

# Intermediate Fluid Mechanics

## Lecture 27: Vorticity Dynamics

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# Chapter Overview

- 1 Chapter Objectives
- 2 Vorticity
- 3 Vorticity Magnitude at a stationary wall
- 4 Vorticity flux at solid walls
- 5 Flux of vorticity at the wall due to pressure gradient
- 6 Role of viscosity in vorticity generation at the wall

# Lecture Objectives

In this new chapter, we will study the concept of Vorticity, Vorticity Dynamics and Vorticity at a solid wall.

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# Vorticity

**Mathematically**, the vorticity is defined as the curl of the velocity,

$$\vec{\omega} = \vec{\nabla} \times \vec{u}. \quad (1)$$

**Physically**, the vorticity represents twice the instantaneous angular rotation rate of a fluid particle.

From mathematics, we also know that the divergence of the curl of any vector is zero, hence

$$\vec{\nabla} \cdot \vec{\omega} = 0. \quad (\text{solenoidal condition of vorticity}) \quad (2)$$

This is analogous to **the condition of incompressibility** of the velocity field.

## Note:

It is important to remember that  $\vec{\nabla} \cdot \vec{\omega} = 0$  for all flows regardless of the type of fluid and regardless whether the flow is incompressible or not.

# Vorticity Solenoidal Condition, ( $\vec{\nabla} \cdot \vec{\omega} = 0$ )

**The solenoidal condition on the vorticity field** sets topological rules for vortex lines.

## Vortex lines:

- These are curves in the fluid such that their corresponding tangents at any point gives the direction of the local vorticity.
- In this way, vortex lines are analogous to streamlines for the velocity field.
- Vortex lines passing through any closed curve form a tubular surface, called a vortex tube.

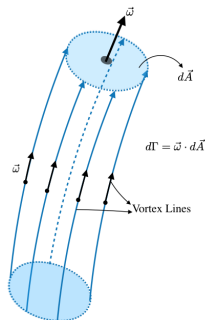


Figure: Illustration of vortex lines conforming a vortex tube.

# Vorticity Solenoidal Condition, ( $\vec{\nabla} \cdot \vec{\omega} = 0$ )

The circulation around a narrow vortex tube is

$$d\Gamma = \vec{\omega} \cdot d\vec{A}, \quad (3)$$

This is also called the **strength of the vortex tube**.

⇒ Based on the solenoidal condition, it results that:

- 1 vortex lines cannot end within a fluid.
- 2 vortex lines form closed loops.
- 3 vortex lines may extend to infinity.
- 4 vortex lines may intersect a wall where the vorticity is zero.

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# Magnitude of vorticity at a stationary wall

Let's examine the connection between the viscous shear stress at the wall and vorticity.

For this, recall that the vorticity vector is

$$\vec{\omega} = \vec{\nabla} \times \vec{u} = \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{bmatrix} = \quad (4)$$

$$= \underbrace{\left( \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \right)}_{\omega_x} \hat{i} + \underbrace{\left( \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \right)}_{\omega_y} \hat{j} + \underbrace{\left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right)}_{\omega_z} \hat{k}. \quad (5)$$

Let's consider two-dimensional flow in the  $x - y$  plane over a solid surface. In this case,  $\vec{\omega} = \omega_z \hat{k}$ .

# Magnitude of vorticity at a stationary wall

From the no-slip condition at the wall, we have that

$$u = v = 0 \quad \text{at} \quad y = 0 \quad (6)$$

and because the wall lies along the x-coordinate at  $y = 0$ ,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial x} = 0 \quad \text{at} \quad y = 0 \quad (7)$$

and

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x^2} = 0 \quad \text{at} \quad y = 0. \quad (8)$$

Also, for an incompressible fluid the continuity equation gives

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \quad \longrightarrow \quad \text{hence,} \quad \frac{\partial v}{\partial y} = 0. \quad (9)$$

# Magnitude of vorticity at a stationary wall

The deviatoric shear stress is defined as,

$$\tau_{ij} = 2\mu e_{ij} = 2\mu \left[ \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \right]. \quad (10)$$

which in the 2D case becomes,

$$\tau = \begin{pmatrix} 2\mu \frac{\partial u}{\partial x} & \mu \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \\ \mu \left( \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) & 2\mu \frac{\partial v}{\partial y} \end{pmatrix} \quad (11)$$

At the wall,  $\tau_w \equiv \tau_{xy}$  is  $\tau_w = \mu \frac{\partial u}{\partial y}$  at  $y = 0$ . In the 2D case,

$$\vec{\omega} = \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) \hat{k} \quad (12)$$

Therefore, at the wall, the vorticity vector is,

$$\vec{\omega}_0 = -\frac{\partial u}{\partial y} \hat{k} = -\mu^{-1} \tau_w \hat{k} \quad \text{at} \quad y = 0. \quad (13)$$

# Magnitude of vorticity at a stationary wall

Notice that  $\tau_w$  is directed along the  $x$ -axis while the vorticity lies along the  $z$ -axis. In general, we find that,

$$\vec{\omega}_0 \cdot \vec{t}_w = 0 \quad \text{along a solid surface,} \quad (14)$$

where  $\vec{t}_w$  is the traction force associated with the wall shear stress, *i.e.*  $\vec{t}_w = \hat{n} \cdot \tau_w$ .

One can also write the above expression as,

$$\hat{n} \cdot \tau_w = -\mu \hat{n} \times \vec{\omega}_0 \quad \text{at a solid surface.} \quad (15)$$

$\Rightarrow$  This states that the vorticity at the wall lies perpendicular to the wall shear stress.

# Magnitude of vorticity at a stationary wall

From the previous results:

- The wall shear stress is indicative of the magnitude of the vorticity at the wall; **but the wall shear stress does not actually generate vorticity.**
- Next, we examine how vorticity is generated or lost at a solid wall.
- One can think of a solid wall as a sink or source of vorticity.
- The flux of vorticity into or out of the flow at a solid wall plays an important role in the overall fluid dynamics, especially in turbulent wall-bounded flows.

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# Vorticity flux at solid walls

We have already showed that the boundary layer equations evaluated at the wall,  $y = 0$ , reduce to

$$\text{x-momentum at wall: } \frac{1}{\rho} \frac{\partial p}{\partial x} \Big|_{y=0} = \nu \frac{\partial^2 u}{\partial y^2} \Big|_{y=0} \quad (16)$$

$$\text{y-momentum at wall: } \frac{1}{\rho} \frac{\partial p}{\partial z} \Big|_{y=0} = \nu \frac{\partial^2 w}{\partial y^2} \Big|_{y=0} \quad (17)$$

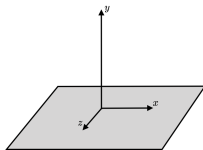


Figure: Illustration of the reference frame.

## Note:

- In the above relations we still assume the boundary layer approximations are valid, so  $\partial p / \partial y \approx 0$  across the boundary layer.
- Also, we assume that the solid wall lies in the  $x - z$  plane, as shown in the Figure. The surface does not necessarily have to be flat; it can be curved as long as the coordinate system follows the contour of the surface.

# Vorticity flux at solid walls

Furthermore, at the wall, the vorticity is

$$\vec{\omega}_0 = \underbrace{\left( \frac{\partial w}{\partial y} \Big|_{y=0} - \frac{\partial v}{\partial z} \Big|_{y=0} \right)}_{\omega_x} \hat{i} + \underbrace{\left( \frac{\partial u}{\partial z} \Big|_{y=0} - \frac{\partial w}{\partial x} \Big|_{y=0} \right)}_{\omega_y} \hat{j} + \underbrace{\left( \frac{\partial v}{\partial x} \Big|_{y=0} - \frac{\partial u}{\partial y} \Big|_{y=0} \right)}_{\omega_z} \hat{k} \quad (18)$$

Since  $\frac{\partial v}{\partial x} \Big|_{y=0} = 0$  and  $\frac{\partial v}{\partial z} \Big|_{y=0} = 0$  due to the non-slip condition, we can rewrite the previous relations as

$$\text{x-momentum at wall: } \frac{1}{\rho} \frac{\partial p}{\partial x} \Big|_{y=0} = -\nu \frac{\partial \omega_z}{\partial y} \Big|_{y=0} \quad (19)$$

$$\text{y-momentum at wall: } \frac{1}{\rho} \frac{\partial p}{\partial z} \Big|_{y=0} = \nu \frac{\partial \omega_x}{\partial y} \Big|_{y=0} \quad (20)$$

# Vorticity flux at solid walls

Lighthill (1963) defined the term diffuse flux density of vorticity outward from the wall as

$$-\nu \frac{\partial \vec{\omega}}{\partial y} \Big|_{y=0} \quad \text{'Vorticity flux at the wall'} \quad (21)$$

We will just refer to the above quantity as the vorticity flux at the wall.

Question: Why is this an appropriate name for this quantity?

# Vorticity flux at solid walls

Answer: Consider a Taylor's series expansion of  $\omega_z$  about the wall. The vorticity at some small distance  $\Delta y$  above the wall is,

$$\omega_z(\Delta y) = \omega_z(y=0) + \left. \frac{\partial \omega_z}{\partial y} \right|_{y=0} \Delta y + \dots \quad (22)$$

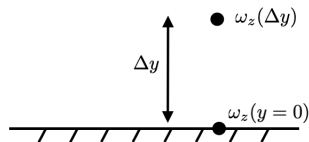


Figure: Finite differences near the wall.

Therefore, the vorticity of the flow in the neighborhood of the wall is the value of the vorticity at the wall plus a contribution from  $\left. \frac{\partial \omega_z}{\partial y} \right|_{y=0}$ .

# Vorticity flux at solid walls

Since the quantity  $\left. \frac{\partial \omega_3}{\partial y} \right|_{y=0}$  increases (or decreases, depending on the sign) the vorticity in the flow relative to the wall value,

$\implies$  One can think of this quantity as acting like a flux of vorticity at the wall, *i.e.* an injection of vorticity at the wall.

By evaluating the  $x$  and  $z$  components of the momentum equation at the wall, we obtained the wall-normal flux of  $\omega_z$  and  $\omega_x$  at the wall.

Question: What about the flux of  $\omega_y$  at the wall?

# Vorticity flux at solid walls

Answer: The y-momentum equation cannot give us this information because it reduces simply to  $\frac{\partial p}{\partial y} = 0$  for boundary layer. Therefore we need some other information. One can use the solenoidal condition,  $\vec{\nabla} \cdot \vec{\omega} = 0$ ,

$$\frac{\partial \omega_x}{\partial x} + \frac{\partial \omega_y}{\partial y} + \frac{\partial \omega_z}{\partial z} = 0 \quad (23)$$

Which evaluated at the wall,

$$\left. \frac{\partial \omega_y}{\partial y} \right|_{y=0} = - \left. \frac{\partial \omega_x}{\partial x} \right|_{y=0} - \left. \frac{\partial \omega_z}{\partial z} \right|_{y=0} \quad (24)$$

$\Rightarrow$  One can see that the flux of  $\omega_y$  from the wall depends on the distribution of  $\omega_x$  and  $\omega_z$  at the wall.

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# Flux of vorticity at the wall due to pressure gradient

- Let's consider a 2D laminar boundary layer wherein the only component of vorticity is  $\omega_z$ .
- Further, since  $\omega_z \approx -\frac{\partial u}{\partial y}$  in the boundary layer and since  $\frac{\partial u}{\partial y} > 0$  in the boundary layer,
- $\implies$  It results in:  $\omega_z < 0$  everywhere in the boundary layer.

Now, let's consider the flux of  $\omega_z$  at the wall due to  $\frac{\partial p}{\partial x}$ ,

$$\frac{\partial p}{\partial x} = -\mu \frac{\partial \omega_z}{\partial y} \Big|_{y=0} \quad (25)$$

+ Note:  $\frac{\partial p}{\partial x}$  at the wall is the same as that at the edge of the boundary layer since pressure is not a function of  $y$  in the boundary layer.

# Flux of vorticity at the wall due to pressure gradient

For the case of a favorable pressure gradient, e.g. flow over a wedge,  $\frac{\partial p}{\partial x} < 0$ ; therefore,

$$\left. \frac{\partial \omega_z}{\partial y} \right|_{y=0} > 0 \quad \text{Favourable pressure gradient} \quad (26)$$

In terms of a Taylor's series expansion about  $y = 0$ ,

$$\omega_z(\Delta y) = \underbrace{\omega_z(0)}_{\text{negative sign}} + \underbrace{\left. \frac{\partial \omega_z}{\partial y} \right|_{y=0} \Delta y}_{\text{positive sign}} + \dots \quad (27)$$

$\Rightarrow \omega_z(\Delta y)$  will be less negative than the vorticity at the wall.

# Flux of vorticity at the wall due to pressure gradient

- Consistent with the notion of gradient transport (e.g. heat flux is 'down' the temperature gradient), vorticity flux is defined as  $-\frac{\partial \omega_z}{\partial y} \Big|_{y=0}$ .
- In a favorable pressure gradient, since  $\frac{\partial \omega_z}{\partial y} \Big|_{y=0} > 0$ , the flux of positive vorticity is toward the wall.
- This means that the wall acts like a sink of positive vorticity or equivalently, a source of negative-sign vorticity.
- This is equivalent to say that there is a higher concentration of negative vorticity in the near wall region for accelerating flows compared to the flat plate boundary layer.
- This extra negative vorticity is introduced at the wall since the wall serves as a source of negative vorticity in accelerating flows.

# Flux of vorticity at the wall due to pressure gradient

For the case of an adverse pressure gradient, e.g. flow in a diverging channel,  $\frac{\partial p}{\partial x} > 0$ ; therefore,

$$\left. \frac{\partial \omega_z}{\partial y} \right|_{y=0} < 0 \quad \text{Adverse pressure gradient} \quad (28)$$

In terms of a Taylor's series expansion about  $y = 0$ ,

$$\omega_z(\Delta y) = \underbrace{\omega_z(0)}_{\text{positive sign}} + \underbrace{\left. \frac{\partial \omega_z}{\partial y} \right|_{y=0} \Delta y}_{\text{negative sign}} + \dots \quad (29)$$

Thus,  $\omega_z(\Delta y)$  is more negative than the vorticity at the wall  $\implies$  The wall acts like a source of positive vorticity or, equivalently, as a sink of negative-sign vorticity.

# Implications for separation

In many external flows (*i.e.* flow over an object such as a cylinder or airfoil) we observe the phenomenon of vortex shedding downstream of the separation point:

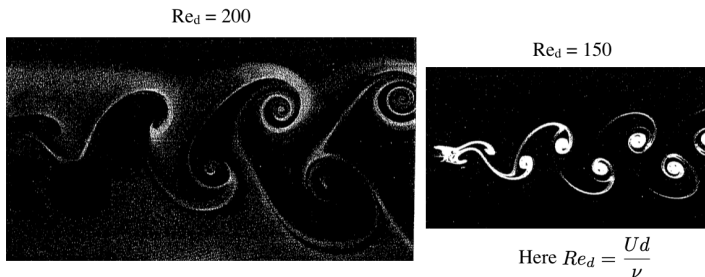


Figure: Illustration of flow separation and vortex shedding. Pairs of counter-rotating vortices that are staggered in a line behind the cylinder. This is referred to as vortex street

# Implications for separation

In the coordinate system shown below,  $x$ -points are tangential along the cylinder surface and  $y$ -points are normal to the surface, the  $+z$ -axis points out of the page in the top half of the cylinder, but in to the page in the bottom half of the cylinder.

⇒ Therefore, the vorticity shed from the top and bottom are both negative relative to their respective coordinate systems.

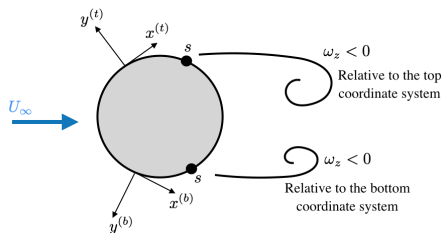


Figure: Local reference frames on the surface of a cylinder.

# Implications for separation

- As the boundary layer approaches separation, the flow decelerates and the boundary layer experiences an adverse pressure gradient.
- In this region, the surface of the cylinder acts as a source of positive sign vorticity.
- This  $+\omega_z$  vorticity diffuses outward and annihilates the existing  $-\omega_z$  vorticity near the surface.
- At a critical point when the value of the vorticity at the wall becomes zero, the  $-\omega_z$  vorticity in the boundary layer can no longer remain attached to the wall, and it is 'shed' from the surface, rolling up into the coherent vortex.

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# Role of viscosity in vorticity generation at the wall

Recall that

$$\frac{\partial p}{\partial x} = -\mu \frac{\partial \omega_z}{\partial y} \Big|_{y=0}. \quad (30)$$

- Although the magnitude of the vorticity flux depends on viscosity, the mechanism of generating vorticity at the wall is driven by an inviscid process and not molecular interactions between fluid particles.
- This follows because the left hand side of the above equation is based entirely on what is happening outside of the boundary layer, where the flow is considered inviscid, *i.e.* viscous effects are negligible.
- Therefore, the generation (flux) of vorticity at the wall is an instantaneous response of the flow to inertial forces that arise primarily due to changes in the boundary conditions (*i.e.* shape of the object over which the fluid is flowing).
- If viscous effects on a molecular level were important in the generation of vorticity at the wall, then this would be a much slower process.

# Comment on diffusion of vorticity

The vorticity transport equation contains the term  $\nu \nabla^2 \vec{\omega}$  (as we will see) which is analogous to the term  $\nu \nabla^2 \vec{u}$  in the momentum equation.

$\Rightarrow$  Vorticity suffers from viscous diffusion. However, we must keep in mind that the primary physical variables are force and momentum.

- At the molecular level, momentum is exchanged between neighboring fluid particles due to random collisions. During the collision, there is an exchange of linear momentum and thermal energy;
- $\Rightarrow$  It is the linear momentum and heat that are diffused.
- Vorticity is not related to molecular spin but to 'mean' velocity gradients averaged over a large number of mean free paths.
- Thus Vorticity is transported not by direct molecular-molecular interactions but as a consequence of the diffusion of linear momentum.