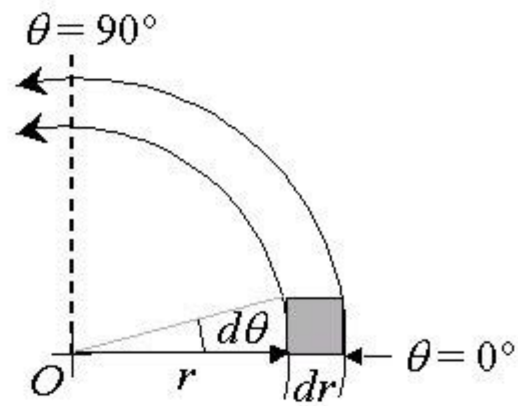
Homework 3

Rotation, Deformation and Kinematics

Your answers to these questions, and what you learn from them, will be greatly enhanced through collaboration and discussion amongst your discussion group and in the recitation. This is actively encouraged. However, once you have decided how to answer these, the final solutions must be prepared individually.

1. Consider the velocity field of a two-dimensional incompressible point vortex at the origin $\boldsymbol{V} = \frac{\Gamma}{2\pi r}\boldsymbol{e_\theta}$ (when written in cylindrical coordinates). (a) Determine the vorticity field of this flow, and thus the average angular velocity of the fluid particles that comprise it. (b) The diagram shows the origin of this flow and coordinate system (O) and an initially rectangular fluid particle located at a radius r and an angle θ =0. Determine the shape of the fluid particle after it has convected one quarter of the way around the vortex (i.e. to θ =90o). Since the particle is infinitesimally small, you may assume that its sides remain straight. (c) Determine by how much the fluid particle has rotated (you will have to choose some rational way of separating the rotation from the distortion). Attempt to explain how your answer is consistent with your answer to part (a).



2. An incompressible flow through a spinning pipe of radius *a* has the velocity field

$$\boldsymbol{V} = \frac{r}{a}\boldsymbol{e_\theta} + \left(1 - \frac{r^2}{a^2}\right)\boldsymbol{e_z}$$

   (a) Derive and sketch an equation for the streamlines of this flow
   (b) Determine two streamfunctions (explicitly, in algebraic form) that together describe this flow. Show that your answer here, is consistent with your answer in part (a) by using the fact that streamlines can be defined by the intersection of streamsurfaces.
   (c) Derive an equation for the vortex lines (using $\boldsymbol{\Omega} \mathrm{x} d\boldsymbol{s} = 0$). Sketch these on the same picture as used in (a).

3. the Cauchy-Stokes decomposition of the change in the velocity vector $d\boldsymbol{V}$ that we performed in class that gave us, in Cartesian coordinates, expressions for the divergence, curl and rate of shear deformation. Repeat this derivation using spherical coordinates and thus find an expression for the rate of shear deformation in sphericals. *Hint: the first step is to determine the components of* $d\boldsymbol{V} = \frac{\partial \boldsymbol{V}}{\partial r}\partial r + \frac{\partial \boldsymbol{V}}{\partial \theta}\partial\theta + \frac{\partial \boldsymbol{V}}{\partial \phi}\partial\phi$*, not forgetting the derivatives of the unit vectors*

4. The sketch shows a supersonic flow past a concave 10° corner. The flow before the turn is uniform and parallel and the velocity and density are 694m/s and 1kg/m^3. The turn is accomplished through an oblique shock wave that makes an angle of 39.3° with the upstream flow direction. The shock is very thin. Flow after the shock is also uniform and parallel with velocity and density of 616m/s and 1.46kg/m^3. (a) Define a looped path that crosses the shock. Use it to demonstrate that, in general, a straight shock in uniform flow does not induce a pressure torque (even though the pressure density relation across a shock is not unique). (b) Turbulent eddies are introduced into the flow upstream of the corner. Estimate the change in magnitude and direction of the vorticity of an eddy as it passes through the shock, if the eddy is initially (i) horizontal, and (ii) vertical. For (ii) compare your answer for the vorticity direction after the shock with that which would be approximately predicted by the solution for small angles described in the class example "Flow around a corner in a channel". (c) Explain your assumptions and how good you think they will be in this case. *Hints: Tracking the actual path of elements of the fluid material (and remembering that the density changes in the shock) will make part (b) easier. Also checkout slides of the kinematics of vorticity class slide set.*