Intermediate Fluid Mechanics

Lecture 14b: More Canonical Type Flows

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ME 5700/6700
September 24, 2025

Chapter Overview

1. Chapter Objectives

2. Flow through a channel with porous walls

3. Streamfunction

Lecture Objectives

- In this lecture, we will examine a few other flow fields where one can obtain an exact solution to the N-S equations.
- Specifically we will explore flow through a channel with porus walls.

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Flow through a channel with porous walls

- Fluid is driven through a 2D channel with infinitely long side walls, via a constant pressure gradient $(dp/dx = -k)$.
- In addition, the side walls are porus such that there is a constant vertical velocity of v_w at both walls.

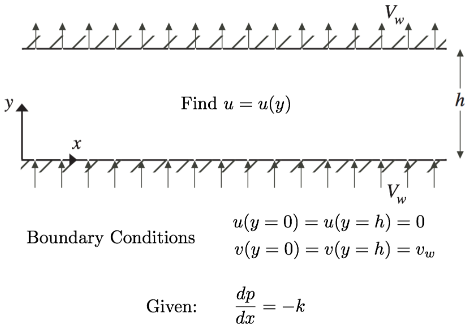

Figure: *Channel flow with porous walls.*

$\Longrightarrow$ The goal is to calculate the streamlines in this flow.

Flow through a channel with porous walls (continued ...)

To get started we will consider the following reasonable assumptions:

- (i) Steady – since there is no explicit time dependence specified in the problem or no initial conditions given.
- (ii) Constant fluid properties – No temperature/pressure variations.
- (iii) Incompressible – This follows from conservation of mass, given assumptions (i) and (ii).
- (iv) Fully developed – Walls are assumed to be infinitely long which means we are considering a section of the channel far from the inlet/outlet.

$+$ Note: Because the geometry is two-dimensional, the solution will be valid only for laminar flow.

Let's start the solution procedure by writing the x-momentum equation and simplifying according to our assumptions:

$$\underbrace{\cancel{\rho\frac{\partial u}{\partial t}}}_{\text{(i)}} + \underbrace{\cancel{\rho\, u\frac{\partial u}{\partial x}}}_{\text{(iv)}} + \rho\, v\frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \underbrace{\cancel{\mu\frac{\partial^2 u}{\partial x^2}}}_{\text{(iv)}} + \mu\frac{\partial^2 u}{\partial y^2}. \tag{1}$$

Hence, this equation reduces to,

$$\rho\, v\frac{\partial u}{\partial y} = +k + \mu\frac{\partial^2 u}{\partial y^2} \tag{2}$$

$\Longrightarrow$ At this point, one cannot yet solve this equation given that we have two unknowns (u & v) and only one equation.

Flow through a channel with porous walls (continued ...)

A second equation is needed that relates u and v. $\Longrightarrow$ The simplest choice is to use the conservation of mass equation.

Since the flow is incompressible, the correct equation is,

$$\underbrace{\cancel{\frac{\partial u}{\partial x}}}_{\text{iv}} + \frac{\partial v}{\partial y} = 0 \qquad \Rightarrow \qquad \frac{\partial v}{\partial y} = 0. \tag{3}$$

This equation can be integrated in y,

$$\int \frac{\partial v}{\partial y}\, dy = \int 0\, dy \qquad \Rightarrow \qquad v = C\,(\text{constant}). \tag{4}$$

Flow through a channel with porous walls (continued ...)

It is now possible to determine the value of the constant C using the boundary conditions.

Plugging in the value of v at the lower wall $(y = 0)$ gives that

$$v(y = 0) = v_w = C \qquad \Rightarrow \qquad C = v_w, \tag{5}$$

which leads to the solution that $v = v_w$.

$\Longrightarrow$ Realize that this means that the vertical velocity is constant everywhere in the channel and equal to v_w.

Flow through a channel with porous walls (continued ...)

This result can now be substituted in the prior simplified momentum equation 2,

$$\rho\, v_w\, \frac{du}{dy} = k + \mu \frac{d^2 u}{dy^2}. \tag{6}$$

(Note: *the partial derivatives of u have been rewritten as ordinary derivatives because the flow is fully-developed and u is a function of y* only.)

Rearranging equation 6 gives that,

$$\mu \frac{d^2 u}{dy^2} - \rho\, v_w\, \frac{du}{dy} = -k. \tag{7}$$

This is a non-homogeneous, 2^{nd} order ODE with constant coefficients. The solution will be the sum of a homogeneous and particular parts,

$$u = u_h + u_p. \tag{8}$$

Flow through a channel with porous walls (continued ...)

Let's now first consider the solution for the particular solution.

For this purpose we guess a solution of the type, $u_p = a\, y$.

Plugging this into the inhomogenous differential equation 7, we find that

$$-\rho\, v_w\, a = -k \qquad \Rightarrow \qquad a = k/(\rho\, v_w). \tag{9}$$

Therefore, the particular solution is

$$u_p = \left(\frac{k}{\rho\, v_w}\right) y. \tag{10}$$

Flow through a channel with porous walls (continued ...)

On the other hand, the homogeneous solution has to satisfy the homogeneous differential equation,

$$\mu \frac{d^2 u}{dy^2} - \rho \, v_w \, \frac{du}{dy} = 0. \tag{11}$$

$\longrightarrow$Assume a solution of the form $u_h = e^{\lambda y}$ that we plug into the homogeneous differential equation (eq. 11) to obtain,

$$\mu\lambda^2 e^{\lambda y} - \rho v_w \lambda e^{\lambda y} = 0 \qquad \Rightarrow \qquad (\mu\lambda^2 - \rho v_w \lambda) e^{\lambda y} = 0. \tag{12}$$

This equation can only be zero if,

$$\mu\lambda^2 - \rho v_w \lambda = 0, \tag{13}$$

which has two roots, $\lambda_1 = 0$ and $\lambda_2 = \rho \, v_w / \mu$.

Flow through a channel with porous walls (continued ...)

Therefore, the homogeneous solution is,

$$u_h = C_1 + C_2 e^{\rho v_w y/\mu}. \tag{14}$$

$\longrightarrow$ At this point one can put together all the different elements of the solution (homogeneous and particular solutions) to obtain,

$$u = C_1 + C_2 e^{\rho v_w y/\mu} + \left(\frac{k}{\rho\, v_w}\right) y. \tag{15}$$

Flow through a channel with porous walls (continued ...)

To determine the values for the integration constants C_1 and C_2 one only needs to apply the boundary conditions,

$$u(y=0) = 0 = C_1 + C_2 + 0 \qquad \Rightarrow \qquad C_1 = -C_2 \tag{16}$$

$$u(y=h) = 0 = C_1 + C_2 e^{\rho v_w h/\mu} + \left(\frac{k}{\rho v_w}\right) h \tag{17}$$

Substituting on the second condition the result of the first one, we can find that,

$$C_2 = \frac{hk}{\rho v_w}\left(1 - e^{\rho v_w h/\mu}\right)^{-1}. \tag{18}$$

The final solution is then,

$$\boxed{u(y) = \frac{h\,k}{\rho\,v_w}\left[\frac{y}{h} - \frac{1 - e^{\rho v_w y/\mu}}{1 - e^{\rho v_w h/\mu}}\right]}. \tag{19}$$

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Streamfunction ψ

To obtain the streamlines, we will plot the iso-contours of the streamfunction ψ, which is defined as,

$$u = \frac{\partial \psi}{\partial y} \qquad\qquad v = -\frac{\partial \psi}{\partial x}. \tag{20}$$

Streamfunction ψ (continued ...)

To obtain first ψ we will integrate the corresponding equations above,

$$\int u\,dy = \int \frac{\partial \psi}{\partial y}\,dy \tag{21}$$

$$\int \frac{h\,k}{\rho\,v_w}\left[\frac{y}{h} - \frac{1 - e^{\rho v_w y/\mu}}{1 - e^{\rho v_w h/\mu}}\right] dy = \psi \tag{22}$$

$$\frac{h\,k}{\rho\,v_w}\left[\frac{y^2}{2h} - \frac{y - \frac{\mu}{\rho v_w} e^{\rho v_w y/\mu}}{1 - e^{\rho v_w h/\mu}}\right] + \underbrace{f(x)}_{*} = \psi. \tag{23}$$

Note that the constant of integration (*) can be a function of x since $\psi = \psi(x, y)$ and the integral is only in y.

Streamfunction ψ (continued ...)

To determine this possible function of x, meaning $f(x)$, one can use the second equation for ψ,

$$v = -\frac{\partial \psi}{\partial x}. \tag{24}$$

Plugging in $v = v_w$ from earlier and differentiating the expression for ψ obtained above with respect to x,

$$-v_w = \frac{\partial}{\partial x}\left[\frac{h\,k}{\rho\, v_w}\left(\frac{y}{h} - \frac{1 - e^{\rho v_w y/\mu}}{1 - e^{\rho v_w h/\mu}}\right) + f(x)\right] \tag{25}$$

$$-v_w = \frac{df}{dx}. \tag{26}$$

To determine f, multiply by dx and integrate,

$$-\int v_w\, dx = \int df \qquad \Rightarrow \qquad f = -v_w\, x + C. \tag{27}$$

In this case, C is another constant of integration.

Streamfunction ψ (continued ...)

This solution for $f(x)$ can be substituted to obtain the final solution for ψ,

$$\psi = \frac{h\,k}{\rho\,v_w}\left[\frac{y^2}{2h} - \frac{y - \frac{\mu}{\rho v_w} e^{\rho v_w y/\mu}}{1 - e^{\rho v_w h/\mu}}\right] - v_w\,x. \tag{28}$$

- Note that we have set $C = 0$ since the actual values of ψ have no physical meaning and are thus arbitrary.
- Recall that it is the difference in ψ that gives the volume flow rate.

Streamfunction ψ (continued ...)

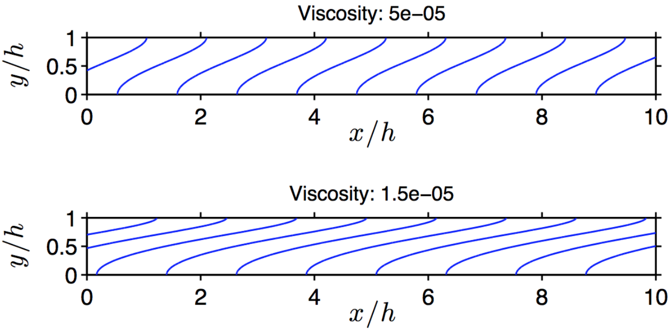

Figure: *Streamline pattern for the case of a channel flow with porus walls for two different values of viscosity,* $\mu = 1.5 \times 10^{-5}$ *and* $\mu = 5 \times 10^{-5}$ $Pa \cdot s$*. Case (a) corresponds to the low viscosity case, and case (b) to the high-viscosity case.*

As the viscosity increases (holding all other parameters constant), the stremalines tend to get a little flatter, a little more horizontal especially in the channel centerline.

Streamfunction ψ (continued ...)

Let's try to predict this behavior directly from the governing equation 6,

$$\rho\, v_w \frac{du}{dy} = k + \mu \frac{d^2 u}{dy^2} \implies \underbrace{\rho \frac{Du}{Dt}}_{I} = \underbrace{k}_{II} + \underbrace{\mu \frac{d^2 u}{dy^2}}_{III}, \tag{29}$$

- $\implies$ The rate of change of horizontal momentum of a fluid particle (I) can increase if either the pressure gradient (II) or viscous diffusion (III) increases.
- An increase in viscosity will increase viscous diffusion, thereby increasing the horizontal momentum.
- If v_w remains constant, the vertical momentum remains unchanged with an increase in μ.

In conclusion, as a particle travels along one of the streamlines if the viscosity is suddenly increased, the streamlines should begin to flatten out since Du/Dt increases but Dv/Dt remains constant no matter what value of μ is chosen.