Intermediate Fluid Mechanics

Lecture 24: Boundary Layer Flows III

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Chapter Overview

Lecture Objectives

In the previous lecture we considered the solution to the laminar boundary layer equations for flow over a flat plate where the pressure gradient is zero.

Question:

What happens in the cases where there is anon-zero pressure gradient?

Chapter Overview

Laminar Boundary Layer with varying freestream velocity

- In flow over a flat plate, the freestream velocity U_∞ remains independent of x.
- In this case, the Blasius solution describes the velocity distribution inside the boundary layer.
- For other types of geometries, such as flow over a wedge or though a converging channel, the freestream velocity varies with x, *i.e.* $U_e = U_e(x)$.

Note: Here we use U_e, with subscript 'e' to denote the velocity at the edge of the boundary layer. We avoid using U_∞ in cases where the freestream velocity varies with x.

Laminar Boundary Layer with varying freestream velocity

In the special case where,

$$U_e = ax^m \tag{1}$$

$\longrightarrow$ One can find a similarity solution to the laminar boundary layer equations (Falkner - Skan solutions).

The boundary layer equations (for steady flow) are,

$$\begin{align} \text{(continuity)} \quad & \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{2} \\ \text{(x-momentum)} \quad & u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = U_e\frac{dU_e}{dx} + \nu\frac{\partial^2 u}{\partial y^2} \tag{3} \\ \text{(Boundary Conditions 1)} \quad & u = v = 0 \text{ at } y = 0 \tag{4} \\ \text{(Boundary Conditions 2)} \quad & u \rightarrow U_e \text{ at } y \rightarrow \infty \tag{5} \\ & \tag{6} \end{align}$$

Rewriting the pressure gradient term

- Based on dimensional analysis of the boundary layer, we found that the pressure gradient across the boundary layer is zero.
- Therefore, the pressure within the boundary layer is equal to the pressure at the edge of the boundary layer.
- Since viscous effects are only important within the boundary layer, the flow outside the boundary layer is consequently assumed to be inviscid.
- In this manner, the pressure is said to be *impressed* on the boundary layer by the outer flow.

Rewriting the pressure gradient term

Using the fact that the outer flow is inviscid, the governing equation for the outer flow is

$$\frac{\partial U_e}{\partial t} + U_e \frac{\partial U_e}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial x}. \tag{7}$$

Note, that the other advection term $V_e \frac{\partial U_e}{\partial y} << U_e \frac{\partial U_e}{\partial x}$, since both V_e and $\partial U_e/\partial y$ are small at the edge of the boundary layer.

For the case of steady flow, the above equation reduces to

$$U_e \frac{\partial U_e}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial x}. \tag{8}$$

The physical interpretation is that advection (or inertia) in the outer flow is driven solely by the pressure gradient. Therefore in the x-momentum equation for the boundary layer, one can substitute $U_e \frac{\partial U_e}{\partial x}$ for the pressure gradient term.

Chapter Overview

Falker-Skan Solution

Following the approach used in the Blasius solution, one can rewrite the BL equations (with an external pressure gradient) using the streamfunction,

$$\frac{\partial \psi}{\partial y}\frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial \psi}{\partial x}\frac{\partial^2 \psi}{\partial y^2} = U_e \frac{dU_e}{dx} + \nu \frac{\partial^3 \psi}{\partial y^3} \tag{9}$$

with boundary conditions,

$$\frac{\partial \psi}{\partial x} = 0 \quad \text{and} \quad \frac{\partial \psi}{\partial y} = 0 \quad \text{at} \quad y = 0 \tag{10}$$

$$\frac{\partial \psi}{\partial y} \to U_e \quad \text{as} \quad y \to \infty. \tag{11}$$

Falker-Skan Solution

We seek a solution of the form

$$\frac{u}{U_e} = g(\eta) \quad \text{where} \quad \eta = y/\delta(x). \tag{12}$$

From the Blasius solution, we found that

$$\frac{\delta}{x} = \frac{1}{\sqrt{Re_x}} \quad \text{where} \quad Re_x = \frac{U_\infty x}{\nu}. \tag{13}$$

In this case, we replace U_∞ with U_e,

$$Re_x = \frac{U_e x}{\nu} = \frac{(ax^m)x}{\nu} = \frac{ax^{m+1}}{\nu}. \tag{14}$$

Falker-Skan Solution

Therefore, one can rewrite δ as:

$$\delta = \sqrt{\frac{\nu\, x}{U_e}} = \sqrt{\frac{\nu\, x}{a\, x^m}} = \sqrt{\frac{\nu}{a}} x^{\frac{1-m}{2}} \tag{15}$$

Then one can rewrite the non-dimensional wall-normal coordinate as,

$$\eta = \frac{y}{\delta} = \frac{y}{x}\sqrt{Re_x} = \frac{y}{x}\sqrt{\frac{a}{\nu}} x^{\frac{m+1}{2}} = \sqrt{\frac{a}{\nu}} y\, x^{\frac{m-1}{2}} \tag{16}$$

Falker-Skan Solution

One can then determine ψ by integrating u,

$$\psi = \int_0^y u\, dy = \int_0^y U_e g(\eta) dy \tag{17}$$

and knowing that $dy = \delta d\eta = \sqrt{\frac{\nu}{a}} x^{\frac{1-m}{2}}\, d\eta$, then

$$\psi = \int_0^\eta a x^m g(\eta) \sqrt{\frac{\nu}{a}} x^{\frac{1-m}{2}}\, d\eta \tag{18}$$

$$\psi = \sqrt{\nu\, a} x^{\frac{m+1}{2}} \int_0^\eta \underbrace{g(\eta)\, d\eta}_{f(\eta)} \tag{19}$$

$$\psi = \sqrt{\nu\, a} x^{\frac{m+1}{2}} f(\eta). \tag{20}$$

Falker-Skan Solution

At the same time, the advection of the outer flow is

$$U_e \frac{dU_e}{dx} = ax^m \frac{d}{dx}(ax^m) = ma^2 x^{2m-1}. \tag{21}$$

One can then apply the chain rule to transform the initial partial differential equation for the streamfunction into an ordinary differential equation for f,

$$f''' + \frac{m+1}{2} f\, f'' + m(1 - f^{'2}) = 0 \tag{22}$$

with the boundary conditions,

$$f(\eta) = 0, \qquad f'(\eta = 0) = 0, \quad \text{and} \quad f'(\eta = \infty) = 1. \tag{23}$$

Note from the above equation that for $m = 0$ the Falker-Skan solution is identical to the Blasius solution for a flat plate. Similarly, the Falker-Skan equation gets resolved numerically.

Solution to the Falkner-Skan equations:

- For $m \geq 0$, all solutions the similarity solutions show a monotonically increasing shear stress ($f''(0)$) as m increases.
- For $m = -0.0904$, then $f''(0) = 0$ and $\implies \tau_0 = 0$ and flow separation is imminent all along the surface.
- For $m < -0.0904$ the solution does not represent boundary layers.

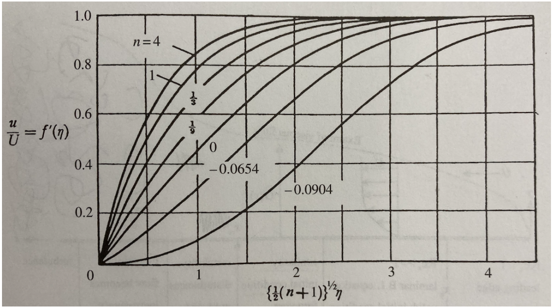

Figure: Velocity distribution in the boundary layer for external stream $U_e = a\,x^m$.

Decelerating flows ($m < 0$)

For $m < 0$ the flows experience an adverse pressure gradient ($\partial p/\partial x > 0$).

$\Longrightarrow$ For $m < 0$ the boundary layer tends toward separation, which occurs when the wall shear stress goes to zero:

$$\left.\frac{\partial u}{\partial y}\right|_{y=0} = 0 \tag{24}$$

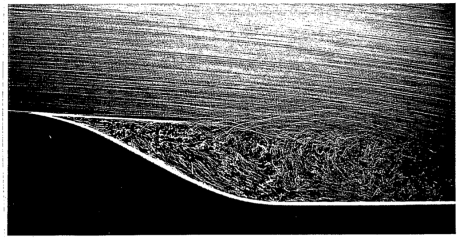

Figure: Example of separation of a laminar boundary layer.

Accelerating flows ($m > 0$)

For $m > 0$ the flows experience a favorable pressure gradient ($\partial p / \partial x < 0$).

In this case, the wall shear stress increases relative to that in a zero-pressure gradient boundary layer.

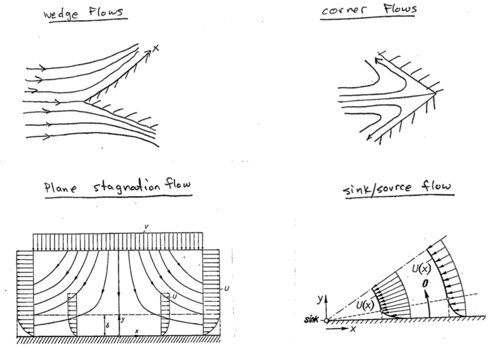

Figure: Flow geometries that may be solved using the Falker-Skan solutions.

Chapter Overview

Breakdown of the Laminar Boundary Layer Solutions

- Agreement of the Blasius solution with experimental data breaks down at large downstream distances where the local Reynolds number (Re_x) is larger than some critical number (Re_{cr})
- At these large Reynolds numbers, the boundary layer becomes unstable and transitions to turbulence.
- The critical Reynolds number varies greatly with the surface roughness, the intensity of existing fluctuations within the outer irrotational flow, and the shape of the leading edge.
- Within a factor of 5, the critical Reynolds number for a boundary layer over a flat plate is found to be $Re_{cr} \sim 10^6$.

Breakdown of the Laminar Boundary Layer Solutions

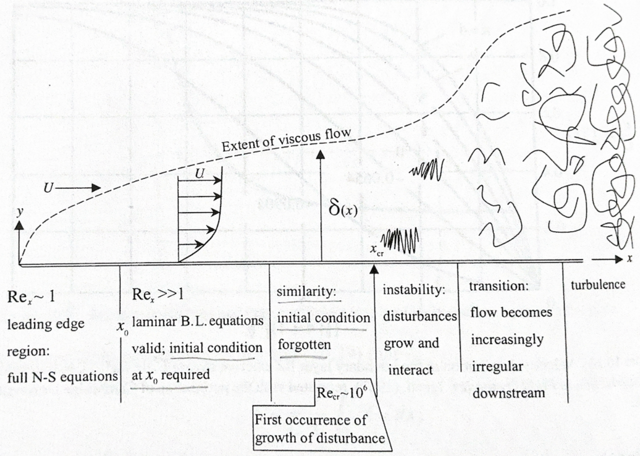

Figure: Schematic depiction of flow over a semiinfinite flat plate.

Breakdown of the Laminar Boundary Layer Solutions

- For finite $Re_x = Ux/\nu \sim 1$, the full Navier-Stokes equations are required to describe the leading edge region properly.

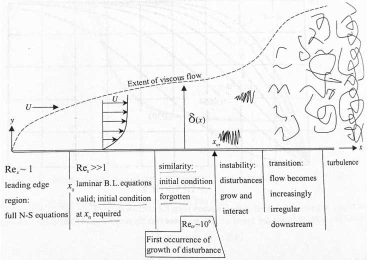

Figure: Schematic depiction of flow over a semiinfinite flat plate.

- As the Re_x gets large at the downstream limit of the leading edge region, one can locate x_0 as the maximal upstream distance of the boundary layer equations.
- For some distance $x > x_0$, the initial condition is remembered.

Breakdown of the Laminar Boundary Layer Solutions

- Finally, the influence of the initial condition maybe neglected and the solution becomes of **similarity form**.
- For some larger Re_x, further downstream the first flow instabilities appear.

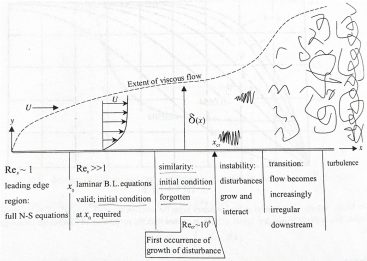

Figure: Schematic depiction of flow over a semiinfinite flat plate.

- As the Re_x increases, the flow becomes increasingly chaotic and irregular in the downstream direction.
- Eventually, the boundary layer becomes fully turbulent with a significant increase in shear stress at the plate τ_0.

Breakdown of the Laminar Boundary Layer Solutions

- After undergoing transition, the boundary layer thickness grows faster than $x^{1/2}$.
- Also, the shear stress increases faster with U in a laminar boundary layer.

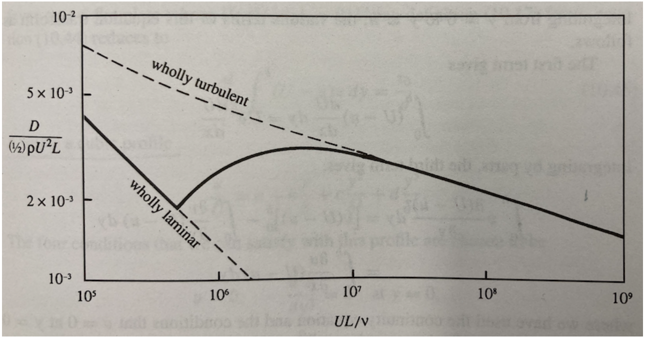

Figure: Measured drag force for a boundary layer over a flat plate, with transition to turbulence starting at $Re_x = 5 \times 10^5$.