# Intermediate Fluid Mechanics

## Lecture 23: Boundary Layer Flows II

Marc Calaf

Department of Mechanical Engineering
*University of Utah*

ME 5700/6700
November 10, 2025

# Chapter Overview

# Lecture Objectives

In this lecture we will continue exploring in more detail the concept of boundary layer. Specifically, we will:

- Introduce specific definitions regarding the boundary layer thickness.
- Derive the Blasius Solution to the Boundary Layer Equations

# Chapter Overview

# General Concepts of BL Theory

Aside from deriving the governing equations for a laminar, two-dimensional, incompressible boundary layer, another important result emerged from the previous lecture, namely a relation describing the growth of the boundary layer,

$$\frac{\delta}{x} = \left[\frac{U_\infty x}{\nu}\right]^{-1/2}. \tag{1}$$

$\Longrightarrow$ if one knows the freestream velocity and distance along the surface relative to the leading edge, then one can predict the local boundary layer thickness, $\delta$.

# General Concepts of BL Theory

This equation,

$$\frac{\delta}{x} = \left[\frac{U_\infty x}{\nu}\right]^{-1/2}. \tag{2}$$

was indirectly obtained by equating the advection time scale to the diffusion time scale in a zero pressure gradient, steady boundary layer.

Hence,

$$\underbrace{T_{adv}}_{\substack{\text{advection} \\ \text{time scale}}} = \underbrace{T_{diff}}_{\substack{\text{diffusion} \\ \text{time scale}}} \tag{3}$$

where,

- $T_{adv} = \frac{x}{U_\infty}$; time for a fluid particle to travel a distance $x$ at velocity $U_\infty$.
- $T_{diff} = \frac{\delta^2}{\nu}$ ; time for a fluid particle to travel across the boundary due to viscous effects.

# General Concepts of BL Theory

By equating,

$$\underbrace{T_{adv}}_{\substack{\text{advection}\\\text{time scale}}} = \underbrace{T_{diff}}_{\substack{\text{diffusion}\\\text{time scale}}} \tag{4}$$

$\Longrightarrow$ That a fluid particle travels a horizontal distance $x$ in the same time it takes to travel a vertical distance $\delta$,

$$\frac{x}{U_\infty} = \frac{\delta}{\nu}$$

$$\Rightarrow \delta = x\left(\frac{U_\infty x}{\nu}\right)^{-1/2} \tag{5}$$

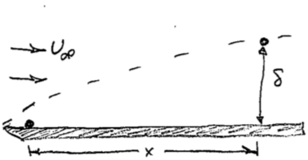


Figure: Two-dimensional flow over a flat plate boundary layer.

# General Concepts of BL Theory

$\longrightarrow$ Regardless, the boundary layer thickness is an ambiguous length scale, because it is difficult to explicitly define the interface between the freestream and the boundary layer.

Typically we estimate $\delta$ by $\delta_{99}$, which is defined as the height above the surface where the horizontal velocity has reached 99% of its free stream value,

$$u(y = \delta_{99}) = 0.99 * U_\infty. \tag{6}$$

# General Concepts of BL Theory

$\longrightarrow$ To avoid having to write the subscript 99, it will be implied henceforth that $\delta$ is to be calculated as $\delta_{99}$.

Two other **integral length scales** are often used to characterize the boundary layer, since quantities based on integral operations tend to be less susceptible to errors from noisy data.

# Chapter Overview

# Displacement Thickness, $\delta^*$

Due to the presence of the boundary layer, there is a loss of both mass flux and momentum flux near the surface.

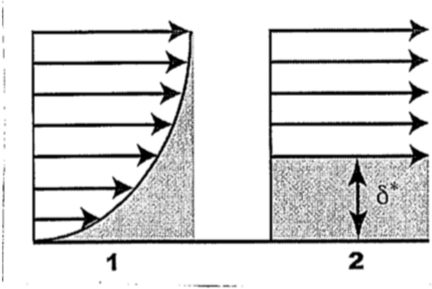


Figure: Comparison between the flow in a boundary layer and the uniform freestream.

Clearly, there is more mass flux and momentum flux accross a vertical plane in case II compared to case I in Figure 2.

# Displacement Thickness, $\delta^*$

**The displacement thickness** $\delta^*$ is the distance the plate would have to be moved so that the loss of mass flux in a uniform flow with velocity $U_\infty$ is equivalent to the loss of mass flux caused by the boundary layer.

If there was no boundary layer, *i.e.* if the flow was inviscid, the mass flux would be

$$\dot{m} = \int_0^\infty \rho\, U_\infty\, dy \qquad \text{no boundary layer.} \tag{7}$$

With the boundary layer, the mass flux is

$$\dot{m} = \int_0^\infty \rho\, U\, dy \qquad \text{with boundary layer.} \tag{8}$$

# Displacement Thickness, $\delta^*$

Therefore, the loss of mass due to the presence of the boundary layer or mass flux deficit is the difference between the two and (by definition of $\delta^*$) is equivalent to $\rho\, U_\infty \delta^*$,

$$\int_0^{\delta^*} \rho\, U_\infty\, dy = \int_0^\infty \rho \Big( U_\infty - u \Big)\, dy \tag{9}$$

$$\rho\, U_\infty\, \delta^* = \rho \int_0^\infty \Big( U_\infty - u \Big)\, dy \tag{10}$$

$$\delta^* = \int_0^\infty \left( 1 - \frac{u}{U_\infty} \right) dy. \tag{11}$$

$+$ Note: since $u \approx U_\infty$ at $y = \delta$, one only needs to integrate up through the boundary layer thickness (or a little beyond this value).

# Chapter Overview

# Momentum Deficit Thickness, $\theta$

Similarly, one can define a length scale $\theta$, that represents the distance the plate would need to be displaced so that the resultant loss in momentum flux of a uniform flow of velocity $U_\infty$ is equivalent to the loss of momentum due to the presence of the boundary layer.

$$\begin{array}{c}\text{momentum deficit flux} \\ \text{of shaded area in case 1}\end{array} =$$

$$= \int_0^\infty \rho \underbrace{u}_{I} \underbrace{\left(U_\infty - u\right)}_{II} dy \qquad (12)$$

- I is the velocity at which this momentum deficit is transported across the vertical plane,
- II is the momentum deficit of shaded are in the figure.

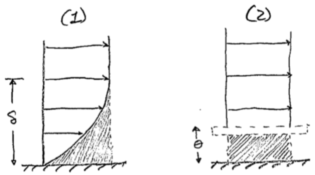


Figure: Momentum deficit in a two-dimensional flat plate boundary layer.

# Momentum Deficit Thickness, $\theta$

Alternatively,

$$\begin{array}{c}\text{momentum deficit flux} \\ \text{of shaded area in case 2}\end{array} = \int_0^{\theta} \rho\, U_\infty^2\, dy \tag{13}$$

Assuming constant density, equating the two integrals gives,

$$\int_0^{\theta} \rho\, U_\infty^2\, dy = \int_0^{\infty} \rho\, u \Big( U_\infty - u \Big)\, dy \tag{14}$$

$$U_\infty^2 \theta = \int_0^{\infty} u \Big( U_\infty - u \Big)\, dy \tag{15}$$

$$\theta = \int_0^{\infty} \frac{u}{U_\infty} \left( 1 - \frac{u}{U_\infty} \right) dy \tag{16}$$

Again, the contribution to the integral for $y \geq \delta$ is small so that we really only need to integrate up to $\delta$ or a little beyond.

# Chapter Overview

# Boundary Layer over a Flat Plate: Blasius Solution

Let's consider the case of flow over a flat plate so that,

$$\frac{\partial p}{\partial x} = 0, \tag{17}$$

$\Longrightarrow$ there is no imposed or geometry induced pressure gradient.

The governing equations and boundary conditions are

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} \qquad \text{x-momentum} \tag{18}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad \text{continuity} \tag{19}$$

with, $u = v = 0$ at $y = 0$ (no-slip at the surface), and $u = U_\infty$ at $y \to \infty$ (far-field boundary condition).

# Boundary Layer over a Flat Plate: Blasius Solution

Since the flow is two-dimensional, it is convenient to work with the streamfunction, instead of the velocity directly.

Recall the definition of the streamfunciton,

$$u \equiv \frac{\partial \psi}{\partial y} \qquad \text{and} \qquad v \equiv -\frac{\partial \psi}{\partial x}. \tag{20}$$

Also, it is convenient to change variables and work with non-dimensional variables.

$\longrightarrow$ So we will seek a solution of the form,

$$\frac{u}{U_\infty} = g(\eta) \qquad \text{where} \qquad \eta = \frac{y}{\delta(x)}. \tag{21}$$

Note, $\delta = \delta(x)$ since the boundary layer grows with the distance along the plate.

# Boundary Layer over a Flat Plate: Blasius Solution

The streamfunction is determined by integrating the u-velocity,

$$\psi = \int_0^y u\, dy, \tag{22}$$

where one needs to introduce the non-dimensional variables, hence

$$\eta = \frac{y}{\delta(x)} \Rightarrow d\eta = \frac{dy}{\delta(x)} \Rightarrow dy = \delta\, d\eta, \tag{23}$$

therefore

$$\psi = \int_0^y u\, dy = \delta \int_0^\eta U_\infty\, g(\eta)\, d\eta = U_\infty\, \delta\, f(\eta), \tag{24}$$

where $g(\eta) = \frac{df}{d\eta}$.

Note that $f(\eta)$ represents the nondimensional streamfunction and $U_\infty \delta$ has units of the volumetric flow rate.

# Boundary Layer over a Flat Plate: Blasius Solution

The x-momentum equation can be rewritten in terms of $\psi$ as

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} \tag{25}$$

$$\frac{\partial \psi}{\partial y}\frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial \psi}{\partial x}\frac{\partial^2 \psi}{\partial y^2} = \nu\frac{\partial^3 \psi}{\partial y^3}, \tag{26}$$

with the corresponding boundary conditions being readjusted as,

- (non-slip on $u$): at $y = 0$, $\partial\psi/\partial y = 0$.
- (far-field of $u$): at $y \to \infty$, $\partial\psi/\partial y \to U_\infty$.
- (non-slip on $v$): at $y = 0$, then $\partial\psi/\partial x = 0$ which upon integration with respect to $x$, $\psi =$ constant; out of which we can choose $\psi = 0$.

# Boundary Layer over a Flat Plate: Blasius Solution

One can now rewrite $\psi$ in terms of the non-dimensional variable $f$,

$$\frac{\partial \psi}{\partial x} = \frac{\partial}{\partial x}\left[U_\infty\, \delta\, f\right] = U_\infty\, f\, \frac{d\delta}{dx} + U_\infty\, \delta\, \underbrace{\frac{\partial f}{\partial x}}_{*}, \tag{27}$$

where: $\frac{\partial f}{\partial x} = \frac{df}{d\eta}\frac{\partial \eta}{\partial x} = \frac{df}{d\eta}\left[\frac{-y}{\delta^2}\frac{d\delta}{dx}\right]$.

Letting, $df/d\eta \equiv f'$,

$$\frac{\partial \psi}{\partial x} = U_\infty f \frac{d\delta}{dx} + U_\infty \delta f' \frac{d\delta}{dx}\left(\frac{-y}{\delta^2}\right) = U_\infty \frac{d\delta}{dx}\left(f - \eta f'\right) \tag{28}$$

$$\frac{\partial^2 \psi}{\partial x \partial y} = \frac{\partial}{\partial y}\left[U_\infty \frac{d\delta}{dx}\left(f - \eta f'\right)\right] = U_\infty \frac{d\delta}{dx}\left[\underbrace{\frac{\partial f}{\partial y}}_{I} - \underbrace{\frac{\partial}{\partial y}\left(\eta f'\right)}_{II}\right] = -\frac{U_\infty \eta f''}{\delta}\frac{d\delta}{dx} \tag{29}$$

where term $I$ and $II$ above were rewritten as,

$$\text{term } I: \quad \frac{\partial f}{\partial y} = \frac{df}{d\eta}\frac{d\eta}{dy} = \frac{f'}{\delta} \tag{30}$$

$$\text{term } II: \quad \frac{\partial}{\partial y}\left(\eta f'\right) = \eta \frac{\partial f'}{\partial y} + f' \frac{\partial \eta}{\partial y} = \frac{\eta f''}{\delta} + \frac{f'}{\delta} \tag{31}$$

# Boundary Layer over a Flat Plate: Blasius Solution

Additionally,

$$\frac{\partial \psi}{\partial y} = \frac{\partial}{\partial y}\left(U_\infty \delta f\right) = U_\infty \delta \frac{\partial f}{\partial y} = U_\infty \delta \frac{df}{d\eta}\frac{d\eta}{dy} = \frac{U_\infty \delta\, f'}{\delta} = U_\infty\, f' \tag{32}$$

$$\frac{\partial^2 \psi}{\partial y^2} = \frac{\partial}{\partial y}\left[U_\infty\, f'\right] = U_\infty \frac{\partial f'}{\partial y} = U_\infty \frac{df'}{d\eta}\frac{d\eta}{dy} = \frac{U_\infty}{\delta} f'' \tag{33}$$

$$\frac{\partial^3 \psi}{\partial y^3} = \frac{\partial}{\partial y}\left[\frac{U_\infty}{\delta} f''\right] = \frac{U_\infty}{\delta}\frac{\partial f''}{\partial y} = \frac{U_\infty}{\delta}\frac{df''}{d\eta}\frac{d\eta}{dy} = \frac{U_\infty}{\delta^2} f''' \tag{34}$$

# Boundary Layer over a Flat Plate: Blasius Solution

Finally, after all these changes in variables, the equivalent x-momentum equation for the streamfunction $\psi$ can be rewritten as,

$$\Big[U_\infty f'\Big]\Big[-\frac{U_\infty \eta\, f''}{\delta}\frac{d\delta}{dx}\Big]-\Big[U_\infty \frac{d\delta}{dx}\Big(f-\eta f'\Big)\Big]\Big[\frac{U_\infty}{\delta}f''\Big]=\nu\Big[\frac{U_\infty}{\delta^2}f'''\Big]. \tag{35}$$

If this one is now divided by $\nu U_\infty/\delta^2$, then,

$$\frac{U_\infty \delta}{\nu}\frac{d\delta}{dx}\Big[-\eta f'\, f''-f\, f''+\eta\, f'\, f''\Big]=f''' \tag{36}$$

$$-\Big(\frac{U_\infty \delta}{\nu}\frac{d\delta}{dx}\Big)f\, f''=f'''. \tag{37}$$

Since $f$ and its derivatives do not explicitly depend on $x$, the only way the above equation can be valid is if

$$\frac{U_\infty\, \delta}{\nu}\frac{d\delta}{dx}=constant. \tag{38}$$

# Boundary Layer over a Flat Plate: Blasius Solution

Let's choose the constant to be equal to $1/2$ for convenience. Then upon integration one obtains,

$$\int \frac{U_\infty \delta}{\nu} d\delta = \int \frac{1}{2} dx \tag{39}$$

$$\frac{U_\infty}{\nu} \frac{\delta^2}{2} = \frac{x}{2} + C. \tag{40}$$

Since $\delta(x = 0)$, *i.e.* the boundary layer has zero thickness at the leading edge of the plate, $C = 0$,

$$\delta = \sqrt{\frac{\nu\, x}{U_\infty}}. \tag{41}$$

# Boundary Layer over a Flat Plate: Blasius Solution

Then, the differential equation for $f$ is,

$$-\frac{1}{2} f\, f'' = f''' \tag{42}$$

and the boundary conditions on $f$ are

$$f'(\infty) = 1 \tag{43}$$

$$f(0) = f'(0) = 0; \tag{44}$$

being the far-field, and non-slip conditions correspondingly.

**Note:** The above ordinary differential equation is nonlinear. In 1908 a series solution to this equation was given by Blasius, who was a student of Prandtl. Today of course, one can solve this numerically using the Runge-Kutta technique.

# Boundary Layer over a Flat Plate: Blasius Solution

From the numerical solution (illustrated in the Figure), one can determine the following length scales,

- $\delta_{99} = 4.9\sqrt{\frac{\nu\, x}{U_\infty}}$
- $\delta^* = 1.72\sqrt{\frac{\nu\, x}{U_\infty}}$
- $\theta = 0.664\sqrt{\frac{\nu\, x}{U_\infty}}$
- $H = \frac{\delta^*}{\theta} = 2.59$

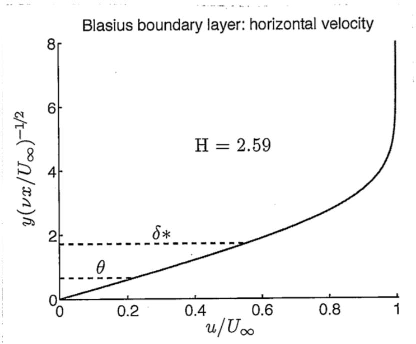


Figure: The Blasius similarity solution of velocity distribution in a laminar boundary layer on a flat plate. The momentum thickness $\theta$, and displacement $\delta^*$ are also indicated.

# Boundary Layer over a Flat Plate: Blasius Solution

From these results, one can see:

- The boundary layer thickness exhibits a parabolic growth in $x$, *i.e.* $\delta \sim \sqrt{x}$.
- For air at standard conditions flowing at a velocity of $U_\infty = 1m/s$, the Reynolds number at a distance of $x = 1m$ from the leading edge of a flat plate is $Re_x = 6.7 \times 10^4$. The flow should be laminar since this $Re_x$ value is below the critical Reynolds number for the onset of turbulence ($Re_c \approx 1 \times 10^5$).
- From the relationship for the boundary layer thickness, one can then predict that $\delta_{99} = 2cm$ at $x = 1m$. Therefore, one can see that indeed the boundary layer is thin, *i.e.* $\delta_{99}/x << 1$.
- Experimental evidence supports the Blasius solution.

# Boundary Layer over a Flat Plate: Blasius Solution

**Vertical Velocity**

The vertical (transverse) velocity, $v$, may be determined from the streamfunction,

$$v = -\frac{\partial \psi}{\partial x} = -\frac{\partial}{\partial x}\Big[U_\infty\,\delta\,f\Big] = -U_\infty f\,\frac{d\delta}{dx} - U_\infty \underbrace{\delta\frac{\partial f}{\partial x}}_{\frac{df}{d\eta}\frac{d\eta}{dx} = -f'\left(\frac{y}{\delta^2}\frac{d\delta}{dx}\right)} \tag{45}$$

$$v = -U_\infty \underbrace{\frac{d\delta}{dx}}_{\frac{d}{dx}\left[\frac{\nu x}{U_\infty}\right]^{1/2} = \frac{1}{2}\left[\frac{\nu}{U_\infty x}\right]^{1/2}} \Big[f - \underbrace{\frac{y}{\delta}}_{\equiv \eta} f'\Big] \tag{46}$$

$$v = -\frac{U_\infty}{2}\left[\frac{\nu}{U_\infty x}\right]^{1/2}\Big[f - \eta f'\Big] \tag{47}$$

$$v = \frac{1}{2}\left(\frac{\nu U_\infty}{x}\right)^{1/2}\Big(\eta\,f' - f\Big) \tag{48}$$

# Boundary Layer over a Flat Plate: Blasius Solution

- The vertical velocity increases from zero at the wall to a maximum value of $v = 0.86\sqrt{\nu U_\infty / x}$ at the edge of the boundary layer.
- One can then see that, whereas $u/U_\infty \sim \mathcal{O}(1)$, the non-dimensional vertical velocity $v/U_\infty \sim (Re_x^{-1/2})$.
- This guarantees that $u/U_\infty >> v/U_\infty$ as long as $Re_x$ is large.

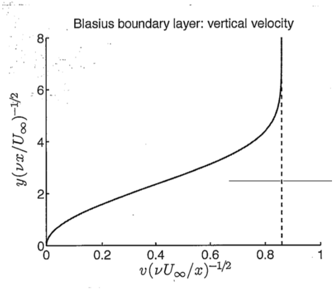


Figure: Vertical profile of the vertical velocity.

## Skin Friction, $C_f$

The skin friction coefficient represents the non-dimensional wall shear stress and is defined as

$$C_f = \frac{\tau_w}{1/2\rho U_\infty^2}, \tag{49}$$

where the wall shear stress (for a Newtonian fluid) is given by

$$\tau_w = \mu \frac{\partial u}{\partial y}\bigg|_{y=0}. \tag{50}$$

# Boundary Layer over a Flat Plate: Blasius Solution

In terms of the streamfunction,

$$\tau_w = \mu \frac{\partial^2 \psi}{\partial y^2}\bigg|_{y=0} = \frac{\mu U_\infty}{\delta} f''(\eta = 0). \tag{51}$$

From the numerical solution, one can find that $f''(\eta = 0) = 0.332$. Therefore,

$$\tau_w = \frac{0.332 \rho U_\infty^2}{\sqrt{Re_x}} \tag{52}$$

and the skin friction coefficient is

$$C_f = \frac{0.664}{\sqrt{Re_x}}. \tag{53}$$

Experimental data compares well to the theoretical prediction above as long as the flow remains laminar. When the flow becomes turbulent the skin friction coefficient increases dramatically (transition occurs around $Re_x \approx 1 \times 10^5$).