Intermediate Fluid Mechanics

Lecture 17: Dimensional Analysis

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Chapter Overview

Lecture Objectives

- In the previous lectures, we learned about fluid dynamics by simplifying the governing equations for some simple flow fields and then solving those equations for the velocities and temperature fields.
- **Dimensional Analysis** is another powerful tool that can give us important information about the flow without us having to actually solve the governing equations.
- In this lecture we will learn about how to use **Dimensional Analysis** in problems related to Fluid Dynamics.

Chapter Overview

Dimensional Analysis

Let's consider the case of an incompressible, planar flow due to an impulsively started flat plate

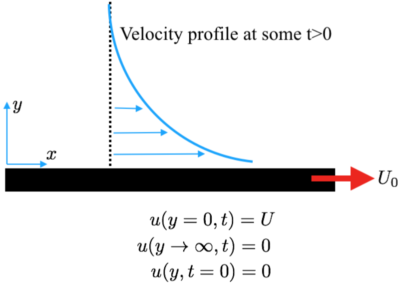

Figure: Sketch of the impulsively started flat plate flow.

$\Longrightarrow$ Since the flow is incompressible conservation of mass reduces to,

$$\underbrace{\cancel{\frac{\partial u}{\partial x}}}_{\substack{\text{fully}\\\text{developed}}} + \frac{\partial v}{\partial y} = 0 \qquad \Rightarrow \qquad \frac{\partial v}{\partial y} = 0 \tag{1}$$

Which leads to: $v = C$

- Applying the non-slip BCs at $y = 0$, *i.e.* $v(y = 0, t) = 0$.
- $\Longrightarrow v = 0$ everywhere.

Let's now consider the x-momentum equation,

$$\text{x-mom:} \qquad \rho \frac{\partial u}{\partial t} + \underbrace{\cancel{\rho\, u \frac{\partial u}{\partial x}}}_{\substack{\text{fully}\\ \text{developed}}} + \underbrace{\cancel{\rho\, v \frac{\partial u}{\partial y}}}_{v=0} = \underbrace{\cancel{-\frac{\partial p}{\partial x}}}_{\substack{\text{No imposed}\\ \text{pressure gradient}}} + \underbrace{\cancel{\mu \frac{\partial^2 u}{\partial x^2}}}_{\substack{\text{fully}\\ \text{developed}}} + \mu \frac{\partial^2 u}{\partial y^2}, \qquad (2)$$

Note: one cannot cancel the time-derivative term because the flow never reaches a steady-state, and hence it is inherently unsteady.

$\Longrightarrow$ In this case then the x-momentum equation reduces to,

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2}, \qquad (3)$$

where $\nu = \mu/\rho$.

This PDE can be solved using Laplace transforms. However, before we proceed in this direction let's consider **Dimensional analysis**.

Chapter Overview

Dimensional Analysis: Example

Let's begin by defining non-dimensional variables.

$\longrightarrow$ These are obtained by dividing the dimensional variables by appropriate characteristic scales.

These characteristic scales may come from the geometry of the flow, the fluid properties, boundary conditions, initial conditions, etc.

Let,

$$\tilde{y} = \frac{y}{L}, \qquad \tilde{t} = \frac{t}{T}, \qquad \tilde{u} = \frac{u}{U}. \tag{4}$$

Dimensional Analysis: Example (continued ...)

- Since there is no physically relevant length scale in this semi-infinite domain ,we will just denote the characteristic length scale as L. (This may represent the vertical distance above the plate at which we want to know the velocity.)
- Similarly, there is no well-defined characteristic time scale, so we just denote the characteristic time scale as T.
- The characteristic velocity scale is chosen from the boundary condition at $y = 0$.

Rearranging the above definitions gives that,

$$y = \tilde{y}L, \qquad t = \tilde{t}T, \qquad u = \tilde{u}U. \tag{5}$$

Dimensional Analysis: Example (continued ...)

Substituting into the x-momentum equation,

$$\underbrace{\frac{\partial(\tilde{u}U)}{\partial(\tilde{t}T)}}_{\text{Inertia}} = \underbrace{\nu\frac{\partial^2(\tilde{u}U)}{\partial(\tilde{y}L)^2}}_{\textit{diffusion}}. \tag{6}$$

Since T, L, U are constants they factor out of the derivatives, hence

$$\frac{\partial\tilde{u}}{\partial\tilde{t}} = \left(\frac{\nu\, T}{L^2}\right)\frac{\partial^2\tilde{u}}{\partial\tilde{y}^2}. \tag{7}$$

Note that:

- The group of parameters in the parenthesis is non-dimensional.
- Because there are only two terms in this equation (inertia and diffusion), the order of magnitude of $\left(\frac{\nu\, T}{L^2}\right)$ must be unity.

Dimensional Analysis: Example (continued ...)

Let's consider a few limit cases, for the sake of discussion:

- If $\left(\frac{\nu T}{L^2}\right) << 1$, then the governing equation basically reduces to $\partial \tilde{u}/\partial \tilde{t} = 0$.
- Conversely, if $\left(\frac{\nu T}{L^2}\right) >> 1$ then the governing equation basically reduces to $\partial^2 \tilde{u}/\partial \tilde{y}^2 = 0$.
- The only solution that satisfies these two limiting cases as well as the boundary conditions is the trivial solution, $\tilde{u} = 0$.
- Therefore, in this flow we must have $\left(\frac{\nu T}{L^2}\right) \approx 1$.
- This gives us a relationship between L and T, $\longrightarrow \frac{L}{\sqrt{\nu T}} \approx 1$.

Dimensional Analysis: Example (continued ...)

Since we didn't define L or T explicitly, we could easily just substitute in the actual independent variables,

$$\frac{y}{\sqrt{\nu\, t}} \approx 1, \qquad \text{or,} \qquad t = y^2/\nu. \tag{8}$$

$\Longrightarrow$ This tells us that the time t required to see a change in the local velocity is equal to the square of the distance above the plate divided by the viscosity.

Dimensional Analysis: Example (continued ...)

Let's go back to the non-dimensional parameter $\left(\frac{\nu\, T}{L^2}\right)$.

One can rewrite this term as,

$$\frac{\nu\, T}{L^2} = \frac{T}{\left(L^2/\nu\right)} = \frac{\text{Time scale of inertia}}{\text{Time scale of diffusion}}. \tag{9}$$

$\Longrightarrow$ This non-dimensional parameter may be thought of as the ratio of two timescales: the **inertia time scale** over the **diffusion time scale**.

- If this ratio must be unity as agreed previously, then we have discovered that the flow can only change as fast as the diffusion time scale allows.
- The further away we are from the plate, the longer we have to wait (exactly L^2/ν) in order to feel a change in our momentum.
- Realize that this is a significant result that was obtained without actually having to solve the governing equation.

Integration of the governing equation

Let's just verify our dimensional analysis by going ahead and solving the governing equation, as written in dimensional form:

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2}. \tag{10}$$

For this purpose we will use the Laplace transforms. Recall that,

$$\mathcal{L}\{u(t)\} = \hat{u}(s) = \int_0^\infty e^{-st}\, u(t)\, dt. \tag{11}$$

Therefore, taking the Laplace transform of both sides of the governing equation gives,

$$\mathcal{L}\left\{\frac{\partial u}{\partial t}\right\} = \mathcal{L}\left\{\nu \frac{\partial^2 u}{\partial y^2}\right\}. \tag{12}$$

Integration of the governing equation (continued ...)

Note that $\mathcal{L}\{\frac{\partial u}{\partial t}\} = s\,\mathcal{L}\{u(y,t)\} - u(y,t=0)$.

$\implies$ Therefore, since $u(y,t=0) = 0$,

$$s\,\hat{u} = \nu \frac{\partial^2 \hat{u}}{\partial y^2}. \tag{13}$$

Realize that we have reduced the PDE for u to an ODE for $\hat{u}$, which we can separate and integrate. However, before one also must transform the boundary conditions,

$$\hat{u}(y=0,s) = \mathcal{L}\{\underbrace{u(y=0,t)}_{=U}\} = \int_0^\infty e^{-st}\,U\,dt = \frac{U}{s}, \tag{14}$$

$$\hat{u}(y\to\infty,s) = \mathcal{L}\{\underbrace{u(y\to\infty,t)}_{=0}\} = \int_0^\infty e^{-st}\,0\,dt = 0. \tag{15}$$

Integration of the governing equation (continued ...)

Rearranging the above ODE,

$$\frac{\partial^2 \hat{u}}{\partial y^2} - \left(\frac{s}{\nu}\right) \hat{u} = 0, \tag{16}$$

One can then assume a solution of the form,

$$\hat{u}(y) = e^{\alpha\, y}. \tag{17}$$

Plugging this solution into the ODE gives,

$$\alpha^2 e^{\alpha\, y} - \left(\frac{s}{\nu}\right) e^{\alpha\, y} = 0 \tag{18}$$

$$\left(\alpha^2 - \frac{s}{\nu}\right) e^{\alpha\, y} = 0 \tag{19}$$

$$\implies \alpha = \pm\sqrt{s/\nu} \tag{20}$$

Integration of the governing equation (continued ...)

The general solution can then be written as,

$$\hat{u}(y) = C_1 \, e^{+\sqrt{s/\nu}\, y} + C_2 \, e^{-\sqrt{s/\nu}\, y}, \tag{21}$$

where C_1, and C_2 are constants.

- From the boundary conditions at $y \to \infty$, we find that $C_1 = 0$.
- From the boundary condition at $y = 0$, $C_2 = U/s$.

Therefore, the solution in the Laplace domain is,

$$\hat{u}(y; s) = \frac{U}{s} e^{-\sqrt{s/\nu} y}. \tag{22}$$

Integration of the governing equation (continued ...)

We now take the inverse Laplace transform to find the solution in the time domain,

$$u(y, t) = \mathcal{L}^{-1}\{\hat{u}(y; s)\}. \tag{23}$$

Using a math book, the inverse transform of the above Laplace solution

$$\frac{u}{U} = erfc\left(\frac{\eta}{2}\right). \tag{24}$$

Where the argument of the complimentary error function is exactly the same ratio of time scales that we obtained from dimensional analysis, $\eta \equiv \frac{y}{\sqrt{\nu t}}$.

Integration of the governing equation (continued ...)

Graphically, the solution looks like:

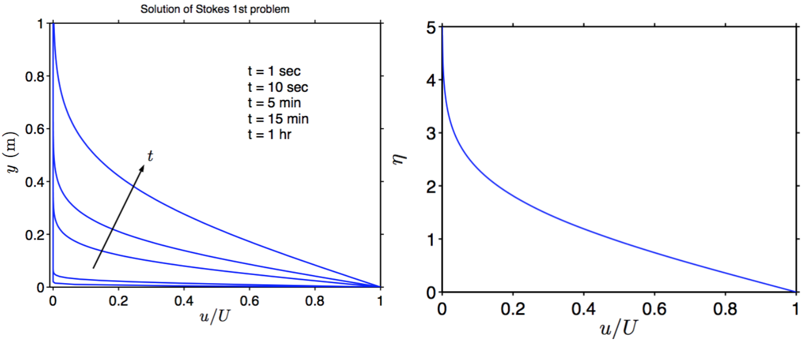

Figure: Graphical representation of the analytical solution.

Chapter Overview

Non-dimensionalizing the Continuity and Navier-Stokes equations

To non-dimensionalize the continuity and the Navier-Stokes equations, we define the following non-dimensional variables,

$$\tilde{x}_i = \frac{x_i}{L}, \qquad \tilde{t} = \frac{t}{T}, \qquad \tilde{u}_i = \frac{u_i}{U}, \qquad \tilde{p} = \frac{p}{p_0}, \qquad \tilde{\rho} = \frac{\rho}{\rho_0}. \tag{25}$$

- Can you write down the non-dimensional form of Conservation of Mass?

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho\, u_i)}{\partial x_i} = 0. \tag{26}$$

- What about the NS equations?

$$\rho \frac{\partial u_i}{\partial t} + \rho\, u_j \frac{\partial u_i}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \rho\, g\, \delta_{i3} + \mu \frac{\partial^2 u_i}{\partial x_j^2}, \tag{27}$$

Non-dimensionalizing the Continuity and Navier-Stokes equations

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Then, the continuity equation,

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho\, u_i)}{\partial x_i} = 0. \tag{29}$$

Can be rewritten in non-dimensional form as:

$$\frac{\partial (\tilde{\rho}\rho_0)}{\partial (\tilde{t}\, T)} + \frac{\partial (\tilde{\rho}\rho_0 \tilde{u}_i U)}{\partial (\tilde{x}_i L)} = 0 \tag{30}$$

$$\frac{\rho_0}{T}\frac{\partial \tilde{\rho}}{\partial \tilde{t}} + \frac{U\,\rho_0}{L}\frac{\partial (\tilde{\rho}\tilde{u}_i)}{\partial \tilde{x}_i} = 0. \tag{31}$$

Non-dimensionalizing Continuity and NS equations (continued ...)

Dividing by $U\rho_0/L$, one finds that

$$S_t \frac{\partial \tilde{\rho}}{\partial \tilde{t}} + \frac{\partial(\tilde{\rho}\tilde{u}_i)}{\partial \tilde{x}_i} = 0, \tag{32}$$

where $S_t \equiv \frac{L}{TU}$ is the **Strouhal number**.

- For now it suffices to say that the **Strouhal number** is a non-dimensional parameter that shows up in the unsteady continuity equation.

- The Strouhal number is particularly relevant in oscillating flows where the characteristic time scale may be represented by an inverse frequency, *i.e.* $T = \omega^{-1}$.

- An important application is associated with the shedding frequency of the vortices behind a circular cylinder or other blunt object.

Non-dimensionalizing Continuity and NS equations (continued ...)

For the Navier Stokes equations,

$$\rho \frac{\partial u_i}{\partial t} + \rho\, u_j \frac{\partial u_i}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \rho\, g\, \delta_{i3} + \mu \frac{\partial^2 u_i}{\partial x_j^2}, \tag{33}$$

when substituting for the non-dimensional variables,

$$\left(\frac{\rho_0\, U}{T}\right) \tilde{\rho} \frac{\partial \tilde{u}_i}{\partial \tilde{t}} + \left(\frac{\rho_0\, U^2}{L}\right) \tilde{\rho} \tilde{u}_j \frac{\partial \tilde{u}_i}{\partial \tilde{x}_j} = -\left(\frac{p_0}{L}\right) \frac{\partial \tilde{p}}{\partial \tilde{x}_i} + (\rho_0\, g) \tilde{\rho}\, \delta_{i3} + \left(\frac{\mu U}{L^2}\right) \frac{\partial^2 \tilde{u}_i}{\partial \tilde{x}_j^2}. \tag{34}$$

If we now divide by $\left(\frac{\rho_0 U^2}{L}\right) \tilde{\rho}$, one obtains that,

$$S_t \frac{\partial \tilde{u}_i}{\partial \tilde{t}} + \tilde{u}_j \frac{\partial \tilde{u}_i}{\partial \tilde{x}_j} = -E_u \frac{1}{\tilde{\rho}} \frac{\partial \tilde{p}}{\partial \tilde{x}_i} + \frac{1}{F_r} \delta_{i3} + \frac{1}{R_e} \frac{\partial^2 \tilde{u}_i}{\partial \tilde{x}_j^2}, \tag{35}$$

Non-dimensionalizing Continuity and NS equations (continued ...)

Note that:

- Strouhal number: $S_t = \frac{L}{T\,U}$, is the ratio betweem the transient inertia and the steady inertia.
- Euler number: $E_u = \frac{p_0}{\rho_0 U^2}$, is the ratio between the pressure force and the inertia force.
- Froude number: $F_r = \frac{U^2}{gL}$, is the ratio between the inertia force and the gravitational force.
- Reynolds number: $R_e = \frac{\rho U L}{\mu}$, is the ratio between the inertia force and the viscous force.

Each of the non-dimensional numbers above may be thought of as representing the ratio of two forces.