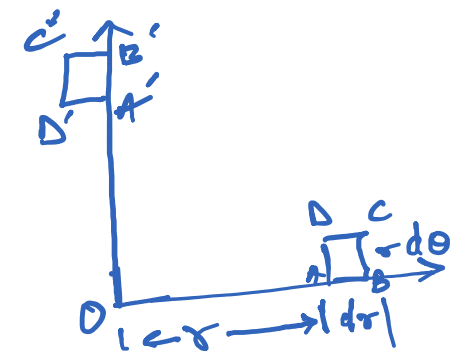
1) $\vec{V} = \frac{\Gamma}{2\pi r}\vec{e}_\theta$

a) $\nabla \times \vec{V} = \frac{1}{r}\begin{vmatrix} \vec{e}_r & r\vec{e}_\theta & \vec{e}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ 0 & \frac{\Gamma}{2\pi} & 0 \end{vmatrix} = \frac{1}{r}\cdot(0) = 0$ except where $r=0$

Since $\vec{\Omega} = \vec{\nabla}\times V = 0$ except at r, this flow is irrotational except at $r=0$, which means the average angular velocity of the particles is zero, (except at the origin)

b) Shape of fluid particle after turning by θ.



INITIAL PARTICLE CORNER POINTS:

POINT	r	θ
A	r	0
B	$r+dr$	0
C	$r+dr$	$d\theta$
D	r	$d\theta$

Now, the time taken by point $A(r,0)$ to travel to point A' is $(r, \pi/2)$

$$t_A = \frac{r\theta}{v_\theta} \; ; \quad \text{where } \theta \text{ is } 90^\circ \text{ or } \pi/2 \text{ and } \vec{v}_\theta = \frac{\Gamma}{2\pi r}$$

$$t_A = \frac{r\,\pi/2}{\Gamma/2\pi r} = \frac{\pi^2 r^2}{\Gamma}$$

Now, Point B of the particle travels at a different velocity $v_{\theta_B} = \frac{\Gamma}{2\pi(r+dr)}\vec{e_\theta}$ and so its new position after time 't_A' will be :-

$$r_{B'} = r + dr, \quad \theta_{B'} = \frac{v_{\theta_{B'}} \times t_A}{(r+dr)} = \frac{\Gamma}{2\pi(r+dr)^2} \cdot \frac{\pi^2 r^2}{\Gamma} = \frac{\pi}{2}\left(1+\frac{dr}{r}\right)^{-2} \approx \frac{\pi}{2}\left(1-2\frac{dr}{r}\right)$$

since $dr \to 0$

Therefor B' is at $\left(r+dr, \frac{\pi}{2}\left(1-2\frac{dr}{r}\right)\right)$

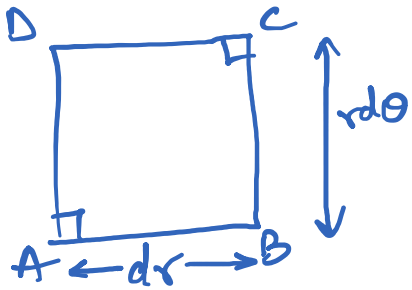
Similarly C' is at $\left(r+dr, d\theta + \frac{\pi}{2}\left(1-2\frac{dr}{r}\right)\right)$

and D' is at $\left(r, d\theta + \frac{\pi}{2}\right)$

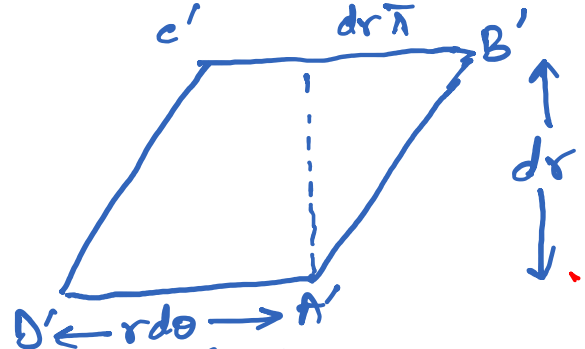
So the final co-ordinates of the particles are :

POINT	r	θ
A'	r	$\pi/2$
B'	$r+dr$	$\pi/2\left(1-2\frac{dr}{r}\right)$
C'	$r+dr$	$d\theta+\frac{\pi}{2}\left(1-2\frac{dr}{r}\right)$
D'	r	$\pi/2+d\theta$

So the initial and final positions are like :-



INITIAL



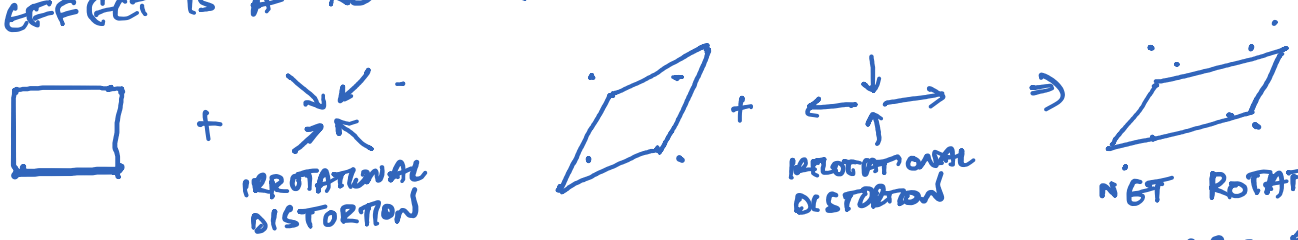
FINAL

c) We can define the total rotation as the average rotation angles of the sides of the particle (could alternately use the diagonals, but the maths could be harder)

so total rotation angle $= \frac{1}{2}$ (Rotation angle of AB + Rotation angle of DA)

$$= \frac{1}{2}\left(\tan^{-1}\left(\frac{dr}{dr\pi}\right) + \frac{\pi}{2}\right) = \underline{53.8^\circ}$$

EXPLANATION : AS THE PARTICLE IS CONNECTED IT SUFFERS AN INFINITE SERIES OF INFINITESIMAL DISTORTIONS. EVEN THOUGH EACH OF THESE DISTORTIONS IS IROTATIONAL THEIR COMBINED EFFECT IS A NET ROTATION. FOR EXAMPLE SEE FIGURE BELOW.



NOTE : SINCE THE QUESTION ASKS FOR AN ATTEMPTED EXPLANATION, ANY REASONABLE RATIONAL EXPLANATION WILL BE ACCEPTED, BUT THIS IS THE CORRECT EXPLANATION.

2) streamlines $\quad \vec{ds} \times \vec{V} = 0$

$$d(r\hat{e}_r + z\hat{e}_z) \times \vec{V} = 0$$

$$\begin{vmatrix} \hat{e}_r & \hat{e}_\theta & \hat{e}_z \\ dr & r\,d\theta & dz \\ 0 & r/a & 1-r^2/a^2 \end{vmatrix} = 0$$

$$\hat{e}_r\left(r\,d\theta(1-r^2/a^2) - r/a\,dz\right) - \hat{e}_\theta\left(dr(1-r^2/a^2)\right) + \hat{e}_z\left(r/a\,dr\right) = 0$$

Since each component should be zero:

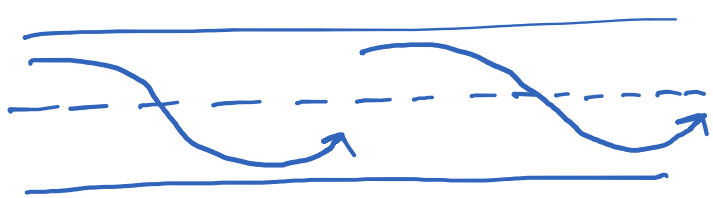
$$r\,d\theta(1-r^2/a^2) = \frac{r}{a}dz$$

$$\left.\begin{aligned} dr(1-r^2/a^2) &= 0 \\ dr(r/a) &= 0 \end{aligned}\right\} \Rightarrow dr = 0 \Rightarrow r = \text{constant is a streamline of the flow}$$

$$\Rightarrow \quad (1-r^2/a^2)\,d\theta = \frac{1}{a}dz$$

Integrating $\Rightarrow \quad z = a(1-r^2/a^2)\theta + \text{constant}$

Thus the streamlines are where $r =$ constant and $z \propto \theta$ i.e <u>spirals</u>.



b) Determine two streamfunctions:

Examining the velocity components we see $v_r = 0$, so we can choose one set of streamfunction $\psi_2 = r$ (this is analogous to our in-class example flow)

$$\underline{\underline{\psi_2 = r}}$$

Since the flow is incompressible, we may take $\alpha = 1$ and compute ψ_1 as

$$\vec{V} = \nabla\psi_1 \times \nabla\psi_2 = \begin{vmatrix} \hat{e}_r & \hat{e}_\theta & \hat{e}_z \\ \frac{\partial\psi_1}{\partial r} & \frac{1}{r}\frac{\partial\psi_1}{\partial\theta} & \partial\psi_1/\partial z \\ 1 & 0 & 0 \end{vmatrix}$$

$$\vec{V} = \hat{e}_\theta \frac{\partial\psi_1}{\partial z} - \frac{\hat{e}_z}{r}\frac{\partial\psi_1}{\partial\theta} = \frac{r}{a}\hat{e}_\theta + \left(1 - \frac{r^2}{a^2}\right)\hat{e}_z$$

Thus

$$\frac{\partial\psi_1}{\partial z} = r/a \longrightarrow \psi_1 = \frac{rz}{a} + f(\theta)$$

$$\frac{-\partial\psi_1}{\partial\theta} = r - \frac{r^3}{a^3} \longrightarrow \psi_1 = -r\theta + \frac{r^3\theta}{a^2} + g(z)$$

finally: $\psi_1 = \frac{rz}{a} - r\theta + \frac{r^3\theta}{a^2} + \text{constant}$

Streamlines are lines of intersection of $\psi_1 = \text{const}$ and $\psi_2 = \text{const}$

hence: $\text{const} = rz/a - r\theta + r^3/a^2\,\theta$ AND $r = \text{const}$

$\Rightarrow \text{const} = z/a - \theta + r^2\theta/a$ (OR) $z = a\theta\left(1 - \frac{r^2}{a^2}\right) + \text{const}$

(c) vorticity $\vec{\Omega} = \vec{\nabla} \times \vec{v} = \frac{1}{r}\begin{vmatrix} \bar{e}_r & r\bar{e}_\theta & e_z \\ \partial/\partial r & \partial/\partial\theta & \partial/\partial z \\ 0 & r/a & \left(1-\frac{r^2}{a^2}\right) \end{vmatrix}$

$$= \frac{1}{r}\left[\frac{\partial}{\partial\theta}(1-r^2/a^2)\hat{e}_r - \frac{\partial}{\partial z} r^2/a\, \hat{e}_r - r\frac{\partial}{\partial r}(1-r^2/a^2)\hat{e}_\theta + \frac{\partial}{\partial r} r^2/a\, \hat{e}_z\right]$$

$$= \frac{1}{r}\left[-r\frac{\partial}{\partial r}(1-r^2/a^2)\hat{e}_\theta + \frac{\partial}{\partial r} r^2/a\, \hat{e}_z\right]$$

$$= \frac{1}{r}\left(2r^2/a^2\, \hat{e}_\theta + 2r/a\, \hat{e}_z\right)$$

$$\vec{\Omega} = \frac{2r}{a^2}\hat{e}_\theta + 2r/a\, \hat{e}_z$$

for vortex lines $\vec{\Omega} \times d\vec{s} = 0 = \begin{vmatrix} \hat{e}_r & \hat{e}_\theta & \hat{e}_z \\ 0 & 2r/a^2 & r/a \\ dr & r\,d\theta & dz \end{vmatrix}$

$$0 = \hat{e}_r\left(\frac{2r}{a^2}dz - \frac{2r}{a}d\theta\right) + \hat{e}_\theta \frac{2}{a}dr - \hat{e}_z \frac{2r}{a^2}dr$$

so vortex lines are constant, $dr = 0$

$$\frac{2r}{a^2}dz = \frac{2r}{a}d\theta \longrightarrow \frac{z}{a^2} = \frac{\theta}{a}$$

thus $\underline{z = a\theta + \text{const}}$

vortex lines are helixes (as in part (a)) but of a different pitch.

3) $$d\vec{v} = \frac{\partial \vec{v}}{\partial r} dr + \frac{\partial \vec{v}}{\partial \theta} d\theta + \frac{\partial \vec{v}}{\partial \phi} d\phi$$

$$\vec{v} = v_r \hat{e}_r + v_\theta \vec{e}_\theta + v_\phi \vec{e}_\phi$$

so $$\frac{\partial \vec{v}}{\partial r} = \frac{\partial v_r}{\partial r}\vec{e}_r + \cancelto{0}{\frac{\partial \vec{e}_r}{\partial r}} v_r + \frac{\partial v_\theta}{\partial r}\vec{e}_\theta + \cancelto{0}{\frac{\partial \vec{e}_\theta}{\partial r}} v_\theta + \frac{\partial v_\phi}{\partial r}\vec{e}_\phi + \cancelto{0}{\frac{\partial \vec{e}_\phi}{\partial r}} v_\phi$$

$$\frac{\partial \vec{v}}{\partial \theta} = \frac{\partial v_r}{\partial \theta}\vec{e}_r + \underbrace{\frac{\partial \vec{e}_r}{\partial \theta}}_{\vec{e}_\theta} v_r + \frac{\partial v_\theta}{\partial \theta}\vec{e}_\theta + \underbrace{\frac{\partial \vec{e}_\theta}{\partial \theta}}_{-\vec{e}_r} v_\theta + \frac{\partial v_\phi}{\partial \theta}\vec{e}_\phi + \cancelto{0}{\frac{\partial \vec{e}_\phi}{\partial \theta}} v_\phi$$

$$= \vec{e}_r\left(\frac{\partial v_r}{\partial \theta} - v_\theta\right) + \vec{e}_\theta\left(\frac{\partial v_\theta}{\partial \theta} + v_r\right) + \vec{e}_\phi\left(\frac{\partial v_\phi}{\partial \theta}\right)$$

AND $$\frac{\partial \vec{v}}{\partial \phi} = \frac{\partial v_r}{\partial \phi}\vec{e}_r + \underbrace{\frac{\partial \vec{e}_r}{\partial \phi}}_{\sin\theta\,\vec{e}_\phi} v_r + \frac{\partial v_\theta}{\partial \phi}\vec{e}_\theta + \underbrace{\frac{\partial \vec{e}_\theta}{\partial \phi}}_{\cos\theta\,\vec{e}_\phi} v_\theta + \frac{\partial v_\phi}{\partial \phi}\vec{e}_\phi + \underbrace{\frac{\partial \vec{e}_\phi}{\partial \phi}}_{-\sin\theta\,\vec{e}_r - \cos\theta\,\vec{e}_\theta} v_\phi$$

$$= \vec{e}_r\left(\frac{\partial v_r}{\partial \phi} - \sin\theta\, v_\phi\right) + \vec{e}_\theta\left(\frac{\partial v_\theta}{\partial \phi} - \cos\theta\, v_\phi\right) + \vec{e}_\phi\left(\frac{\partial v_\phi}{\partial \phi} + \sin\theta\, v_r + \cos\theta\, v_\theta\right)$$

so, $$(d\vec{v})\cdot\vec{e}_r = \frac{\partial v_r}{\partial r} dr + \left(\frac{\partial v_r}{\partial \theta} - v_\theta\right) d\theta + \left(\frac{\partial v_r}{\partial \phi} - \sin\theta\, v_\phi\right) d\phi$$

$$(d\vec{v})\cdot\vec{e}_\theta = \frac{\partial v_\theta}{\partial r} dr + \left(\frac{\partial v_\theta}{\partial \theta} + v_r\right) d\theta + \left(\frac{\partial v_\theta}{\partial \phi} - \cos\theta\, v_\phi\right) d\phi$$

$$(d\vec{v})\vec{e}_\phi = \frac{\partial v_\phi}{\partial r} dr + \left(\frac{\partial v_\phi}{\partial \theta}\right) d\theta + \left(\frac{\partial v_\phi}{\partial \phi} + \sin\theta\, v_r + \cos\theta\, v_\theta\right) d\phi$$

So

$$\begin{bmatrix} d\vec{v} \end{bmatrix} = \begin{bmatrix} \partial v_r/\partial r & \frac{1}{r}\left(\frac{\partial v_r}{\partial \theta} - v_\theta\right) & \frac{1}{r\sin\theta}\left(\frac{\partial v_r}{\partial \phi} - \sin\theta\, v_\phi\right) \\ \partial v_\theta/\partial r & \frac{1}{r}\left(\frac{\partial v_\theta}{\partial \theta} + v_r\right) & \frac{1}{r\sin\theta}\left(\frac{\partial v_\theta}{\partial \phi} - \cos\theta\, v_\phi\right) \\ \partial v_\phi/\partial r & \frac{1}{r}\frac{\partial v_\phi}{\partial \theta} & \frac{1}{r\sin\theta}\left(\frac{\partial v_\phi}{\partial \phi} + \sin\theta\, v_r + \cos\theta\, v_\theta\right) \end{bmatrix} \begin{bmatrix} dr \\ r\,d\theta \\ r\sin\theta\, d\phi \end{bmatrix}$$

$(r\sin\theta\, d\phi)$

The matrix above can be broken into 3 matrices, symmetric, diagonal & anti-symmetric ones.

(Rates of shear strain)

<u>Symmetric</u>:

$$\begin{pmatrix} \partial v_r/\partial r & \frac{1}{2}\left(\frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r} + \frac{1}{r}\frac{\partial v_r}{\partial \theta}\right) & \frac{1}{2}\left[\frac{\partial v_\phi}{\partial r} - \frac{v_\phi}{r} + \frac{1}{r\sin\theta}\frac{\partial v_r}{\partial \phi}\right] \\ \cdot & \frac{1}{2}\left(\frac{\partial v_\theta}{\partial \theta} + v_r\right) & \frac{1}{2}\left(\frac{1}{r}\frac{\partial v_\theta}{\partial \theta} - \frac{v_\phi}{r\sin\theta} + \frac{1}{r\sin\theta}\frac{\partial v_\theta}{\partial \phi}\right) \\ \cdot & \cdot & \frac{1}{r\sin\theta}\left[\frac{\partial v_\phi}{\partial \phi} + \sin\theta\, v_r + \cos\theta\, v_\theta\right] \end{pmatrix}$$

So

$$\operatorname{div}\vec{v} = \frac{\partial v_r}{\partial r} + \frac{v_r}{r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi} + \frac{v_r}{r} + \frac{v_\theta}{r\tan\theta}$$

$$= \frac{1}{r^2}\frac{\partial r^2 v_r}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial v_\theta \sin\theta}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi}$$

$\boxed{\text{DIVERGENCE}}$

Anti-symmetric part :

$$\begin{pmatrix} 0 & \frac{1}{2}\left(\frac{\partial v_r}{\partial \theta}\frac{1}{r} - \frac{v_\theta}{r} - \frac{\partial v_\theta}{\partial r}\right) & \frac{1}{2}\left[\frac{1}{r\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{v_\phi}{r} - \frac{\partial v_\phi}{\partial r}\right] \\ \cdot & 0 & \frac{1}{2}\left[\frac{1}{r\sin\theta}\frac{\partial v_\theta}{\partial \phi} - \frac{v_\phi}{r\tan\theta} - \frac{1}{r}\frac{\partial v_\phi}{\partial \theta}\right] \\ \cdot & \cdot & 0 \end{pmatrix}$$

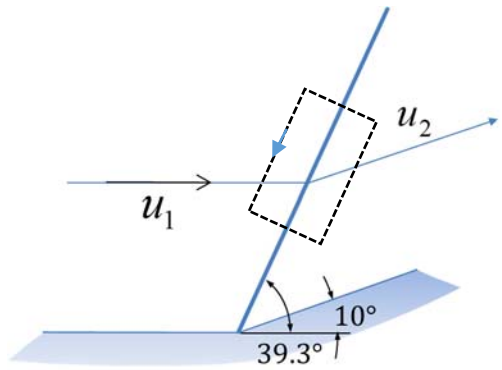
So:

$$\text{curl}\,\vec{V} = \left\{ \vec{e}_r \left[\frac{1}{r\sin\theta}\frac{\partial v_\theta}{\partial \phi} - \frac{v_\phi}{r\tan\theta} - \frac{1}{r}\frac{\partial v_\phi}{\partial \theta}\right] + \vec{e}_\theta \left[\frac{1}{r\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{v_\theta}{r} - \frac{\partial v_\phi}{\partial r}\right] + \vec{e}_\phi \left[\frac{1}{r}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta}{r} - \frac{\partial v_\theta}{\partial r}\right] \right\} \times (-1)$$

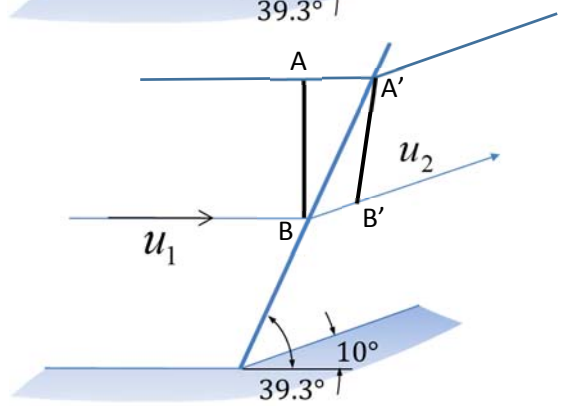
$$= \frac{1}{r^2\sin\theta} \begin{vmatrix} \vec{e}_r & r\vec{e}_\theta & r\sin\theta\,\vec{e}_\phi \\ \partial/\partial r & \partial/\partial\theta & \partial/\partial\phi \\ v_r & r v_\theta & r\sin\theta\, v_\phi \end{vmatrix}$$

$\boxed{\text{curl}}$

4. (a) Consider the loop shown, and the pressure-force torque integral $-\oint \frac{\nabla p.ds}{\rho}$ evaluated around this loop. In the uniform regions the integrand is zero since $\nabla p = 0$. At the shock crossing points, ∇p will be very large but, since we cross the shock in opposite directions on the two sides of the loop, the contributions of $\nabla p.\, d\boldsymbol{s}$ from these two crossings will be equal and opposite, and thus will cancel. The net pressure torque will thus be zero.



(b) If the vorticity is being carried with the fluid material, then we can track it by tracking what happens to the fluid lines. Consider the vertical fluid line AB as it just arrives at the shock, and the same line A'B' a short time later just as it leaves the shock. Note that:



> the distance BB' is 616/694=0.888 times the length of AA'. The length of AA'=tan(50.7°) times the length of AB, so BB'=0.888*tan(50.7°)=1.074 AB

> the distance BA' is equal to the distance AB/cos(50.7°)= 1.579 AB

> by the cosine rule the length of A'B' = $\sqrt{1.074^2 + 1.579^2 - 2 \times 1.074 \times 1.579 \cos(29.3°)} = 0.83$ times the length of AB

>the angle of BB'A' is (by the sine rule) =$\sin^{-1}(\frac{1.579}{0.83}\sin(29.3°)) = 111.41°$ so A'B' is rotated 10+68.59 =78.59 degrees counter-clockwise from horizontal.

(i) An initially horizontal eddy/fluid line will be coincident with a streamline of the flow. The fluid line, and thus the eddy it represents will thus rotate counter-clockwise by 10 degrees with the turning of the flow as it passes through the shock. The length of the fluid line/eddy will reduce by the ratio BB'/AA'=0.888. The cross sectional area of the eddy will thus be multiplied by the factor $\frac{1}{0.888} \times \frac{\rho_1}{\rho_2} = \frac{1}{0.888} \times \frac{1}{1.46} = \frac{1}{1.296} = 0.771$ to maintain the same mass of fluid in the eddy. Thus, the vorticity will increase in strength by the factor of $\frac{1}{0.771} = 1.296$.

(ii) The initially vertical fluid line/eddy will rotate clockwise by an angle of 11.41 degrees. (Note that this is 1.41 degrees greater than the angle of rotation implied by the small angle approximation argument, of requiring that the sum of rotation rates of perpendicular fluid lines remain zero). The length of the fluid line/eddy will decrease by the factor A'B'/AB=0.83. The cross sectional area of the eddy will thus be multiplied by the factor $\frac{1}{0.83} \times \frac{\rho_1}{\rho_2} = \frac{1}{0.83} \times \frac{1}{1.46} = \frac{1}{1.212} = 0.825$ to maintain the same mass. Thus, the vorticity will increase in strength by the factor of $\frac{1}{0.825} = 1.212$.

(c) We must assume

1. an absence of

    a. viscous,
    b. body force and
    c. pressure-force torques.
2. That the turbulence is weak and doesn't significantly alter the mean flow of the vortex

Assumptions 1b and 1c are valid by definition of the problem. Again 2 and 1a are linked, but in this case the shock is such a rapid and violent change that these assumptions will likely be very good also.