Second-order Systems

$$T(s) = \frac{b_0}{a_2 s^2 + a_1 s + a_0} = \frac{b_0}{a_2\left(s^2 + \frac{a_1}{a_2}s + \frac{a_0}{a_2}\right)} \cdot \underbrace{\frac{a_0/a_2}{a_0/a_2}}_{=1}$$

$$= \frac{b_0}{a_0}\left(\frac{a_0/a_2 \,*}{s^2 + \frac{a_1}{a_2}s + \frac{a_0}{a_2}\,*}\right)$$

same

$$\boxed{T(s) = \bar{K}\left(\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}\right)}$$

} General form for 2^{nd}-order system

$\bar{K}$ = DC gain for the transfer function, when $s=0$ (frequency goes to zero)

ζ = damping coefficient $0 \le \zeta < \infty$; ω_n

Ex

$$T(s) = \bar{K} \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

General form of 2^{nd}-order system.

$$T(s) = \frac{68}{s^2 + 9s + 10}$$

Find ζ, $\bar{K}$, and ω_n need to put $T(s)$ into General form:

$$T(s) = \frac{68}{s^2+9s+10} \cdot \frac{10}{10} =$$

Time response for a second-order system

2nd-order system

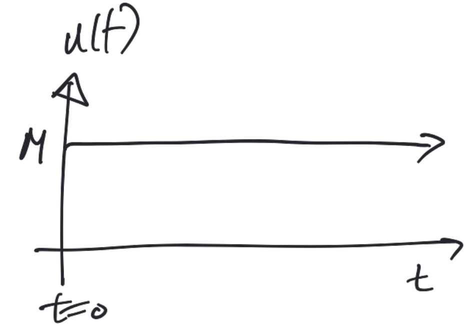
$$U(s) \longrightarrow \boxed{T(s)} \longrightarrow Y(s) \longrightarrow \underbrace{y(t)}_{\text{Time response.}}$$

step input

$$T(s) = \bar{K}\left(\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}\right)$$

$$T(s) = \frac{Y(s)}{U(s)} = \frac{\text{output}}{\text{input}}$$

what is $y(t)$ when input is a step



Input (step of magnitude M):

$$u(t) = \begin{cases} 0 & t \le 0 \\ M & t > 0 \end{cases}$$

$$\mathcal{L}\{u(t)\} = \frac{M}{s} \quad \text{(step of magnitude M)}$$

So, the output equation becomes:

$$T(s) = \frac{Y(s)}{U(s)} \Rightarrow Y(s) = \underline{\underline{T(s)}}\,\underline{\underline{U(s)}}$$

$$\Rightarrow Y(s) = \bar{K}\left(\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}\right)\left(\frac{M}{s}\right)$$

what is the output $y(t)$?

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\bar{K}\left(\frac{w_n^2}{s^2+2\zeta w_n s+w_n^2}\right)\left(\frac{M}{s}\right)\right\}$$

$$y(t) = \bar{K}M\,\mathcal{L}^{-1}\left\{\frac{K_1}{s} + \frac{K_2 s + K_3}{s^2+2\zeta w_n s+w_n^2}\right\}$$

$$y(t) = \bar{K}M\,\mathcal{L}^{-1}\left\{\frac{1}{s} + \frac{(s+2\zeta w_n) + \frac{\zeta}{\sqrt{1-\zeta^2}}\,w_n\sqrt{1-\zeta^2}}{(s+\zeta w_n)^2 + w_n^2(1-\zeta^2)}\right\}$$

$$y(t) = \bar{K}M\left[1 - e^{-\zeta w_n t}\left(\cos w_n\sqrt{1-\zeta^2}\,t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin w_n\sqrt{1-\zeta^2}\,t\right)\right]$$

$$y(t) = \bar{K}M\left(1 - \frac{1}{\sqrt{1-\xi^2}} e^{-\xi \omega_n t} \cos\left(\omega_n \sqrt{1-\xi^2}\, t - \phi\right)\right)$$

where $\phi = \tan^{-1}\left(\dfrac{\xi}{\sqrt{1-\xi^2}}\right)$

Note:

* $y(t)$ is a function of time
* Contains information about the system.

$$\xi = \text{damping} \begin{cases} \xi = 0 \Rightarrow \text{no damping} \\ 0 < \xi < 1 \Rightarrow \text{underdamped system} \\ \xi > 1 \Rightarrow \text{overdamped system} \end{cases}$$

Response for 2nd-order system for different ζ:

$$T(s) = \bar{K} \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{N(s)}{D(s)}$$

Poles: $D(s) = 0$

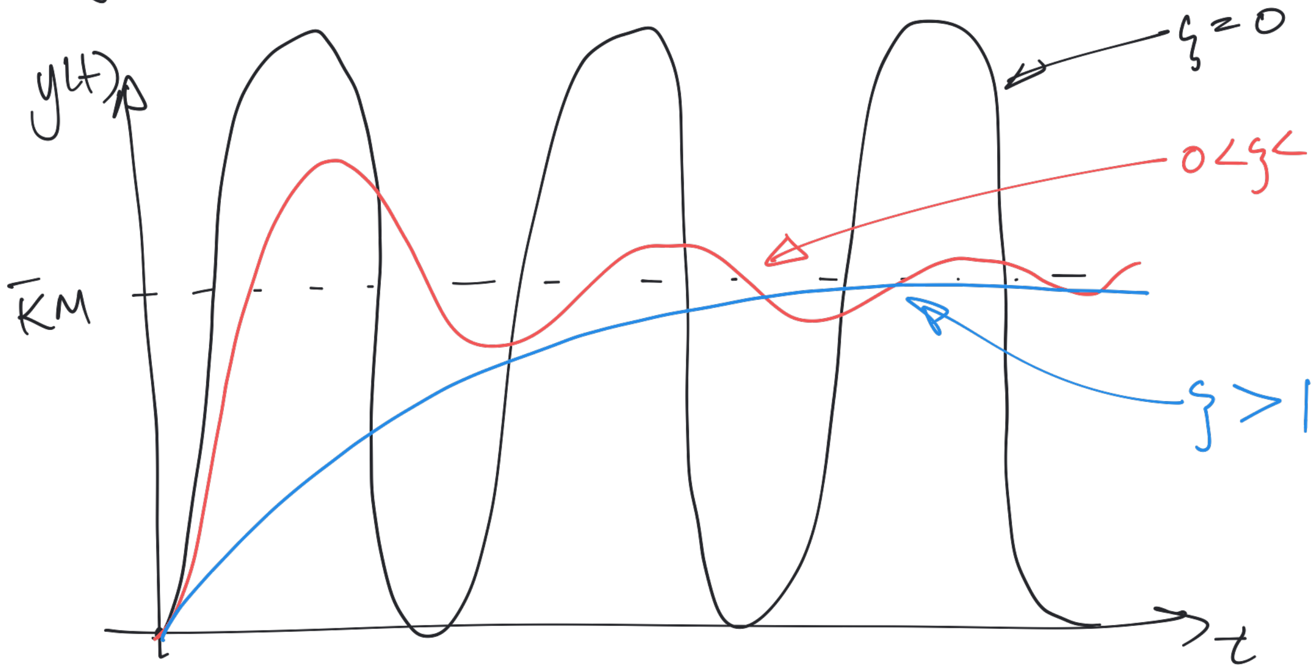
zeros: $N(s) = 0 \Rightarrow$ no zeros

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$$

$$\Rightarrow s_{1,2} = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$$

$$= \underbrace{-\sigma}_{\text{real part}} \pm j \underbrace{\omega_d}_{\text{imaginary part}}$$

$y(t)$ output $\rightarrow$ plot for different ζ



Time domain specifications for 2nd order systems

$$U(s) \longrightarrow \boxed{T(s)} \longrightarrow Y(s)$$

$$u(t) = \begin{cases} 0 & t \le 0 \\ 1 & t > 0 \end{cases}$$

Step input

$y(t) =$ function depends $\zeta, \omega_n, \bar{K}, M$

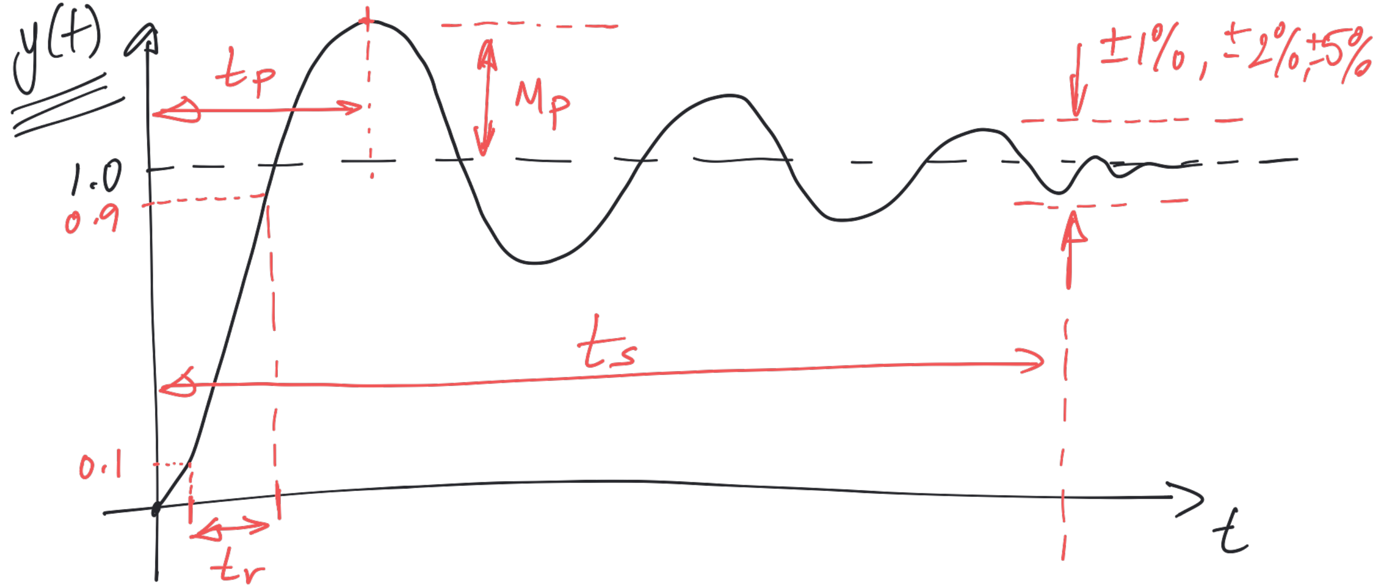
time domain specification.

Settling time

rise time

Time to peak

% over shoot, etc.



Assume : $K = 1$ and $M = 1$; $0 < \xi < 1$ (under damp syst)

$$\underline{\underline{y(t)}} = \underbrace{1 - e^{-\xi w_n t}\left(\cos w_n\sqrt{1-\xi^2}\,t + \frac{\xi w_n}{w_n\sqrt{1-\xi^2}}\sin w_n\sqrt{1-\xi^2}\,t\right)}$$

taking $\mathcal{L}^{-1}\{Y(s)\}$ when input is a step Input.

Rise time (t_r):

$$t_r \cong \frac{1.8}{\omega_n}$$
(2nd-order)

$$t_r \cong \frac{2.2}{a}$$
(first-order system)

Settling time:

(2nd order system) $$t_s \cong \frac{4}{\zeta \omega_n} = \frac{4}{\sigma}$$ (2% settling time)

(first-order system) $$t_s = \frac{4}{a}$$

overshoot (M_p):

$$M_p = e^{-\pi \xi / \sqrt{1-\xi^2}} \qquad 0 \leq \xi < 1$$

percent overshoot (%OS):

$$\underline{\underline{\%OS}} = M_p * 100 = 100 e^{\underline{\underline{-\pi \xi / \sqrt{1-\xi^2}}}}$$

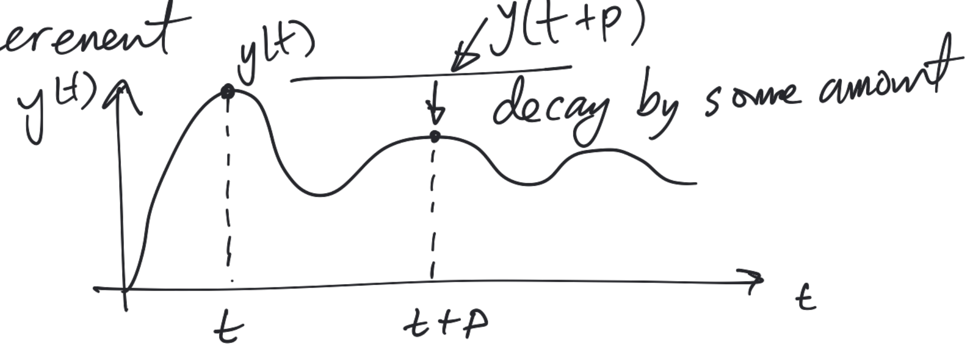
Damping ratio from the %OS:

$$\xi = \frac{-\ln(\%OS/100)}{\sqrt{\pi^2 + \ln^2(\%OS/100)}}$$

Time-to-peak (t_p):

$$t_p = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n\sqrt{1-\xi^2}}$$

Log decrement



$$\delta = \ln\left(\frac{y(t)}{y(t+p)}\right)$$ — log decrement

$$\Rightarrow \quad \xi = \frac{\delta}{\sqrt{4\pi^2 + \delta^2}}$$

Allows you to find ξ from δ.