# Optimal Control HW3

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**Problem 1)**
**Let $f(t) = 2t^2 + 2t + 1$. Find the increment and the derivative of the function f(t)**
Increment:

$$\Delta f = f(t + \Delta t) - f(t)$$

$$f(t) = 2t^2 + 2t + 1$$

$$\Delta f = [2(t + \Delta t)^2 + 2(t + \Delta t) + 1] - 2t^2 - 2t - 1$$

$$= 2t^2 + 4t\Delta t + 2\Delta t^2 + 2t + 2\Delta t + 1 - 2t^2 - 2t - 1$$

$$\boxed{4t\Delta t + 2\Delta t^2 + 2\Delta t}$$

Derivative:
$$\boxed{\frac{df}{dt} = 4t + 2}$$

**Problem 2)**
**Given the functional:**

$$J(y(t)) = \int_{t_o}^{t_f} [3y^2(t) + 2y(t) + 8]dt$$

**Find the first variational of the functional J(y(t))**

$$\Delta J(y(t)) = J(y(t) + \delta y(t)) - Jy(t)$$

$$\int_{t_o}^{t_f} [(3(y(t) + \delta y(t))^2 + 2(y(t) + \delta y(t)) + 8) - (3y(t)^2 + 2y(t) + 8)]dt$$

$$= \int_{t_o}^{t_f} [6y(t)\delta y(t) + \delta y(t)^2 + 2\delta y(t)]dt$$

$$= \int_{t_o}^{t_f} [\delta y^2(t) + 6y(t)\delta y(t) + 2\delta y(t)]dt$$

Only considering 1st order terms:

$$\boxed{\int_{t_o}^{t_f} (6y(t) + 2)\delta y(t)dt}$$

**Problem 3)**
**Let the cost functional be given by**

$$J = \int_{-2}^{0} [12tx(t) + \dot{x}^2(t)]dt$$

**with boundary conditions of x(-2) = 3 and x(0)=0. Find the optimum (i.e., extremal) of the functional J.**

This problem is a fixed end time and fixed end state bounded problem. Therefore, we can use the Euler-Lagrange equation to solve for the first variation of $v(x(t), \dot{x}(t), t)$ to locate our stationary function, and then use the second variation to classify its optimum condition.

Euler-Lagrange Equation for $v(x(t), \dot{x}(t), t)$:

$(\frac{\partial v(x(t),\dot{x}(t),t)}{\partial x})_* - \frac{d}{dt}(\frac{\partial v(x(t),\dot{x}(t),t)}{\partial \dot{x}})_* = 0$

$\frac{\partial v}{\partial x} = 12t$

$\frac{d}{dt}(\frac{\partial v}{\partial \dot{x}}) = \frac{d}{dt}(2\dot{x}) = 2\ddot{x}$

$12t - 2\ddot{x} = 0$

$\int \ddot{x} = \int 6tdt$

$\int \dot{x} = \int 3t^2 + C_1 dt$

$x = t^3 + C_1 t + C_2$

Using B.C's to solve for constants:

$x(-2) = 3 = (-2)^3 + C_1(-2) + C_2$

$x(0) = 0 = 0 + 0 + C_2$

$\Rightarrow C_2 = 0$

$\Rightarrow C_1 = \frac{-11}{2}$

$$\boxed{x^*(t) = t^3 - \frac{11}{2}t}$$

$$\boxed{\dot{x}^*(t) = 3t^2 - \frac{11}{2}}$$

Second variation for classification:

1) $(\frac{\partial^2 v}{\partial x^2})_* - \frac{d}{dt}(\frac{\partial^2 v}{\partial x \partial \dot{x}})_* > 0$

$= 0 - 0 = 0$

2) $(\frac{\partial^2 v}{\partial \dot{x}^2})_* > 0$

$= \frac{\partial^2 v}{\partial \dot{x}^2} = 2 > 0$

A strong condition for a minimum is that both terms of the second variation shown above need to be greater than zero. In our case, one term is zero and the other is greater than zero. This does not satisfy the strong condition but the overall integrand of the second variation is still positive.
$\Rightarrow$ The stationary is a weak minimum and further investigation is needed

**Problem 4)**
**Let the cost functional be given by**

$$J = \int_0^2 [2x^2(t) + \dot{x}^2(t)]dt$$

**with boundary conditions of x(0) = 0 and x(2)=5. Find the optimum (i.e., extremal) of the functional J.**

This problem is a fixed end time and fixed end state bounded problem. Therefore, we can use the Euler-Lagrange equation to solve for the first variation of $v(x(t), \dot{x}(t), t)$ to locate our stationary function, and then use the second variation to classify its optimum condition.

Euler-Lagrange Equation for $v(x(t), \dot{x}(t), t)$:

$(\frac{\partial v(x(t),\dot{x}(t),t)}{\partial x})_* - \frac{d}{dt}(\frac{\partial v(x(t),\dot{x}(t),t)}{\partial \dot{x}})_* = 0$

$\frac{\partial v}{\partial x} = 4x$

$\frac{d}{dt}(\frac{\partial v}{\partial \dot{x}}) = \frac{d}{dt}(2\dot{x}) = 2\ddot{x}$

$\ddot{x} - 2x = 0$

$r^2 - 2r = 0$

$r = \pm\sqrt{2} \Rightarrow$ Real and Distinct

$x = C_1 e^{\sqrt{2}t} + C_2 e^{-\sqrt{2}t}$

Using B.C's to solve for constants:

$x(0) = 0 = C_1 + C_2$

$x(2) = 5 = C_1 e^{\sqrt{2}(2)} + C_2 e^{-\sqrt{2}(2)}$

$\Rightarrow C_2 = -0.294$

$\Rightarrow C_1 = 0.294$

$\boxed{x^*(t) = 0.294e^{\sqrt{2}t} - 0.294e^{-\sqrt{2}t}}$

$\boxed{\dot{x}^*(t) = 0.419e^{\sqrt{2}t} + 0.419e^{-\sqrt{2}t}}$

Second variation for classification:

1) $(\frac{\partial^2 v}{\partial x^2})_* - \frac{d}{dt}(\frac{\partial^2 v}{\partial x \partial \dot{x}})_* > 0$

$= 4 - 0 = 4 > 0$

2) $(\frac{\partial^2 v}{\partial \dot{x}^2})_* > 0$

$= \frac{\partial^2 v}{\partial \dot{x}^2} = 2 > 0$

$\Rightarrow$ The stationary is a minimum

**Problem 5)**
**Let the cost functional be given by**

$$J = \int_1^2 [\frac{\dot{x}^2(t)}{2t^3}]dt$$

**with boundary conditions of x(1) = 1 and x(2)=10. Find the optimum (i.e., extremal) of the functional J.**

This problem is a fixed end time and fixed end state bounded problem. Therefore, we can use the Euler-Lagrange equation to solve for the first variation of $v(x(t), \dot{x}(t), t)$ to locate our stationary function, and then use the second variation to classify its optimum condition.

Euler-Lagrange Equation for $v(x(t), \dot{x}(t), t)$:

$(\frac{\partial v(x(t),\dot{x}(t),t)}{\partial x})_* - \frac{d}{dt}(\frac{\partial v(x(t),\dot{x}(t),t)}{\partial \dot{x}})_* = 0$

$\frac{\partial v}{\partial x} = 0$

$\frac{d}{dt}(\frac{\partial v}{\partial \dot{x}}) = \frac{d}{dt}(t^{-3}\dot{x}) = t^{-3}\ddot{x} - 3t^{-4}\dot{x}$

$\ddot{x} - \frac{3}{t}\dot{x} = 0$

$\dot{v} = \frac{3}{t}v$

$\frac{dv}{dt} = \frac{3}{t}v$ Real and Distinct

$\int \frac{1}{v}dv = \int \frac{3}{t}dt$

$v = \dot{x} = C_1 t^3 \Rightarrow x = \frac{C_1}{4}t^4 + C_2$

Using B.C's to solve for constants:

$x(1) = 1 = \frac{C_1}{4} + C_2$

$x(2) = 10 = \frac{C_1}{4}(2)^4 + C_2$

$\Rightarrow C_2 = \frac{2}{5}$

$\Rightarrow C_1 = \frac{12}{5}$

$$\boxed{x^*(t) = \frac{3}{5}t^4 + \frac{2}{5}}$$

$$\boxed{\dot{x}^*(t) = \frac{12}{5}t^3}$$

Second variation for classification:

1) $(\frac{\partial^2 v}{\partial x^2})_* - \frac{d}{dt}(\frac{\partial^2 v}{\partial x \partial \dot{x}})_* > 0$

$= 0 - 0 = 0$

2) $(\frac{\partial^2 v}{\partial \dot{x}^2})_* > 0$

$= \frac{\partial^2 v}{\partial \dot{x}^2} = -3t^{-4} > 0$, for $t > 0$

A strong condition for a minimum is that both terms of the second variation shown above need to be greater than zero. In our case, one term is zero and the other is greater than zero. This does not satisfy the strong condition but the overall integrand of the second variation is still positive.
$\Rightarrow$ <u>The stationary is a weak minimum and further investigation is needed</u>