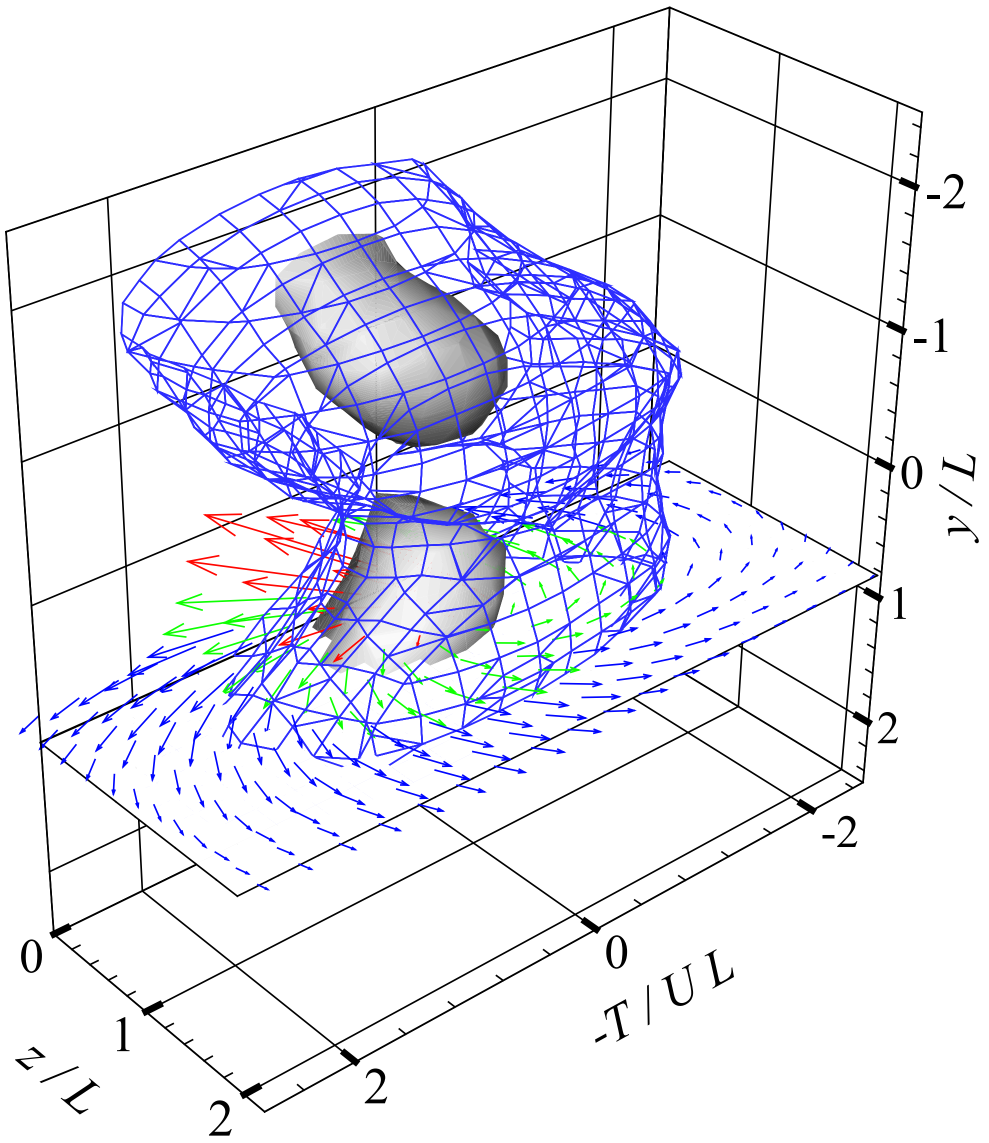
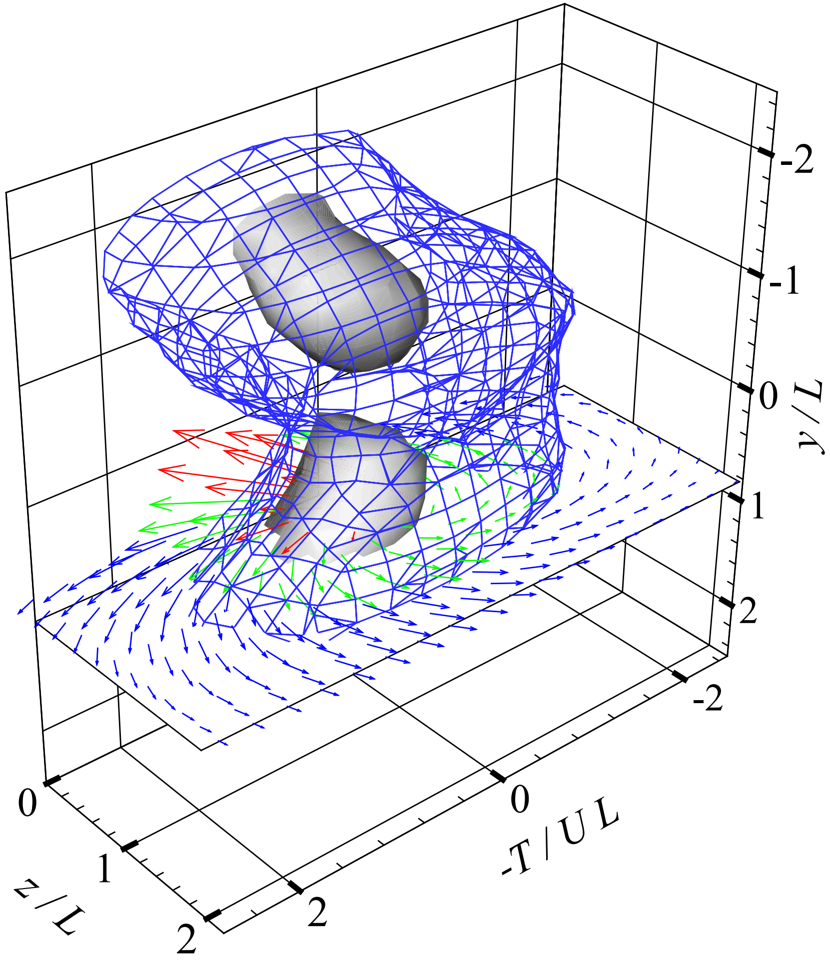
3. Vector Calculus

Fluid particle: Differentially Small Piece of the Fluid Material

- Concept of differential change in a vector, vector field
- Changes in unit vectors
- Calculus w.r.t. time
- Integral calculus w.r.t. space
- Differential calculus w.r.t. space
- Integral theorems, second order operators



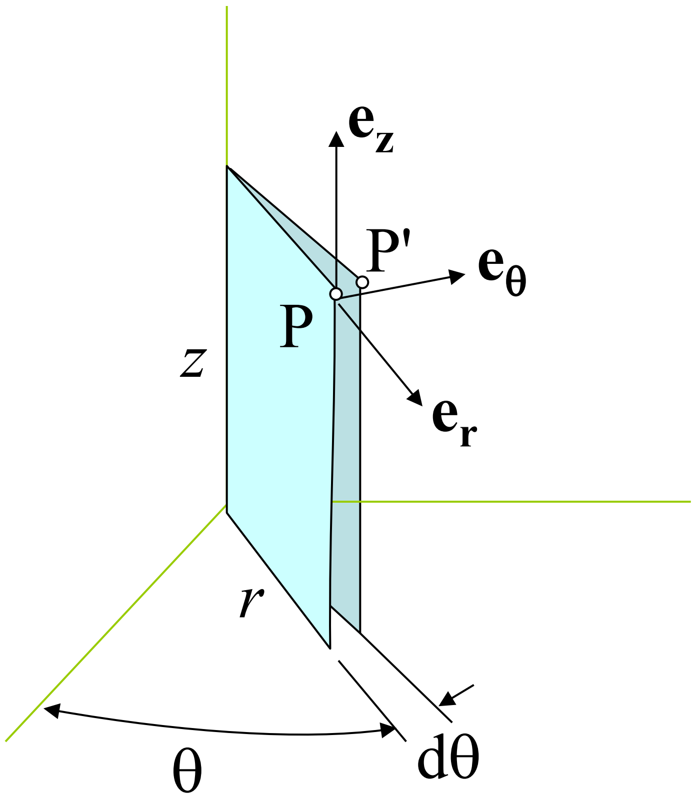
Concept of Differential Change In a Vector. The Vector Field.

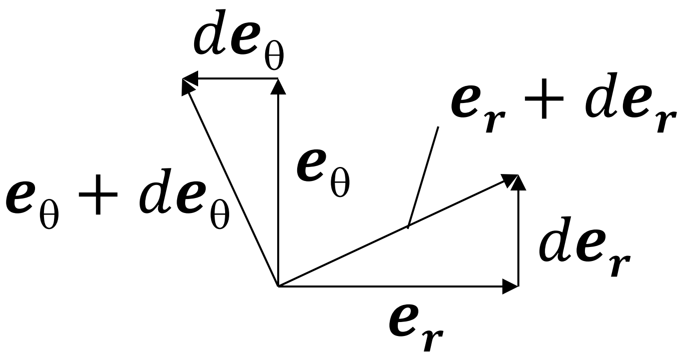


Scalar field $\phi=\phi(\mathbf{r},t)$

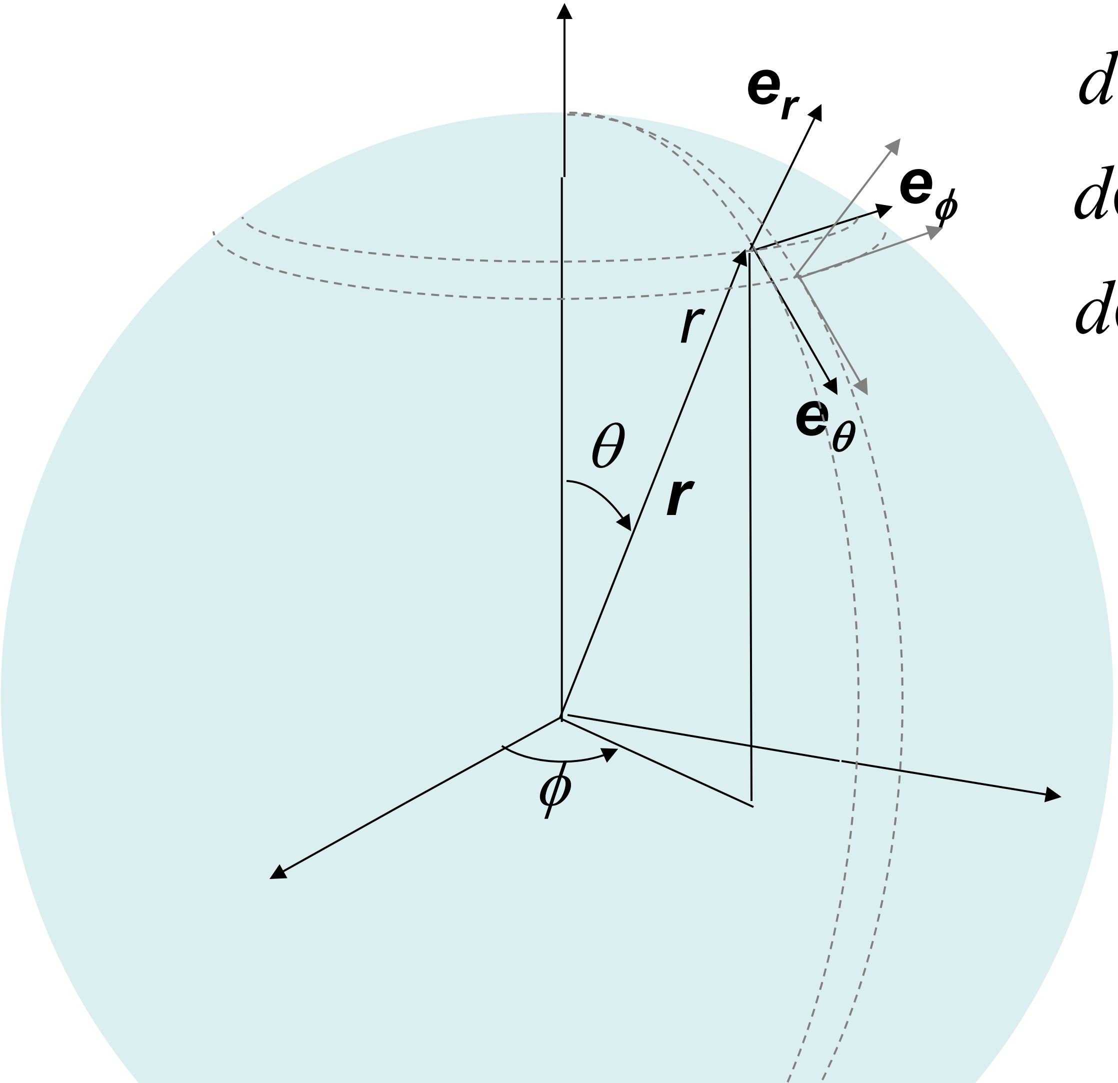
Vector field $\mathbf{V}=\mathbf{V}(\mathbf{r},t)$

Change in Unit Vectors – Cylindrical System





Change in Unit Vectors – Spherical System



$$d\mathbf{e}_r = d\theta\mathbf{e}_\theta + d\phi\sin\theta\mathbf{e}_\phi$$

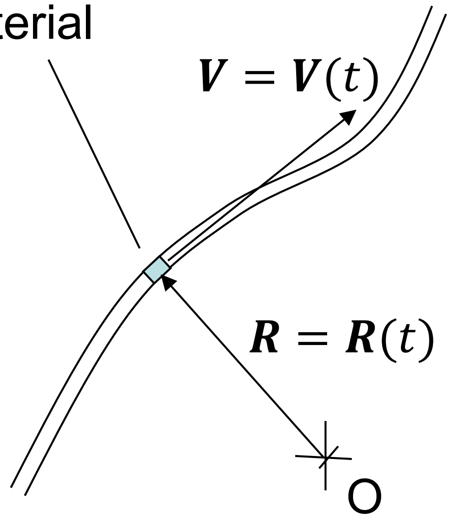
$$d\mathbf{e}_\theta = -d\theta\mathbf{e}_r + d\phi\cos\theta\mathbf{e}_\phi$$

$$d\mathbf{e}_\phi = -d\phi\sin\theta\mathbf{e}_r - d\phi\cos\theta\mathbf{e}_\theta$$

See “Formulae for Vector Algebra and Calculus”

Example

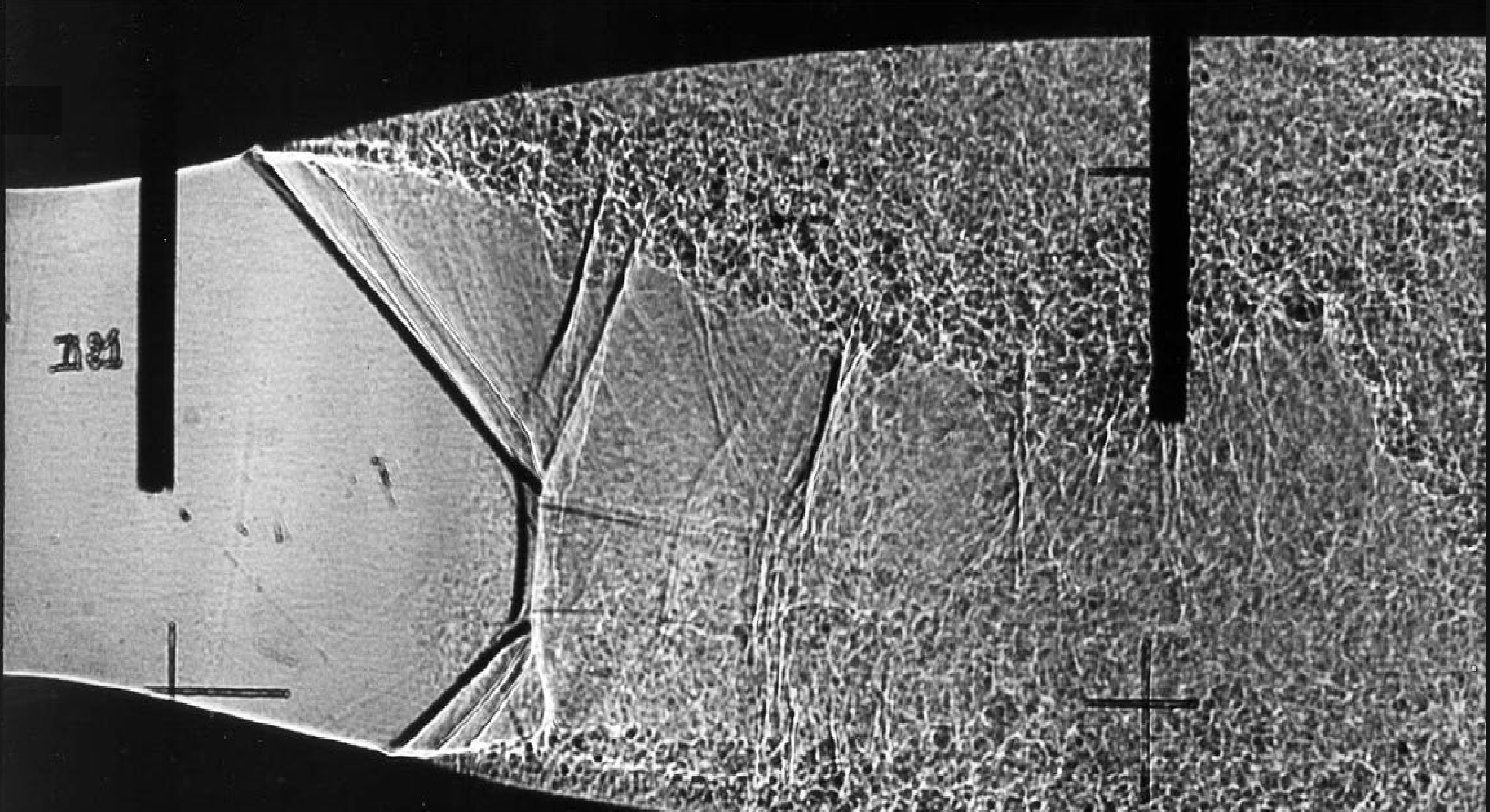
Fluid particle
Differentially small piece of the fluid material



The position of fluid particle moving in a flow varies with time. Working in different coordinate systems write down expressions for the position and, by differentiation, the velocity vectors.

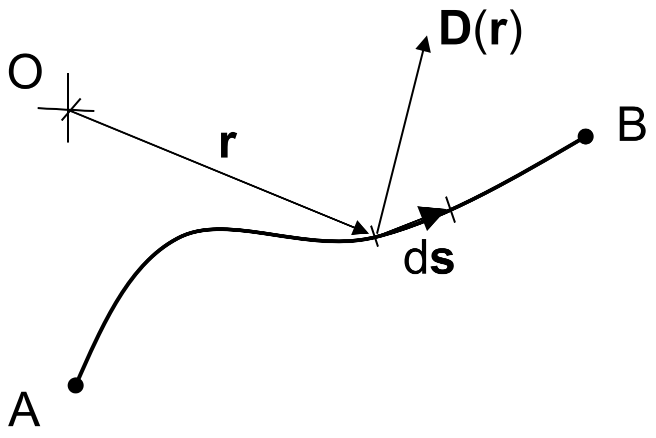
Vector Calculus w.r.t. Time

- Since *any* vector may be decomposed into scalar components, calculus w.r.t. time, only involves *scalar* calculus of the components

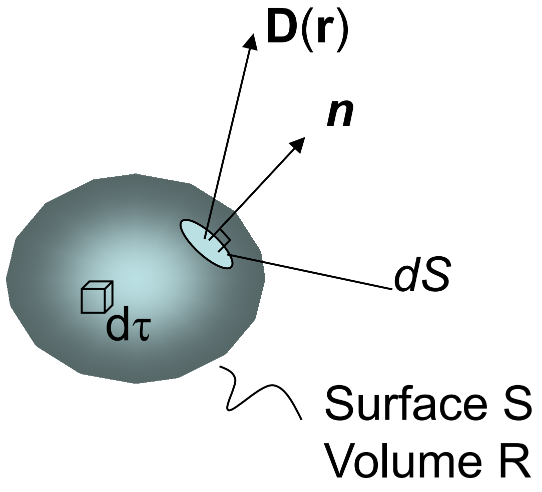


Bourgoing & Benay (2005), ONERA, France
Schlieren visualization - Sensitive to in-plane index of ref. gradient

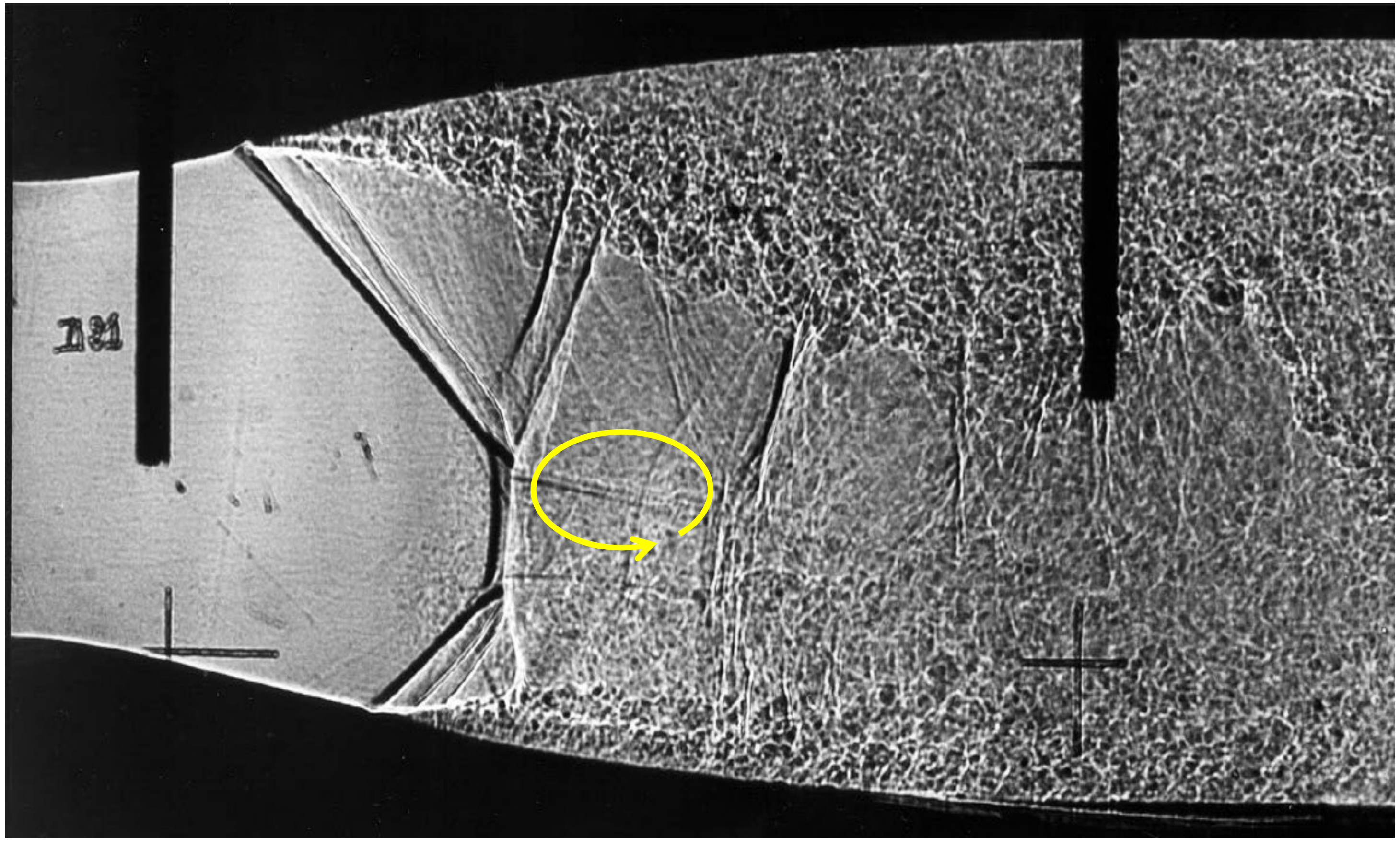
Integral Calculus With Respect to Space



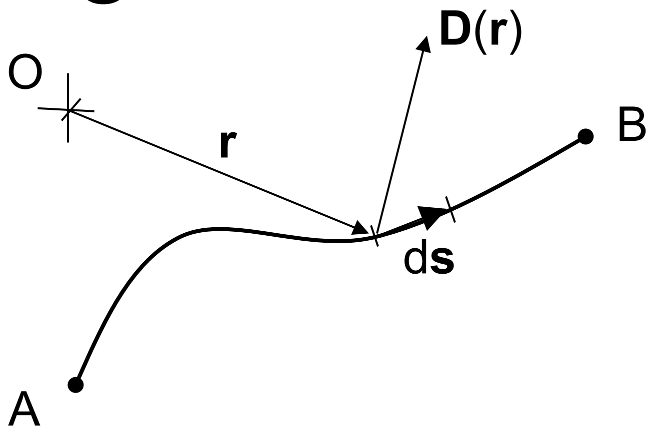
$\mathbf{D}=\mathbf{D}(\mathbf{r})$, $\phi = \phi(\mathbf{r})$



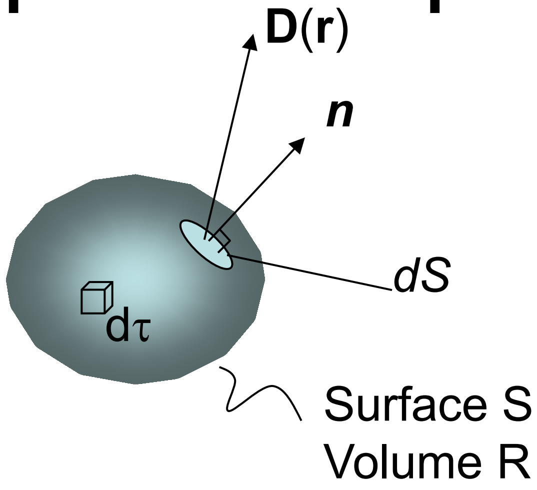
<u>Line Integrals</u>



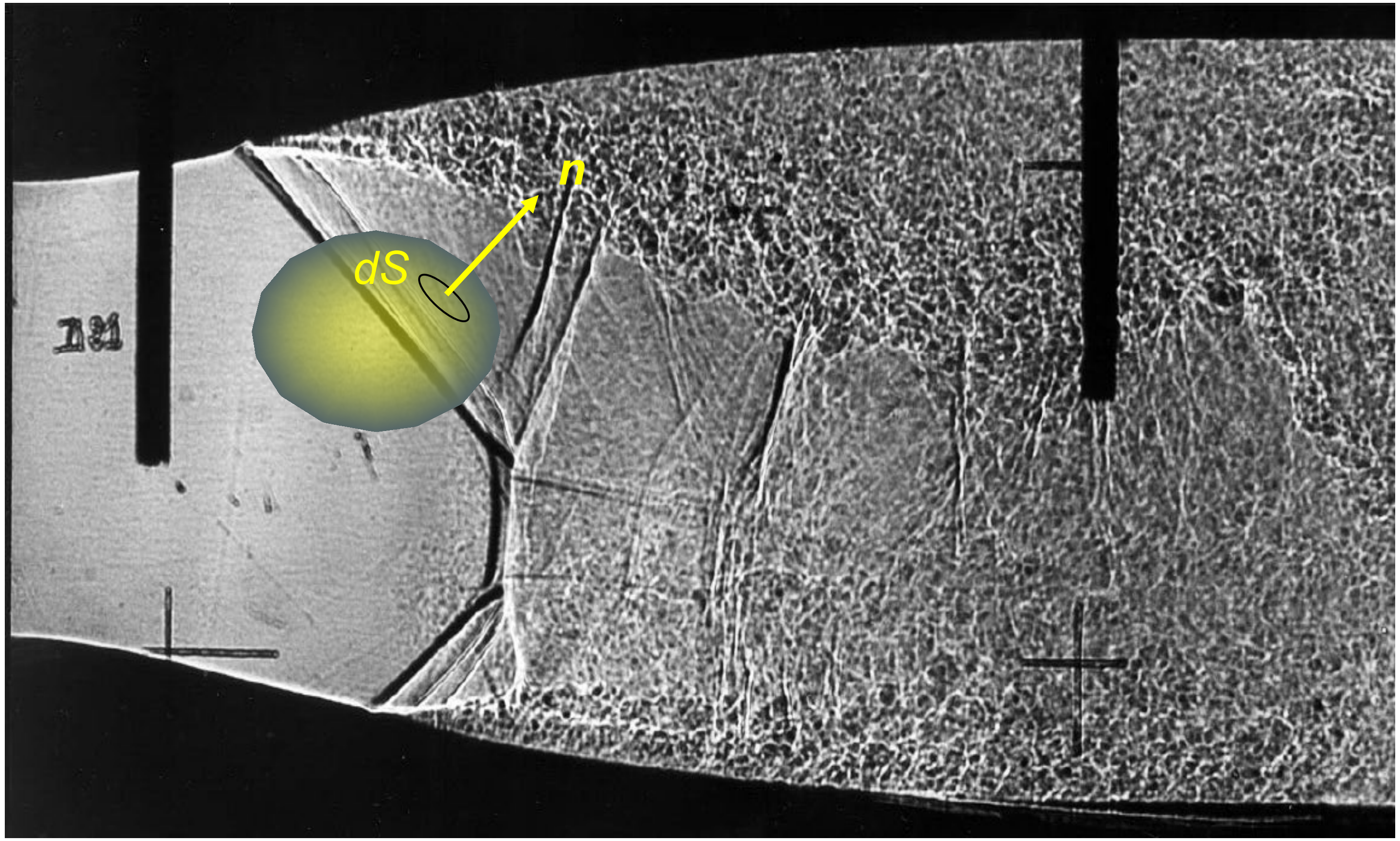
Integral Calculus With Respect to Space



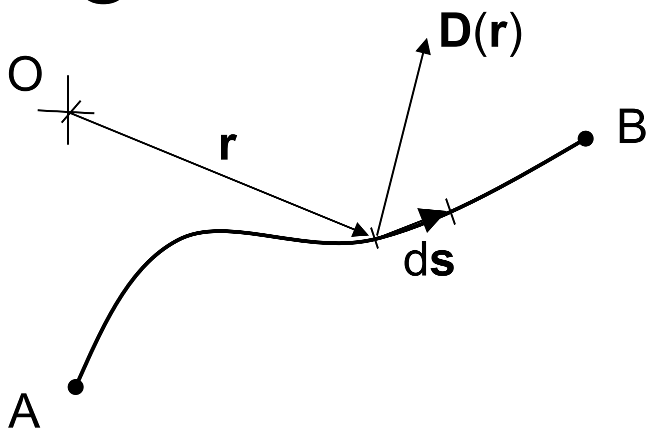
D=**D**(**r**), $\phi = \phi(\mathbf{r})$



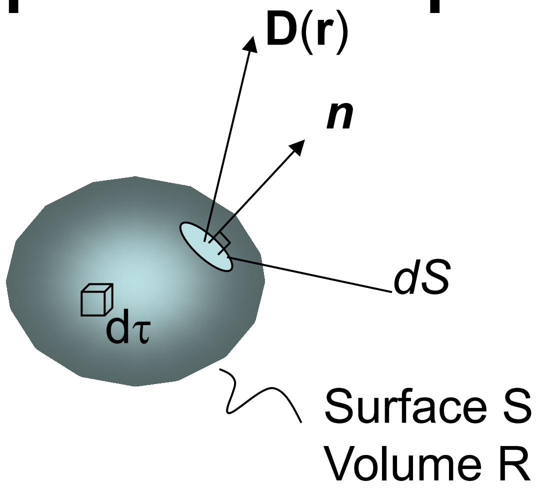
<u>Surface Integrals</u>


dS
n

Integral Calculus With Respect to Space

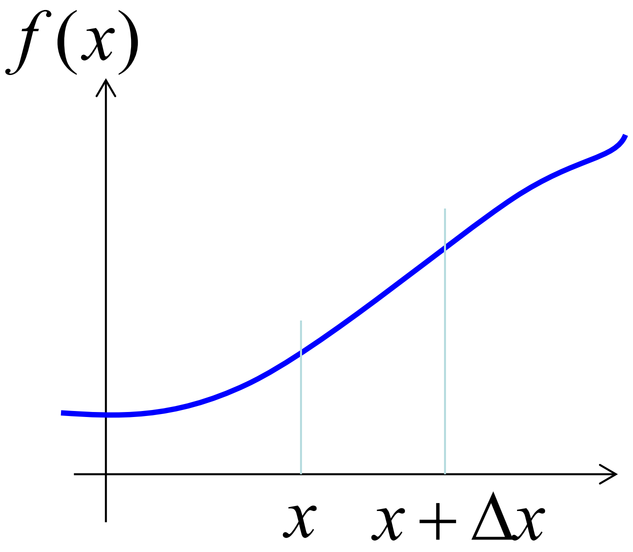


$\mathbf{D}=\mathbf{D}(\mathbf{r})$, $\phi = \phi(\mathbf{r})$

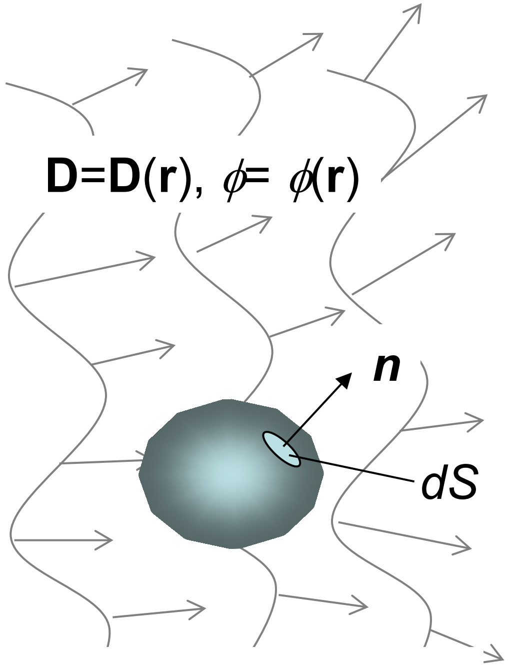


<u>Volume Integrals</u>

Differential Calculus w.r.t. Space in 1D

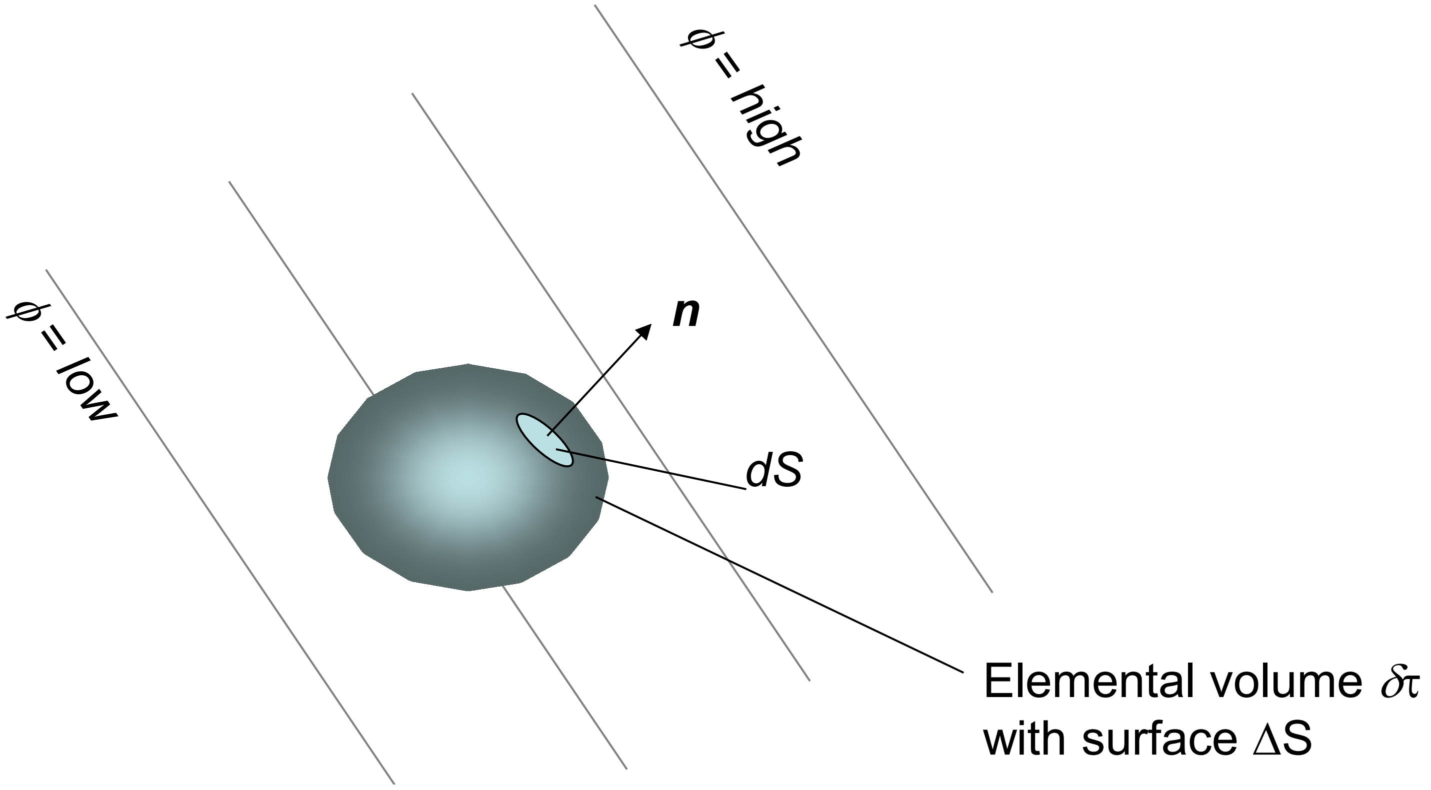


Differential Calculus w.r.t. Space in 3D

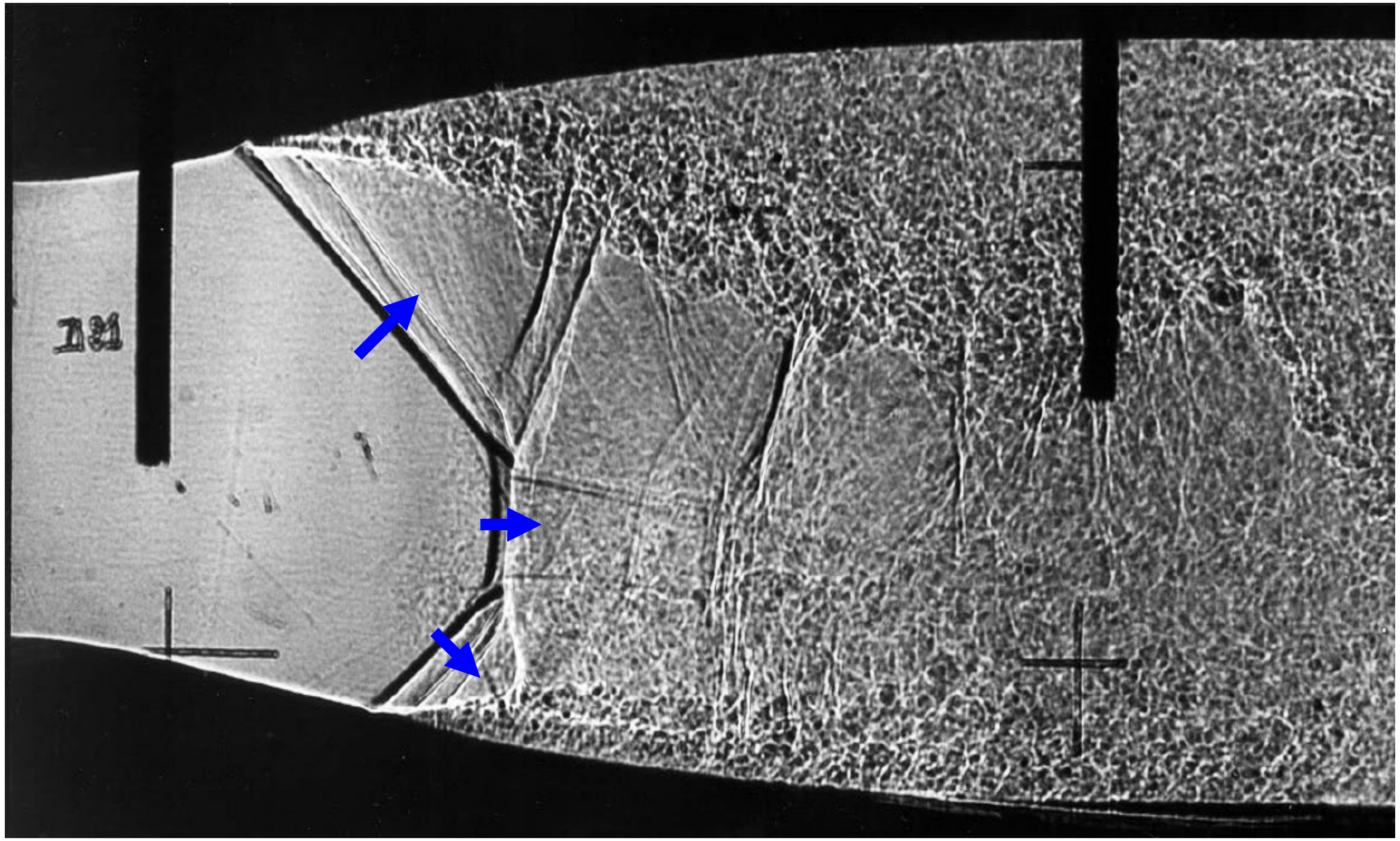


Elemental volume $\delta\tau$
with surface ΔS

Gradient



$$\mathrm{grad}\phi \equiv \mathrm{Lim}_{\delta\tau\to 0} \frac{1}{\delta\tau} \oint_{\Delta S} \phi \mathbf{n} \mathrm{dS}$$



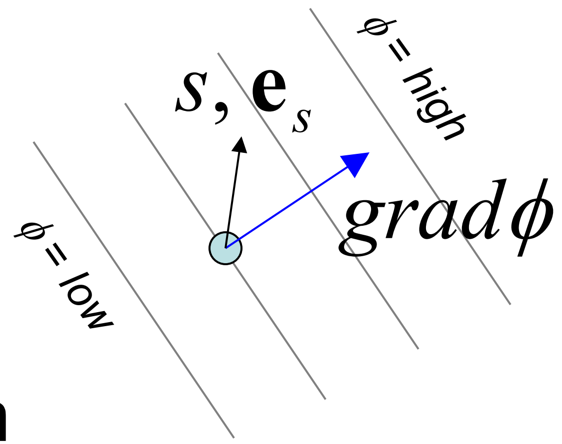
Gradient

$$\mathrm{grad}\phi \equiv \mathrm{Lim}_{\delta\tau\to 0} \frac{1}{\delta\tau} \oint_{\Delta S} \phi \mathbf{n} dS$$

Magnitude and direction of the slope in the scalar field at a point

Gradient

- Component of gradient is the partial derivative in the direction of that component



- Fourier´s Law of Heat Conduction

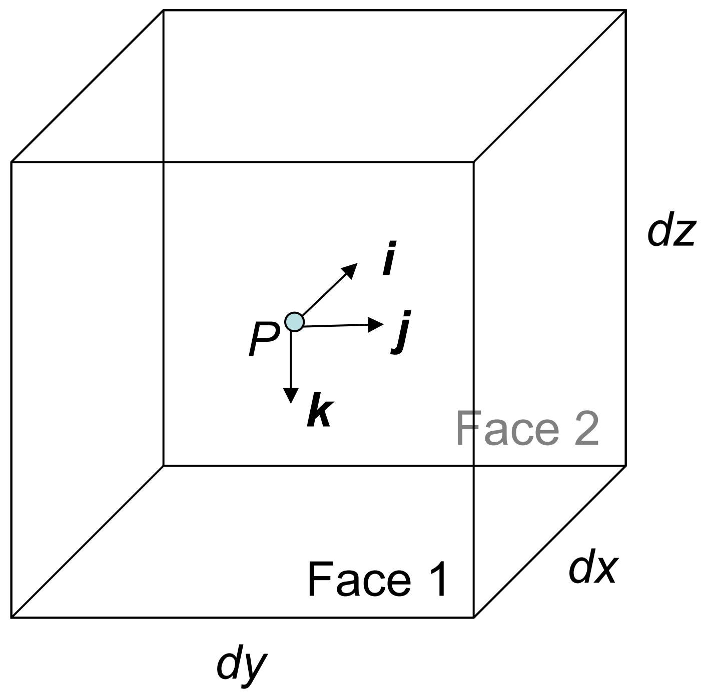
Differential form of the Gradient

Cartesian system

Evaluate integral by expanding the variation in ϕ about a point P at the center of an elemental Cartesian volume. Consider the two x faces:

$$\operatorname{grad}\phi \equiv \mathrm{Lim}_{\delta\tau\to 0}\frac{1}{\delta\tau}\oint_{\Delta S}\phi\mathbf{n}\mathrm{dS}$$

$\phi = \phi(x,y,z)$



$$\int_{Face\,1}\phi\mathbf{n}\mathrm{dS} \approx \left(\phi - \frac{\partial\phi}{\partial x}\frac{dx}{2}\right)(-\mathbf{i})dydz$$

$$\int_{Face\,2}\phi\mathbf{n}\mathrm{dS} \approx \left(\phi + \frac{\partial\phi}{\partial x}\frac{dx}{2}\right)(+\mathbf{i})dydz$$

adding these gives $\mathbf{i}\dfrac{\partial\phi}{\partial x}dxdydz$

Proceeding in the same way for y and z

we get $\mathbf{j}\dfrac{\partial\phi}{\partial y}dxdydz$ and $\mathbf{k}\dfrac{\partial\phi}{\partial z}dxdydz$, so

$$\operatorname{grad}\phi \equiv \mathrm{Lim}_{\delta\tau\to 0}\frac{1}{\delta\tau}\oint_{\Delta S}\phi\mathbf{n}\mathrm{dS}$$

$$= \mathrm{Lim}_{\delta\tau\to 0}\frac{1}{\delta\tau}\left(\mathbf{i}\frac{\partial\phi}{\partial x}dxdydz + \mathbf{j}\frac{\partial\phi}{\partial y}dxdydz + \mathbf{k}\frac{\partial\phi}{\partial z}dxdydz\right) = \underline{\underline{\mathbf{i}\frac{\partial\phi}{\partial x} + \mathbf{j}\frac{\partial\phi}{\partial y} + \mathbf{k}\frac{\partial\phi}{\partial z}}}$$

Differential Forms of the Gradient

$$grad\Phi = \nabla\Phi$$

Cartesian

$$\vec{i}\,\frac{\partial\Phi}{\partial x}+\vec{j}\,\frac{\partial\Phi}{\partial y}+\vec{k}\,\frac{\partial\Phi}{\partial z} = \left(\vec{i}\,\frac{\partial}{\partial x}+\vec{j}\,\frac{\partial}{\partial y}+\vec{k}\,\frac{\partial}{\partial z}\right)\Phi$$

Cylindrical

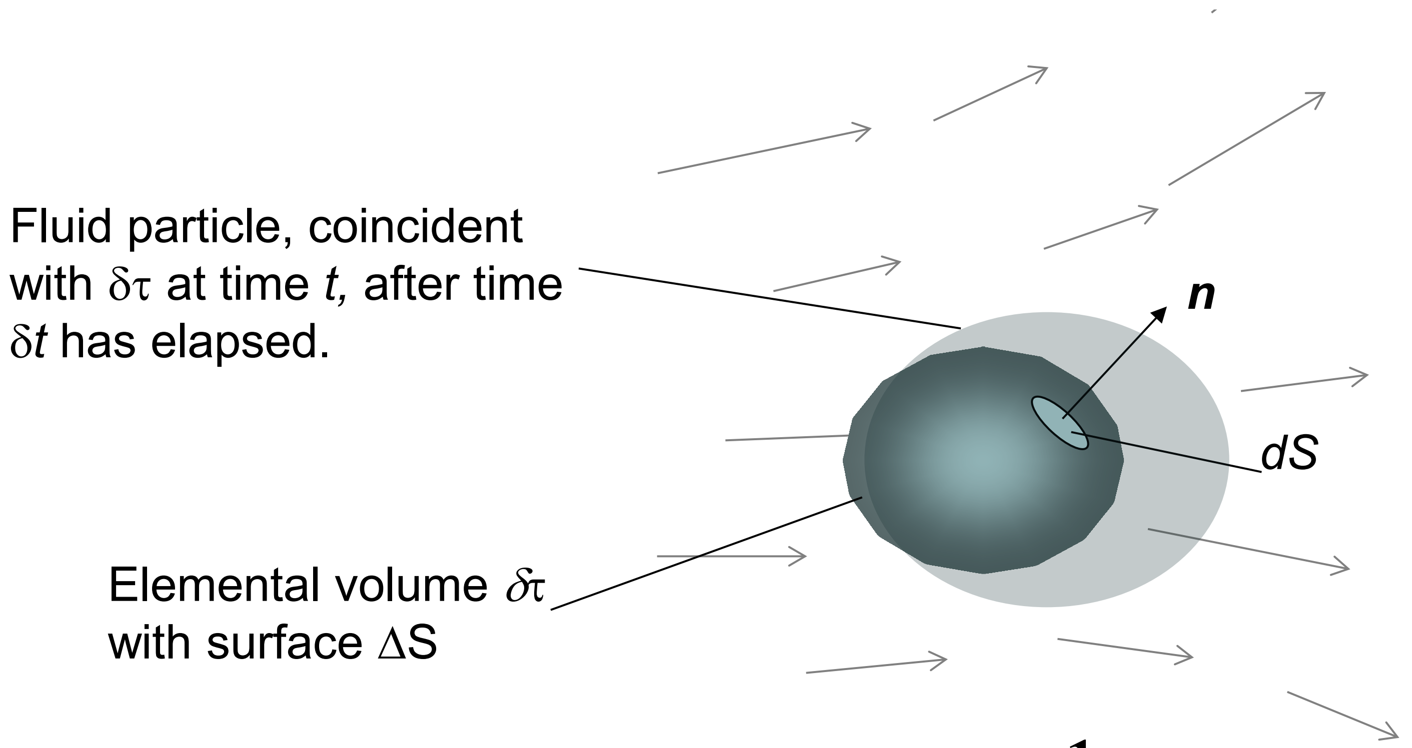
$$\vec{e}_r\frac{\partial\Phi}{\partial r}+\frac{\vec{e}_\theta}{r}\frac{\partial\Phi}{\partial\theta}+\vec{e}_z\frac{\partial\Phi}{\partial z} = \left(\vec{e}_r\frac{\partial}{\partial r}+\frac{\vec{e}_\theta}{r}\frac{\partial}{\partial\theta}+\vec{e}_z\frac{\partial}{\partial z}\right)\Phi$$

Spherical

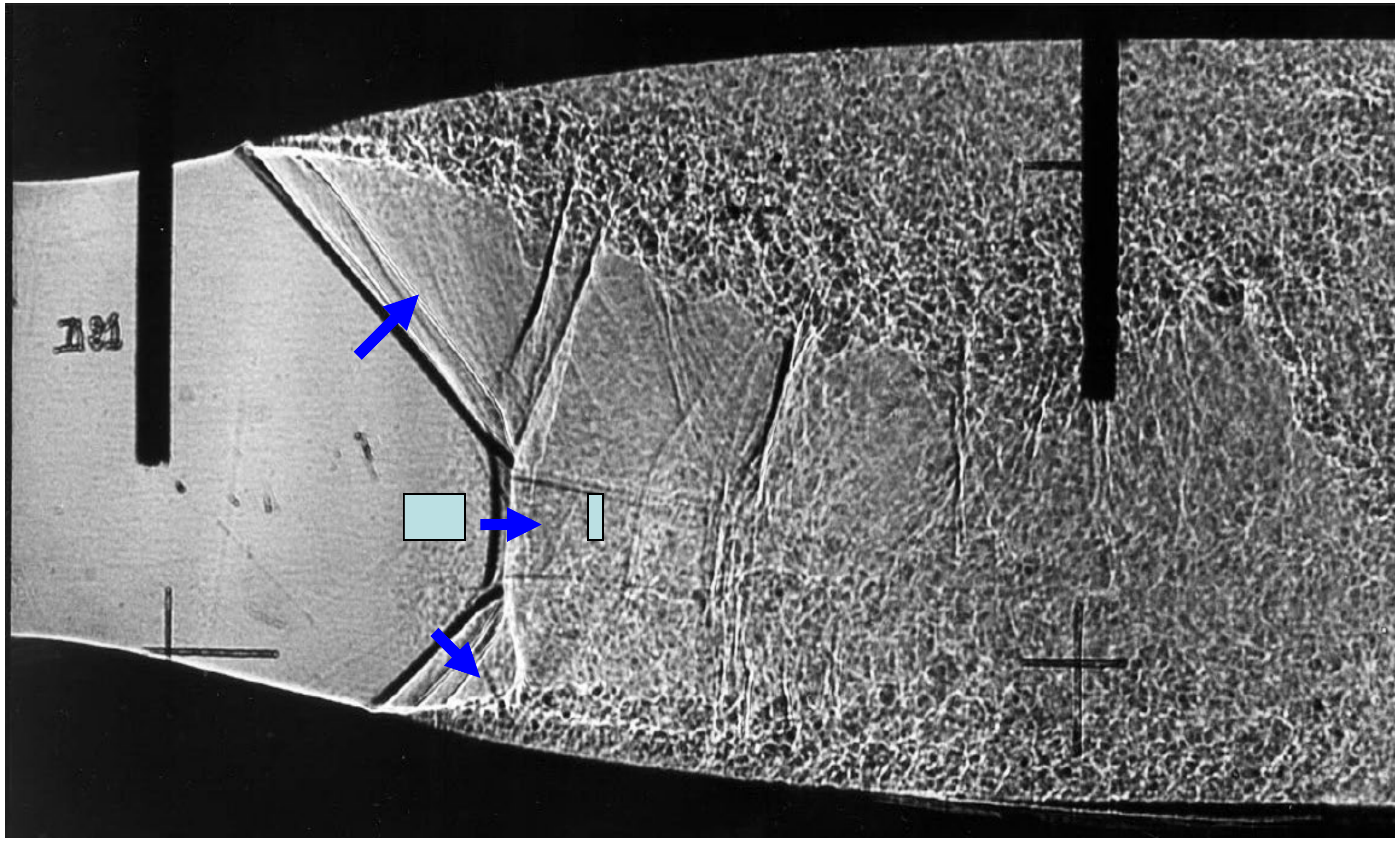
$$\vec{e}_r\frac{\partial\Phi}{\partial r}+\frac{\vec{e}_\theta}{r}\frac{\partial\Phi}{\partial\theta}+\frac{\vec{e}_\phi}{r\sin\theta}\frac{\partial\Phi}{\partial\phi} = \left(\vec{e}_r\frac{\partial}{\partial r}+\frac{\vec{e}_\theta}{r}\frac{\partial}{\partial\theta}+\frac{\vec{e}_\phi}{r\sin\theta}\frac{\partial}{\partial\phi}\right)\Phi$$

These differential forms define the vector operator ∇

Divergence



$$div\mathbf{V} \equiv \mathrm{Lim}_{\delta\tau\to 0} \frac{1}{\delta\tau} \oint_{\Delta S} \mathbf{V}.\mathbf{n}\mathrm{dS}$$



Gradient

$$\mathrm{grad}\phi \equiv \mathrm{Lim}_{\delta\tau\to 0}\frac{1}{\delta\tau}\oint_{\Delta S}\phi\mathbf{n}dS$$

Magnitude and direction of the slope in the scalar field at a point

Divergence

$$div\mathbf{V} \equiv \mathrm{Lim}_{\delta\tau\to 0}\frac{1}{\delta\tau}\oint_{\Delta S}\mathbf{V}.\mathbf{n}dS$$

For velocity: proportionate rate of change of volume of a fluid particle

Differential Forms of the Divergence

$$div\vec{A} = \nabla.\vec{A}$$

Cartesian

$$\frac{\partial A_x}{\partial x}+\frac{\partial A_y}{\partial y}+\frac{\partial A_z}{\partial z} = \left(\vec{i}\,\frac{\partial}{\partial x}+\vec{j}\,\frac{\partial}{\partial y}+\vec{k}\,\frac{\partial}{\partial z}\right).\vec{A}$$

Cylindrical

$$\frac{1}{r}\frac{\partial\, rA_r}{\partial r}+\frac{1}{r}\frac{\partial A_\theta}{\partial \theta}+\frac{\partial A_z}{\partial z} = \left(\vec{e}_r\frac{\partial}{\partial r}+\frac{\vec{e}_\theta}{r}\frac{\partial}{\partial \theta}+\vec{e}_z\frac{\partial}{\partial z}\right).\vec{A}$$

Spherical

$$\frac{1}{r^2}\frac{\partial\, r^2 A_r}{\partial r}+\frac{1}{r\sin\theta}\frac{\partial A_\theta \sin\theta}{\partial \theta}+\frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial \phi} = \left(\vec{e}_r\frac{\partial}{\partial r}+\frac{\vec{e}_\theta}{r}\frac{\partial}{\partial \theta}+\frac{\vec{e}_\phi}{r\sin\theta}\frac{\partial}{\partial \phi}\right).\vec{A}$$

Differential Forms of the Curl

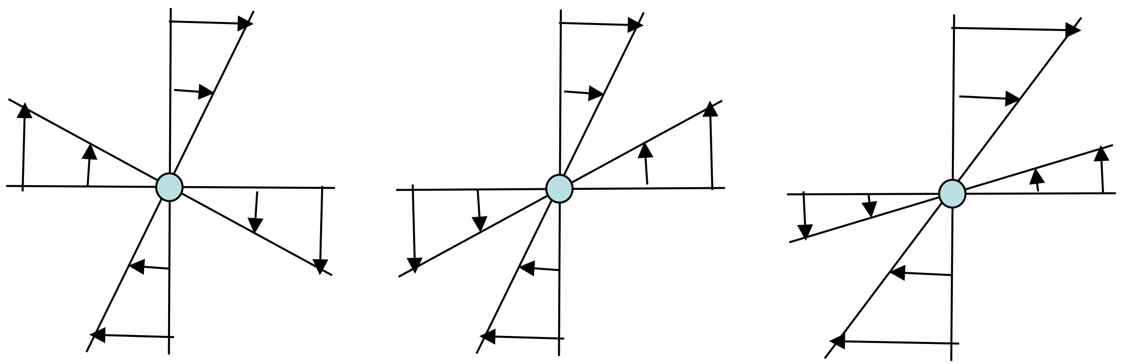
$$curl\vec{A} \equiv -\mathrm{Lim}_{\delta\tau\to 0}\frac{1}{\delta\tau}\oint_{\Delta S}\vec{A}\times\vec{n}\,\mathrm{dS}$$

$$curl\vec{A} = \nabla\times A = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = \frac{1}{r}\begin{vmatrix} \vec{e}_r & r\,\vec{e}_\theta & \vec{e}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ A_r & rA_\theta & A_z \end{vmatrix} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \vec{e}_r & r\,\vec{e}_\theta & r\sin\theta\,\vec{e}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ A_r & rA_\theta & r\sin\theta\,A_\phi \end{vmatrix}$$

Cartesian Cylindrical Spherical

Physical Interpretation of the Curl

$\nabla \times \boldsymbol{V}$ **=**



Pure rotation　　No rotation　　Rotation

Curl

$$curl\mathbf{V} \equiv -\mathrm{Lim}_{\delta\tau\to 0}\frac{1}{\delta\tau}\oint_{\Delta S}\mathbf{V}\times\mathbf{n}\mathrm{d}S$$

$$\mathbf{e}.curl\mathbf{V} \equiv -\mathrm{Lim}_{\delta\tau\to 0}\frac{1}{\delta\tau}\oint_{\Delta S}\mathbf{e}.\mathbf{V}\times\mathbf{n}\mathrm{d}S$$

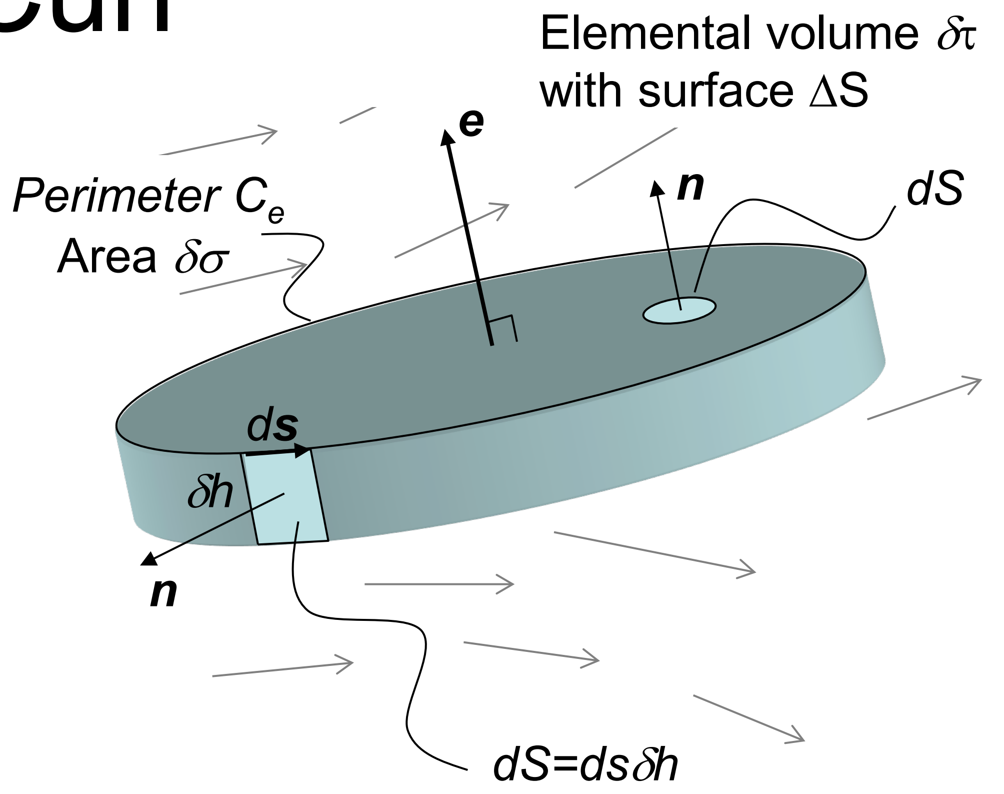
$$\mathbf{e}.curl\mathbf{V} \equiv \mathrm{Lim}_{\delta\tau\to 0}\frac{1}{\delta\sigma\delta h}\oint_{\Delta S}\mathbf{V}.\mathbf{e}\times\mathbf{n}\mathrm{d}S$$

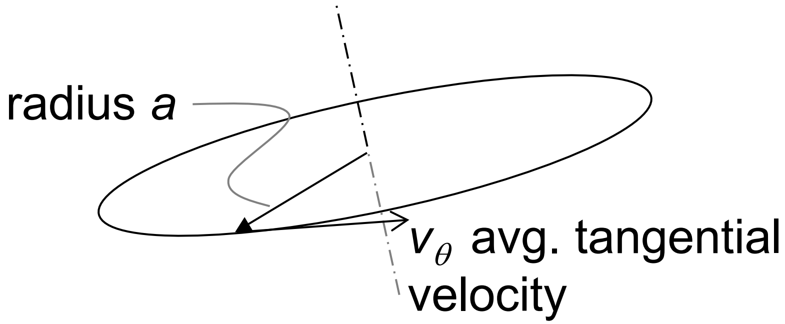
$$\mathbf{e}.curl\mathbf{V} \equiv \mathrm{Lim}_{\delta\tau\to 0}\frac{1}{\delta\sigma\delta h}\oint_{\Delta S}\mathbf{V}.\mathbf{e}\times\mathbf{n}\mathrm{d}s\,\delta h$$

$$\boxed{\mathbf{e}.curl\mathbf{V} \equiv \mathrm{Lim}_{\delta\sigma\to 0}\frac{1}{\delta\sigma}\oint_{C_e}\mathbf{V}.\mathrm{d}\mathbf{s} = \mathrm{Lim}_{\delta\sigma\to 0}\frac{\Gamma_{Ce}}{\delta\sigma}}$$

$$\mathbf{e}.curl\mathbf{V} \equiv \mathrm{Lim}_{a\to 0}\frac{1}{\pi a^2}v_\theta 2\pi a = 2\mathrm{Lim}_{a\to 0}\frac{v_\theta}{a}$$

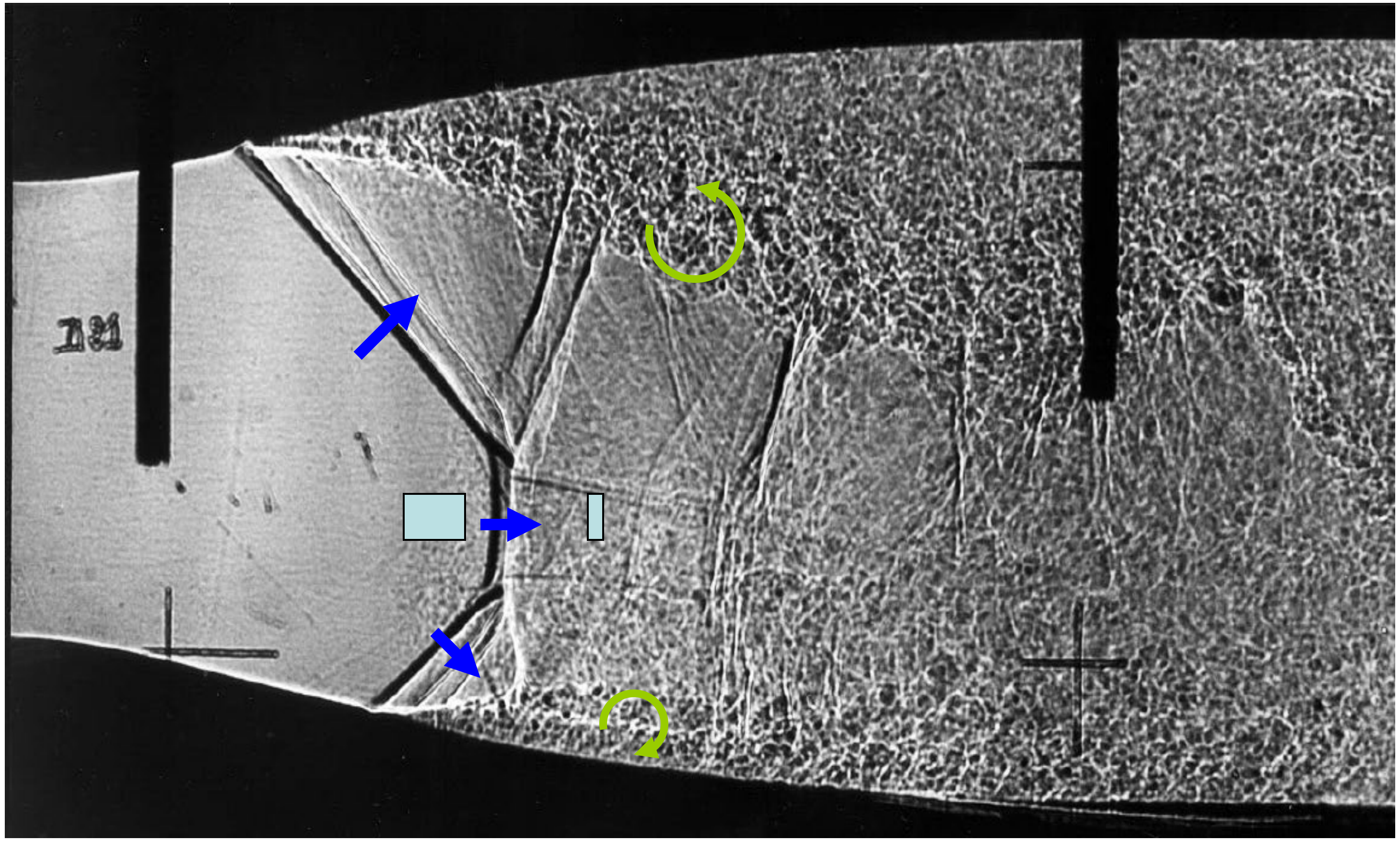
= twice the avg. angular velocity about ***e***





Curl

$$\boxed{\mathbf{e}.curl\mathbf{V} \equiv \mathrm{Lim}_{\delta\sigma\to 0}\frac{1}{\delta\sigma}\oint_{\mathrm{C_e}}\mathbf{V}.\mathrm{d}\mathbf{s} = \mathrm{Lim}_{\delta\sigma\to 0}\frac{\Gamma_{\mathrm{Ce}}}{\delta\sigma}}$$



Gradient

$$\text{grad}\phi \equiv \text{Lim}_{\delta\tau\to 0}\frac{1}{\delta\tau}\oint_{\Delta S}\phi\mathbf{n}dS$$

Magnitude and direction of the slope in the scalar field at a point

Divergence

$$div\mathbf{V} \equiv \text{Lim}_{\delta\tau\to 0}\frac{1}{\delta\tau}\oint_{\Delta S}\mathbf{V}.\mathbf{n}dS$$

For velocity: proportionate rate of change of volume of a fluid particle

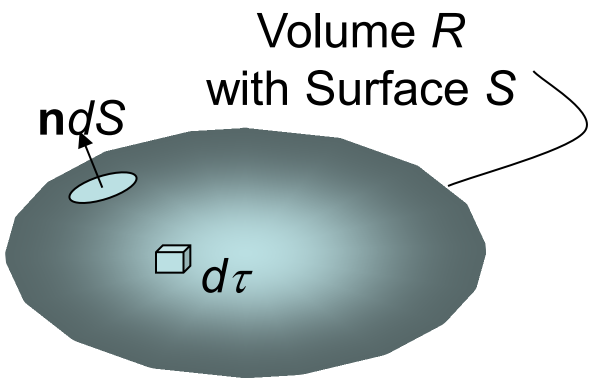
Curl

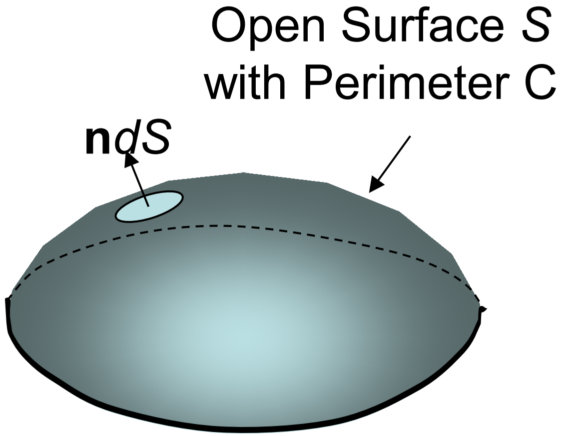
$$curl\mathbf{V} \equiv -\text{Lim}_{\delta\tau\to 0}\frac{1}{\delta\tau}\oint_{\Delta S}\mathbf{V}\times\mathbf{n}dS$$

For velocity: twice the circumferentially averaged angular velocity of a fluid particle = Vorticity $\mathbf{\Omega}$

1st Order Integral Theorems

- Gradient theorem
- Divergence theorem
- Curl theorem
- Stokes' theorem



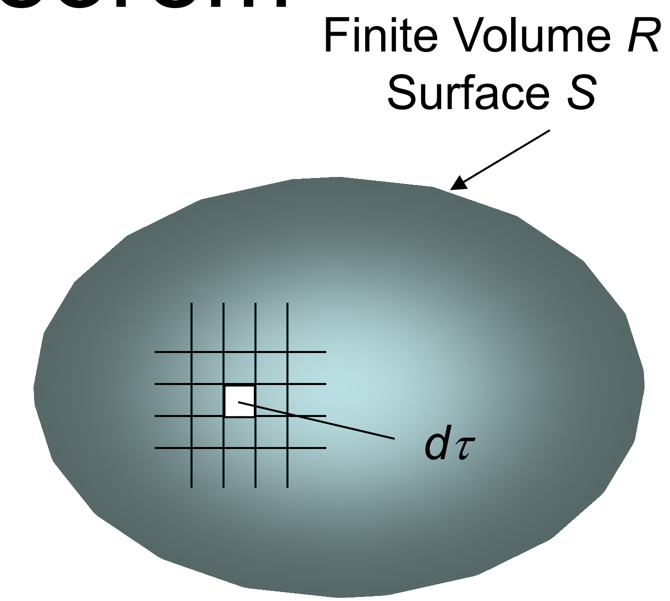


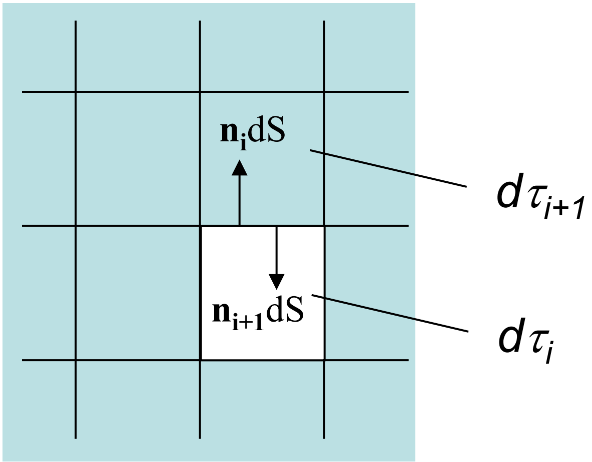
The Gradient Theorem

Begin with the definition of grad:

$$\mathrm{grad}\phi \equiv \mathrm{Lim}_{\delta\tau\to 0}\frac{1}{\delta\tau}\oint_{\Delta S}\phi\mathbf{n}\mathrm{dS}$$

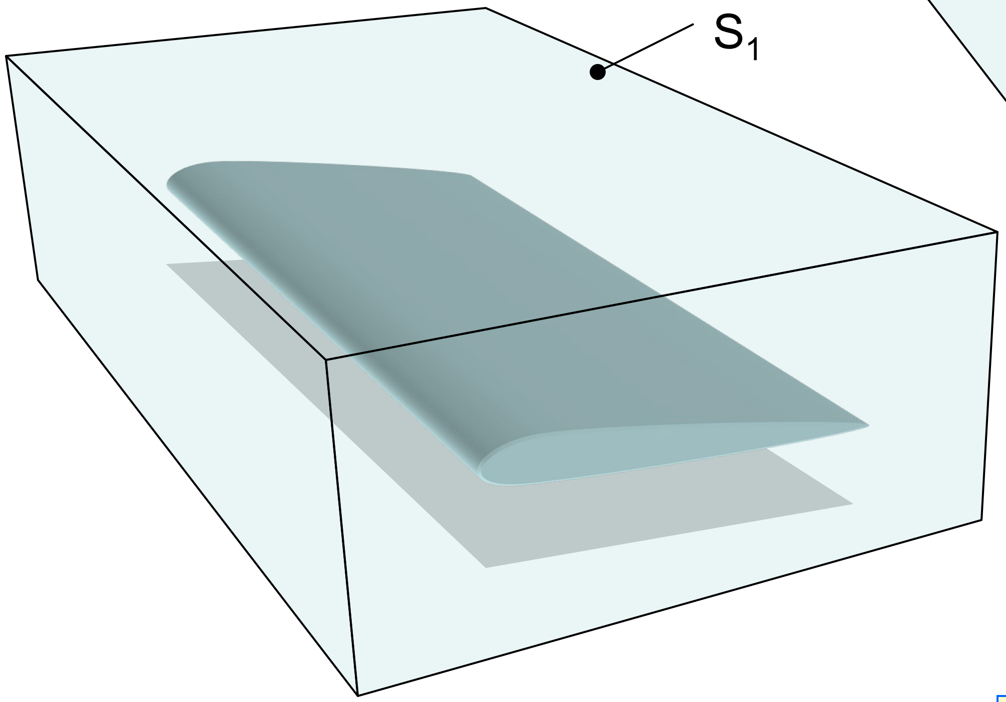
Sum over all the $d\tau$ in R:

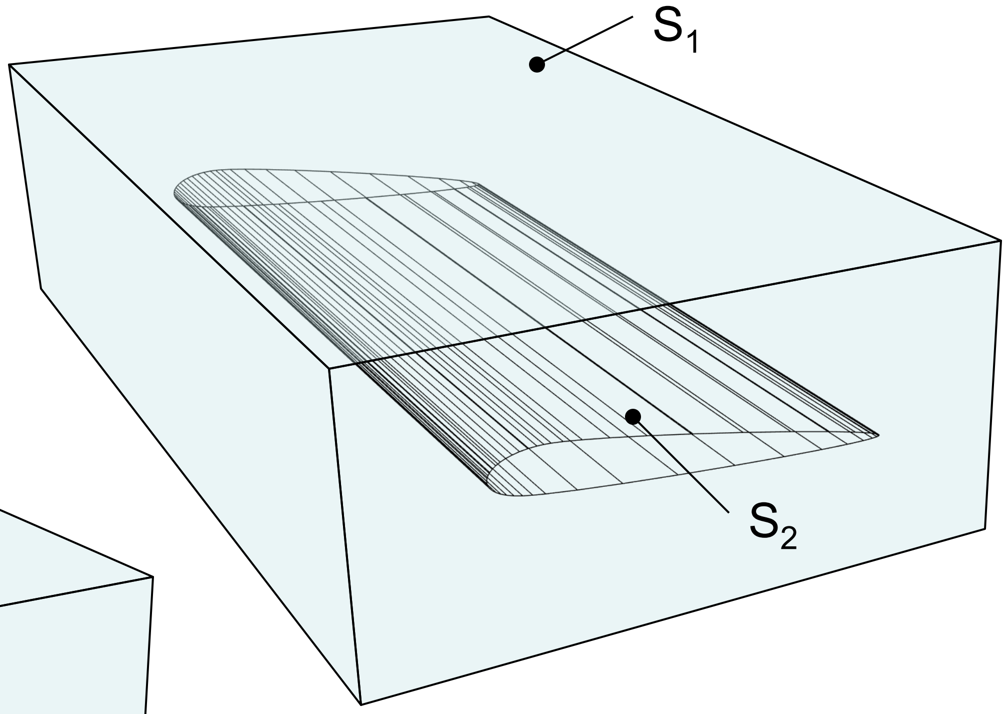




Assumptions in Gradient Theorem

Flow over a finite wing





$S = S_1 + S_2$

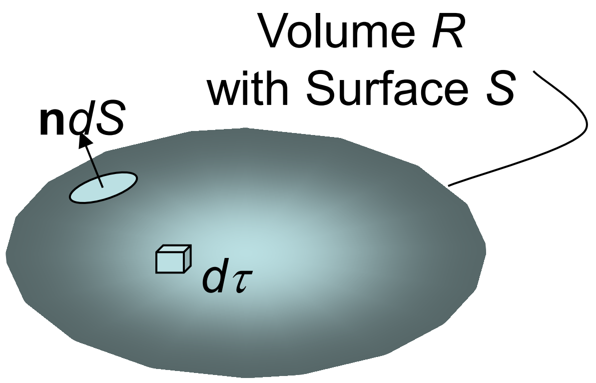
R is the volume of fluid enclosed between S_1 and S_2

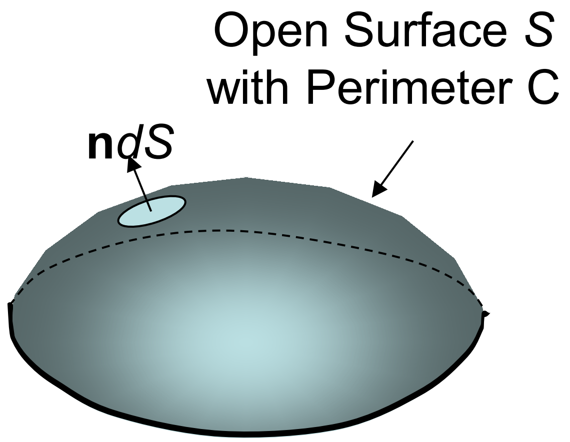
$$\int_R \nabla p \, d\tau = \int_S p\mathbf{n} dS$$

p is not defined inside the wing so the wing itself must be excluded from the integral

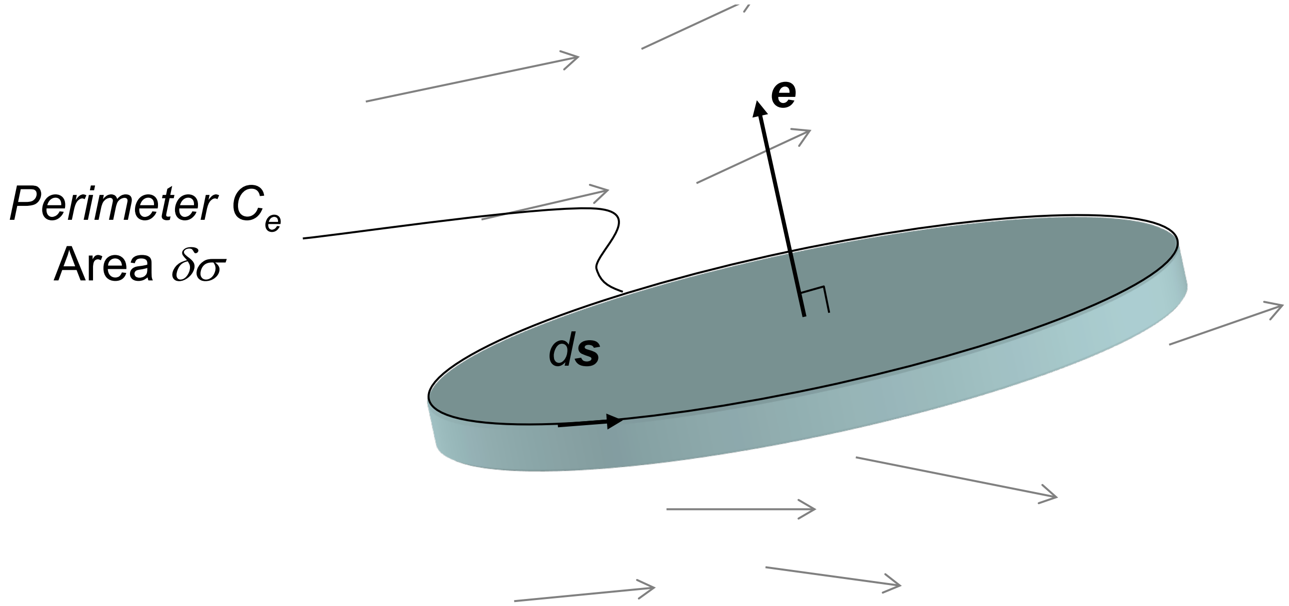
1st Order Integral Theorems

- Gradient theorem $$\int_R \nabla\phi\, d\tau = \oint_S \phi\mathbf{n}\, dS$$
- Divergence theorem $$\int_R \nabla.\mathbf{A}\, d\tau = \oint_S \mathbf{A}.\mathbf{n}\, dS$$
- Curl theorem $$\int_R \nabla\times\mathbf{A}\, d\tau = -\oint_S \mathbf{A}\times\mathbf{n}\, dS$$
- Stokes' theorem $$\int_S \nabla\times\mathbf{A}.\mathbf{n}\, dS = \oint_C \mathbf{A}.\mathbf{ds}$$





Alternative Definition of the Curl



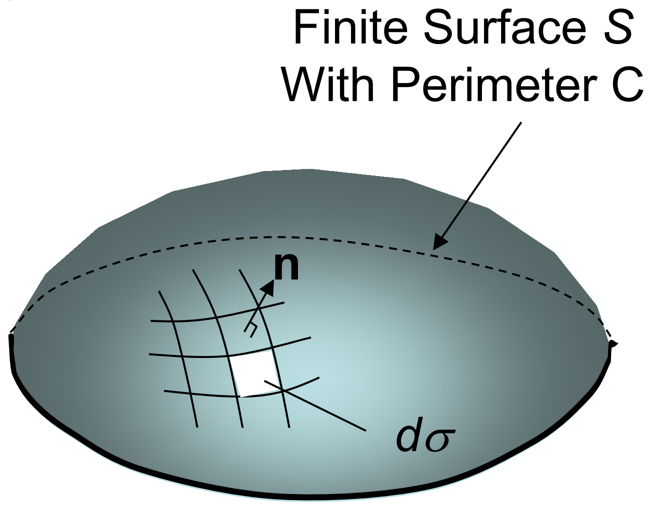
$$\mathbf{e}.curl\mathbf{A} \equiv \mathrm{Lim}_{\delta\sigma\to 0}\frac{1}{\delta\sigma}\oint_{\mathrm{C_e}}\mathbf{A}.\mathrm{d}\mathbf{s} = \mathrm{Lim}_{\delta\sigma\to 0}\frac{\Gamma_{\mathrm{Ce}}}{\delta\sigma}$$

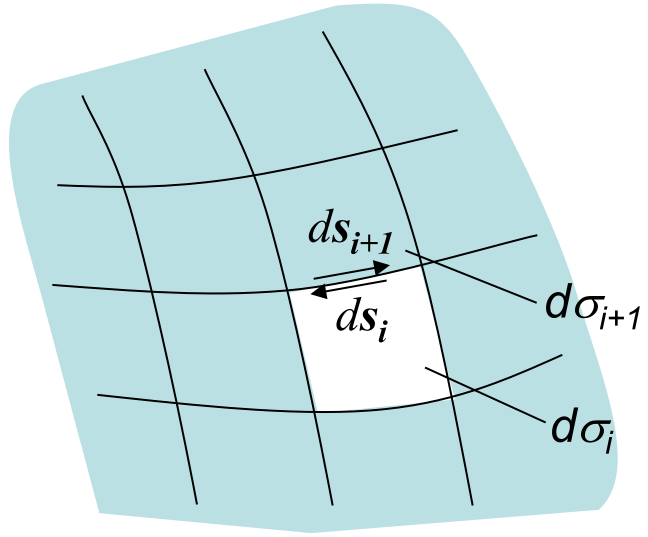
Stokes' Theorem

Begin with the alternative definition of curl, choosing the direction **e** to be the outward normal to the surface **n**:

$$\mathbf{n}.\nabla \times \mathbf{A} \equiv \mathrm{Lim}_{\delta\sigma \to 0} \frac{1}{\delta\sigma} \oint_{\mathrm{C_e}} \mathbf{A}.\mathrm{d}\mathbf{s}$$

Sum over all the $d\sigma$ in S:





Stokes´ Theorem and Velocity

- Apply Stokes´ Theorem to a velocity field

$$\int_S \nabla \times \mathbf{V}.\mathbf{n}\,dS = \oint_C \mathbf{V}.\mathrm{d}\mathbf{s}$$

- Or, in terms of vorticity and circulation

$$\int_S \boldsymbol{\Omega}.\mathbf{n}\,dS = \oint_C \mathbf{V}.\mathrm{d}\mathbf{s} = \Gamma_C$$

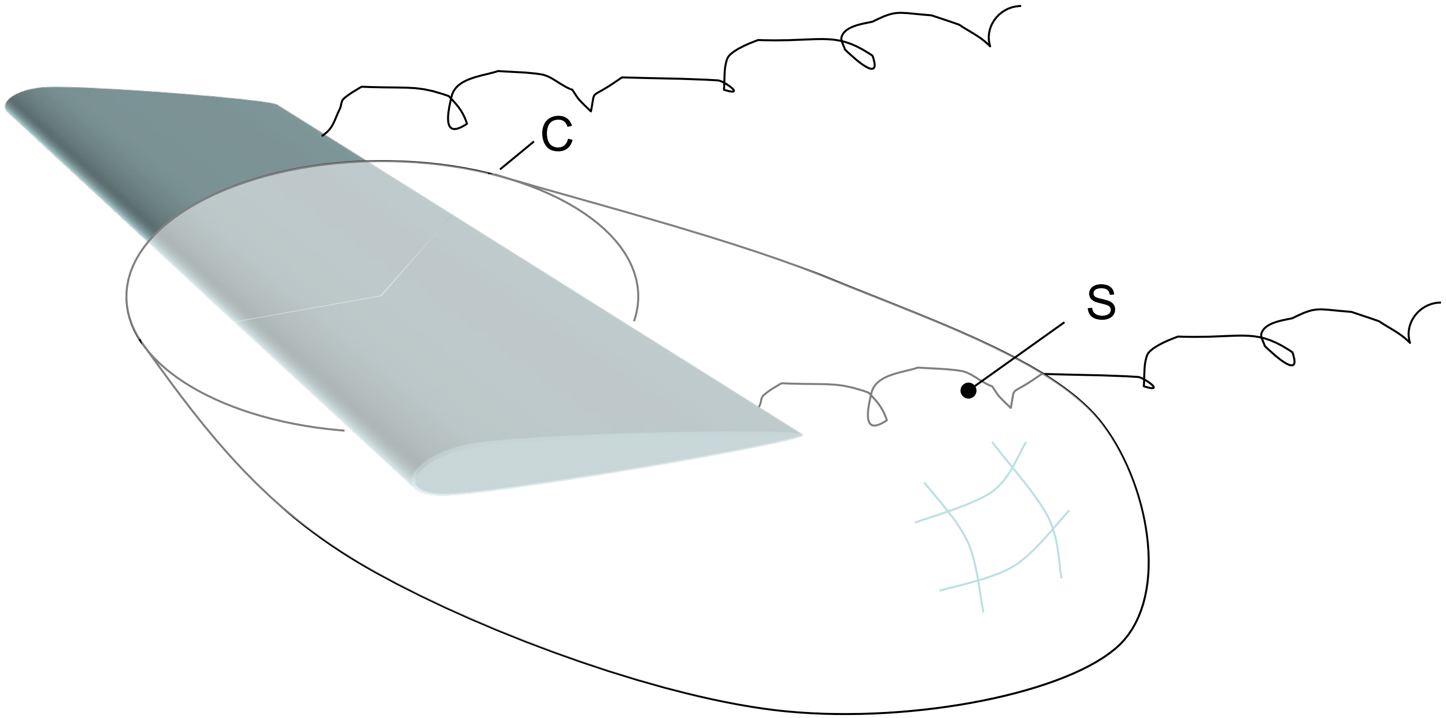
- What about a closed surface?

$$\oint_S \boldsymbol{\Omega}.\mathbf{n}\,dS = 0$$

Assumptions of Stokes´ Theorem

Flow over a finite wing

$$\int_S \nabla \times \mathbf{V}.\mathbf{n}dS = \oint_C \mathbf{V}.\mathrm{d}\mathbf{s}$$



Wing with circulation must trail vorticity. *Always.*

Vector Operators of Vector Products

$$\nabla(\psi\Phi) = \psi\nabla\Phi + \Phi\nabla\psi$$

$$\nabla.(\Phi\vec{A}) = \Phi\nabla.\vec{A} + \nabla\Phi.\vec{A}$$

$$\nabla\times(\Phi\vec{A}) = \Phi\nabla\times\vec{A} + \nabla\Phi\times\vec{A}$$

$$\nabla(\vec{A}.\vec{B}) = (\vec{A}.\nabla)B + (\vec{B}.\nabla\vec{A}) + \vec{A}\times(\nabla\times\vec{B}) + \vec{B}\times(\nabla\times\vec{A})$$

$$\nabla.(\vec{A}\times\vec{B}) = \vec{B}.\nabla\times\vec{A} - \vec{A}.\nabla\times\vec{B}$$

$$\nabla\times(\vec{A}\times\vec{B}) = \vec{A}(\nabla.\vec{B}) + (\vec{B}.\nabla)\vec{A} - \vec{B}(\nabla.\vec{A}) - (\vec{A}.\nabla)\vec{B}$$

Convective Operator

$$(\vec{A}.\nabla)\Phi = \left(A_x \frac{\partial}{\partial x} + A_y \frac{\partial}{\partial y} + A_z \frac{\partial}{\partial z} \right)\phi$$

$$= \vec{A}.(\nabla\Phi)$$

$\mathbf{V}.\nabla\rho$ = change in density in direction of **V**, multiplied by magnitude of **V**

$$(\vec{A}.\nabla)\vec{B} = \left(A_x \frac{\partial}{\partial x} + A_y \frac{\partial}{\partial y} + A_z \frac{\partial}{\partial z} \right)\vec{B}$$

$$= \tfrac{1}{2}[\nabla(\vec{A}.\vec{B}) - \vec{A}\times(\nabla\times\vec{B}) - \vec{B}\times(\nabla\times\vec{A})$$

$$-\nabla\times(\vec{A}\times\vec{B}) + \vec{A}(\nabla.B) - \vec{B}(\nabla.\vec{A})]$$

Second Order Operators

$$\nabla.\nabla\phi = \nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}$$

The Laplacian, may also be applied to a vector field.

$$\nabla(\nabla.\mathbf{A})$$

$$\nabla\times\nabla\times\mathbf{A} = \nabla(\nabla.\mathbf{A}) - \nabla^2\mathbf{A}$$

$$\nabla\times\nabla\phi \equiv 0$$

- So, any vector differential equation of the form $\nabla\times$**B**=0 can be solved identically by writing **B**=$\nabla\phi$.
- We say **B** is ***irrotational***.
- We refer to ϕ as the ***scalar potential***.

$$\nabla.\nabla\times\mathbf{A} \equiv 0$$

- So, any vector differential equation of the form ∇.**B**=0 can be solved identically by writing **B**=$\nabla\times$**A**.
- We say **B** is ***solenoidal*** or ***incompressible***.
- We refer to **A** as the ***vector potential***.

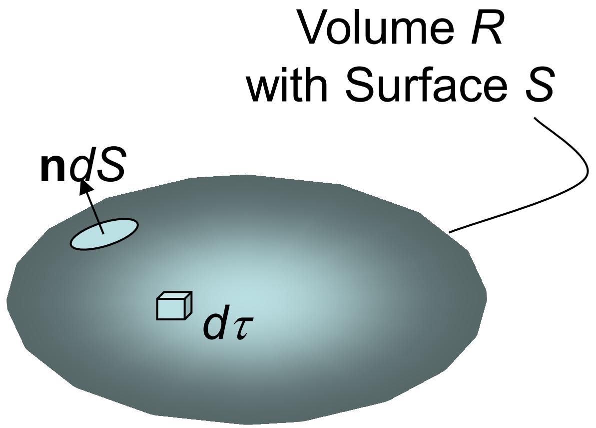
2^{nd} Order Integral Theorems

- Green's theorem (1^{st} form)

$$\int_R \psi\nabla^2\phi + \nabla\psi\nabla\phi \, d\tau = \oint_S \psi \frac{\partial\phi}{\partial \mathrm{n}} \mathrm{d}S$$

- Green's theorem (2^{nd} form)

$$\int_R \psi\nabla^2\phi - \phi\nabla^2\psi \; d\tau = \oint_S \psi \frac{\partial\phi}{\partial \mathrm{n}} - \phi \frac{\partial\psi}{\partial \mathrm{n}} \mathrm{d}S$$



These are both re-expressions of the divergence theorem.

Helmholz Decomposition Theorem

- Any vector field may be expressed as the sum of a gradient vector field and a curl vector field.

$$\mathbf{B} = \nabla \times \mathbf{A} + \nabla \phi$$

Vector Potential

Scalar Potential