ME 5700/6700: Intermediate Fluid Dynamics
EXAM - I
October 18, 2023

NAME: ______________________

The exam has two portions:

1- **In-class (50 min). The first part (questions 1-7) should be completed during the in-class exam time.**
2- **Take-home (*till 10am tomorrow Oct 19th*). The second part (questions 8-10) should be completed at home and submitted on canvas. <u>You can also re-work just 1 in-class problem for increased credit.</u> You are NOT allowed to talk with anyone about the take-home exam and must sign the honor policy below.**

<u>In-class portion (65/100) You can use your 1-page equation sheet:</u>

1. [**4 points**] Explain the difference between Lagrangian and Eulerian approaches in fluid flow analysis.

In Lagrangian view, we tag fluid particles (tracers) and we track them as they move in the flow (like a pathline). In Eulerian view, we consider a region of interest and we observe the flow as the flow passes through the region (e.g., streamlines plotted in a region of interest).

2. [**4 points**] Strain rate interpretation: Describe what the trace of the strain rate tensor physically represents.

Lec 6: Tr(S) = Div(velocity) = Dilation and represents the volumetric strain rate.

3. [**4 points**] Provide an example where the flow is incompressible but density is not constant. Mathematically state the conservation of mass equation in terms of density for this case.

Lec10: Density stratified flows (e.g, the atmosphere). $\frac{D\rho}{Dt} = 0$ Material derivative.

4. [**3 points**] What is the definition of wall shear stress (WSS) in terms of traction?

Lec11: WSS = Tangential component of traction at the wall.

5. [**10 points**] In magnetohydrodynamics, we have a PDE that relates the magnetic field **B** (a vector) to the velocity vector $\boldsymbol{v}$. The equation in vectorial notation is (∇^2 is Laplacian and $\times$ is cross-product):

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{v} \times \boldsymbol{B}) + \nabla^2 \boldsymbol{B}$$

Write the equation in index notation (make sure the indices are consistent and follow the index laws).

$$(\boldsymbol{v} \times \boldsymbol{B})_i = \epsilon_{ipq} v_p B_q$$
$$\nabla \times (\boldsymbol{v} \times \boldsymbol{B}) = \epsilon_{kji} \partial_j \epsilon_{ipq} v_p B_q$$

Now the free index for this entire cross product term is k, so the other two terms must also have free index of k:

$$\frac{\partial B_k}{\partial t} = \epsilon_{kji} \partial_j \epsilon_{ipq} v_p B_q + \partial_l \partial_l B_k$$
$$\frac{\partial B_k}{\partial t} = \epsilon_{kji} \epsilon_{ipq} \partial_j v_p B_q + \partial_l \partial_l B_k$$

6. [**20 points**] Consider a 2D velocity vector field given by $u=x^2$ and $v = -2xy-1$

   a. Compute the components of the strain rate tensor.

e notation instead of S is used in this solution for strain rate tensor:

By definition

$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$ and for a 2D flow only 4 components can exist (3 independent)

$$e_{11} = \frac{1}{2}\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial x}\right) = \frac{\partial u}{\partial x} = \frac{\partial (x^2)}{\partial x} = 2x$$
$$e_{22} = \frac{1}{2}\left(\frac{\partial v}{\partial y} + \frac{\partial v}{\partial y}\right) = \frac{\partial v}{\partial y} = \frac{\partial (-2xy-1)}{\partial y} = -2x$$
$$e_{12} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = \frac{1}{2}\left(\frac{\partial (x^2)}{\partial y} + \frac{\partial (-2xy-1)}{\partial x}\right) = -y$$

Note that $e_{12} = e_{21}$

   b. By just looking at the strain rate tensor (without any other calculation) is the flow incompressible? Why?

Trace(S) is the same as divergence of velocity so it must be zero for the flow to be incompressible. Trace(S) = 2x – 2x = 0 → It is incompressible

c. Compute the vorticity vector. Is the flow irrotational?

For a 2D flow only 1 component of the vorticity vector is possible

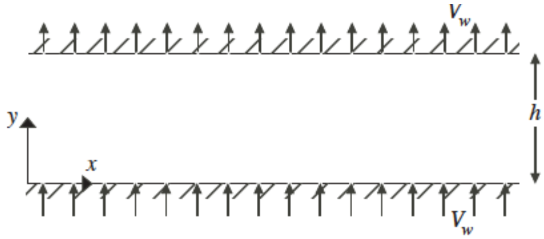
$$\omega_z = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

and for our flow

$$\omega_z = \left(\frac{\partial(-2xy-1)}{\partial x} - \frac{\partial(x^2)}{\partial y}\right) = -2y$$

It is not zero, so the flow is not irrotational.

7. [**20 points**] Consider flow in a channel (similar to Couette flow) but with porous walls. A constant vertical velocity V_w exists in the y direction at the top/bottom porous walls (instead of no-penetration). The flow is driven by a given pressure gradient $\frac{\partial P}{\partial x}$. The flow is assumed fully developed.



x-component of the incompressible Navier–Stokes equation:

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) = -\frac{\partial P}{\partial x} + \rho g_x + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

y-component of the incompressible Navier–Stokes equation:

$$\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right) = -\frac{\partial P}{\partial y} + \rho g_y + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right)$$

z-component of the incompressible Navier–Stokes equation:

$$\rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right) = -\frac{\partial P}{\partial z} + \rho g_z + \mu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right)$$

a) Use the continuity equation to find the y component of velocity v everywhere. (state assumptions)

Continuity:

$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$ we have $\frac{\partial u}{\partial x} = 0\ (fully\ developed)$ → $\frac{\partial v}{\partial y} = 0 \rightarrow v = Constant$

Since v=V_w at the top/bottom walls, then v=V_w everywhere

b) Write the x-component of momentum equation and simplify it (state all assumptions). Write the boundary conditions needed to solve the equation.

This is exactly same as the Couette flow we did in class but only one extra term survives (in the convective acceleration part) because v is not zero:

$$\rho\left(V_W\,\frac{\partial u}{\partial y}\right) = -\frac{\partial P}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial y^2}\right)$$

BC: u=0 at y=0 and y=h

c) **[Extra credit: +6]** Solve the equation to find the x-component of velocity u. To make the algebra easy assume $\rho = \mu = V_w = \frac{\partial P}{\partial x} = 1$.

The above equation becomes:

$$u' = -1 + u''$$

From our undergrad ODE class we can solve this to get:

$$u(y) = c_1 e^y + c_2 - y$$

Applying the BCs we can find the unknown constants and we get:

$$c_1 = \frac{h}{e^h - 1}$$

$$c_2 = \frac{-h}{e^h - 1}$$

Take-home portion (35/100):

In order to protect the integrity of this course, I am requiring all students taking this exam to sign a pledge that you will not cheat on this exam (Exam 1 ME EN 5700/6700). Please note that the take-home part is open book and open notes, however, all of the work that you submit must be your own. You may consult material and problems in the book, in your notes, but you are not allowed to discuss the material on this exam with your peers in the class or anyone else prior to the deadline (10 am Oct 19th). Furthermore, you are not allowed to post questions to online forums. **Anyone found to have cheated on this exam will automatically fail the exam**.

I affirm that I, _____________________ (write your name) am submitting my own work for this exam. I have neither given nor received help on this exam. I have not consulted with anyone (other than the instructor) on questions nor looked at or copied another student's work. I understand the penalties for academic dishonesty on this exam is failure.

____

Signature

____

Date/Time

Submit on a separate paper (upload to Canvas by 10am Thurs Oct 19th)
<u>You can also submit one of the in-class questions that you thought you performed not so great and get increased partial credit based on your new submission.</u>

8. **[10 points]** Pressure Poisson equation (PPE)
   Consider the pressure Poisson equation

$$\frac{\partial}{\partial x_i}\left(\frac{\partial P}{\partial x_i}\right) = -\rho \frac{\partial u_j}{\partial x_i}\frac{\partial u_i}{\partial x_j}$$

Write down this equation for a **<u>3D</u>** <u>flow</u> in terms of P, and 3D velocity components u,v,w. (i.e., convert from index notation to a typical equation). I DID NOT subtract grades for not showing the index summation work.

The RHS is a bit time-consuming. You need to sum over i and j each going between 1-3 (think about a double loop in programming).

First over i:

$$\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2} + \frac{\partial^2 P}{\partial z^2} = -\rho\left(\frac{\partial u_j}{\partial x_1}\frac{\partial u_1}{\partial x_j} + \frac{\partial u_j}{\partial x_2}\frac{\partial u_2}{\partial x_j} + \frac{\partial u_j}{\partial x_3}\frac{\partial u_3}{\partial x_j}\right)$$

Now expand over j

$$\begin{aligned}\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2} + \frac{\partial^2 P}{\partial z^2} & \\ = -\rho\left(\frac{\partial u_1}{\partial x_1}\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_1}\frac{\partial u_1}{\partial x_2} + \frac{\partial u_3}{\partial x_1}\frac{\partial u_1}{\partial x_3}\right) & \\ -\rho\left(\frac{\partial u_1}{\partial x_2}\frac{\partial u_2}{\partial x_1} + \frac{\partial u_2}{\partial x_2}\frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_2}\frac{\partial u_2}{\partial x_3}\right) & \\ -\rho\left(\frac{\partial u_1}{\partial x_3}\frac{\partial u_3}{\partial x_1} + \frac{\partial u_2}{\partial x_3}\frac{\partial u_3}{\partial x_2} + \frac{\partial u_3}{\partial x_3}\frac{\partial u_3}{\partial x_3}\right) &\end{aligned}$$

Where $u_1 = u$, $u_2 = v$, $u_3 = w$ and $x_1 = x$ $x_2 = y$ $x_3 = z$

9. **[10 points]** Rotation tensor:
   In the kinematics lectures, we learned that a line element evolves according to the velocity gradient as

$$\frac{d}{dt}(\delta x_j) = \delta x_i \frac{\partial V_i}{\partial x_j}$$

Let's look into the role of rotation tensor part of the velocity gradient. Plug in the rotation tensor R_{ij} in the velocity gradient in this equation and demonstrate that the action of rotation tensor does not change the length of the line element. That is,
$\frac{d}{dt}(\delta x_j^2) = 0$

The hint as to how to solve this problem was provided in Lec 7.

Consider R_{ij} as the velocity gradient and multiply both sides by δx_j:

LHS: $\delta x_j \frac{d}{dt}(\delta x_j) = \frac{d}{dt}\left(\frac{\delta x_j^2}{2}\right)$

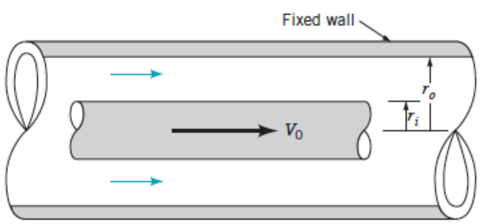
RHS: $\delta x_j \delta x_i R_{ij}$ Note that $\delta x_j \delta x_i$ is symmetric (change i and j to see it is the same) and R is anti-symmetric therefore this product is zero (sym * anti-sym tensor =0)

RHS =0

Therefore, we have proved that desired expression.

10. **[15 points]** Consider the Poiseuille flow we derived in class with the same assumptions (steady, fully developed, driven by a given pressure gradient, etc.). We plug in another cylinder inside the tube and the inner cylinder is moving with a velocity V_0 as shown. The inner moving cylinder has radius r_i and the outer fixed cylinder has radius r_0. Use cylindrical equation form of Navier-stokes (just like how we derived Poiseuille flow) and find the velocity profile. Note the fluid domain here is $r_i < r < r_0$

Hint: The main difference with Poiseuille flow here is the boundary conditions.



First, you need to start from the N.S. equations, simplify, and then you get to this equation (exactly same procedure as in Lec15 Poiseuille flow problem). Now you consider the different BCs and solve:

$$v_z = \frac{1}{4\mu}\left(\frac{\partial p}{\partial z}\right)r^2 + c_1 \ln r + c_2 \qquad (1)$$

With boundary conditions, $r = r_o$, $v_z = 0$, and $r = r_i$, $v_z = V_0$, it follows that:

$$0 = \frac{1}{4\mu}\left(\frac{\partial p}{\partial z}\right)r_o^2 + c_1 \ln r_o + c_2 \qquad (2)$$

$$V_0 = \frac{1}{4\mu}\left(\frac{\partial p}{\partial z}\right)r_i^2 + c_1 \ln r_i + c_2 \qquad (3)$$

Subtract Eq. (2) from Eq. (3) to obtain

$$V_0 = \frac{1}{4\mu}\left(\frac{\partial p}{\partial z}\right)(r_i^2 - r_o^2) + c_1 \ln \frac{r_i}{r_o}$$

so that

$$c_1 = \frac{V_0 - \frac{1}{4\mu}\left(\frac{\partial p}{\partial z}\right)(r_i^2 - r_o^2)}{\ln \frac{r_i}{r_o}}$$

Similarly, use the other BC to find the other constant…..

I was NOT very picky in grading your EXACT mathematical expression and just subtracted points if the mathematical form of the eqn and constants or BCs were wrong.

11. [OPTIONAL] Redo one of the in-class questions that you think you did not perform that well (you will not get penalized if this is worse solution!).