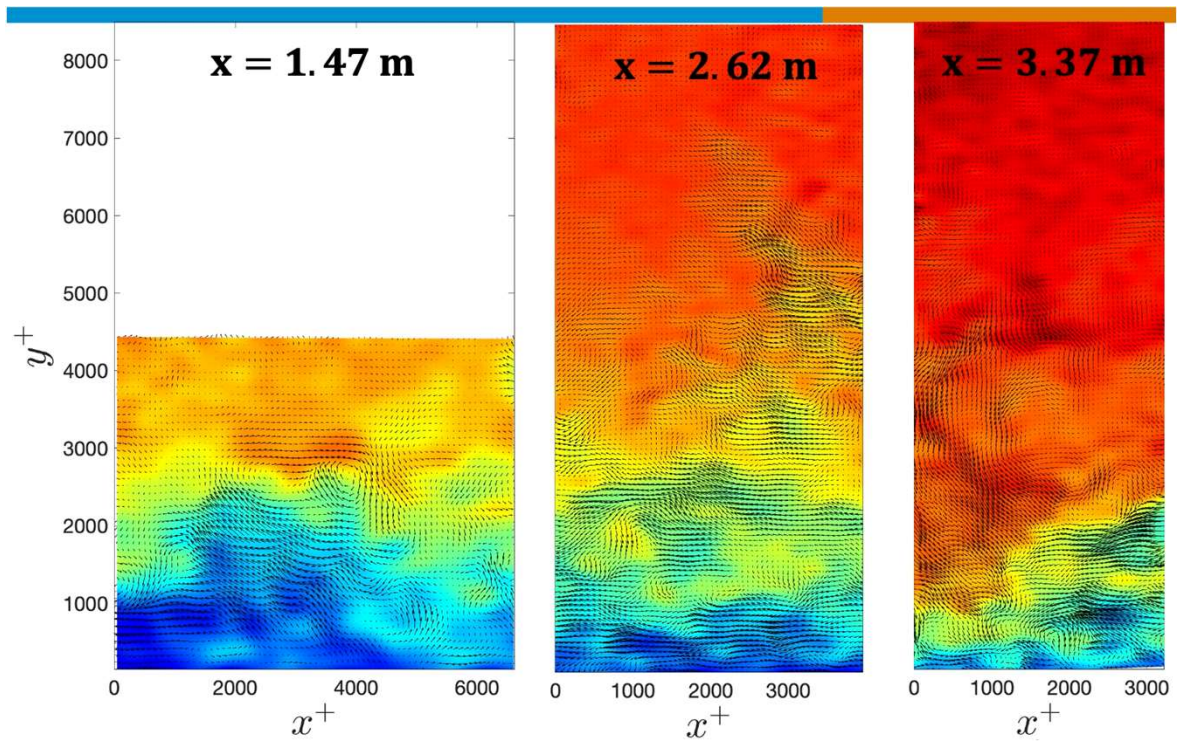
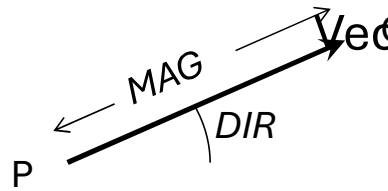
2. Vector Algebra

Boundary Layer Turbulence in Pressure Gradient.

V. Vishwanathan *et al*. 2021. Analysis of Coherent Structures over a Smooth Wall Turbulent Boundary Layer in Pressure Gradient Using Spectral Proper Orthogonal Decomposition. in *AIAA Aviation 2021 Forum*.



Vector basics



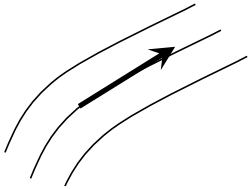
Vector: $\boldsymbol{A}$, A $\vec{A}$

Magnitude: $|\boldsymbol{A}|$, A

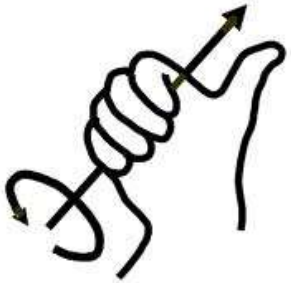
Scalar: p, ϕ

Types

- *Polar vector*
  - VELOCITY, $\vec{V}$, FORCE $\vec{F}$, PRESSURE GRADIENT $\vec{\nabla} P$



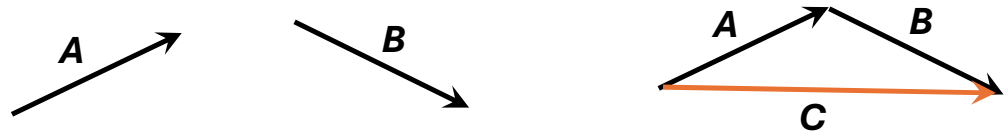
- *Axial vector*
  - ANGULAR VELOCITY, $\vec{\omega}$
    VORTICITY, $\vec{\Omega}$
    AREA $\vec{A}$



- *Unit vector*
  - $\hat{i}, \hat{j}, \hat{k}, \vec{e}_s, \vec{e}_\theta, \hat{n}$

Vector Algebra

- Addition



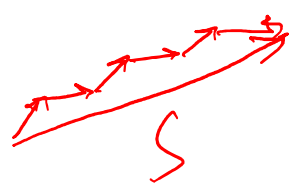
$$\boldsymbol{A} + \boldsymbol{B} = \boldsymbol{C}$$

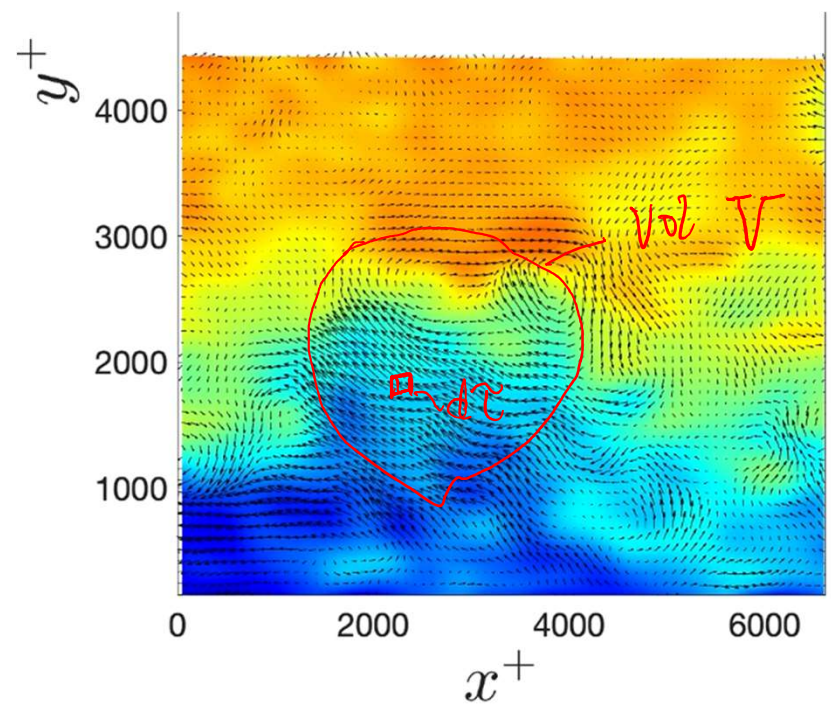
MOMENTUM IN VOLUME $d\tau$,

$dm\vec{V} = \rho\, d\tau\, \vec{V}$ (or) $\rho\, \vec{V}\, d\tau$

MOMENTUM ~~of~~ IN VOLUM V

$$\int_V \rho\, \vec{V}\, d\tau$$

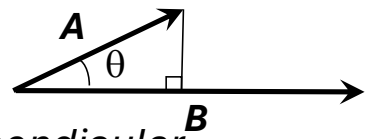




Vector Algebra

- Dot, or scalar, product

$\boldsymbol{A}.\boldsymbol{B} = AB\cos\theta$

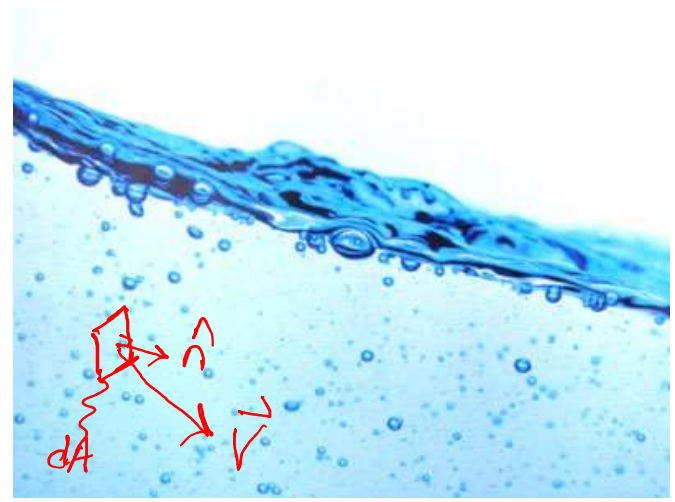


$\boldsymbol{A}.\boldsymbol{B}=\boldsymbol{B}.\boldsymbol{A}$ $\boldsymbol{A}.\boldsymbol{A}=A^2$ $\boldsymbol{A}.\boldsymbol{B}=0$ *if perpendicular*

WORK $= \vec{F}\cdot\vec{S}$

FLOW RATE THROUGH AN AREA

$= \vec{v}\cdot\hat{n}\,dA$

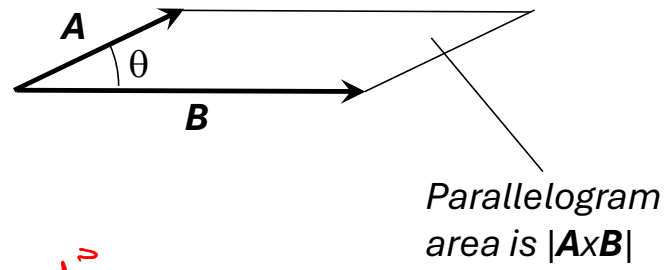


Vector Algebra

- Cross, or vector, product

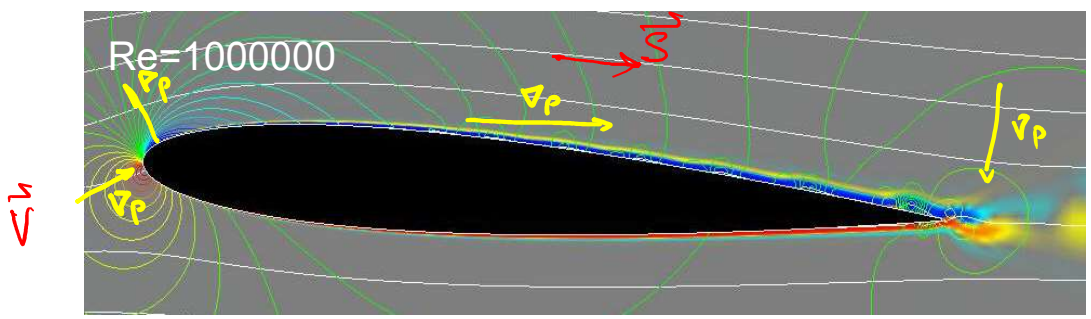
  Ax***B***=*AB*sinθ***e***

  ↳ RH RULE ROTATION FROM A to B



$\vec{V} \times d\vec{S} = 0$ IS THE EQN OF A STREAMLINE

- ***A***x***B***=-***B***x***A***
- ***A***x***A***=*0*
- ***A***x***B***=*0 if **A** and **B** parallel*



Vector Algebra – Triple Products

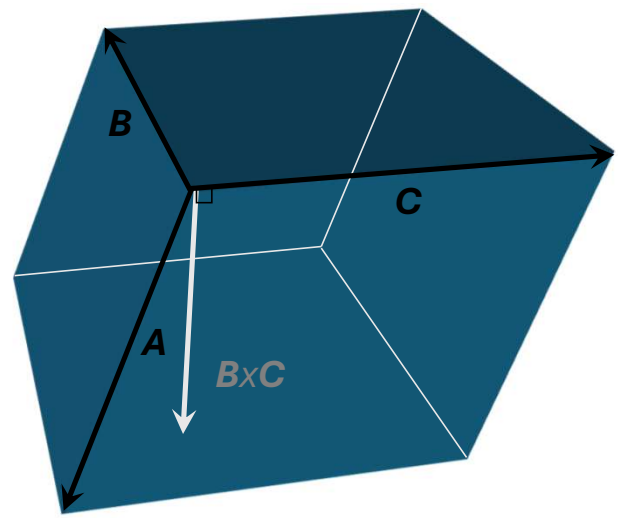
1. ***(A.B)C*** = ***(B.A)C***

2. Mixed product ***A.BxC***

VOLUME OF THE PARALLELOPIPED

$\vec{A}\cdot\vec{B}\times\vec{C} = \vec{C}\cdot\vec{A}\times\vec{B} = \vec{B}\cdot\vec{C}\times\vec{A}$

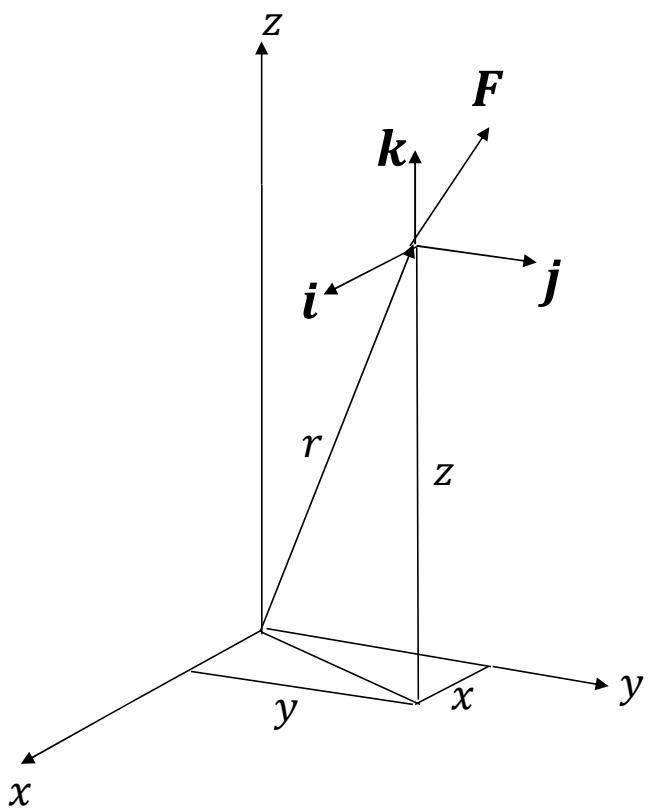
$\vec{A}\cdot\vec{B}\times\vec{C} = -\vec{A}\cdot\vec{C}\times\vec{B}$



3. Vector triple product ***Ax(BxC)***

$= (\vec{A}\cdot\vec{C})\vec{B} - (\vec{A}\cdot\vec{B})\vec{C}$

Cartesian Coordinates



- Coordinates x, y, z
- Unit vectors $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ (in directions of increasing coordinates) are constant
- Position vector

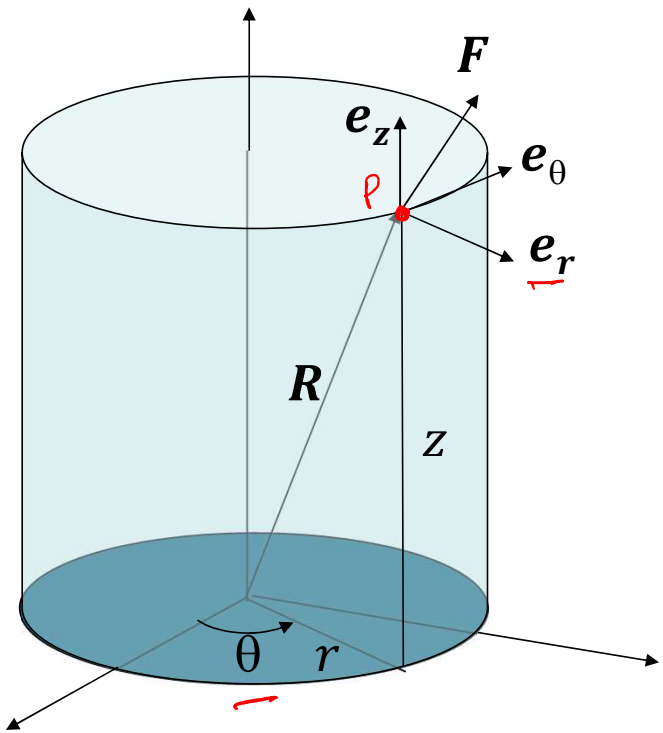
$$\vec{r} = x\hat{i} + y\hat{j} + z\vec{k}$$

- Vector components

$$\vec{F} = F_x\hat{i} + F_y\hat{j} + F_z\vec{k}$$

$$\vec{F}\cdot\hat{i}$$

Cylindrical Coordinates



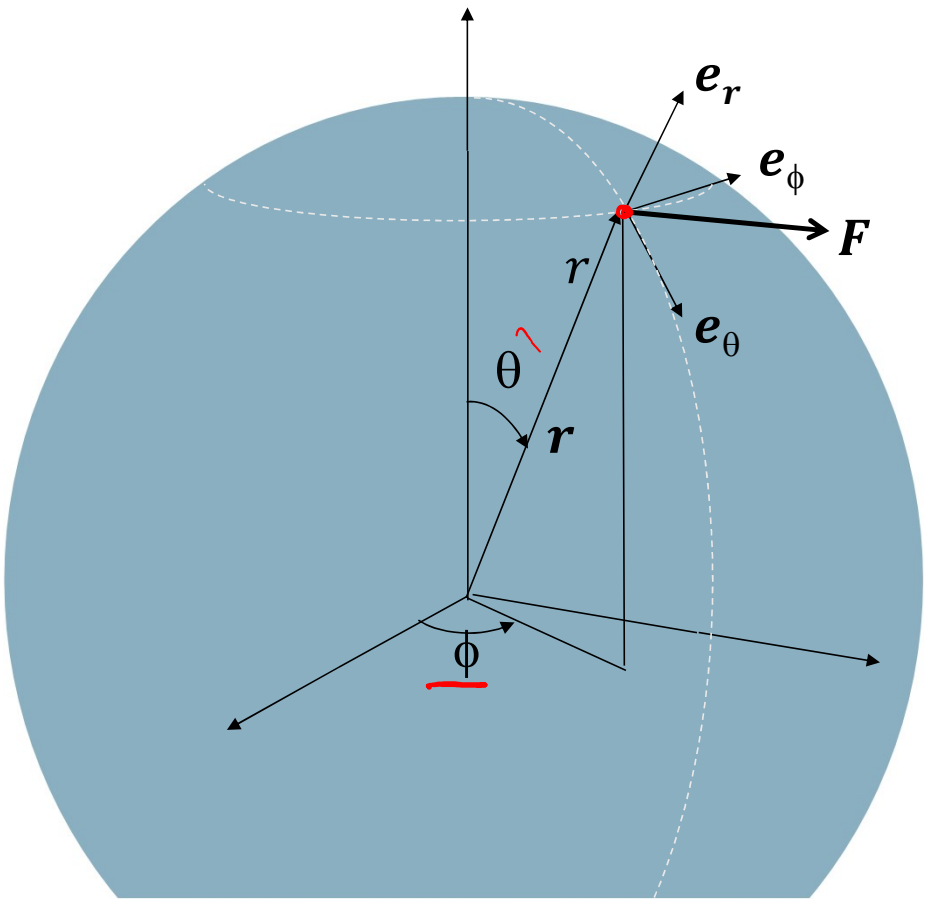
- Coordinates r, θ, z
- Unit vectors $\boldsymbol{e_r}, \boldsymbol{e_\theta}, \boldsymbol{e_z}$ (in directions of increasing coordinates)
- Position vector

$$\vec{R} = r\,\vec{e_r} + z\,\vec{e_z}$$

- Vector components

$$\vec{F} = F_r\,\vec{e_r} + F_\theta\,\vec{e_\theta} + F_z\,\vec{e_z}$$

Spherical Coordinates



- Coordinates r, θ, ϕ
- Unit vectors $\boldsymbol{e}_r, \boldsymbol{e}_\theta, \boldsymbol{e}_\phi$ (in directions of increasing coordinates)
- Position vector

$$\vec{r} = r\vec{e_r}$$

- Vector components

$$\vec{F} = F_r\vec{e_r} + F_\theta\vec{e_\theta} + F_\phi\vec{e_\phi}$$

Vector Algebra in Components

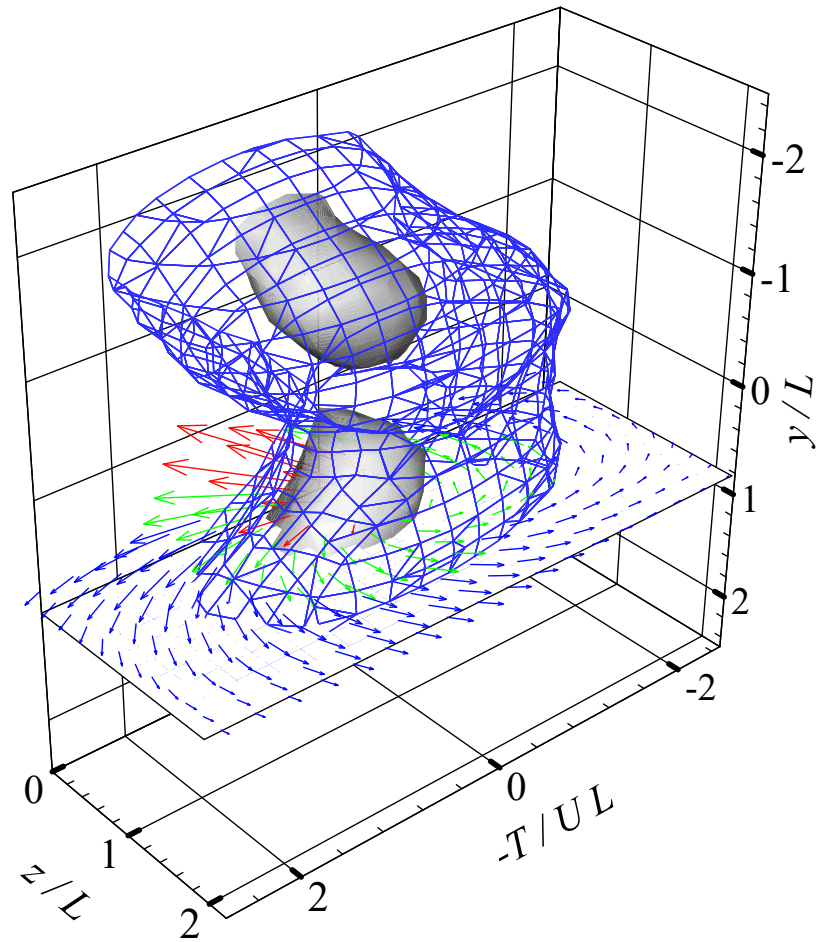
$$\mathbf{A}.\mathbf{B} = A_1 B_1 + A_2 B_2 + A_3 B_3$$

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix}$$

...works for any orthogonal coordinate system

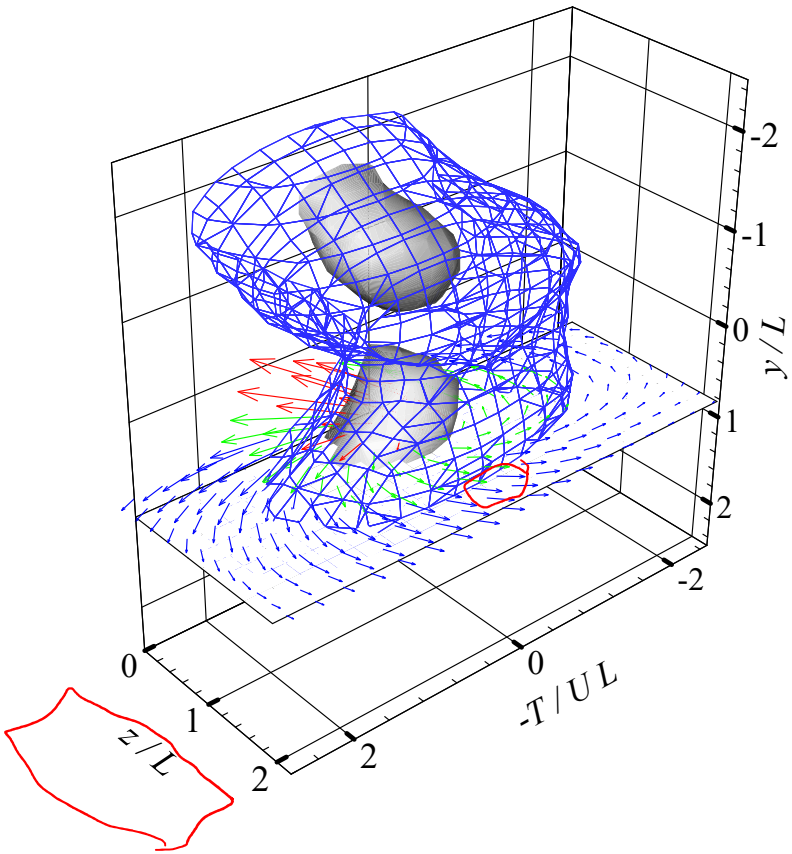
3. Vector Calculus

Fluid particle: Differentially Small Piece of the Fluid Material



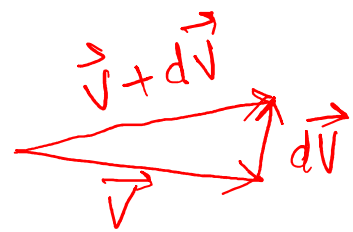
- Concept of differential change in a vector, vector field
- Changes in unit vectors
- Calculus w.r.t. time
- Integral calculus w.r.t. space
- Differential calculus w.r.t. space
- Integral theorems, second order operators

Concept of Differential Change In a Vector. The Vector Field.

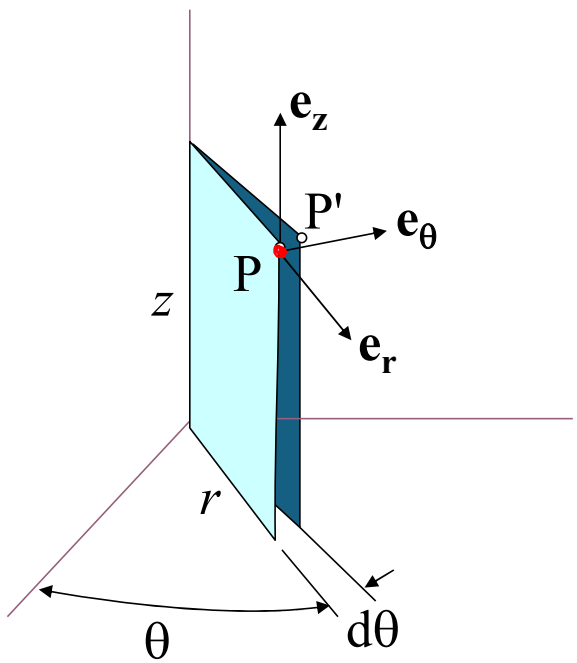


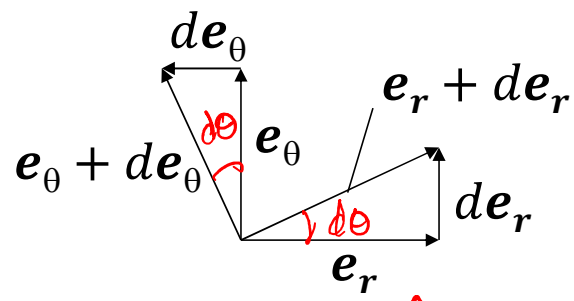
Scalar field $\phi=\phi(\mathbf{r},t)$

Vector field $\mathbf{V}=\mathbf{V}(\mathbf{r},t)$



Change in Unit Vectors – Cylindrical System



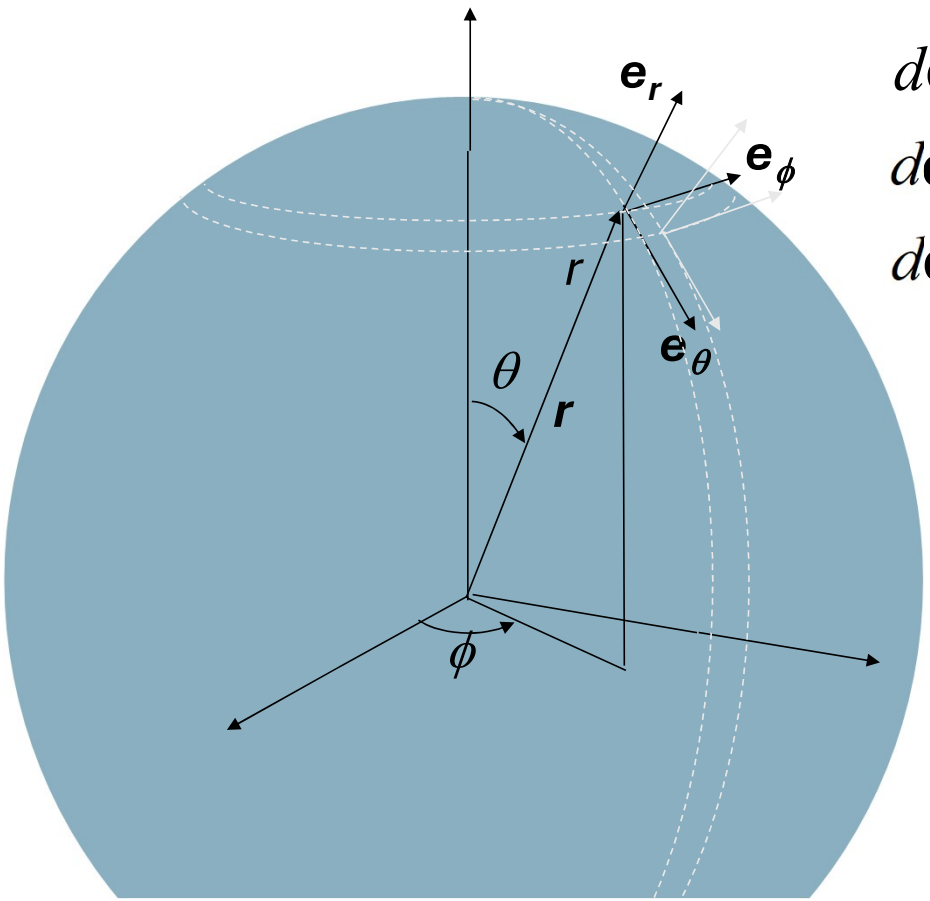


$$d\vec{e_r} = d\theta\,\vec{e_\theta}$$

$$d\vec{e_\theta} = -d\theta\,\vec{e_r}$$

$$d\vec{e_z} = 0$$

Change in Unit Vectors – Spherical System



$$d\mathbf{e}_r = d\theta\,\mathbf{e}_\theta + d\phi\sin\theta\,\mathbf{e}_\phi$$

$$d\mathbf{e}_\theta = -d\theta\,\mathbf{e}_r + d\phi\cos\theta\,\mathbf{e}_\phi$$

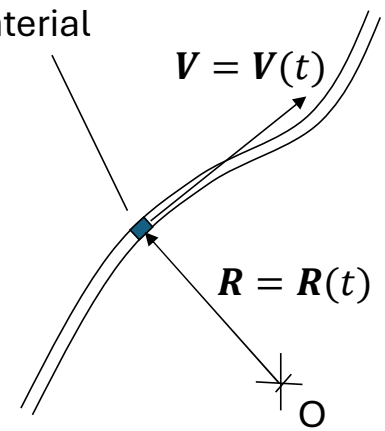
$$d\mathbf{e}_\phi = -d\phi\sin\theta\,\mathbf{e}_r - d\phi\cos\theta\,\mathbf{e}_\theta$$

See "Formulae for Vector Algebra and Calculus"

Example

Fluid particle

Differentially small piece of the fluid material



The position of fluid particle moving in a flow varies with time. Working in different coordinate systems write down expressions for the position and, by differentiation, the velocity vectors.

CARTESIAN $\vec{R} = x\vec{i} + y\vec{j} + z\vec{k}$

$$\vec{V} = \frac{d\vec{R}}{dt} = \frac{dx}{dt}\vec{i} + \frac{dy}{dt}\vec{j} + \frac{dz}{dt}\vec{k}$$

CYLINDRICAL $\vec{R} = r\vec{e}_r + z\vec{e}_z$

$$\vec{V} = \frac{dr}{dt}\vec{e}_r + \frac{dz}{dt}\vec{e}_z + r\frac{d\vec{e}_r}{dt}$$

$$\frac{d\vec{e}_r}{dt} = \frac{d\theta}{dt}\vec{e}_\theta$$

$$\Rightarrow \vec{V} = \frac{dr}{dt}\vec{e}_r + r\frac{d\theta}{dt}\vec{e}_\theta + \frac{dz}{dt}\vec{e}_z$$

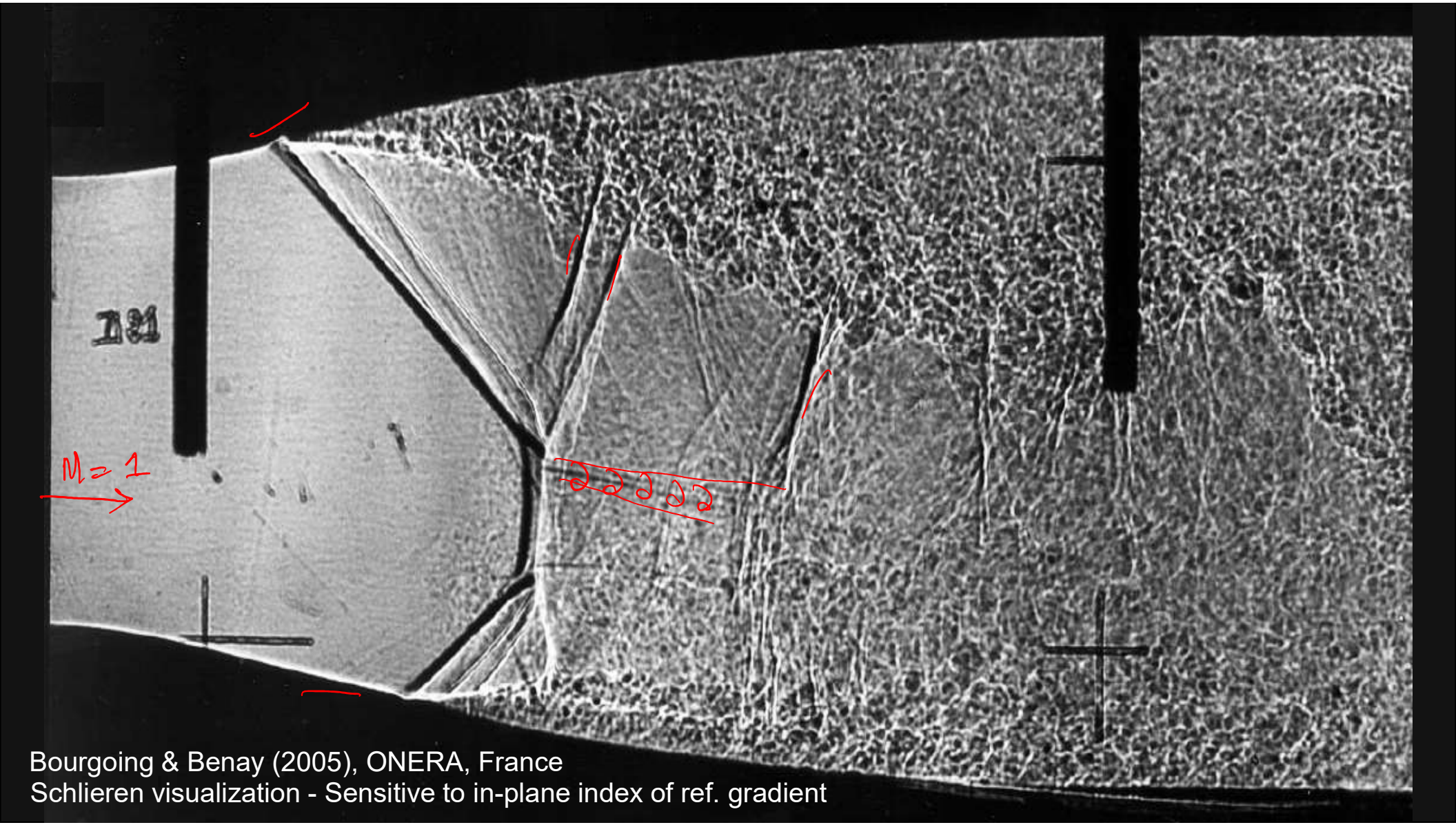
Vector Calculus w.r.t. Time

- Since *any* vector may be decomposed into scalar components, calculus w.r.t. time, only involves *scalar* calculus of the components

$$\frac{\partial(\vec{A}+\vec{B})}{\partial t} = \frac{\partial \vec{A}}{\partial t} + \frac{\partial \vec{B}}{\partial t}$$

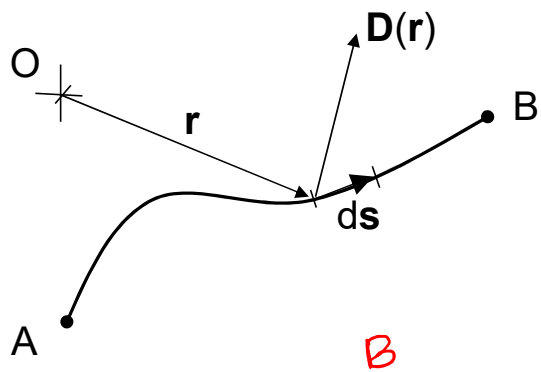
$$\frac{\partial(\vec{A}\cdot\vec{B})}{\partial t} = A\cdot\frac{\partial \vec{B}}{\partial t} + \frac{\partial A}{\partial t}\cdot B$$

$$\frac{\partial}{\partial t}(\vec{A}\times\vec{B}) = \vec{A}\times\frac{\partial \vec{B}}{\partial t} + \frac{\partial \vec{A}}{\partial t}\times B$$

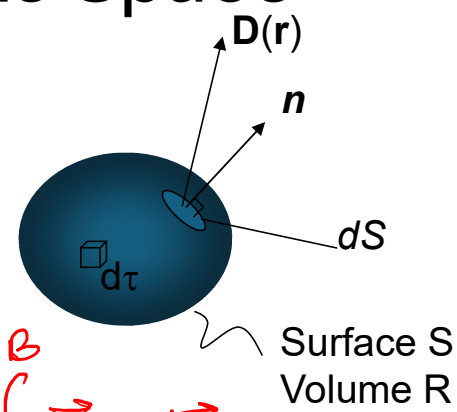


Bourgoing & Benay (2005), ONERA, France
Schlieren visualization - Sensitive to in-plane index of ref. gradient

Integral Calculus With Respect to Space



$\mathbf{D}=\mathbf{D}(\mathbf{r})$, $\phi = \phi(\mathbf{r})$



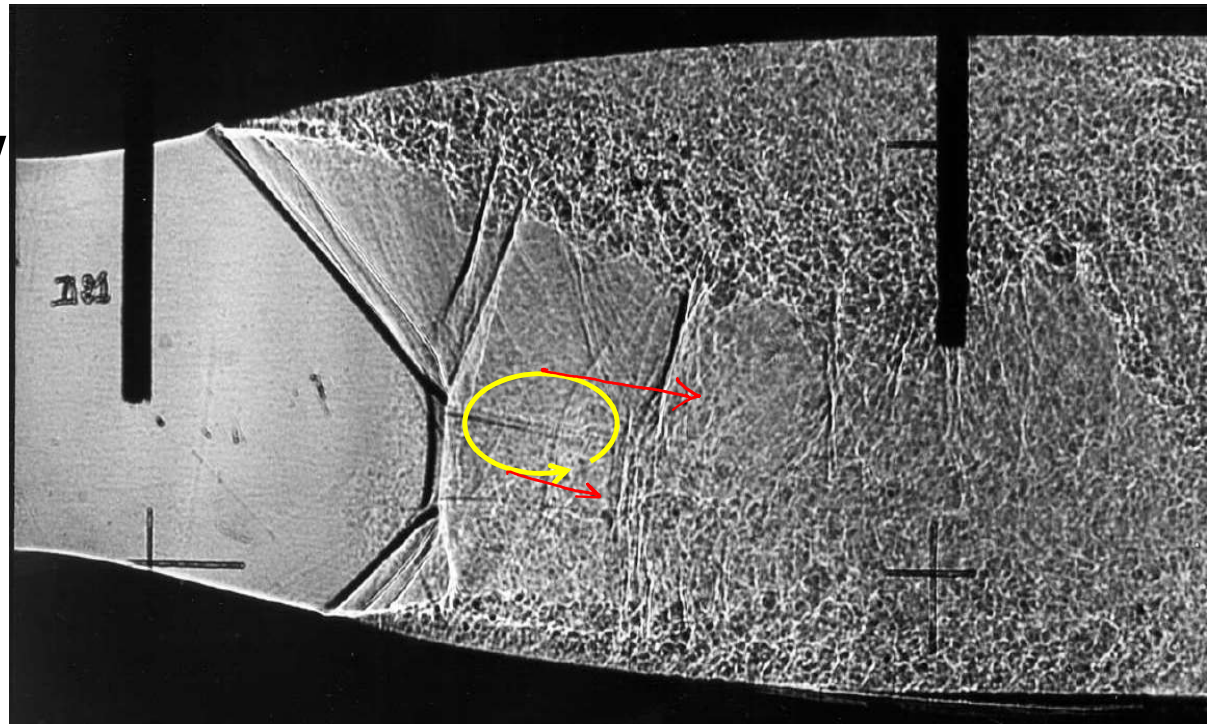
Line Integrals

$$\int_A^B \phi \, d\vec{s} \qquad \int_A^B \vec{D} \cdot d\vec{s} \qquad \int_A^B \vec{D} \times d\vec{s}$$

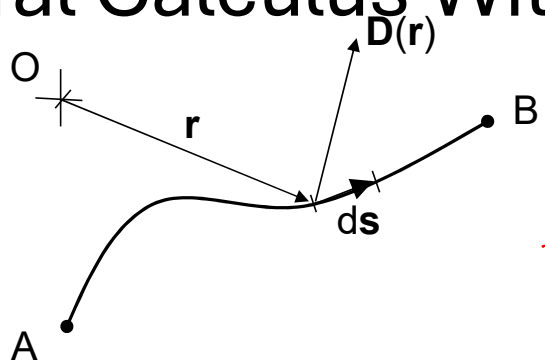
$\oint_A^B$ FOR CLOSED LOOP

EG. CIRCULATION $\Gamma \equiv \oint \vec{V} \cdot d\vec{s}$

Review



Integral Calculus With Respect to Space



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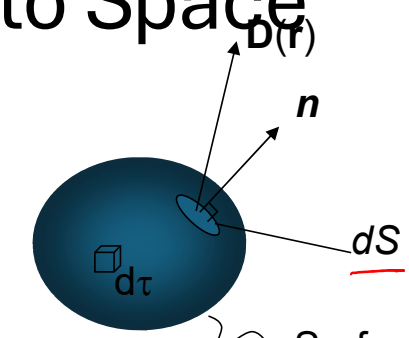


Surface S
Volume R

Surface Integrals

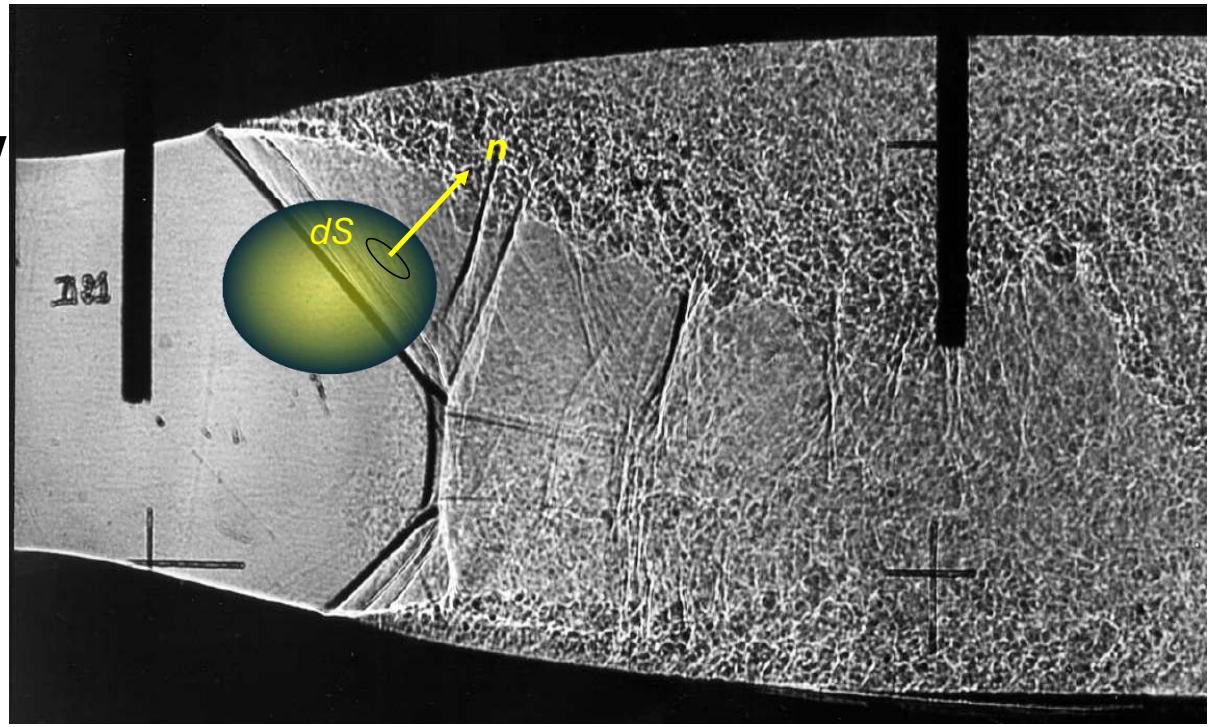
$$\oint_S \phi \vec{n}\, dS \qquad \oint_S \vec{D} \cdot \vec{n}\, dS \qquad \oint_S \vec{D} \times \vec{n}\, dS$$

$\oint$ FOR A CLOSED SURFACE

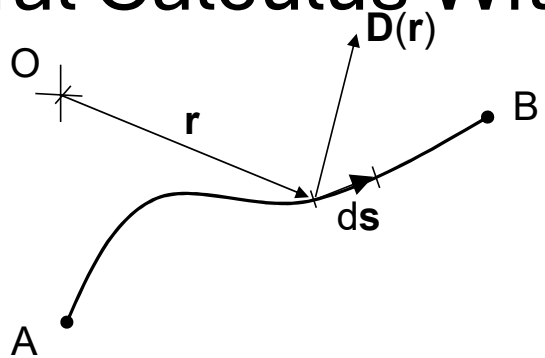
~~Eg~~ Eg. FLOW RATE OUT OF A VOLUME

$$\oint \vec{V} \cdot \vec{n}\, dS$$

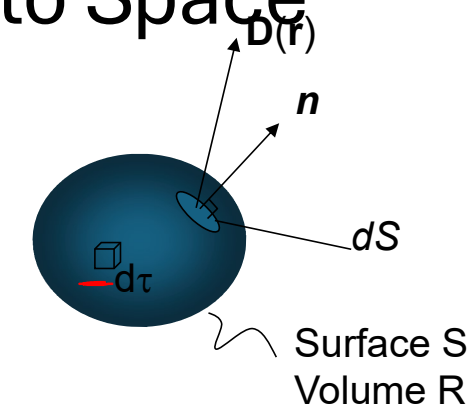
Review



Integral Calculus With Respect to Space



$\mathbf{D}=\mathbf{D}(\mathbf{r})$, $\phi = \phi(\mathbf{r})$

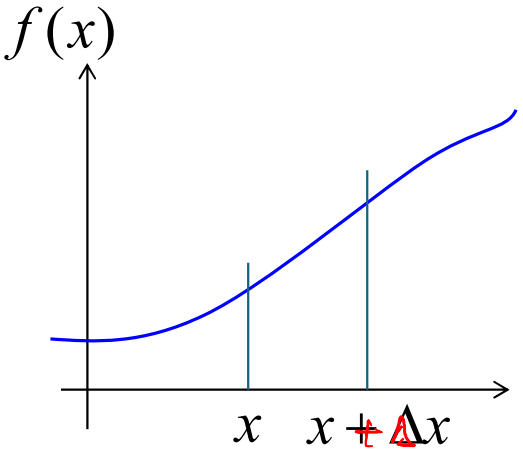


<u>Volume Integrals</u>

$$\int_R \phi \, d\tau \qquad \int_R \vec{D} \, d\tau$$

Differential Calculus w.r.t. Space in 1D

$$\frac{df}{dx} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

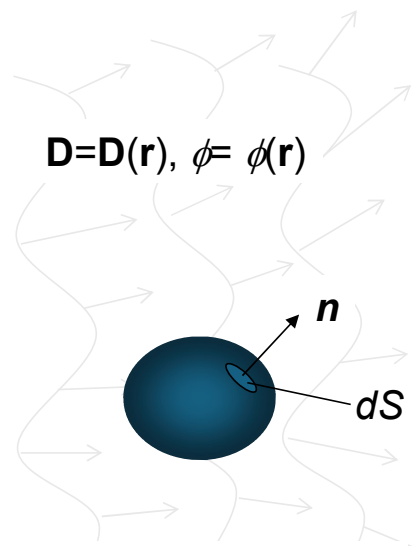


Differential Calculus w.r.t. Space in 3D

$$\text{grad}\,\phi \equiv \lim_{\delta\tau \to 0} \frac{1}{\delta\tau} \oint_S \phi\, \vec{n}\, dS$$

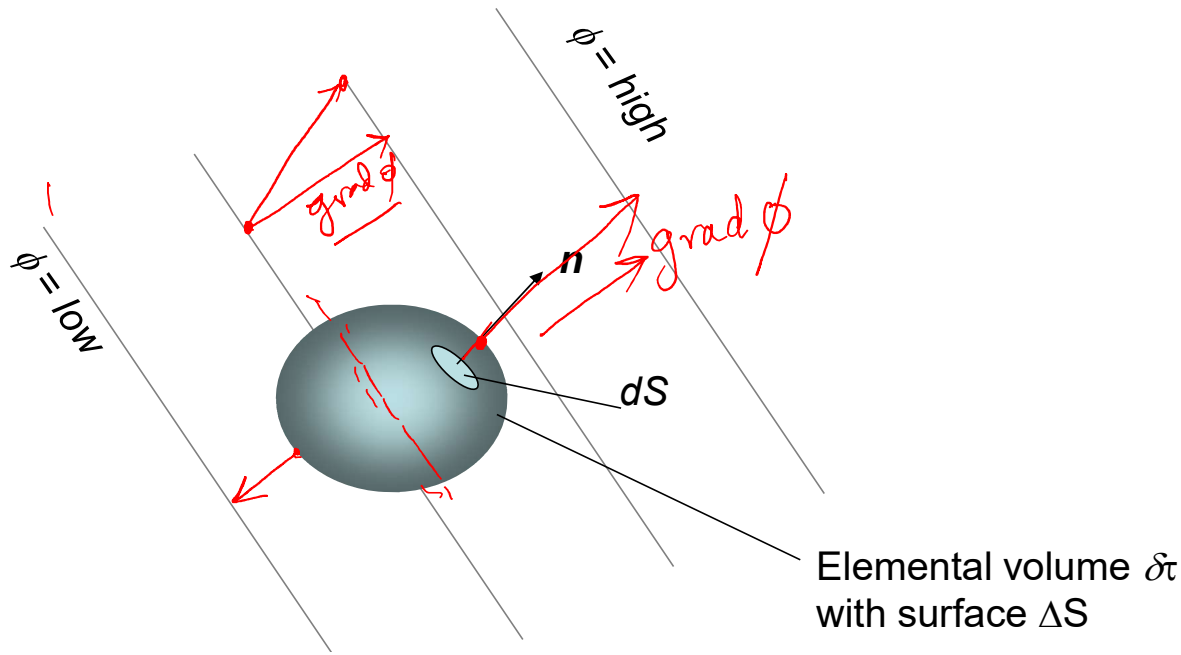
$$\text{div}\,\vec{D} \equiv \lim_{\delta\tau \to 0} \frac{1}{\delta\tau} \oint_S \vec{D} \cdot \vec{n}\, dS$$

$$\text{curl}\,\vec{D} \equiv -\lim_{\delta\tau \to 0} \frac{1}{\delta\tau} \oint \vec{D} \times \vec{n}\, dS$$



Elemental volume $\delta\tau$
with surface ΔS

Gradient

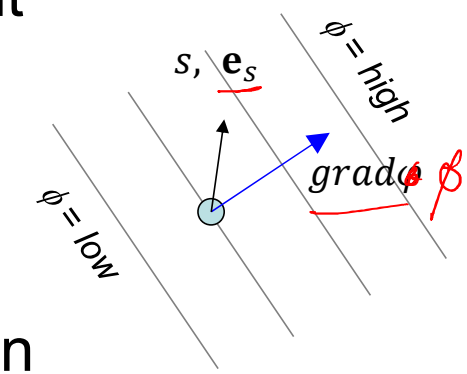


$$\mathrm{grad}\phi \equiv \mathrm{Lim}_{\delta\tau\to 0}\frac{1}{\delta\tau}\oint_{\Delta S}\phi\mathbf{n}\mathrm{d}S$$

Gradient

- Component of gradient is the partial derivative in the direction of that component

$$\frac{\partial \phi}{\partial s} = \vec{e}_s \cdot \text{grad}\,\phi$$



- Fourier´s Law of Heat Conduction

$$q = -k \frac{\partial T}{\partial n} \quad ; \quad \vec{q} = -k \nabla T$$

Differential form of the Gradient

Cartesian system

Evaluate integral by expanding the variation in ϕ about a point *P* at the center of an elemental Cartesian volume. Consider the two *x* faces:

$$\int_{Face\ 1} \phi \mathbf{n} dS \approx \left(\phi - \frac{\partial \phi}{\partial x}\frac{dx}{2}\right)(-\mathbf{i})dydz$$

$$\int_{Face\ 2} \phi \mathbf{n} dS \approx \left(\phi + \frac{\partial \phi}{\partial x}\frac{dx}{2}\right)(+\mathbf{i})dydz$$

adding these gives $\mathbf{i}\frac{\partial \phi}{\partial x}dxdydz$

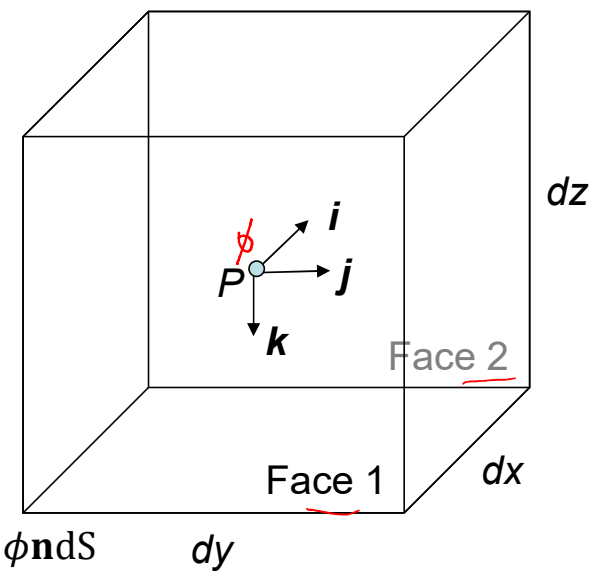
Proceeding in the same way for *y* and *z*

we get $\mathbf{j}\frac{\partial \phi}{\partial y}dxdydz$ and $\mathbf{k}\frac{\partial \phi}{\partial z}dxdydz$, so

$$\mathrm{grad}\phi \equiv \mathrm{Lim}_{\delta\tau\to 0}\frac{1}{\delta\tau}\oint_{\Delta S}\phi \mathbf{n} dS$$

$$\mathrm{grad}\phi \equiv \mathrm{Lim}_{\delta\tau\to 0}\frac{1}{\delta\tau}\oint_{\Delta S}\phi \mathbf{n} dS$$

$\phi = \phi(x,y,z)$



$$\nabla\phi = \mathrm{Lim}_{\delta\tau\to 0}\frac{1}{\delta\tau}\left(\mathbf{i}\frac{\partial \phi}{\partial x}dxdydz + \mathbf{j}\frac{\partial \phi}{\partial y}dxdydz + \mathbf{k}\frac{\partial \phi}{\partial z}dxdydz\right) = \mathbf{i}\frac{\partial \phi}{\partial x} + \mathbf{j}\frac{\partial \phi}{\partial y} + \mathbf{k}\frac{\partial \phi}{\partial z}$$

Differential Forms of the Gradient

$$grad\Phi = \nabla\Phi$$

Cartesian

$$\vec{i}\frac{\partial\Phi}{\partial x} + \vec{j}\frac{\partial\Phi}{\partial y} + \vec{k}\frac{\partial\Phi}{\partial z} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\Phi$$

Cylindrical

$$\vec{e}_r\frac{\partial\Phi}{\partial r} + \frac{\vec{e}_\theta}{r}\frac{\partial\Phi}{\partial\theta} + \vec{e}_z\frac{\partial\Phi}{\partial z} = \left(\vec{e}_r\frac{\partial}{\partial r} + \frac{\vec{e}_\theta}{r}\frac{\partial}{\partial\theta} + \vec{e}_z\frac{\partial}{\partial z}\right)\Phi$$

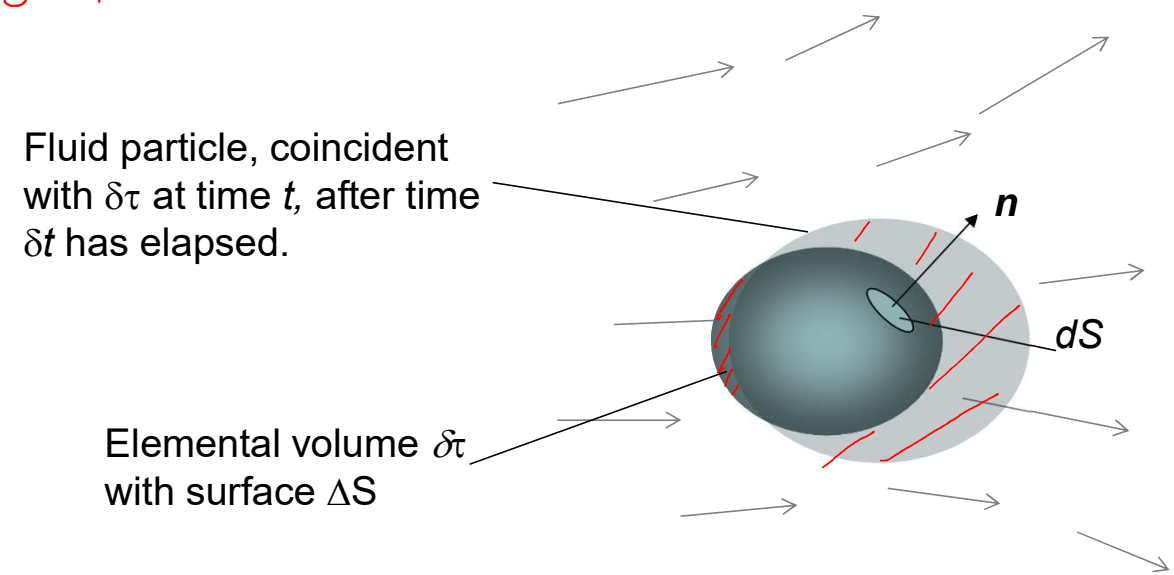
Spherical

$$\vec{e}_r\frac{\partial\Phi}{\partial r} + \frac{\vec{e}_\theta}{r}\frac{\partial\Phi}{\partial\theta} + \frac{\vec{e}_\phi}{r\sin\theta}\frac{\partial\Phi}{\partial\phi} = \left(\vec{e}_r\frac{\partial}{\partial r} + \frac{\vec{e}_\theta}{r}\frac{\partial}{\partial\theta} + \frac{\vec{e}_\phi}{r\sin\theta}\frac{\partial}{\partial\phi}\right)\Phi$$

These differential forms define the vector operator ∇

Divergence

ROC OF VOLUME OF THE FLUID (PARTICLE) PER UNIT VOLUME



$$div\mathbf{V} \equiv \mathrm{Lim}_{\delta\tau\to 0}\frac{1}{\delta\tau}\oint_{\Delta S}\mathbf{V}.\mathbf{n}\mathrm{d}S$$

Differential Forms of the Divergence

$$\boxed{div\vec{A} = \nabla.\vec{A}}$$

Cartesian

$$\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right).\vec{A}$$

Cylindrical

$$\frac{1}{r}\frac{\partial rA_r}{\partial r} + \frac{1}{r}\frac{\partial A_\theta}{\partial \theta} + \frac{\partial A_z}{\partial z} = \left(\vec{e}_r\frac{\partial}{\partial r} + \frac{\vec{e}_\theta}{r}\frac{\partial}{\partial \theta} + \vec{e}_z\frac{\partial}{\partial z}\right).\vec{A}$$

Spherical

$$\frac{1}{r^2}\frac{\partial r^2 A_r}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial A_\theta \sin\theta}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial \phi} = \left(\vec{e}_r\frac{\partial}{\partial r} + \frac{\vec{e}_\theta}{r}\frac{\partial}{\partial \theta} + \frac{\vec{e}_\phi}{r\sin\theta}\frac{\partial}{\partial \phi}\right).\vec{A}$$

Differential Forms of the Curl

$$curl\vec{A} \equiv -\mathrm{Lim}_{\delta\tau\to 0}\frac{1}{\delta\tau}\oint_{\Delta S}\vec{A}\times\vec{n}\mathrm{dS}$$

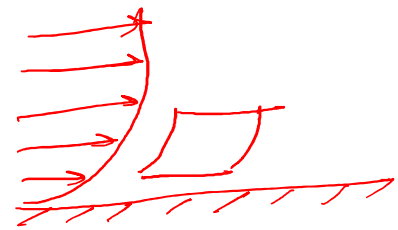
$$curl\vec{A} = \nabla\times A = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = \frac{1}{r}\begin{vmatrix} \vec{e}_r & r\vec{e}_\theta & \vec{e}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ A_r & rA_\theta & A_z \end{vmatrix} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \vec{e}_r & r\vec{e}_\theta & r\sin\theta\,\vec{e}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ A_r & rA_\theta & r\sin\theta\,A_\phi \end{vmatrix}$$

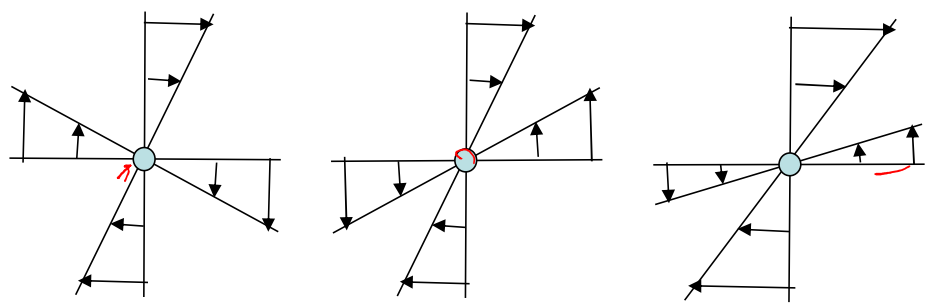
Cartesian Cylindrical Spherical

Physical Interpretation of the Curl

$\nabla \times \boldsymbol{V} =$ TWICE THE CIRCUMFERENTIALLY AVERAGED ANGULAR VELOCITY OF THE FLUID (PARTICLES)

$=$ VORTICITY $\vec{\Omega}$





Pure rotation No rotation Rotation

Curl

1) $$curl\mathbf{V} \equiv -\mathrm{Lim}_{\delta\tau\to 0}\frac{1}{\delta\tau}\oint_{\Delta S}\mathbf{V}\times\mathbf{n}\mathrm{dS}$$

2) $$\mathbf{e}.curl\mathbf{V} \equiv -\mathrm{Lim}_{\delta\tau\to 0}\frac{1}{\delta\tau}\oint_{\Delta S}\mathbf{e}.\mathbf{V}\times\mathbf{n}\mathrm{dS}$$

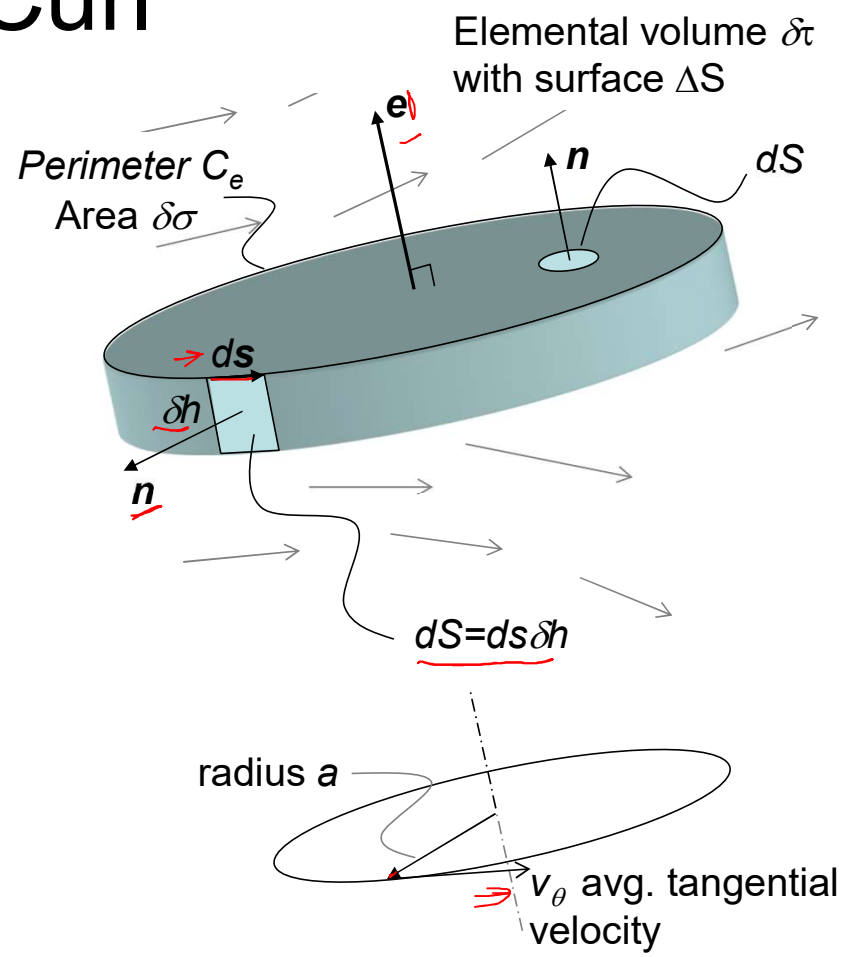
3) $$\mathbf{e}.curl\mathbf{V} \equiv \mathrm{Lim}_{\delta\tau\to 0}\frac{1}{\delta\sigma\delta h}\oint_{\Delta S}\mathbf{V}.\mathbf{e}\times\mathbf{n}\mathrm{dS}$$

4) $$\mathbf{e}.curl\mathbf{V} \equiv \mathrm{Lim}_{\delta\tau\to 0}\frac{1}{\delta\sigma\cancel{\delta h}}\oint_{\Delta S}\mathbf{V}.\mathbf{e}\times\mathbf{n}ds\cancel{\delta h}$$

5) $$\boxed{\mathbf{e}.curl\mathbf{V} \equiv \mathrm{Lim}_{\delta\sigma\to 0}\frac{1}{\delta\sigma}\oint_{C_e}\mathbf{V}.\mathrm{d}\mathbf{s} = \mathrm{Lim}_{\delta\sigma\to 0}\frac{\Gamma_{Ce}}{\delta\sigma}}$$

$$\mathbf{e}.curl\mathbf{V} \equiv \mathrm{Lim}_{a\to 0}\frac{1}{\pi a^2}v_\theta 2\pi a = 2\mathrm{Lim}_{a\to 0}\frac{v_\theta}{a}$$

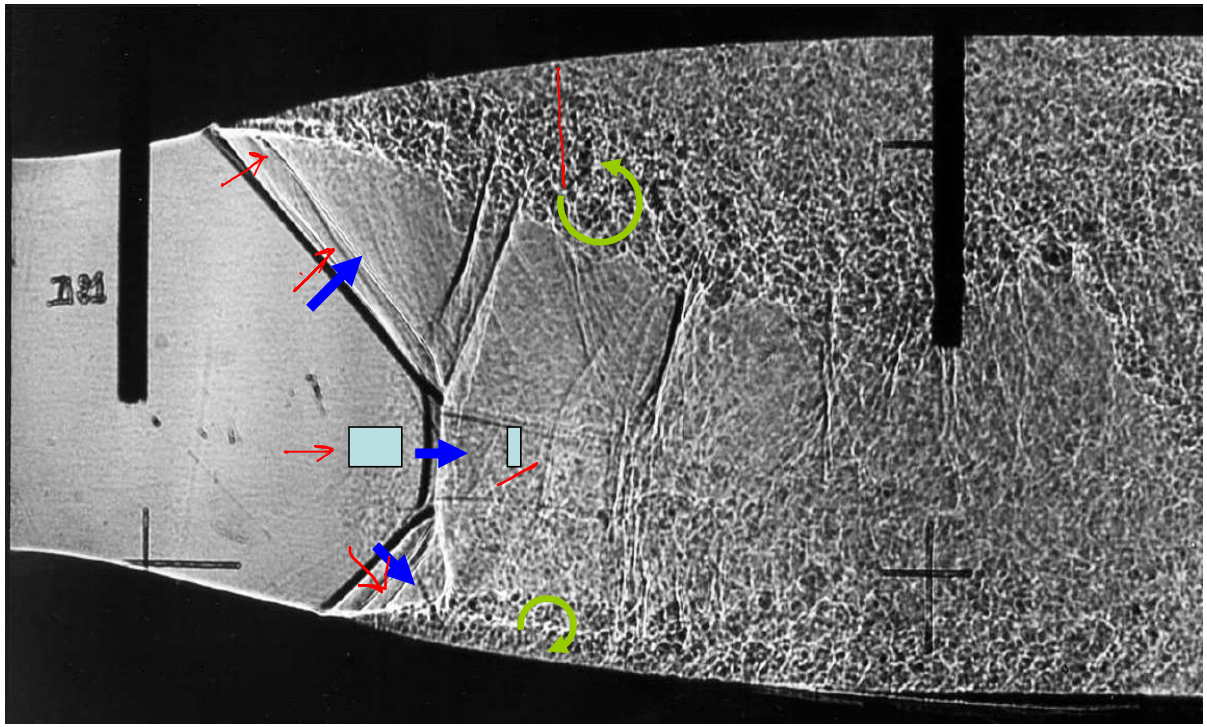
= twice the avg. angular velocity about **e**



Curl

$$\mathbf{e}.curl\mathbf{V} \equiv \mathrm{Lim}_{\delta\sigma\to 0}\frac{1}{\delta\sigma}\oint_{C_e}\mathbf{V}.\mathrm{d}\mathbf{s} = \mathrm{Lim}_{\delta\sigma\to 0}\frac{\Gamma_{Ce}}{\delta\sigma}$$

Component of the curl is the circulation / unit area around the axis defining the component.



Gradient

$$\mathrm{grad}\phi \equiv \mathrm{Lim}_{\delta\tau\to 0}\frac{1}{\delta\tau}\oint_{\Delta S}\phi\mathbf{n}\mathrm{d}S$$

Magnitude and direction of the slope in the scalar field at a point

Divergence

$$div\mathbf{V} \equiv \mathrm{Lim}_{\delta\tau\to 0}\frac{1}{\delta\tau}\oint_{\Delta S}\mathbf{V}.\mathbf{n}\mathrm{d}S$$

For velocity: proportionate rate of change of volume of a fluid particle

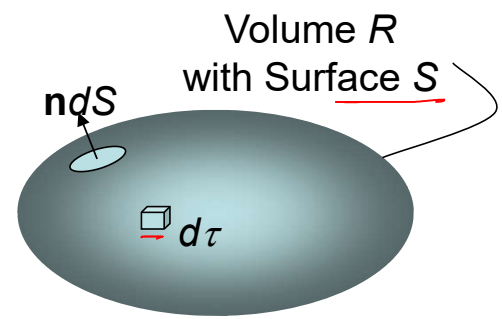
Curl

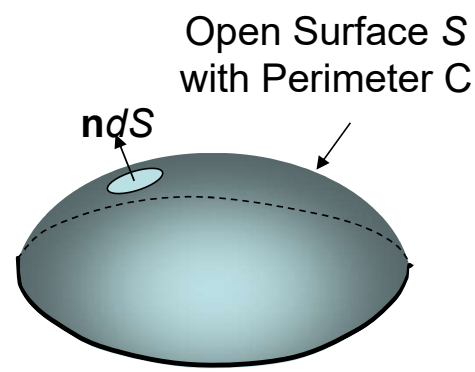
$$curl\mathbf{V} \equiv -\mathrm{Lim}_{\delta\tau\to 0}\frac{1}{\delta\tau}\oint_{\Delta S}\mathbf{V}\times\mathbf{n}\mathrm{d}S$$

For velocity: twice the circumferentially averaged angular velocity of a fluid particle = Vorticity $\mathbf{\Omega}$

1st Order Integral Theorems

- Gradient theorem $$\int_R \nabla \phi d\tau = \oint_S \phi \mathbf{n} dS$$
- Divergence theorem $$\int_R \nabla . \mathbf{A} d\tau = \oint_S \mathbf{A}.\mathbf{n} dS$$
- Curl theorem $$\int_R \nabla \times \mathbf{A} d\tau = -\oint_S \mathbf{A} \times \mathbf{n} dS$$
- Stokes' theorem $$\int_S \nabla \times \mathbf{A}.\mathbf{n} dS = \oint_C \mathbf{A}.\mathrm{d}\mathbf{s}$$



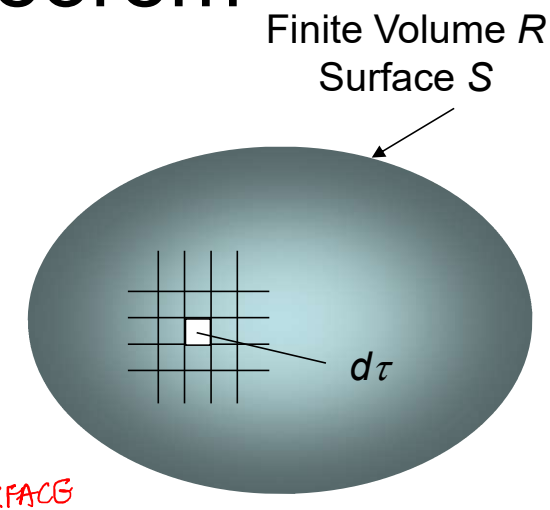


The Gradient Theorem

Begin with the definition of grad:

$$\rightarrow \quad \mathrm{grad}\phi \equiv \mathrm{Lim}_{\delta\tau\to 0}\frac{1}{\delta\tau}\oint_{\Delta S}\phi\mathbf{n}\mathrm{d}S$$

Sum over all the $d\tau$ in R:

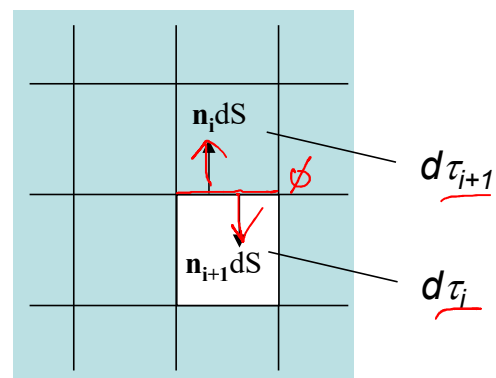


$$\sum_R \mathrm{grad}\,\phi\,\delta\tau = \sum_R \oint_{\Delta S}\phi\,\vec{n}\,dS$$

RHS ONLY HAS NET CONTRIBUTIONS FROM THE EXTERIOR SURFACE OF R

$$\sum_R \mathrm{grad}\,\phi\,d\tau = \sum_S \phi\,\vec{n}\,dS$$

$$= \int_R \mathrm{grad}\,\phi\,d\tau = \oint_S \phi\,\vec{n}\,dS$$

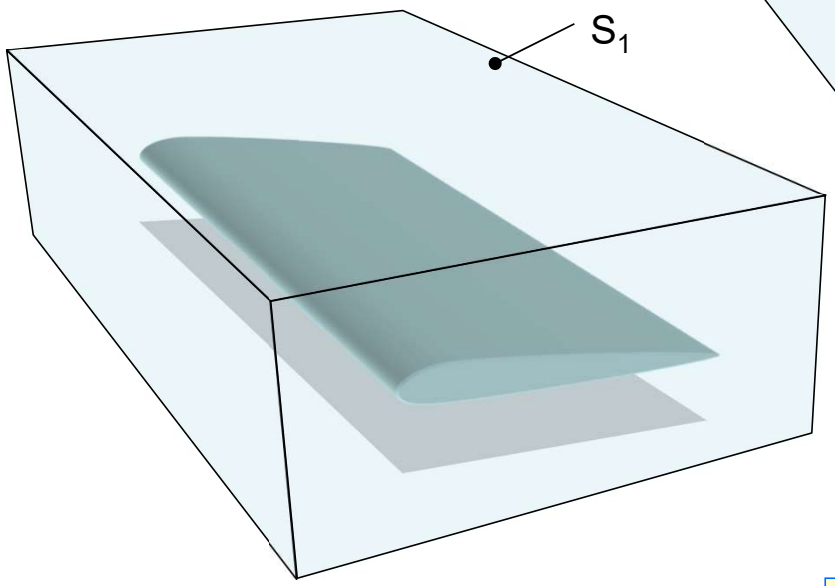


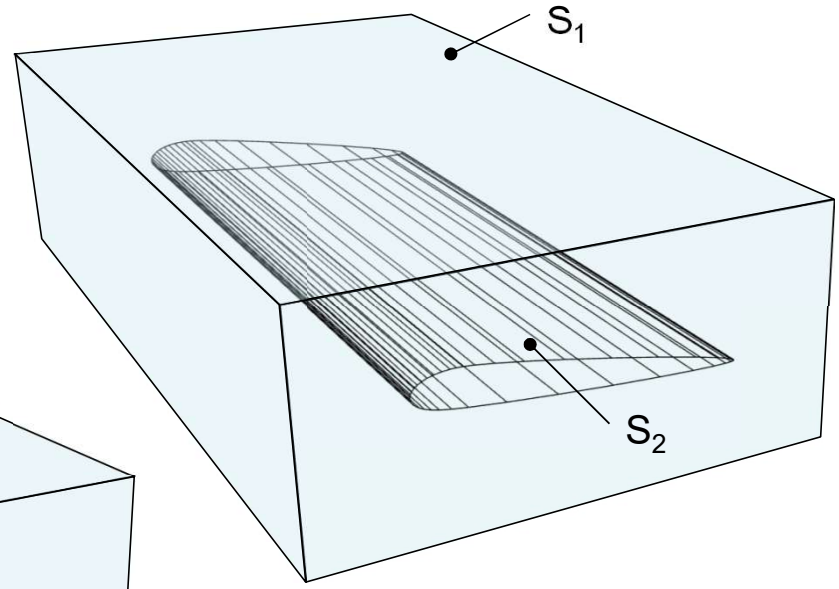
Assumptions in Gradient Theorem

NONE!

PURE MATH RESULT!!

Flow over a finite wing





$S = S_1 + S_2$

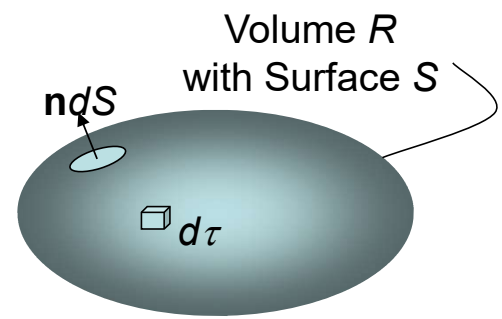
R is the volume of fluid enclosed between S_1 and S_2

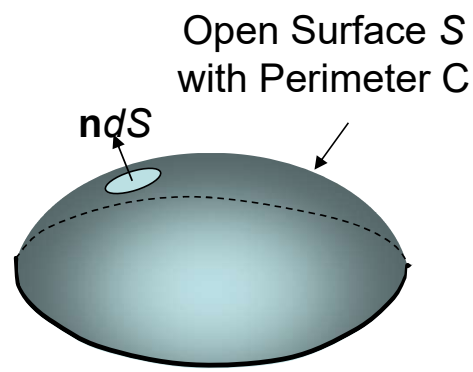
$$\int_R \nabla p d\tau = \oint_S p\mathbf{n}\mathrm{dS}$$

p is not defined inside the wing so the wing itself must be excluded from the integral

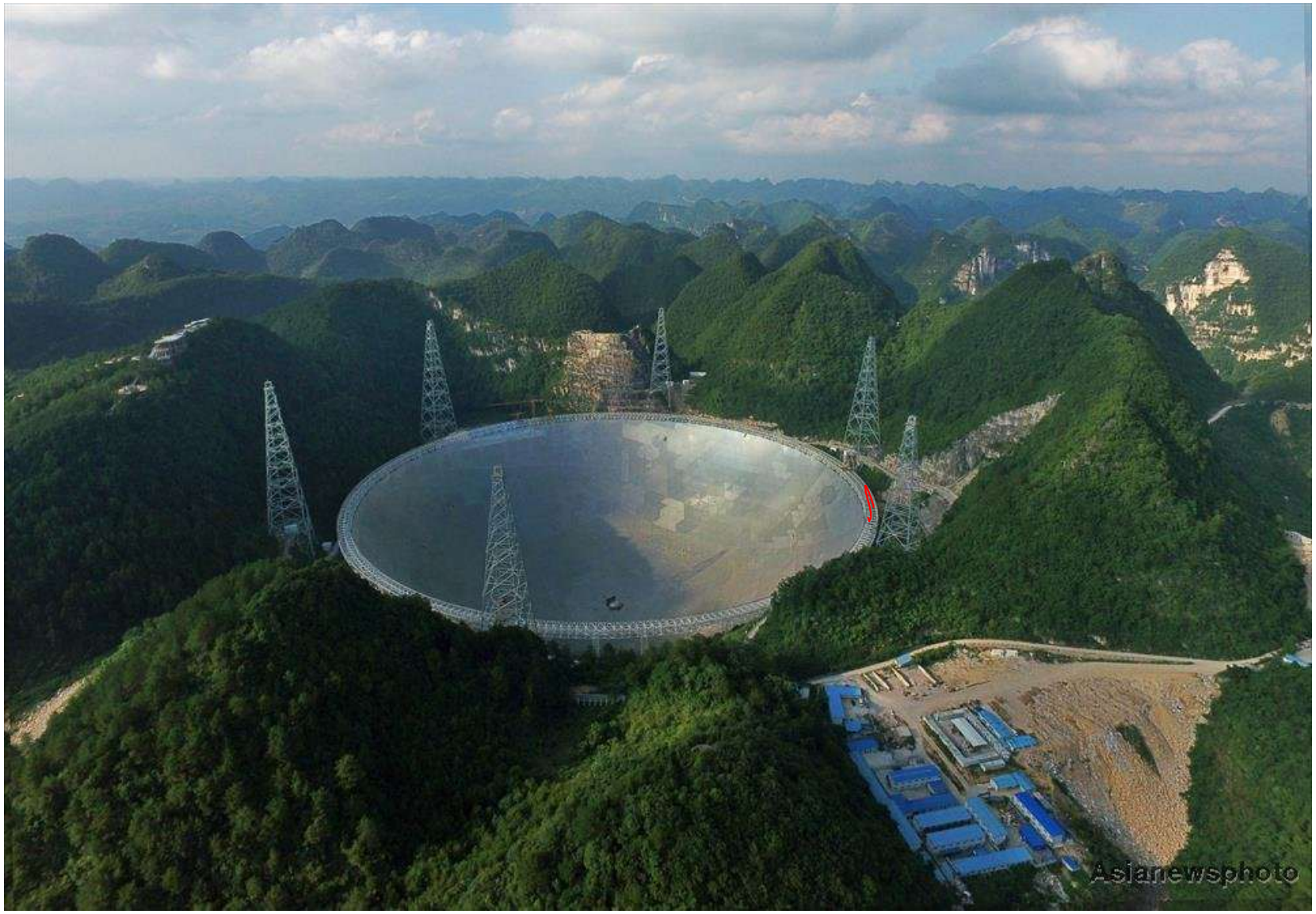
1st Order Integral Theorems

- Gradient theorem $$\int_R \nabla\varphi d\tau = \oint_S \varphi \mathbf{n} dS$$
- Divergence theorem $$\int_R \nabla . \mathbf{A} d\tau = \oint_S \mathbf{A} . \mathbf{n} dS$$
- Curl theorem $$\int_R \nabla \times \mathbf{A} d\tau = -\oint_S \mathbf{A} \times \mathbf{n} dS$$
- Stokes' theorem $$\int_S \nabla \times \mathbf{A} . \mathbf{n} d\,S = \oint_C \mathbf{A} . \mathrm{d}\mathbf{s}$$





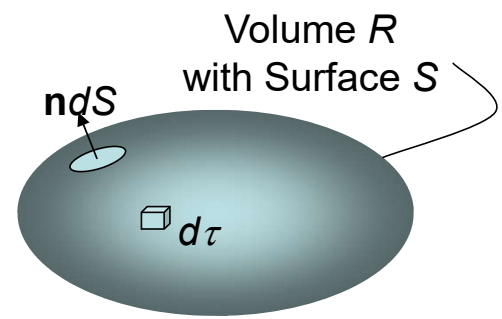
500m aperture
Radio Telescope.

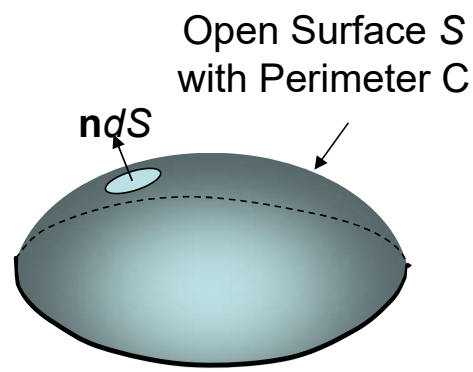


Guizhou Province,
southwest China.

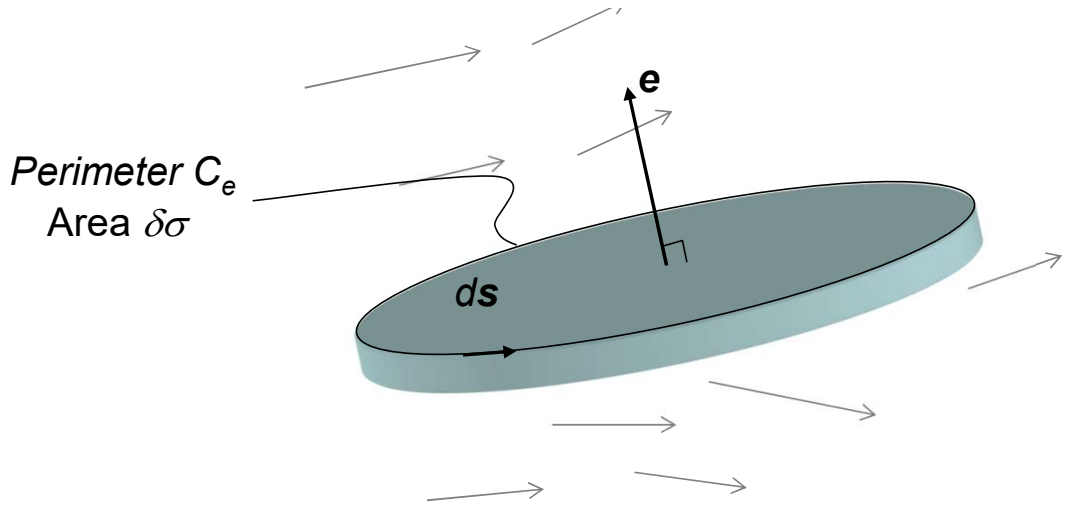
1st Order Integral Theorems

- Gradient theorem $$\int_R \nabla\varphi d\tau = \oint_S \varphi \mathbf{n} dS$$
- Divergence theorem $$\int_R \nabla . \mathbf{A} d\tau = \oint_S \mathbf{A} . \mathbf{n} dS$$
- Curl theorem $$\int_R \nabla \times \mathbf{A} d\tau = -\oint_S \mathbf{A} \times \mathbf{n} dS$$
- Stokes' theorem $$\int_S \nabla \times \mathbf{A} . \mathbf{n} d\,S = \oint_C \mathbf{A} . \mathrm{d}\mathbf{s}$$





Alternative Definition of the Curl



$$\mathbf{e}.\,curl\mathbf{A} \equiv \mathrm{Lim}_{\delta\sigma\to 0}\frac{1}{\delta\sigma}\oint_{\mathrm{C_e}}\mathbf{A}.\,\mathrm{d}\mathbf{s} = \mathrm{Lim}_{\delta\sigma\to 0}\frac{\Gamma_{\mathrm{Ce}}}{\delta\sigma}$$

Stokes' Theorem

Begin with the alternative definition of curl, choosing the direction **e** to be the outward normal to the surface **n**:

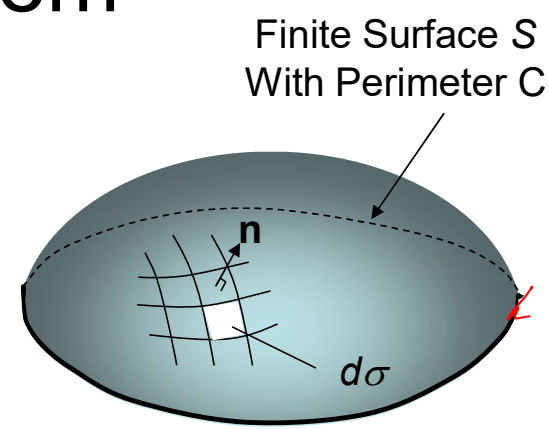
$$\mathbf{n}.\nabla \times \mathbf{A} \equiv \mathrm{Lim}_{\delta\sigma \to 0} \frac{1}{\delta\sigma} \oint_{C_e} \mathbf{A}.\mathrm{d}\mathbf{s}$$

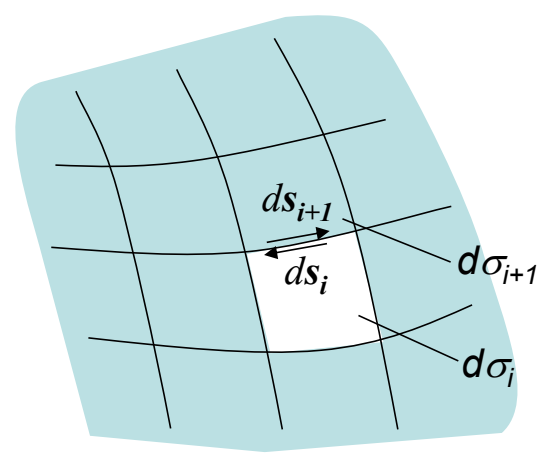
Sum over all the $d\sigma$ in S:

$$\sum n \cdot \nabla \times \vec{A} = \sum \oint_C \vec{A} \cdot d\vec{s}$$

CONTRIBUTIONS TO RHS FROM INTERIOR INTERFACES CANCEL OUT, GIVING,

$$\int_S \vec{\nabla} \times \vec{A} \cdot n \, dS = \oint_C \vec{A} \cdot ds$$





Stokes´ Theorem and Velocity

- Apply Stokes´ Theorem to a velocity field

$$\int_S \nabla \times \mathbf{V}.\mathbf{n} d\,S = \oint_C \mathbf{V}.\mathrm{d}\mathbf{s}$$

- Or, in terms of vorticity and circulation

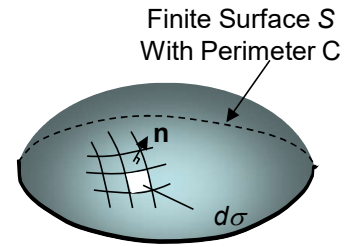
$$\int_S \mathbf{\Omega}.\mathbf{n} d\,S = \oint_C \mathbf{V}.\mathrm{d}\mathbf{s} = \Gamma_C$$

- What about a closed surface?

$$\oint_S \mathbf{\Omega}.\mathbf{n}\, dS = 0$$

Assumptions of Stokes´ Theorem

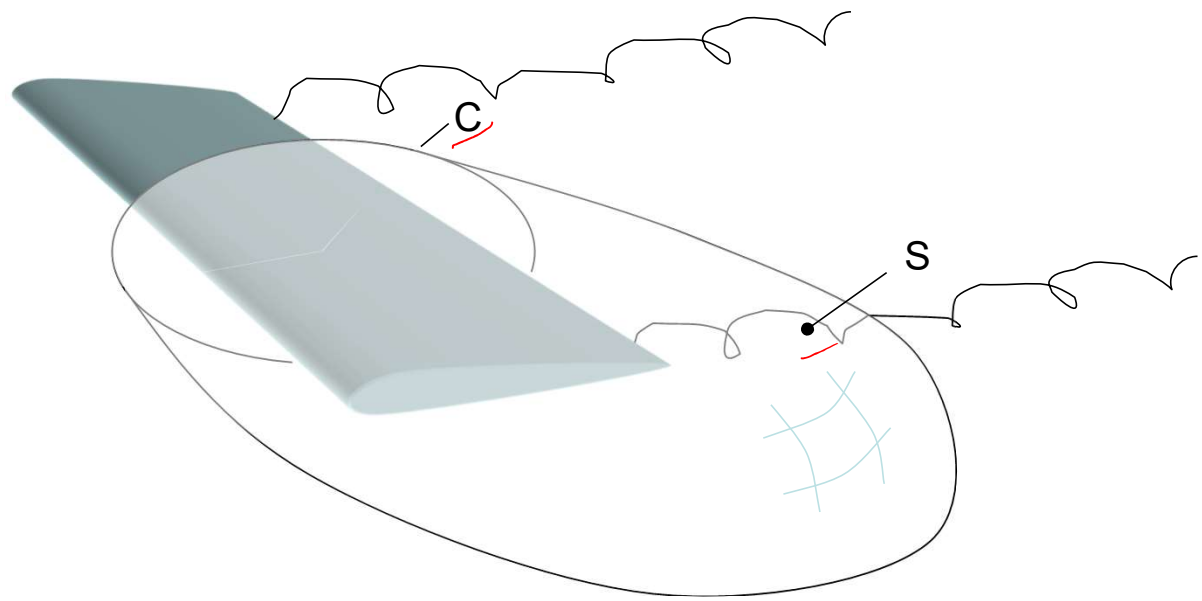
$$\int_S \nabla \times \mathbf{A}.\mathbf{n} d\,S = \oint_C \mathbf{A}.\mathrm{d}\mathbf{s}$$



NONE

Flow over a finite wing

$$\int_S \nabla \times \mathbf{V}.\mathbf{n} d\,S = \oint_C \mathbf{V}.\mathrm{d}\mathbf{s}$$



Wing with circulation must trail vorticity. *Always.*

Vector Operators of Vector Products

$$\nabla(\psi\Phi) = \psi\nabla\Phi + \Phi\nabla\psi$$

$$\nabla.(\Phi\vec{A}) = \Phi\nabla.\vec{A} + \nabla\Phi.\vec{A}$$

$$\nabla\times(\Phi\vec{A}) = \Phi\nabla\times\vec{A} + \nabla\Phi\times\vec{A}$$

$$\nabla(\vec{A}.\vec{B}) = (\vec{A}.\nabla)B + (\vec{B}.\nabla)\vec{A} + \vec{A}\times(\nabla\times\vec{B}) + \vec{B}\times(\nabla\times\vec{A})$$

$$\nabla.(\vec{A}\times\vec{B}) = \vec{B}.\nabla\times\vec{A} - \vec{A}.\nabla\times\vec{B}$$

$$\nabla\times(\vec{A}\times\vec{B}) = \vec{A}(\nabla.\vec{B}) + (\vec{B}.\nabla)\vec{A} - \vec{B}(\nabla.\vec{A}) - (\vec{A}.\nabla)\vec{B}$$

Convective Operator

$$(\vec{A}.\nabla)\Phi = \left(A_x\frac{\partial}{\partial x} + A_y\frac{\partial}{\partial y} + A_z\frac{\partial}{\partial z}\right)\varphi$$
$$= \vec{A}.(\nabla\Phi)$$

$\mathbf{V}.\nabla\rho$ = change in density in direction of **V**, multiplied by magnitude of **V**

$$(\vec{A}.\nabla)\vec{B} = \left(A_x\frac{\partial}{\partial x} + A_y\frac{\partial}{\partial y} + A_z\frac{\partial}{\partial z}\right)\vec{B}$$
$$= \frac{1}{2}[\nabla(\vec{A}.\vec{B}) - \vec{A}\times(\nabla\times\vec{B}) - \vec{B}\times(\nabla\times\vec{A})$$
$$-\nabla\times(\vec{A}\times\vec{B}) + \vec{A}(\nabla.B) - \vec{B}(\nabla.\vec{A})]$$

Second Order Operators

$$\nabla.\nabla\phi = \nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}$$

The Laplacian, may also be applied to a vector field.

$$\nabla(\nabla.\mathbf{A})$$

$$\nabla \times \nabla \times \mathbf{A} = \nabla(\nabla.\mathbf{A}) - \nabla^2\mathbf{A}$$

$$\nabla \times \nabla\phi \equiv 0$$

- So, any vector differential equation of the form $\nabla\times\mathbf{B}=0$ can be solved identically by writing $\mathbf{B}=\nabla\phi$.
- We say **B** is ***irrotational***.
- We refer to ϕ as the ***scalar potential***.

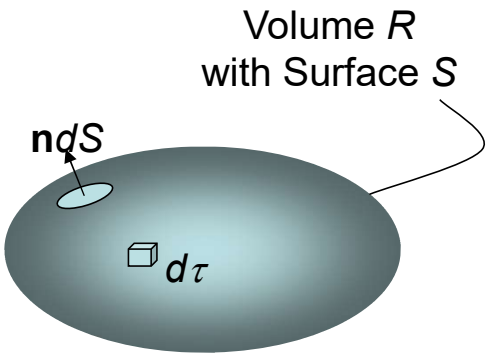
$$\nabla.\nabla \times \mathbf{A} \equiv 0$$

- So, any vector differential equation of the form $\nabla.\mathbf{B}=0$ can be solved identically by writing $\mathbf{B}=\nabla\times\mathbf{A}$.
- We say **B** is ***solenoidal*** or ***incompressible***.
- We refer to **A** as the ***vector potential***.

2nd Order Integral Theorems

- Green's theorem (1st form)

$$\int_R \psi \nabla^2 \phi + \nabla\psi \nabla\phi d\tau = \oint_S \psi \frac{\partial \phi}{\partial \mathrm{n}} \mathrm{d}S$$



- Green's theorem (2nd form)

$$\int_R \psi \nabla^2 \phi - \phi \nabla^2 \psi \; d\tau = \oint_S \psi \frac{\partial \phi}{\partial \mathrm{n}} - \phi \frac{\partial \psi}{\partial \mathrm{n}} \mathrm{d}S$$

These are both re-expressions of the divergence theorem.