ME 5200/6200 Classical Controls

Homework 02 Solutions

Do the following problems and show all your work for full credit. Note: not all problems will be graded, but you must complete all problems to get full credit.

Problem 1

Find the time function corresponding to each of the following Laplace transforms using partial-fraction expansion and Laplace tables:

(a) $F(s) = \frac{1}{s(s+1)}$

(b) $F(s) = \frac{5}{s(s+1)(s+5)}$

(c) $F(s) = \frac{3s+2}{s^2+2s+10}$

(d) $F(s) = \frac{3s^2+6s+6}{(s+1)(s^2+6s+10)}$

(a) Use partial fraction expansion (real roots):

$$F(s) = \frac{1}{s(s+1)} = \frac{C_1}{s} + \frac{C_2}{s+1}$$

$$\Rightarrow \frac{1}{s(s+1)} = \frac{C_1 s + C_1 + C_2 s}{s(s+1)}$$

$$\Rightarrow 1 = C_1 s + C_1 + C_2 s$$ Find C_1 + C_2

Let $s=0$: $\underline{\underline{1 = C_1}}$

Let $s=-1$: $1 = -C_1 + C_1 - C_2 \Rightarrow \underline{\underline{C_2 = -1}}$

Thus: $F(s) = \frac{C_1}{s} + \frac{C_2}{s+1} = \frac{1}{s} - \frac{1}{s+1}$

Using the L.T. table, we get:

$f(t) = \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} \Rightarrow \boxed{f(t) = 1 - e^{-t}}$

(b) $F(s) = \dfrac{5}{s(s+1)(s+5)}$ use partial fractions expansion as the roots are real

$$\Rightarrow F(s) = \frac{C_1}{s} + \frac{C_2}{s+1} + \frac{C_3}{s+5}$$ Find C_1, C_2, C_3

we can solve for C_1, C_2, C_3 using method in Part(a) or use the short cut described in the notes:

$$C_1 = \left.\frac{5}{(s+1)(s+5)}\right|_{s=0} = \frac{5}{5} = \underline{\underline{1}}$$

$$C_2 = \left.\frac{5}{s(s+5)}\right|_{s=-1} = \underline{\underline{-\frac{5}{4}}}$$

$$C_3 = \left.\frac{5}{s(s+1)}\right|_{s=-5} = \underline{\underline{\frac{1}{4}}}$$

Hence $F(s) = \dfrac{1}{s} - \dfrac{5/4}{s+1} + \dfrac{1/4}{s+5}$

$$\Rightarrow f(t) = \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \mathcal{L}^{-1}\left\{\frac{5/4}{s+1}\right\} + \mathcal{L}^{-1}\left\{\frac{1/4}{s+5}\right\}$$

Using Tables, we get:

$$\boxed{f(t) = 1 - \frac{5}{4}e^{-t} + \frac{1}{4}e^{-5t}}$$

(c) $$F(s) = \frac{3s+2}{\underbrace{s^2+2s+10}}$$ ← using quadratic formula or Matlab, roots are complex, so this is Case 3 in Notes. We will complete the square.

$$s^2 + \frac{2s}{2} + 1 + 10 - 1$$

$$\Rightarrow F(s) = \frac{3s+2}{(s+1)^2+3^2}$$

$$\Rightarrow F(s) = \frac{3s}{(s+1)^2+3^2} + \frac{2}{(s+1)^2+3^2}$$

using L.T. table, $\mathcal{L}\{e^{at}\sin(bt)\} = \frac{b}{(s-a)^2+b^2}$

where $a=-1$ and $b=3$, so

$$\mathcal{L}^{-1}\left\{\frac{2\cdot\frac{3}{3}}{(s+1)^2+3^2}\right\} = \frac{2}{3}\left[e^{-t}\sin(3t)\right] = f_1(t)$$

Note that $\mathcal{L}^{-1}\left\{\frac{3s}{(s+1)^2+3^2}\right\} = \frac{d}{dt}\mathcal{L}^{-1}\left\{\frac{3}{(s+1)^2+3^2}\right\}$

$$\Rightarrow f_2(t) = \frac{d}{dt}\left(e^{-t}\sin(3t)\right) = 3e^{-t}\cos(3t) + e^{-t}\sin(3t)$$

combine:

$$f(t) = f_2(t) + f_1(t) = 3e^{-t}\cos(3t) + e^{-t}\sin(3t) + \frac{2}{3}e^{-t}\sin(3t)$$

$$\Rightarrow \boxed{f(t) = e^{-t}\sin(3t) + 3e^{-t}\cos(3t)}$$

(d) $$F(s) = \frac{3s^2+6s+6}{\underbrace{(s+1)}_{\substack{\text{real}\\\text{root}\\\Rightarrow \text{case } 1}}\underbrace{(s^2+6s+10)}_{\text{roots are complex} \Rightarrow \text{case } 3}}$$

partial fraction expansion is:

$$F(s) = \frac{c_1}{s+1} + \frac{c_2 s + c_3}{s^2+6s+10} \qquad (1)$$

we can easily find c_1 as follows:

$$c_1 = \left.\frac{3s^2+6s+6}{s^2+6s+10}\right|_{s=-1} = \underline{\underline{\frac{3}{5}}}$$

Now, we can sub $c_1 = 3/5$ back into (1) and equate the numerator as follows:

$$\frac{3/5}{s+1} + \frac{c_2 s + c_3}{s^2+6s+10} = \frac{3s^2+6s+6}{(s^2+6s+10)(s+1)}$$

$$\Rightarrow \tfrac{3}{5}(s^2+6s+10) + (s+1)(c_2 s + c_3) = 3s^2+6s+6$$

$$\Rightarrow \frac{3}{5}s^2 + \frac{18}{5}s + 6 + c_2 s^2 + c_3 s + c_2 s + c_3 = 3s^2+6s+6$$

combine like powers of s:

$$(c_2 + 3/5)s^2 + (18/5 + c_3)s + 6 + c_2 + c_3 = 3s^2+6s+6$$

Now we equate:

$$c_2 + 3/5 = 3 \Rightarrow c_2 = \frac{12}{5}$$

$$18/5 + c_3 + c_2 = 6 \Rightarrow c_3 = 0$$

Thus:

$$F(s) = \frac{\frac{3}{5}}{s+1} + \frac{12/5\, s}{s^2 + 6s + 10}$$

complete the square and use table!

$$F(s) = \frac{3/5}{s+1} + \frac{12/5\, s}{(s^2+6s+9)+10-9}$$

$$= \frac{3/5}{s+1} + \frac{12/5\, s \cdot 1}{(s+3)^2 + 1}$$

$$= \frac{3}{5}\frac{1}{s+1} + \frac{12}{5}\frac{s}{(s+3)^2+1}$$

derivative operator!

$$\Rightarrow f(t) = \mathcal{L}^{-1}\left\{\frac{3}{5}\frac{1}{s+1}\right\} + \mathcal{L}^{-1}\left\{\frac{12}{5}\cdot\frac{s}{(s+3)^2+1}\right\}$$

Using tables

$$\Rightarrow f(t) = \frac{3}{5}e^{-t} + \frac{12}{5}\frac{d}{dt}\left\{e^{-3t}\sin(t)\right\}$$

$$\Rightarrow \boxed{f(t) = \frac{3}{5}e^{-t} - \frac{36}{5}e^{-3t}\sin(t) + \frac{12}{5}e^{-3t}\cos(t)}$$

Problem 2

Solve the following ordinary differential equations using Laplace transforms. You can use tables to solve as needed.

(a) $\ddot{y}(t) + \dot{y}(t) + 3y(t) = 0;\ y(0) = 1,\ \dot{y}(0) = 2$
(b) $\ddot{y}(t) - 2\dot{y}(t) + 4y(t) = 0;\ y(0) = 1,\ \dot{y}(0) = 2$
(c) $\ddot{y}(t) + \dot{y}(t) = \sin t;\ y(0) = 1,\ \dot{y}(0) = 2$

(a) Take Laplace Transform w/ I.C. use derivative property:

$$\mathcal{L}\{\ddot{y}(t) + \dot{y}(t) + 3y(t)\} = \mathcal{L}\{0\} \quad \text{w/ } y(0)=1,\ \dot{y}(0)=2$$

$$\Rightarrow s^2 Y(s) - s y(0) - \dot{y}(0) + sY(s) - y(0) + 3Y(s) = 0$$

$$\Rightarrow s^2 Y(s) + sY(s) - s - 2 - 1 + 3Y(s) = 0$$

$$[s^2 + s + 3]\,Y(s) = s + 3$$

$$\Rightarrow Y(s) = \frac{s+3}{s^2+s+3}$$

complex root, so case 3 $\Rightarrow$ complete square.

$$\Rightarrow Y(s) = \frac{s+3}{(s^2+s+\frac{1}{4}) + 3 - \frac{1}{4}}$$

derivative!

$$Y(s) = \frac{s+3}{(s+1/2)^2 + 11/4} = \frac{s}{(s+1/2)^2 + 11/4} + \frac{3}{(s+1/2)^2 + 11/4}$$

use the Laplace table:

$$f(t) = \mathcal{L}^{-1}\{Y(s)\} \Rightarrow \boxed{f(t) = e^{-1/2 t}\cos\frac{\sqrt{11}}{2}t + \frac{5\sqrt{11}}{11}e^{-1/2 t}\sin\frac{\sqrt{11}}{2}t}$$

(b) Using same steps as in Part (a), we get:

$$\mathcal{L}\{\ddot{y}(t) - 2\dot{y}(t) + 4y(t)\} = \mathcal{L}\{0\} \quad \text{w/ } y(0) = 1, \ \dot{y}(0) = 2$$

$$\Rightarrow s^2 Y(s) - s y(0) - \dot{y}(0) - 2sY(s) - 2y(0) + 4Y(s) = 0$$

$$\Rightarrow Y(s) = \frac{s}{s^2 - 2s + 4}$$

complex roots, so we complete the square and use table.

$$\Rightarrow Y(s) = \frac{s}{(s-1)^2 + 3}$$

$$\Rightarrow y(t) = \mathcal{L}^{-1}\{Y(s)\} = \frac{d}{dt}\left[e^t \sin\sqrt{3}\,t\right]$$

$$\Rightarrow \boxed{y(t) = \frac{1}{\sqrt{3}} e^t \sin\sqrt{3}\,t + e^t \cos\sqrt{3}\,t}$$

(c) Again, we use the same procedures as above

$$\mathcal{L}\{\ddot{y}(t) + \dot{y}(t) = \sin(t)\} \quad w/ \quad y(0) = 1 \quad \dot{y}(0) = 2$$

$$\Rightarrow s^2 Y(s) - s y(0) - \dot{y}(0) + sY(s) - y(0) = \frac{1}{s^2+1}$$

$$\Rightarrow Y(s) = \frac{s^3 + 3s^2 + s + 4}{s(s+1)(s^2+1)} \leftarrow \text{complex roots!}$$

Partial expansion

$$Y(s) = \frac{c_1}{s} + \frac{c_2}{s+1} + \frac{c_3 s + c_4}{s^2+1}$$

Using process in notes:

$$c_1 = \left. \frac{s^3 + 3s^2 + s + 4}{(s+1)(s^2+1)} \right|_{s=0} \Rightarrow \underline{\underline{c_1 = 4}}$$

$$c_2 = \left. \frac{s^3 + 3s^2 + s + 4}{(s)(s^2+1)} \right|_{s=-1} \Rightarrow \underline{\underline{c_2 = -\frac{5}{2}}}$$

Now equate:

$$\frac{4}{s} + \frac{-5/2}{s+1} + \frac{c_3 s + 4}{s^2+1} = \frac{s^3 + 3s^2 + s + 4}{s(s+1)(s^2+1)}$$

$$\Rightarrow s^3\left(\frac{3}{2} + c_3\right) + s^2\left(4 + c_3 + c_4\right) + s\left(\frac{3}{2} + c_4\right) + 4 = s^3 + 3s^2 + s + 4$$

Matching coefficients:

$$C_4 + \frac{3}{2} = 1 \Rightarrow \underline{\underline{C_4 = -1/2}}$$

$$C_3 + \frac{3}{2} = 1 \Rightarrow \underline{\underline{C_3 = -1/2}}$$

Therefore:

$$Y(s) = \frac{4}{s} + \frac{-5/2}{s+1} + \frac{-1/2\,s - 1/2}{s^2+1} = \frac{4}{s} + \frac{-\frac{5}{2}}{s+1} - \frac{1}{2}\frac{s}{s^2+1} - \frac{1}{2}\frac{1}{s^2+1}$$

Using the table again, we get:

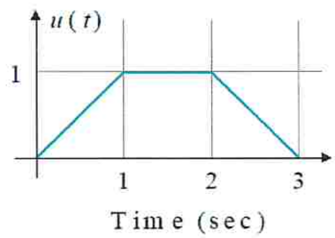
$$\boxed{y(t) = \mathcal{L}^{-1}\{Y(s)\} = 4 - \frac{5}{2}e^{-t} - 1/2\cos(t) - \frac{1}{2}\sin(t)}$$

Problem 3

Consider the following second order system with a transfer function given by:

$$G(s) = \frac{K\,\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}.$$

Let the natural frequency be 60 Hz, the damping constant be 0.001, and the constant K=5.2. Suppose the input to the system is given by:



Using Matlab, simulate the output response of the system and provide the following:

(a) Show a plot of output vs. time, label all axes.
(b) Briefly describe the response based on the input from Part (a) -- what's happening?
(c) Now suppose the input is a unit step instead of the input u(t) shown above. Simulate the response and provide a plot of output vs. time. Label all axes appropriately.
(d) For Part (c), what is the final value?
(e) Rather than get the final value from the plot in Part (c), how else could you have done it?
(f) Provide print out of your Matlab code (m-file, Simulink model, etc.) and submit it with your homework for grading.

(a) See plot

(b) the response shows slight oscillations because the system is 2nd order and has light (low) damping, $0 < \zeta < 1$. System is under damped.

(c) See plot

(d) From plot, it's about $\underline{5}$

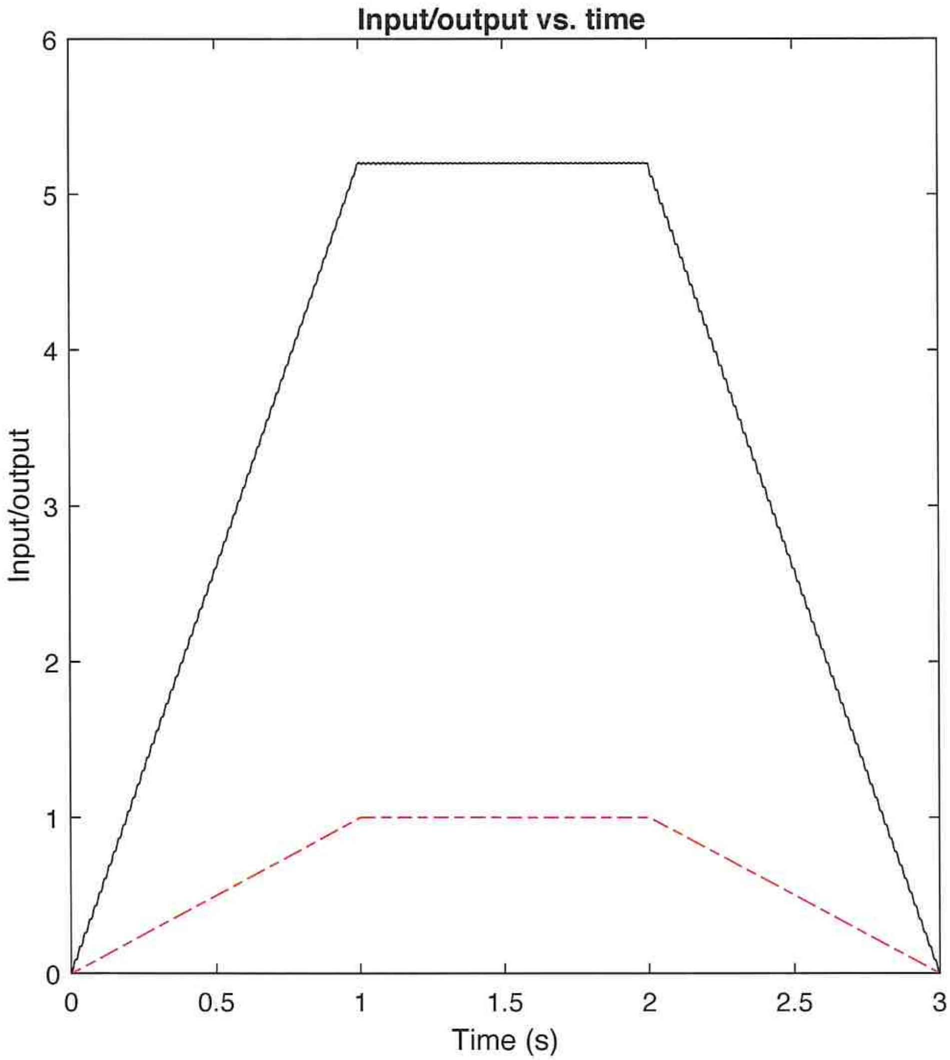
(e) Use final value theorem:

$$y_{ss} = \lim_{s\to 0} s\,Y(s) = \lim_{s\to 0} s\,\big(G(s)\,U(s)\big) = \lim_{s\to 0} \cancel{s}\left(\frac{K\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}\right)\cdot\frac{1}{\cancel{s}}$$

$$y_{ss} = K = \underline{\underline{5.2}}$$

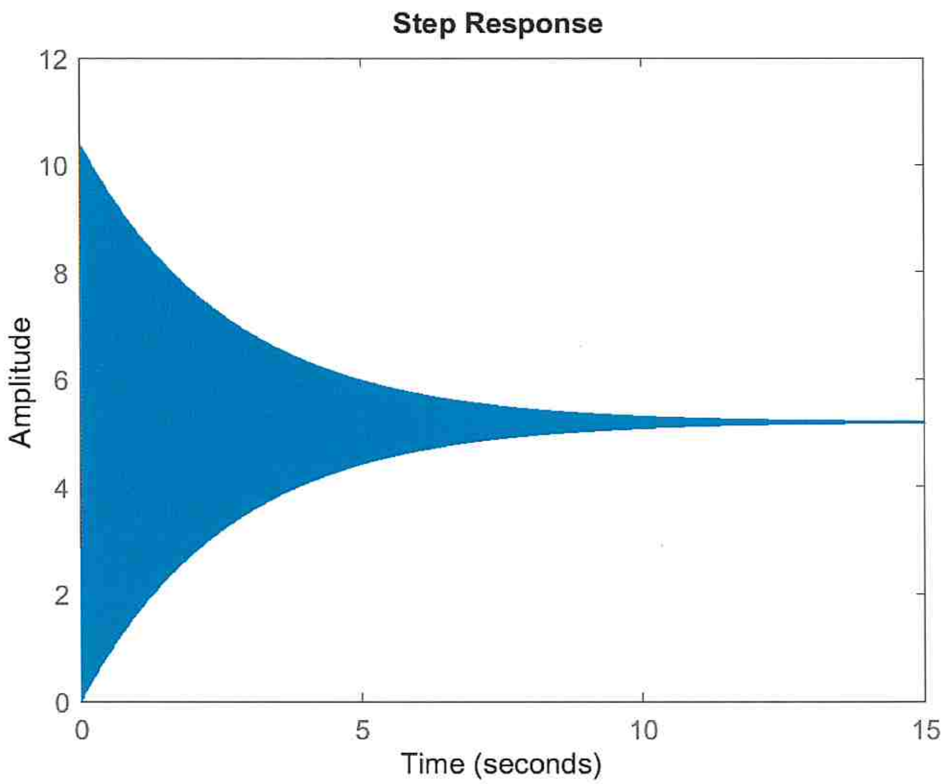
(f) see code.

Problem 3a



Input/output vs. time
Input/output
Time (s)
0
1
2
3
4
5
6
0
0.5
1
1.5
2
2.5
3

(c)



(f)


```
% ME 5200/6200 Homework 02, Problem 3
% Kam K. Leang, Copyright 2024

clear all

% define system
K = 5.2;
zeta = 0.001;
wn = 60*2*pi;

G = tf([K*wn^2],[1 2*zeta*wn wn^2]);

% Defining the input
dt = 0.001; % delta t
t1 = [0:dt:1-dt];
t2 = [1:dt:2-dt];
t3 = [2:dt:3-dt];
t = [t1 t2 t3];

u1 = t1; % linear function with slope 1
u2 = ones(size(t2));
u3 = -t1 + 1;
u = [u1 u2 u3];

% using linear simulator to find output
[y,t] = lsim(G,u,t);

% response for given input
figure(1); clf;
plot(t,u,'r--',t,y,'k');
xlabel('Time (s)'); ylabel('Input/output');
title('Input/output vs. time')

% step response
figure(2); clf;
step(G)
```