Intermediate Fluid Mechanics

Lecture 9: Conservation of Mass

Marc Calaf

Department of Mechanical Engineering

University of Utah

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Chapter Overview

Lecture Objectives

In this lecture we will:

- Learn about the mathematical representation of the Conservation of Mass law in an Eulerian framework.
- Introduce a more robust definition to conservation of Mass in comparison to that of earlier courses,...

Chapter Overview

Leibnitz Theorem of Calculus

Leibnitz's theorem tells us how to differentiate an integral with respect to a variable (*e.g.* t), when the limits of integration are also functions of this same variable (t).

• Let's consider the following example, where function F varies with both variables x and t.

$\Longrightarrow$ Leibnitz's theorem states that

$$\frac{d}{dt}\int_{x=a(t)}^{x=b(t)} F(x,t)dx = \int_a^b \frac{\partial F}{\partial t} + \frac{d\ b}{dt}F(b,t) - \frac{d\ a}{dt}F(a,t). \qquad (1)$$

Note, if a and b were constants (independent of time), then equation 1 reduces to,

$$\frac{d}{dt}\int_{x=a(t)}^{x=b(t)} F(x,t)dx = \int_a^b \frac{\partial F}{\partial t}. \qquad (2)$$

(Therefore, the last two terms on the right hand side of equation 1 account for the fact that the limits of integration are changing with respect to t).

Leibnitz Theorem of Calculus (continued ...)

Figure below describes why the additional terms $\frac{d\,b}{dt}F(b,t) - \frac{d\,a}{dt}F(a,t)$are necessary.

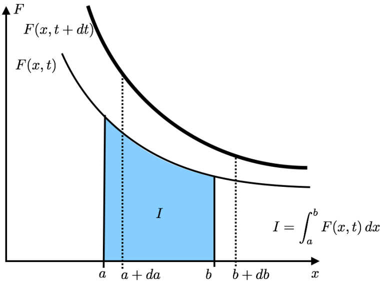

Figure: *Graphical representation of the Leibnitz theorem.*

Basically, the area under the curve changes not only due to the fact that $F(x, t)$ is changing in time but also because the limits of the integration are changing which creates two additional areas that must be counted.

Leibnitz Theorem of Calculus (continued ...)

Looking at this in a more symbolic form,

$$\frac{d}{dt}\int_{a(t)}^{b(t)} F(x,t)\,dx = \lim_{\Delta t \to 0} \frac{\int_{a+da}^{b+db} F(x,t+dt)\,dx - \int_a^b F(x,t)\,dx}{\Delta t} \tag{3}$$

$$= \lim_{\Delta t \to 0} \frac{C + B - A}{\Delta t} \tag{4}$$

These three areas correspond to:

- $A \equiv da \times F(a,t) \rightarrow$ loss of F due to inner boundary moving at rate da/dt.
- $B \equiv db \times F(b,t) \rightarrow$ gain of F due to outer boundary moving at rate db/dt.
- $C \equiv \left[\int \frac{\partial F}{\partial t} dx\right] dt \rightarrow$ local change in F integrated over a region.

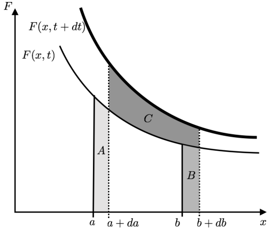

Figure: Graphical representation of the Leibnitz theorem.

Chapter Overview

Generalization of Leibnitz's Theorem in three dimensions:

In general,

$$\frac{d}{dt}\int_{\forall(t)} F(\vec{x},t)\, d\forall = \int_{\forall(t)} \frac{\partial F}{\partial t}\, d\forall + \int_{A(t)} F(\vec{u}_A \cdot d\vec{A}) \tag{5}$$

where:

- $\forall(t)$ is the volume over which the integral is being performed,
- $A(t)$ is the surface area bounding the volume $\forall(t)$,
- $\vec{u}_A$ is the velocity of the boundary $A(t)$,
- $d\vec{A}$ is the differential area on the surface having a direction corresponding to the outward normal to the surface at that point.

Note that the surface integral accounts for the contribution due to the changes in the bounding surface (or integral limits).

Generalization of Leibnitz's Theorem in three dimensions (continued ...)

Fixed Volume

For a fixed volume, $\vec{u}_A = 0$ and hence equation 5 gets reduced to,

$$\frac{d}{dt}\int_{\forall(t)} F(\vec{x},t)\, d\forall = \int_{\forall(t)} \frac{\partial F}{\partial t}\, d\forall \tag{6}$$

Note:

- In this case one has an ordinary derivative on the left hand side, because after performing the integration over the volume, the resultant function only depends on time t.
- That is, integrating over the spatial coordinate removes the $\vec{x}$-dependence.

Generalization of Leibnitz's Theorem in three dimensions (continued ...)

Material Volume

A material volume is one in which the surface moves with the fluid, so that $\vec{u}_A = \vec{u}$, where $\vec{u}$ denotes the fluid velocity.

In this case, the volume is a function of time and Leibnitz's theorem may be written as

$$\frac{D}{Dt}\int_{\forall(t)} F(\vec{x},t)\, d\forall = \int_{\forall(t)} \frac{\partial F}{\partial t}\, d\forall + \int_{A(t)} F\,(\vec{u}\cdot d\vec{A}) \tag{7}$$

+ Note: In the equation above, we use the capital D/Dt to remind ourselves that we are talking about the time rate of change following the material (*i.e. fluid*).

Generalization of Leibnitz's Theorem in three dimensions (continued ...)

Given,

$$\frac{D}{Dt}\int_{\forall(t)} F(\vec{x},t)\, d\forall = \int_{\forall(t)} \frac{\partial F}{\partial t}\, d\forall + \int_{A(t)} F\left(\vec{u}\cdot d\vec{A}\right) \tag{8}$$

One can use the Gauss' theorem to rewrite the surface integral in the last term as a volume integral (using index notation),

$$\frac{D}{Dt}\int_{\forall(t)} F\, d\forall = \int_{\forall}\left[\frac{\partial F}{\partial t} + \frac{\partial (F\, u_j)}{\partial x_j}\right] d\forall \tag{9}$$

Note:

The LHS represents a Lagrangian description since the volume follows the material; whereas the RHS represents an Eulerian description since the integral is evaluated assuming the volume remains constant (at any given instant in time).

Chapter Overview

Conservation of Mass (Continuity Equation)

A system of fluid particles is defined such that the mass of the system remains constant for all time.

Therefore, conservation of mass can be defined as

$$\frac{dm}{dt}|_{sys} = 0, \tag{10}$$

where the time derivative is evaluated following the system.

In fluid dynamics, however, we do not deal with the quantity of mass, since the fluid is a continuum. Rather, we prefer to use density. Therefore, conservation of mass can be rewritten as

$$\frac{d}{dt} \int_{\forall(t)} \rho(\vec{x}, t) \, d\forall = 0. \tag{11}$$

Conservation of Mass (continued ...)

At this point one can use the Leibnitz's theorem,

$$\frac{d}{dt}\int_{\forall(t)} \rho(\vec{x},t)\, d\forall = \int_{\forall} \left[\frac{\partial \rho}{\partial t} + \frac{\partial(\rho\, u_i)}{\partial x_i}\right] d\forall = 0. \tag{12}$$

Since the volume is arbitrary, the equality above must hold for the integrand as well as the integral. Therefore, one can find that,

$$\boxed{\frac{\partial \rho}{\partial t} + \frac{\partial(\rho\, u_i)}{\partial x_i} = 0.} \tag{13}$$

(One can reach the same conservation of mass law in an Eulerian framework using a slightly more physical derivation.)

Conservation of Mass: Physical Derivation

For this, consider a volume fixed in space as shown in the Figure.

The rate of increase of mass inside the volume is

$$\frac{d}{dt}\int_{\forall} \rho \, d\forall = \int_{\forall} \frac{\partial \rho}{\partial t} d\forall \qquad (14)$$

(Note that in this case, the time derivative can move inside the integral because the limits of integration remain fixed in time.)

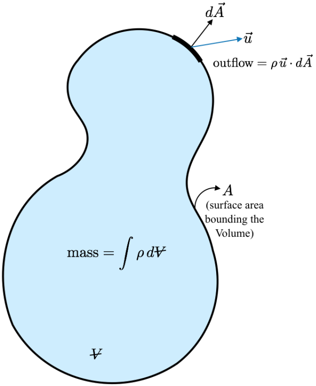

Figure: Conservation of Mass.

Conservation of Mass: Physical Derivation (continued ...)

At the same time, the net **mass flux** out of the volume is

$$\dot{M} = \int_A \rho \, \vec{u} \cdot d\vec{A}, \tag{15}$$

where $d\vec{A} = dA \, \hat{n}$ with $\hat{n}$ denoting the unit outward normal to the surface.

(Note that only, the component of the velocity perpendicular to the surface produces an outflow.)

Conservation of Mass: Physical Derivation (continued ...)

The conservation of mass principle states that the rate of increase of mass within a fixed volume must be equal to the net inflow through the boundaries.

Using the Gauss theorem (divergence theorem) to recast the surface/area integral on the RHS in terms of an equivalent volume integral,

$$\int_{\forall} \left[\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{u}) \right] d\forall = 0 \tag{16}$$

One can now use the same argument to get rid of the volume integral; and recover the same result as that obtained from Leibnitz's theorem

$$\boxed{\frac{\partial \rho}{\partial t} + \frac{\partial (\rho\, u_i)}{\partial x_i} = 0.} \tag{17}$$

Conservation of Mass

Regardless Conservation of Mass can be rewritten as,

$$\frac{\partial \rho}{\partial t} + \rho \frac{\partial u_i}{\partial x_i} + u_i \frac{\partial \rho}{\partial x_i} = 0 = \underbrace{\frac{1}{\rho}\frac{D\rho}{Dt}}_{I} + \underbrace{\frac{\partial u_i}{\partial x_i}}_{II} \tag{18}$$

- Term I: the rate of change of the density of a fluid particle over its original density.
- Term II: dilation of a fluid particle, which represents how fast the volume of the fluid particle is expanding/contracting.

$\Longrightarrow$ If a fluid particle's density changes, then its volume must compensate accordingly to ensure that the mass of the fluid particle remains constant.

Chapter Overview

Incompressible Flow

There is a special case, which arises when the volume of individual fluid particles remains constant. In this case, the flow is referred to as **incompressible**.

Mathematically, an incompressible flow satisfied the relation

$$\frac{\partial u_i}{\partial x_i} = 0 \tag{19}$$

Because of the continuity equation, we see that in an incompressible flow

$$\frac{D\rho}{Dt} = 0, \tag{20}$$

Note that his requirement does not necessarily mean that the density must remain constant in space, as is discussed below.

Density of a Stratified Flow:

Consider an incompressible flow, wherein we know that

$$\frac{D\rho}{Dt} = 0. \tag{21}$$

Using the definition of the material derivative, one can write this as

$$\frac{\partial \rho}{\partial t} + u_i \frac{\partial \rho}{\partial x_i} = 0. \tag{22}$$

Further consider that the flow is steady so that $\partial \rho / \partial t = 0$, this leaves

$$u_i \frac{\partial \rho}{\partial x_i} = 0 \tag{23}$$

Density of a Stratified Flow (continued ...)

$$u_i \frac{\partial \rho}{\partial x_i} = 0 \tag{24}$$

1. This relation can certainly be satisfied if $\partial \rho / \partial x_i = 0$, *i.e.* the density is the same everywhere in space. If this is true, it means that the density is uniform.
2. The above relation can also be satisfied if the velocity is perpendicular to the density gradient. When this happens, one can say that the flow is density stratified.

Density of a Stratified Flow (continued ...)

When does scenario (2) happen in real life? (In the ocean and in the atmosphere). For the case of the atmosphere, the temperature decreases as one moves away from the earth's surface. This creates a vertically stratified density field as shown in the Figure,

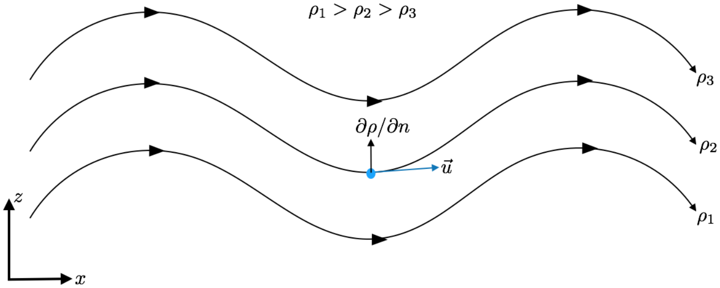

Figure: *Stratification in the atmosphere.*

Density of a Stratified Flow (continued ...)

- When the isocontour lines of density are coincident with the streamlines, then $\vec{u} \cdot \vec{\nabla}\rho = 0$ even though the density is non-uniform.
- This happens in the atmosphere because $\vec{\nabla}\rho$ points in the vertical direction (*i.e.* density decreases with height) however, the predominant mean flow is in the horizontal direction.
- A similar situation occurs in the ocean, whereby the density increases with depth due to both temperature and salinity gradients.
- However, again, the predominant mean current is horizontal.
- Therefore, in both cases, incompressibility is a very good assumption.

+ Note: The above discussion of density stratified flow assumes the flow is steady. Assuming steady flow may be valid depending on the time scales of interest.