Intermediate Fluid Mechanics

Lecture 12: Constitutive Equation

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Chapter Overview

Lecture Objectives

In the previous lecture we obtained Cauchy's equation of motion,

$$\rho \frac{\partial u_i}{\partial t} + \rho\, u_j \frac{\partial u_i}{\partial x_j} = \rho\, g_i + \frac{\partial \tau_{ij}}{\partial x_j}. \tag{1}$$

- This is a vector equation that represents three separate equations.
- With the continuity equation, one has a total of four equations.
- However, one has a total of nine unknowns (three velocity components and six components of the stress tensor).

To solve this system of equations, one needs to develop a constitutive equation to relate the shear stress to the velocity gradients.

The **Objective** of this lecture is to develop a **Constitutive Equation**, to replace the six unknowns of the stress tensor with one unknown, the pressure. This will allow us to solve the system of equations (4 unknowns + 4 diff. equations).

Chapter Overview

Constitutive Equation for Newtonian Fluids:

For the purposes of developing the constitutive equation it is convenient to write the stress tensor as the sum of two parts,

$$\tau_{ij} = \underbrace{-p\delta_{ij}}_{\text{isotropic tensor}} + \underbrace{\sigma_{ij}}_{\text{non-isotropic or deviatoric tensor}} \tag{2}$$

or in matrix form,

$$\begin{pmatrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \tau_{33} \end{pmatrix} = \begin{pmatrix} -p & 0 & 0 \\ 0 & -p & 0 \\ 0 & 0 & -p \end{pmatrix} + \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix} \tag{3}$$

An isotropic tensor is defined as one whose components do not change under rotations of the coordinate system.

That is, no matter how one rotates the coordinate system, an isotropic tensor always has only diagonal components that have identical magnitude and sign.

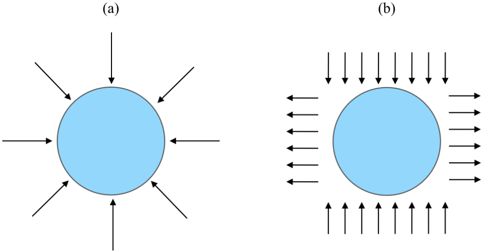

Figure: *(a) Isotropic stress field; (b) non-isotropic normal stresses.*

Fluid in Equilibrium

- When a fluid is in equilibrium (*i.e* at rest and not moving), it can only sustain normal stresses.
- It can be further argued that these normal stresses must be isotropic in order for the fluid to be in equilibrium.

Why? $\Longrightarrow$ In the case of a non-isotropic stress field (case II shown in Figure 1), the stress field would tend to deform the sphere into an ellipsoid, without any necessary change in volume.

This deformation cannot be balanced by a Body force, because body forces act over the entire volume.

$\Longrightarrow$ The only force that could balance this stress is inertia (Cauchy's equation of motion), which is inconsistent with the fluid being in a state of equilibrium!

Chapter Overview

Pressure

Therefore, one can conclude that an isotropic stress field is the only type of stress that can exist for the case of a fluid in equilibrium,

$$\tau_{ij} = -p\delta_{ij}. \tag{4}$$

Note: The negative sign is required because pressure acts to compress the material, while the sign convention associated with τ_{ij} is that a positive stress acts in the same direction as the outward unit normal to the surface (*i.e.* a positive stress acts in tension.)

Here, 'p' represents the thermodynamic pressure, which maybe related to the density ρ, and temperature T, of the fluid through an equation of state, *i.e.* $p = p(\rho, T)$.

$\Longrightarrow$ For an ideal gas, the equation of state is given by $\boxed{p = \rho\, R\, T}$.

Pressure (continued ...)

For a fluid in motion, one generally writes the stress as

$$\boxed{\tau_{ij} = \underbrace{-p\delta_{ij}}_{\text{Term I}} + \underbrace{\sigma_{ij}}_{\text{Term II}}} \tag{5}$$

- **Term I** represents the thermodynamic pressure (or isotropic part)
- **Term II** represents the deviatoric part (*i.e.* the part deviating from the isotropic part.)

Eventhough, this is a fluid in motion, p is still assumed to represent the thermodynamic pressure. Because the relaxation time for fluids is very small under ordinary conditions, classical thermodynamic relations are applicable to most fluid flows (and certainly to the ones that we will discuss in this course).

Chapter Overview

Modeling the deviatoric stress σ_{ij}

• The task ahead is to determine a **constitutive law** relating σ_{ij} to the velocity field u_i and/or its derivatives $\partial u_i/\partial x_j$.

$\Longrightarrow$ We will first make an argument on why σ_{ij} should not depend on u_i.

Modeling the deviatoric stress (continued ...)

Consider the simple example of a fluid particle immersed in a uniform flow field, where the fluid is driven by the same velocity u everywhere,

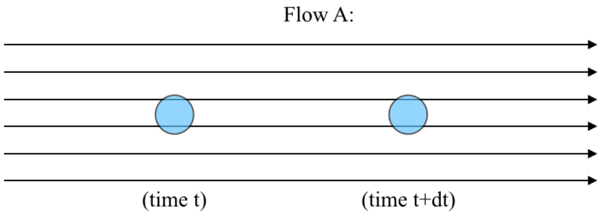

Figure: *Uniform flow in the streamwise direction.*

$\Longrightarrow$ It is intuitive to expect that the fluid particle would not experience any other stresses acting on it aside from the pressure because the particle is simply translating and not deforming.

Modeling the deviatoric stress (continued ...)

On the other hand, we already know that in a shear flow, a spherical fluid particle would tend to deform into an ellipsoid,

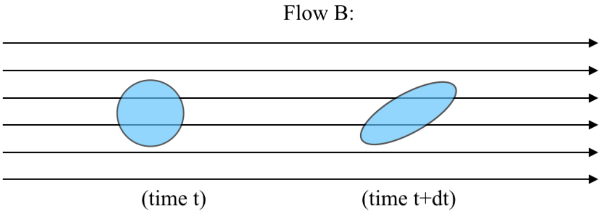

Figure: *Flow in the streamwise direction.*

In this case, one expects that there must be some stresses acting on the fluid particle to make it deform in this way.

These stresses (or the deformation) are the result of a velocity gradient, as one can intuitively deduce from the figure.

Stress as a function of the strain rate tensor

$\Longrightarrow$ Based on our intuition, we seek to model the deviatoric stress σ_{ij} as a function of the velocity gradient, $\partial u_i/\partial x_j$.

In the past, we had seen how $\partial u_i/\partial x_j$ can be written as the sum of a **symmetric** and **anti-symmetric** part,

$$\frac{\partial u_i}{\partial x_j} = \underbrace{\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)}_{\text{Strain-rate tensor, } e_{ij}} + \underbrace{\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)}_{\text{spin-tensor, } r_{ij}}. \tag{6}$$

- Since the anti-symmetric part, r_{ij} represents fluid rotation without deformation, it cannot generate stress.
- Therefore, one can assume that σ_{ij} is a function of only the strain-rate tensor e_{ij}.

Stress as a function of the strain rate tensor (continued ...)

What is the simplest function we can think of, aside from σ_{ij} being constant?

(*Answer:* Linear.)

Stress as a function of the strain rate tensor (continued ...)

Since both σ_{ij} and e_{ij} are tensors, a linear relation between them in general would involve a fourth-order tensor,

$$\sigma_{ij} = k_{ijmn} e_{mn}, \tag{7}$$

- In this case, each stress component is linearly related to all nine components of e_{ij}.
- This means that k_{ijmn} is comprised of $3^4 = 81$ components, each of which represents a different material property that depends on the thermodynamic state of the medium.

Stress as a function of the strain rate tensor (continued ...)

However, **if the medium is isotropic**:

- It has no directional preference (*i.e* the stress-strain rate relation is independent of a rotation of the coordinate system),
- The material deforms in the same manner regardless of the direction of applied stress,

$\implies k_{ijmn}$ reduces to the following:

$$k_{ijmn} = \lambda\delta_{ij}\delta_{mn} + \mu\delta_{im}\delta_{jn} + \gamma\delta_{in}\delta_{jm}, \tag{8}$$

where λ, μ, γ are scalars that depend on the local thermodynamic state of the fluid.

Note: the assumption of isotropy has allowed us to reduce the 81 components of k_{ijmn} down to a mere 3 (see '*Methods of Mathematical Physics*' by H. Jeffreys and B. Jeffreys, 1972, pp. 87-88 for more details.)

Stress as a function of the strain rate tensor (continued ...)

One can do even better, though, because we know that σ_{ij} is symmetric, therefore, k_{ijmn} must also be symmetric with respect to i and j.

This means that:

$$k_{ijmn} = k_{jimn} \tag{9}$$

$$\lambda\delta_{ij}\delta_{mn} + \mu\delta_{im}\delta_{jn} + \gamma\delta_{in}\delta_{jm} = \lambda\delta_{ji}\delta_{mn} + \mu\delta_{jm}\delta_{in} + \gamma\delta_{jn}\delta_{im} \tag{10}$$

$$(\mu - \gamma)\delta_{jn}\delta_{im} = (\mu - \gamma)\delta_{jm}\delta_{in} \tag{11}$$

since $\delta_{jn}\delta_{im} \neq \delta_{jm}\delta_{in}$, the only way this equality can hold is if

$$\mu - \gamma = 0 \rightarrow \mu = \gamma. \tag{12}$$

Therefore, the stress-strain rate relation is

$$\sigma_{ij} = [\lambda \delta_{ij} \delta_{mn} + \mu \delta_{im} \delta_{jn} + \gamma \delta_{in} \delta_{jm}] e_{mn} \tag{13}$$
$$= \lambda \delta_{ij} \delta_{mn} e_{mn} + \mu \delta_{im} \delta_{jn} e_{mn} + \mu \delta_{in} \delta_{jm} e_{mn} \tag{14}$$
$$= \lambda e_{mm} \delta_{ij} + 2\mu e_{ij} \tag{15}$$

Note that, $e_{mm} = \partial u_m / \partial x_m$, which is the rate of change of the volume per unit volume (or dilation).

Therefore, the total stress is,

$$\boxed{\tau_{ij} = -p\delta_{ij} + \lambda e_{mm} \delta_{ij} + 2\mu e_{ij}.} \tag{16}$$

Chapter Overview

Mechanical Pressure, $\overline{p}$

The mechanical pressure is defined as the average normal stress, and is different than the thermodynamic pressure,

$$\overline{p} = -\frac{1}{3}\tau_{ii} = -\frac{1}{3}(\tau_{11} + \tau_{22} + \tau_{33}). \tag{17}$$

From equation 16,

$$\overline{p} = -\frac{1}{3}[-p\delta_{ii} + \lambda e_{mm}\delta_{ii} + 2\mu e_{ii}] \tag{18}$$

$$= -\frac{1}{3}[-3p + (3\lambda + 2\mu)e_{mm}] \tag{19}$$

$$p - \overline{p} = \underbrace{(3\lambda + 2\mu)}_{\kappa}\, e_{mm} \tag{20}$$

where κ is a coefficient of bulk viscosity.

Mechanical Pressure, $\overline{p}$ (continued ...)

Note that since $e_{mm} = \partial u_m / \partial x_m$, one can use conservation of mass to rewrite

$$p - \overline{p} = \underbrace{(3\lambda + 2\mu)}_{\kappa} e_{mm} \tag{21}$$

as,

$$p - \overline{p} = \frac{\kappa}{\rho} \frac{D\rho}{Dt}. \tag{22}$$

This says that the difference in the thermodynamic and mechanical pressures is related to how fast the density of a fluid particle is changing.

$\Longrightarrow$ the thermodynamic and mechanical pressures will be equal $(p = \overline{p})$ when:

- $\kappa = 0$, the bulk viscosity is zero $($*i.e.* $\lambda = 2/3\mu)$
- $\frac{D\rho}{Dt} = 0$, *i.e.* the flow is incompressible.

Chapter Overview

Newtoninan Fluid

Assuming $\kappa = 0$, the stress-strain relation derived above takes the form

$$\tau_{ij} = -p\delta_{ij} + \mu(2e_{ij} - \frac{2}{3}e_{mm}\delta_{ij}). \tag{23}$$

- Fluids that obey this relation are referred to as Newtonian Fluids.
- Note, μ is the dynamic or absolute viscosity, the value of which depends on the local thermodynamic state of the fluid, *i.e.*, $\mu = \mu(p, T)$.
- Note that the dependence on p is very weak, meaning that μ changes only very little with large changes in p.

Newtoninan Fluid (continued ...)

The important **assumptions** regarding a **Newtonian Fluid** are:

- (i) the bulk viscosity is negligible (*i.e.* the thermodynamic and mechanical pressures are equal)
- (ii) the material properties are isotropic
- (iii) the deviatoric stress is a linear function of the strain-rate.

Newtoninan Fluid (continued ...)

Other types of fluid fall into the category of **non-Newtonian fluids**

- These include polymer solutions (which do not display isotropic material properties) and emulsions (such as blood).
- In these fluids, **the stress is a non-linear function of the strain-rate**.
- In addition, Non-Newtonian fluids typically also depend on the history of the strain-rate. This memory effect gives some non-Newtonian fluids an elastic property.