ME 5200/6200 and ECE 5615/6615 Classical Controls

Homework 07 Solutions

Problem 1

Consider the following loop transfer functions where K is a constant. Sketch the root locus for each for K>0 by hand, showing the details for all steps to draw root locus discussed in class.

$$\text{(a)}\quad G_c(s)G(s) = \frac{K}{s(s+10)(s+8)}$$

$$\text{(b)}\quad G_c(s)G(s) = \frac{K}{(s^2+2s+2)(s+2)}$$

$$\text{(c)}\quad G_c(s)G(s) = \frac{K(s+5)}{s(s+1)(s+10)}$$

$$\text{(d)}\quad G_c(s)G(s) = \frac{K(s^2+4s+8)}{s^2(s+1)}$$

(a)

Step 1

zeros: none $\qquad m=0$

poles: $s(s+10)(s+8)=0 \qquad n=3$

$\Rightarrow \quad s=0$

$s=-10$

$s=-8$

Step 2

Draw locus on real axis to left of odd number of poles + zeros (see plot).

Step 3

Find asymptotes and α:

# of asymptotes: $h = n-m = 3-0 = \underline{\underline{3}}$

Angle of asymptotes:

$$\phi_1 = \frac{180^\circ + 360^\circ(1-1)}{3} = 60^\circ$$

$$\phi_2 = \frac{180^\circ + 360^\circ(2-1)}{3} = 180^\circ$$

$$\phi_3 = \frac{180^\circ + 360^\circ(3-1)}{3} = 300^\circ$$

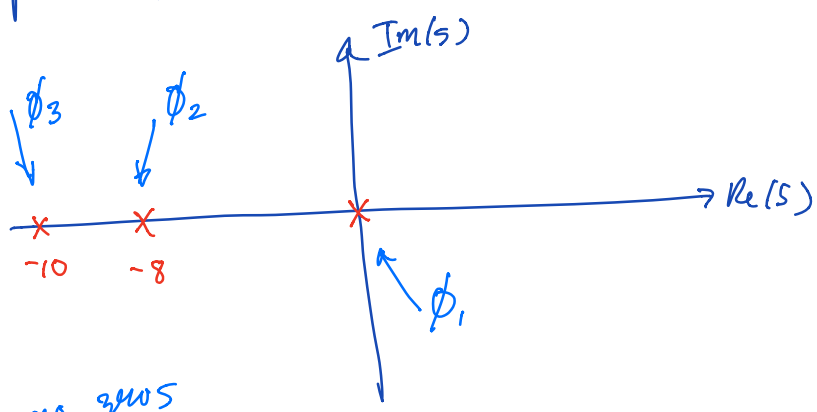
asymptotes centered about α:

$$\alpha = \frac{\sum Re(P_i) - \sum Re(Z_i)}{n-m} = \frac{-10-8}{3}$$

$$\Rightarrow \underline{\underline{\alpha = -6}}$$

<u>Step 4</u> Find angle of departure for poles

$q = 1$, no repeated poles



no zeros

$$\phi_1 = \cancel{\sum \psi_i}^{=0} - \sum \phi_j - 180^\circ - 360^\circ(h-1), \quad h=1$$

$$\phi_1 = -(0^\circ + 0^\circ) - 180^\circ - 0 = -180^\circ = \underline{\underline{180^\circ}}$$

$$\phi_2 = -(180^\circ + 0^\circ) - 180^\circ - 0 = \underline{\underline{0^\circ}}$$

$$\phi_3 = -(+180^\circ + 180^\circ) - 180^\circ - 0 = -180^\circ = \underline{\underline{180^\circ}}$$

No arrival angle to zeros b/c no zeros!

<u>Step 5</u> find $j\omega$-axis crossing

characteristic equation:

$$1 + \frac{K}{s(s+8)(s+10)} = 0$$

$$\Rightarrow s(s+8)(s+10) + K = 0$$

$$s(s^2 + 18s + 80) + K = 0$$

$$s^3 + 18s^2 + 80s + K = 0 \qquad (\#1)$$

Use R-H approach to find K:

s^3	1	80
s^2	18	K
s	b_1	b_2
s^0	c_1	

$$b_1 = \frac{-\begin{vmatrix} 1 & 80 \\ 18 & K \end{vmatrix}}{18} = \frac{-K + 80 \cdot 18}{18}$$

$$b_2 = \frac{-\begin{vmatrix} 1 & 0 \\ 18 & 0 \end{vmatrix}}{18} = 0$$

$$c_1 = \frac{-\begin{vmatrix} 18 & K \\ b_1 & 0 \end{vmatrix}}{b_1} = K$$

for stability:

$$b_1 > 0 \Rightarrow b_1 = -\frac{K}{18} + 80 = 0$$

$$K > 0 \qquad \Rightarrow -\frac{K}{18} = -80$$

$$\underline{\underline{K = 1440}}$$

Now we find ω at $j\omega$-axis crossing:

Look at auxillary equation:

$$18s^2 + 1440 = 0$$

solve for s:

$$18s^2 = -1440$$

$$\Rightarrow s^2 = \pm 80 = \pm 4\sqrt{5}\,j$$

$$\Rightarrow \omega = 4\sqrt{5} = \underline{\underline{8.9}}$$

<u>Step 6</u> Find breakaway points / break in points

$$\frac{d}{ds}(K) = \frac{d}{ds}\left(-\frac{1}{\hat{G}(s)}\right)\Bigg|_{s=s_0} = \frac{1}{b^2}\left[b\frac{da}{ds} - a\frac{db}{ds}\right]\Bigg|_{s=s_0} = 0$$

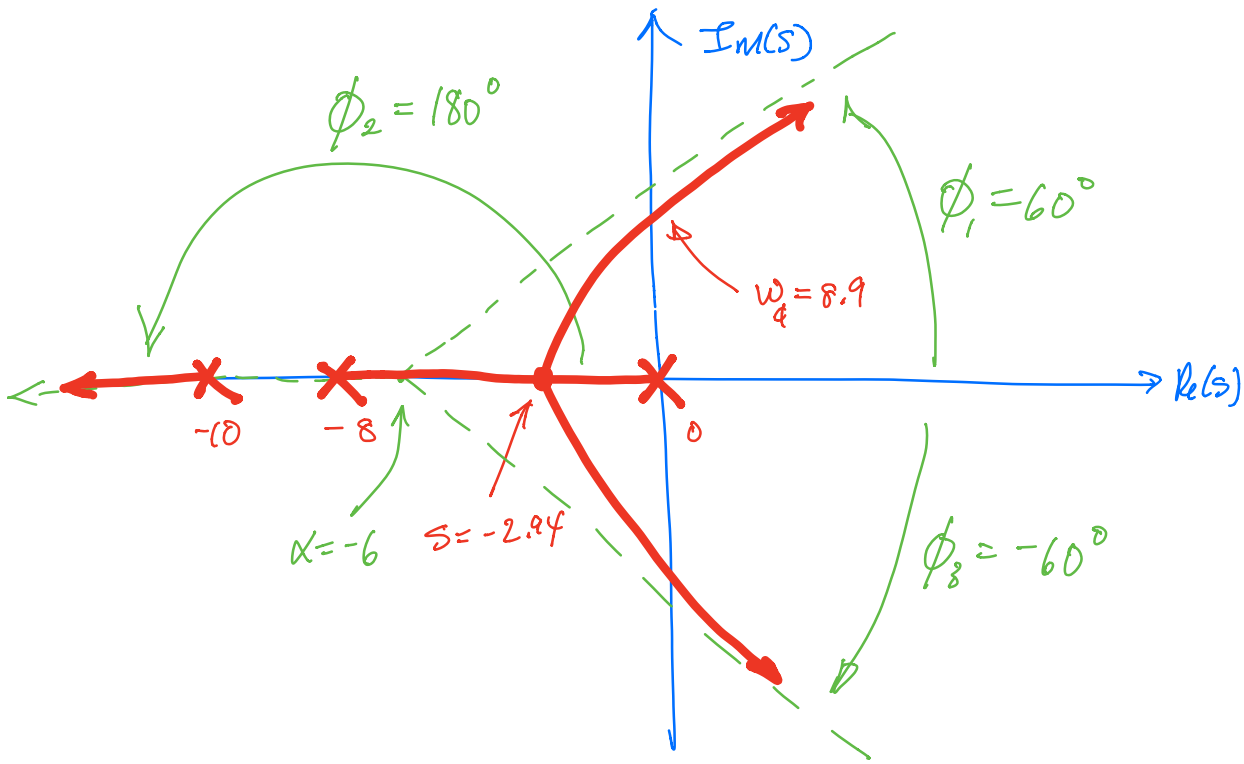
$$\hat{G}(s) = \frac{1}{s(s+8)(s+10)} = \frac{a(s)}{b(s)}$$

$$\Rightarrow \frac{d}{ds}(K) = \frac{1}{[s(s+8)(s+10)]^2}\left[\cancel{(s(s+8)(s+10))\cdot 0}^{\,0} - 1\left(3s^2+36s+80\right)\right] = 0$$

$$\Rightarrow 3s^2 + 36s + 80 = 0$$ solve using Matlab

$$\Rightarrow s = \underline{\underline{-2.94}} \text{ and } -9.05$$

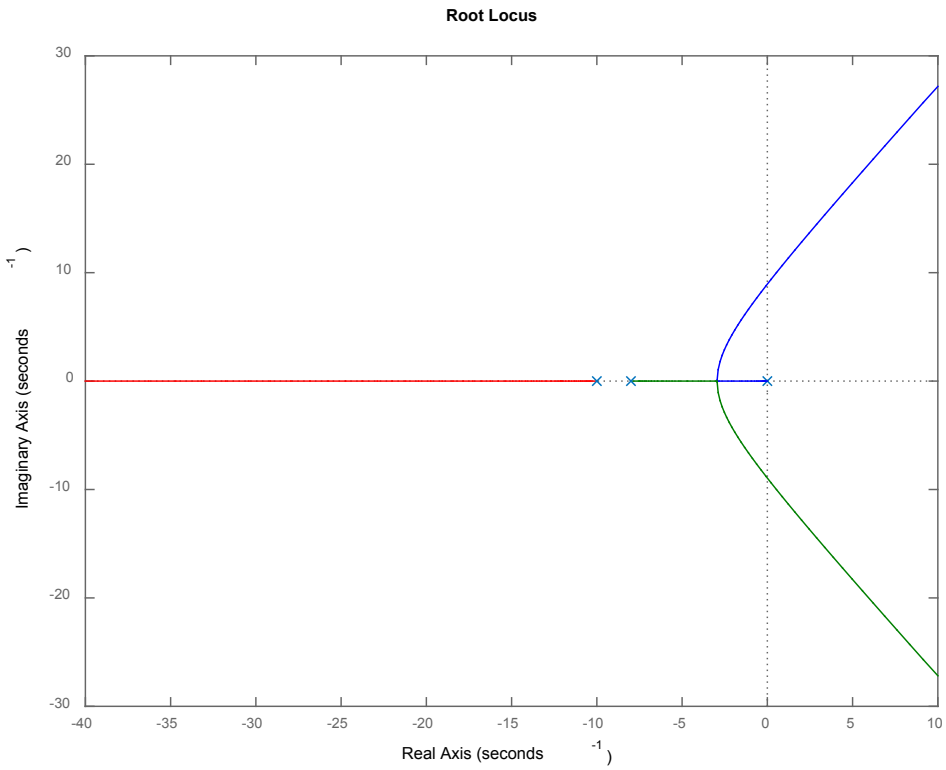
How we draw Root locus:



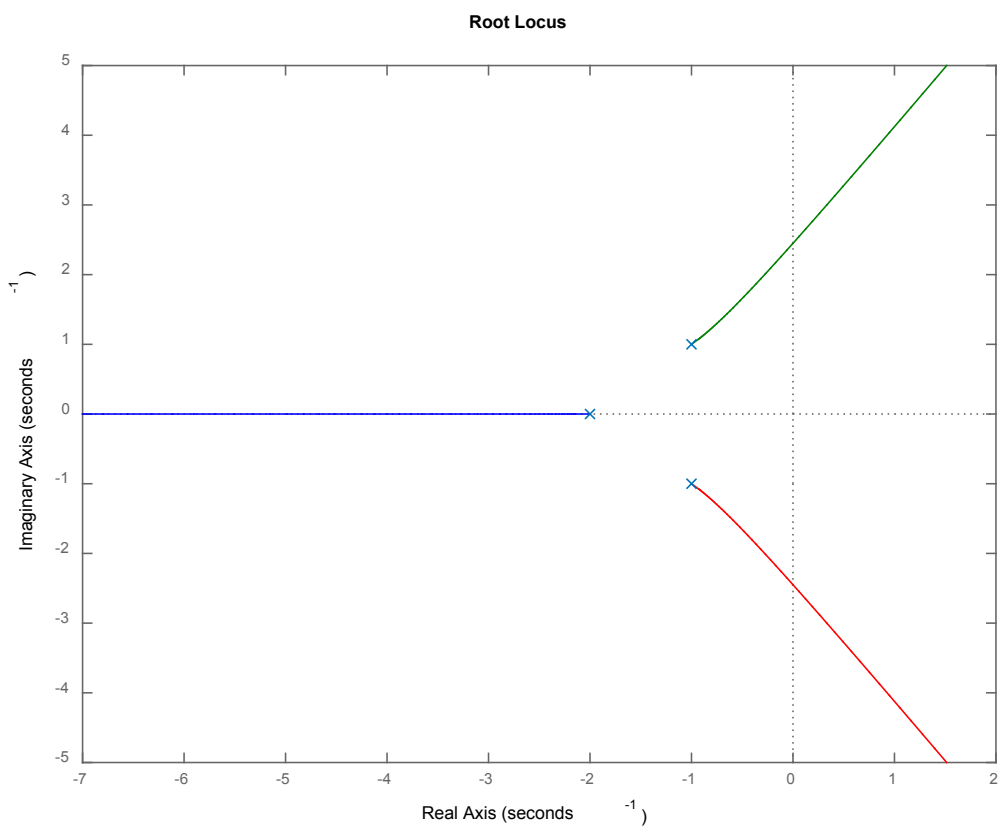
See Matlab plot to confirm above plot is correct!

Parts (b) – (d) are solved using same steps as above! Matlab results are provided below as solutions.

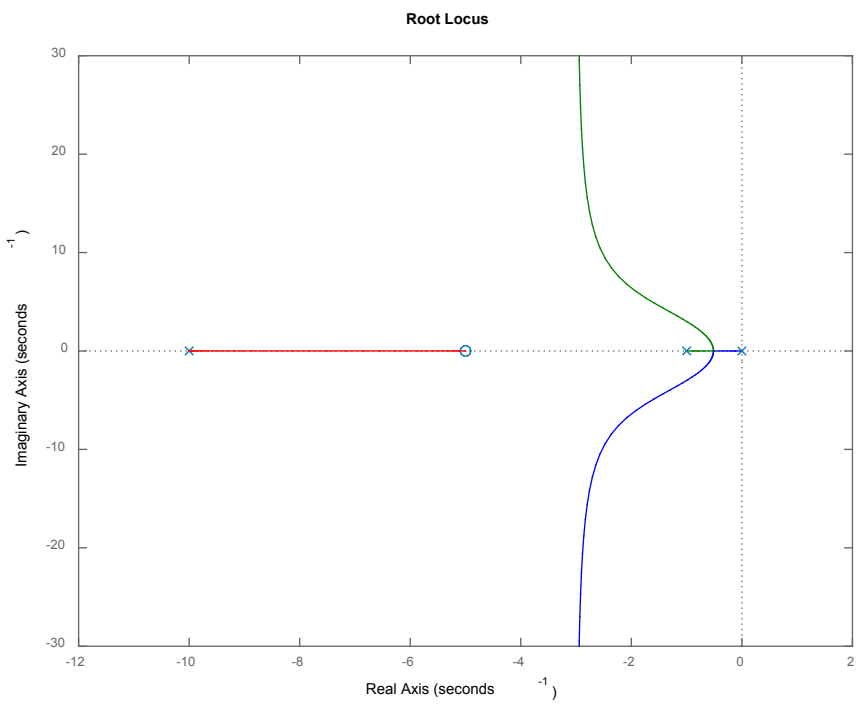
(a)


Root Locus
Imaginary Axis (seconds -1)
Real Axis (seconds -1)
30
20
10
0
-10
-20
-30
-40
-35
-30
-25
-20
-15
-10
-5
0
5
10

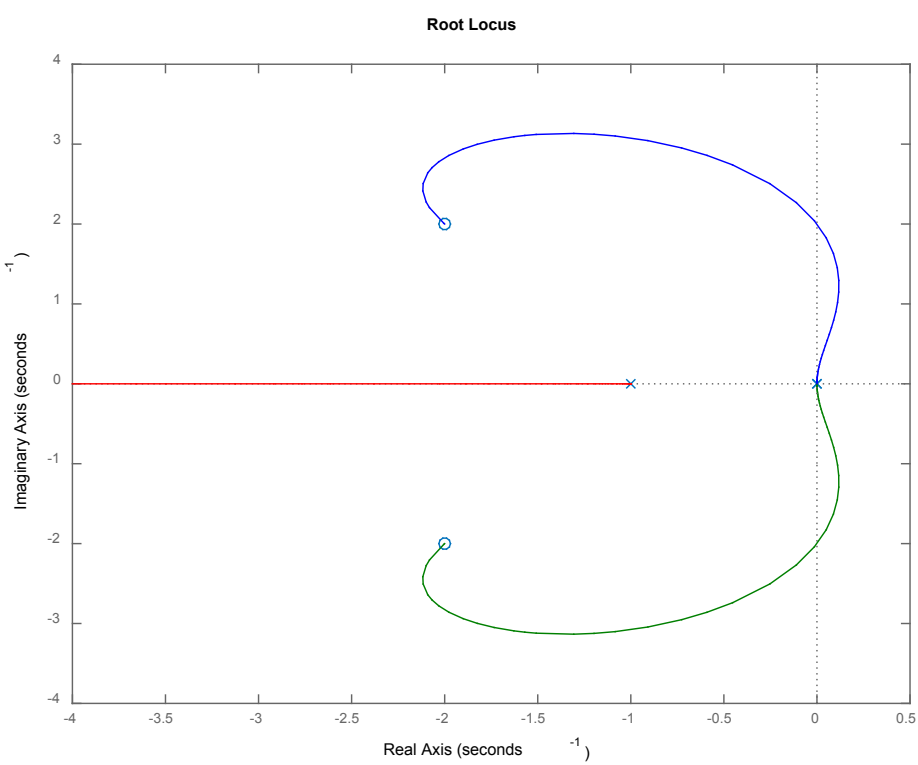
(b)


Root Locus
Imaginary Axis (seconds -1)
Real Axis (seconds -1)
5
4
3
2
1
0
-1
-2
-3
-4
-5
-7
-6
-5
-4
-3
-2
-1
0
1
2

(c)


Root Locus
Imaginary Axis (seconds -1)
Real Axis (seconds -1)
30
20
10
0
-10
-20
-30
-12
-10
-8
-6
-4
-2
0
2

(d)


Root Locus
Imaginary Axis (seconds -1)
Real Axis (seconds -1)
4
3
2
1
0
-1
-2
-3
-4
-4
-3.5
-3
-2.5
-2
-1.5
-1
-0.5
0
0.5

Problem 2

The yaw-control dynamics for an attack jet has an open-loop transfer function given by

$$G(s) = \frac{1}{s(s+3)(s^2+2s+2)}$$

Suppose you use negative feedback using proportional gain K. (a) sketch the root locus by hand and show all of the steps including finding the break-away points. (b) Find the location where the root locus crosses the imaginary axis (marginally stable case) and the gain K required for those roots to be at that location. Show all your steps!

Step 1

zeros: none $\quad m = 0$

poles: $s(s+3)(s^2+2s+2) = 0$

$$\left.\begin{aligned} s &= 0 \\ s &= -3 \\ s &= -1 \pm j \end{aligned}\right\} n = 4$$

Step 2 see plot below

Step 3 # asymptotes: 4

$$\phi_1 = \frac{180^\circ + 360(1-1)}{4} = 45^\circ$$

$$\phi_2 = \frac{180^\circ + 360(2-1)}{4} = 135^\circ$$

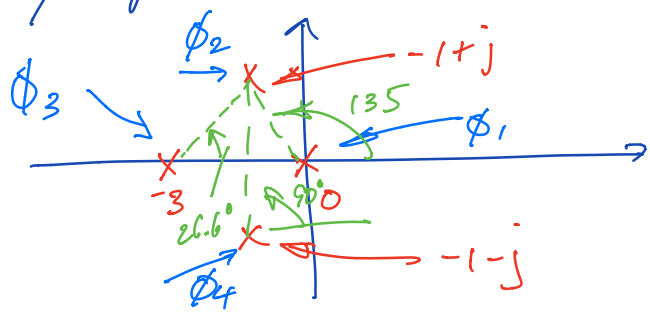
$$\phi_3 = \frac{180^\circ + 360^\circ(3-1)}{4} = 225^\circ$$

$$\phi_4 = \frac{180^\circ + 360^\circ(4-1)}{4} = 315^\circ$$

$$\alpha = \frac{\sum Re(P_i) - \sum Re(z_j)}{4} = \frac{-3-1-1}{4}$$

$$\alpha = -5/4$$

Step 4 Angle of departure: $h=1$ (no repeats)



$$\phi_1 = \sum \psi_i - \sum \phi_j - 180^\circ - 360^\circ (h-1), \quad h=1$$

$$\Rightarrow \phi_1 = 0 - (0) - 180^\circ - 0 = \underline{-180^\circ}$$

$$\phi_2 = 0 - (135^\circ + 26.6^\circ + 90) - 180 - 0 = \underline{-71.6^\circ}$$

$$\phi_3 = 0 - (180^\circ) - 180^\circ - 0 = \underline{0^\circ}$$

$$\phi_4 = -\phi_2 = \underline{+71.6^\circ}$$

Step 5 Apply R-H approach, we find that

$$K \approx 7.8$$

$$\omega_d \approx 1.08 \text{ rad/s}$$

Step 6 Find breakaway points / break in points

$$\frac{d}{ds}(K) = \frac{d}{ds}\left(-\frac{1}{\hat{G}(s)}\right)\Bigg|_{s=s_0} = \frac{1}{b^2}\left[b\frac{da}{ds} - a\frac{db}{ds}\right]\Bigg|_{s=s_0} = 0$$

$$\hat{G}(s) = \frac{1}{s(s+3)(s^2+2s+2)} = \frac{a(s)}{b(s)} \; ; \; b(s) = s^4 + 5s^3 + 8s^2 + 6s$$

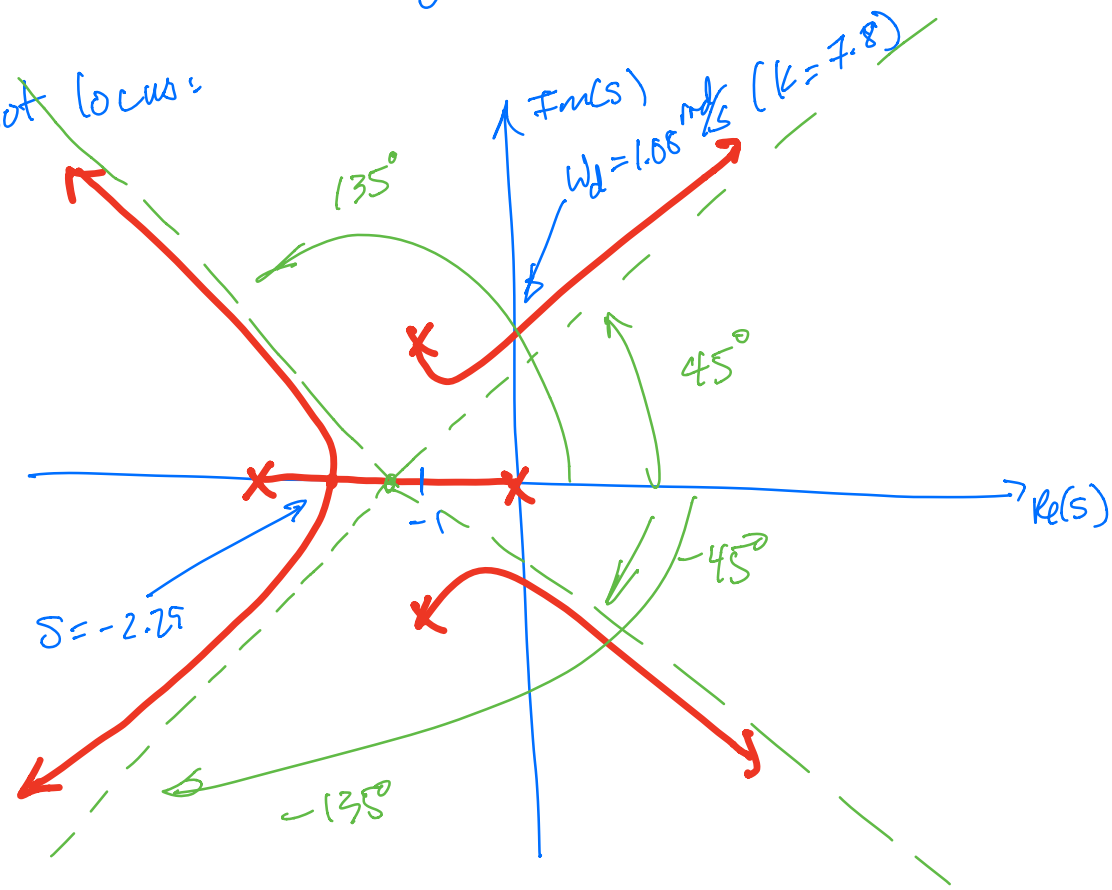
$$\Rightarrow \frac{d}{ds}(K) = \frac{1}{b^2}\left[0 - 1\cdot\left(4s^3 + 15s^2 + 16s + 6\right)\right] = 0$$

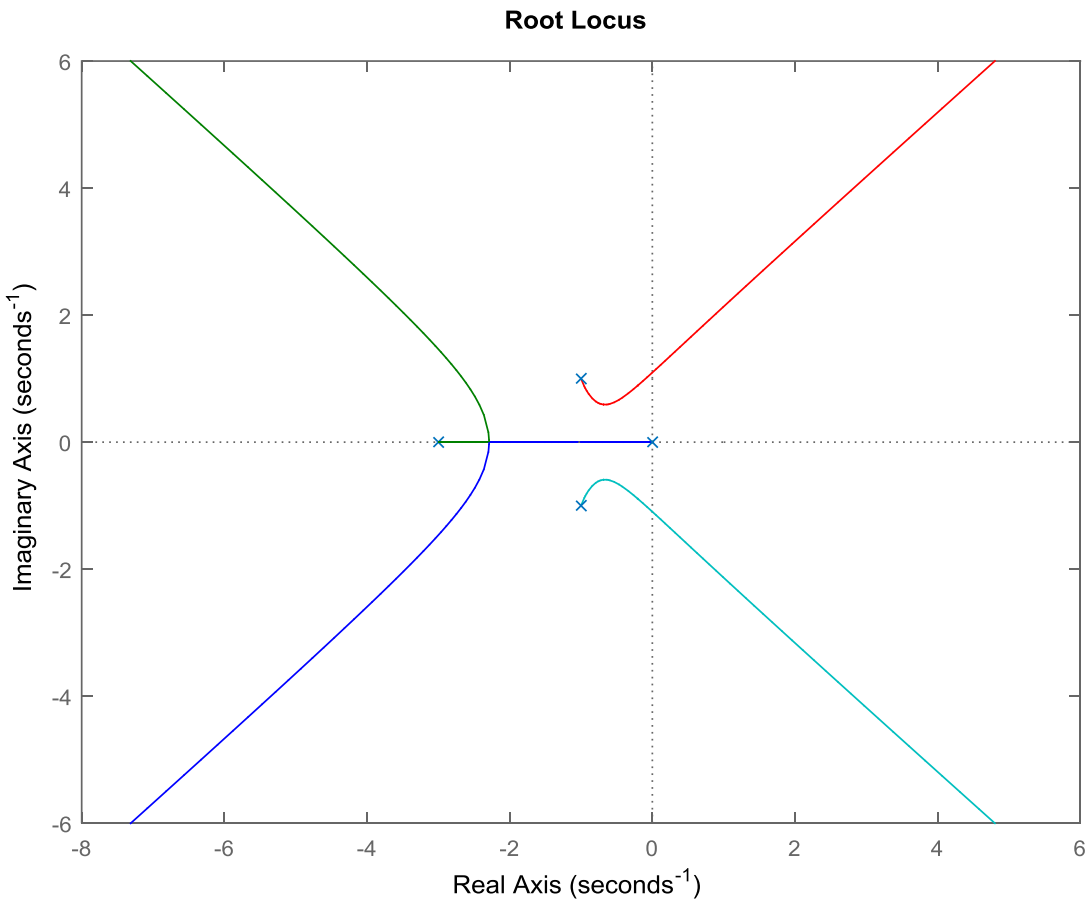
$$\Rightarrow \quad 4s^3 + 15s^2 + 16s + 6 = 0 \qquad \text{solve using Matlab}$$

$$\Rightarrow \quad \underline{\underline{s = -2.29}} \quad \text{and} \quad s = -0.73 \pm 0.349j$$

Break Away pt.!

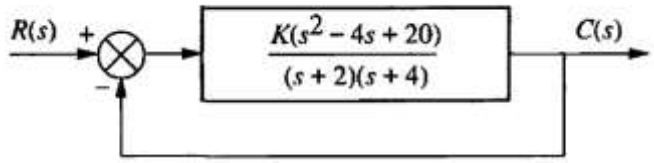
Root locus:




Root Locus
Imaginary Axis (seconds^{-1})
Real Axis (seconds^{-1})
6
4
2
0
-2
-4
-6
-8
-6
-4
-2
0
2
4
6

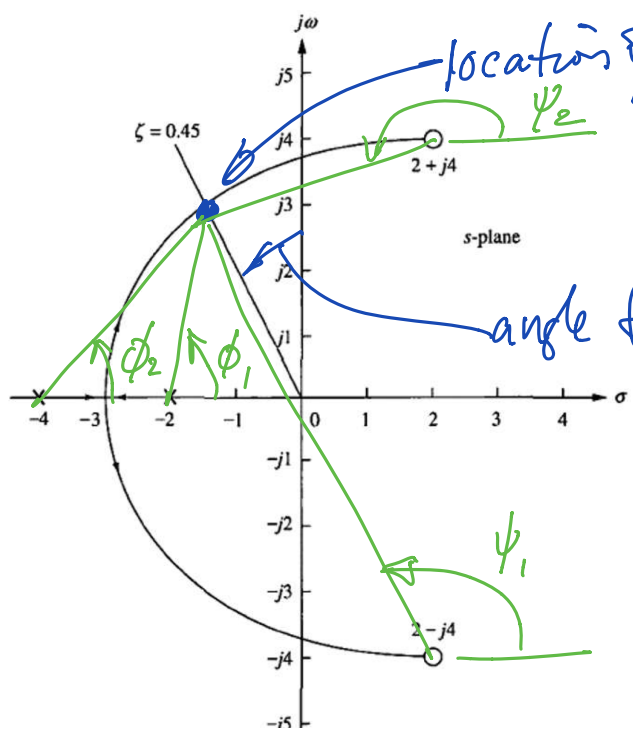
Problem 3 (Design problem)

Consider the following closed-loop system. Sketch the root locus by hand and then find:



a. The exact point and gain where the locus crosses the 0.45 damping ratio line
b. The exact point and gain where the locus crosses the $j\omega$-axis
c. The breakaway point on the real axis
d. The range of K within which the system is stable

(a) First, sketch root locus as shown below:



$\theta = \sin^{-1}(0.45)$

$\theta = 26.74^\circ$

To find the point where line cross root locus, we use trial and error w/ angle condition. Search along the line by trying to satisfy:

$$\angle \hat{G}(s) = -180^\circ + 360^\circ(l) \quad \text{for } l = 0, \pm 1, \ldots$$

$$\Rightarrow \sum \psi_i - \sum \phi_i = 180^\circ + 360^\circ(l)$$

As you move along the line, determine whether

$$\psi_1 + \psi_2 + \phi_1 + \phi_2 = 180^\circ + 360^\circ(\ell)$$

This process is iterative and solution is:

$$\underline{\underline{s_0 = -1.53 + 3.84j}}$$

(b) Solve using R-H table or you can search along the $j\omega$-axis like in part (a):

$$\Rightarrow \quad s = \pm 3.9j \quad \text{where} \quad \underline{\underline{\omega_d = 3.9 \text{ rad/s}}}$$

and $\underline{\underline{K = 1.5}}$

(c) Break away points:

$$\frac{d}{ds}(K) = \frac{d}{ds}\left(-\frac{1}{\hat{G}(s)}\right)\Bigg|_{s=s_0} = \frac{1}{b^2}\left[b\frac{da}{ds} - a\frac{db}{ds}\right]\Bigg|_{s=s_0} = 0$$

$$a(s) = s^2 - 4s + 20$$

$$b(s) = s^2 + 6s + 8$$

$$\Rightarrow \frac{1}{b^2}\left[b\left[2s-4\right] - \left[s^2 - 4s + 20\right](2s+6)\right] = 0$$

$$\frac{2s-4}{b} - \frac{(s^2-4s+20)(2s+6)}{b^2} = 0$$

$$(2s-4)(s^2+6s+8) - (s^2-4s+20)(2s+6) = 0$$

Solving for s using Matlab, we get:

$$\cancel{2s^3} + \cancel{8s^2} - \cancel{8s} - 32 - \cancel{2s^3} + \cancel{2s^2} - \cancel{16s} - 120 = 0$$

$$10s^2 - 24s - 152 = 0$$

$$\boxed{s = -2.88} \text{ and } 5.28$$

↑ Break away point

(d) Solve using R-H approach:

answer is

$0 < K < 1.5$ for stability!