velocity of the rain drop. [Assume the density of the glass bead is 1500 kg/m^3; the density and viscosity of water are 1000 kg/m^3 and 1×10^{-3} kg/m s, respectively; and, the density and viscosity of air are 1 kg/m^3 and 1.5×10^{-5} kg/m s.

8pt

First we begin by matching π_2,

$$\frac{\mu_g}{\Delta\rho_g D_{p_g}\sqrt{gD_{p_g}}} = \frac{\mu_r}{\Delta\rho_r D_{p_r}\sqrt{gD_{p_r}}} \tag{2}$$

$$\Longrightarrow D_{p_g} = \left[\left(\frac{\mu_g}{\mu_r}\right)\left(\frac{\Delta\rho_r}{\Delta\rho_g}\right)\right]^{2/3} D_{p_r} = 26.1mm \tag{3}$$

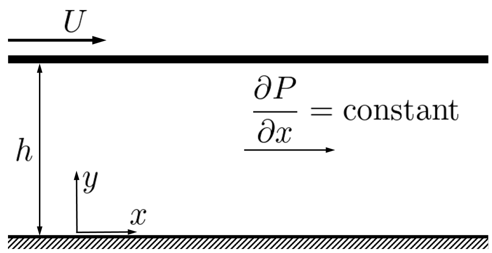
Next one matches the first Pi group, π_1,

$$\frac{V_{t_g}}{\sqrt{gD_{p_g}}} = \frac{V_{t_r}}{\sqrt{gD_{p_r}}} \tag{4}$$

$$\Longrightarrow V_{t_r} = V_{t_g}\sqrt{\frac{D_{p_r}}{D_{p_g}}} = 2.7m/s \tag{5}$$

P4. [20 points] **Conservation equations** An incompressible viscous fluid is between two infinite plates, as shown in the figure below. The bottom plate is fixed, and the top plate moves at a constant velocity of U. The distance between the two plates is h. There is a constant pressure gradient along the x-direction. The flow is fully developed. Ignore gravity.

5pt per question.



(a) Write down the stream-wise component of the Navier-Stokes equations in non-dimensional form using characteristic variables of the problem.

To non-dimensionalize the continuity and the Navier-Stokes equations, we define the following non-dimensional variables,

$$\tilde{x}_i = \frac{x_i}{L}, \qquad \tilde{t} = \frac{t}{T}, \qquad \tilde{u}_i = \frac{u_i}{U}, \qquad \tilde{p} = \frac{p}{\Delta p}, \qquad \tilde{\rho} = \frac{\rho}{\rho_0}. \tag{6}$$

For the Navier Stokes equations,

$$\rho \frac{\partial u_i}{\partial t} + \rho\, u_j \frac{\partial u_i}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \rho\, g\, \delta_{i3} + \mu \frac{\partial^2 u_i}{\partial x_j^2}, \tag{7}$$

when substituting for the non-dimensional variables,

$$\left(\frac{\rho_0\, U}{T}\right) \tilde{\rho} \frac{\partial \tilde{u}_i}{\partial \tilde{t}} + \left(\frac{\rho_0 U^2}{L}\right) \tilde{\rho} \tilde{u}_j \frac{\partial \tilde{u}_i}{\partial \tilde{x}_j} = -\left(\frac{\Delta p}{L}\right) \frac{\partial \tilde{p}}{\partial \tilde{x}_i} + (\rho_0\, g) \tilde{\rho}\, \delta_{i3} + \left(\frac{\mu U}{L^2}\right) \frac{\partial^2 \tilde{u}_i}{\partial \tilde{x}_j^2}. \tag{8}$$

If we now divide by $\left(\frac{\rho_0 U^2}{L}\right) \tilde{\rho}$, one obtains that,

$$S_t \frac{\partial \tilde{u}_i}{\partial \tilde{t}} + \tilde{u}_j \frac{\partial \tilde{u}_i}{\partial \tilde{x}_j} = -E_u \frac{1}{\tilde{\rho}} \frac{\partial \tilde{p}}{\partial \tilde{x}_i} + \frac{1}{F_r} \delta_{i3} + \frac{1}{R_e} \frac{\partial^2 \tilde{u}_i}{\partial \tilde{x}_j^2}. \tag{9}$$

(b) Write down the stream-wise component of the Navier-Stokes equations, in dimensional form, simplify it based on the problem configuration and solve for the velocity distribution of the flow.

Fully developed flow gives $\partial u/\partial x = 0$, and $w = 0$ assuming 2-D flow. The continuity equation then reduces to

$$\frac{\partial v}{\partial y} = 0.$$

Excluding w-terms and gravity, and with $\partial p/\partial x = \partial P/\partial x$ the x-direction momentum equation is

$$\rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{\partial P}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y}\right)$$
$$\frac{\partial^2 u}{\partial y} = \frac{1}{\mu}\frac{\partial P}{\partial x}.$$

Integrating twice gives

$$u = \frac{1}{2\mu}\frac{\partial P}{\partial x}y^2 + C_1 y + C_2.$$

Boundary conditions:

$$u(y=0) = 0 \Rightarrow C_2 = 0; \qquad u(y=h) = U \Rightarrow C_1 = \frac{U}{h} - \frac{1}{2\mu}\frac{\partial P}{\partial x}h.$$

Thus,

$$u(y) = -\frac{h^2}{2\mu}\frac{\partial P}{\partial x}\left[\frac{y}{h} - \left(\frac{y}{h}\right)^2\right] + U\frac{y}{h}.$$

(c) What is the volumetric flow rate of the flow in the x-direction.

$$Q = \int_0^h u\mathrm{d}y = \int_0^h \left(-\frac{h^2}{2\mu}\frac{\partial P}{\partial x}\left(\frac{y}{h} - \left(\frac{y}{h}\right)^2\right) + U\frac{y}{h}\right)\mathrm{d}y = -\frac{h^3}{12\mu}\frac{\partial P}{\partial x} + \frac{Uh}{2}$$

(d) Can the velocity profile be an algebraic sum of velocity distributions of two simpler flows, and why? If it can, what are the simpler flows?

The resultant velocity profile is an algebraic sum of the solutions to the Poiseuille flow,

$$-\frac{h^2}{2\mu}\frac{\partial P}{\partial x}\left[\frac{y}{h} - \left(\frac{y}{h}\right)^2\right],$$

and the Couette flow,

$$U\frac{y}{h}.$$

They can be added up to obtain the resultant velocity profile because the reduced momentum equation and its boundary conditions are linear.

P5. [20 points] **Laminar boundary layers:**

a. Write the von Karman integral equation for a boundary layer over a solid wall with no suction/blowing.

3pt

In the case that there is no injection of fluid at the wall the von Karman integral equation can be written as:

$$(U\delta^*)\frac{dU}{dx} + \frac{d}{dx}(U^2\theta) = \frac{\tau_w}{\rho}. \tag{10}$$

b. Under what assumptions is the above equation valid?

3pt

$\Longrightarrow$ For two-dimensional, incompressible, steady flows,

c. Define mathematically (and in words) the displacement thickness.

3pt

$\Longrightarrow$ The displacement thickness (δ^*) is the distance the plate would have to be moved so that the loss of mass flux in a uniform flow with velocity U_∞ is equivalent to the loss of mass flux caused by the boundary layer.

$$\delta^* = \int_0^\infty \left(1 - \frac{u}{U_\infty}\right) dy. \tag{11}$$

d. Define mathematically (and in words) the momentum thickness.

3pt

$\Longrightarrow$ The momentum thickness (θ) is a length scale that represents the distance the plate would need to be displaced so that the resultant loss in momentum flux of a uniform flow of velocity U_∞ is equivalent to the loss of momentum due to the presence of the boundary layer.

$$\theta = \int_0^\infty \frac{u}{U_\infty}\left(1 - \frac{u}{U_\infty}\right) dy \tag{12}$$

e. Simplify the von Karman integral for flow over a flat (very large) plate (don't solve just simplify).

3pt

In this case, $\frac{dU}{dx}$ and hence the von Karman integral equation can be simplified to:

$$\frac{d}{dx}(U^2\theta) = \frac{\tau_w}{\rho}. \tag{13}$$

f. Assume a laminar boundary layer stream wise velocity profile given by:

5pt

$$\frac{u}{U_\infty} = \left(\frac{y}{\delta}\right) = \eta \text{ where } \eta = y/\delta.$$

Using this and your reduced form of the von Karman integral from part [e.], solve for $\delta(x)$ [Hint: change the integration limits in the momentum thickness definition from $0 \to \delta$ to $\infty \to 1$].

$$\theta = \int_0^\infty \frac{u}{U_\infty}\Big(1 - \frac{u}{U_\infty}\Big)\, dy \Longrightarrow \theta = \int_0^\infty \eta\Big(1-\eta\Big)\,\delta d\eta = \delta\Big(\frac{\eta^2}{2} - \frac{\eta^3}{3}\Big)\Big|_0^1 = \delta\Big(\frac{1}{2} - \frac{1}{3}\Big) = \frac{\delta}{6} \tag{14}$$

Hence, upon substitution in the simplified Von Karman equation,

$$\frac{d}{dx}(U_\infty^2 \frac{\delta}{6}) = \frac{\tau_w}{\rho} \tag{15}$$

$$\frac{U_\infty^2}{6}\frac{d\delta}{dx} = \frac{\tau_w}{\rho}. \tag{16}$$

At the same time,

$$\tau_w = \mu \frac{du}{dy}\Big|_{y=0} = \mu \frac{U_\infty}{\delta}\frac{dy}{dy} = \mu\frac{U_\infty}{\delta} \tag{17}$$

Therefore, upon substitution in the previous equation,

$$\frac{U_\infty^2}{6}\frac{d\delta}{dx} = \frac{\mu}{\rho}\frac{U_\infty}{\delta} \tag{18}$$

which upon rearranging and integrating,

$$\delta\, d\delta = 6\frac{\nu}{U_\infty}dx \tag{19}$$

$$\frac{\delta^2}{2} = 6\frac{\nu}{U_\infty}x \tag{20}$$

$$\boxed{\delta = \sqrt{12\frac{\nu\, x}{U_\infty}}} \tag{21}$$

P6. [20 points] **Vorticity Transport**

a. Consider the vorticity transport equation shown below.

$$\underbrace{\frac{\partial \vec{\omega}}{\partial t}}_{I} + \underbrace{\left(\vec{u} \cdot \vec{\nabla}\right) \vec{\omega}}_{II} = \underbrace{\left(\vec{\omega} \cdot \vec{\nabla}\right) \vec{u}}_{III} + \underbrace{\frac{\vec{\nabla}\rho \times \vec{\nabla}P}{\rho^2}}_{IV} + \underbrace{\nu \vec{\nabla}^2 \vec{\omega}}_{V}$$

1. Give the name and a short description of the physical meaning of each term I–V.

5pt

I – Local time rate of change of vorticity.

II – Advection of vorticity.

III – Vortex stretching and reorientation.

IV – Baroclinic torque.

V – Viscous difussion of voritcity.

2. Which term in the vorticity transport equation is linked to the turbulent energy "cascade" process?

3pt $\Longrightarrow$ Term III

b. Define (in words) what a <u>vortex line</u> is.

3pt

$\Longrightarrow$ A vortex line is a curve in the fluid such that their corresponding tangents at any point gives the direction of the local vorticity. In this way, vortex lines are analogous to streamlines for the velocity field. Vortex lines passing through any closed curve form a tubular surface, called vortex tubes.

c. What is the <u>solenoidal</u> condition for vorticity?

3pt

$\Longrightarrow \vec{\nabla} \cdot \vec{\omega} = 0$

d. Assume you have a steady flow with a velocity field that only varies in the vertical direction ($\vec{V} = f(y)$) and for which the density ρ is constant everywhere. Simplify the original vorticity equation given at the top.

5pt

From the given we know that $\vec{V} = f(y)$, hence d/dx of $\vec{V} = 0$. Also density is constant every where. Therefore, the vorticity itself can be written as:

$$\vec{\omega} = \vec{\nabla} \times \vec{u} = \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{bmatrix} = \tag{22}$$

$$= \underbrace{\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)}_{\omega_x} \hat{i} + \underbrace{\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)}_{\omega_y} \hat{j} + \underbrace{\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)}_{\omega_z} \hat{k} \tag{23}$$

$$= -\frac{\partial u}{\partial y}\hat{k}. \tag{24}$$

This means that for this 2D flow field there is only vorticity in the z-direction. With this result and the previous assumptions from the given, each individual term of the vorticity equation can be rewritten as follows:

$$\underbrace{\frac{\partial \vec{\omega}}{\partial t}}_{I} + \underbrace{\left(\vec{u} \cdot \vec{\nabla}\right) \vec{\omega}}_{II} = \underbrace{\left(\vec{\omega} \cdot \vec{\nabla}\right) \vec{u}}_{III} + \underbrace{\frac{\vec{\nabla}\rho \times \vec{\nabla}P}{\rho^2}}_{IV} + \underbrace{\nu \vec{\nabla}^2 \vec{\omega}}_{V}$$

given that:

Term I: $\frac{\partial \vec{\omega}}{\partial t} = 0 \Longrightarrow$ The velocity field does not depend on time, and hence the vorticity field does not either.

Term II: $\left(\vec{u} \cdot \vec{\nabla}\right) \vec{\omega} = \left(\vec{u} \cdot \vec{\nabla}\right) \omega_z = -(u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y})\frac{\partial u}{\partial y} = -v\frac{\partial^2 u}{\partial y^2}\,\hat{k}$

Term III: $\left(\vec{\omega} \cdot \vec{\nabla}\right) \vec{u} = \left(\omega_x \frac{\partial}{\partial x} + \omega_y \frac{\partial}{\partial y} + \omega_z \frac{\partial}{\partial z}\right) \vec{u} = \omega_z \frac{\partial \vec{u}}{\partial z} = 0$

Term IV: $\frac{\vec{\nabla}\rho \times \vec{\nabla}P}{\rho^2} = 0 \Longrightarrow$ Because the density is constant everywhere.

Term V: $\nu \vec{\nabla}^2 \vec{\omega}_z = \nu \left(\frac{\partial^2 \omega_z}{\partial x^2} + \frac{\partial^2 \omega_z}{\partial y^2} + \frac{\partial^2 \omega_z}{\partial z^2}\right) = -\nu \left(\frac{\partial}{\partial y}\frac{\partial^2 u}{\partial x^2} + \frac{\partial}{\partial y}\frac{\partial^2 u}{\partial y^2} + \frac{\partial}{\partial y}\frac{\partial^2 u}{\partial z^2}\right) = -\nu \frac{\partial^3 u}{\partial y^3}\,\hat{k}$

e. Does vorticity exist at a solid wall? Why or why not [support your position with equations or words]

1pt

Vorticity may or may not exist at a solid wall. What can not happen at the wall is that vorticity is created without an external forcing process. Vorticity maybe created for example when an unperturbed flow collides with a flat plate and the non-slip condition forces the fluid to attach at the surface of the given plate.