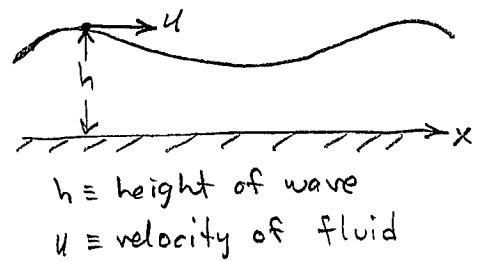


1. One-dimensional shallow wave equation is shown below.

$$\frac{\partial h}{\partial x} = -\frac{u}{g} \frac{\partial u}{\partial x}$$



(a) Use a length scale  $L$  and velocity scale  $V_0$  to nondimensionalize this equation

ANSWER

The nondimensional variables are

$$\tilde{h} = \frac{h}{L}, \quad \tilde{x} = \frac{x}{L}, \quad \tilde{u} = \frac{u}{V_0}$$

Rearranging,

$$h = \tilde{h}L, \quad x = \tilde{x}L, \quad u = \tilde{u}V_0$$

Substituting these variables into the differential equation gives

$$\frac{\frac{\partial \tilde{h}}{\partial \tilde{x}}}{\frac{\partial \tilde{x}}{\partial \tilde{x}}} = -\frac{V_0^2}{gL} \tilde{u} \frac{\partial \tilde{u}}{\partial \tilde{x}}$$

$$\rightarrow \frac{\partial \tilde{h}}{\partial \tilde{x}} = -\frac{V_0^2}{gL} \tilde{u} \frac{\partial \tilde{u}}{\partial \tilde{x}}$$

(b) What is the nondimensional parameter that characterizes this flow?

ANSWER

$\frac{V_0^2}{gL}$  which is the square of the Froude number.

check units:  $\frac{(m/s)^2}{m/s^2} \checkmark$

2. Consider the simplified Navier-Stokes equation

$$\underbrace{\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y}}_{\text{I}} = -\underbrace{\frac{\partial p}{\partial x}}_{\text{II}} + \underbrace{\mu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)}_{\text{III}}.$$

The flow has a characteristic length scale  $L$  and a velocity scale  $U$ .

(a) Determine the proper nondimensionalization of the pressure gradient (term II) if term I is zero.

ANSWER

If term I is zero, then advection/inertia does not play a significant role in the dynamics of the problem.

The nondimensional variables are

$$\tilde{u} = \frac{u}{U}, \quad \tilde{x} = \frac{x}{L}, \quad \tilde{y} = \frac{y}{L}, \quad \tilde{p} = \frac{p}{P} \quad \text{don't know what this is yet}$$

Plug in nondimensional variables into equation omitting term I:

$$0 = -\frac{P}{L} \frac{\partial \tilde{p}}{\partial \tilde{x}} + \frac{\mu U}{L^2} \left( \frac{\partial^2 \tilde{u}}{\partial \tilde{x}^2} + \frac{\partial^2 \tilde{u}}{\partial \tilde{y}^2} \right)$$

Divide by  $\mu U / L^2$ :

$$0 = -\frac{P L}{\mu U} \frac{\partial \tilde{p}}{\partial \tilde{x}} + \left( \frac{\partial^2 \tilde{u}}{\partial \tilde{x}^2} + \frac{\partial^2 \tilde{u}}{\partial \tilde{y}^2} \right)$$

In order to satisfy the equality, i.e. in order for both terms to remain important in the dynamics, the non-dimensional group of parameters in front of  $\frac{\partial \tilde{p}}{\partial \tilde{x}}$  must be order 1:

$$\frac{PL}{\mu U} \sim O(1) \Rightarrow \boxed{P = \frac{\mu U}{L}}$$

2. (cont.)

- (b) Determine the proper nondimensionalization of the pressure gradient (term II) if term III is zero.

ANSWER

If term III is zero, then viscous diffusion is unimportant in the dynamics of the flow.

The nondimensional variables are the same as in part (a)

$$\tilde{u} = \frac{u}{U}, \quad \tilde{x} = \frac{x}{L}, \quad \tilde{y} = \frac{y}{L}, \quad \tilde{P} = \frac{P}{\bar{P}}$$

Plug these into differential equation

$$\frac{\rho U^2}{L} \left( \tilde{u} \frac{\partial \tilde{u}}{\partial \tilde{x}} + \tilde{v} \frac{\partial \tilde{u}}{\partial \tilde{y}} \right) = - \frac{\bar{P}}{L} \frac{\partial \tilde{P}}{\partial \tilde{x}}$$

Divide by  $\rho U^2 / L$

$$\tilde{u} \frac{\partial \tilde{u}}{\partial \tilde{x}} + \tilde{v} \frac{\partial \tilde{u}}{\partial \tilde{y}} = - \frac{\bar{P}}{\rho U^2} \frac{\partial \tilde{P}}{\partial \tilde{x}}$$

In order to satisfy this equality so that both terms (inertia and pressure gradient) contribute to the dynamics, we again need the nondimensional group of parameters in front of the pressure gradient to be order 1:

$$\frac{\bar{P}}{\rho U^2} \sim O(1) \Rightarrow \boxed{\bar{P} = \rho U^2}$$

- (c) Provide real world examples when the two cases would be valid

ANSWER

case a: inertia is insignificant in creeping flow (low Reynolds number flow, i.e. slow velocity, small length scale, high viscosity)

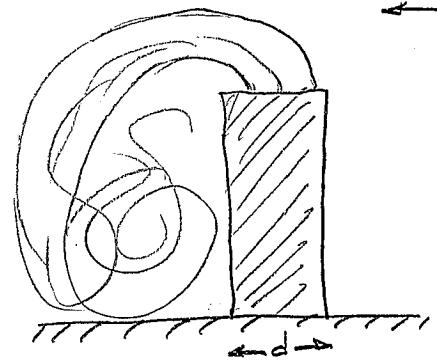
case b: diffusion is insignificant in inviscid flows, i.e. flow far from any solid surface where the Reynolds number is high.

3. Passengers on a ship cruising at 30 knots experience smoke pollution on the deck from the stack exhaust. A 20% scale model is to be studied.

- (a) Should a water channel or wind tunnel be used? State the relevant nondimensional parameter in this problem.

ANSWER

The problem is the air pollution from the stack exhaust. In terms of the fluid dynamics, it doesn't matter whether the stack is on a ship that is moving at speed  $U$  or whether the stack is stationary with a flow of speed  $U$  moving across it.



The important nondimensional parameter is the Reynolds number:

$$Re = \frac{Ud}{\nu}$$

In order to answer whether a water channel or wind tunnel should be used, we need to consider how matching the Reynolds number in the two cases affects  $U_m$  (velocity of the model study).

- (b) what velocity should be used in the model study?

ANSWER

We need to match the Reynolds numbers.

$$Re_m = Re_p$$

$$\frac{U_m d_m}{\nu_m} = \frac{U_p d_p}{\nu_p}$$

$$U_m = U_p \left( \frac{d_p}{d_m} \right) \left( \frac{\nu_m}{\nu_p} \right) = 308.6 \left( \frac{\nu_m}{\nu_p} \right) \text{ m/s}$$

3. (cont.)

Wind Tunnel  
If the model study was conducted in a wind tunnel at standard atmospheric conditions ( $V_m = V_p$ ), then the required velocity of the model would be  $U_m = 308.6 \text{ m/s}$ . Note, the speed of sound is about  $340 \text{ m/s}$ , so compressibility effects would definitely be present causing incomplete similarity.

Water Channel

On the other hand, if the model studies were performed in a water channel,  $V_m = 0.9 \times 10^{-6} \text{ m}^2/\text{s}$  and  $V_p = 1.5 \times 10^{-5} \text{ m}^2/\text{s}$ ,

so

$$U_m = 308.6 \left( \frac{0.9 \times 10^{-6}}{1.5 \times 10^{-5}} \right)$$

$$\rightarrow U_m = 18.5 \text{ m/s}$$

This is still pretty fast for a water channel, but it's possible.

4. A 25:1 scale model of a submarine is tested in a wind tunnel with  $\rho = 200 \text{ kPa}$  and  $T = 300 \text{ K}$ . The expected speed of the prototype is  $30 \text{ km/hr}$  ( $8.33 \text{ m/s}$ ).

(a) What should be the speed of the wind tunnel?

ANSWER

Since the submarine is completely submerged, the important nondimensional parameter is the Reynolds number. Matching the Reynolds number gives

$$Re_m = Re_p$$

$$\frac{\rho_m U_m L_m}{\mu_m} = \frac{\rho_p U_p L_p}{\mu_p}$$

$$U_m = U_p \left( \frac{L_p}{L_m} \right) \left( \frac{\rho_p}{\rho_m} \right) \left( \frac{\mu_m}{\mu_p} \right)$$

Use the ideal gas law to determine  $\rho_m$ :

$$\rho_m = \frac{P_m}{R T_m} = \frac{200 \times 10^3 \text{ Pa}}{(287 \text{ J/kg}\cdot\text{K})(300 \text{ K})} = 2.32 \text{ kg/m}^3$$

The viscosity of air at  $T=300 \text{ K}$  is  $1.846 \times 10^{-5} \text{ kg/m}\cdot\text{s}$ . The viscosity of water at  $20^\circ\text{C}$  is  $1 \times 10^{-3} \text{ kg/m}\cdot\text{s}$  and the density is about  $1000 \text{ kg/m}^3$ . Plugging in the numbers

$$U_m = (8.33)(25) \left( \frac{1000}{2.32} \right) \left( \frac{1.846 \times 10^{-5}}{1 \times 10^{-3}} \right)$$

$$\rightarrow U_m = 1.66 \times 10^3 \text{ m/s} \quad \text{(unrealistic)}$$

4. (cont.)

(b) What is the ratio of the model drag force to the prototype drag force?

ANSWER

If dynamic similarity is satisfied, then we know

$$C_{Dm} = C_{Dp}$$

$$\frac{F_{Dm}}{\frac{1}{2} \rho_m U_m^2 L_m^2} = \frac{F_{Dp}}{\frac{1}{2} \rho_p U_p^2 L_p^2}$$

$$\frac{F_{Dm}}{F_{Dp}} = \left( \frac{l_m}{l_p} \right) \left( \frac{U_m^2}{U_p^2} \right) \left( \frac{L_m^2}{L_p^2} \right)$$

$$\frac{F_{Dm}}{F_{Dp}} = \left( \frac{2.32}{1000} \right) \left( \frac{1.66 \times 10^3}{8.33} \right)^2 \left( \frac{1}{25} \right)^2$$

$$\frac{F_{Dm}}{F_{Dp}} = 0.00232 \times 39,712 \times 0.0016$$

$$\rightarrow \frac{F_{Dm}}{F_{Dp}} = 0.147$$

5. A windmill is designed to operate at 20 rpm in a 15 mph wind to produce 300 kW of power. The blades are 175 ft in diameter. A 10:1 model is to be tested at 90 mph wind velocity. HW3

- (a) What rotor speed should be used in the model experiment?

ANSWER

We should apply the Buckingham  $\pi$  theorem to determine the important dimensionless parameters so that we properly ensure dynamic similarity.

$$\pi(P, \omega, d, U, \rho, \mu) = 0$$

$N = 6$  total variables

blade diameter  
 wind speed  
 power  
 rotor speed  
 air density  
 air viscosity

The dimensional matrix is

	$P$	$\omega$	$d$	$U$	$\rho$	$\mu$
$M$	1	0	0	0	1	1
$L$	2	0	1	1	-3	-1
$T$	3	-1	0	-1	0	-1

The rank is 3, so we need three repeating variables. We choose  $\omega, d, \rho$ . Note, we need to choose either  $\rho$  or  $\mu$  because these are the only two that contain the fundamental unit of mass. We do not choose  $\mu$ , because that would mean that we would have to include viscous effects in all of our  $\pi$  groups. Based on problem 3-2, we know that this may not be appropriate for high Reynolds numbers. Since  $d$  is our only length scale, we choose that to satisfy the fundamental unit of length. The last fundamental unit we need to include is time. Here, we can choose either  $U$  or  $\omega$ . Because we are ultimately interested in power, I chose  $\omega$  since torque times rotor speed directly yields power output.

5. (cont.)

○ We will have  $\underbrace{6-3}_{\text{rank}}^N = 3$   $\Pi$  groups

$$\Pi_1 = \cup \omega^a d^b \rho^c$$

$$M^0 L^0 T^0 = (L^1 T^{-1})^a (T^{-1})^b (L^1)^c (M^1 L^{-3})^c$$

$$M: 0 = c$$

$$L: 0 = 1 + b - 3c \Rightarrow b = -1$$

$$T: 0 = -1 - a \Rightarrow a = -1$$

$$\rightarrow \Pi_1 = \cup / \omega d$$

$$\Pi_2 = \cup \omega^a d^b \rho^c$$

$$M^0 L^0 T^0 = (M^1 L^{-1} T^{-1}) (T^{-1})^a (L^1)^b (M^1 L^{-3})^c$$

$$M: 0 = 1 + c \Rightarrow c = -1$$

$$L: 0 = -1 + b - 3c \Rightarrow b = -2$$

$$T: 0 = -1 - a \Rightarrow a = -1$$

$$\rightarrow \Pi_2 = \frac{1}{\rho \omega d^2}$$

$$\Pi_3 = \cup \omega^a d^b \rho^c$$

$$M^0 L^0 T^0 = (M^1 L^2 T^{-3}) (T^{-1})^a (L^1)^b (M^1 L^{-3})^c$$

$$M: 0 = 1 + c \Rightarrow c = -1$$

$$L: 0 = 2 + b - 3c \Rightarrow b = -5$$

$$T: 0 = -3 - a \Rightarrow a = -3$$

$$\rightarrow \Pi_3 = \frac{1}{\rho \omega^3 d^5}$$

5. (cont.)

Buckingham Pi Theorem tells us that

$$\Pi_3 = f(\Pi_1, \Pi_2)$$

To determine the rotor speed in the wind tunnel, we need to match  $\Pi_1$ , because viscous effects in  $\Pi_2$  will not be important for the particular case of flow over a windmill.

$$(\Pi_1)_m = (\Pi_1)_p$$

$$\frac{U_m}{d_m \omega_m} = \frac{U_p}{d_p \omega_p}$$

$$\omega_m = \left( \frac{U_m}{U_p} \right) \left( \frac{d_p}{d_m} \right) \omega_p$$

$$\omega_m = \left( \frac{90}{15} \right) \left( \frac{10}{1} \right) (20 \text{ rpm})$$

$$\omega_m = 1200 \text{ rpm}$$

(b) What power should be expected in the model experiment?

ANSWER

We realize that viscous effects are not important in this application, so that in terms of the appropriate dynamics,

$$\Pi_3 = f(\Pi_1).$$

Since we have already match  $\Pi_1$  in part a, we are guaranteed that  $\Pi_3$  will be matched:

$$(\Pi_3)_m = (\Pi_3)_p$$

$$\frac{P_m}{\rho_m \omega_m^3 d_m^5} = \frac{P_p}{\rho_p \omega_p^3 d_p^5}$$

$\rho_m = \rho_p$  (fluid the same in both)

$$P_m = P_p \left( \frac{\rho_m}{\rho_p} \right) \left( \frac{\omega_m}{\omega_p} \right)^3 \left( \frac{d_m}{d_p} \right)^5 = 300 \text{ kW} \left( \frac{1200}{20} \right)^3 \left( \frac{1}{10} \right)^5$$

$$\rightarrow P_m = 648 \text{ kW}$$

6. Consider a circular cylinder of diameter  $D$  and length  $l$  that is being towed in a water tank at velocity  $U$ .

- (a) Express the drag force in dimensionless form as a function of all the relevant nondimensional parameters.

ANSWER

The dimensional drag force is represented by

$$F_D = f(\rho, \mu, U, l, D)$$

$N = 6$  total variables

The dimensional matrix is

	$F_D$	$\rho$	$\mu$	$U$	$l$	$D$
$M$	1	1	1	0	0	0
$L$	1	-3	-1	1	1	1
$T$	-2	0	-1	-1	0	0

rank = 3

We choose the repeating variables:  $\rho, U, D$

We expect  $6-3 = 3$  groups:

$$\Pi_1 = F_D \rho^a U^b D^c$$

$$M^0 L^0 T^0 = (M^1 L^1 T^{-2})(M^1 L^{-3})^a (L^1 T^{-1})^b (L^1)^c$$

$$M: 0 = 1 + a \Rightarrow a = -1$$

$$L: 0 = 1 - 3a + b + c \Rightarrow c = -2$$

$$T: 0 = -2 + b \Rightarrow b = -2$$

$$\rightarrow \Pi_1 = \frac{F_D}{\rho U^2 D^2}$$

$$\Pi_2 = \mu \rho^a U^b D^c$$

$$M^0 L^0 T^0 = (M^1 L^1 T^{-1})(M^1 L^{-3})^a (L^1 T^{-1})^b (L^1)^c$$

$$M: 0 = 1 + a \Rightarrow a = -1$$

$$L: 0 = -1 - 3a + b + c \Rightarrow c = -1$$

$$T: 0 = -1 - b \Rightarrow b = -1$$

$$\rightarrow \Pi_2 = \frac{\mu}{\rho U D}$$

$$\Pi_3 = l \rho^a U^b D^c$$

$$M^0 L^0 T^0 = (L') (M^1 L^{-3})^a (L^1 T^{-1})^b (L')^c$$

$$M: 0 = a$$

$$L: 0 = 1 - 3a + b + c \Rightarrow c = -1$$

$$T: 0 = -b$$

$$\rightarrow \Pi_3 = \frac{l}{D}$$

Buckingham's Pi theorem states that  $\Pi_1 = f(\Pi_2, \Pi_3)$  or

$$\frac{F_D}{\rho U^2 D^2} = f\left(\frac{l}{D}, \frac{U}{\rho D}\right)$$

(b) Estimate the maximum speed that the cylinder can be towed in the water tank without causing cavitation.

ANSWER

Assume cavitation occurs at a cavitation number of  $C_a = 0.5$ . We also know that the pressure coefficient in this flow reaches a minimum value of  $C_p = -2.4$ . Therefore, we have two conditions that we need to match simultaneously.

$$C_a = \frac{P - P_\infty}{\frac{1}{2} \rho U_{max}^2} = 0.5 \quad \text{and} \quad C_p = \frac{P - P_\infty}{\frac{1}{2} \rho U_{max}^2} = -2.4.$$

$$P = 0.25 \rho U_{max}^2 + P_\infty \quad \text{and} \quad P = -1.2 \rho U_{max}^2 + P_\infty$$

Setting the two equations equal gives

$$0.25 \rho U_{max}^2 + P_\infty = -1.2 \rho U_{max}^2 + P_\infty \Rightarrow 1.45 \rho U_{max}^2 = P_\infty - P_\infty$$

The vapor pressure of water at standard conditions ( $T = 20^\circ C$ ) is  $P_v = 2.34 \text{ kPa}$ . Standard atmospheric pressure is  $P_\infty = 101.3 \text{ kPa}$

$$U_{max} = \sqrt{\frac{P_\infty - P_v}{1.45 \rho}} = \sqrt{\frac{101.3 \times 10^3 - 2.34 \times 10^3}{1.45 (1000 \text{ kg/m}^3)}} \Rightarrow U_{max} = 8.26 \text{ m/s}$$