Optimal Control HW1

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Problem 1) What do you expect to get out of this class? Please be specific about your expectations.

I expect to gain the knowledge to apply optimal control to unmanned and manned aerial vehicles, specifically for fixed wing platforms in my own future industry work aspirations. I also hope to learn the skills in optimal control to apply to my own research in the FPC lab with Dr. He where we aim to achieve control of slender delta wings using active flow control rather than traditional control surfaces. If we achieve this goal I want it to be the optimal solution to our research interest not just a solution.

Problem 2) In class, Dr. Leang described open and closed-loop control and showed the basic block diagrams for the two structures. Prove the benefits of closed-loop control over open-loop control through the sensitivity of each structure with respect to a change in a system parameter. Hint: You may recall discussions in your prior classical control course about sensitivity analysis and how it can be used to show the benefits of closed-loop control.

let $G(s) = \frac{kB}{s+1}$ (1st order system with proportional gain k)

Open Loop

$S_P^F = \frac{P}{F}\frac{\partial F}{\partial P}$

$S_B^{G(s)} = \frac{B}{G(s)}\frac{\partial G(s)}{\partial B}$

$S_B^{G(s)} = \frac{B}{\frac{kB}{s+1}}(\frac{k}{s+1})$

$S_B^{G(s)} = \frac{B}{B} = 1$

$\Rightarrow$ For open loop control, the system is 100% sensitive to parameter B regardless of whatever controller gain k might be.

Closed Loop

$S_P^F = \frac{P}{F}\frac{\partial F}{\partial P}$

$S_B^{G_{cl}(s)} = \frac{B}{G_{cl}(s)}\frac{\partial G_{cl}(s)}{\partial B}$

$G_{cl}(s) = \frac{kB}{s+1+kB}$

$S_B^{G_{cl}(s)} = \frac{B}{\frac{kB}{s+1+kB}}(\frac{k(s+1+kB)-kB(k)}{(s+1+kB)^2})$

$S_B^{G_{cl}(s)} = \frac{(s+1+kB)-Bk}{(s+1+kB)}$

$S_B^{G_{cl}(s)} = \frac{(s+1+kB)-Bk}{(s+1+kB)} = \frac{s+1+kB}{s+1+kB} - \frac{Bk}{s+1+kB} = 1 - \frac{Bk}{s+1+kB}$

Evaluated at s = 0 to understand steady state behavior,

$S_B^{G_{cl}(0)} = 1 - \frac{Bk}{1+kB}$

$\Rightarrow$ For closed loop control $S \in [0,1], k > 0$, meaning we can effect the system sensitivity to parameter B with some controller gain k.

Problem 3) In class, the transfer function for a transformed state $\overline{x} = Px$ from a state-space representation was by:

$$Y(s) = [\hat{C}(sI - \hat{A})^{-1}\hat{B} + D]U(s) = \hat{G}(s)U(s)$$

where $\hat{A} = PAP^{-1}$ and $\hat{B} = PB$ Note that the matrix P is invertible. Show that $\hat{G}(s) = C(sI - A)^{-1}B + D$. What does this mean exactly in terms of the poles of the transformed and original system?

$\hat{G} = \hat{C}(sI - \hat{A})^{-1}\hat{B} + D$

$\hat{A} = PAP^{-1}$

$\hat{B} = PB$

$\hat{G} = \hat{C}(sI - PAP^{-1})^{-1}PB + D$

$\hat{G} = \hat{C}(PP^{-1}sI - PAP^{-1})^{-1}PB + D$

$\hat{G} = \hat{C}(PsIP^{-1} - PAP^{-1})^{-1}PB + D$

$\hat{G} = \hat{C}(P(sI - A)P^{-1})^{-1}PB + D$

Identity: $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$

$\hat{G} = \hat{C}(P(sI - A)^{-1}P^{-1})PB + D$

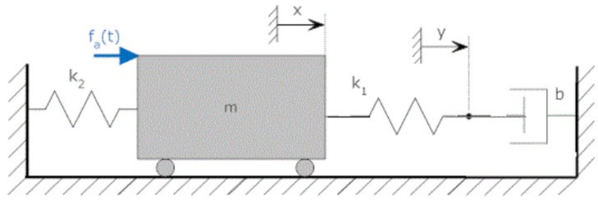
$\hat{G} = \hat{C}P(sI - A)^{-1}IB + D$

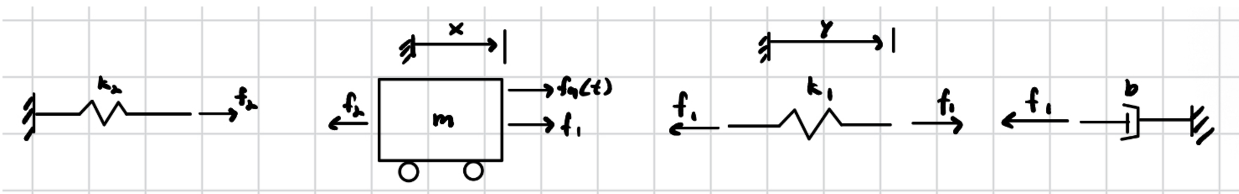
$\hat{G} = C(sI - A)^{-1}B + D$, $C = \hat{C}P$

$\hat{G} = C(sI - A)^{-1}B + D$

The transfer functions will be the same after the transformation which means their poles are also the same.

Problem 4) Consider the following mechanical system where the input is fa(t) and let the states be x, dx/dt, and y. Derive the state-space equation for this system. Show all your work, including relevant free body diagrams





$(1) f_2 = k_2 x$

$(2) f_a(t) + f_1 - f_2 = m\ddot{x}$

$(3) f_1 = k_1(y - x)$

$(4) f_1 = -b\dot{y}$

$(3) = (4) \Rightarrow -b\dot{y} = k_1(y - x)$

$0 = b\dot{y} + k_1(y - x)$

$(1),(2),(3) \Rightarrow f_a + k_1(y - x) - k_2 x = m\ddot{x}$

$f_a + k_1 y - (k_1 + k_2)x = m\ddot{x}$

$States : \dot{x}, x, y$

$\ddot{x} = \frac{k_1}{m}y - \frac{(k_1+k_2)}{m}x + \frac{f_a}{m}$

$\dot{y} = -\frac{k_1}{b}y + \frac{(k_1)}{b}x$

let $\dot{q_1} = \ddot{x} \Rightarrow q_1 = \dot{x}$

let $\dot{q_2} = \dot{y} \Rightarrow q_2 = y$

let $q_3 = x \Rightarrow \dot{q_3} = \dot{x} = q1$

$\dot{q_1} = \frac{k_1}{m}q_2 - \frac{(k_1+k_2)}{m}q_3 + \frac{f_a}{m}$

$\dot{q_2} = -\frac{k_1}{b}q_2 + \frac{(k_1)}{b}q_3$

$\dot{q_3} = q_1$

$$\begin{bmatrix}\dot{q_1}\\ \dot{q_2}\\ \dot{q_3}\end{bmatrix} = \begin{bmatrix}0 & \frac{k_1}{m} & -\frac{(k_1+k_2)}{m}\\ 0 & -\frac{k_1}{b} & \frac{k_1}{b}\\ 1 & 0 & 0\end{bmatrix}\begin{bmatrix}q_1\\ q_2\\ q_3\end{bmatrix} + \begin{bmatrix}\frac{1}{m}\\ 0\\ 0\end{bmatrix} f_a \quad \text{or} \quad \begin{bmatrix}\ddot{x}\\ \dot{y}\\ \dot{x}\end{bmatrix} = \begin{bmatrix}0 & \frac{k_1}{m} & -\frac{(k_1+k_2)}{m}\\ 0 & -\frac{k_1}{b} & \frac{k_1}{b}\\ 1 & 0 & 0\end{bmatrix}\begin{bmatrix}\dot{x}\\ y\\ x\end{bmatrix} + \begin{bmatrix}\frac{1}{m}\\ 0\\ 0\end{bmatrix} f_a$$

Problem 5)

(a) Determine whether the system given by the following state-space representation is controllable:

$$A = \begin{bmatrix} -2 & 1 \\ -1 & 3 \end{bmatrix}; B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

(b) Verify this result using MATLAB and include your Matlab results.

a)

n = 2

$$C = \begin{bmatrix} B & AB \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & -2 \\ 0 & -1 \end{bmatrix}$$

$\Rightarrow$ Rank(C) = 2 $\Rightarrow$ Full rank $\Rightarrow$ Controllable

b)

```
Co =

     1    -2
     0    -1


rank_Co =

     2
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(Code in Appendix)

Problem 6) Consider the following system with state-space representation:

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}; B = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

(a) What are the poles of the system?
(b) Design a full-state feedback regulator for the system such that the closed-loop poles are at s = -0.5 ± i. Hint: use pole placement technique, such as Ackermann's approach.
(c) Simulate the response of the closed-loop system using Matlab and show plots of the evolution of the states, x1 and x2, vs. time for initial conditions x1(0)=1 and x2(0)=0. Label all plots and include all Matlab code, etc.

a)

sI - A

$$= \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} s-1 & 0 \\ 0 & s+2 \end{bmatrix}$$

$$(s-1)(s+2) = 0$$

$$s = 1 \Rightarrow Unstable$$

$$s = -2 \Rightarrow Stable$$

$$C = \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix} \Rightarrow Rank(C) = 2 \Rightarrow Controllable$$

b)

Desired Poles: $s = -0.5 \pm i$

$$(s - (-0.5 + i))(s - (-0.5 - i))$$

$s^2 + s + 1.25 = 0 \Rightarrow$ Desired Characteristic Equation

$$A_{des}(A) = A^2 + A + 1.25I$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} + \begin{bmatrix} 1.25 & 0 \\ 0 & 1.25 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} + \begin{bmatrix} 1.25 & 0 \\ 0 & 1.25 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} 1.25 & 0 \\ 0 & 1.25 \end{bmatrix}$$

$$= \begin{bmatrix} 3.25 & 0 \\ 0 & 3.25 \end{bmatrix}$$

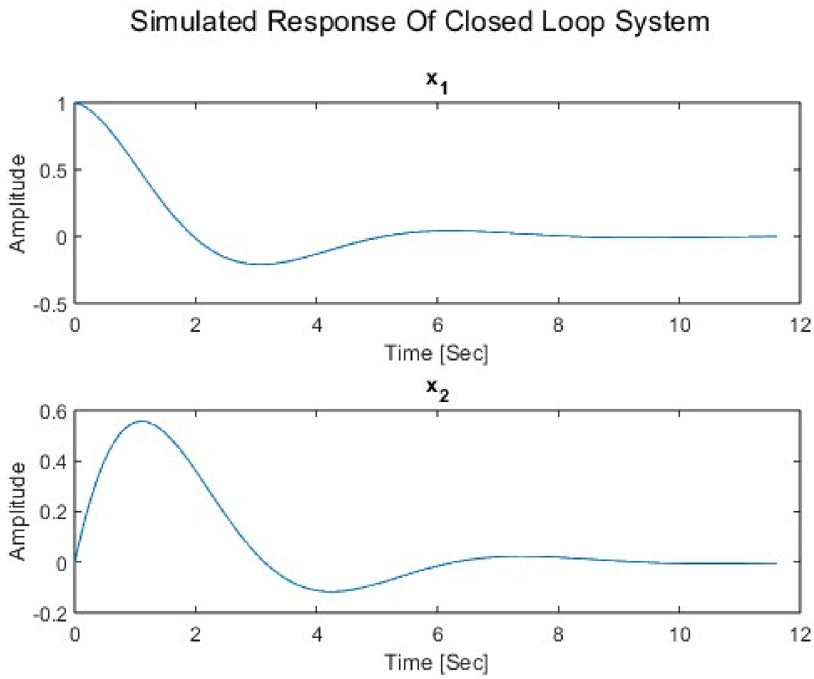
$$K = \begin{bmatrix} 0 & 1 \end{bmatrix} C^{-1} A_{des}$$

$$C^{-1} = \frac{1}{2-(-1)} \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2/3 & -1/3 \\ 1/3 & 1/3 \end{bmatrix}$$

$$K = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 2/3 & -1/3 \\ 1/3 & 1/3 \end{bmatrix} \begin{bmatrix} 3.25 & 0 \\ 0 & 3.25 \end{bmatrix}$$

$$K = \begin{bmatrix} 1.083 & 1.083 \end{bmatrix}$$

c)



(Code in Appendix)

Problem 7)
(a) Do this Problem by hand:

Calculate the 1-norm of $B = \begin{bmatrix} 5 & -4 & 2 \\ -1 & 2 & 3 \\ -2 & 1 & 0 \end{bmatrix}$

(b) Do this problem by hand:

Calculate the infinity-norm of $B = \begin{bmatrix} 5 & -4 & 2 \\ -1 & 2 & 3 \\ -2 & 1 & 0 \end{bmatrix}$

a)

$$||B||_1 =_{1\leq j\leq n}^{Max} \textstyle\sum_{i=1}^{m} |b_{ij}|$$

$$= \begin{bmatrix} (5+1+2) & (4+2+1) & (2+3+0) \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 7 & 5 \end{bmatrix}$$

$$||B||_1 = 8$$

b)

$$||B||_\infty =_{1\leq j\leq m}^{Max} \textstyle\sum_{i=1}^{n} |b_{ij}|$$

$$= \begin{bmatrix} (5+4+2) \\ (1+2+3) \\ (2+1+0) \end{bmatrix}$$

$$= \begin{bmatrix} 11 \\ 6 \\ 3 \end{bmatrix}$$

$$||B||_\infty = 11$$

Problem 8) Determine the definiteness of the form $Q(x_1, x_2, x_3) = 3x_1^2 + 2x_2^2 + 3x_3^2 - 2x_1x_2 - 2x_2x_3$

$Q = x^T Ax$

$$x^T Ax = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= \begin{bmatrix} ax_1 + dx_2 + gx_3 & bx_1 + ex_2 + hx_3 & cx_1 + fx_2 + ix_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= ax_1^2 + dx_1x_2 + gx_3x_1 + bx_1x_2 + ex_2^2 + hx_3x_2 + cx_1x_3 + fx_2x_3 + ix_3^2$$

$$3x_1^2 + 2x_2^2 + 3x_3^2 - 2x_1x_2 - 2x_2x_3 = ax_1^2 + dx_1x_2 + gx_3x_1 + bx_1x_2 + ex_2^2 + hx_3x_2 + cx_1x_3 + fx_2x_3 + ix_3^2$$

$a = 3, e = 2, i = 3$

$(d + b)(x_1x_2)$

$d = b$

$2b(x_1x_2)$

$\Rightarrow b = d = -1$

$(h + f)(x_2x_3)$

$f = h$

$2h(x_2x_3)$

$\Rightarrow f = h = -1$

$$A = \begin{bmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{bmatrix}$$

$eig(A) = 1, 3, 4 > 0 \Rightarrow$ Positive Definite

Appendix

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clear, clc, close all

%% Probelm 5
A = [-2 1; -1 -3];
B = [1; 0];

Co = ctrb(A,B)
rank_Co = rank(Co)

%% Problem 6
A = [1 0; 0 -2];
B = [1;-1];
K = [0 1]* [(2/3) (-1/3); (1/3) (1/3)] * [3.25 0; 0 3.25];

ACL = (A-B*K);
c = [1 0; 0 1];
x0 = [1;0];
sys = ss(ACL,[],c,[]);

[y,tOut,x] = initial(sys,x0);

figure
subplot(2,1,1)
plot(tOut, y(:,1))
xlabel("Time [Sec]"); ylabel("Amplitude"); title("x_1")
subplot(2,1,2)
plot(tOut, y(:,2))
xlabel("Time [Sec]"); ylabel("Amplitude"); title("x_2")
sgtitle("Simulated Response Of Closed Loop System")

%% Problem 8
eig([3 -1 0; -1 2 -1; 0 -1 3])
```