Intermediate Fluid Mechanics

Lecture 14: Canonical Type Flows

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Chapter Overview

1. Chapter Objectives
2. Poiseuille Flow
3. Non-dimensional for of the Poiseuille Flow
4. Practical Applications
5. Shear Stress and Vorticity
6. Flow Development

Lecture Objectives

- In this lecture, we will examine flow through a channel with infinitely long parallel plates. This is often called a **Poiseuille flow** after the person who first solved this problem.
- This is another type of canonical-type flow, that is helpful to understand the contributions from the different terms in the NS-equations.

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Poiseuille Flow (flow through a channel)

This flow geometry has practical relevance, such that it can be used as an analogy to different types of flows:

- Flow through a large aspect ratio duct,
- Flow induced in a journal bearing where the radius of curvature of the bearing is large compared to the gap height h
- Flow through a circular pipe, which admits a solution analogous to the channel flow.
- ...

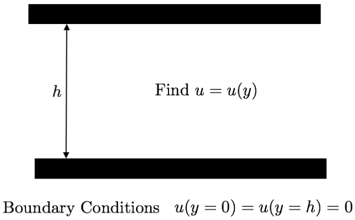

Figure: *Illustration of a Poiseuille Flow.*

Poiseuille Flow - Conservation of Mass

To solve this problem, let's consider the following assumptions:

- Steady-state flow solution,
- Far from the inlet or outlet of the channel so that the flow does not appear to vary in the x-direction (i.e. the flow is fully developed in the x-direction).
- Assume an incompressible flow.

Under these considerations, the continuity equation can be written as,

$$\frac{\partial u_i}{\partial x_i} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{1}$$

Poiseuille Flow - Conservation of Mass (continued ...)

- Since the fluid is *planar*, the third velocity component is taken to be zero.
- From the fully developed flow assumption, $\partial/\partial x = 0$,

$\Longrightarrow$ The continuity equation reduces to,

$$\frac{\partial v}{\partial y} = 0. \tag{2}$$

Upon integration, one obtains again that $v = C$.

$\Longrightarrow$ Using the non-slip conditions at the top and bottom walls of the channel for all velocity components $v(y = h) = v(y = 0) = 0$, leads to the fact that $C = 0$.

$\Longrightarrow$ The vertical velocity $v = 0$ throughout the channel flow.

Poiseuille Flow - Momentum Eq.

Next we consider the x-momentum equation,

$$\rho \underbrace{\cancel{\frac{\partial u}{\partial t}}}_{\text{steady}} + \rho\, u \underbrace{\cancel{\frac{\partial u}{\partial x}}}_{\text{fully developed}} + \rho \underbrace{\cancel{v \frac{\partial u}{\partial y}}}_{v=0} = -\frac{\partial p}{\partial x} + \underbrace{\cancel{\mu \frac{\partial^2 u}{\partial x^2}}}_{\text{fully developed}} + \mu \frac{\partial^2 u}{\partial y^2}, \qquad (3)$$

which upon simplification results in

$$\mu \frac{\partial^2 u}{\partial y^2} = \frac{\partial p}{\partial x}. \qquad (4)$$

Poiseuille Flow - Momentum Eq. (continued)

What happens if one gets rid of the pressure gradient?

$$\frac{\partial p}{\partial x} = 0 \tag{5}$$

Poiseuille Flow - Momentum Eq. (continued)

What happens if one gets rid of the pressure gradient?

If one assumes $\frac{\partial p}{\partial x} = 0$ then,

$$\mu \frac{\partial^2 u}{\partial y^2} = 0, \tag{6}$$

which upon integration leads to $u = A\,y + B$. Upon use of the boundary conditions,

$$u(y=0) = 0 = B \qquad \text{and} \qquad u(y=h) = 0 = Ah \Rightarrow A = 0. \tag{7}$$

$\Longrightarrow$ Upon use of the boundary equations one obtains that the solution is

$$u = 0, \longrightarrow \textbf{ the trivial solution}.$$

Poiseuille Flow - Momentum Eq. (continued)

Note:

- Physically, it is important to realize that 'something' needs to drive the flow.
- In the Couette flow case, the motion of the upper wall drives the flow.
- In this case, the flow is driven by a pressure gradient, which could be established by having a fan or pump at one of the ends of the channel.
- The pressure gradient must be prescribed, *i.e.* it is considered to be a known parameter.

Poiseuille Flow - Momentum Eq. (continued)

Before we try to solve the x-momentum equation for the channel flow,

$$\mu \frac{\partial^2 u}{\partial y^2} = \frac{\partial p}{\partial x}. \tag{8}$$

let's consider for an instant the y-momentum equation too,

$$\rho \underbrace{\cancel{\frac{\partial v}{\partial t}}}_{\text{steady}} + \rho\, u \underbrace{\cancel{\frac{\partial v}{\partial x}}}_{\text{fully developed}} + \rho \underbrace{\cancel{v \frac{\partial v}{\partial y}}}_{v=0} = -\frac{\partial p}{\partial y} + \underbrace{\cancel{\mu \frac{\partial^2 v}{\partial x^2}}}_{\text{fully developed}} + \underbrace{\cancel{\mu \frac{\partial^2 v}{\partial y^2}}}_{v=0}. \tag{9}$$

Therefore, one obtains that the pressure is only a function of x.

$$-\frac{\partial p}{\partial y} = 0 \qquad \Longrightarrow \qquad p = p(x). \tag{10}$$

Poiseuille Flow - Momentum Eq. (continued)

If we now solve

$$\mu \frac{\partial^2 u}{\partial y^2} = \frac{\partial p}{\partial x}, \tag{11}$$

knowing that $p = p(x)$ only, the equations is easier to integrate,

$$\int \frac{d^2 u}{dy^2} dy = \int \frac{1}{\mu} \frac{dp}{dx}\, dy \tag{12}$$

$$\int \frac{du}{dy} dy = \int \left[\frac{1}{\mu} \frac{dp}{dx}\, y + A\right] dy \tag{13}$$

$$u(y) = \frac{1}{\mu} \frac{dp}{dx} \frac{y^2}{2} + A\, y + B, \tag{14}$$

where A and B are integration constants.

Poiseuille Flow - Momentum Eq. (continued)

To determine the integration constants one needs to apply the boundary conditions. At the lower wall,

$$u(y=0)=0 \qquad \Rightarrow \qquad 0=\frac{1}{\mu}\frac{dp}{dx}\frac{(0)^2}{2}+A\,(0)+B \qquad \Rightarrow \qquad B=0, \tag{15}$$

$$u(y=h)=0 \qquad \Rightarrow \qquad 0=\frac{1}{\mu}\frac{dp}{dx}\frac{h^2}{2}+A\,(0) \qquad \Rightarrow \qquad A=\frac{-h}{2\mu}\frac{dp}{dx}, \tag{16}$$

hence the final solution is,

$$\boxed{u=\frac{1}{2\mu}\frac{dp}{dx}\left(y^2-h\,y\right)}. \tag{17}$$

This solution is simple enough that it can be seen that the velocity profile is parabolic along the y-coordinate.

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Poiseuille Flow - Non-dimensionalizing the solution

It is always better if one can plot the solution without having to specify numeric values for the parameters. $\Longrightarrow$ For this reason, one needs to non-dimensionalize the velocity and distance,

To non-dimensionalize the solution one needs to pick characteristic length scales. It makes sense to pick the gap size h as the characteristic length scale.

Then one can define a new variable $\tilde{y}$ (that has no units) as

$$\tilde{y} = \frac{y}{h} \tag{18}$$

As for a characteristic velocity scale, one should pick the maximum velocity, the average velocity or some other quantity?

Poiseuille Flow - Non-dimensionalizing (continued ...)

To answer this question, let's first compute both:

- (a) The maximum velocity occurs when $du/dy = 0$.

Taking the derivative of equation 17 and setting equal to zero, one finds

$$\frac{du}{dy} = \frac{1}{2\mu}\frac{dp}{dx}\Big[2y - h\Big] = 0 \Rightarrow y = \frac{h}{2}, \tag{19}$$

which means that the maximum velocity occurs along the channel centerline, hence

$$u(y_{max}) = \frac{1}{2\mu}\frac{dp}{dx}\left[\left(\frac{h}{2}\right)^2 - h\left(\frac{h}{2}\right)\right] = \frac{1}{2\mu}\frac{dp}{dx}\left[\frac{h^2}{4} - \frac{h^2}{2}\right] = \frac{-h^2}{8\mu}\frac{dp}{dx}, \tag{20}$$

Note: The negative sign makes sense when one realizes that the pressure gradient must be negative for the flow to move towards the right (positive x-direction).

Poiseuille Flow - Non-dimensionalizing (continued ...)

- (b) The average velocity $\overline{u}$ is defined as: $Q = \frac{\overline{u}}{A}$,

where Q is the volume flow rate, and A is the cross-sectional area.

$\Longrightarrow$ Since the geometry is infinite in the z-direction (into the page), one can calculate the volume flow rate per unit depth, which means one only has to integrate over y,

$$Q = \int_0^h u\,dy = \int_0^h \frac{1}{2\mu}\frac{dp}{dx}\left(y^2 - h\,y\right) = \frac{1}{2\mu}\frac{dp}{dx}\int_0^h \left(y^2 - h\,y\right) dy \tag{21}$$

$$Q = \frac{1}{2\mu}\frac{dp}{dx}\left[\frac{y^3}{3} - h\frac{y^2}{2}\right]_0^h = \frac{1}{2\mu}\frac{dp}{dx}\left[\frac{h^3}{3} - \frac{h^3}{2} - 0 + 0\right] = \frac{-h^3}{12\mu}\frac{dp}{dx}. \tag{22}$$

Since in this case the cross sectional area per unit depth is just h,

$$\overline{u} = \frac{Q}{A} = \frac{\frac{-h^3}{12\mu}\frac{dp}{dx}}{h} = \frac{-h^2}{12\mu}\frac{dp}{dx} \tag{23}$$

Poiseuille Flow - Characteristic velocity scale

Now let's try to non-dimensionalize the solution and see what happens.

- (1) Introduce the non-dimensional length by multiplying both sides by h^2,

$$u = \frac{1}{2\mu}\frac{dp}{dx}\left[y^2\left(\frac{h^2}{h^2}\right) - h\,y\left(\frac{h^2}{h^2}\right)\right] = \frac{h^2}{2\mu}\frac{dp}{dx}\left[\left(\frac{y}{h}\right)^2 - \left(\frac{y}{h}\right)\right] = \frac{h^2}{2\mu}\frac{dp}{dx}\left[\tilde{y}^2 - \tilde{y}\right]. \tag{24}$$

- (2) Use u_{max} to non-dimensionalize u, (divide both sides of the previous equation by $\frac{-h^2}{8\mu}\frac{dp}{dx}$),

$$\frac{u}{\frac{-h^2}{8\mu}\frac{dp}{dx}} = \left[\frac{1}{\frac{-h^2}{8\mu}\frac{dp}{dx}}\right]\frac{h^2}{2\mu}\frac{dp}{dx}\left[\tilde{y}^2 - \tilde{y}\right] \tag{25}$$

which can be simplified to,

$$\tilde{u} = -4(\tilde{y}^2 - \tilde{y}), \qquad \text{where} \qquad \tilde{u} = \frac{u}{u_{max}} \tag{26}$$

Poiseuille Flow - Characteristic velocity scale

In parallel, let's determine the average velocity in terms of a percentage of the max velocity,

$$\frac{\overline{u}}{u_{max}} = \frac{\frac{-h^2}{12\mu}\frac{dp}{dx}}{\frac{-h^2}{8\mu}\frac{dp}{dx}} = \frac{8}{12} = \frac{2}{3}$$

$$\implies \qquad \overline{u} = \frac{2}{3}u_{max}. \tag{27}$$

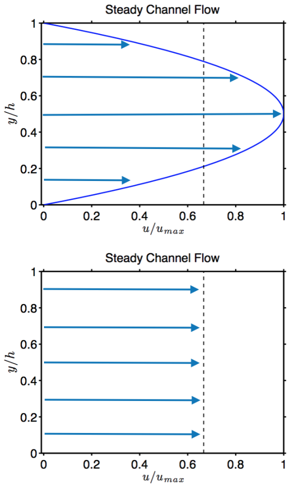

Figure: Non-dimenionalized Poiseuille Flow and the average.

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Poiseuille Flow - Practical Applications

There are a couple of practical engineering applications for our knowledge of the functional form of u_{max} and $\overline{u}$.

- **(i)** First, we want to model the flow as being uniform (*i.e.* independent of y), then the analysis tells us that a uniform flow, having an equivalent volume flow rate as the actual flow, must have a velocity $u = \frac{2}{3} u_{max}$.

- **(ii)** One can use the knowledge about the functional form of $\overline{u}$ for design purposes.

For example, if one wants to design a duct in an HVAC system, one might be limited in the choice of h (due to the fact that ducts maybe only manufactured in a set of fixed sizes).

Additionally, we might have requirements on the desired volume flow rate, Q as determined by the size of the room one needs to heat/cool.

Using the relation for $\overline{u}$, one can solve for the required pressure gradient,

$$\frac{dp}{dx} = \frac{-12\mu\overline{u}}{h^2} = \frac{-12\mu\, Q}{h^3} \tag{28}$$

Poiseuille Flow - Practical Applications (continued ...)

Knowing the length of the duct, L, and using the fact that one end of the duct exits to atmospheric pressure, one can estimate a pressure at the inlet of

$$\frac{p_{atm} - p_{inlet}}{L} = \frac{-12\mu Q}{h^3}, \tag{29}$$

hence,

$$p_{inlet} = p_{atm} + \frac{12\mu Q\, L}{h^3}. \tag{30}$$

$\Longrightarrow$ Using this knowledge, one can appropriately size the fan necessary for this simple HVAC system.

Poiseuille Flow - Practical Applications

Note:

Realize that although the logic used in deriving the above equation for the pressure at the inlet is correct, the equation itself cannot be used in practice.

$\Longrightarrow$ The reason is that in HVAC systems, the flow is likely to be turbulent and hence three-dimensional. Therefore, the original assumptions used to simplify the N-S equations are not quite right.

Now it is instructive to calculate two other quantities, namely the shear stress and the vorticity.

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Poiseuille Flow - Shear Stress

Since we are dealing with an incompressible, Newtonian fluid, the shear stress is given by,

$$\sigma_{ij} = 2\mu e_{ij} = \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right). \tag{31}$$

Since the flow is assumed to be planar, we only have one non-zero component of the shear stress, namely σ_{12} or σ_{xy},

$$\sigma_{12} = \mu\left(\frac{\partial u}{\partial y} + \cancel{\frac{\partial v}{\partial x}}\right). \tag{32}$$

Next, substituting in for $u(y)$ of the channel flow,

$$\sigma_{12} = \mu\frac{\partial}{\partial y}\left[\frac{1}{2\mu}\frac{dp}{dx}\left(y^2 - h\,y\right)\right] = \frac{1}{2}\frac{dp}{dx}\Big[2y - h\Big]. \tag{33}$$

Poiseuille Flow - Shear Stress (continued)

If we also non-dimensionalize this,

$$\sigma_{xy} = \frac{h}{2}\frac{dp}{dx}\left(2\frac{y}{h} - 1\right) = \frac{h}{2}\frac{dp}{dx}\left(2\tilde{y} - 1\right). \tag{34}$$

- The shear stress will be non-dimensionalized by $h\left(\frac{dp}{dx}\right)$ since the units will cancel.
- Also, since we know that $dp/dx < 0$, and in order to get the sign correct for the direction of the shear stress, we will normalize by the magnitude of dp/dx and explicitly include a negative sign in the expression for the non-dimensional shear stress,

$$\tilde{\sigma}_{xy} = -\frac{1}{2}\left(2\tilde{y} - 1\right) \tag{35}$$

where $\tilde{\sigma}_{xy} = \frac{\sigma_{xy}}{h|\frac{dp}{dx}|}$.

Poiseuille Flow - Shear Stress (continued)

This is represented in the following figure,

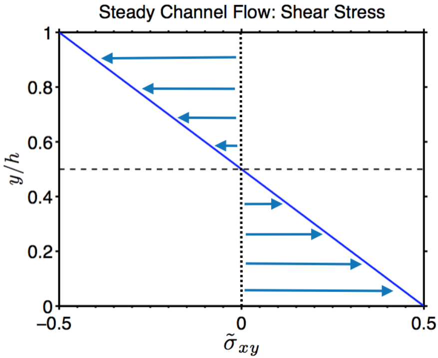

Figure: *Non-dimensionalized representation of the shear stress for a Poiseuille Flow.*

Poiseuille Flow - Shear Stress (continued)

There are a few observations that one can make about the shear stress:

- (i) the actual magnitude of the shear stress does not depend on the viscosity of the fluid, only on the imposed pressure gradient and gap.
- (ii) the magnitude of the shear stress is greatest at the walls. This is because the velocity gradient is the largest there.
- (iii) the shear stress acts in opposite directions on the top and bottom walls.
- (iv) the shear stress is zero at the center of the channel.

From an engineering perspective, the shear stress is important because it causes friction and hence drag. It costs money (in the way of pump/fan power) to overcome this friction.

Poiseuille Flow - Shear Stress (continued)

What is the traction force on the walls due to the shear stress?

Recall that $t_i = \sigma_{ij}\, n_j$, where n_j is the outward unit normal to the surface,

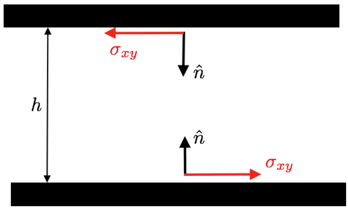

Figure: *Non-dimensionalized representation of the shear stress for a Poiseuille Flow.*

Poiseuille Flow - Shear Stress (continued)

- At the top surface:

$$t_x^{top} = \sigma_{xy}\, n_y = \Big(-\frac{1}{2}\Big)(-1) = \frac{1}{2} \tag{36}$$

- At the bottom surface:

$$t_x^{bot} = \sigma_{xy}\, n_y = \Big(\frac{1}{2}\Big)(1) = \frac{1}{2} \tag{37}$$

Therefore, the total force on the walls is in the t_x-direction. TO prevent a duct from moving, one needs a reaction force in the $-x$ direction.

Poiseuille Flow - Vorticity

If we now recall the definition of vorticity,

$$\omega_i = \varepsilon \frac{\partial u_k}{\partial x_j}, \tag{38}$$

and apply it to the specific case of the Poiseuille flow, since the flow is planar, we only have one component of vorticity in the z-direction (in/out of the page),

$$\omega_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}, \tag{39}$$

substituting in for the derivative,

$$\omega_z = \frac{-1}{2\mu}\frac{dp}{dx}(2y - h). \tag{40}$$

Poiseuille Flow - Vorticity (continued)

Again, non-dimensionalizing gives that,

$$\tilde{\omega}_z = \frac{-1}{2}(2\tilde{y} - 1) \tag{41}$$

where $\tilde{\omega}_z = \omega_z/((h/\mu)(dp/dx))$.

$\rightarrow$In this case the graphical representation looks identical to that for the shear stress.

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Poiseuille Flow - Development

Velocity profile in the entry region of a channel

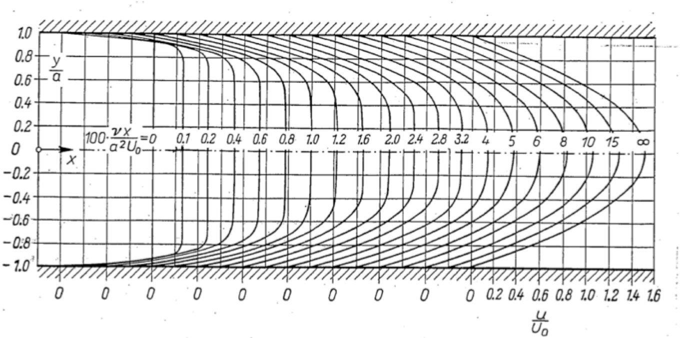

Figure: *Downstream evolution of the velocity profile in the entry region of a channel. a is the channel half height.*

Poiseuille Flow - Development

- The flow enter the channel with uniform momentum over the cross section.
- Once inside the channel, the momentum of the fluid in direct contact with the walls goes to zero due to the non-slip condition.
- In this way the wall act as a momentum sink. This loss of momentum diffuses in the y-direction toward the centerline from both the top and bottom walls.
- This process requires some time since diffusion is slow, or in the case of the entry length problem, this diffusion process takes place over some length L of the channel.
- Once the diffusion of momentum has reached the channel centerline there is no further diffusion of momentum and we say that the flow has become fully-developed.