Intermediate Fluid Mechanics

Lecture 21: Dimensional Analysis V

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ME 5700/6700

October 22, 2025

Chapter Overview

Lecture Objectives

In this lecture we will:

- Focus on the use of the Buckingham-Pi Theorem
- We will consider a couple of practice examples,...

Chapter Overview

What happens if one neglects relevant variables?

In the previous lecture we found that,

$$\frac{\Delta p}{\rho U^2} = \phi\left(\frac{l}{d},\, \frac{e}{d},\, \frac{\mu}{\rho U d}\right) \tag{1}$$

$\Longrightarrow$ where the function ϕ is determined empirically through laboratory experiments and/or numerical simulations.

What happens if one neglects relevant variables? (continued ...)

To determine ϕ, we could:

- Design an experiment where we could easily vary the Reynolds number (by adjusting U, for example) while keeping $\frac{e}{d}$ and $\frac{l}{d}$ constant and measuring the pressure coefficient, $\frac{\Delta p}{\rho U^2}$.
- Then, we would plot the data in non-dimensional variables and perform a curve fit to estimate ϕ.
- We could also run other experiments to determine the dependence of $\frac{\Delta p}{\rho U^2}$ on $\frac{l}{d}$ (while keeping Re and $\frac{e}{d}$ constant)
- As well as changing $\frac{e}{d}$ (while keeping Re and $\frac{l}{d}$ constant).

What happens if one neglects relevant variables? (continued ...)

Suppose that after plotting the results in non-dimensional form, one finds the results illustrated below,

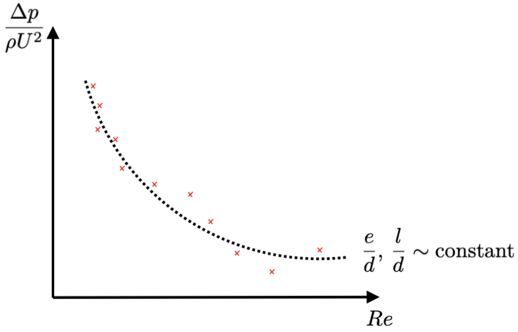

Figure: Pressure coefficient versus Re number keeping $\frac{e}{d}$ and $\frac{l}{d}$ constant. Symbols denote the experimental data.

What happens if one neglects relevant variables? (continued ...)

Later, if the experiment is repeated with a different pipe, having a different non-dimensional roughness, one might see the results illustrated below,

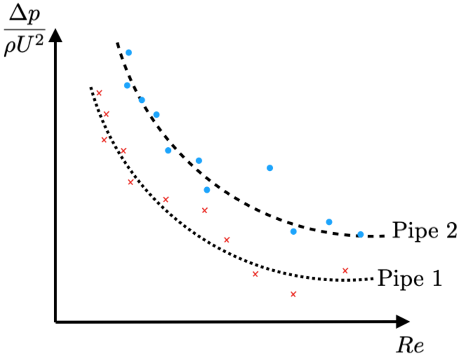

Figure: Pressure coefficient versus Re number keeping $\frac{e}{d}$ and $\frac{l}{d}$ constant. Symbols denote the experimental data.

What happens if one neglects relevant variables? (continued ...)

Answer: The ramifications depend on which variable is neglected.

For example, if we neglected to include the roughness e, then we simply loose a π-group. In this case we would predict that,

$$\frac{\Delta p}{\rho U^2} = \phi\left(\frac{l}{d},\ Re\right) \tag{2}$$

regardless of the roughness.

Question: How would we know that this is incorrect?

What happens if one neglects relevant variables? (continued ...)

Answer: we would observe disparity between the experimental data and the above relation!

This might happen as follows:

- One selects a section of pipe and run experiments to determine $\frac{\Delta p}{\rho U^2}$ vs Re.
- One obtains the data labeled as 'Pipe 1'.
- Based on your (erroneous) dimensional analysis, you would predict that all pipes would behave similarly to 'Pipe 1'.
- When you try to test a different pipe and observe that the data from the new pipe does not follow 'Pipe 1', but rather follow 'Pipe 2' results!

$\Longrightarrow$ Then you would have to go back and question your dimensional analysis and ask whether you forgot an important variable.

What happens if one neglects relevant variables? (continued ...)

If on the other hand, we inadvertently omitted the density from the dimensional analysis, the ramifications would be more dramatic because both, the pressure coefficient, as well as the Reynolds number depends on ρ.

In this scenario, we would have

$$f(\Delta p, d, l, e, U, \mu), \tag{3}$$

where $n = 6$. The dimensional matrix is,

	Δp	d	l	e	U	μ
M	1	0	0	0	0	1
L	-1	1	1	1	1	-1
T	-2	0	0	0	-1	-1

and the rank is still $r = 3$. The repeating variables are now U, d, μ and we only expect 3 Pi groups.

What happens if one neglects relevant variables? (continued ...)

$$\pi_1 = U^a \, d^b \, \mu^c \, \Delta p \tag{4}$$

which in terms of fundamental units,

$$M^0 \, L^0 \, T^0 = (M^1 \, L^{-1} \, T^{-2})(L^1 \, T^{-1})^a (L^1)^b (M^1 \, L^{-1} \, T^{-1})^c \tag{5}$$

In order for this equality to hold, the powers on M, L, and T must be identical,

$$M: \qquad 0 = 1 + c \qquad \Rightarrow c = -1 \tag{6}$$

$$T: \qquad 0 = -1 + a + b - c \qquad \Rightarrow b = 1 \tag{7}$$

$$L: \qquad 0 = -2 - a - c \qquad \Rightarrow a = -1 \tag{8}$$

What happens if one neglects relevant variables? (continued ...)

Therefore,

$$\pi_1 = \frac{\Delta p}{Ud/\mu}. \tag{9}$$

Similarly,

$$\pi_2 = U^a\, d^b\, \mu^c\, l = \frac{l}{d} \tag{10}$$

$$\pi_3 = U^a\, d^b\, \rho^c\, e = \frac{e}{d} \tag{11}$$

What happens if one neglects relevant variables? (continued ...)

These results lead to,

$$\frac{\Delta p}{Ud/\mu} = \phi(\frac{l}{d}, \frac{e}{d}) \tag{12}$$

This says that given a fixed section of pipe, so that $l/d =$ constant and $e/d =$ constant, the above relation predicts that,

$$\frac{\Delta p}{Ud/\mu} = \text{constant.} \tag{13}$$

$\Longrightarrow$ One would quickly find that the experimental data does NOT support this prediction, alerting us of the fact that we must have forgotten something in the dimensional analysis.

Chapter Overview

Example 1: Drag Coefficient ($C_D = \frac{F_D}{1/2\,\rho U^2\,A}$)

An airship operates at 20 m/s in air at standard conditions. A model is constructed to 1/20 scale and tested in a wind tunnel at the same air temperature to determine the drag.

Question 1: If the model is tested in at 75 m/s, what pressure should be used in the wind tunnel?

Question 2: If the model drag force is 250 N, what is the expected drag of the prototype?

Example 1 (continued ...)

Answer: We use Buckingham's Pi theorem.

We assume that the drag force (F_D) depends on density (ρ), viscosity (μ), airspeed (U) (or equivalently the speed of the vehicle), and some characteristic length scale (L) of the vehicle,

$$f(F_D,\, \rho,\, \mu,\, U,\, L) = 0 \tag{14}$$

Example 1 (continued ...)

Let's consider first the dimensions of all the variables/parameters represented in the form of a Dimensional Matrix:

	F_D	ρ	μ	U	L
M	1	1	1	0	0
L	1	-3	-1	1	1
T	-2	0	-1	-1	0

In this example case, $n = 5$, and $r = 3$.

$\Longrightarrow n - r = 2$ and we expect two non-dimensional π-groups.

Example 1 (continued ...)

To form the π groups, one first needs to select $r = 3$ '*repeating variables*', which will be represented in each of the π-groups.

$\longrightarrow$ In this case, based on experience, and ensuring that at least each of the dimensions $(M,\ L,\ T)$ are represented by at least one of the repeating variables, we select U, ρ, and L as repeating variables.

Therefore,

$$\pi_1 = F_D\, U^a\, L^b\, \rho^c \tag{15}$$

which in terms of fundamental units,

$$M^0\, L^0\, T^0 = (M^1\, L^1\, T^{-2})(L^1\, T^{-1})^a (L^1)^b (M^1\, L^{-3})^c \tag{16}$$

Example 1 (continued ...)

For this equality to hold, the powers on M, L, and T must be identical,

$$M: \qquad 0 = 1 + c \qquad \Rightarrow c = -1 \tag{17}$$
$$T: \qquad 0 = 1 + a + b - 3c \qquad \Rightarrow b = -2 \tag{18}$$
$$L: \qquad 0 = -2 - a \qquad \Rightarrow a = -2 \tag{19}$$

Therefore,

$$\pi_1 = \frac{F_D}{\rho U^2 L^2}. \tag{20}$$

Note that in the denominator, ρU^2 represents a dynamic pressure, and L^2 represents an area.

Example 1 (continued ...)

To find the second non-dimensional π-group we proceed similarly,

$$\pi_2 = \mu\, U^a\, L^b\, \rho^c \tag{21}$$

which in terms of fundamental units,

$$M^0\, L^0\, T^0 = (M^1\, L^{-1}\, T^{-1})(L^1\, T^{-1})^a (L^1)^b (M^1\, L^{-3})^c \tag{22}$$

In order for this equality to hold, the powers on M, L, and T must be identical,

$$M: \qquad 0 = 1 + c \qquad \Rightarrow c = -1 \tag{23}$$

$$T: \qquad 0 = -1 + a + b - 3c \qquad \Rightarrow b = -1 \tag{24}$$

$$L: \qquad 0 = -1 - a \qquad \Rightarrow a = -1 \tag{25}$$

Example 1 (continued ...)

Which results in,

$$\pi_2 = \frac{\mu}{\rho U L} = \frac{1}{Re}. \tag{26}$$

Hence, Buckingham's Pi theorem tells us that

$$C_D = f(Re), \tag{27}$$

where $C_D = \frac{F_D}{\rho U^2 L^2}$ and $Re = \frac{\rho U L}{\mu}$.

Example 1 (continued ...)

The above equation tells us that if we match the Reynolds numbers between the model and the prototype, then the corresponding drag coefficients will also be identical.

Therefore, matching Re,

$$Re_m = Re_p \tag{28}$$

$$\frac{\rho_m U_m L_m}{\mu_m} = \frac{\rho_p U_p L_p}{\mu_p} \tag{29}$$

But since the temperature is the same, $\mu_m = \mu_p$ because viscosity is only a very weak function of pressure.

Example 1 (continued ...)

Upon substitution,

$$\rho_m = \rho_p \left(\frac{U_p}{U_m}\right)\left(\frac{L_p}{L_m}\right) \tag{30}$$

$$\rho_m = \rho_p \left(\frac{20}{75}\right)\left(\frac{20}{1}\right) = 5.33\rho_p. \tag{31}$$

From the ideal gas law, $p = \rho R T$. The ratio between p_m/p_p is determined from the ideal gas law:

$$\frac{p_m}{p_p} = \frac{\rho_m\, R_m\, T_m}{\rho_p\, R_p\, T_p} \tag{32}$$

$$\frac{p_m}{p_p} = \frac{\rho_m}{\rho_p} \tag{33}$$

$$p_m = 5.33\, p_p \tag{34}$$

$$\tag{35}$$

Example 1 (continued ...)

Since $p_p = 101.3kPa$ (given), then $p_m = 5.39 \times 10^5$ Pa.

In addition, using again Buckingham's Pi theorem one can determine $(F_D)_m$ by equating the model and prototype C_D values.

$$(C_D)_m = (C_D)_p \tag{36}$$

$$(F_D)_p = (F_D)_m \left(\frac{U_p}{U_m}\right)^2 \left(\frac{L_p}{L_m}\right)^2 \left(\frac{\rho_p}{\rho_m}\right) = 5.34 (F_D)_m = 1.34\,kN. \tag{37}$$

Chapter Overview

Example 2: Model Propeller

A model propeller 600mm in diameter is tested in a wind tunnel. Air that approaches the propeller at 45 m/s causes it to spin at 2000 rpm. The thrust and torque measured under these conditions are 110 N and 10 $N \cdot m$, respectively. A prototype 10 times as large as the model is to be built.

Question: Calculate the angular velocity of the prototype propeller and the thrust generated for an approach air speed of 120 m/s. Neglect the effect of viscosity.

Example 2: (continued ...)

Let's keep viscosity in the problem initially. The functional relationship between the dimensional variables is

$$f(F_t,\, d,\, \omega,\, U,\, \mu,\, \rho) = 0, \tag{38}$$

where ω is the angular velocity of the propeller, and F_t is the thrust force. In this case, the dimensional matrix is,

	F_t	d	ω	U	μ	ρ
M	1	0	0	0	1	1
L	1	1	0	1	-1	-3
T	-2	0	-1	-1	-1	0

Example 2: (continued ...)

Given that the rank of the matrix is $r = 3$ and the number of variables is $n = 6$, one should expect three non-dimensional groups.

Select d, ω, and ρ as repeating variables, then

$$\pi_1 = F_t \, d^a \, \omega^b \, \rho^c \tag{39}$$

which in terms of fundamental units,

$$M^0 \, L^0 \, T^0 = (M^1 \, L^1 \, T^{-2})(L^1)^a(T^{-1})^b(M^1 \, L^{-3})^c \tag{40}$$

Example 2: (continued ...)

In order for this equality to hold, the powers on M, L, and T must be identical,

$$M: \quad 0 = 1 + c \quad \Rightarrow c = -1 \tag{41}$$
$$L: \quad 0 = 1 + a - 3c \quad \Rightarrow a = -4 \tag{42}$$
$$T: \quad 0 = -2 - b \quad \Rightarrow b = -2 \tag{43}$$

Therefore,

$$\pi_1 = \frac{F_t}{\rho \omega^2 d^4}. \tag{44}$$

Example 2: (continued ...)

For the second Pi group,

$$\pi_2 = U\, d^a\, \omega^b\, \rho^c \tag{45}$$

which in terms of fundamental units,

$$M^0\, L^0\, T^0 = (L^1\, T^{-1})(L^1)^a(T^{-1})^b(M^1\, L^{-3})^c \tag{46}$$

In order for this equality to hold, the powers on M, L, and T must be identical,

$$M: \qquad 0 = c \qquad \Rightarrow c = 0 \tag{47}$$
$$L: \qquad 0 = 1 + a - 3c \qquad \Rightarrow a = -1 \tag{48}$$
$$T: \qquad 0 = -1 - b \qquad \Rightarrow b = -1 \tag{49}$$

Therefore,

$$\pi_2 = \frac{U}{d\omega}. \tag{50}$$

Example 2: (continued ...)

Finally, for the third Pi group,

$$\pi_3 = \mu\, d^a\, \omega^b\, \rho^c \tag{51}$$

which in terms of fundamental units,

$$M^0\, L^0\, T^0 = (M^1 L^{-1}\, T^{-1})(L^1)^a (T^{-1})^b (M^1\, L^{-3})^c \tag{52}$$

In order for this equality to hold, the powers on M, L, and T must be identical,

$$M: \qquad 0 = 1 + c \qquad \Rightarrow c = -1 \tag{53}$$

$$L: \qquad 0 = -1 + a - 3c \qquad \Rightarrow a = -2 \tag{54}$$

$$T: \qquad 0 = -1 - b \qquad \Rightarrow b = -1 \tag{55}$$

Therefore,

$$\pi_3 = \frac{\mu}{\rho d^2 \omega}. \tag{56}$$

As a result, Buckingham Pi theorem tells us that

$$\pi_1 = f(\pi_2, \pi_3) \tag{57}$$

or otherwise said,

$$\frac{F_t}{\rho \omega^2 d^4} = f\Big(\frac{U}{d\omega}, \frac{\mu}{\rho d^2 \omega}\Big). \tag{58}$$

$\Longrightarrow$ If the viscous effects are not important, then we can write $\pi_1 = g(\pi_2)$, or

$$\frac{F_t}{\rho \omega^2 d^4} = g\Big(\frac{U}{d\omega}\Big). \tag{59}$$

Example 2: (continued ...)

To achieve dynamic similarity, we must match π_2,

$$(\pi_2)_m = (\pi_2)_p \tag{60}$$

$$\frac{U_m}{d_m \omega_m} = \frac{U_p}{d_p \omega_p}, \tag{61}$$

and hence the angular velocity of the prototype is,

$$\omega_p = \omega_m \left(\frac{U_p}{U_m}\right)\left(\frac{d_m}{d_p}\right) = 2000\left(\frac{120}{45}\right)\left(\frac{1}{10}\right) = 533rpm. \tag{62}$$

Example 2: (continued ...)

Once dynamic similarity is established by matching π_2, we know that

$$(\pi_1)_m = (\pi_1)_p \tag{63}$$

$$\frac{(F_t)_m}{\rho_m \omega_m^2 d_m^4} = \frac{(F_t)_p}{\rho_p \omega_p^2 d_p^4} \tag{64}$$

$$(F_t)_p = (F_t)_m \left(\frac{\omega_p}{\omega_m}\right)^2 \left(\frac{d_p}{d_m}\right)^4 = 78.1\ kN \tag{65}$$

(In the above result we have assumed that $\rho_m = \rho_p$.)