Intermediate Fluid Mechanics

Lecture 15: The Energy Equation

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Chapter Overview

Lecture Objectives

- In this lecture, we will explore the rate of change of energy in a fluid flow.
- For this purpose we will consider both, the mechanical energy (*i.e* kinetic energy) and thermal energy separately.

Chapter Overview

Mechanical Energy Equation

Let's take the dot product of u_i with the momentum equation (in the form of the Cauchy's equation of motion) to get an equation for energy,

$$
\begin{aligned}
u_i \left[\rho \frac{Du_i}{Dt} = \rho\, g_i + \frac{\partial \tau_{ij}}{\partial x_j} \right] \\
\underbrace{\rho \frac{D}{Dt}\left(\frac{1}{2} u_i u_i \right)}_{I} = \underbrace{\rho u_i g_i}_{II} + \underbrace{u_i \frac{\partial \tau_{ij}}{\partial x_j}}_{III} \qquad (1)
\end{aligned}
$$

where,

- **Term I:** time rate of change of the kinetic energy following a fluid particle.
- **Term II:** rate of work done by the body force $\vec{g}$.
- **Term III:** rate of work done by the net surface force.

Mechanical Energy Equation (continued ...)

Recall that work is done by moving a force through a distance:

$$W = \vec{F} \cdot \vec{r}, \quad [Joules] \tag{2}$$

and work rate (or power) is equal to a force being moved at some speed:

$$\dot{W} = \vec{F} \cdot \vec{u}, \quad [Watts] \qquad \text{(Work Rate).} \tag{3}$$

Mechanical Energy Equation (continued ...)

Therefore, the total work rate due to body force $\vec{g}$ is

$$\dot{W}_b = \int_{\forall} \rho\, u_i\, g_i\, d\forall \tag{4}$$

and the total work rate due to the surface forces is given by,

$$\dot{W}_s = \int_S u_i\, F_{s_i}\, dS = \int_s u_i\, n_j\, \tau_{ij}\, dS = \int_{\forall} \frac{\partial}{\partial x_j}\left(u_i\, \tau_{ij}\right) d\forall \tag{5}$$

Realize that this last term can we further rewritten as,

$$\frac{\partial}{\partial x_j}\left(u_i\, \tau_{ij}\right) = \underbrace{u_i\, \frac{\partial \tau_{ij}}{\partial x_j}}_{\text{(a)}} + \underbrace{\tau_{ij}\, \frac{\partial u_i}{\partial x_j}}_{\text{(b)}}\,. \tag{6}$$

Mechanical Energy Equation (continued ...)

From the last equation,

$$\frac{\partial}{\partial x_j}\left(u_i\,\tau_{ij}\right) = \underbrace{u_i\,\frac{\partial \tau_{ij}}{\partial x_j}}_{\text{(a)}} + \underbrace{\tau_{ij}\,\frac{\partial u_i}{\partial x_j}}_{\text{(b)}}\,. \tag{7}$$

- **Term (a)** is the same appearing in equation 1, (i.e. rate of work done by the net surface force).
- **Term (b)** is new.
- The reason why Term b doesn't appear in equation 1 is because it doesn't induce to a net rate of work.
- We will study this terms more in detail later on.

Mechanical Energy Equation (continued ...)

In parallel, using the material derivative definition, one can rewrite the left hands side of equation 1, such that,

$$\underbrace{\rho \frac{\partial}{\partial t}\left(\frac{1}{2}u_i u_i\right)}_{\text{Local Rate of Change of KE}} + \underbrace{\rho\, u_j \frac{\partial}{\partial x_j}\left(\frac{1}{2}u_i u_i\right)}_{\text{Advection of KE}} = \rho u_i g_i + u_i\, \frac{\partial \tau_{ij}}{\partial x_j}. \tag{8}$$

One can rewrite the above equation by multiplying the continuity eq. by $\frac{1}{2}u_i u_i$,

$$\frac{1}{2}(u_i u_i)\left[\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_j}(\rho u_j) = 0\right] \tag{9}$$

$$\frac{1}{2}(u_i u_i)\frac{\partial \rho}{\partial t} + \frac{1}{2}(u_i u_i)\frac{\partial}{\partial x_j}(\rho u_j) = 0 \tag{10}$$

and adding it to the above equation (since it is equivalent to adding 0),

$$\underbrace{\rho \frac{\partial}{\partial t}\left(\frac{1}{2}u_i u_i\right)}_{I} + \underbrace{\rho\, u_j \frac{\partial\left(\frac{1}{2}u_i u_i\right)}{\partial x_j}}_{II} + \underbrace{\frac{1}{2}(u_i u_i)\frac{\partial \rho}{\partial t}}_{III} + \underbrace{\frac{1}{2}(u_i u_i)\frac{\partial}{\partial x_j}(\rho u_j)}_{IV} = \rho u_i g_i + u_i\, \frac{\partial \tau_{ij}}{\partial x_j}. \tag{11}$$

Mechanical Energy Equation (continued ...)

If we now combine terms (terms I & III, and terms II & IV) using the product rule of differentiation,

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho u_i u_i\right) + \frac{\partial}{\partial x_j}\left(u_j \frac{1}{2}\rho u_i u_j\right) = \rho u_i g_i + u_i \frac{\partial \tau_{ij}}{\partial x_j}, \tag{12}$$

which can be rewritten using $E \equiv \frac{1}{2}\rho\, u_i\, u_i$,

$$\boxed{\frac{\partial E}{\partial t} + \underbrace{\frac{\partial}{\partial x_j}(u_j\, E)}_{(*)} = \rho\, u_i\, g_i + u_j\, \frac{\partial \tau_{ij}}{\partial x_j}} \tag{13}$$

Note that term (*) represents the divergence of the kinetic energy flux $(\vec{\nabla} \cdot (\vec{u}\, E))$.

Remember that, the flux divergence term tells us how much energy is being transported from one region to another.

$\Longrightarrow$One can see this using the Gauss' theorem,

$$\int_{\forall} \vec{\nabla} \cdot (\vec{u}\, E)\, d\forall = \int_{A} E\, \vec{u} \cdot \vec{A}. \tag{14}$$

The second term in the above equation represents the net flux of kinetic energy across the surface A bounding the volume $\forall$.

Chapter Overview

Deformation work and viscous dissipation

Next let's consider the RHS of equation 13, which tells us about how kinetic energy is lost to internal energy of deformation of fluid particles.

$$\underbrace{\frac{\partial}{\partial x_j}(u_i\,\tau_{ij})}_{I} = \underbrace{\tau_{ij}\frac{\partial u_i}{\partial x_j}}_{II} + \underbrace{u_i\frac{\partial \tau_{ij}}{\partial x_j}}_{III}, \tag{15}$$

where:

- **Term I** is the total work rate,
- **Term II** is the deformation work,
- **Term III** results in an increase of kinetic energy of the fluid particle.

Note that from the derivation of the kinetic energy equation the only part of the work rate (from the surface forces) that affects the kinetic energy of the fluid is the last term (III).

Deformation work rate

The deformation work rate can be rewritten as,

$$\tau_{ij}\,\frac{\partial u_i}{\partial x_j} = \tau_{ij}\,(\underbrace{e_{ij}}_{\substack{\text{Symmetric}\\ \text{part of } \partial_j u_i}} + \underbrace{r_{ij}}_{\substack{\text{Anti-Symmetric}\\ \text{part of } \partial_j u_i}}). \tag{16}$$

However, given that $\tau_{ij}\,r_{ij} = 0$ because τ_{ij} is symmetric and r_{ij} is anti-symmetric, the deformation work rate can be simply written as,

$$\tau_{ij}\,\frac{\partial u_i}{\partial x_j} = \tau_{ij}\,e_{ij}. \tag{17}$$

If additionally, one only considers **Newtonian-type fluids**, from the constitutive law we have that,

$$\tau_{ij} = -p\delta_{ij} + 2\mu e_{ij} - \frac{2}{3}\mu\left(\frac{\partial u_i}{\partial x_i}\right)\delta_{ij}, \tag{18}$$

and hence substituting in the deformation work rate expression,

$$\tau_{ij}\,\frac{\partial u_i}{\partial x_j} = -p\underbrace{\delta_{ij}\,e_{ij}}_{e_{ii}} + 2\mu e_{ij}\,e_{ij} - \frac{2}{3}\mu\underbrace{\left(\frac{\partial u_i}{\partial x_i}\right)}_{e_{ii}}\underbrace{\delta_{ij}\,e_{ij}}_{e_{ii}}. \tag{19}$$

Viscous Dissipation, ϕ

We define **viscous dissipation** as the last two terms in the deformation work rate,

$$\phi = 2\,\mu\,e_{ij}\,e_{ij} - \frac{2}{3}\mu\,(e_{mm})^2 \tag{20}$$

This term can be shown to always be a positive definite quantity, which has important implications.

The fact that this term is always positive can be seen from the fact that this term can be rewritten as,

$$\phi = 2\mu\left[e_{ij} - \frac{1}{3}e_{mm}\delta_{ij}\right]\left[e_{ij} - \frac{1}{3}e_{mm}\delta_{ij}\right]. \tag{21}$$

Viscous Dissipation, ϕ

Next we illustrate how equation 21 is equivalent to equation 20.

$$\phi = 2\mu\left[e_{ij} - \frac{1}{3}e_{mm}\delta_{ij}\right]\left[e_{ij} - \frac{1}{3}e_{mm}\delta_{ij}\right] \tag{22}$$

$$\phi = 2\mu\left[e_{ij}\,e_{ij} - \frac{2}{3}e_{mm}\,\delta_{ij}\,e_{ij} + \frac{1}{9}(e_{mm})^2\delta_{ij}\delta_{ij}\right] \tag{23}$$

$$\phi = 2\mu\left[e_{ij}\,e_{ij} - \frac{2}{3}(e_{mm})^2 + \frac{1}{3}(e_{mm})^2\right] \tag{24}$$

$$\phi = 2\,\mu\,e_{ij}\,e_{ij} - \frac{2}{3}\mu\,(e_{mm})^2 \tag{25}$$

Deformation work rate (continued ...)

Therefore, the deformation work can be written at this point as,

$$\tau_{ij} \frac{\partial u_i}{\partial x_j} = -p e_{ii} + \phi \tag{26}$$

Substituting this back into the expression for the total work rate due to the surface forces (equation 15) gives,

$$\frac{\partial}{\partial x_j}(u_i \, \tau_{ij}) = -p \, e_{ii} + \phi + u_i \, \frac{\partial \tau_{ij}}{\partial x_j} \tag{27}$$

or rearranging,

$$u_i \, \frac{\partial \tau_{ij}}{\partial x_j} = \frac{\partial}{\partial x_j}(u_i \, \tau_{ij}) + p \, e_{ii} - \phi. \tag{28}$$

Deformation work rate (continued ...)

Substituting equation 28 back into the governing equation for the time rate of change of the kinetic energy gives,

$$\boxed{\underbrace{\rho \frac{D}{Dt}\left(\frac{1}{2} u_i u_i\right)}_{I} = \underbrace{\rho\, g_i\, u_i}_{II} + \underbrace{\frac{\partial}{\partial x_j}(u_i\, \tau_{ij})}_{III} + \underbrace{p e_{ii}}_{IV} - \underbrace{\phi}_{IV}} \tag{29}$$

- **Term I:** time rate of change of kinetic energy following a fluid particle.
- **Term II:** work rate due to body forces acting on fluid particle.
- **Term III:** total rate of work done by surface forces acting on the fluid particle.
- **Term IV:** rate of work done by volume expansion/contraction
- **Term V:** Viscous dissipation

Note: since viscous dissipation ϕ is positive definite, it represents a loss of mechanical energy. The magnitude of ϕ is high in regions with large velocity gradient, *i.e.* shear.

Energy Equation (continued ...)

Question: Where does the lost power go?

Energy Equation (continued ...)

Question: Where does the lost power go?

Answer: To increase the thermal energy.

Chapter Overview

Potential Energy

For the case when the body force is conservative, like for instance gravity, which can be written as the gradient of a potential function $\pi = g\, z$

$\Longrightarrow$ Then the work rate term becomes,

$$u_i\, g_i = -u_i \frac{\partial}{\partial x_i}(g\, z) = -\frac{D}{Dt}(gz). \tag{30}$$

Note , the above equation is true because,

$$\frac{D}{Dt}(g\, z)) = \underbrace{\cancel{\frac{\partial}{\partial t}(g\, z)}}_{\substack{\text{neither } g \text{ or } z \\ \text{depend on } t}} + u_i \frac{\partial}{\partial x_i}(g\, z) = u_i \frac{\partial}{\partial x_i}(g\, z). \tag{31}$$

Potential Energy

Finally, substituting equation 30 into equation 29, gives an alternative form of the mechanical energy equation when gravity is the only body force,

$$\boxed{\rho \frac{D}{Dt}\left(\frac{1}{2}\, u_i u_i + gz\right) = \frac{\partial}{\partial x_j}(u_i\, \tau_{ij}) + pe_{ii} - \phi} \tag{32}$$