Intermediate Fluid Mechanics

Lecture 5: Introduction to Index Notation

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Lecture Objectives

In this lecture we will learn about index notation.

- This is a shorthand notation invented by Einstein and that is very useful in treating differential equations in vectorial space.
- In this notation one uses *indices* as subscripts to represent the different components of vectors and tensors (which will be defined more explicitly later).

Chapter Overview

Index Notation

Let's consider the vector $\vec{a}$ as

$$\vec{a} = a_1\,\hat{e}_1 + a_2\,\hat{e}_2 + a_3\,\hat{e}_3, \tag{1}$$

where a_1, a_2, a_3 represent the components of vector $\vec{a}$ along the axis of an Euclidean coordinate system represented by the unit vectors $\hat{e}_1$, $\hat{e}_2$, $\hat{e}_3$.

+ Note that the same vector can be written more compactly as

$$\vec{a} = a_1\,\hat{e}_1 + a_2\,\hat{e}_2 + a_3\,\hat{e}_3 = \sum_{1=1}^{3} a_i \hat{e}_i. \tag{2}$$

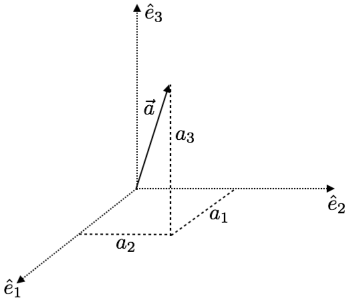

Figure: Representation of a given vectorial space.

Index Notation (continued ...)

$$\vec{a} = a_1\,\hat{e}_1 + a_2\,\hat{e}_2 + a_3\,\hat{e}_3 = \sum_{1=1}^{3} a_i \hat{e}_i. \tag{3}$$

In index notation, the summation symbol is dropped as well as the unit vectors, and the vector is simply represented by its component values,

$$\vec{a} \Longleftrightarrow a_i, \quad \text{for } i = 1,\, 2,\, 3, \tag{4}$$

where 'i' is referred as a *free index* and can take on three values in order to specify the components of the vector in three-dimensional space.

Chapter Overview

Arithmetic operations using index notation

(i) Multiplication of a vector ($\vec{a}$) by a a scalar m:

$$m\,\vec{a} = \vec{c} \tag{5}$$

$$m(a_1\hat{e}_1 + a_2\hat{e}_2 + a_3\hat{e}_3) = c_1\hat{e}_1 + c_2\hat{e}_2 + c_3\hat{e}_3 \tag{6}$$

$$m\,a_1\hat{e}_1 + m\,a_2\hat{e}_2 + m\,a_3\hat{e}_3) = c_1\hat{e}_1 + c_2\hat{e}_2 + c_3\hat{e}_3 \tag{7}$$

Which can be written in three different equations, one for each vectorial direction,

$$\hat{e}_1 : \quad m\,a_1 = c_1 \tag{8}$$

$$\hat{e}_2 : \quad m\,a_2 = c_2 \tag{9}$$

$$\hat{e}_3 : \quad m\,a_3 = c_3 \tag{10}$$

$$\tag{11}$$

or in short as,

$$m\,a_i = c_i \quad \text{for } i = 1,\, 2,\, 3, \tag{12}$$

(ii) Addition of two vectors ($\vec{a}$ and $\vec{b}$)

$$\vec{a} + \vec{b} = \vec{c} \tag{13}$$

$$(a_1\hat{e}_1 + a_2\hat{e}_2 + a_3\hat{e}_3) + (b_1\hat{e}_1 + b_2\hat{e}_2 + b_3\hat{e}_3) = (c_1\hat{e}_1 + c_2\hat{e}_2 + c_3\hat{e}_3) \tag{14}$$

which again can be splitted in three different equations,

$$\begin{aligned} \hat{e}_1 : \quad & a_1 + b_1 = c_1 && (15) \\ \hat{e}_2 : \quad & a_2 + b_2 = c_2 && (16) \\ \hat{e}_3 : \quad & a_3 + b_3 = c_3 && (17) \\ & && (18) \end{aligned}$$

or in short as,

$$a_i + b_i = c_i \quad \text{for } i = 1,\ 2,\ 3, \tag{19}$$

Arithmetic operations using index notation (continued)

+ Note: Observations from (i) and (ii) lead to the following rules.

Rule 1: A free index can be represented by any letter. For example, the following equation

$$a_i + b_i = c_i \tag{20}$$

is identical to

$$a_j + b_j = c_j \tag{21}$$

which is identical to

$$a_p + b_p = c_p \tag{22}$$

and so on ...

Arithmetic operations using index notation (continued)

Rule 2: If a free index appears in one term of an equation, then that same index must appear as a free index in ALL terms of the equation.

For example, the following equation is invalid,

$$a_i + b_i = c_j \tag{23}$$

Arithmetic operations using index notation (continued)

(iii) Subtraction of two vectors ($\vec{u}$ and $\vec{v}$)

$$a\vec{u} - b\vec{v} = c\vec{w} \tag{24}$$

- Here, each vector can be represented in index notation using a free index as a subscript *i.e.* u_k, v_k and w_k.
- In index notation there are no free indices associated with the scalar quantities a, b, and c.
- To satisfy Rule 2, one must ensure that all terms have the same free index.

Hence:

$$a\, u_k - b\, v_k = c\, w_k. \tag{25}$$

Note:

Remember that it is inherently implied that the free index can take on the values $k = 1, 2, 3$, which in turn yields three different equations. Therefore, from now on one can cease to write 'for $k = 1, 2, 3$' following the equation.

Arithmetic operations using index notation (continued)

(iv) Dot product of two vectors ($\vec{a}$ and $\vec{b}$)

$$\vec{a} \cdot \vec{b} = c \tag{26}$$

$$(a_1 \hat{e}_1 + a_2 \hat{e}_2 + a_3 \hat{e}_3) \cdot (b_1 \hat{e}_1 + b_2 \hat{e}_2 + b_3 \hat{e}_3) = c \tag{27}$$

$$a_1\, b_1 + a_2\, b_2 + a_3\, b_3 = c. \tag{28}$$

This can be written compactly using a summation symbol,

$$\sum_i^3 a_i\, b_i = c. \tag{29}$$

In practice the summation symbol is dropped, and the above equation can be simply be rewritten as

$$a_i\, b_i = c. \tag{30}$$

In this case, i is the repeating index because it appears twice in one term.

Arithmetic operations using index notation (continued)

Rule 3: When the same index appears twice in a single term, then it is referred to as a *repeating index*, and summation over each value of the index (1, 2, 3) is implied.

For example, suppose one encounters the following term in an equation,

$$m\, d_k\, c_k\, u_j \tag{31}$$

Above, k is repeated index, and j is a free index. This means that one would need to sum the expression for $k = 1,\, 2,\, 3$, or equivalently

$$m\, d_k\, c_k\, u_j = \sum_{k=1}^{3} m\, d_k\, c_k\, u_j = m\, u_j\, (d_1\, c_1 + d_2\, c_2 + d_3\, c_3) \tag{32}$$

Arithmetic operations using index notation (continued)

Rule 4: An index cannot repeat more than twice in any given term.

For example, the following index expression is invalid,

$$u_j \, v_j \, w_j = p_j \tag{33}$$

Arithmetic operations using index notation (continued)

(v) Tensors

- Zeroth-order tensor: This is a scalar, because there are zero free indices, and hence, only one component associated with this quantity.
- First-order tensor: This is a vector. In this case there is one free index and hence three components associated with this quantity in a 3D-vectorial space.

$$\text{vector notation: } \vec{u} \iff \text{index notation: } u_i \tag{34}$$

  where i takes on values from 1 to 3 in a 3D vectorial space.
- Second-order tensor: This is analogous to a 3×3 matrix because there are two free indices, and hence nine components associated with this quantity in a 3D vectorial space. In this case, the index notation is given by A_{ij}. For a second-order tensor expressed as A_{ij}, for example, the convention is that the first index represents the row, while the second index represents the column in the matrix representation.
- Third-order tensor: This quantity has three free indices and hence a total of 27 components in a 3D vectorial space. An example would be $C_{i,j,k}$ where $i,\ j,\ k = 1,\ 2,\ 3$ respectively.

Arithmetic operations using index notation (continued)

(vi) The Kronecker delta tensor (δ_{ij})

This matrix represents the identity matrix (I) such that its matrix representation corresponds to

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{35}$$

This can also be written in more compact form as,

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \tag{36}$$

The identity matrix exhibits a special property. Namely that given any vector $\vec{a}$, one can write

$$I\,\vec{a} = \vec{a}, \tag{37}$$

or expressed in index notation

$$\delta_{ij}\,a_i = a_j. \tag{38}$$

Arithmetic operations using index notation (continued)

(vii) Permutation tensor (ε_{ijk}):

This quantity represents a special thrid-order tensor defined as

$$\varepsilon_{ijk} = \begin{cases} 1, & \text{if } \ i, j, k \ \text{ permute in a cyclic order} \\ 0, & \text{if any of the two indices are identical} \\ -1, & \text{if they permut anticylcically.} \end{cases} \tag{39}$$

Depending on the value and order of the three indices, the permutation tensor will take on a value of 1, 0, and/or -1.

Arithmetic operations using index notation (continued)

Useful properties of the permutation tensor:

1. an index of ε_{ijk} can be moved two places (either to the right or to the left) without changing its value. That is,

$$\varepsilon_{ijk} = \varepsilon_{jki} = \varepsilon_{kij} \tag{40}$$

2. swapping two adjacent indices reverses the sign, that is,

$$\varepsilon_{ijk} = -\varepsilon_{ikj} \quad \textit{or} \quad \varepsilon_{ijk} = -\varepsilon_{jik} \tag{41}$$

3. the epsilon-delta relation allows us to write the product of two permutation tensors having one repeating index in terms of four different kronecker delta tensors as follows,

$$\varepsilon_{ijk}\,\varepsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl} \tag{42}$$

Arithmetic operations using index notation (continued)

(viii) Cross product of two vectors ($\vec{a}$ and $\vec{b}$)

$$\vec{a} \times \vec{b} = \vec{c} \tag{43}$$

$$(a_2\, b_3 - a_3\, b_2)\hat{e}_1 + (a_3\, b_1 - a_1\, b_3)\hat{e}_2 + (a_1\, b_2 - a_2 b_1)\hat{e}_3 = \vec{c} \tag{44}$$

$$\sum_{i=1}^{3} \sum_{j=1}^{3} a_i\, b_j\, \varepsilon_{ijk} = c_k \tag{45}$$

$$a_i\, b_j\, \varepsilon_{ijk} = c_k \tag{46}$$

(ix) The Curl of the velocity field

To decipher the physical meaning of the curl, one can use the **Stokes Theorem**.

$$(\vec{\nabla} \times \vec{u}) \cdot \hat{e}_3 = \lim_{s_3 \to 0} \frac{1}{s_3} \oint \vec{u} \cdot d\vec{s}, \qquad (47)$$

where s_3 represents the surface area having an outward unit normal in the $\hat{e}_3$ direction.

• The line integral is performed by integrating the component of $\vec{u}$ aligned with $d\vec{s}$ at every point along the contour C_3 that bounds the surface area s_3.

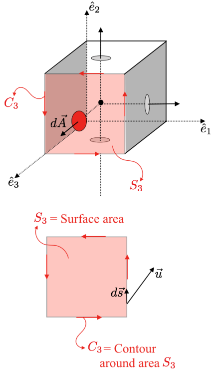

Figure: Representation of a fluid element, and the corresponding vectorial representation.

Arithmetic operations using index notation (continued)

(ix) The Curl of the velocity field (continued)

$$(\vec{\nabla} \times \vec{u}) \cdot \hat{e}_3 = \lim_{s_3 \to 0} \frac{1}{s_3} \oint \vec{u} \cdot d\vec{s}, \tag{48}$$

- If $\vec{u}$ represents the velocity and if $(\vec{\nabla} \times \vec{u}) \cdot \hat{e}_3 \neq 0$, then this means there is some net rotation about the $\hat{e}_3$ axis.
- A similar argument can be made for the other two components of the curl.
- That is, if $(\vec{\nabla} \times \vec{u}) \cdot \hat{e}_2 \neq 0$ then there is a net rotation around the $\hat{e}_2$ axis
- If $(\vec{\nabla} \times \vec{u}) \cdot \hat{e}_1 \neq 0$ then there is a net rotation around the $\hat{e}_1$ axis.

Arithmetic operations using index notation (continued)

Practice Exercise:

$\Longrightarrow$ Use index notation to demonstrate that $\delta_{mi}\delta_{jm} = \delta_{ij}$.

Arithmetic operations using index notation (continued)

Practice Exercise:

$\Longrightarrow$ Use index notation to demonstrate that $\delta_{mi}\delta_{jm} = \delta_{ij}$.

To demonstrate this equality we will explicitly expand out the implied summation over m (the repeating index):

$$a_{ij} = \sum_{m=1}^{3} \delta_{mi}\delta_{jm} = \delta_{1i}\delta_{j1} + \delta_{2i}\delta_{j2} + \delta_{3i}\delta_{j3}. \tag{49}$$

Further i, $j = 1, 2, 3$ given that we are treating a 3D vectorial space. Hence,

$$\begin{aligned}
i = 1, j = 1: &\quad a_{11} = \delta_{11}\delta_{11} + \delta_{21}\delta_{12} + \delta_{31}\delta_{13} = 1 && (50)\\
i = 1, j = 2: &\quad a_{12} = \delta_{11}\delta_{21} + \delta_{21}\delta_{22} + \delta_{31}\delta_{23} = 0 && (51)\\
i = 1, j = 3: &\quad a_{13} = \delta_{11}\delta_{31} + \delta_{21}\delta_{32} + \delta_{31}\delta_{33} = 0 && (52)\\
& && (53)
\end{aligned}$$

(and so on ...)

Arithmetic operations using index notation (continued)

Practice Exercise:

$\Longrightarrow$ Use index notation to demonstrate that $\delta_{mi}\delta_{jm} = \delta_{ij}$.

$$i = 1,\, j = 1: \quad a_{11} = \delta_{11}\delta_{11} + \delta_{21}\delta_{12} + \delta_{31}\delta_{13} = 1 \tag{54}$$

$$i = 1,\, j = 2: \quad a_{12} = \delta_{11}\delta_{21} + \delta_{21}\delta_{22} + \delta_{31}\delta_{23} = 0 \tag{55}$$

$$i = 1,\, j = 3: \quad a_{13} = \delta_{11}\delta_{31} + \delta_{21}\delta_{32} + \delta_{31}\delta_{33} = 0 \tag{56}$$

$$\tag{57}$$

Until obtain the full matrix form for a_{ij} that looks like,

$$a_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \delta_{ij}. \tag{58}$$

Arithmetic operations using index notation (continued)

Practice Exercise:

$\Longrightarrow$ Use index notation to demonstrate that $\delta_{mi}\delta_{jm} = \delta_{ij}$.

Note that one could have proceed to this a bit more directly by acknowledging the fact that $\delta_{mi}\delta_{jm}$ represents a tensor given that there are two free indices i and j.

Thus one can write,

$$a_{ij} = \delta_{ij}\delta_{jm}. \tag{59}$$

But this will only be not zero if $m = j$, in which case, $a_{ij} = \delta_{ij}$.