Intermediate Fluid Mechanics

Lecture 13: The Navier-Stokes Equations

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Chapter Overview

Lecture Objectives

Following up from Lecture 11 and 12, if one substitutes the stress-strain rate relation for a Newtonian fluid back into Cauchy's equation of motion, one can now obtain a closed system of equations.

$\Longrightarrow$ The formulation and resolution of these equations will be explored in this lecture.

Chapter Overview

1. Chapter Objectives
2. The Navier-Stokes Equations:
3. Couette Flow:
4. Transient Couette Flow:

Navier-Stokes Equation

By substituting the stress-strain rate relation for a Newtonian fluid back into Cauchy's equation of motion, leads to the so-called **Navier-Stokes** equations:

$$\rho \frac{Du_i}{Dt} = \rho g_i + \frac{\partial}{\partial x_j}\Big[-p\delta_{ij} + \mu\Big(2e_{ij} - \frac{2}{3}e_{mm}\delta_{ij}\Big)\Big], \tag{1}$$

$$\rho \frac{Du_i}{Dt} = \rho g_i - \frac{\partial}{\partial x_j}\Big(p\delta_{ij}\Big) + \frac{\partial}{\partial x_j}\Big(2\mu e_{ij}\Big) - \frac{2}{3}\frac{\partial}{\partial x_j}\Big(\mu e_{mm}\delta_{ij}\Big), \tag{2}$$

$$\rho \frac{Du_i}{Dt} = \rho g_i - \frac{\partial p}{\partial x_i} + 2\frac{\partial}{\partial x_j}\Big[\frac{\mu}{2}\Big(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\Big)\Big] - \frac{2}{3}\frac{\partial}{\partial x_j}\Big[\mu\delta_{ij}\frac{\partial u_m}{\partial x_m}\Big] \tag{3}$$

Navier-Stokes Equation (continued...)

Further, if temperature differences in the fluid are small, then one can assume that μ remains constant and hence it can be pulled out of the derivatives (recall that μ depends only very weakly on p),

$$\rho \frac{Du_i}{Dt} = \rho g_i - \frac{\partial p}{\partial x_i} + \mu \frac{\partial^2 u_i}{\partial x_j \partial x_j} + \mu \frac{\partial}{\partial x_i} \underbrace{\frac{\partial u_j}{\partial x_j}}_{\text{Term I}} - \frac{2}{3}\mu \frac{\partial}{\partial x_i} \underbrace{\frac{\partial u_m}{\partial x_m}}_{\text{Term II}} \tag{4}$$

and given that Term I and Term II are the same, the above equation gets reduced to the final form of the Navier-Stokes (N-S) equation,

$$\boxed{\rho \frac{Du_i}{Dt} = \rho g_i - \frac{\partial p}{\partial x_i} + \mu \frac{\partial^2 u_i}{\partial x_j \partial x_j} + \frac{\mu}{3} \frac{\partial}{\partial x_i} \left(\frac{\partial u_m}{\partial x_m} \right)} \tag{5}$$

Navier-Stokes Equation (continued...)

Note that for an incompressible flow, the last term $\partial u_m / \partial x_m$ is zero, and the N-S equation gets reduced to,

$$\boxed{\underbrace{\rho \frac{\partial u_i}{\partial t}}_{\text{Term I}} + \underbrace{u_j \frac{\partial u_i}{\partial x_j}}_{\text{Term II}} = \underbrace{\rho g_i}_{\text{Term III}} \underbrace{- \frac{\partial p}{\partial x_i}}_{\text{Term IV}} + \underbrace{\mu \frac{\partial^2 u_i}{\partial x_j \partial x_j}}_{\text{Term V}}}, \tag{6}$$

- Term I: Local rate of change of momentum.
- Term II: Advection of momentum.
- Term III: Body force.
- Term IV: Pressure gradient.
- Term V: Viscous diffusion. This is the term that we will use to describe the molecular transport of momentum between neighboring fluid particles due to the material property of viscosity.

Note: We will see in the example below that the time scale for momentum transport by molecular diffusion is quite long, *i.e.* viscous diffusion is a slow process.

Chapter Overview

Example: Couette Flow

Couette Flow,

- the fluid moves between two infinite parallel plates,
- the lower one is stationary and the upper plate moves at a constant velocity of U_0.
- The flow is one-dimensional because the plates are infinitely long.

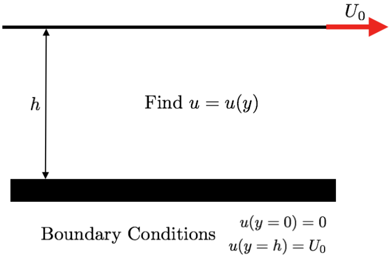

Figure: *Illustration of a Couette Flow.*

Example: Couette Flow (continued ...)

In this problem we will first solve for the steady flow solution and then go back and discuss the unsteady case.

$\Longrightarrow$To effectively solve this problem one needs to consider both, the **continuity equation** and the **N-S equation**.

(1) The Continuity equation

- Assuming an incompressible flow
- the flow is planar (or 2D), the third component of the velocity is directly zero.

$$\frac{\partial u_i}{\partial x_i} = 0 \rightarrow \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \tag{7}$$

Example: Couette Flow (continued ...)

Further, since the plates are infinitely long, $\partial/\partial x = 0$, *i.e.* the flow is independent of the x-direction,

$$\frac{\partial v}{\partial y} = 0 \Rightarrow v = C \qquad \text{where } C \text{ is a constant.} \tag{8}$$

Additionally, because of the non-slip condition at the lower wall,

$$v(y = 0) = 0, \tag{9}$$

therefore $C = 0$, and so $v = 0$ everywhere.

Example: Couette Flow (continued ...)

(2) The x-Momentum equation

$$\underbrace{\cancel{\rho\frac{\partial u}{\partial t}}}_{\text{Steady}} + \rho\, u \underbrace{\cancel{\frac{\partial u}{\partial x}}}_{\text{indep. of x}} + \underbrace{\cancel{\rho v\frac{\partial u}{\partial y}}}_{v=0} = \underbrace{\cancel{-\frac{\partial p}{\partial x}}}_{\text{No ext.}\partial p/\partial x} + \mu \underbrace{\cancel{\frac{\partial^2 u}{\partial x^2}}}_{\text{indep. of x}} + \mu\,\frac{\partial^2 u}{\partial^2 y}, \tag{10}$$

which reduces to,

$$\mu\,\frac{\partial^2 u}{\partial^2 y} = 0. \tag{11}$$

$+$ Note: The pressure gradient term is zero in this case because there is nothing driving a pressure difference. A pressure difference can occur, for example, in pipes where the flow is driven by a pump at one end. Pressure differences can also occur when there are obstacles in the flow or the cross-sectional area of a channel changes.

This simplified diff. equation can be solved for by integrating twice, which leads to

$$u = A\,y + B, \tag{12}$$

with A and B being integration constants.

The constants are determined by applying boundary conditions,

$$u(y=0) = 0 \Rightarrow 0 = A(0) + B \Rightarrow B = 0 \tag{13}$$
$$u(y=h) = U_0 \Rightarrow U_0 = A(h) \Rightarrow A = U_0/h. \tag{14}$$

Therefore, the solution (*i.e.* the velocity) is

$$\boxed{u(y) = \frac{U_0}{h}\,y.} \tag{15}$$

Example: Couette Flow (continued ...)

Next, if one divides both sides by U_0,

$$\frac{u}{U_0} = \frac{y}{h}. \tag{16}$$

Note that both sides of the equation are dimensionless (*i.e.* have no units). Hence, one can next define new non-dimensional variables,

$$\tilde{u} = \frac{u}{U_0} \qquad \text{and} \qquad \tilde{y} = \frac{y}{h}. \tag{17}$$

This is convenient because now one can plot the non-dimensional solution without having to specify values for the physical parameters h and U_0 (See Figure 2).

Physical interpretation of the steady solution

- In this case we saw that the N-S equations (momentum) reduced to the statement that viscous diffusion equals zero.
- We also saw that all of the inertia terms (local rate of change of momentum and advection) are zero as well, and the vertical velocity is zero.
- Therefore, in the steady-state flow, fluid particles are simply translating in the horizontal direction with a velocity that is directly proportional to their distance below the moving plate.

Physical interpretation of the steady solution

But, what does it mean that the solution is governed by viscous diffusion being zero?

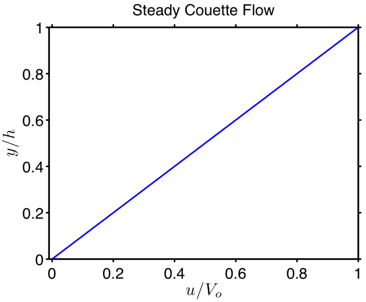

Figure: *Resultant Couette Flow.*

Answer: *It means that the flow reaches steady state when there is no more momentum transport between neighboring fluid particles. All of the momentum of the top plate has diffused through the gap to the lower plate. When this happens the flow reaches its steady-state behavior. This will be seen more clearly next when considering the transient Couette flow.*

Chapter Overview

Transient Couette Flow

In this case, we seek the solution (*i.e.* the velocity) as a function of both the vertical coordinate y and time t.

- At $t = 0$, the upper plate is at rest. $\Longrightarrow$ Next, there is an infinite acceleration to obtain a constant velocity U_0 for any $t > 0$.
- As the upper plate begins to move, because of non-slip condition, the adjacent fluid begins to move.
- Fluid particles transfer some of their momentum to the fluid particles below them that have no momentum. '*Chain Effect*'.
- The rate of transfer of momentum in this way is determined by the viscosity of the fluid (i.e. higher viscosity fluids will exhibit a faster rate).

Transient Couette Flow (continued)

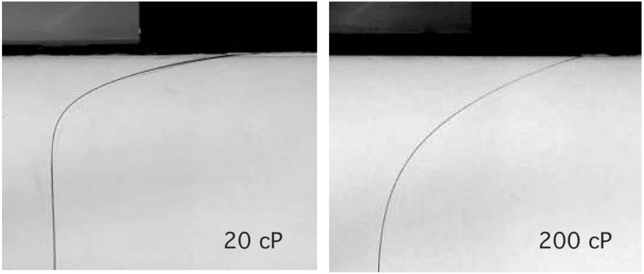

Figure: *Transient Couette Flow. The fluid on the left has a viscosity of* 20 *cp while the fluid on the right has a viscosity of* 200 *cp. Both plates are moving at the same speed. Clearly momentum penetrates deeper in case of higher viscosity.*

Solution to the N-S equations for transient Couette Flow

For the transient case:

- One still has that $\frac{\partial}{\partial x} = 0$ because the plates are still infinite.
  $\Longrightarrow$The continuity equation remains unchanged.
- Hence, from the continuity equation that $v = 0$.
- In the x-momentum equation, the unsteady term must now be retained, in addition to the viscous diffusion,

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2}, \tag{18}$$

(*Recall that* $\nu = \frac{\mu}{\rho}$ *is the kinematic viscosity.*)

Solution to the N-S equations for transient Couette Flow

The boundary and initial conditions for the problem remain unchanged:

$$u(y = 0, t) = 0 \tag{19}$$
$$u(y = h, t) = U_0 \tag{20}$$
$$u(y, t = 0) = 0. \tag{21}$$

Solution to the N-S equations for transient Couette Flow

To solve this problem, it is easier if one considers a change of variables:

$$\xi = \frac{y}{h} \qquad \text{and} \qquad \tau = \frac{\nu\, t}{h^2}, \tag{22}$$

- Here, τ here has nothing to do with the stress tensor.
- Also note that both ξ and τ represent non-dimensional independent variables.
- Similarly we can also define a non-dimensional velocity,

$$\phi(\xi, \tau) = \frac{u(y,t)}{U_0} - \frac{y}{h}. \tag{23}$$

Solution to the N-S equations for transient Couette Flow

In the new non-dimensional variables, the boundary and initial conditions become,

$$
\begin{aligned}
&\phi(\xi = 0, \tau) = 0 \\
&\phi(\xi = 1, \tau) = 0 \\
&\phi(\xi, \tau = 0) = \frac{u(y, t = 0)}{U_0} - \frac{y}{h} = \phi_0 - \xi = -\xi.
\end{aligned} \tag{24}
$$

Solution to the N-S equations for transient Couette Flow

Next, one must rewrite the x-momentum equation in terms of the new variables.

From equation 23 it follows that

$$u(u,t) = U_0\Big[\phi(\xi,\tau) + \frac{y}{h}\Big]. \tag{25}$$

And using the chain rule,

$$\frac{\partial u}{\partial t} = U_0 \frac{\partial \phi}{\partial \tau}\frac{\partial \tau}{\partial t} = \frac{U_0\,\nu}{h^2}\,\frac{\partial \phi}{\partial \tau},$$

$$\frac{\partial u}{\partial y} = U_0 \frac{\partial \phi}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{U_0}{h} = \frac{U_0}{h}\Big[\frac{\partial \phi}{\partial \xi} + 1\Big],$$

$$\frac{\partial}{\partial y}\Big(\frac{\partial u}{\partial y}\Big) = \frac{\partial}{\partial \xi}\Big(\frac{\partial u}{\partial y}\Big)\frac{\partial \xi}{\partial y} = \frac{U_0}{h}\frac{\partial^2 \phi}{\partial \xi^2}\Big(\frac{1}{h}\Big) = \frac{U_0}{h^2}\frac{\partial^2 \phi}{\partial \xi^2}. \tag{26}$$

Solution to the N-S equations for transient Couette Flow

Substituting equations 26 into equation 18, we find that

$$\frac{U_0 \nu}{h^2}\frac{\partial \phi}{\partial \tau} = \nu \left[\frac{U_0}{h^2}\frac{\partial^2 \phi}{\partial \xi^2} \right], \tag{27}$$

which reduces to,

$$\frac{\partial \phi}{\partial \tau} = \frac{\partial^2 \phi}{\partial \xi^2} \tag{28}$$

with

$$\phi(\xi = 0, \tau) = \phi(\xi = 1, \tau) = 0 \tag{29}$$

$$\phi(\xi, \tau = 0) = -\xi. \tag{30}$$

Solution to the N-S equations for transient Couette Flow

Because the boundary conditions in the new variables are conveniently homogeneous, the new differential equation,

$$\frac{\partial \phi}{\partial \tau} = \frac{\partial^2 \phi}{\partial \xi^2} \tag{31}$$

can be solved using separation of variables.

That is, we assume $\phi(\xi, \tau)$ to be the product of two functions, each of which is only dependent on one of the variables, hence:

$$\phi(\xi, \tau) = F(\xi) G(\tau), \tag{32}$$

Solution to the N-S equations for transient Couette Flow

Upon substitution in the normalized differential equation

$$\frac{\partial \phi}{\partial \tau} = \frac{\partial^2 \phi}{\partial \xi^2} \tag{33}$$

Results in:

$$F(\xi)G'(\tau) = F''(\xi)G(\tau) \tag{34}$$

which can be rearranged to,

$$\frac{G'(\tau)}{G(\tau)} = \frac{F''(\xi)}{F(\xi)}. \tag{35}$$

Note

Realize that $G' = \frac{dG}{d\tau}$ and that $F'' = \frac{d^2F}{d\xi^2}$ where the derivatives are ordinary since F and G are only functions of one variable.

Solution to the N-S equations for transient Couette Flow

In equation 35, both sides must be equal to the same constant since the left hand side is a function of τ only, while the right hand side is a function of ξ only.

Therefore, they can only be equal if both terms are independent of τ and ξ, *i.e.* a constant,

$$\frac{G'(\tau)}{G(\tau)} = \frac{F''(\xi)}{F(\xi)} = -\lambda^2. \tag{36}$$

It is for convenience only that the constant is chosen as $-\lambda^2$.

Solving now for $G(\tau)$,

$$\frac{dG}{d\tau} = -\lambda^2 G \qquad \longrightarrow \qquad \int \frac{dG}{G} = -\int \lambda^2 \, d\tau, \tag{37}$$

which upon integration one obtains that,

$$ln(G) = -\lambda^2 \tau + \tilde{C} \qquad \longrightarrow \qquad G(\tau) = C\, e^{-\lambda^2 \tau}. \tag{38}$$

Solution to the N-S equations for transient Couette Flow

Similarly, one can solve for F,

$$\frac{d^2 F}{d\xi^2} + \lambda^2 F = 0. \tag{39}$$

This is a second order ODE with constant coefficients, hence to solve it we assume a solution of the form,

$$F(\xi) = e^{r\xi}. \tag{40}$$

By substituting this solution back into the ODE one obtains that,

$$(r^2 + \lambda^2)\, e^{r\xi} = 0 \qquad \longrightarrow \qquad r = \pm i\,\lambda. \tag{41}$$

Therefore,

$$F(\xi) = A\, cos(\lambda\xi) + B\, sin(\lambda\xi), \tag{42}$$

where A and B are constants of integration.

Solution to the N-S equations for transient Couette Flow

At this point, one can use the boundary conditions to determine A and B,

$$\phi(\xi = 0, \tau) = F(\xi = 0)G(\tau) = 0 \longrightarrow F(\xi = 0) = 0 \tag{43}$$

$$\phi(\xi = 1, \tau) = F(\xi = 1)G(\tau) = 0 \longrightarrow F(\xi = 1) = 0 \tag{44}$$

Substituting the boundary conditions at $\xi = 0$ one obtains that

$$F(0) = A\, cos(0) + B\, \cancel{sin(0)} \qquad \Longrightarrow \qquad A = 0 \tag{45}$$

and for $\xi = 1$,

$$F(1) = B\, \cancel{sin(\lambda)} = 0. \tag{46}$$

At this point one cannot pick $B = 0$ because then $F(\xi) = 0$ would result in a trivial solution. Therefore we must have,

$$sin(\lambda) = 0 \qquad \Longrightarrow \qquad \lambda_n = n\,\pi \qquad \text{for } n = 1, 2, \ldots. \tag{47}$$

There are an infinite number of λ's that satisfy this criterion. The λ_n are termed the eigenvalues while $sin(\lambda_n\, \xi)$ are the eigenfunctions.

Solution to the N-S equations for transient Couette Flow

At this point we can join the solutions obtained independently, such that,

$$\phi_n(\xi,\tau) = C\,e^{-\lambda_n^2\,\tau}\,B\,sin(\lambda_n\,\xi), \qquad \text{with} \qquad \lambda_n = n\phi, \qquad n = 1, 2, \ldots . \tag{48}$$

Since the original differential equation is linear, by principle of superposition, we know that any linear combination of $\phi_n(\xi,\tau)$ is also a valid solution.

$\Longrightarrow$ the general solution is the sum of all possible combinations.

$$\phi(\xi,\tau) = \sum_{n=1}^{\infty}\phi_n(\xi,\tau) = \sum_{n}^{\infty} A_n\,sin(n\pi\,\xi)\,e^{-n^2\,\pi^2\tau} \tag{49}$$

Note, it is convenient to just combine the product of B and C into a single constant A_n.

Solution to the N-S equations for transient Couette Flow

Next we use the initial conditions to determine A_n,

$$\phi(\xi, \tau = 0) = \sum_{n}^{\infty} A_n \, sin(n\pi\,\xi)\, e^{-n^2\,\pi^2\,0} = -\xi$$
$$= \sum_{n}^{\infty} A_n \, sin(n\pi\,\xi) = -\xi. \tag{50}$$

At this point we determine A_n by taking advantage of the fact that the eigenfunctions are orthogonal. That is, $sin(n\pi\xi)$ and $cos(n\pi\xi)$ for $n = 1, 2, \ldots$ form a mutually orthogonal set of functions on the interval $0 \leq \xi \leq 1$.

Realize that the principle of orthogonality guarantees that,

$$\int_0^1 cos(n\pi\,\xi)\,cos(m\pi\xi)\,d\xi = \begin{cases} 0, & \text{if } m \neq n \\ 1, & \text{if } m = n \end{cases} \tag{51}$$

$$\int_0^1 cos(n\pi\,\xi)\,sin(m\pi\xi)\,d\xi = 0 \qquad \text{for all } m,\ n \tag{52}$$

$$\int_0^1 sin(n\pi\,\xi)\,sin(m\pi\xi)\,d\xi = \begin{cases} 0, & \text{if } m \neq n \\ 1, & \text{if } m = n \end{cases}. \tag{53}$$

Solution to the N-S equations for transient Couette Flow

Therefore, if we multiply both sides of equation 50 by $sin(m\pi\xi)$ and integrate over the interval $0 \leq \xi \leq 1$,

$$\int_0^1 sin(m\pi\xi) \sum_n^\infty A_n \, sin(n\pi\,\xi)\, d\xi = \int_0^1 -\xi \, sin(m\pi\xi) \tag{54}$$

$$\sum_n^\infty A_n \int_{0^1} sin(m\pi\xi)\, sin(n\pi\,\xi)\, d\xi = \left. \frac{-sin(m\pi\xi) + m\pi\xi \, cos(m\pi\xi)}{m^2\pi} \right|_{\xi=0}^{\xi=1} \tag{55}$$

$$A_m \int_0^1 sin^2(m\pi\xi)\, d\xi = \frac{m\pi\xi \, cos(m\pi\xi)}{m^2\pi} \tag{56}$$

$$\frac{1}{2} A_m = \frac{(-1)^m}{m\phi} \tag{57}$$

$$A_m = \frac{2\,(-1)^m}{m\pi} \tag{58}$$

Therefore, the solution is

$$\phi(\xi, \tau) = \sum_{n=1}^\infty \frac{2\,(-1)^m}{m\pi} \, sin(n\pi\xi)\, e^{-n^2\pi^2\tau} \tag{59}$$

Solution to the N-S equations for transient Couette Flow

This solution can also be written in terms of u,

$$\boxed{\frac{u(\xi,\tau)}{U_0} = \underbrace{\left[\sum_{n=1}^{\infty} \frac{2\,(-1)^m}{m\pi}\, sin(n\pi\xi)\, e^{-n^2\pi^2\tau}\right]}_{\text{Transient Correction}} + \underbrace{\xi}_{\text{Steady-State Sol.}}} \qquad (60)$$

From the above result, it can be seen that the unsteady solution of the Couette flow corresponds to the sum of the steady-state solution plus a transient correction.

Solution to the N-S equations for transient Couette Flow

The non-dimensional velocity profile is illustrated below for different non-dimensional times τ. Again this method of visualizing the solution is convenient because we do not have to specify any of the physical parameters directly.

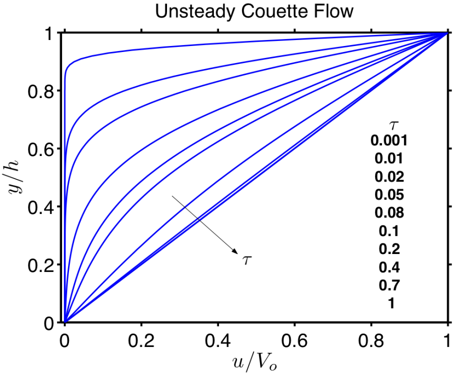

Figure: *Solution for the Non-Steady Couette flow at different time intervals.*

Solution to the N-S equations for transient Couette Flow

- Note that $\tau = 1$ corresponds to the steady-state condition. Let's define the actual time required to reach steady-state as t_s. Thus, from the definition $\tau = \frac{\nu t}{h^2}$, we have that

$$\frac{\nu t_s}{h^2} = 1 \qquad \Rightarrow \qquad t_s = \frac{h^2}{\nu}. \tag{61}$$

  Note that for a gap of $h = 1$cm and the fluid being water with $\nu = 1 \times 10^{-6}\, m^2/2$, then $t_s = 100\, s$. This shows how viscous diffusion is a slow process.

- (ii) As we sit at any fixed y position, the flow velocity appears to continuously increase in time (this is the $\partial u/\partial t$ term) toward the steady-state velocity. This happens because the momentum from the top plate continues to penetrate across the gap, until this momentum transfer reaches the bottom plate, at which point no further momentum transport can happen, which defines the steady state condition.