Root Locus (Part I)

Graphical technique for qualitatively assesing performance of a closed-loop system.

What information can we get from r-locus:

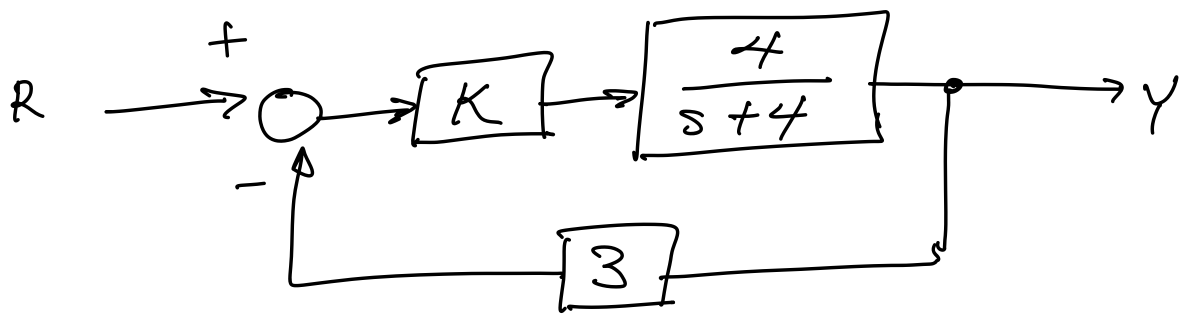
- How a controller gain affects transient response, such as t_s, t_r, $\%OS$, ξ, ω_n and ω_d.

- Tells us when the closed-loop system goes unstable.

what is it?

Root locus is a plot of how the closed-loop poles for a basic feedback system changes in the s-plane when a parameter varies, typically from $0 \longrightarrow \infty$.

Consider simple example of root locus



want to create root locus:

1. Find equation for poles of C-loop system:

$$\frac{Y(s)}{R(s)} = \frac{K\left(\frac{4}{s+4}\right)}{\boxed{1 + K(3)\left(\frac{4}{s+4}\right)}}$$

solve for poles:

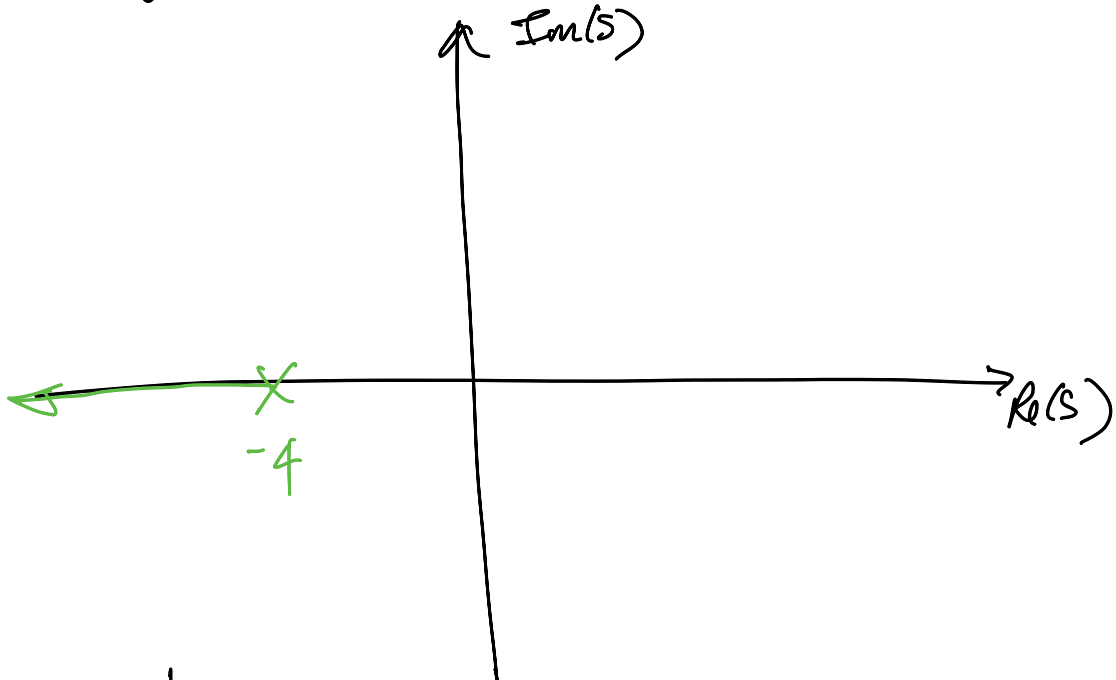
$$1 + K(3)\left(\frac{4}{s+4}\right) = 0$$

$$\tau = \frac{1}{4+12K}$$

$$\Rightarrow s + 4 + 3 \cdot 4 \cdot K = 0$$

$$\Rightarrow \boxed{s = -(4 + 12K)}$$ c-loop pole

Plot of s vs. $K > 0$

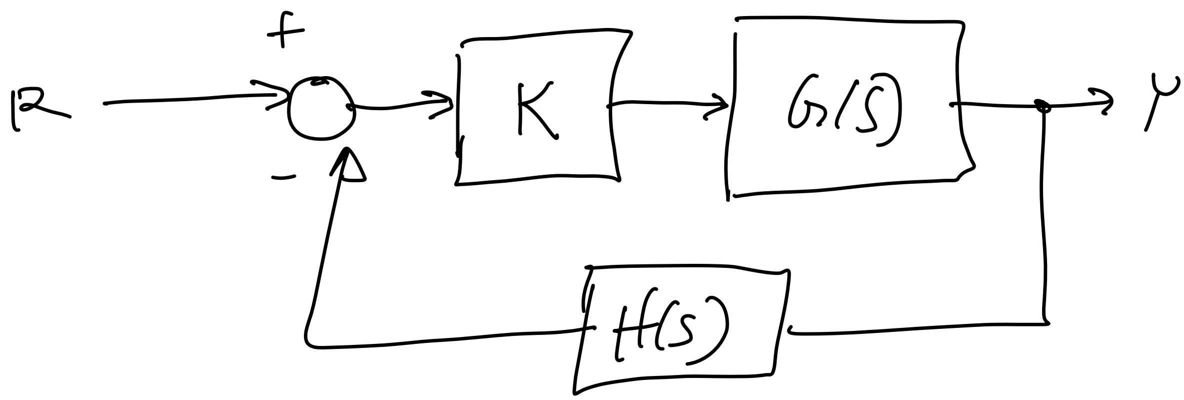


Observations

1) as $K \to \infty$, $t_s \to 0$

2) C-loop system stays stable!

<u>Generalize</u>



Assume $K > 0$

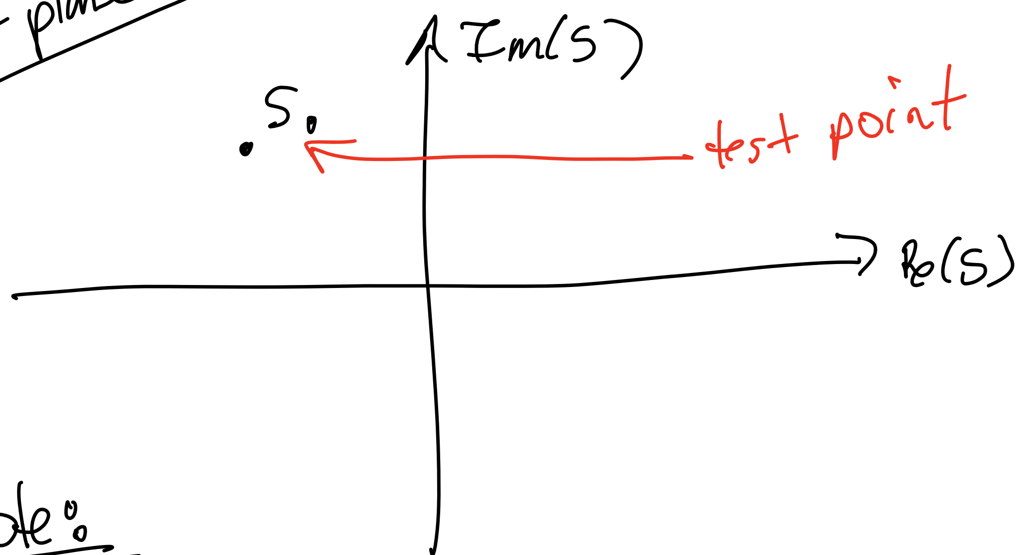
Governing equation:

$$\frac{Y(s)}{R(s)} = \frac{K\,G(s)}{1 + K\,H(s)\,G(s)} \rightarrow \text{poles}$$

$$\Rightarrow \boxed{1 + K\,H(s)\,G(s) = 0}$$

To draw a root locus, we use this expression & satisfy it.

s-plane



Note:

If a test point, s_0, in the s-plane belongs on the root locus for a particular closed-loop system, then that test point s_0 will satisfy the above equation!

Now let's take a closer look at the conditions for drawing a root locus.

Poles: $\boxed{1 + K G(s) H(s) = 0}$

This is the governing equation to draw a root locus.

Look closer again (Research)

$$1 + K G(s) H(s) = 0$$

solve for $G(s) H(s)$:

$$K G(s) H(s) = -1$$

Assume $K > 0$, so:

$$G(s) H(s) = \frac{-1}{K} < 0$$

Condition 1: Angle condition

$$\angle [G(s) H(s)] = 180^\circ + 360^\circ l$$

angle

$$l = 0, \pm 1, \pm 2 \ldots$$

Condition 2 : Magnitude condition

magnitude $G(s)H(s) = -\frac{1}{K}$

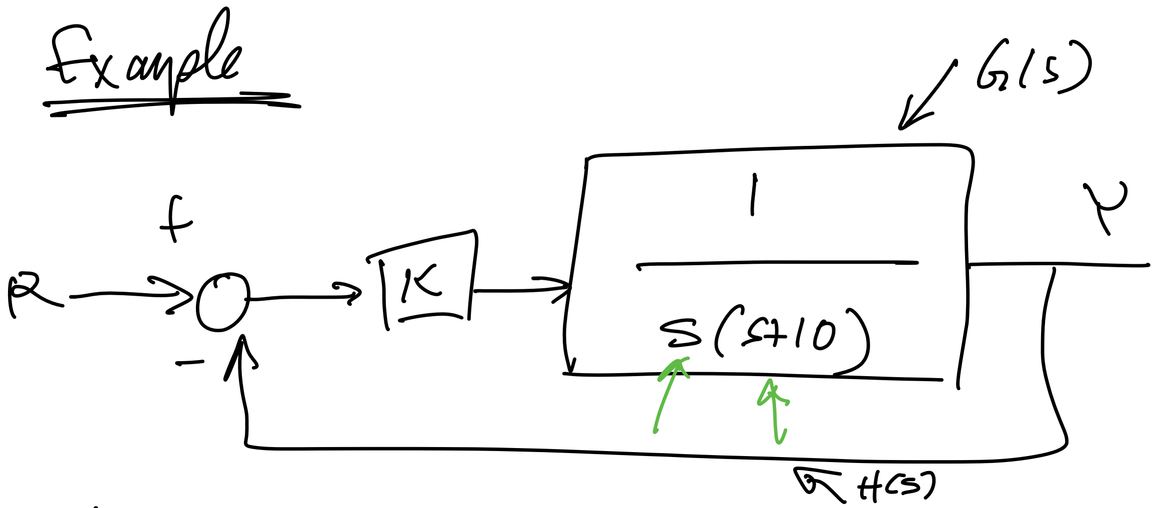
$$|G(s)H(s)| = \left|-\frac{1}{K}\right|$$

solve for K:

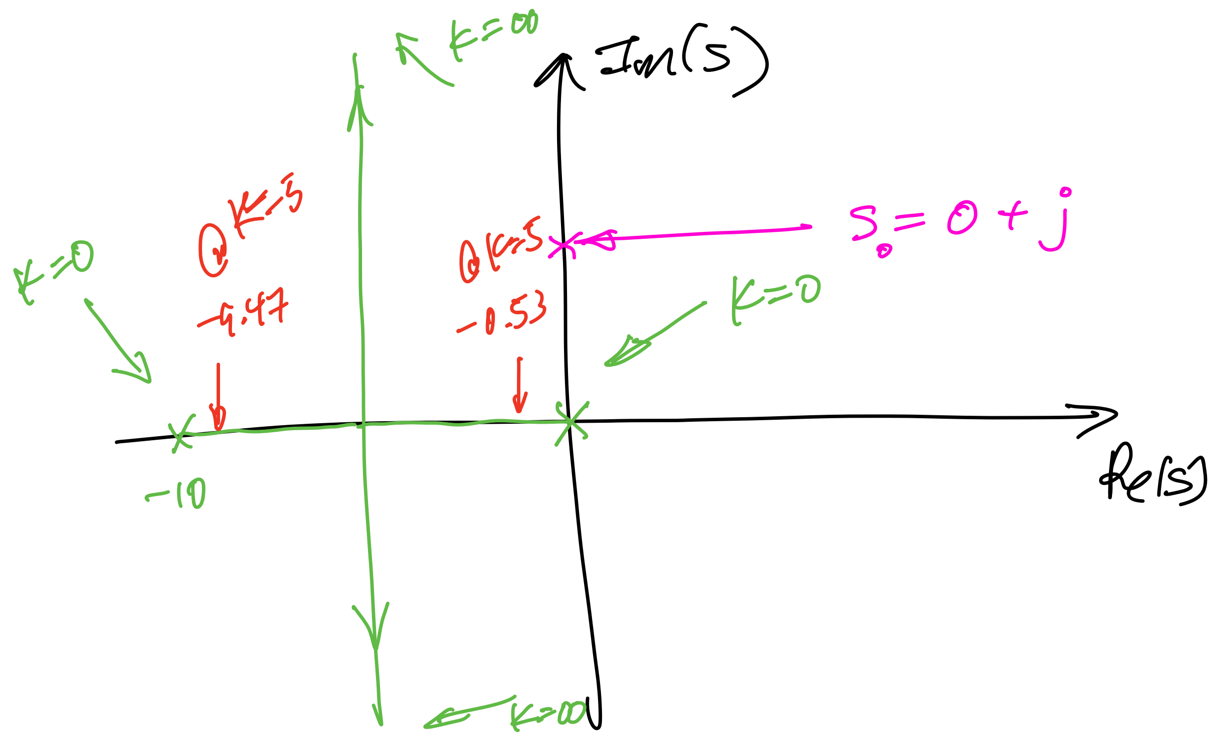
$$K = \left|\frac{-1}{G(s)H(s)}\right| = \frac{1}{|G(s)H(s)|}$$

Note that $G(s)H(s)$ is a complex number, so it will have magnitude and angle associated with it.

Example



show that $s = -9.47$ and -0.53 with $K=5$ statisfies the 2 conditions.



$S_0 = -9.47$ @ $K=5$

$S_0 = -0.53$ @ $K=5$

Apply Angle & Magnitude condition

(1) Angle cond:

$$\angle G(s_0)H(s_0) \stackrel{?}{=} 180^\circ + 360^\circ l \quad l = 0, \pm 1, \dots$$

$$\angle \left(\frac{1}{s_0(s_0+10)} \right) \stackrel{?}{=} 180^\circ + 360^\circ l$$

$$\angle \frac{1}{(-9.47)(-9.47+10)} = \angle \frac{1}{-5} = \angle -0.2$$

$$\Rightarrow 180^\circ \stackrel{?}{=} 180^\circ + 360^\circ l \quad \underline{\underline{l=0}}$$

$\Rightarrow$ <u>Yes</u>

Now let $s_0 = +j$

$$\angle G(s_0)H(s_0) = ?$$

$$\angle \frac{1}{s_0(s_0+10)} = \angle \frac{1}{j(j+10)}$$

$$\Rightarrow \angle \frac{1}{-1+10j} \cdot \frac{-1-j10}{-1-j10}$$

$$\Rightarrow \angle \frac{-1-10j}{1+100} = -\frac{1}{101} - \frac{10}{101}j$$

$$\Rightarrow \angle G(s_0)H(s_0) \neq 180^\circ + 360^\circ l$$

$\Rightarrow$ Not on root locus!

check magnitude cond:

$$K = \frac{1}{|G(s_0)\,H(s_0)|} = \frac{1}{\left|\frac{1}{(-9.47)(-9.47+10)} \cdot 1\right|}$$

$$= \frac{1}{|-1/5|}$$

$\underline{5 = 5}$

Yes $s_0 = -9.47$
is on root locus!