Conservation Laws

- Conservation of mass

  R.O.C of mass = 0

- Conservation of momentum

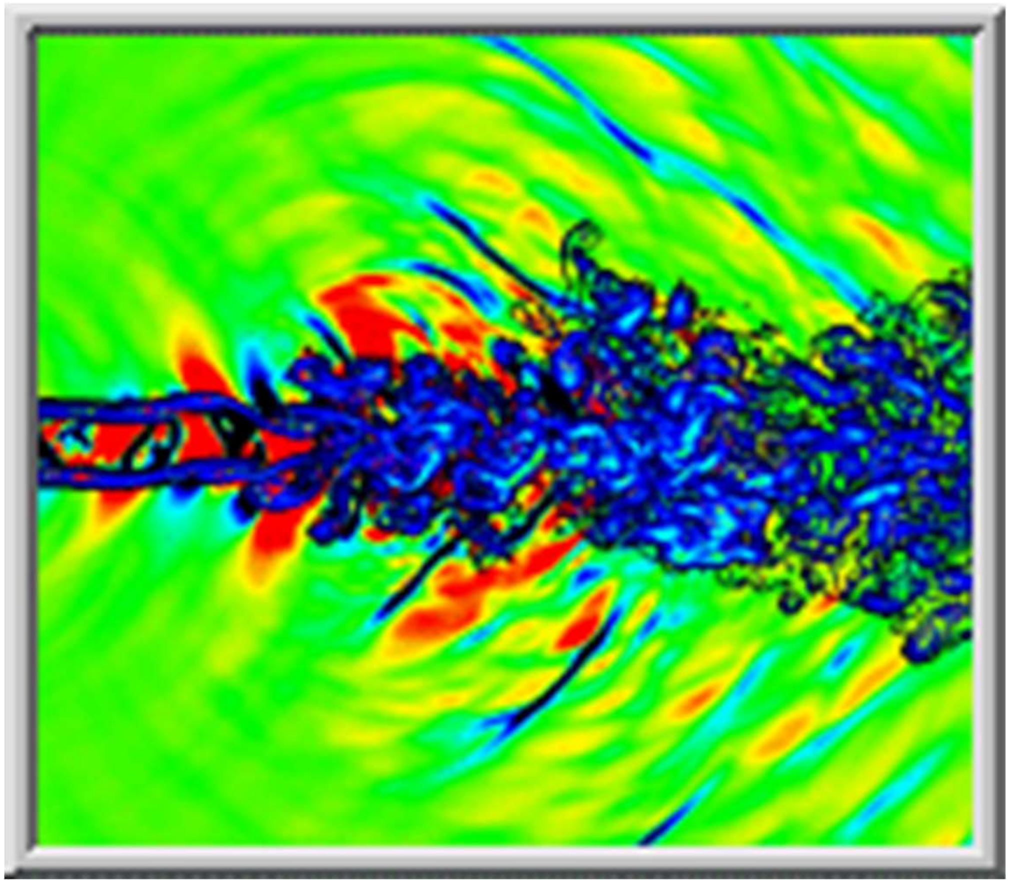
  $$\text{R.O.C of momentum} = \vec{F}_{pressure} + \vec{F}_{viscous} + \vec{F}_{BODY}$$

- Conservation of energy

  $$\text{R.O.C of energy} = W_{pressure} + W_{viscous} + W_{Body} + Q$$

APPLY TO FLUID MATERIAL, NOT THE SPACE THROUGH WHICH IT FLOWS

Supersonic Turbulent Jet Flow and Near Acoustic Field

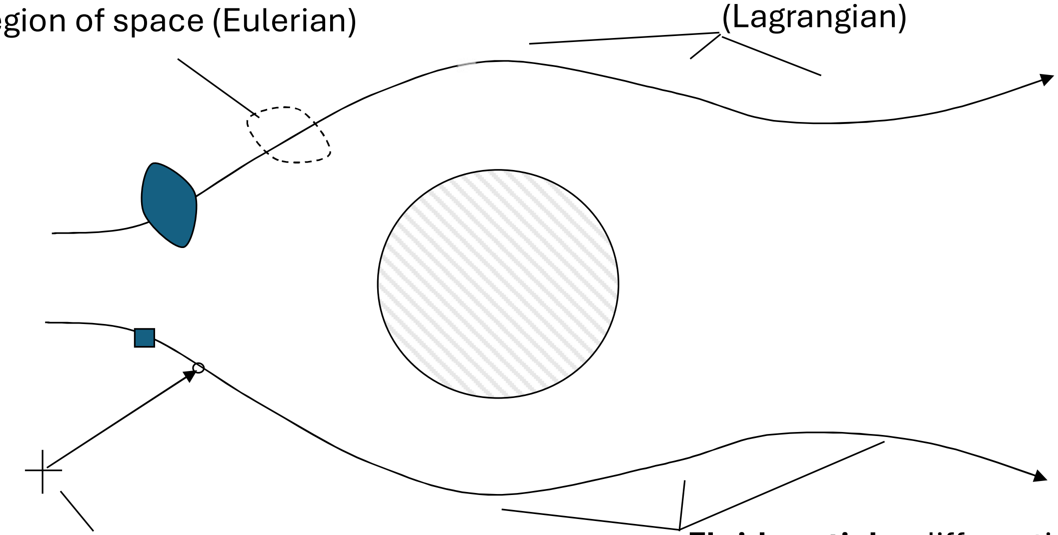


Freund *at al*. (1997)
Stanford Univ.
DNS

Perspectives

Eulerian Perspective – FLOW AS SEEN AT FIXED LOCATIONS IN SPACE OR OVER FIXED VOLUMES

Lagrangian Perspective – THE FLOW AS SEEN BY FLUID MATERIAL



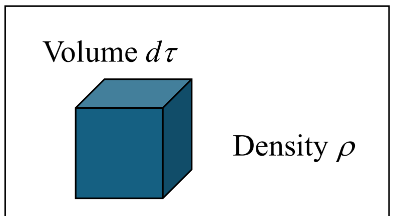
Conservation of Mass

From a Lagrangian Perspective

$$\Rightarrow \vec{\nabla}\cdot\vec{V} = \frac{\text{ROC vol}}{\text{unit vol}} = \frac{\frac{\partial \delta\tau}{\partial t}}{\delta\tau}$$

Law: Rate of Change of Mass of Fluid Material = 0

For a Fluid Particle:



ρ, ρ

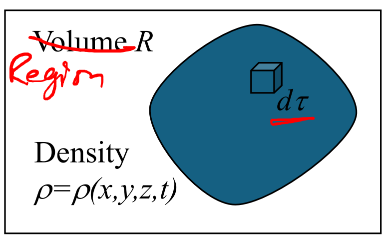
$$\left.\frac{\partial(\rho\, d\tau)}{\partial t}\right|_{PARTICLE} = 0$$

$$\Rightarrow \rho \left.\frac{\partial(d\tau)}{\partial t}\right|_{PART} + d\tau \left.\frac{\partial \rho}{\partial t}\right|_{PART} = 0$$

$$\Rightarrow \rho\, \vec{\nabla}\cdot\vec{V} + \left.\frac{\partial \rho}{\partial t}\right|_{PART} = 0$$

$$\Rightarrow \rho\, \vec{\nabla}\cdot\vec{V} + \frac{D\rho}{Dt} = 0 \qquad (3)$$

For a Fluid System:



$$\left.\frac{\partial}{\partial t}\int_R \rho\, d\tau\right|_{PART} = 0$$

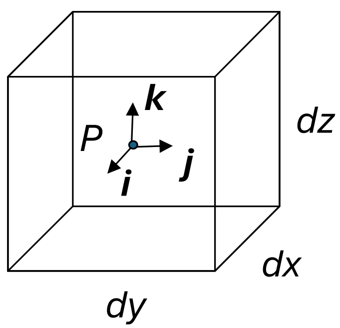
$$\frac{D}{Dt}\int_R \rho\, d\tau = 0$$

$\frac{D}{Dt} = \left.\frac{\partial}{\partial t}\right|_{part}$ is referred to as

the SUBSTANTIAL DERIVATIVE
(or total, or material, or Lagrangian...)

Conservation of Momentum

From a Lagrangian Perspective (Fluid Particle)



Law: Rate of Change of Momentum = $\mathbf{F}_{\text{body}}+\mathbf{F}_{\text{pressure}}+\mathbf{F}_{\text{viscous}}$

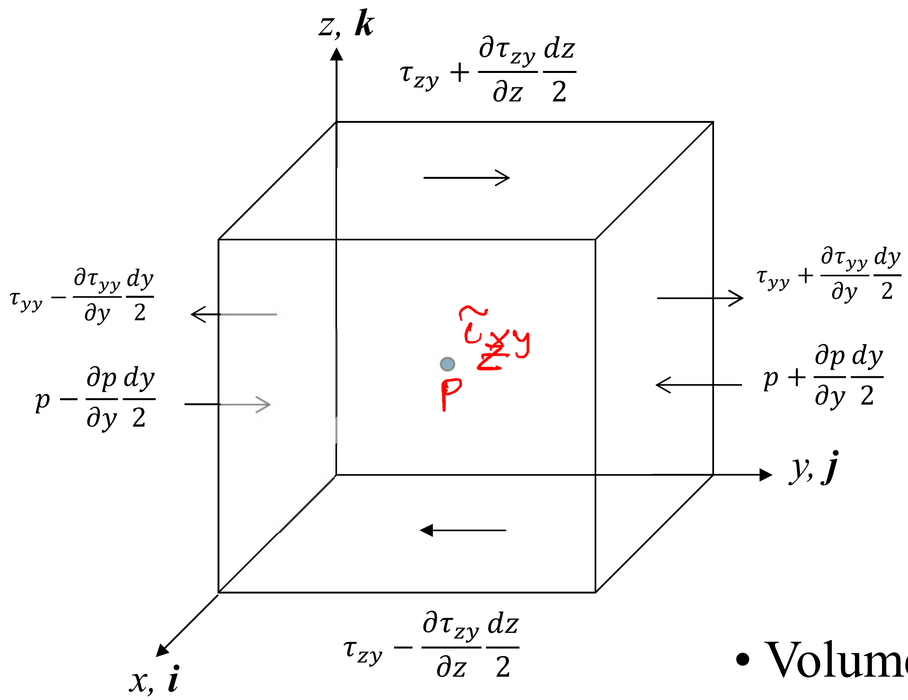
ROC of Momentum $= \left.\dfrac{\partial \rho d\tau \mathbf{V}}{\partial t}\right|_{part} = \rho d\tau \left.\dfrac{\partial \mathbf{V}}{\partial t}\right|_{part} = \rho d\tau \dfrac{D\mathbf{V}}{Dt}$

$\mathbf{F}_{\text{body}}$: $= \mathbf{f}\rho d\tau$

Elemental Volume, Surface Forces

Sides of volume have lengths dx, dy, dz

y-component



$\tau_{xy} + \frac{\partial \tau_{xy}}{\partial y}\frac{dx}{2}$

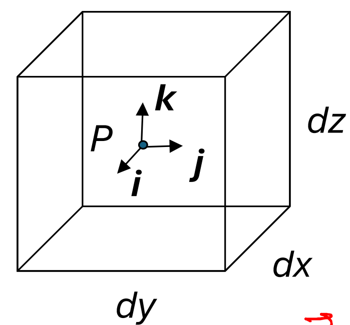
$\tau_{xy} - \frac{\partial \tau_{xy}}{\partial y}\frac{dx}{2}$

On front and rear faces

- Volume dτ = dxdydz
- Density ρ
- Velocity $\mathbf{V}$

Conservation of Momentum

From a Lagrangian Perspective (Fluid Particle)



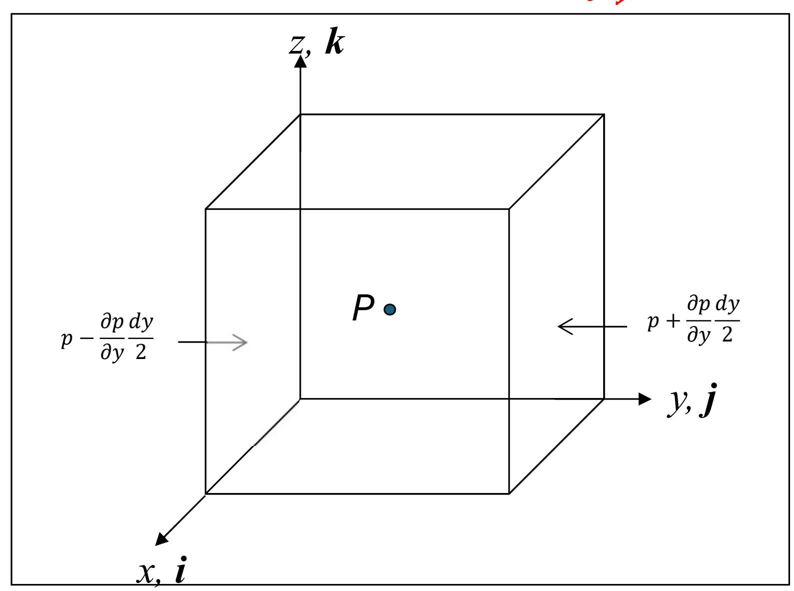
Law: Rate of Change of Momentum = $\mathbf{F}_{body}$+$\mathbf{F}_{pressure}$+$\mathbf{F}_{viscous}$

ROC of Momentum $= \left.\dfrac{\partial \rho d\tau \mathbf{V}}{\partial t}\right|_{part} = \rho d\tau \left.\dfrac{\partial \mathbf{V}}{\partial t}\right|_{part} = \underline{\underline{\rho d\tau \dfrac{D\mathbf{V}}{Dt}}}$

$\mathbf{F}_{body}$: $= \underline{\underline{\mathbf{f}\rho d\tau}}$

$\mathbf{F}_{pressure}$: $y\ component =$ - - -

$-\dfrac{\partial p}{\partial x} d\tau\, \vec{i}\ ;\quad -\dfrac{\partial p}{\partial z} d\tau\, \vec{k}$



Conservation of M

From a Lagrangian Perspective (Fluid Pa

Law: Rate of Change of Momentum = $\mathbf{F}_{bo}$

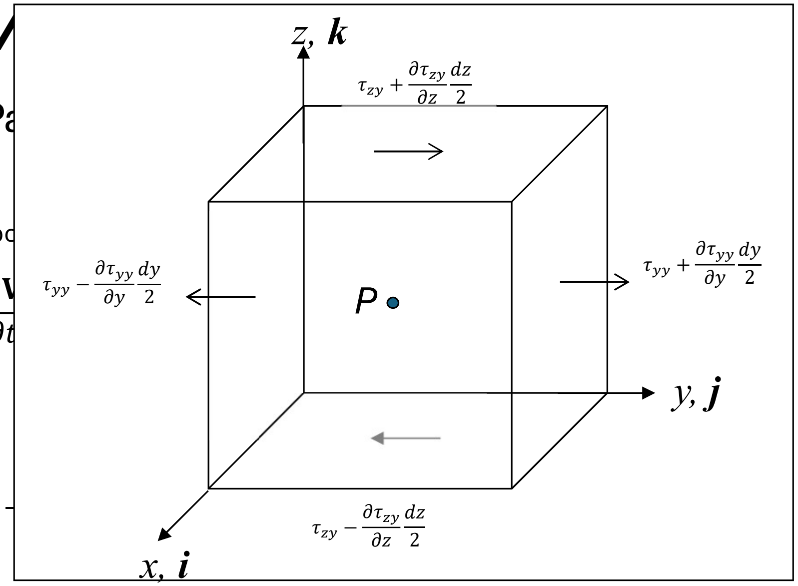
ROC of Momentum $= \left.\dfrac{\partial \rho d\tau \mathbf{V}}{\partial t}\right|_{part} = \rho d\tau \dfrac{\partial \mathbf{V}}{\partial t}$

$\mathbf{F}_{body}$: $= \mathbf{f}\rho d\tau$

$\mathbf{F}_{pressure}$: $y\ component = \left(p - \dfrac{\partial p}{\partial y}\dfrac{1}{2}dy\right)dxdz\mathbf{j}\ -$

$so\ \mathbf{F}_{pressure} = -\nabla p\ d\tau$

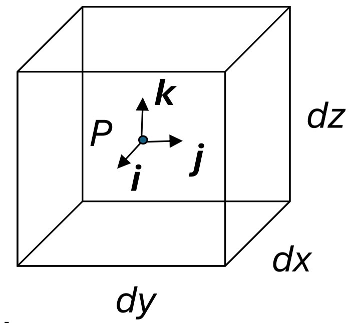
$\mathbf{F}_{viscous}$: $y\ component..$



Conservation of Momentum

From a Lagrangian Perspective (Fluid Particle)

Law: Rate of Change of Momentum = $\mathbf{F}_{\text{body}}+\mathbf{F}_{\text{pressure}}+\mathbf{F}_{\text{viscous}}$



ROC of Momentum $\quad = \left.\dfrac{\partial \rho d\tau \mathbf{V}}{\partial t}\right|_{part} = \rho d\tau \left.\dfrac{\partial \mathbf{V}}{\partial t}\right|_{part} = \rho d\tau \dfrac{D\mathbf{V}}{Dt}$

$\mathbf{F}_{\text{body}}$: $\quad = \mathbf{f}\rho d\tau$

$\mathbf{F}_{\text{pressure}}$: $\quad y\ component = \left(p - \dfrac{\partial p}{\partial y}\dfrac{1}{2}dy\right)dxdz\mathbf{j} - \left(p + \dfrac{\partial p}{\partial y}\dfrac{1}{2}dy\right)dxdz\mathbf{j} = -\dfrac{\partial p}{\partial y}d\tau\mathbf{j}$

$so \quad \mathbf{F}_{pressure} = -\nabla p\ d\tau$

$\mathbf{F}_{\text{viscous}}$: $\quad y\ component\ldots \quad \left(\tau_{yy} + \dfrac{\partial \tau_{yy}}{\partial y}\dfrac{1}{2}dy\right)dxdz\mathbf{j} - \left(\tau_{yy} - \dfrac{\partial \tau_{yy}}{\partial y}\dfrac{1}{2}dy\right)dxdz\mathbf{j} +$

$$\left(\tau_{xy} + \frac{\partial \tau_{xy}}{\partial x}\frac{1}{2}dx\right)dydz\mathbf{j} - \left(\tau_{xy} - \frac{\partial \tau_{xy}}{\partial x}\frac{1}{2}dx\right)dydz\mathbf{j} +$$

$$\left(\tau_{zy} + \frac{\partial \tau_{zy}}{\partial z}\frac{1}{2}dz\right)dxdy\mathbf{j} - \left(\tau_{zy} - \frac{\partial \tau_{zy}}{\partial z}\frac{1}{2}dz\right)dxdy\mathbf{j} = \overbrace{\left(\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z}\right)\mathbf{j}d\tau}^{\text{Likewise for } x \text{ and } z}$$

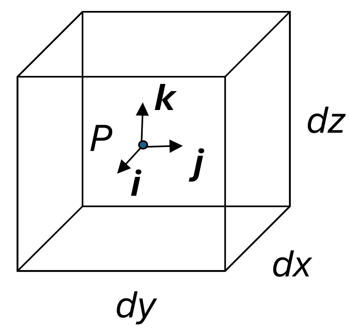
So, $\boxed{\rho\dfrac{D\mathbf{V}}{Dt} = \rho\mathbf{f} - \nabla p + (\nabla.\boldsymbol{\tau}_x)\mathbf{i} + (\nabla.\boldsymbol{\tau}_y)\mathbf{j} + (\nabla.\boldsymbol{\tau}_z)\mathbf{k}}$ where

$$\boldsymbol{\tau}_x = \tau_{xx}\mathbf{i} + \tau_{yx}\mathbf{j} + \tau_{zx}\mathbf{k}$$
$$\boldsymbol{\tau}_y = \tau_{xy}\mathbf{i} + \tau_{yy}\mathbf{j} + \tau_{zy}\mathbf{k}$$
$$\boldsymbol{\tau}_z = \tau_{xz}\mathbf{i} + \tau_{yz}\mathbf{j} + \tau_{zz}\mathbf{k}$$

Conservation of Energy

From a Lagrangian Perspective (Fluid Particle)

Law: Rate of Change of Energy = W_{body}+$W_{pressure}$+$W_{viscous}$+Q



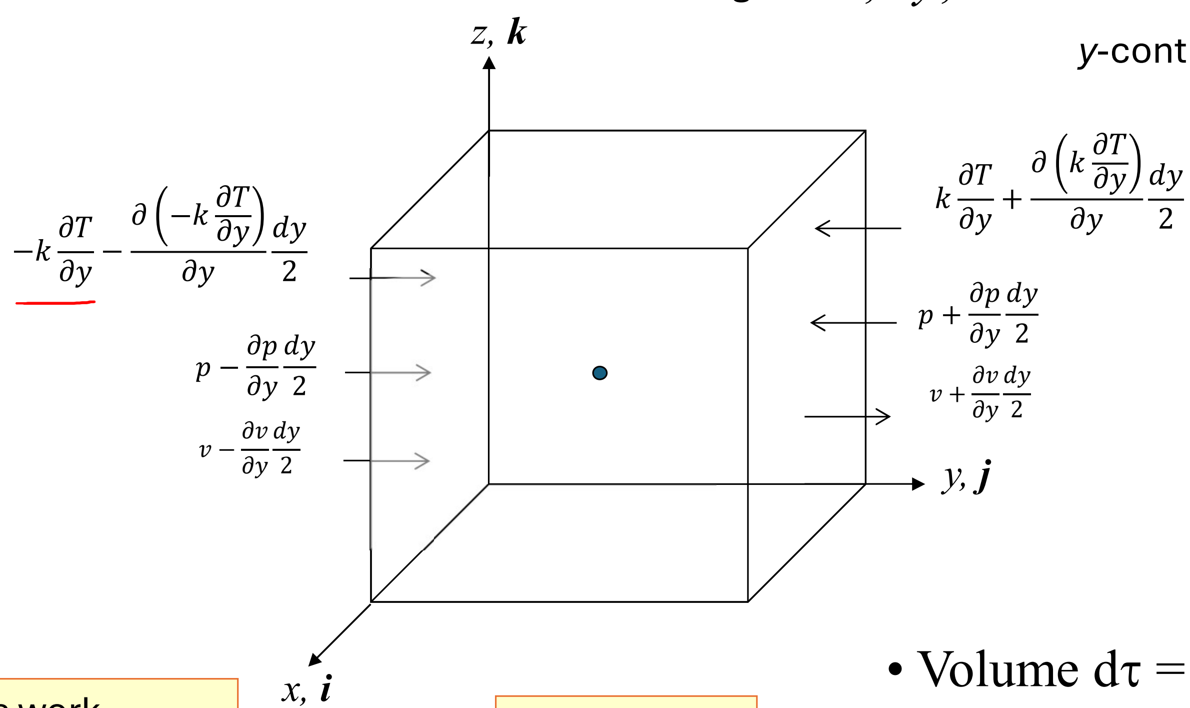
ROC of Energy $$= \left.\frac{\partial \rho d\tau (e + \frac{1}{2}V^2)}{\partial t}\right|_{part} = \underline{\underline{\rho d\tau \frac{D(e + \frac{1}{2}V^2)}{Dt}}}$$

W_{body} $$= \underline{\underline{\mathbf{f}.\mathbf{V}\rho d\tau}}$$

Elemental Volume, Surface Force Work and Heat Transfer

Sides of volume have lengths $\mathrm{d}x$, $\mathrm{d}y$, $\mathrm{d}z$

y-contributions



Viscous work requires expansion of *v* velocity on all six sides

Velocity components *u, v, w*

- Volume $\mathrm{d}\tau = \mathrm{d}x\mathrm{d}y\mathrm{d}z$
- Density ρ
- Velocity $\mathbf{V}$

Conservation of Energy

From a Lagrangian Perspective (Fluid Particle)

Law: Rate of Change of Energy = W_{body}+$W_{pressure}$+$W_{viscous}$+Q

ROC of Energy $= \left.\dfrac{\partial \rho d\tau(e+\frac{1}{2}V^2)}{\partial t}\right|_{part} = \underline{\underline{\rho d\tau \dfrac{D(e+\frac{1}{2}V^2)}{Dt}}}$

W_{body} $= \underline{\underline{\mathbf{f}.\mathbf{V}\rho d\tau}}$

$W_{pressure}$ $= \underline{\underline{-\nabla.(p\mathbf{V})\ d\tau}}$

$W_{viscous}$ $= \underline{\underline{(\nabla.(u\boldsymbol{\tau}_x)+\nabla.(v\boldsymbol{\tau}_y)+\nabla.(w\boldsymbol{\tau}_z)\ )d\tau}}$

Q: $y\ contribution = \left(k\dfrac{\partial T}{\partial y}+\dfrac{\partial}{\partial y}\left(k\dfrac{\partial T}{\partial y}\right)\dfrac{1}{2}dy\right)dxdz-\left(k\dfrac{\partial T}{\partial y}-\dfrac{\partial}{\partial y}\left(k\dfrac{\partial T}{\partial y}\right)\dfrac{1}{2}dy\right)dxdz=\dfrac{\partial}{\partial y}\left(k\dfrac{\partial T}{\partial y}\right)d\tau$

$so\quad \underline{\underline{Q=\nabla.(k\nabla T)\ d\tau}}$

So, $$\boxed{\rho\frac{D(e+\frac{1}{2}V^2)}{Dt}=\rho\mathbf{f}.\mathbf{V}-\nabla.(p\mathbf{V})+\nabla.(u\boldsymbol{\tau}_x)+\nabla.(v\boldsymbol{\tau}_y)+\nabla.(w\boldsymbol{\tau}_z)+\nabla.(k\nabla T)}$$

Equations for Changes Seen From a Lagrangian Perspective

Differential Form (for a particle)

$$\frac{D\rho}{Dt} = -\rho\nabla.\mathbf{V}$$

$$\rho\frac{D\mathbf{V}}{Dt} = \rho\mathbf{f} - \nabla p + (\nabla.\boldsymbol{\tau}_x)\mathbf{i} + (\nabla.\boldsymbol{\tau}_y)\mathbf{j} + (\nabla.\boldsymbol{\tau}_z)\mathbf{k}$$

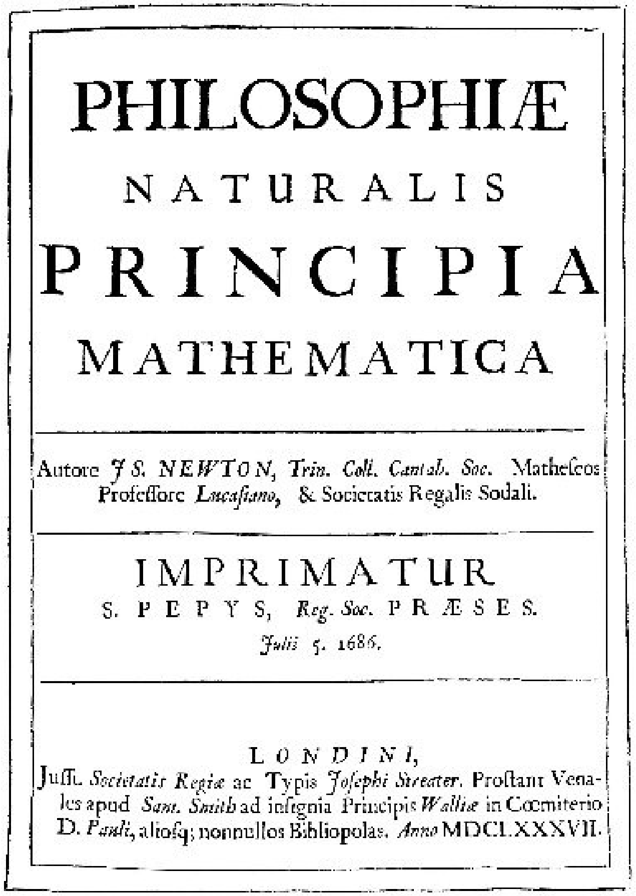
$$\rho\frac{D(e+\frac{1}{2}V^2)}{Dt} = \rho\mathbf{f}.\mathbf{V} - \nabla.(p\mathbf{V}) + \nabla.(u\boldsymbol{\tau}_x) + \nabla.(v\boldsymbol{\tau}_y) + \nabla.(w\boldsymbol{\tau}_z) + \nabla.(k\nabla T)$$

Integral Form (for a system)

$$\frac{D}{Dt}\int_R \rho\, d\tau = 0$$

$$\frac{D}{Dt}\int_R \rho\mathbf{V}\, d\tau = \int_R \mathbf{f}\rho\, d\tau - \oint_S p\mathbf{n}\, dS + \oint_S (\boldsymbol{\tau}_x.\mathbf{n})\,\mathbf{i} + (\boldsymbol{\tau}_y.\mathbf{n})\mathbf{j} + (\boldsymbol{\tau}_z.\mathbf{n})\mathbf{k}\; dS$$

$$\frac{D}{Dt}\int_R (e+\frac{V^2}{2})\rho\, d\tau = \int_R \mathbf{V}.\mathbf{f}\rho\, d\tau + \oint_S \left[-p\mathbf{n} + (\boldsymbol{\tau}_x.\mathbf{n})\,\mathbf{i} + (\boldsymbol{\tau}_y.\mathbf{n})\mathbf{j} + (\boldsymbol{\tau}_z.\mathbf{n})\,\mathbf{k}\right].\mathbf{V}\, dS + \oint_S k(\nabla T).\mathbf{n}\, dS$$



PHILOSOPHIÆ
NATURALIS
PRINCIPIA
MATHEMATICA

Autore *J S. NEWTON*, *Trin. Coll. Cantab. Soc.* Matheſeos Profeſſore *Lucaſiano*, & Societatis Regalis Sodali.

IMPRIMATUR
S. PEPYS, *Reg. Soc.* PRÆSES.
Julii 5. 1686.

LONDINI,
Juſſu *Societatis Regiæ* ac Typis *Joſephi Streater*. Proſtant Venales apud *Sam. Smith* ad inſignia Principis *Walliæ* in Cœmiterio D. *Pauli*, aliosq; nonnullos Bibliopolas. *Anno* MDCLXXXVII.

Isaac Newton
1642-1727

Equations for Changes Seen From a Lagrangian Perspective

Differential Form (for a particle)

$$\frac{D\rho}{Dt} = -\rho\nabla.\mathbf{V}$$

$$\rho\frac{D\mathbf{V}}{Dt} = \rho\mathbf{f} - \nabla p + (\nabla.\boldsymbol{\tau}_x)\mathbf{i} + (\nabla.\boldsymbol{\tau}_y)\mathbf{j} + (\nabla.\boldsymbol{\tau}_z)\mathbf{k}$$

$$\rho\frac{D(e+\frac{1}{2}V^2)}{Dt} = \rho\mathbf{f}.\mathbf{V} - \nabla.(p\mathbf{V}) + \nabla.(u\boldsymbol{\tau}_x) + \nabla.(v\boldsymbol{\tau}_y) + \nabla.(w\boldsymbol{\tau}_z) + \nabla.(k\nabla T)$$

Integral Form (for a system)

$$\frac{D}{Dt}\int_R \rho\, d\tau = 0$$

$$\frac{D}{Dt}\int_R \rho\mathbf{V}\, d\tau = \int_R \mathbf{f}\rho\, d\tau - \oint_S p\mathbf{n}\, dS + \oint_S (\boldsymbol{\tau}_x.\mathbf{n})\,\mathbf{i} + (\boldsymbol{\tau}_y.\mathbf{n})\mathbf{j} + (\boldsymbol{\tau}_z.\mathbf{n})\mathbf{k}\ dS$$

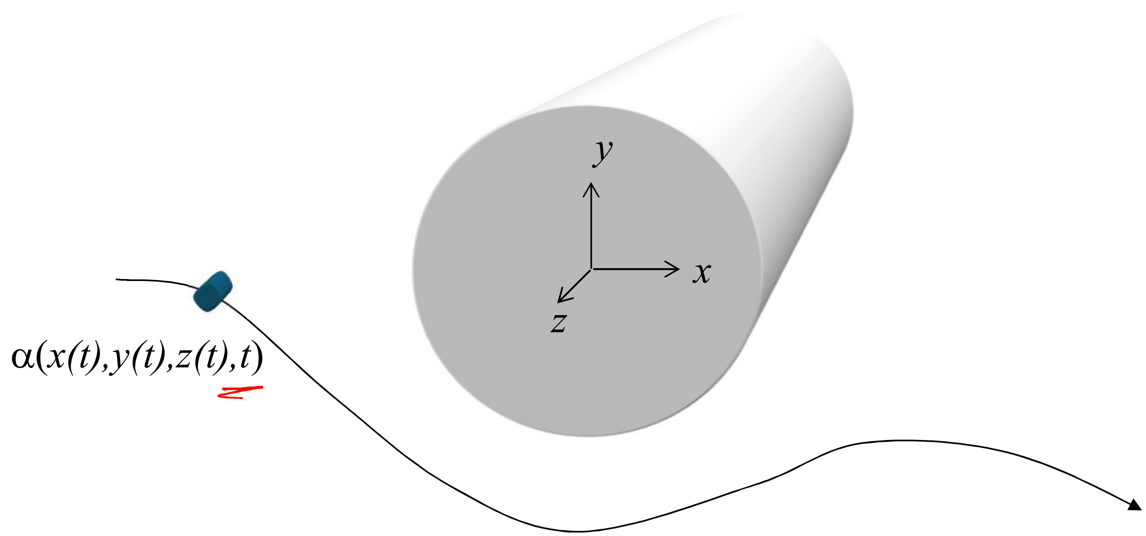
$$\frac{D}{Dt}\int_R (e+\frac{V^2}{2})\rho\, d\tau = \int_R \mathbf{V}.\mathbf{f}\rho\, d\tau + \oint_S \left[-p\mathbf{n} + (\boldsymbol{\tau}_x.\mathbf{n})\,\mathbf{i} + (\boldsymbol{\tau}_y.\mathbf{n})\mathbf{j} + (\boldsymbol{\tau}_z.\mathbf{n})\,\mathbf{k}\right].\mathbf{V}\, dS + \oint_S k(\nabla T).\mathbf{n}\, dS$$

Conversion from Lagrangian to Eulerian rate of change - Derivative

$$\frac{D\alpha}{Dt} = \frac{\partial\alpha}{\partial x}\cdot\overbrace{\frac{\partial x}{\partial t}}^{u} + \frac{\partial\alpha}{\partial y}\cdot\overbrace{\frac{\partial y}{\partial t}}^{v} + \frac{\partial\alpha}{\partial z}\cdot\overbrace{\frac{\partial z}{\partial t}}^{w} + \frac{\partial\alpha}{\partial t}$$

$$= \frac{\partial\alpha}{\partial t} + u\frac{\partial\alpha}{\partial x} + v\frac{\partial\alpha}{\partial y} + w\frac{\partial\alpha}{\partial z}$$

$$\frac{D\alpha}{Dt} = \frac{\partial\alpha}{\partial t} + (\vec{V}\cdot\vec{\nabla})\alpha$$



Conversion from Lagrangian to Eulerian rate of change - Integral

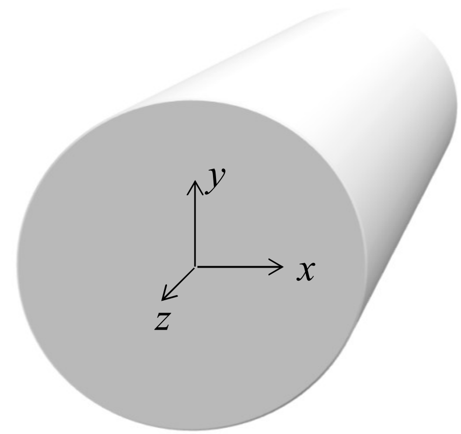
$$\frac{D}{Dt}\int_R \alpha\, d\tau = \int_R \frac{D(\alpha\, d\tau)}{Dt} = \int_R \frac{D\alpha}{Dt}\, d\tau + \alpha \cancel{\frac{D\, d\tau}{Dt}}\ \vec{\nabla}\cdot\vec{V}\, d\tau$$

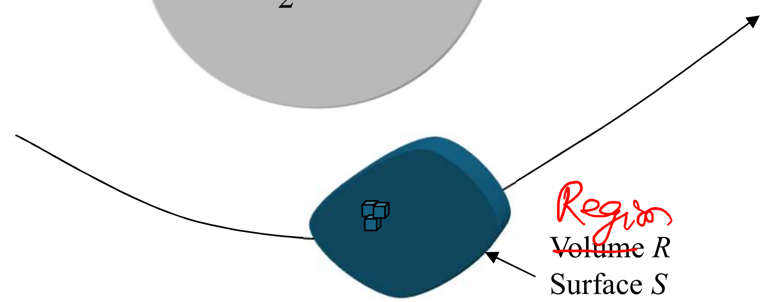
$$= \int_R \left(\frac{\partial \alpha}{\partial t} + \underbrace{\vec{V}\cdot\vec{\nabla}\alpha + \alpha\, \vec{\nabla}\cdot\vec{V}}_{\vec{\nabla}\cdot(\alpha\vec{V})} \right) d\tau$$

$$= \int_R \left(\frac{\partial \alpha}{\partial t} + \vec{\nabla}\cdot(\alpha\vec{V}) \right) d\tau$$

$$\frac{D}{Dt}\int_R \alpha\, d\tau = \int_R \frac{\partial \alpha}{\partial t}\, d\tau + \int_S \alpha\, \vec{V}\cdot\vec{n}\, dS$$

REYNOLDS TRANSPORT THEOREM





Equations for Changes Seen From a Lagrangian Perspective

$$\boxed{\frac{D}{Dt} = \frac{\partial}{\partial t}\bigg|_{part}}$$

Differential Form (for a particle)

$$\frac{D\rho}{Dt} = -\rho\nabla.\mathbf{V}$$

$$\rho\frac{D\mathbf{V}}{Dt} = \rho\mathbf{f} - \nabla p + (\nabla.\boldsymbol{\tau}_x)\mathbf{i} + (\nabla. \ldots$$

$$\frac{D\alpha}{Dt} = \frac{\partial\alpha}{\partial t} + \mathbf{V}.\nabla\alpha$$

$$\rho\frac{D(e+\frac{1}{2}V^2)}{Dt} = \rho\mathbf{f}.\mathbf{V} - \nabla.(p\mathbf{V}) + \nabla.(u\boldsymbol{\tau}_x) + \nabla.(v\boldsymbol{\tau}_y) + \nabla.(w\boldsymbol{\tau}_z) + \nabla.(k\nabla T)$$

Integral Form (for a system)

$$\frac{D}{Dt}\int_R \rho\, d\tau = 0$$

$$\frac{D}{Dt}\int_R \rho\mathbf{V}\, d\tau = \int_R \mathbf{f}\rho \ldots$$

$$\frac{D}{Dt}\int_R \alpha\, d\tau = \int_R \frac{\partial\alpha}{\partial t} d\tau + \oint_S \alpha\mathbf{V}.\mathbf{n}dS$$

$$\frac{D}{Dt}\int_R (e+\frac{V^2}{2})\rho\, d\tau = \int_R \mathbf{V}.\mathbf{f}\rho\, d\tau + \oint_S [-p\mathbf{n} + (\boldsymbol{\tau}_x.\mathbf{n})\mathbf{i} + (\boldsymbol{\tau}_y.\mathbf{n})\mathbf{j} + (\boldsymbol{\tau}_z.\mathbf{n})\mathbf{k}].\mathbf{V}\, dS + \oint_S k(\nabla T).\mathbf{n}\, dS$$

Equations for Changes Seen From an Eulerian Perspective

$$\boxed{\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{V}.\nabla}$$

Differential Form (for a fixed volume element)

$$\frac{D\rho}{Dt} = -\rho\nabla.\mathbf{V}$$

$$\rho\frac{D\mathbf{V}}{Dt} = \rho\mathbf{f} - \nabla p + (\nabla.\boldsymbol{\tau}_x)\mathbf{i} + (\nabla.\boldsymbol{\tau}_y)\mathbf{j} + (\nabla.\boldsymbol{\tau}_z)\mathbf{k}$$

$$\rho\frac{D(e+\frac{1}{2}V^2)}{Dt} = \rho\mathbf{f}.\mathbf{V} - \nabla.(p\mathbf{V}) + \nabla.(u\boldsymbol{\tau}_x) + \nabla.(v\boldsymbol{\tau}_y) + \nabla.(w\boldsymbol{\tau}_z) + \nabla.(k\nabla T)$$

Integral Form (for a system)

$$\int_R \frac{\partial\rho}{\partial t}\,d\tau + \oint_S \rho\mathbf{V}.\mathbf{n}dS = 0$$

$$\int_R \frac{\partial\rho\mathbf{V}}{\partial t}\,d\tau + \oint \rho\mathbf{V}(\mathbf{V}.\mathbf{n})dS = \int_R \mathbf{f}\rho\,d\tau - \oint_S p\mathbf{n}\,dS + \oint_S (\boldsymbol{\tau}_x.\mathbf{n})\,\mathbf{i} + (\boldsymbol{\tau}_y.\mathbf{n})\mathbf{j} + (\boldsymbol{\tau}_z.\mathbf{n})\mathbf{k}\;dS$$

$$\int_R \frac{\partial\rho(e+\frac{1}{2}V^2)}{\partial t})\,d\tau + \oint_S \rho(e+\tfrac{1}{2}V^2)\mathbf{V}.\mathbf{n}dS = \int_R \mathbf{V}.\mathbf{f}\rho\,d\tau + \oint_S \left[-p\mathbf{n} + (\boldsymbol{\tau}_x.\mathbf{n})\,\mathbf{i} + (\boldsymbol{\tau}_y.\mathbf{n})\mathbf{j} + (\boldsymbol{\tau}_z.\mathbf{n})\,\mathbf{k}\right]\mathbf{V}\,dS + \oint_S k(\nabla T).\mathbf{n}\,dS$$

Equivalence of Integral and Differential Forms

Cons. of mass (Integral form)	$\int_R \frac{\partial \rho}{\partial t}\, d\tau + \oint_S \rho \mathbf{V}.\mathbf{n} dS \;=\; 0$
Divergence Theorem	$\oint_S \rho \mathbf{V}.\mathbf{n} dS = \int_R \nabla.(\rho \mathbf{V})\, d\tau$
Conservation of mass for any volume R	$\int_R \left(\frac{\partial \rho}{\partial t} + \nabla.(\rho \mathbf{V}) \right) d\tau = 0$
Then we get	$\frac{\partial \rho}{\partial t} + \nabla.(\rho \mathbf{V}) = 0$ or $\frac{\partial \rho}{\partial t} + \mathbf{V}.\nabla \rho + \rho \nabla.\mathbf{V} = 0$
Cons. of mass (Differential form)	$\frac{D\rho}{Dt} = -\rho \nabla.\mathbf{V}$

Constitutive Relations - Closing the Equations of Motion

Could we solve, in principle, the equations we have derived for a particular flow?

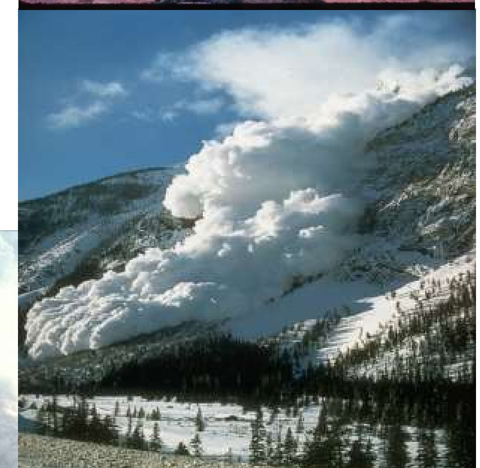
Constitutive Relations

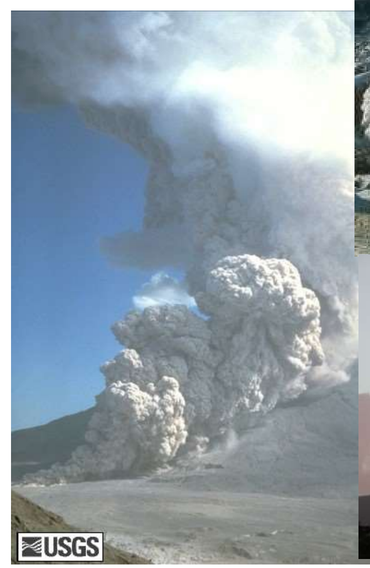
- Equations of motion
  - 5 eqns: Mass (1), Momentum (3), Energy (1)
  - Unknowns: p, ρ, u, v, w…

- THERMODYNAMICS. (RELATIONS FOR p, ρ, T, e)

2 MORE EQNS
$$\begin{cases} p = p(\rho, T) \\ e = e(\rho, T) \end{cases}$$

– VISCOUS STRESSES





Newtonian (Isotropic) Fluid

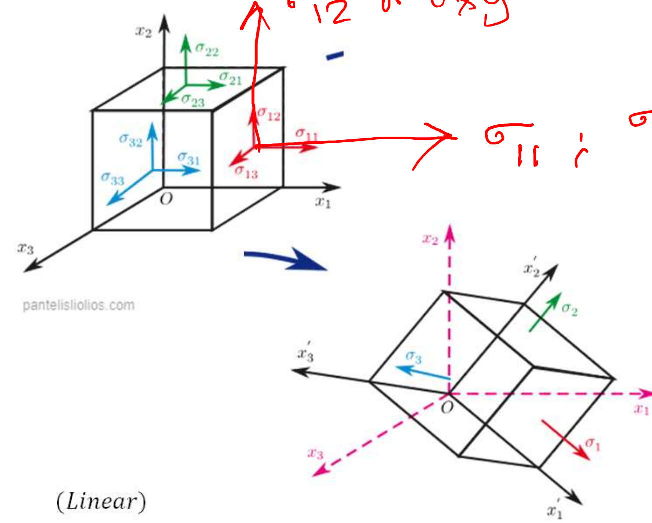
- VISCOUS STRESS IS PROPORTIONAL TO THE ~~SHEAR~~ STRAIN RATE
- RELATIONSHIP IS ISOTROPIC

DEFORMATION RATE
DISTORTION RATE

Stress, is a tensor… $\begin{pmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yx} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{pmatrix} = \tau_{ij}$

σ_{12} or τ_{xy}

σ_{11} ; σ_{xx} (or) τ_{xx}



- Principal axes… axis directions for which all off shear stresses are zero

$$\begin{pmatrix} \overline{\tau_{xx}} & 0 & 0 \\ 0 & \overline{\tau_{yy}} & 0 \\ 0 & 0 & \overline{\tau_{zz}} \end{pmatrix}$$

- Tensor invariants… combinations of elements that don't change with the axis directions

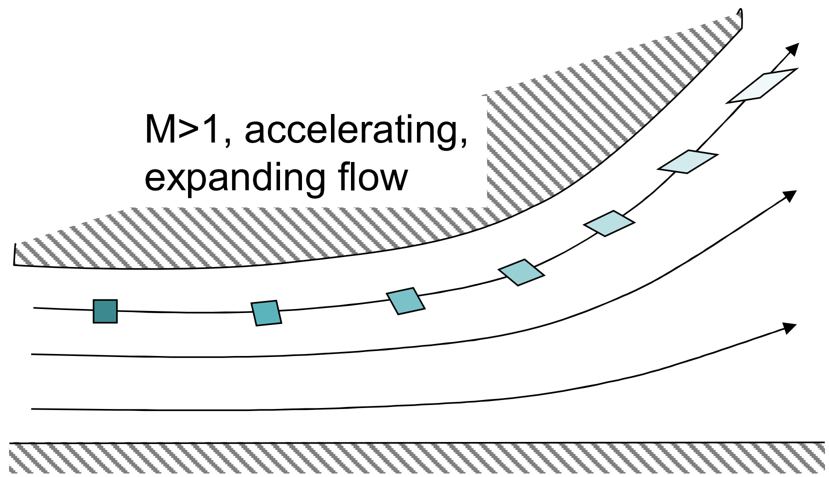
$$I = \sum_{i=1}^{3} \tau_{ii} \qquad (Linear)$$

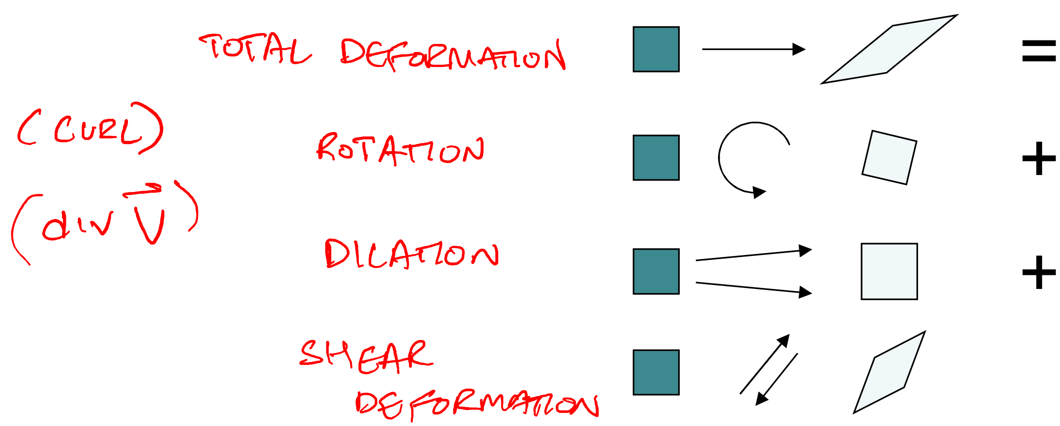
$$II = \sum_{i=1}^{3}\sum_{j=1}^{3} \tau_{ii}\tau_{jj} - \tau_{ij}\tau_{ji} \qquad (Quadratic)$$

$$III = Determinant \qquad (Cubic)$$

Distortion of a Particle in a Flow

Physically

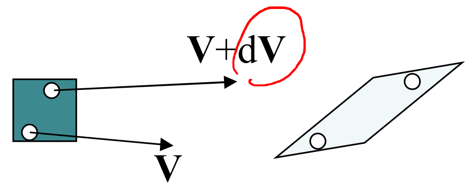




Distortion of a Particle in a Flow

Mathematically

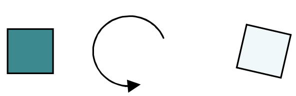
$$d\mathbf{V} = \begin{pmatrix} du \\ dv \\ dw \end{pmatrix} = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{pmatrix} \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix} = \frac{d\mathbf{V}}{d\mathbf{r}} d\mathbf{r}$$

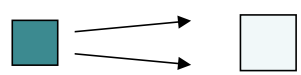


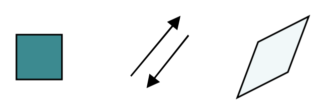
Deformation is represented by $d\mathbf{V}$×time so rate of deformation is given by $d\mathbf{V}$

$$d\mathbf{V} = \begin{pmatrix} 0 & -\frac{1}{2}\left[\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right] & \frac{1}{2}\left[\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right] \\ \frac{1}{2}\left[\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right] & 0 & -\frac{1}{2}\left[\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right] \\ -\frac{1}{2}\left[\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right] & \frac{1}{2}\left[\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right] & 0 \end{pmatrix} d\mathbf{r} + \begin{pmatrix} \frac{\partial u}{\partial x} & 0 & 0 \\ 0 & \frac{\partial v}{\partial y} & 0 \\ 0 & 0 & \frac{\partial w}{\partial z} \end{pmatrix} d\mathbf{r} + \begin{pmatrix} 0 & \frac{1}{2}\left[\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right] & \frac{1}{2}\left[\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right] \\ \frac{1}{2}\left[\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right] & 0 & \frac{1}{2}\left[\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right] \\ \frac{1}{2}\left[\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right] & \frac{1}{2}\left[\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right] & 0 \end{pmatrix} d\mathbf{r}$$

Total change = rotation + dilation + shear deformation







Rate of deformation or **strain rate**

Newtonian (Isotropic) Fluid

$$\begin{pmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yx} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{pmatrix} \underbrace{proportional\ to}_{ISOTROPICALLY} \begin{pmatrix} \dfrac{\partial u}{\partial x} & \dfrac{1}{2}\left[\dfrac{\partial v}{\partial x}+\dfrac{\partial u}{\partial y}\right] & \dfrac{1}{2}\left[\dfrac{\partial u}{\partial z}+\dfrac{\partial w}{\partial x}\right] \\ \dfrac{1}{2}\left[\dfrac{\partial v}{\partial x}+\dfrac{\partial u}{\partial y}\right] & \dfrac{\partial v}{\partial y} & \dfrac{1}{2}\left[\dfrac{\partial w}{\partial y}+\dfrac{\partial v}{\partial z}\right] \\ \dfrac{1}{2}\left[\dfrac{\partial u}{\partial z}+\dfrac{\partial w}{\partial x}\right] & \dfrac{1}{2}\left[\dfrac{\partial w}{\partial y}+\dfrac{\partial v}{\partial z}\right] & \dfrac{\partial w}{\partial z} \end{pmatrix}$$

So

$$\begin{pmatrix} \overline{\tau_{xx}} & 0 & 0 \\ 0 & \overline{\tau_{yy}} & 0 \\ 0 & 0 & \overline{\tau_{zz}} \end{pmatrix} \underbrace{proportional\ to}_{ISOTROPICALLY} \begin{pmatrix} \overline{\dfrac{\partial u}{\partial x}} & 0 & 0 \\ 0 & \overline{\dfrac{\partial v}{\partial y}} & 0 \\ 0 & 0 & \overline{\dfrac{\partial w}{\partial z}} \end{pmatrix}$$

So ***Each stress comp.*** = *Const.* × ***Corresponding strain rate component*** + *Const.* × ***First invariant of strain rate tensor***

Or

$$\overline{\tau_{xx}} = 2\mu\overline{\frac{\partial u}{\partial x}} + \lambda\left(\overline{\frac{\partial u}{\partial x}}+\overline{\frac{\partial v}{\partial y}}+\overline{\frac{\partial w}{\partial z}}\right) = 2\mu\overline{\frac{\partial u}{\partial x}} + \lambda\nabla.\mathbf{V}$$

And likewise for y and z.

Stokes' Hypothesis

$$\overline{\tau_{xx}} = 2\mu \overline{\frac{\partial u}{\partial x}} + \lambda \nabla . \mathbf{V}$$

$$\tau_{xx} + \tau_{yy} + \tau_{zz} = 0$$

$$2\mu \overbrace{\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right)}^{\vec{\nabla} . \vec{V}} + 3\lambda \left(\vec{\nabla} . \vec{V} \right) = 0$$

$$\Rightarrow \quad \lambda = -\frac{2}{3}\mu$$

Stokes' Hypothesis implies, in general (non-principal) axes:

$$\tau_{xx} = 2\mu \frac{\partial u}{\partial x} - \frac{2}{3}\mu \nabla . \mathbf{V}$$ and likewise for τ_{yy} and τ_{zz}

$$\tau_{xy} = \mu \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right)$$ and likewise for τ_{yz} and τ_{xz}

The Equations of Motion

Differential Form (for a fixed volume element)

The Continuity equation

$$\frac{D\rho}{Dt} = -\rho\nabla.\mathbf{V}$$

These form a closed set when two thermodynamic relations are specified

The Navier Stokes' equations

$$\rho\frac{Du}{Dt} = \rho\,\mathrm{f}_x - \frac{\partial p}{\partial x} + \frac{\partial}{\partial x}\left(2\mu(\frac{\partial u}{\partial x} - \tfrac{1}{3}\nabla.\mathbf{V})\right) + \frac{\partial}{\partial y}\left(\mu(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y})\right) + \frac{\partial}{\partial z}\left(\mu(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x})\right)$$

$$\rho\frac{Dv}{Dt} = \rho\,\mathrm{f}_y - \frac{\partial p}{\partial y} + \frac{\partial}{\partial x}\left(\mu(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y})\right) + \frac{\partial}{\partial y}\left(2\mu(\frac{\partial v}{\partial y} - \tfrac{1}{3}\nabla.\mathbf{V})\right) + \frac{\partial}{\partial z}\left(\mu(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y})\right)$$

$$\rho\frac{Dw}{Dt} = \rho\,\mathrm{f}_z - \frac{\partial p}{\partial z} + \frac{\partial}{\partial x}\left(\mu(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x})\right) + \frac{\partial}{\partial y}\left(\mu(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y})\right) + \frac{\partial}{\partial z}\left(2\mu(\frac{\partial w}{\partial z} - \tfrac{1}{3}\nabla.\mathbf{V})\right)$$

The Viscous Flow Energy Equation

$$\rho\frac{D(e + \frac{1}{2}V^2)}{Dt} = \rho\mathbf{f}.\mathbf{V} - \nabla(p\mathbf{V}) + \nabla.(k\nabla T) + \frac{\partial}{\partial x}\left[2\mu u(\frac{\partial u}{\partial x} - \tfrac{1}{3}\nabla.\mathbf{V}) + \mu v(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}) + \mu w(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x})\right]$$

$$+\frac{\partial}{\partial y}\left[\mu u(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}) + 2\mu v(\frac{\partial v}{\partial y} - \tfrac{1}{3}\nabla.\mathbf{V}) + \mu w(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y})\right] + \frac{\partial}{\partial z}\left[\mu u(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}) + \mu v(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}) + 2\mu w(\frac{\partial w}{\partial z} - \tfrac{1}{3}\nabla.\mathbf{V})\right]$$

Assumptions made / Info encoded

Assumption/Law	**Mass**	**NS**	**VFEE**
Conservation of mass	✓		
Conservation of momentum		✓	
Conservation of energy			✓
Continuum	✓	✓	✓
Newtonian fluid		✓	✓
Isotropic viscosity		✓	✓
Stokes´ Hypothesis		✓	✓
Fourier´s Law of Heat conduction			✓
No heat addition except by conduction			✓

Fluid Statics and Dynamics

$$\frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + \cancel{V.\nabla \rho} \to 0$$

The Continuity equation

$$\frac{d\rho_x}{dt} = \frac{D\rho}{Dt} = -\rho \cancel{\nabla.\mathbf{V}}$$

The Navier Stokes' equations

$$\rho\frac{Du}{Dt} = \rho \mathrm{f}_x - \frac{\partial p}{\partial x} + \cancel{\frac{\partial}{\partial x}\left(2\mu(\frac{\partial u}{\partial x} - \tfrac{1}{3}\nabla.\mathbf{V})\right) + \frac{\partial}{\partial y}\left(\mu(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y})\right) + \frac{\partial}{\partial z}\left(\mu(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x})\right)}$$

$$\rho\frac{Dv}{Dt} = \rho \mathrm{f}_y - \frac{\partial p}{\partial y} + \cancel{\frac{\partial}{\partial x}\left(\mu(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y})\right) + \frac{\partial}{\partial y}\left(2\mu(\frac{\partial v}{\partial y} - \tfrac{1}{3}\nabla.\mathbf{V})\right) + \frac{\partial}{\partial z}\left(\mu(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y})\right)}$$

$$\rho\frac{Dw}{Dt} = \rho \mathrm{f}_z - \frac{\partial p}{\partial z} + \cancel{\frac{\partial}{\partial x}\left(\mu(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x})\right) + \frac{\partial}{\partial y}\left(\mu(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y})\right) + \frac{\partial}{\partial z}\left(2\mu(\frac{\partial w}{\partial z} - \tfrac{1}{3}\nabla.\mathbf{V})\right)}$$

The Viscous Flow Energy Equation

$\nabla\cdot(k\nabla T)$

$$\rho\frac{D(e+\frac{1}{2}V^2)}{Dt} = \cancel{\rho\mathbf{f}.\mathbf{V}} - \cancel{\nabla.(p\mathbf{V})} + \nabla.(k\nabla T) + \cancel{\frac{\partial}{\partial x}\left[2\mu u(\frac{\partial u}{\partial x} - \tfrac{1}{3}\nabla.\mathbf{V}) + \mu v(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}) + \mu w(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x})\right]}$$

$$\cancel{+ \frac{\partial}{\partial y}\left[\mu u(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}) + 2\mu v(\frac{\partial v}{\partial y} - \tfrac{1}{3}\nabla.\mathbf{V}) + \mu w(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y})\right] + \frac{\partial}{\partial z}\left[\mu u(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}) + \mu v(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}) + 2\mu w(\frac{\partial w}{\partial z} - \tfrac{1}{3}\nabla.\mathbf{V})\right]}$$

Fluid Statics (***V*** = 0)

- Continuity

$$\frac{d\rho}{dt} = 0$$

- Momentum

$$\nabla p = \rho \vec{f} \qquad \nabla \times \nabla p = \nabla \times \rho \vec{f} = 0$$

- Energy (Equation of Heat Conduction)

$$\rho \frac{de}{dt} = \nabla \cdot (k \nabla T)$$

Second Order Operators

$$\nabla.\nabla\phi = \nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}$$

The Laplacian, may also be applied to a vector field.

$$\nabla(\nabla.\mathbf{A})$$

$$\nabla\times\nabla\times\mathbf{A} = \nabla(\nabla.\mathbf{A}) - \nabla^2\mathbf{A}$$

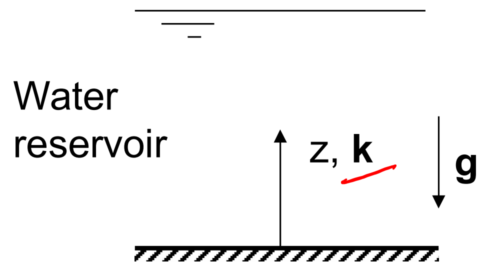
$$\nabla\times\nabla\phi \equiv 0$$

- So, any vector differential equation of the form $\nabla\times$**B**=0 can be solved identically by writing **B**=$\nabla\phi$.
- We say **B** is ***irrotational***.
- We refer to ϕ as the ***scalar potential***.

$$\nabla.\nabla\times\mathbf{A} \equiv 0$$

- So, any vector differential equation of the form ∇.**B**=0 can be solved identically by writing **B**=$\nabla\times$**A**.
- We say **B** is ***solenoidal*** or ***incompressible***.
- We refer to **A** as the ***vector potential***.

Example: Liquid at Rest Under Gravity



$$\boxed{\nabla P = \rho \vec{f}} = -\rho g \vec{k}$$

$$\vec{i}\frac{\partial P}{\partial x} + \vec{j}\frac{\partial P}{\partial y} + \vec{k}\frac{\partial P}{\partial z} = -\rho g \vec{k}$$

$$\frac{\partial P}{\partial z} = -\rho g \qquad \frac{\partial P}{\partial x} = \frac{\partial P}{\partial y} = 0 \quad \Rightarrow \quad P = P(z)$$

$$\Rightarrow \quad \frac{dP}{dz} = -\rho g \quad \Rightarrow \quad \int dP = -\int \rho g \, dz$$

Assume ρ, g are constant

$$P = -\rho g z + \text{CONST}$$

Fluid Statics and Dynamics

The Continuity equation

$$\frac{D\rho}{Dt} = -\rho \nabla . \mathbf{V}$$

These form a closed set when two thermodynamic relations are specified

The Navier Stokes' equations

$$\rho \frac{Du}{Dt} = \rho \, \mathrm{f}_x - \frac{\partial p}{\partial x} + \frac{\partial}{\partial x}\left(2\mu (\frac{\partial u}{\partial x} - \tfrac{1}{3}\nabla . \mathbf{V}) \right) + \frac{\partial}{\partial y}\left(\mu (\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}) \right) + \frac{\partial}{\partial z}\left(\mu (\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}) \right)$$

$$\rho \frac{Dv}{Dt} = \rho \, \mathrm{f}_y - \frac{\partial p}{\partial y} + \frac{\partial}{\partial x}\left(\mu (\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}) \right) + \frac{\partial}{\partial y}\left(2\mu (\frac{\partial v}{\partial y} - \tfrac{1}{3}\nabla . \mathbf{V}) \right) + \frac{\partial}{\partial z}\left(\mu (\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}) \right)$$

$$\rho \frac{Dw}{Dt} = \rho \, \mathrm{f}_z - \frac{\partial p}{\partial z} + \frac{\partial}{\partial x}\left(\mu (\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}) \right) + \frac{\partial}{\partial y}\left(\mu (\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}) \right) + \frac{\partial}{\partial z}\left(2\mu (\frac{\partial w}{\partial z} - \tfrac{1}{3}\nabla . \mathbf{V}) \right)$$

The Viscous Flow Energy Equation

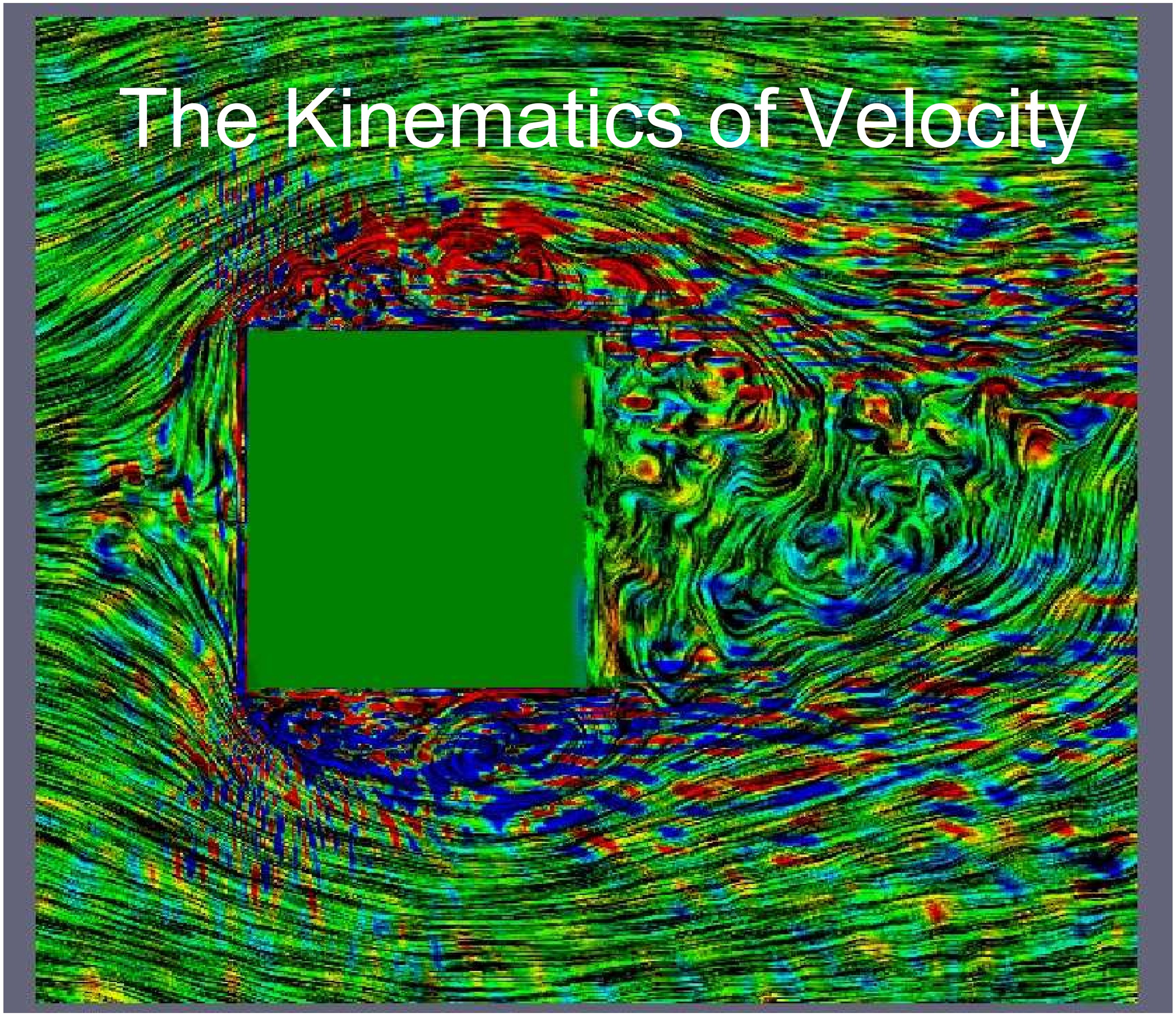
$$\rho \frac{D(e + \tfrac{1}{2}V^2)}{Dt} = \rho \mathbf{f}.\mathbf{V} - \nabla .(p\mathbf{V}) + \nabla .(k\nabla T) + \frac{\partial}{\partial x}\left[2\mu u (\frac{\partial u}{\partial x} - \tfrac{1}{3}\nabla . \mathbf{V}) + \mu v (\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}) + \mu w (\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}) \right]$$

$$+ \frac{\partial}{\partial y}\left[\mu u (\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}) + 2\mu v (\frac{\partial v}{\partial y} - \tfrac{1}{3}\nabla . \mathbf{V}) + \mu w (\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}) \right] + \frac{\partial}{\partial z}\left[\mu u (\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}) + \mu v (\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}) + 2\mu w (\frac{\partial w}{\partial z} - \tfrac{1}{3}\nabla . \mathbf{V}) \right]$$

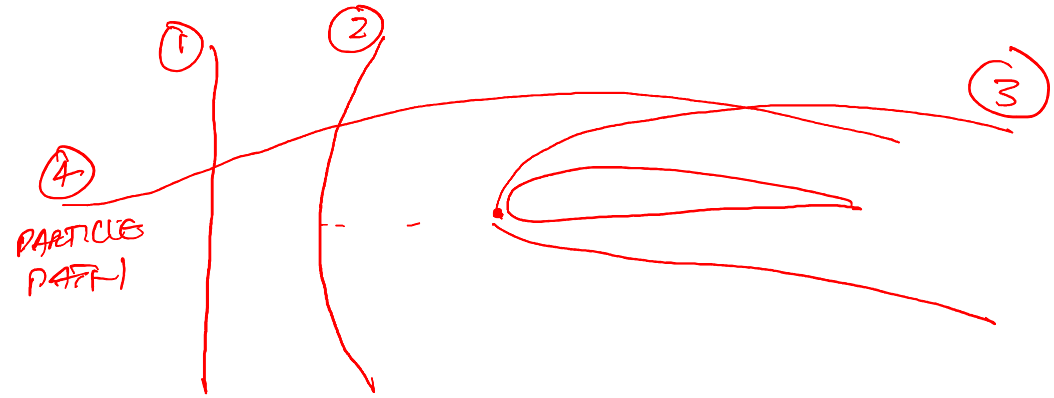
Kinematics

- Kinematic Concepts
  - Velocity: Fluid lines, particle paths, streamlines, etc.
  - Vorticity: Vortex lines, sheets and tubes
- Helmholtz´s Vortex Theorems
- Kelvin´s Circulation Theorem
- Some applications and examples

The Kinematics of Velocity



Kinematic Concepts - Velocity



1. Fluid Line. ANY CONTINOUS STRING OF FLUID PARTICLES.

   MOVES WITH FLOW

   CANNOT BE BROKEN

2. Particle Path. PATH TRAVERESED BY FLUID PARTICLE