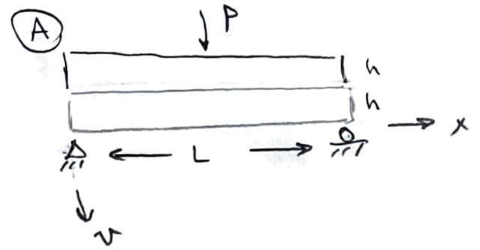
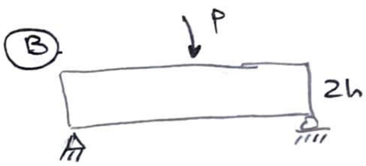
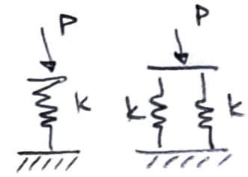
Shear Stress Constrains Deflection

Compare 

to 

- For single beam, $v(x = L/2) = \dfrac{PL^3}{48EI}$, where $I = \frac{1}{12}bh^3$

- For A, v is half what it would be for a single beam of height h
  - v is linear function of P, so can use superposition (with spring analogy)



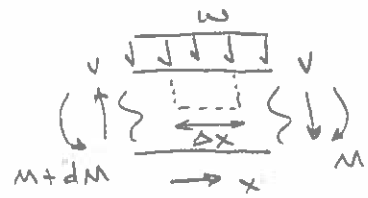
$$\Rightarrow \quad v_A = \frac{1}{2}\,\frac{PL^3}{48EI}$$

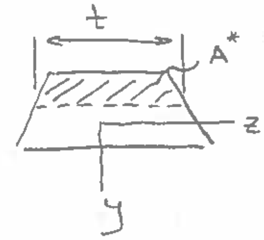
- For B, $v_B = \dfrac{PL^3}{48EI_B}$, but $I_B = \frac{1}{12}b(2h)^3 = \frac{1}{12}bh^3\,(8)$

$$\Rightarrow \quad v_B = \frac{1}{8}\,\frac{PL^3}{48EI}$$

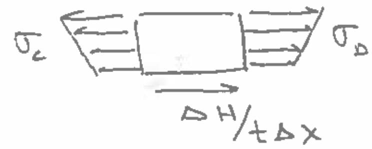
$$\Rightarrow \quad v_B = \frac{1}{4}v_A$$

Shear in Bending





- Quantifying shear stress in bending due to transverse load
  - Consider beam of arbitrary cross-section
  - Equilibrium of element in x-dir



    - Note $\sigma_c > \sigma_D$ since moment changes along beam... thus
    - $\Sigma F_x = 0 = \Delta H + \int \sigma_D \, dA^* - \int \sigma_c \, dA^*$

$$\Rightarrow \quad \Delta H = \int (\sigma_c - \sigma_D)\, dA^* = \int \left(\frac{M_c y}{I} - \frac{M_D y}{I} \right) dA^*$$

$$= \frac{1}{I}(M_c - M_D) \underbrace{\int y \, dA^*}_{} \rightarrow Q \text{ (1}^{\text{st}} \text{ moment of area about N.A.)}$$

  - Recall $V = \dfrac{dM}{dx} \Rightarrow \Delta M = V \Delta x$

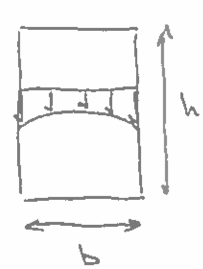
$$\Rightarrow \quad \Delta H = \frac{\Delta M}{I} Q = \frac{VQ}{I} \Delta x$$

$$\Rightarrow \quad \tau_{yx} = \frac{\Delta H}{t \Delta x} = \frac{VQ}{It} \quad \left(\text{sometimes } \frac{VQ}{Ib}\right)$$

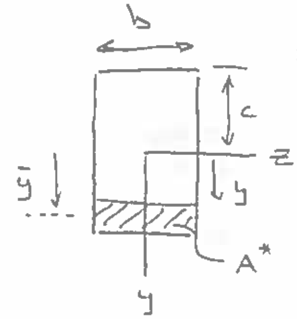
  - Define $q = \dfrac{\Delta H}{\Delta x} = \dfrac{VQ}{I}$ (shear force gradient ≡ shear flow)

* Note we assume here that τ_{yx} is uniform across the beam width; this is not true but is a reasonable approximation for widths that are small, i.e. b/h is small

b/h	τ_{max}/τ_{avg}
0.25	1.008
50	15.65



- Consider rectangular beam → find $\tau_{yx}(y)$ w/ transverse load V



- $\tau_{yx} = \dfrac{VQ}{It}$
  - $I = \frac{1}{12} b h^3 = \frac{1}{12} b (2c)^3 = \frac{2}{3} b c^3$
  - $t = b$
  - $Q = \bar{y} A^*$
    - $\bar{y} = \frac{1}{2}(y + c)$
    - $A^* = b(c - y)$

  $\Rightarrow Q = \frac{b}{2}(c^2 - y^2)$

$$\Rightarrow \tau_{yx} = \frac{V \frac{\cancel{b}}{2}(c^2 - y^2)}{\frac{2}{3} b c^3 (\cancel{b})} = \frac{3}{4} V \frac{c^2 - y^2}{bc^3} = \frac{3V}{4bc^3}(c^2 - y^2)$$

$$= \frac{3V}{4bc}\left(1 - \frac{y^2}{c^2}\right)$$

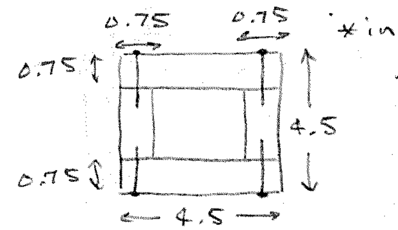
- τ_{yx} is parabolic

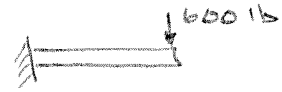
- $\tau_{yx} = 0$ @ $y = \pm c$
- $\tau_{yx,max} = \dfrac{3V}{4bc}$ (for rect beam)

  (* sometimes written as $\dfrac{3V}{2A}$)

Example

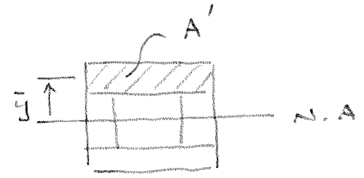
A box beam is constructed by nailing four wooden planks together, as shown. The nails are spaced 3 in. The beam is cantilevered and needs to support a 600 lb transverse load @ its end. Determine whether it would be best to use the beam in the orientation shown or to rotate it 90°.





Sol'n

- How does rotation matter?
  - Flexure stresses not affected
  - * Shear @ nails will be different



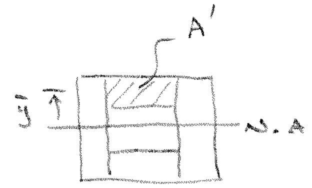
- $\tau = \dfrac{VQ}{It}$
  - $Q = A'\bar{y} = (4.5\,in)(0.75\,in)(1.875\,in) = 6.328\,in^3$
  - $I = \frac{1}{12}b_o h_o^3 - \frac{1}{12}b_i h_i^3 = \frac{1}{12}\left[(4.5\,in)^4 - (3\,in)^4\right] = 27.421\,in^4$
  - $t = 2\,(0.75\,in) = 1.5\,in$

$$\Rightarrow \tau = \frac{(600\,lb)(6.328\,in^3)}{(27.421\,in^4)(1.5\,in)} = 92.3\,psi$$

- But need force / length $\Rightarrow$ use <u>shear flow</u>

$$\Rightarrow q = \frac{VQ}{I} = \tau t = (92.3\,psi)(1.5\,in) = 138.5\,lb/in$$

- What do nails experience?
  - $q/\text{side} = 138.5/2 = 69.2\,lb/in$
  - $F_N = (69.2\,lb/in)(3\,in) = \underline{207.7\,lb}$



- $\tau' = \dfrac{VQ'}{It}$ * See top of next page before proceeding with this part of the problem.
  - $Q' = A'\bar{y} = (3\,in)(0.75\,in)(1.875\,in) = 4.219\,in^3$
  - I is the same ; t is the same

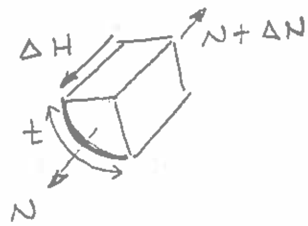
$$\Rightarrow \tau' = \frac{(600\,lb)(4.219\,in^3)}{(27.421\,in^4)(1.5\,in)} = 61.5\,psi \Rightarrow q = 92.3\,lb/in$$

$$\Rightarrow F_N' = \left(\frac{92.3\,lb/in}{2}\right)(3\,in) = \underline{138.5\,lb}$$

- $$\frac{q}{q'} = \frac{VQ/I}{VQ'/I} = \frac{Q}{Q'} = \frac{A'\bar{y}}{A''\bar{y}} = \frac{A'}{A''} = \frac{4.5\,in}{3\,in}$$

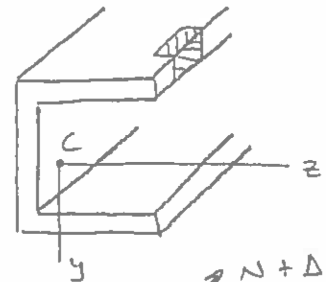
* only difference between the two is area isolated by shear surface of interest!

* In our derivation of τ, we assumed a horizontal shear surface . This was convenient but not necessary. We could have done this . In both cases, the imbalance in x-dir force due to flexure stresses is fixed by shear load on the available surface; it need not be horizontal or vertical.

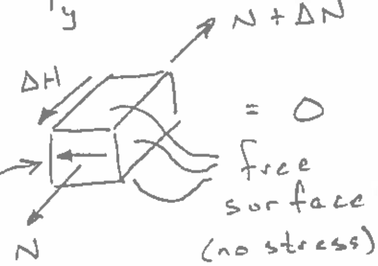


* Note that the theory tells us that Q is defined using the area that is isolated from the neutral axis based on the location where we are interested in the shear stress. It makes no requirements on the orientation of the surface of interest.

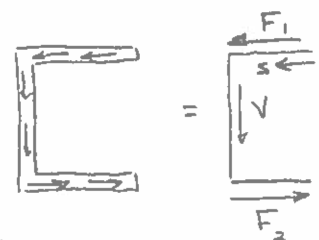
- Shear Center
  - Transverse loading of thin-walled beams w/o vertical plane symmetry produces bending and twisting unless force acts through shear center.



  - For application of force @ shear center, what does stress distribution look like?
    - Consider isolated element
      - $\Delta H = N + \Delta N - N$ (from $\Sigma F_x = 0$)
      - provides direction of stress on x-y face

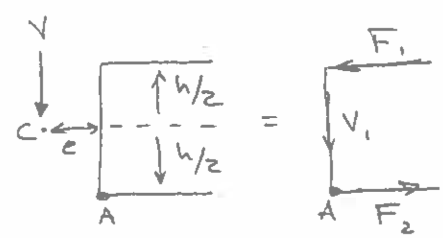


    - Similar analysis over full section shows direction of shear throughout



    - The force acting through each section is determined as, e.g. $F_1 = \iint \tau \, dA = \int q \, ds$

  - If V is applied at the shear center, C, of a beam, its moment about any point must be equivalent to the sum of the moments created by the stresses in the beam



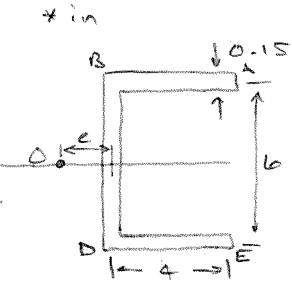
$$\text{e.g. } \Sigma M_A = Ve = F_1 h$$

$$\text{Then, } e = \frac{F_1 h}{V}$$

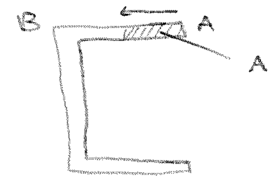
* Don't confuse combined action of shear stresses with torsion. There is no torsion if V acts at the shear center.

<u>Example</u> (C.A. 6.5, 6.6.; Beer 7; pp. 456-7)

Determine the shear center location e for the open section shown. Assuming a vertical transverse shear load of 2.5 kip, determine the shear stress distribution.



<u>Sol'n</u>



- Find force in flange regions: $F_{AB} = \int_A^B q\,ds$

  - $q = \frac{VQ}{I}$

  - $Q = A'\bar{y} = st\frac{h}{2} \Rightarrow q = \frac{V}{I}st\frac{h}{2}$

$$\Rightarrow F_{AB} = \frac{Vth}{2I}\int_0^b s\,ds = \frac{Vth}{2I}\frac{b^2}{2} = \frac{Vthb^2}{4I}$$

- Find e: $Ve = 2F_{AB}\frac{h}{2} = F_{AB}h$

$$\Rightarrow e = \frac{F_{AB}}{V}h = \frac{Vthb^2}{4I}\frac{h}{V} = \frac{th^2b^2}{4I}$$

- $I = I_{web} + 2I_{flange} = \frac{1}{12}th^3 + 2\left[\frac{1}{12}bt^3 + bt\left(\frac{h}{2}\right)^2\right]$

  * t^3 very small $\Rightarrow I = \frac{1}{12}th^2(6b+h)$

$$\Rightarrow e = \frac{\cancel{t}\cancel{h^2}b^2\,\cancel{12}^{3}}{\cancel{4}\,\cancel{t}\cancel{h^2}(6b+h)} = \frac{3b^2}{6b+h} = \underline{1.6 \text{ in}}$$

- Shear stress in flanges

$$\tau = \frac{q}{t} = \frac{Vh}{2I}s \Rightarrow \text{linear}$$

$$\tau_B = \frac{Vh}{2I}b = \frac{V\cancel{h}b}{\cancel{2}}\frac{\cancel{12}^{6}}{th^{\cancel{2}}(6b+h)} = \underline{2.22 \text{ ksi}}$$

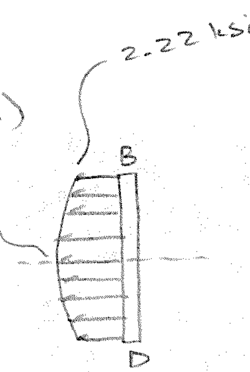
- Shear stress in web

  $\tau_B = 2.22$ ksi from flange calc

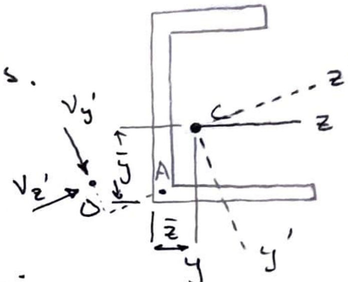
  τ_{max} @ N.A. $= \frac{VQ}{It}$

  - $Q = A'\bar{y} = bt\left(\frac{h}{2}\right) + t\frac{h}{2}\left(\frac{h}{4}\right) = \frac{1}{8}ht(4b+h)$

$$\Rightarrow \tau_{max} = \frac{(2.5\text{ kip})\left(\frac{1}{8}ht(4b+h)\right)}{\frac{1}{12}th^2(6b+h)t} = \underline{3.06 \text{ ksi}}$$

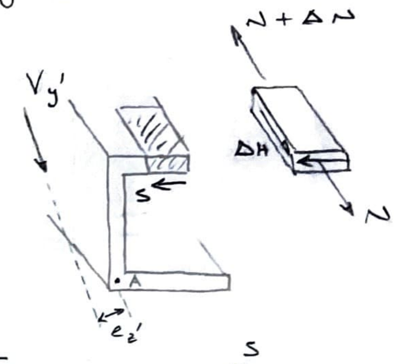


Shear Flow - Asymmetrical Section



__Example:__ Find shear center for section shown

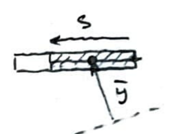
- See slide for dimensions, cross-sectional prop's.
- Separately assume shear loads in each of the principal directions. Note that $\tau = \frac{VQ}{It}$ requires that V acts perpendicular to N.A.; need to use principal directions so we have just one contributing inertia parameter.
  (no eq'n for τ involving I_{yz})

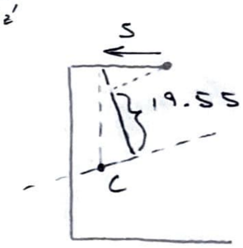


- Consider $V_{y'}$ first
  - Consider $\Sigma M_A = V_{y'} e_{z'} = F_1 (37.5)$

    $\Rightarrow e_{z'} = 37.5 \frac{F_1}{V_{y'}}$ (1)

  - Find $F_1 = \int_0^{12.5} \tau_{xz} t \, ds$

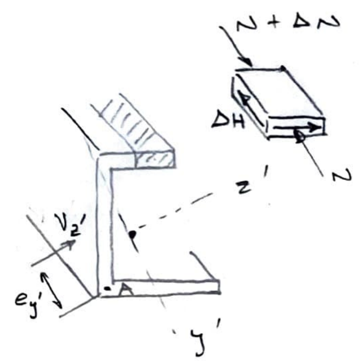




    - $\tau_{xy} = \frac{V_{y'} Q_{z'}}{I_{z'} t}$
    - $Q_{z'} = A_z^* \bar{y}'$
      - $A_z^* = st$
      - $\bar{y}_{z'} = 19.55 + \frac{1}{2} s (\sin 13.05)$

$$\Rightarrow F_1 = \frac{V_{y'} t}{I_{z'}} \int_0^{12.5} s \left(19.55 + \tfrac{1}{2} s \sin 13.05\right) ds = 0.0912 V_{y'}$$

$$\Rightarrow e_{z'} = 37.5 \frac{0.0912 V_{y'}}{V_{y'}} = \underline{3.42 \text{ mm}}$$



- Now $V_{z'}$:
  - Again, consider $\Sigma M_A = V_{z'} e_{y'} = F_1 (37.5)$

    $\Rightarrow e_{y'} = 37.5 \frac{F_1}{V_{z'}}$

  - $F_1 = \int_0^{12.5} \tau_{xz} t \, ds$

    - $\tau_{xy} = \frac{V_{z'} Q_{y'}}{I_{y'} t}$
    - $Q_{y'} = A_y^* \bar{z}'$
      - $A_y^* = st$
      - $\bar{z}' = 12.05 - \frac{1}{2} s \cos 13.05$

$$\Rightarrow F_1 = \frac{V_{z'} t}{I_{y'}} \int_0^{12.5} s \left(12.05 - \tfrac{1}{2} s \cos 13.05\right) ds = 0.204 V_{z'}$$

$$\Rightarrow e_{y'} = \underline{7.65 \text{ mm}}$$