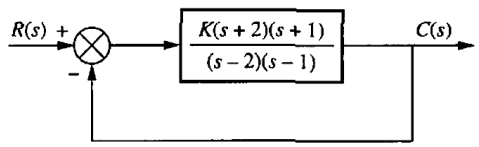
ME 5200/6200 and ECE 5615/6615 Classical Controls

Homework 08 Solutions

Problem 1

Consider the following closed-loop system. Answer the following questions using the root locus technique:

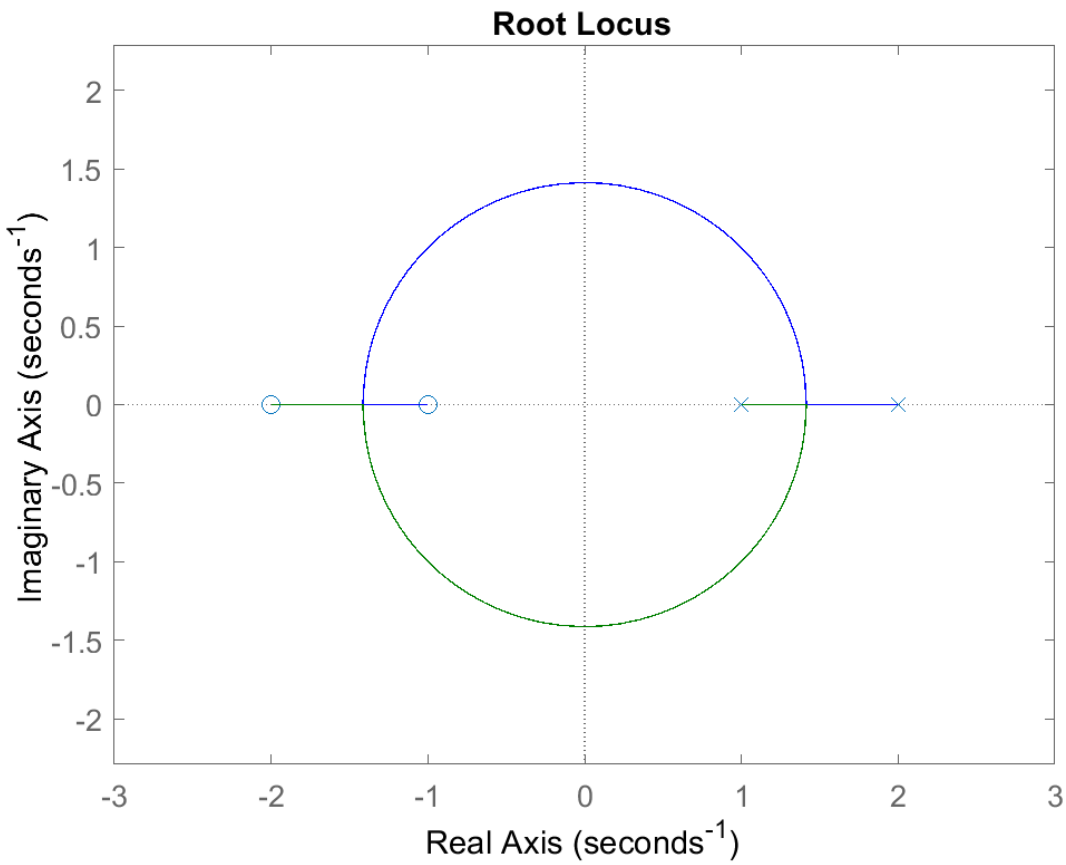


(a) Draw the root locus plot for K>0 using the Matlab function "rlocus". Label the axes.

Solution: Matlab code for using 'rlocus'

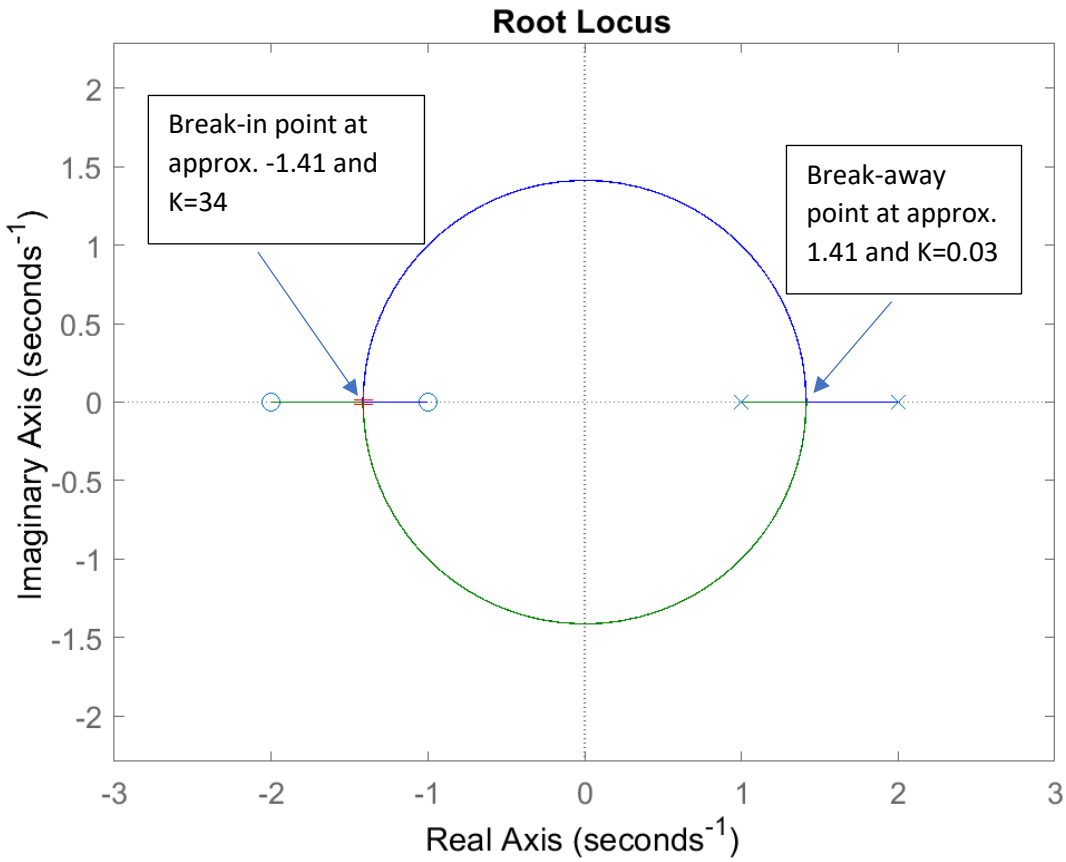
```
>> n=conv([1 2],[1 1])

n =

    1     3     2

>> d = conv([1 -2],[1 -1])

d =

    1    -3     2

>> G=tf(n,d)

G =

  s^2 + 3 s + 2
  -------------
  s^2 - 3 s + 2


Continuous-time transfer function.

Model Properties
```

>> rlocus(G)



(b) Show on the root locus plot the break-away and break-in points, label them appropriately. What is the value of the gain K at these points? You can use the Matlab function "rlocfind" to do this.

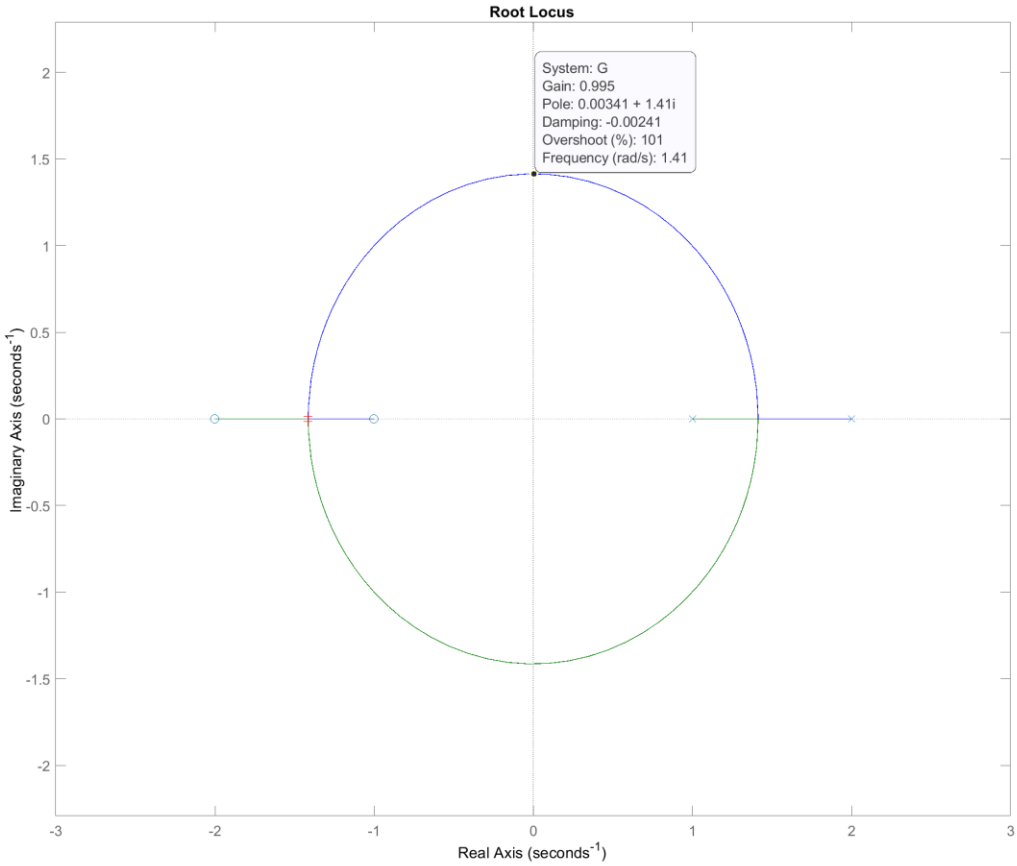
See plot for break-away and break-in points on the real axis. The values of K are shown.



(c) Show on the plot the range of K to ensure that the system is stable. You can use the Matlab function "rlocfind" to do this. Alternatively, you can check your answer using Routh-Hurwitz stability criterion, but you should show how you used the root locus plot and "rlocfind" function to find the range of K for stability. Indicate on the plot the location of the poles when K=0 and indicate the location of the poles when the closed-loop system become unstable.

See plot where values were found using rlocfind or rlocus by clicking on the points of interest:

For stability, from the plot, K>0.995. The poles are located at s=+1 and s=+2 when K=0. When the closed-loop system becomes unstable, the poles are on the jw-axis, hence s=+/-1.41 as shown in the plot.

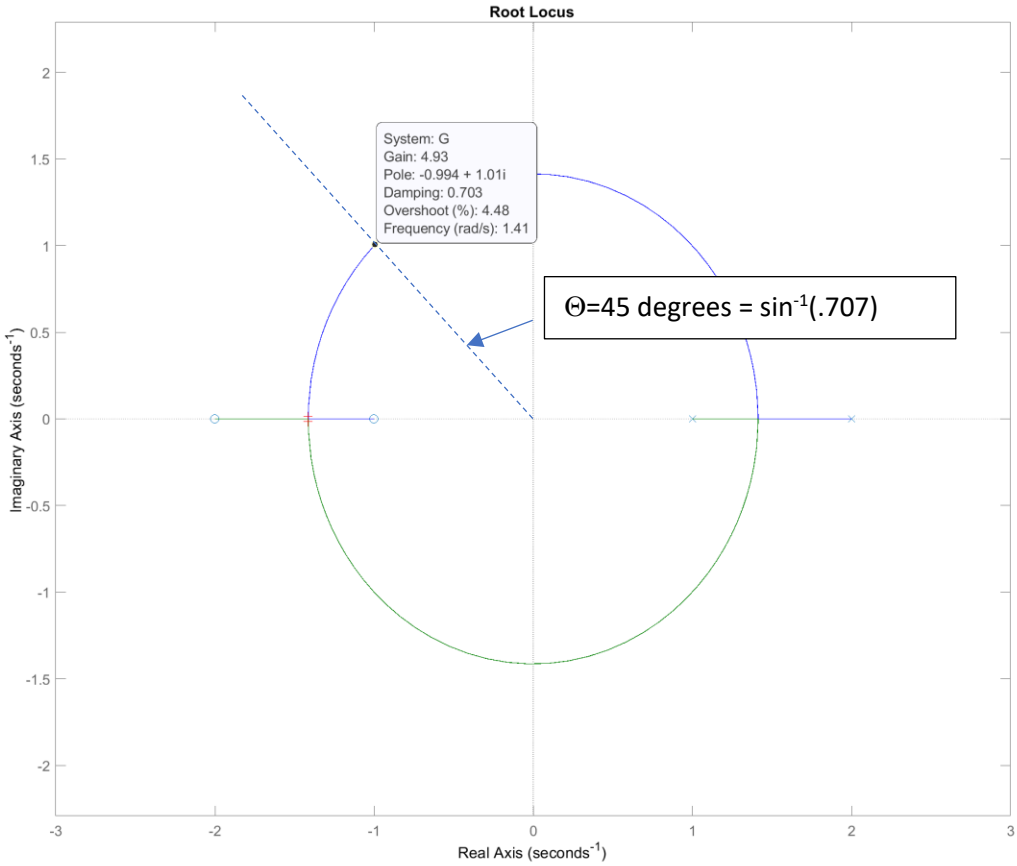


(d) Find the value of K that yields a stable system with critically damped second-order poles. Again, you can use the Matlab function "rlocfind" to do this and also check your answer with the concept of root locus and how the closed-loop poles on the root locus must satisfy the magnitude and angle condition.

See the plot: When the poles both land on the real axis in the left-hand plane, K=34 and the system is critically damped. See plot in Part (b) above.

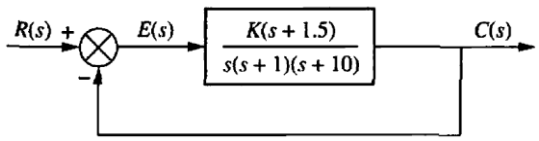
(e) Find the value of K that yields a stable system with a pair of second-order poles that have a damping ratio of 0.707. Again, you can use the "rlocfind" function and also your knowledge of second-order systems and how the damping value maps into the s-plane.

Searching along the line made by the angle of 135 degrees, K is approximately 4.93.


Root Locus
System: G
Gain: 4.93
Pole: -0.994 + 1.01i
Damping: 0.703
Overshoot (%): 4.48
Frequency (rad/s): 1.41
Θ=45 degrees = sin-1(.707)
Imaginary Axis (seconds-1)
Real Axis (seconds-1)
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-3
-2
-1
0
1
2
3

Problem 2

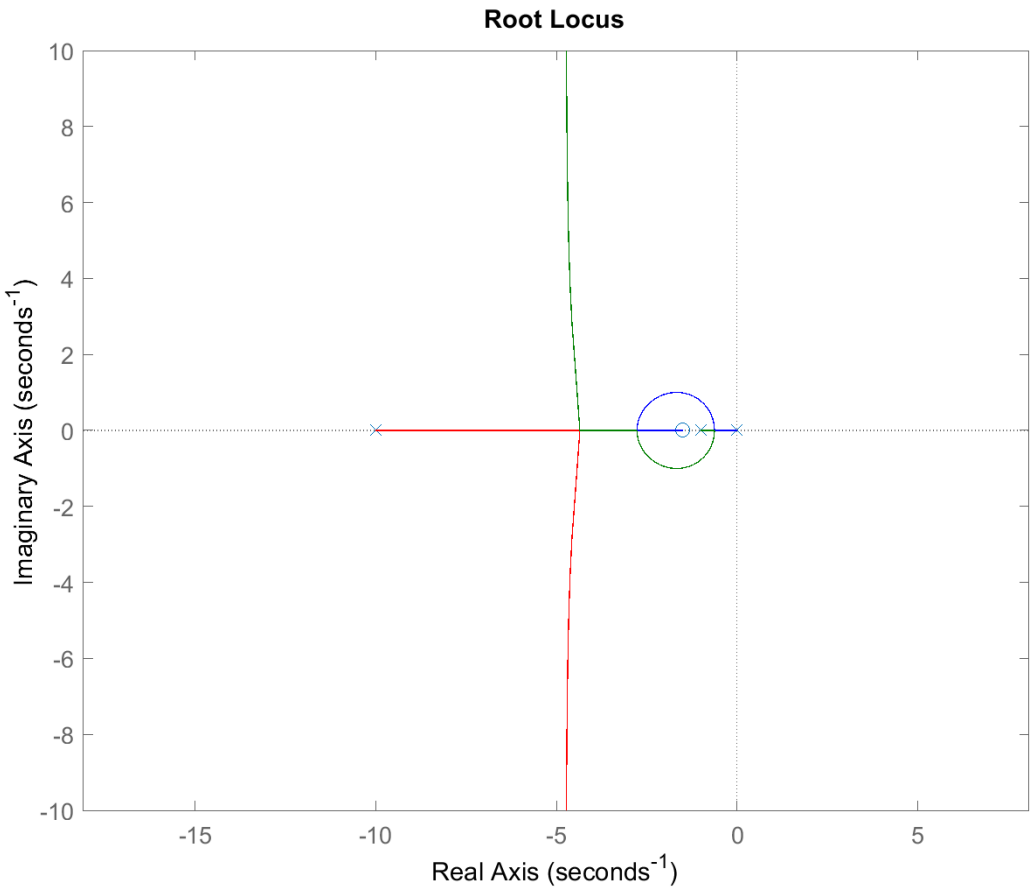
Consider the following proportional control feedback system where the objective is to design the controller (i.e., find value of K) such that the percent overshoot for the closed-loop system is 1.52%. Find the value of K to achieve this by using the root-locus approach, that is, drawing the root locus and using the plot to determine the value of K to satisfy the requirement of 1.52% overshoot. You can use the Matlab functions "rlocus" and "rlocfind", as well as "step" and "stepinfo" to assist you in finding K. You are not required to draw the root locus by hand, but encouraged to use the tools in Matlab and your knowledge of root locus, time responses, etc., to find K.



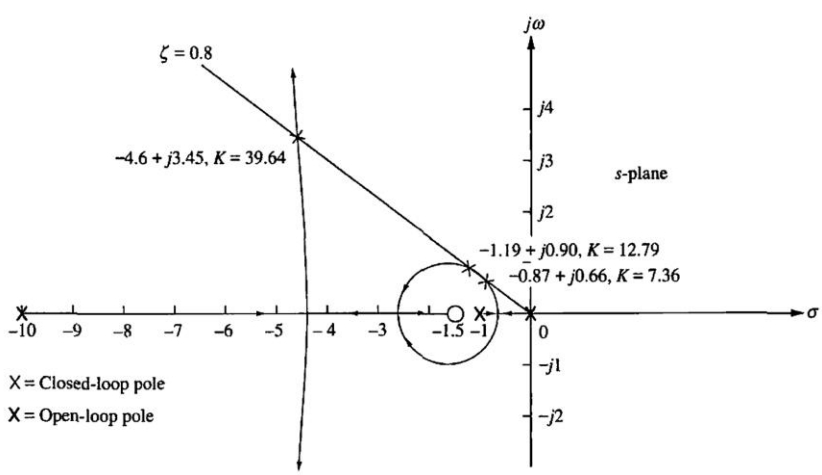
Please provide the following information:

(a) Using Matlab to draw root locus for K>0. Show the plot and label the axes.

```
>> n=[1 1.5]

n =

   1.0000    1.5000

>> d = conv([1 1 0],[1 10])

d =

    1    11    10     0

>> G=tf(n,d)

G =

      s + 1.5
  -------------------
  s^3 + 11 s^2 + 10 s

Continuous-time transfer function.

Model Properties

>> rlocus(G)
```

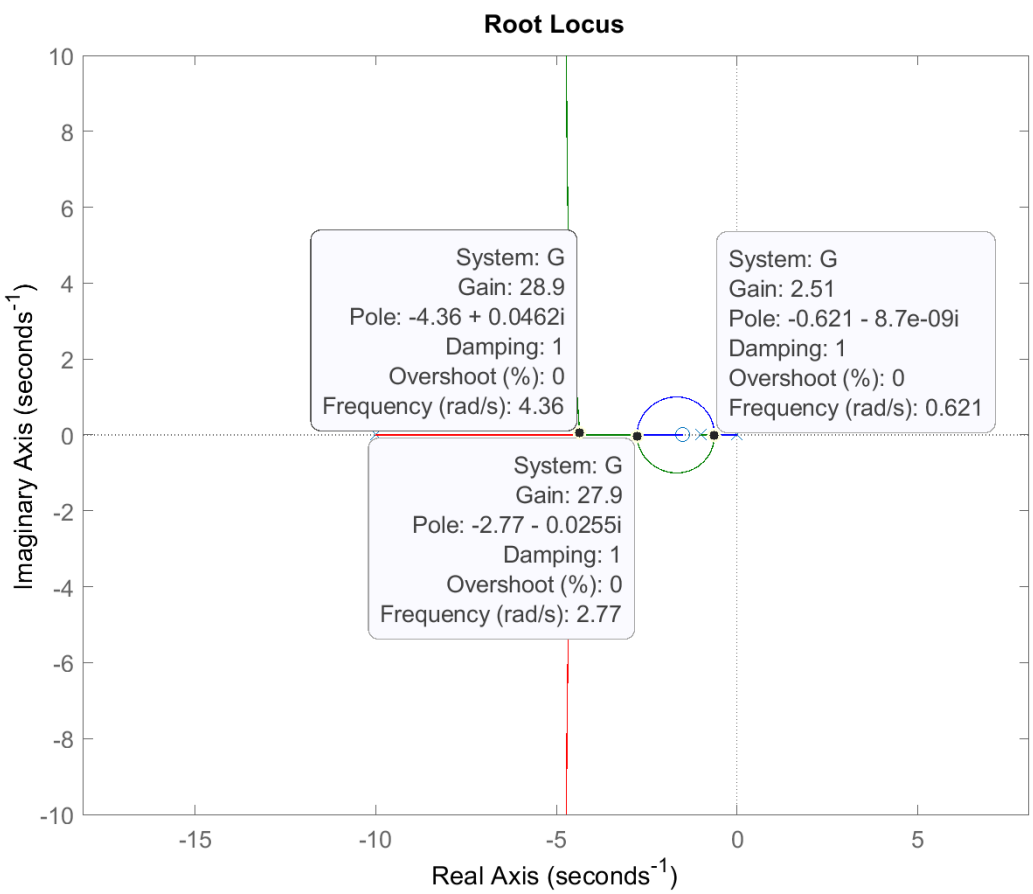


A closer look at the root locus showing key points:



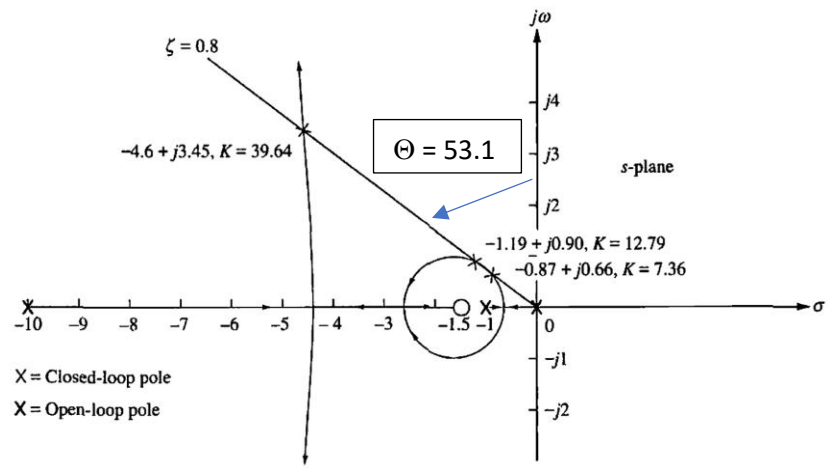
Notice that this is a third-order system with one zero. Break-away points on the real axis can occur between 0 and -1 and between -1.5 and -10, where the gain reaches a peak value.

Using the root-locus plot and in Matlab by clicking on key points, break-away points are found at -0.621 with gain of 2.51 and at -4.36 with gain 28.9. The break-in point is at -2.77 with a gain of 27.9 (see plot below).



(a) On your plot from (a), show where the closed-loop poles should be in order to satisfy the 1.52% overshoot requirement. Show analysis to justify your answer(s).

Assume the system response can be approximated by a second-order system, thus a 1.52% overshoot will correspond to a damping ratio of 0.8, which can be found from solving for the damping ratio using the percent overshoot equation. The 0.8 damping ratio value is indicated by the line associated with an angle of 53.1 degrees:



Use the root locus plot in Matlab and searching along the 0.8 damping ratio line for the point where the angles from the open-loop poles and zeros add up to an odd multiple of 180°. This is the point where the root locus crosses the 0.8 damping ratio or 1.52 percent overshoot line. Three points satisfy this criterion: —0.87 ± /0.66, — 1.19 ± /0.90, and -4.6 ±j 3.45 with respective gains of 7.36,12.79, and 39.64. For each point the settling time and peak time are evaluated using

$$T_s = \frac{4}{\xi\omega_n}$$

where $\xi\omega_n$ is the real part of the closed-loop pole, and also using

$$T_p = \frac{\pi}{\omega_n\sqrt{1-\xi^2}}$$

where $\omega_n\sqrt{1-\xi^2}$ is the imaginary part of the closed-loop pole.

(b) Determine the possible values of K that are associated with your answers in Part (b).

To test our assumption of a second-order system, we must calculate the location of the third pole. Using the root locus program, search along the negative extension of the real axis between the zero at -1.5 and the pole at -10 for points that match the value of gain found at the second-order dominant poles. For each of the three crossings of the 0.8 damping ratio line, the third closed-loop pole is at —9.25, -8.6, and —1.8, respectively. The results are summarized in the table below:

Case	Closed-loop poles	Closed-loop zero	Gain	Third closed-loop pole	Settling time	Peak time	K_v
1	$-0.87 \pm j0.66$	$-1.5 + j0$	7.36	−9.25	4.60	4.76	1.1
2	$-1.19 \pm j0.90$	$-1.5 + j0$	12.79	−8.61	3.36	3.49	1.9
3	$-4.60 \pm j3.45$	$-1.5 + j0$	39.64	−1.80	0.87	0.91	5.9

(c) Determine the steady-state error for the chosen K.

*Finally, let us examine the steady-state error produced in each case. Note that we have little control over the steady-state error at this point. When the gain is set to meet the transient response, we have also designed the steady-state error. For the example, the steady-state error specification is given by Kv and is calculated as Kv = $\lim_{s\to 0} sG(s)$ = K(1.5)/(1*10). The results for each case is shown above in the table.*

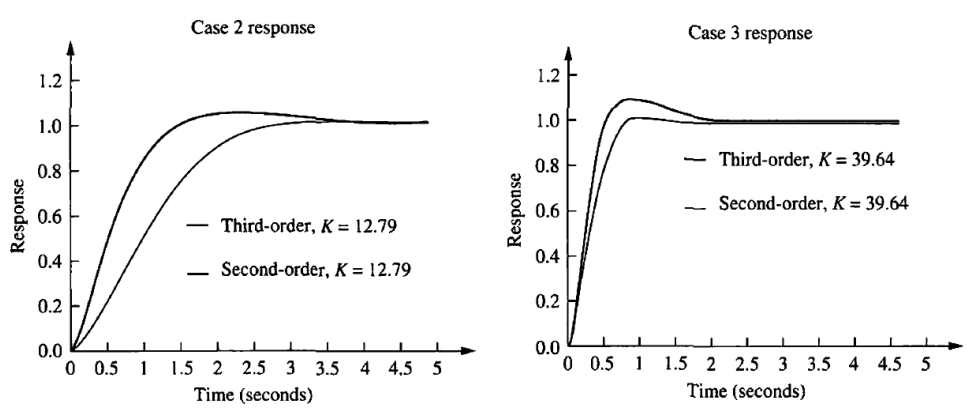
(c) Analyze the results from Part (c) to determine the K that will work. Note that you may have to invoke the assumption of a second-order system in order to force the closed-loop system to have a 1.52% overshoot.

How valid are the second-order assumptions? From the table above, Cases 1 and 2 yield third closed-loop poles that are relatively far from the closed-loop zero. For these two cases there is no pole-zero cancellation, and a second-order system approximation is not valid. In Case 3, the third closed-loop pole and the closed-loop zero are relatively close to each other, and a second-order system approximation can be considered valid. In order to show this, let us make a partial-fraction expansion of the closed-loop step response of Case 3 and see that the amplitude of the exponential decay is much less than the amplitude of the underdamped sinusoid. The closed-loop step response, 03(5), formed from the closed-loop poles and zeros of Case 3 is

$$
\begin{aligned}
C_3(s) &= \frac{39.64(s+1.5)}{s(s+1.8)(s+4.6+j3.45)(s+4.6-j3.45)} \\
&= \frac{39.64(s+1.5)}{s(s+1.8)(s^2+9.2s+33.06)} \\
&= \frac{1}{s} + \frac{0.3}{s(s+18)} - \frac{1.3(s+4.6)+1.6(3.45)}{(s+4.6)^2+3.45^2}
\end{aligned}
$$

Thus, the amplitude of the exponential decay from the third pole is 0.3, and the amplitude of the underdamped response from the dominant poles is $\sqrt{1.3^2+1.6^2}$ *= 2.06. Hence, the dominant pole response is 6.9 times as large as the nondominant exponential response, and we assume that a second-order approximation is valid.*

Using Matlab to create the step responses for the difference cases, we obtain:

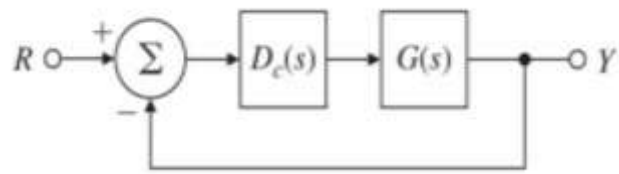


which shows comparisons of step responses for the problem we have just solved. Cases 2 and 3 are plotted for both the third-order response and a second-order response, assuming just the dominant pair of poles calculated in the design problem. Again, the second-order approximation was justified for Case 3, where there is a small difference in percent overshoot. The second-order approximation is not valid for Case 2. Other than the excess overshoot, Case 3 responses are similar.

Problem 3

Consider the following unity feedback control system shown below. Using root locus techniques, design the following two controllers:

(a) Let the open-loop plant transfer function be given as $G(s) = 1/s^2$. Design a lead compensator (controller) $Dc(s) = K*(s+z)/(s+p)$ to be added in series with the plant so that the dominant closed-loop poles are located at $s = -2 \pm 2j$. Use Matlab to create step response plots for the new closed-loop system with the lead compensator. How does the response differ between the open-loop and closed-loop systems?

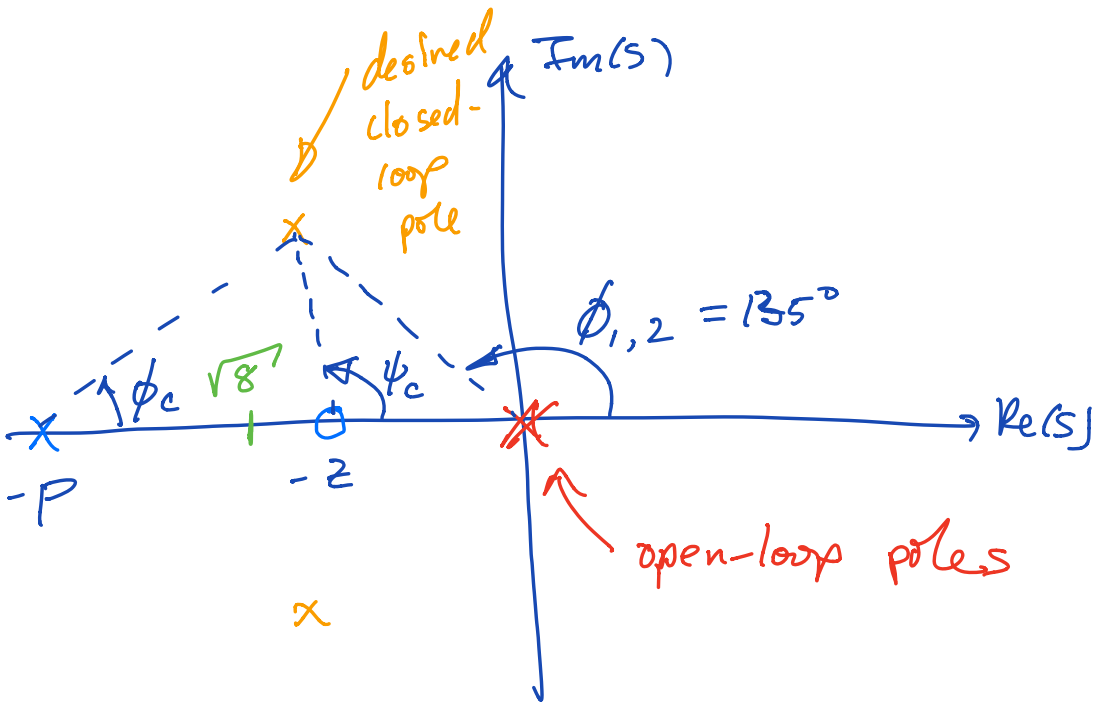


Desired closed-loop pole are:

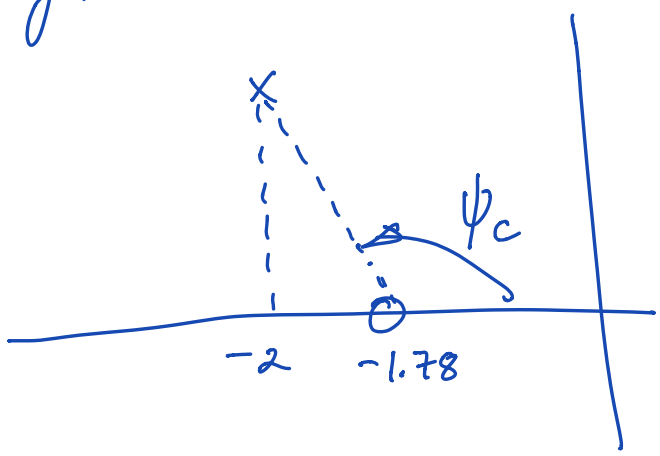
$(s+2-2j)(s+2+2j) = s^2 + 4s + 8$

hence closed loop $\omega_n = \sqrt{8} \cong 2.8$ rad/s

we need to pick location of the zero, but trick is zero should be to the right of $s = \sqrt{8}$ in order for angle condition to work.



pick for example: $z = -1.78$, so by geometry:



$$\psi_c = 90° + \tan^{-1}\left(\frac{.22}{2}\right) = \underline{\underline{96.3°}}$$

Now we find ϕ_c:

Angle condition:

$$\sum \psi_i - \sum \phi_i = 180^\circ + 360^\circ(l-1)$$

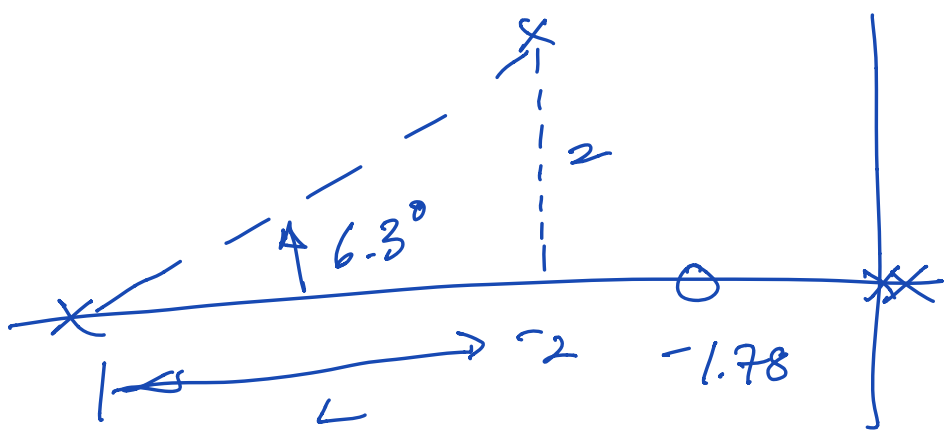
$$\psi_c - (135^\circ + 135^\circ + \phi_c) = 180^\circ + 360^\circ(l-1)$$

$$96.3^\circ - 270^\circ - \phi_c = 180^\circ + 360^\circ(l-1)$$

$$96.3^\circ - \phi_c = 360^\circ + 360^\circ(l-1)$$

$$96.3^\circ - \phi_c = 450^\circ + 360^\circ(l-1)$$

$$\Rightarrow \underline{\underline{\phi_c = 6.3^\circ}}$$



$$\tan 6.3^\circ = \frac{2}{2} \Rightarrow \underline{\underline{L = 18.1}}$$

So, we put pole of lead at:

$$p = 2 + 18.1 \simeq \underline{\underline{20.1}}$$

lead controller is:

$$C(s) = K\left(\frac{s + 1.78}{s + 20.1}\right)$$

Now, to find K:

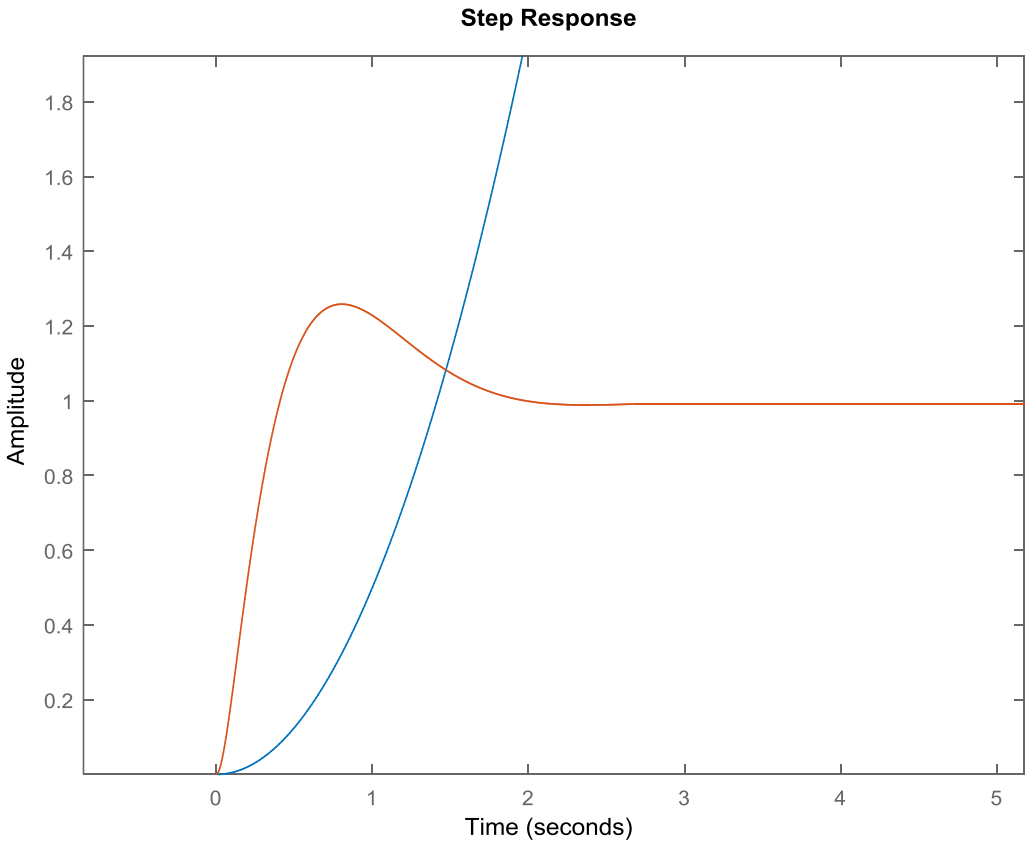
$$K = \left|\frac{1}{C(s)\,G(s)}\right|_{s=-2+2j}$$

$$\Rightarrow \underline{\underline{K \simeq 72}}$$

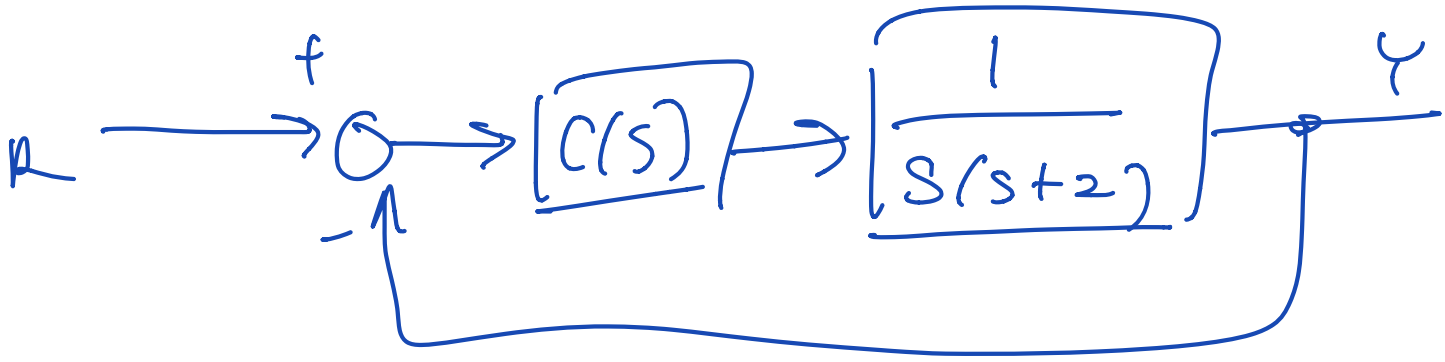
$$\Rightarrow \boxed{C(s) = 72\left(\frac{s + 1.78}{s + 20.1}\right)}$$

See attached step responses comparing open-loop and closed-loop.

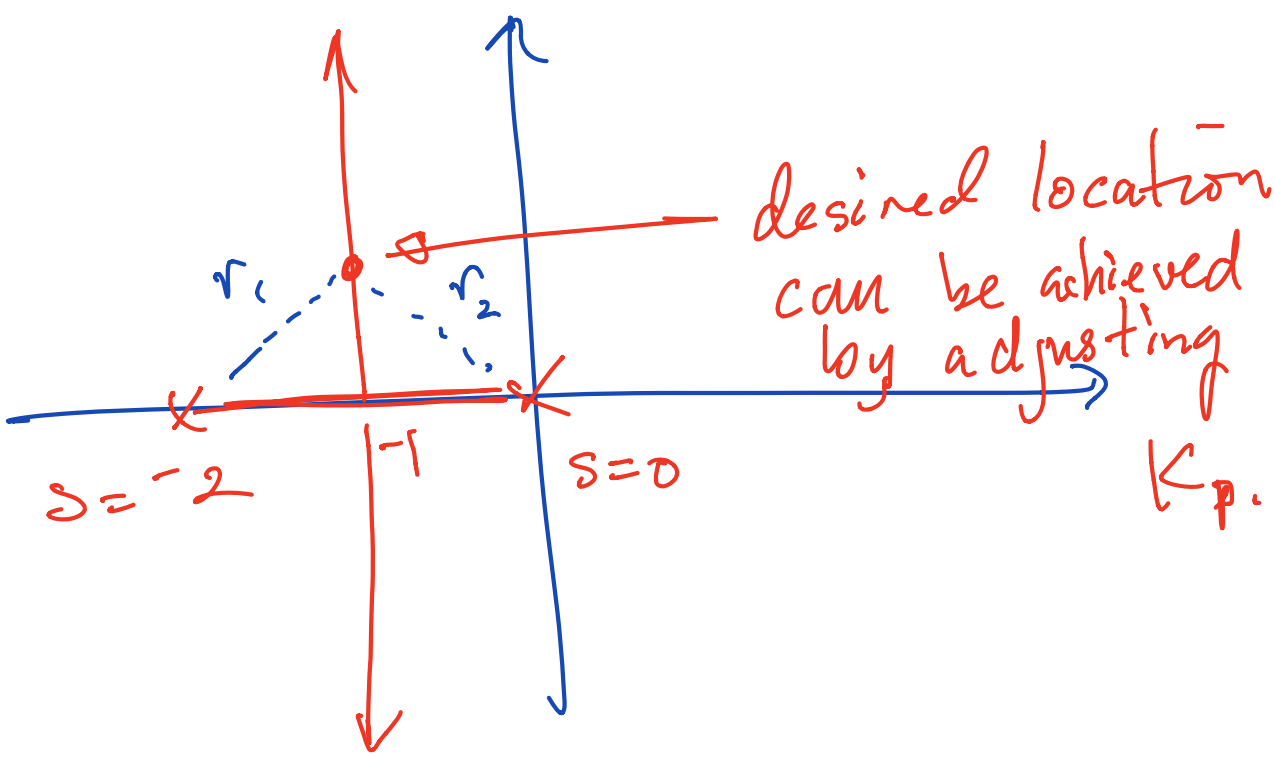
open-loop is simply drifts off to infinity, but lead helps to stabilize but still has a little overshoot.


Step Response
Amplitude
Time (seconds)
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
1
2
3
4
5

(b) Now let G(s) = 1/s(s+2). Design a lag compensator so that the dominant closed-loop poles are located at s = -1 ± j and the steady-state error due to a ramp is less than 0.2. Use Matlab to generate plots that compare the open-loop and closed-loop responses. Does the closed-loop system achieve the desired steady-state behavior? Comment on this and show proof by providing plots.



Using just $C(s) = K_p$, we can put poles in the desired location:



$$K_p = \left| \frac{-1}{G(s)} \right|_{s=-1+j} = r_1 * r_2$$

$$K_p = \sqrt{2} \cdot \sqrt{2} = \underline{\underline{2}}$$

With $K_p = 2$, e_{ss} for ramp is

$$e_{ss} = \lim_{s \to 0} \frac{1}{K_v} \qquad K_v = \lim_{s \to 0} s\,G(s)K_p$$

$$\Rightarrow \quad K_v = \lim_{s \to 0} s\,K_p \left(\frac{1}{s(s+2)} \right)$$

$$\Rightarrow \quad K_v = 1$$

But, we need $e_{ss} = 0.2$, so

$K_v = 5$.

to get $K_v = 5$, we want:

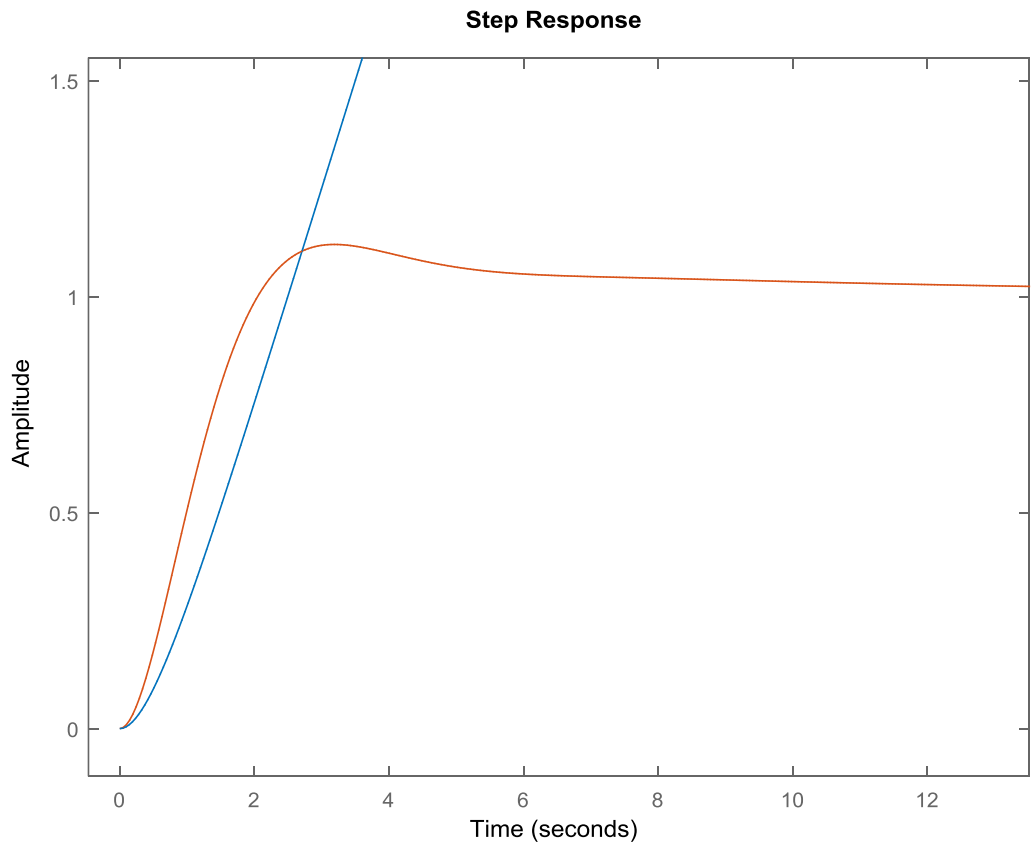
$$K_v = \lim_{s \to 0} s\, G(s)\, C(s)$$

$$= \lim_{s \to 0} s\, K_p \left(\frac{s+z}{s+p}\right)\left(\frac{1}{s(s+2)}\right) = 5$$

so, we pick:

$$\boxed{C(s) = 2\left(\frac{s+0.1}{s+0.02}\right)}$$

see plot below comparing open-loop & closed-loop behavior.


Step Response
Amplitude
Time (seconds)
1.5
1
0.5
0
0
2
4
6
8
10
12

Problem 4

Consider a unity feedback control system, where the open-loop plant transfer function be given by

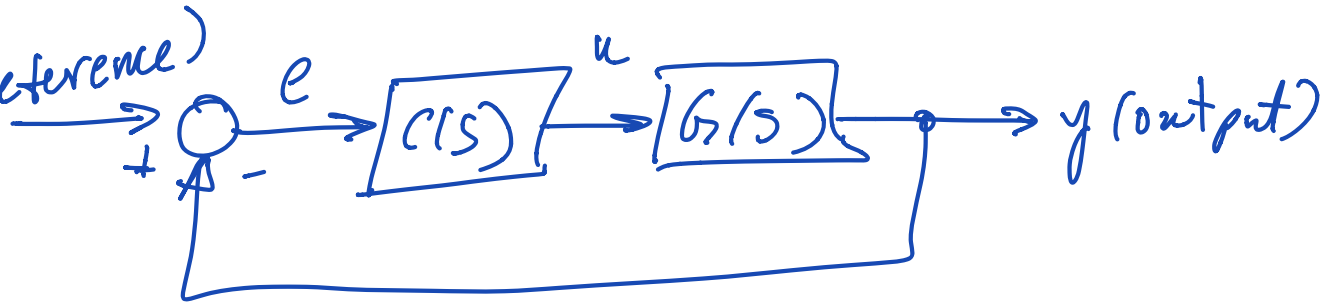
$$G(s) = \frac{1}{s(s+b)(s\text{^}2 + 2\xi\omega_n s + \omega_n\text{^}2)}$$

with b=10, ξ = 0.707, and ω_n = 4 rad/s.

Design a PID controller to yield good step response by using the Ziegler-Nichols tuning method:

(a) Sketch the closed-loop block diagram and label all signals and blocks.
(b) Write the transfer function for the closed-loop system in terms of the controller parameters, Kp, Ki, and Kd and the open-loop system.
(c) Using Matlab, make a plot of the step response (unit step input) for the open-loop system G(s). What is the steady-state value? What is the settling time?
(d) Now design a PID controller and provide the parameters for Kp, Ki, and Kd based on the Ziegler-Nichols tuning method.
(e) Using Matlab, make a plot of the step response for the closed-loop system using the PID gains you found from Part (d). Determine the percent overshoot, setting time, and steady-state error.
(f) Next, try to tune the PID gains to further improve the transient and steady-state response, for example, try to reduce the settling time by a factor of 10 if possible, minimize overshot, and achieve as close as possible zero steady-state error. Provide the values you found. This process is trial and error, and you use as a starting point the values from Part (d).

(a)



$$C(s) = \text{PID controller} = K_p + \frac{K_i}{s} + K_d s$$

$$G(s) = \text{plant} = \frac{1}{s(s+b)(s^2 + 2\xi\omega_n s + \omega_n^2)}$$

(b) $\dfrac{Y(s)}{R(s)} = T(s) = \dfrac{C(s)\,G(s)}{1+C(s)\,G(s)}$

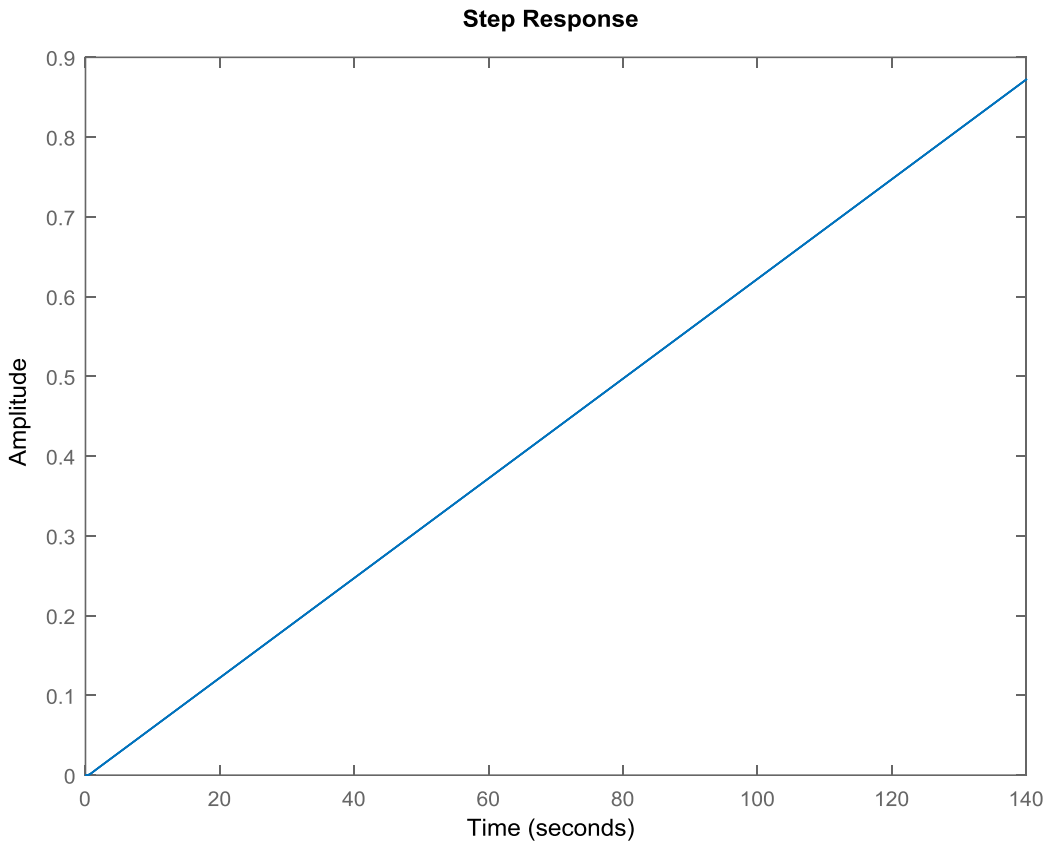
$$T(s) = \frac{K_d s^2 + K_p s + K_i}{s^2(s+b)(s^2+2\zeta\omega_n s+\omega_n^2) + K_d s^2 + K_p s + K_i}$$

This can also be simplified.

(c) See plot below of step response for open-loop system.

Because there's a pole at $s=0$, system simply drifts off to infinity! So, $t_s = \infty$ and there's no steady-state value!

Step response for open-loop system for Part (c)



(b) we use ultimate sensitivity method:

$$C(s) = K_p \left(1 + \frac{1}{T_i s} + T_d s\right)$$

Closed-loop with K_p = small and use $R = 0$ (Reference input)

First find K_p^* - gain that creates oscillations, then find K_i + K_d.

$$K_p = 1.6 K_p^*$$
$$T_i = 0.5 P_u$$
$$T_d = 0.125 P_u$$

see plot below for $K_p^* \approx 637$

where $P_u = 2$ seconds

So we get:

$$K_p = 1.6\,(637) = \underline{\underline{1019.2}}$$

$$K_i = \frac{K_p}{T_i} = \frac{1019.2}{0.5(2)} = \underline{\underline{1019.2}}$$

$$K_d = K_p T_d = 1019.2\,(0.125(2)) = \underline{\underline{254.8}}$$

So:

$$\boxed{C(s) = 1019.2 + \frac{1019.2}{s} + 254.8\,s}$$

This is just the starting point - may need to tune.

(e) See step response using PID from part (d).

$t_s \approx 20\,s$

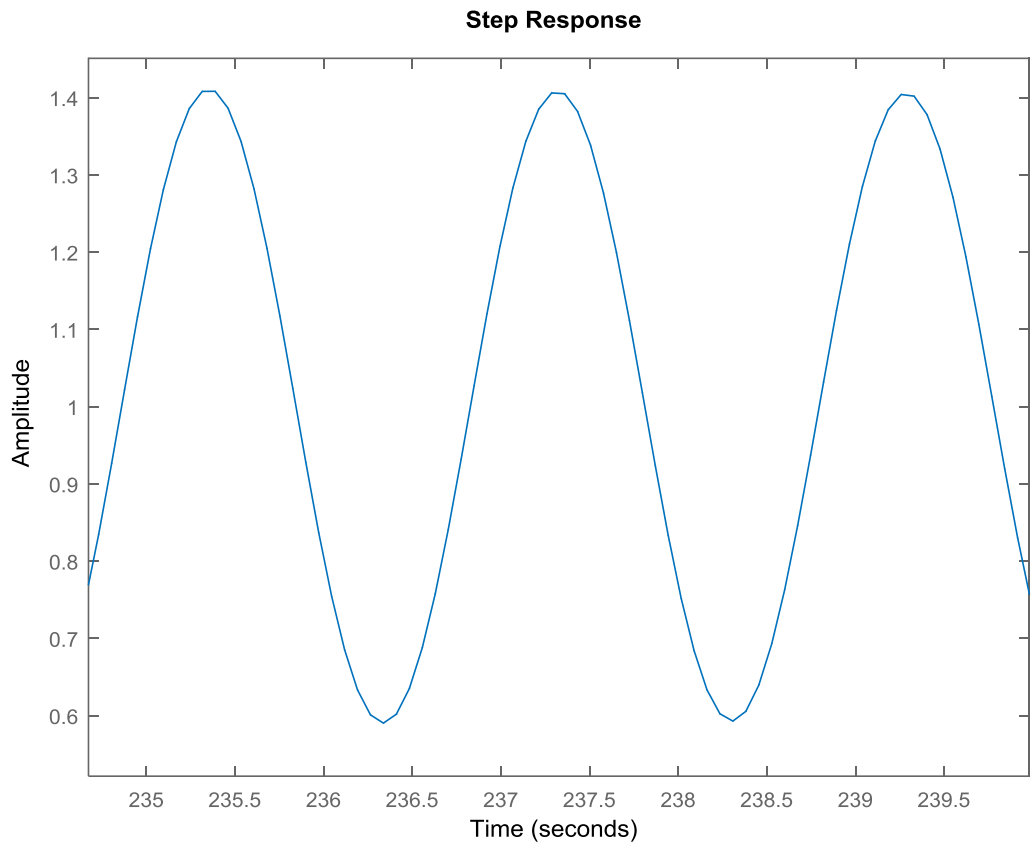
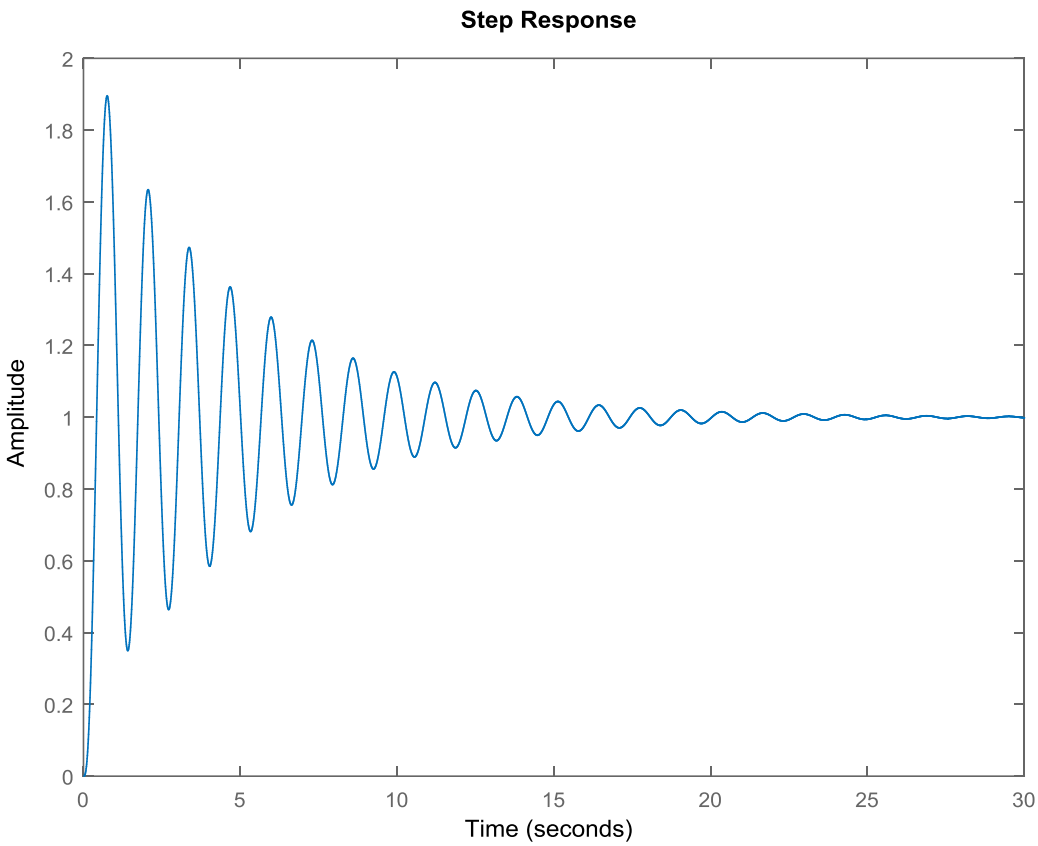
$\%OS \approx 90\%$

Steady-state error $= 0$

(f) More tuning can be done to improve performance.

Your results will vary.

Step response of Kp controller to find Kp* = 637:



Step response for PID design using Kp=1019.2, Ki=1019.2 and Kd=254.8