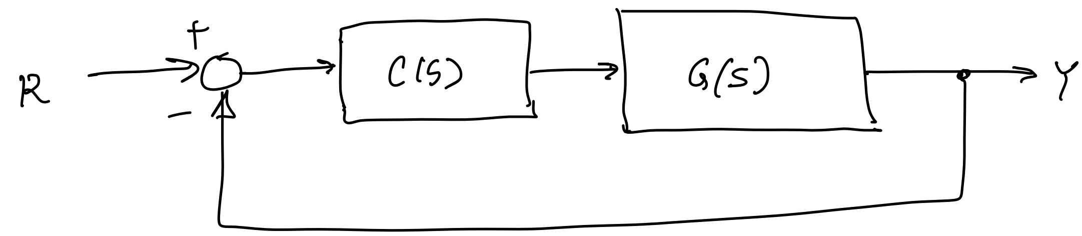
Lead Compensator Design Example



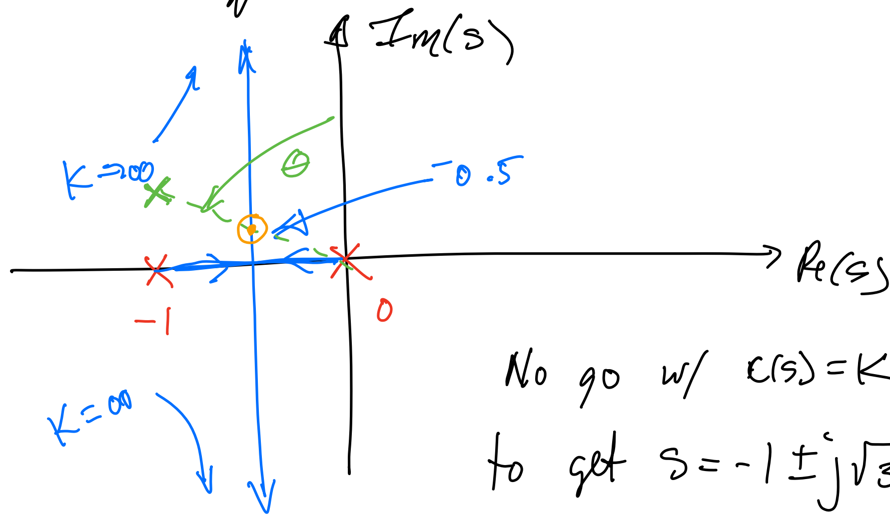
$$G(s) = \frac{1}{s(s+1)} \rightarrow s=0,\ s=-1 \qquad (2^{nd} \text{ order})$$

Design a lead compensator

$$C(s) = K\left(\frac{s+z}{s+p}\right)$$ s.t. Dominant closed loop poles are $s = -1 \pm j\sqrt{3}$

Case 1 proportional control $C(s) = K$

characteristic equation $1 + KG(s) = 0$



No go w/ $C(s)=K$ to get $s=-1 \pm j\sqrt{3}$

Case 2 lead compensator

$$C(s) = K\frac{s+z}{s+p} \qquad z<p \qquad G(s) = \frac{1}{s(s+1)}$$

want dominant closed-loop pole $\underbrace{s = -1 \pm j\sqrt{3}}$

$$(s+1-\sqrt{3}j)(s+1+\sqrt{3}j)$$

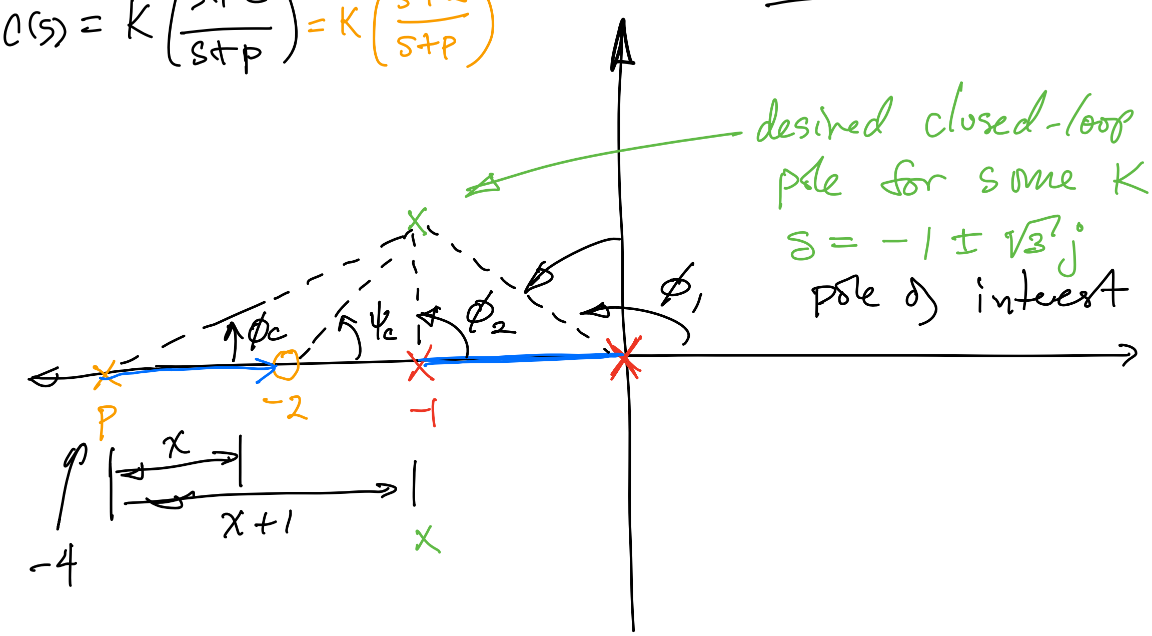
$$\Rightarrow \underbrace{s^2 + 2s + 4}$$

Dominant closed-loop dynamics

$$s^2 + 2\zeta\omega_n s + \omega_n^2$$

$$\Rightarrow \omega_n^2 = 4 \Rightarrow \underline{\underline{\omega_n = 2}}$$

$$C(s) = K\left(\frac{s+z}{s+p}\right) = K\left(\frac{s+2}{s+p}\right)$$



$$\phi_1 = 120^\circ \qquad \phi_2 = 90^\circ \qquad \psi_c = 60^\circ$$

Apply angle conditions:

$$\sum \psi_i - \sum \phi_j = 180^\circ + 360^\circ(l-1)$$

$$\Rightarrow \psi_c - (\phi_c + \phi_1 + \phi_2) = 180^\circ + 360^\circ(l-1)$$

$$\Rightarrow 60^\circ - \phi_c - 120^\circ - 90^\circ = 180^\circ + 360^\circ(l-1)$$

$$\Rightarrow \phi_c = -330^\circ \quad \text{or} \quad \phi_c = 30^\circ$$

Let $p = x+1$ distance from -1 pole

$$\tan \phi_c = \frac{\sqrt{3}}{x+1} \Rightarrow x+1 = \frac{\sqrt{3}}{\tan(30^\circ)}$$

$$\Rightarrow x = 2 \Rightarrow p = 2+2 = 4$$

Thus:

$$C(s) = K\left(\frac{s+2}{s+4}\right)$$

Gain / magnitude condition:

$$1 + K\left(\frac{s+2}{s+4}\right)G(s) = 0$$

$$\Rightarrow K = -\frac{1}{\left(\frac{s+2}{s+4}\right)G(s)}\Bigg|_{s=-1\pm\sqrt{3}j}$$

$$K = \left|\frac{s(s+1)(s+4)}{s+2}\right|_{s=-1+\sqrt{3}j}$$

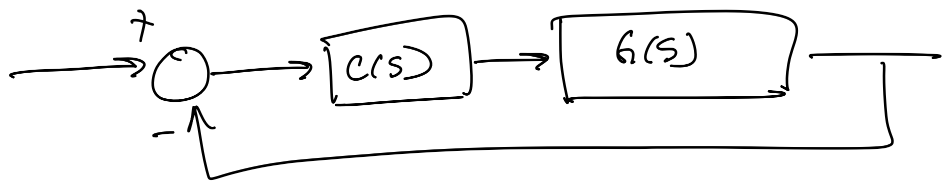
$$K = \left|\frac{(-1+\sqrt{3}j)(-1+\sqrt{3}j+1)(-1+\sqrt{3}j+4)}{-1+\sqrt{3}j+2}\right|$$

$$\underline{\underline{K = 6}}$$

Controller: $\boxed{C(s) = 6\left(\frac{s+2}{s+4}\right)}$

Lag Example

$$G(s) = \frac{1}{s(s+2)}$$



$$C(s) = K\,\frac{s+z}{s+p} \qquad z > p$$

Design lag controller so that $s = -1 \pm j$ and s.s. error to a unit ramp is less than 0.2.

Pick small $z = 0.1$, then we find pole p that satisfies our requirements.

Look at s.s. error:

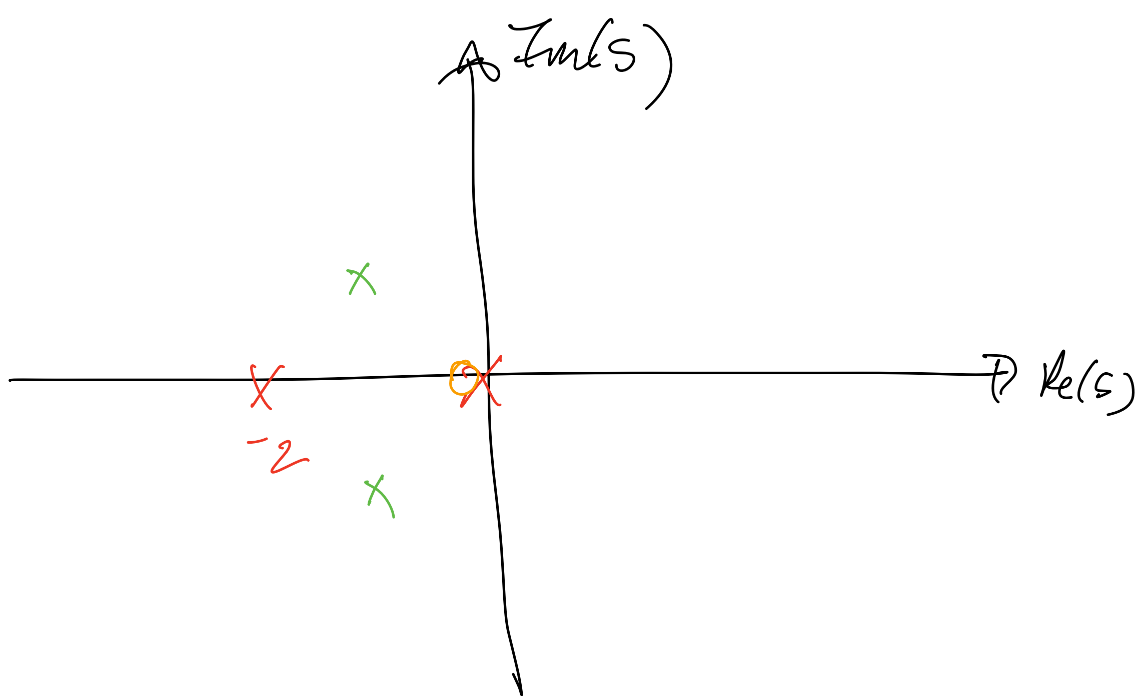
$$e_{ss} \leq 0.2 = \frac{1}{K_v}$$

$$\Rightarrow \frac{1}{K_v} \le 0.2 \quad \text{where } K_v = \lim_{s \to 0} sC(s)G(s)$$

$$\Rightarrow K_v = \lim_{s \to 0} s \left(\frac{s+0.1}{s+p}\right) K \left(\frac{1}{s(s+2)}\right)$$

$$\Rightarrow K_v = \frac{0.1K}{2p}$$

$$\Rightarrow e_{ss} = \frac{1}{K_v} = \frac{2p}{0.1K} \le 0.2 \quad \#1$$



$$K\left(\frac{s+0.1}{s+p}\right)$$

$$1 + C(s)\,G(s) = 0$$

$$\Rightarrow K = \left.\frac{-1}{\left(\frac{s+0.1}{s+p}\right)G(s)}\right|_{s=-1+j}$$

$$\#2 \quad K = \left.\frac{s(s+2)(s+p)}{(s+0.1)}\right|_{s=-1+j}$$

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solve simultaneously:

$$p \simeq 0.02$$

$$K \simeq 100p \cong 2$$

$$\boxed{C(s) = 2\left(\frac{s+0.1}{s+0.02}\right)}$$