Chapter 3

Dynamic Response

Problems and Solutions for Section 3.1: Review of Laplace Transforms

1. Show that, in a partial-fraction expansion, complex conjugate poles have coefficients that are also complex conjugates. (The result of this relationship is that whenever complex conjugate pairs of poles are present, only one of the coefficients needs to be computed.)

   Solution:

   Consider the second-order system with poles at $-\alpha \pm j\beta$,

   $$H(s) = \frac{1}{(s+\alpha+j\beta)(s+\alpha-j\beta)}.$$

   Perform Partial Fraction Expansion:

   $$\begin{aligned} H(s) &= \frac{C_1}{s+\alpha+j\beta} + \frac{C_2}{s+\alpha-j\beta}. \\ C_1 &= \frac{1}{s+\alpha-j\beta}|_{s=-\alpha-j\beta} = \frac{1}{2\beta}j, \\ C_2 &= \frac{1}{s+\alpha+j\beta}|_{s=-\alpha+j\beta} = -\frac{1}{2\beta}j, \\ &\therefore \quad C_1 = C_2^*. \end{aligned}$$

2. Find the Laplace transform of the following time functions:

   (a) $f(t) = 1 + 2t$

   (b) $f(t) = 3 + 7t + t^2 + \delta(t)$, where $\delta(t)$ is the unit impulse function

   (c) $f(t) = e^{-t} + 2e^{-2t} + te^{-3t}$

   (d) $f(t) = (t+1)^2$

(e) $f(t) = \sinh t$

Solution:

(a)

$$
\begin{aligned}
f(t) &= 1 + 2t. \\
L\{f(t)\} &= L\{1(t)\} + L\{2t\}, \\
&= \frac{1}{s} + \frac{2}{s^2}, \\
&= \frac{s+2}{s^2}.
\end{aligned}
$$

We can verify the answer using MATLAB:
> > laplace(1+2*t)
ans =
(2+s)/sˆ2

(b)

$$
\begin{aligned}
f(t) &= 3 + 7t + t^2 + \delta(t), \\
L\{f(t)\} &= L\{3\} + L\{7t\} + L\{t^2\} + L\{\delta(t)\}, \\
&= \frac{3}{s} + \frac{7}{s^2} + \frac{2!}{s^3} + 1, \\
&= \frac{s^3 + 3s^2 + 7s + 2}{s^3}.
\end{aligned}
$$

We can verify the answer using MATLAB:
> > syms s t
> > laplace(3+7*t+tˆ2+dirac(t))
ans =
1+3/s+7/sˆ2+2/sˆ3

(c)

$$
\begin{aligned}
f(t) &= e^{-t} + 2e^{-2t} + te^{-3t}, \\
L\{f(t)\} &= L\{e^{-t}\} + L\{2e^{-2t}\} + L\{te^{-3t}\}, \\
&= \frac{1}{s+1} + \frac{2}{s+2} + \frac{1}{(s+3)^2}.
\end{aligned}
$$

We can verify the answer using MATLAB:
> > syms s t
> > laplace(exp(-t)+2*exp(-2*t)+t*exp(-3*t))
ans =
1/(1+s)+2/(2+s)+1/(s+3)ˆ2

(d)

$$\begin{aligned} f(t) &= (t+1)^2, \\ &= t^2 + 2t + 1. \\ L\{f(t)\} &= L\{t^2\} + L\{2t\} + L\{1\}, \\ &= \frac{2!}{s^3} + \frac{2}{s^2} + \frac{1}{s}, \\ &= \frac{s^2+2s+2}{s^3}. \end{aligned}$$

We can verify the answer using MATLAB:

> > laplace((t+1)ˆ2)

ans =

(2+2*s+sˆ2)/sˆ3

(e) Using the trigonometric identity,

$$\begin{aligned} f(t) &= \sinh t, \\ &= \frac{e^t - e^{-t}}{2}, \\ L\{f(t)\} &= L\{\frac{e^t}{2}\} - L\{\frac{e^{-t}}{2}\}, \\ &= \frac{1}{2}(\frac{1}{s-1}) - \frac{1}{2}(\frac{1}{s+1}), \\ &= \frac{1}{s^2-1}. \end{aligned}$$

We can verify the answer using MATLAB:

> > syms s t

> > laplace(sinh(t))

ans =

1/(sˆ2-1)

Remark: A useful reference for this problem and the next several problems is: B. R. Hunt, R. L. Lipsman, J. E. Osborn, J. M. Rosenberg, *Differential Equations with* MATLAB, Wiley, Third Edition, 2012.

3. Find the Laplace transform of the following time functions:

(a) $f(t) = 3\cos 6t$

(b) $f(t) = \sin 2t + 2\cos 2t + e^{-t}\sin 2t$

(c) $f(t) = t^2 + e^{-2t}\sin 3t$

Solution:

(a)

$$\begin{aligned} f(t) &= 3\cos 6t \\ L\{f(t)\} &= L\{3\cos 6t\} \\ &= 3\frac{s}{s^2+36}. \end{aligned}$$

We can verify the answer using Matlab:
```
> > syms s t
> > laplace(3*cos(6*t))
ans =
3*s/(s^2+36)
```

(b)

$$\begin{aligned} f(t) &= \sin 2t + 2\cos 2t + e^{-t}\sin 2t \\ &= L\{f(t)\} = L\{\sin 2t\} + L\{2\cos 2t\} + L\{e^{-t}\sin 2t\}, \\ &= \frac{2}{s^2+4} + \frac{2s}{s^2+4} + \frac{2}{(s+1)^2+4}. \end{aligned}$$

We can verify the answer using Matlab:
```
> > syms s t
> > laplace(sin(2*t)+2*cos(2*t)+exp(-t)*sin(2*t))
ans =
2*(4*s^2+7*s+9+s^3)/(s^2+4)/(s^2+2*s+5)
```

(c)

$$\begin{aligned} f(t) &= t^2 + e^{-2t}\sin 3t, \\ &= L\{f(t)\} = L\{t^2\} + L\{e^{-2t}\sin 3t\}, \\ &= \frac{2!}{s^3} + \frac{3}{(s+2)^2+9}, \\ &= \frac{2}{s^3} + \frac{3}{(s+2)^2+9}. \end{aligned}$$

We can verify the answer using Matlab:
```
> > syms s t
> > laplace(t^2+exp(-2*t)*sin(3*t))
ans =
2/s^3+3/(s^2+4*s+13)
```

4. Find the Laplace transform of the following time functions:

   (a) $f(t) = t\sin t$

(b) $f(t) = t\cos 3t$
(c) $f(t) = te^{-t} + 2t\cos t$
(d) $f(t) = t\sin 3t - 2t\cos t$
(e) $f(t) = 1(t) + 2t\cos 2t$

Solution:

(a)

$$\begin{aligned} f(t) &= t\sin t \\ L\{f(t)\} &= L\{t\sin t\} \end{aligned}$$

Use multiplication by time Laplace transform property (Table A.1, entry #11),

$$\begin{aligned} L\{tg(t)\} &= -\frac{d}{ds}G(s). \\ \text{Let } g(t) &= \sin t \quad \text{and use} \quad L\{\sin at\} = \frac{a}{s^2+a^2}. \\ L\{t\sin t\} &= -\frac{d}{ds}(\frac{1}{s^2+1^2}), \\ &= \frac{2s}{(s^2+1)^2}, \\ &= \frac{2s}{s^4+2s^2+1}. \end{aligned}$$

We can verify the answer using MATLAB:
> > syms s t
> > laplace(t*sin(t))
ans =
2*s/(s^2+1)^2

(b)

$$f(t) = t\cos 3t$$

Use multiplication by time Laplace transform property (Table A.1, entry #11),

$$\begin{aligned} L\{tg(t)\} &= -\frac{d}{ds}G(s). \\ \text{Let } g(t) &= \cos 3t \quad \text{and use} \quad L\{\cos at\} = \frac{s}{s^2+a^2}. \\ L\{t\cos 3t\} &= -\frac{d}{ds}(\frac{s}{s^2+9}), \\ &= \frac{-[(s^2+9)-(2s)s]}{(s^2+9)^2}, \\ &= \frac{s^2-9}{s^4+18s^2+81}. \end{aligned}$$

We can verify the answer using MATLAB:

> > syms s t

> > laplace(t*cos(3*t))

ans =

(s^2-9)/(s^2+9)^2

(c)

$$f(t) = te^{-t} + 2t\cos t$$

Use the following Laplace transforms and properties (Table A.1, entries 4, 11, and 3),

$$\begin{aligned}
L\{te^{-at}\} &= \frac{1}{(s+a)^2}, \\
L\{tg(t)\} &= -\frac{d}{ds}G(s), \\
L\{\cos at\} &= \frac{s}{s^2+a^2}, \\
L\{f(t)\} &= L\{te^{-t}\} + 2L\{t\cos t\}, \\
&= \frac{1}{(s+1)^2} + 2\left(-\frac{d}{ds}\frac{s}{s^2+1}\right), \\
&= \frac{1}{(s+1)^2} - 2\left[\frac{(s^2+1)-(2s)s}{(s^2+1)^2}\right], \\
&= \frac{1}{(s+1)^2} + \frac{2(s^2-1)}{(s^2+1)^2}.
\end{aligned}$$

We can verify the answer using MATLAB:

> > syms s t

> > laplace(t*exp(-t)+2*t*cos(t))

ans =

1/(1+s)^2+2*(s^2-1)/(s^2+1)^2

(d)

$$f(t) = t\sin 3t - 2t\cos t.$$

Use the following Laplace transforms and properties (Table A.1, entries 11, 3),

$$\begin{aligned}
L\{tg(t)\} &= -\frac{d}{ds}G(s), \\
L\{\sin at\} &= \frac{a}{s^2+a^2}, \\
L\{\cos at\} &= \frac{s}{s^2+a^2}, \\
L\{f(t)\} &= L\{t\sin 3t\} - 2L\{t\cos t\}, \\
&= -\frac{d}{ds}[\frac{3}{s^2+9}] - 2(-\frac{d}{ds}[\frac{s}{s^2+1}]), \\
&= \frac{(2s\times 3)}{(s^2+9)^2} + 2\frac{[(s^2+1)-(2s)s]}{(s^2+1)^2}, \\
&= \frac{6s}{(s^2+9)^2} - \frac{2(s^2-1)}{(s^2+1)^2}.
\end{aligned}$$

We can verify the answer using MATLAB:

> > syms s t

> > laplace(t*sin(3*t)-2*t*cos(t))

ans =

6*s/(sˆ2+9)ˆ2-2*(sˆ2-1)/(sˆ2+1)ˆ2

(e)

$$\begin{aligned}
f(t) &= 1(t) + 2t\cos 2t, \\
L\{1(t)\} &= \frac{1}{s}, \\
L\{tg(t)\} &= -\frac{d}{ds}G(s), \\
L\{\cos at\} &= \frac{s}{s^2+a^2}, \\
L\{f(t)\} &= L\{1(t)\} + 2L\{t\cos 2t\}, \\
&= \frac{1}{s} + 2(-\frac{d}{ds}\frac{s}{s^2+4}), \\
&= \frac{1}{s} - 2[\frac{(s^2+4)-(2s)s}{(s^2+4)^2}], \\
&= \frac{1}{s} - 2\frac{(-s^2+4)}{(s^2+4)^2}.
\end{aligned}$$

We can verify the answer using MATLAB:

> > syms s t

> > laplace(1+2*t*cos(2*t))

ans =

1/s+2*(sˆ2-4)/(sˆ2+4)ˆ2

5. Find the Laplace transform of the following time functions (* denotes convolution):

   (a) $f(t) = \sin t \sin 3t$

   (b) $f(t) = \sin^2 t + 3\cos^2 t$

   (c) $f(t) = (\sin t)/t$

   (d) $f(t) = \sin t * \sin t$

   (e) $f(t) = \int_0^t \cos(t-\tau)\sin\tau d\tau$

Solution:

(a)

$$f(t) = \sin t \sin 3t.$$

Use the trigonometric relation,

$$\begin{aligned}
\sin\alpha t \sin\beta t &= \frac{1}{2}\cos(|\alpha-\beta|t) - \frac{1}{2}\cos(|\alpha+\beta|t), \\
\alpha &= 1 \quad \text{and} \quad \beta = 3. \\
f(t) &= \frac{1}{2}\cos(|1-3|t) - \frac{1}{2}\cos(|1+3|t), \\
&= \frac{1}{2}\cos 2t - \frac{1}{2}\sin 4t. \\
L\{f(t)\} &= \frac{1}{2}L\{\cos 2t\} - \frac{1}{2}L\{\cos 4t\}, \\
&= \frac{1}{2}[\frac{s}{s^2+4} - \frac{s}{s^2+16}]. \\
&= \frac{6s}{(s^2+4)(s^2+16)}.
\end{aligned}$$

We can verify the answer using MATLAB:

```
>> syms s t
>> laplace(sin(t)*sin(3*t))
ans =
6*s/(s^2+16)/(s^2+4)
```

(b)

$$f(t) = \sin^2 t + 3\cos^2 t.$$

Use the trigonometric formulas,

$$\begin{aligned}
\sin^2 t &= \frac{1-\cos 2t}{2}, \\
\cos^2 t &= \frac{1+\cos 2t}{2}, \\
f(t) &= \frac{1-\cos 2t}{2} + 3(\frac{1+\cos 2t}{2}), \\
&= 2+\cos 2t. \\
L\{f(t)\} &= L\{2\} + L\{\cos 2t\} \\
&= \frac{2}{s} + \frac{s}{s^2+4}, \\
&= \frac{3s^2+8}{s(s^2+4)}.
\end{aligned}$$

We can verify the answer using MATLAB:

```
> > syms s t
> > laplace(sin(t)^2+3*cos(t)^2)
ans =
(8+3*s^2)/s/(s^2+4)
```

(c) We first show the result that division by time is equivalent to integration in the frequency domain. This can be done as follows,

$$\begin{aligned}
F(s) &= \int_0^\infty e^{-st} f(t)dt, \\
\int_s^\infty F(s)ds &= \int_s^\infty [\int_0^\infty e^{-st} f(t)dt]ds,
\end{aligned}$$

Interchanging the order of integration,

$$\begin{aligned}
\int_s^\infty F(s)ds &= \int_0^\infty [\int_s^\infty e^{-st} ds] f(t)dt, \\
\int_s^\infty F(s)ds &= \int_0^\infty [-\frac{1}{t}e^{-st}]_s^\infty f(t)dt, \\
&= \int_0^\infty \frac{f(t)}{t} e^{-st} dt.
\end{aligned}$$

Using this result then,

$$\begin{aligned}
L\{\sin t\} &= \frac{1}{s^2+1}, \\
L\{\left\{\frac{\sin t}{t}\right\}\} &= \int_s^\infty \frac{1}{\xi^2+1} d\xi, \\
&= \tan^{-1}(\infty) - \tan^{-1}(s), \\
&= \frac{\pi}{2} - \tan^{-1}(s), \\
&= \tan^{-1}(\left(\frac{1}{s}\right)).
\end{aligned}$$

where a table of integrals was used and the last simplification follows from the related trigonometric identity.

(d)

$$f(t) = \sin t * \sin t.$$

Use the convolution Laplace transform property (Table A.1, entry # 7),

$$\begin{aligned} L\{\sin t * \sin t\} &= (\frac{1}{s^2+1})(\frac{1}{s^2+1}), \\ &= \frac{1}{s^4+2s^2+1}. \end{aligned}$$

(e)

$$\begin{aligned} f(t) &= \int_0^t \cos(t-\tau)\sin\tau d\tau. \\ L\{f(t)\} &= L\{\int_0^t \cos(t-\tau)\sin\tau d\tau\} = L\{\cos(t) * \sin(t)\}. \end{aligned}$$

This is just the definition of the convolution theorem,

$$\begin{aligned} L\{f(t)\} &= \frac{s}{s^2+1}\frac{1}{s^2+1}, \\ &= \frac{s}{s^4+2s^2+1}. \end{aligned}$$

6. Given that the Laplace transform of *f(t)* is *F(s)*, find the Laplace transform of the following:

(a) $g(t) = f(t)\cos t$

(b) $g(t) = \int_0^t \int_0^{t_1} f(\tau)d\tau dt_1$

Solution:

(a) First write $\cos t$ in terms of the related Euler identity (Appendix WA, Eq. 33),

$$g(t) = f(t)\cos t = f(t)\frac{e^{jt} + e^{-jt}}{2} = \frac{1}{2}f(t)e^{jt} + \frac{1}{2}f(t)e^{-jt}.$$

Then using entry #4 of Table A.1 we have,

$$G(s) = \frac{1}{2}F(s-j) + \frac{1}{2}F(s+j) = \frac{1}{2}[F(s-j) + F(s+j)].$$

(b) Let us define

$$\tilde{f}(t_1) = \int_0^{t_1} f(\tau)d\tau,$$

then

$$g(t) = \int_0^{t} \tilde{f}(t_1)dt_1,$$

and from entry #6 of Table A.1 we have

$$L\{\tilde{f}(t)\} = \tilde{F}(s) = \frac{1}{s}F(s)$$

and using the same result again, we have

$$G(s) = \frac{1}{s}\tilde{F}(s) = \frac{1}{s}[\frac{1}{s}F(s)] = \frac{1}{s^2}F(s).$$

7. Find the time function corresponding to each of the following Laplace transforms using partial fraction expansions:

(a) $F(s) = \frac{2}{s(s+2)}$

(b) $F(s) = \frac{10}{s(s+1)(s+10)}$

(c) $F(s) = \frac{3s+2}{s^2+4s+20}$

(d) $F(s) = \frac{3s^2+9s+12}{(s+2)(s^2+5s+11)}$

(e) $F(s) = \frac{1}{s^2+4}$

(f) $F(s) = \frac{2(s+2)}{(s+1)(s^2+4)}$

(g) $F(s) = \frac{s+1}{s^2}$

(h) $F(s) = \frac{1}{s^6}$

(i) $F(s) = \frac{4}{s^4+4}$

(j) $F(s) = \frac{e^{-s}}{s^2}$

Solution:

(a) Perform partial fraction expansion,

$$
\begin{aligned}
F(s) &= \frac{2}{s(s+2)}, \\
&= \frac{C_1}{s} + \frac{C_2}{s+2}. \\
C_1 &= \frac{2}{s+2}|_{s=0} = 1, \\
C_2 &= \frac{2}{s}|_{s=-1} = -2, \\
F(s) &= \frac{1}{s} - \frac{2}{s+1}. \\
L^{-1}\{F(s)\} &= L^{-1}\{\frac{1}{s}\} - L^{-1}\{\frac{2}{s+1}\}, \\
f(t) &= 1(t) - 2e^{-t}1(t).
\end{aligned}
$$

We can verify the answer using MATLAB:

```
> > syms s t
> > ilaplace(2/(s*(s+2)))
ans =
1-exp(-2*t)
```

(b) Perform partial fraction expansion,

$$
\begin{aligned}
F(s) &= \frac{10}{s(s+1)(s+10)}, \\
&= \frac{C_1}{s} + \frac{C_2}{s+1} + \frac{C_3}{s+10}. \\
C_1 &= \frac{10}{(s+1)(s+10)}|_{s=0} = 1, \\
C_2 &= \frac{10}{s(s+10)}|_{s=-1} = -\frac{10}{9}, \\
C_3 &= \frac{10}{s(s+1)}|_{s=-10} = \frac{1}{9}, \\
F(s) &= \frac{1}{s} - \frac{\frac{10}{9}}{s+1} + \frac{\frac{1}{9}}{s+10}, \\
f(t) &= L^{-1}\{F(s)\} = 1(t) - \frac{10}{9}e^{-t}1(t) + \frac{1}{9}e^{-10t}1(t).
\end{aligned}
$$

We can verify the answer using MATLAB:

```
> > syms s t
> > ilaplace(10/(s*(s+1)*(s+10)))
ans =
-10/9*exp(-t)+1+1/9*exp(-10*t)
```

(c) Re-write and carry out partial fraction expansion,

$$\begin{aligned}
F(s) &= \frac{3s+2}{s^2+4s+20}, \\
&= 3\frac{(s+2)-4/3}{(s+2)^2+4^2}, \\
&= \frac{3(s+2)}{(s+2)^2+4^2} - \frac{4}{(s+2)^2+4^2}, \\
f(t) &= L^{-1}\{F(s)\} = (3e^{-2t}\cos 4t - e^{-2t}\sin 4t)1(t).
\end{aligned}$$

We can verify the answer using MATLAB:
```
>> syms s t
>> ilaplace((3*s+2)/(s^2+4*s+20))
ans =
exp(-2*t)*(3*cos(4*t)-sin(4*t))
```

(d) Perform partial fraction expansion,

$$\begin{aligned}
F(s) &= \frac{3s^2+9s+12}{(s+2)(s^2+5s+11)} \\
&= \frac{C_1}{s+2} + \frac{C_2 s + C_3}{s^2+5s+11} \\
C_1 &= \frac{(3s^2+9s+12)}{(s^2+5s+11)}|_{s=-2} = \frac{6}{5}.
\end{aligned}$$

Equate numerators:

$$\begin{gathered}
\frac{\frac{6}{5}}{(s+2)} + \frac{C_2 s + C_3}{(s^2+5s+11)} = \frac{3s^2+9s+12}{(s+2)(s^2+5s+11)}, \\
(C_2 + \frac{6}{5})s^2 + (6 + C_3 + 2C_2)s + (2C_3 + \frac{66}{5}) = 3s^2 + 6s + 6.
\end{gathered}$$

Equate like powers of s to find C_2 and C_3:

$$\begin{aligned}
C_2 + \frac{6}{5} &= 3 \Rightarrow C_2 = \frac{9}{5}, \\
2C_3 + \frac{66}{5} &= 12 \Rightarrow C_3 = \frac{-3}{5}, \\
F(s) &= \frac{\frac{6}{5}}{(s+2)} + \frac{\frac{9}{5}s - \frac{3}{5}}{(s^2+5s+11)}, \\
&= \frac{\frac{6}{5}}{(s+2)} + \frac{9}{5}\frac{s+2.5}{(s+2.5)^2+4.75} + \frac{2.34\sqrt{4.75}}{(s+2.5)^2+4.75}. \\
f(t) &= L^{-1}\{F(s)\} = (\frac{6}{5}e^{-2t} + \frac{9}{5}e^{-2.5t}cos(\sqrt{4.75}t) - 3.9e^{-2.5t}sin(\sqrt{4.75}t)1(t) \\
&= (\frac{6}{5}e^{-2t} + \frac{9}{5}e^{-2.5t}cos(\sqrt{4.75}t) - 2.34e^{-2.5t}sin(\sqrt{4.75}t)])1(t).
\end{aligned}$$

We can verify the answer using MATLAB:

> > syms s t

> > ilaplace((3*s^2+9*s+12)/((s+2)*(s^2+5*s+11)))

ans =

6/5*exp(-2*t)+3/95*(57*cos(1/2*19^(1/2)*t)-17*19^(1/2)*sin(1/2*19^(1/2)*t))*exp(-5/2*t)

(e) Re-write and use entry #17 of Table A.2,

$$\begin{aligned} F(s) &= \frac{1}{s^2+4}. \\ &= \frac{1}{2}\frac{2}{(s^2+2^2)}. \\ f(t) &= \frac{1}{2}\sin 2t. \end{aligned}$$

We can verify the answer using MATLAB:

> > syms s t

> > ilaplace(1/(s^2+4))

ans =

1/2*sin(2*t)

(f)

$$\begin{aligned} F(s) &= \frac{2(s+2)}{(s+1)(s^2+4)}. \\ &= \frac{C_1}{(s+1)} + \frac{C_2 s + C_3}{(s^2+4)}. \\ C_1 &= \frac{2(s+2)}{(s^2+4)}|_{s=-1} = \frac{2}{5}. \end{aligned}$$

Equate numerators and like powers of s terms:

$$\begin{aligned} (\frac{2}{5}+C_2)s^2 + (C_2+C_3)s + (\frac{8}{5}+C_3) &= 2s+4, \\ \frac{8}{5}+C_3 &= 4 \qquad \Rightarrow C_3 = \frac{12}{5}, \\ C_2+C_3 &= 2 \qquad \Rightarrow C_2 = -\frac{2}{5}, \\ \frac{2}{5}+C_2 &= 0. \end{aligned}$$

$$
\begin{aligned}
F(s) &= \frac{\frac{2}{5}}{(s+1)} + \frac{-\frac{2}{5}s + \frac{12}{5}}{(s^2+4)}, \\
&= \frac{\frac{2}{5}}{(s+1)} + \frac{-\frac{2}{5}s}{(s^2+2^2)} + \frac{6}{5}\frac{2}{(s^2+2^2)}. \\
f(t) &= \frac{2}{5}e^{-t} - \frac{2}{5}\cos 2t + \frac{6}{5}\sin 2t.
\end{aligned}
$$

We can verify the answer using MATLAB:

```
> > syms s t
> > ilaplace(2*(s+2)/((s+1)*(s^2+4)))
ans =
-4/5*cos(t)^2+12/5*sin(t)*cos(t)+2/5+2/5*exp(-t)
```

(g) Perform partial fraction expansion,

$$
\begin{aligned}
F(s) &= \frac{s+1}{s^2}, \\
&= \frac{1}{s} + \frac{1}{s^2}. \\
f(t) &= (1+t)1(t).
\end{aligned}
$$

We can verify the answer using MATLAB:

```
> > syms s t
> > ilaplace((s+1)/(s^2))
ans =
t+1
```

(h) Use entry #6 of Table A.2,

$$
\begin{aligned}
F(s) &= \frac{1}{s^6}, \\
f(t) &= L^{-1}\{\frac{1}{s^6}\} = \frac{t^5}{5!} = \frac{t^5}{120}.
\end{aligned}
$$

We can verify the answer using MATLAB:

```
> > syms s t
> > ilaplace(1/s^6)
ans =
1/120*t^5
```

(i) Re-write as,

$$\begin{aligned}
F(s) &= \frac{4}{s^4+4}, \\
&= \frac{\frac{1}{2}s+1}{s^2+2s+2} + \frac{-\frac{1}{2}s+1}{s^2-2s+2}, \\
&= \frac{(s+1)-\frac{1}{2}s}{(s+1)^2+1} - \frac{(s-1)-\frac{1}{2}s}{(s-1)^2+1}.
\end{aligned}$$

Use Table A.2 entry #19 and Table A.1 entry #5,

$$\begin{aligned}
f(t) &= L^{-1}\{F(s)\} = e^{-t}\cos(t) - \frac{1}{2}\frac{d}{dt}\{e^{-t}\sin(t)\} - e^{t}\cos(t), \\
&\quad -\frac{1}{2}\frac{d}{dt}\{e^{t}\sin(t)\}, \\
&= e^{-t}\cos(t) - \frac{1}{2}\{-e^{-t}\sin(t) + \cos(t)e^{-t}\} \\
&\quad -e^{t}\cos(t) + \frac{1}{2}\{e^{t}\sin(t) + \cos(t)e^{t}\}, \\
&= -\cos(t)\{\left\{\frac{-e^{-t}+e^{t}}{2}\right\}\} + \sin(t)\{\left\{\frac{-e^{-t}+e^{t}}{2}\right\}\}, \\
f(t) &= -\cos(t)\sinh(t) + \sin(t)\cosh(t).
\end{aligned}$$

We can verify the answer using MATLAB:
> > syms s t
> > ilaplace(4/(s^4+4))
ans =
cosh(t)*sin(t)-sinh(t)*cos(t)

(j) Using entry #2 of Table A.1,

$$\begin{aligned}
F(s) &= \frac{e^{-s}}{s^2}. \\
f(t) &= L^{-1}\{F(s)\} = (t-1)1(t-1).
\end{aligned}$$

We can verify the answer using MATLAB:
> > syms s t
> > ilaplace(exp(-s)/(s^2))
ans =
heaviside(t-1)*(t-1)

8. Find the time function corresponding to each of the following Laplace transforms:

(a) $F(s) = \frac{1}{s(s+2)^2}$

(b) $F(s) = \frac{2s^2+s+1}{s^3-1}$

(c) $F(s) = \frac{2(s^2+s+1)}{s(s+1)^2}$

(d) $F(s) = \frac{s^3+2s+4}{s^4-16}$

(e) $F(s) = \frac{2(s+2)(s+5)^2}{(s+1)(s^2+4)^2}$

(f) $F(s) = \frac{(s^2-1)}{(s^2+1)^2}$

(g) $F(s) = \tan^{-1}((\frac{1}{s}))$

Solution:

(a) Perform partial fraction expansion,

$$
\begin{aligned}
F(s) &= \frac{1}{s(s+2)^2}, \\
&= \frac{C_1}{s} + \frac{C_2}{(s+1)} + \frac{C_3}{(s+2)^2}. \\
C_1 &= sF(s)|_{s=0} = \frac{1}{(s+2)^2}|_{s=0} = \frac{1}{4}, \\
C_3 &= (s+2)^2F(s)|_{s=-2} = \frac{1}{s}|_{s=-2} = -\frac{1}{2}, \\
C_2 &= \frac{d}{ds}[(s+2)^2F(s)]_{s=-2}, \\
&= \frac{d}{ds}[s^{-1}]_{s=-2}, \\
&= -\frac{1}{s^2}|_{s=-2}, \\
&= -\frac{1}{4}, \\
F(s) &= \frac{1/4}{s} + \frac{-1/4}{(s+2)} + \frac{-1/2}{(s+2)^2}. \\
f(t) &= L^{-1}\{F(s)\} = (1/4 - 1/4e^{-2t} - 1/2te^{-2t})1(t).
\end{aligned}
$$

We can verify the answer using MATLAB:

$>>$ syms s t

$>>$ ilaplace(1/(s*(s+2)^2))

ans =

1/4-1/4*exp(-2*t)*(1+2*t)

(b) Perform partial fraction expansion,

$$\begin{aligned}F(s) &= \frac{2s^2+s+1}{s^3-1},\\ &= \frac{2s^2+s+1}{(s-1)(s^2+s+1)},\\ &= \frac{4/3}{s-1}+\frac{C_2s+C_3}{s^2+s+1}\\ &= \frac{4/3}{s-1}+\frac{2/3s+1/3}{s^2+s+1}\end{aligned}$$

$$= \frac{4/3}{s-1}+\frac{2/3(s+0.5)}{(s+0.5)^2+3/4}$$

$$f(t) = L^{-1}\{F(s)\} = 4/3e^t + 2/3e^{-0.5t}cos(\frac{\sqrt{3}t}{2}).$$

We can verify the answer using MATLAB:

```
> > syms s t
> > ilaplace((2*s^2+s+1)/(s^3-1))
ans =
4/3*exp(t)+2/3*exp(-1/2*t)*cos(1/2*3^(1/2)*t)
```

(c) Carry out partial fraction expansion,

$$\begin{aligned}F(s) &= \frac{2(s^2+s+1)}{s(s+1)^2},\\ &= \frac{C_1}{s}+\frac{C_2}{(s+1)}+\frac{C_3}{(s+1)^2}.\\ C_1 &= sF(s)|_{s=0} = \frac{2(s^2+s+1)}{(s+1)^2}|_{s=0} = 2,\\ C_3 &= (s+1)^2F(s)|_{s=-1} = \frac{2(s^2+s+1)}{s}|_{s=-1} = -2,\\ C_2 &= \frac{d}{ds}[(s+1)^2F(s)]_{s=-1},\\ &= \frac{d}{ds}[\frac{2(s^2+s+1)}{s}]_{s=-1},\\ &= \frac{2(2s+1)s-2(s^2+s+1)}{s^2}|_{s=-1},\\ &= 0.\end{aligned}$$

$$\begin{aligned}F(s) &= \frac{2}{s}+\frac{0}{(s+1)}+\frac{-2}{(s+1)^2}.\\ f(t) &= L^{-1}\{F(s)\} = 2\{1-te^{-t}\}1(t).\end{aligned}$$

We can verify the answer using MATLAB:

> > syms s t

> > ilaplace((2*s^2+2*s+2)/(s*(s+1)^2));ans =2-2*t*exp(-t)

(d) Carry out partial fraction expansion,

$$
\begin{aligned}
F(s) &= \frac{s^3+2s+4}{s^4-16} = \frac{As+B}{s^2-4} + \frac{Cs+D}{s^2+4} = \frac{\frac{3}{4}s+\frac{1}{2}}{s^2-4} + \frac{\frac{1}{4}s-\frac{1}{2}}{s^2+4}, \\
&= \frac{1}{4}\sinh(2t) + \frac{3}{4}\cosh(2t) - \frac{1}{2}\sin(2t) + \frac{1}{4}cos(2t).
\end{aligned}
$$

We can verify the answer using MATLAB:

> > syms s t

> > ilaplace((s^3+2*s+4)/(s^4-16))

ans =

-1/4*sin(2*t)+1/2*exp(2*t)+1/4*exp(-2*t)+1/4*cos(2*t)

(e) Expand in partial fraction expansion and compute the residues using the results from Appendix A,

$$
\begin{aligned}
F(s) &= \frac{2(s+2)(s+5)^2}{(s+1)(s^2+4)^2}, \\
&= \frac{C_1}{s+1} + \frac{C_2}{s-2j} + \frac{C_3}{s+2j} + \frac{C_4}{(s-2j)^2} + \frac{C_5}{(s+2j)^2}. \\
C_1 &= (s+1)F(s)|_{s=-1} = \frac{32}{25} = 1.280, \\
C_4 &= (s-2j)^2F(s)|_{s=2j} = \frac{-83-39j}{20} = -4.150 - j1.950, \\
C_5 &= C_4^* = -4.150 + j1.950, \\
C_2 &= \frac{d}{ds}[(s-2j)^2F(s)]_{s=2j} = \frac{-128-579j}{200}, \\
&= -0.64 - j2.895, \\
C_3 &= C_2^* = -0.64 + j2.895.
\end{aligned}
$$

These results can also be verified with the MATLAB residue command,

```
a =[1 1 8 8 16 16];
b =[2 24 90 100];
[r,p,k]=residue(b,a)
```

r =

-0.64000000000000 - 2.89500000000002i

-4.15000000000002 - 1.95000000000000i

-0.64000000000000 + 2.89500000000002i

-4.15000000000002 + 1.95000000000000i

1.28000000000001

p =
0.00000000000000 + 2.00000000000000i
0.00000000000000 + 2.00000000000000i
0.00000000000000 - 2.00000000000000i
0.00000000000000 - 2.00000000000000i
-1.00000000000000
k =
[]
We then have,

$$\begin{aligned} f(t) &= 1.28e^{-t} + 2|C_2|\cos(2t + \arg C_2) + 2|C_4|t\cos(2t + \arg C_4), \\ &= 1.28e^{-t} + 5.92979\cos(2t - 1.788) + 9.1706t\cos(2t - 2.702). \end{aligned}$$

where

$$|C_2| = 2.96489, \quad |C_4| = 4.5853, \quad \arg C_2 = \tan^{-1}(\left(\frac{-2.895}{-0.64}\right)) = -1.788,$$

using the atan2 command in MATLAB, and

$$\arg C_4 = \tan^{-1}(\left(\frac{-1.950}{-4.150}\right)) = -2.702,$$

also using the atan2 command in MATLAB.

(f)

$$F(s) = \frac{(s^2 - 1)}{(s^2 + 1)^2}.$$

Using the multiplication by time Laplace transform property (Table A.1 entry #11):

$$-\frac{d}{ds}G(s) = L\{tg(t)\}.$$

We can see that

$$-\frac{d}{ds}[\frac{s}{(s^2 + 1)}] = \frac{s^2 - 1}{(s^2 + 1)^2}.$$

So the inverse Laplace transform of $F(s)$ is:

$$L^{-1}\{F(s)\} = t\cos t.$$

We can verify the answer using MATLAB:
> > syms s t
> > ilaplace((s^2-1)/(s^2+1)^2)
ans =
t*cos(t)

(g) Follows from Problem 5 (c), or expand in series,

$$\tan^{-1}(\left(\frac{1}{s}\right)) = \frac{1}{s} - \frac{1}{3s^3} + \frac{1}{5s^5} - \dots$$

Then,

$$L^{-1}\{\tan^{-1}(\left(\frac{1}{s}\right))\} = 1 - \frac{t^2}{3!} + \frac{t^4}{5!} - \dots.. = \frac{\sin(t)}{t}.$$

Alternatively, let us assume

$$L^{-1}\{\tan^{-1}(\left(\frac{1}{s}\right))\} = f(t).$$

We use the identity

$$\frac{d}{ds}[\tan^{-1} s] = \frac{1}{1+s^2},$$

which means that

$$L^{-1}\{-\frac{1}{s^2+1}\} = -tf(t) = -\sin(t).$$

Therefore,

$$f(t) = \frac{\sin(t)}{t}.$$

We can verify the answer using MATLAB:

```
> > syms s t
> > ilaplace(atan(1/s))
ans =
1/t*sin(t)
```

9. Solve the following ordinary differential equations using Laplace transforms:

   (a) $\ddot{y}(t) + \dot{y}(t) + 3y(t) = 0; y(0) = 1, \dot{y}(0) = 2$
   (b) $\ddot{y}(t) - 2\dot{y}(t) + 4y(t) = 0; y(0) = 1, \dot{y}(0) = 2$
   (c) $\ddot{y}(t) + \dot{y}(t) = \sin t; y(0) = 1, \dot{y}(0) = 2$
   (d) $\ddot{y}(t) + 3y(t) = \sin t; y(0) = 1, \dot{y}(0) = 2$
   (e) $\ddot{y}(t) + 2\dot{y}(t) = e^t; y(0) = 1, \dot{y}(0) = 2$
   (f) $\ddot{y}(t) + y(t) = t; y(0) = 1, \dot{y}(0) = -1$

Solution:

(a)
$$\ddot{y}(t)+\dot{y}(t)+3y(t)=0; \qquad y(0)=1, \qquad \dot{y}(0)=2$$

Using Table A.1 entry #5, the differentiation Laplace transform property,
$$s^2Y(s)-sy(0)-\dot{y}(0)+sY(s)-y(0)+3Y(s)=0$$

$$\begin{aligned} Y(s) &= \frac{s+3}{s^2+s+3}, \\ &= \frac{(s+\frac{1}{2})+\frac{5}{2}}{(s+\frac{1}{2})^2+\frac{11}{4}}, \\ &= \frac{(s+\frac{1}{2})}{(s+\frac{1}{2})^2+\frac{11}{4}}+\frac{5\sqrt{11}}{11}\frac{\sqrt{\frac{11}{4}}}{(s+\frac{1}{2})^2+\frac{11}{4}}. \end{aligned}$$

Using Table A.2 entries #19 and #20,

$$y(t)=e^{-\frac{1}{2}t}\cos\frac{\sqrt{11}}{2}t+\frac{5\sqrt{11}}{11}e^{-\frac{1}{2}t}\sin\frac{\sqrt{11}}{2}t.$$

We can verify the answer using MATLAB:
\> > dsolve('D2y+Dy+3*y=0','y(0)=1','Dy(0)=2','t')
ans =
5/11*11ˆ(1/2)*exp(-1/2*t)*sin(1/2*11ˆ(1/2)*t)+exp(-1/2*t)*cos(1/2*11ˆ(1/2)*t)

(b)
$$\ddot{y}(t)-2\dot{y}(t)+4y(t)=0; y(0)=1, \dot{y}(0)=2.$$
$$s^2Y(s)-sy(0)-\dot{y}(0)-2sY(s)+2y(0)+4Y(s)=0.$$

$$\begin{aligned} Y(s) &= \frac{s}{s^2-2s+4}, \\ &= \frac{s}{(s-1)^2+3}, \end{aligned}$$

Using Table A.1 entry #5 and Table A.2 entry #20,

$$\begin{aligned} y(t) &= \frac{d}{dt}[e^t\sin\sqrt{3}t] \\ y(t) &= \frac{1}{\sqrt{3}}e^t\sin\sqrt{3}t+e^t\cos\sqrt{3}t \end{aligned}$$

We can verify the answer using MATLAB:
\> > dsolve('D2y-2*Dy+4*y=0','y(0)=1','Dy(0)=2','t')
ans =
1/3*3ˆ(1/2)*exp(t)*sin(3ˆ(1/2)*t)+exp(t)*cos(3ˆ(1/2)*t)

(c)

$$\ddot{y}(t) + \dot{y}(t) = \sin t; \quad y(0) = 1, \quad \dot{y}(0) = 2$$

$$s^2Y(s) - sy(0) - \dot{y}(0) + sY(s) - y(0) = \frac{1}{s^2+1}$$

$$\begin{aligned} Y(s) &= \frac{s^3 + 3s^2 + s + 4}{s(s+1)(s^2+1)}, \\ &= \frac{C_1}{s} + \frac{C_2}{s+1} + \frac{C_3 s + C_4}{s^2+1}. \end{aligned}$$

$$\begin{aligned} C_1 &= \frac{s^3 + 3s^2 + s + 4}{(s+1)(s^2+1)}|_{s=0} = 4, \\ C_2 &= \frac{s^3 + 3s^2 + s + 4}{s(s^2+1)}|_{s=-1} = -\frac{5}{2}. \end{aligned}$$

$$\begin{aligned} \frac{4}{s} + \frac{-\frac{5}{2}}{s+1} + \frac{C_3 s + C_4}{s^2+1} &= \frac{s^3 + 3s^2 + s + 4}{s(s+1)(s^2+1)} \\ s^3(\frac{3}{2} + C_3) + s^2(4 + C_3 + C_4) + s(\frac{3}{2} + C_4) + 4 &= s^3 + 3s^2 + s + 4. \end{aligned}$$

Match coefficients of like powers of s

$$\begin{aligned} C_4 + \frac{3}{2} &= 1 \qquad \Longrightarrow C_4 = -\frac{1}{2}, \\ C_3 + \frac{3}{2} &= 1 \qquad \Longrightarrow C_3 = -\frac{1}{2}. \end{aligned}$$

$$\frac{4}{s} + \frac{-\frac{5}{2}}{s+1} + \frac{-\frac{1}{2}s - \frac{1}{2}}{s^2+1} = \frac{4}{s} + \frac{-\frac{5}{2}}{s+1} - \frac{1}{2}\frac{s}{s^2+1} - \frac{1}{2}\frac{1}{s^2+1}.$$

Using Table A.2 entries #2, #7, #17, and #18

$$y(t) = 4 - \frac{5}{2}e^{-t} - \frac{1}{2}\cos t - \frac{1}{2}\sin t.$$

We can verify the answer using MATLAB:

```
>> dsolve('D2y+Dy-sin(t)=0','y(0)=1','Dy(0)=2','t')
ans =
-1/2*sin(t)-1/2*cos(t)-5/2*exp(-t)+4
```

(d)

$$\ddot{y}(t) + 3y(t) = \sin t; \quad y(0) = 1, \quad \dot{y}(0) = 2,$$

$$s^2Y(s) - sy(0) - \dot{y}(0) + 3Y(s) = \frac{1}{s^2+1},$$

$$\begin{aligned} Y(s) &= \frac{s^3+2s^2+s+3}{(s^2+3)(s^2+1)}, \\ &= \frac{C_1 s + C_2}{s^2+3} + \frac{C_3 s + C_4}{s^2+1}. \end{aligned}$$

$$\frac{(C_1 s + C_2)(s^2+1) + (C_3 s + C_4)(s^2+3)}{(s^2+3)(s^2+1)} = \frac{s^3+2s^2+s+3}{(s^2+3)(s^2+1)}.$$

Match coefficients of like powers of s:

$$s^3(C_1+C_3)+s^2(C_2+C_4)+s(C_1+3C_3)+(C_2+3C_4) = s^3+2s^2+s+3,$$

$$\begin{aligned} C_1 + C_3 &= 1 \quad \Longrightarrow C_1 = -C_3 + 1, \\ C_2 + C_4 &= 2 \quad \Longrightarrow C_2 = 2 - C_4, \\ C_1 + 3C_3 &= 1 \quad \Longrightarrow -C_3 + 1 + 3C_3 = 1 \quad \Longrightarrow C_3 = 0, \\ &\Longrightarrow C_1 = 1, \\ C_2 + 3C_4 &= 3 \quad \Longrightarrow (2 - C_4) + 3C_4 = 3 \quad \Longrightarrow C_4 = \frac{1}{2}, \\ &\Longrightarrow C_2 = \frac{3}{2}, \end{aligned}$$

$$\begin{aligned} Y(s) &= \frac{\frac{1}{2}s + \frac{3}{2}}{s^2+3} + \frac{\frac{1}{2}}{s^2+1}, \\ &= \frac{\frac{1}{2}}{s^2+3} + \frac{\sqrt{3}}{2}\frac{\sqrt{3}}{s^2+3} + \frac{1}{2}\frac{1}{s^2+1}. \end{aligned}$$

$$y(t) = \frac{1}{2}\cos\sqrt{3}t + \frac{\sqrt{3}}{2}\sin\sqrt{3}t + \frac{1}{2}\sin t.$$

We can verify the answer using MATLAB:

> > dsolve('D2y+3*y-sin(t)=0','y(0)=1','Dy(0)=2','t')

ans =

1/2*sin(3^(1/2)*t)*3^(1/2)+cos(3^(1/2)*t)+1/2*sin(t)

(e)

$$\ddot{y}(t) + 2\dot{y}(t) = e^t; \quad y(0) = 1, \quad \dot{y}(0) = 2$$

$$s^2Y(s) - sy(0) - \dot{y}(0) + 2sY(s) - 2y(0) = \frac{1}{s-1}$$

$$\begin{aligned} Y(s) &= \frac{s^2+3s-3}{s(s-1)(s+2)}, \\ &= \frac{C_1}{s} + \frac{C_2}{s-1} + \frac{C_3}{s+2}. \end{aligned}$$

$$\begin{aligned}
C_1 &= \frac{s^2+3s-3}{(s-1)(s+2)}|_{s=0} = \frac{3}{2}, \\
C_2 &= \frac{s^2+3s-3}{s(s+2)}|_{s=1} = \frac{1}{3}, \\
C_3 &= \frac{s^2+3s-3}{s(s-1)}|_{s=-2} = -\frac{5}{6},
\end{aligned}$$

$$Y(s) = \frac{\frac{3}{2}}{s} + \frac{1}{3}\frac{1}{s-1} - \frac{5}{6}\frac{1}{s+2}.$$

$$y(t) = \frac{3}{2} + \frac{1}{3}e^t - \frac{5}{6}e^{-2t}.$$

We can verify the answer using MATLAB:

> > dsolve('D2y+2*Dy-exp(t)=0','y(0)=1','Dy(0)=2','t')

ans =

1/3*exp(t)-5/6*exp(-2*t)+3/2

(f) Using the results from Appendix A,

$$\ddot{y}(t) + y(t) = t; \quad y(0) = 1, \quad \dot{y}(0) = -1,$$

$$s^2Y(s) - sy(0) - \dot{y}(0) + Y(s) = \frac{1}{s^2},$$

$$\begin{aligned}
Y(s) &= \frac{s^3 - s^2 + 1}{s^2(s^2+1)}, \\
&= \frac{C_1}{s} + \frac{C_2}{s^2} + \frac{C_3 s + C_4}{s^2+1}.
\end{aligned}$$

$$\begin{aligned}
C_1 &= \frac{d}{ds}\frac{(s^3 - s^2 + 1)}{(s^2+1)}|_{s=0} = 0, \\
C_2 &= \frac{(s^3 - s^2 + 1)}{(s^2+1)}|_{s=0} = 1.
\end{aligned}$$

$$\begin{aligned}
\frac{1}{s^2} + \frac{C_3 s + C_4}{s^2+1} &= \frac{s^3 - s^2 + 1}{s^2(s^2+1)}, \\
\frac{(s^2+1) + (C_3 s + C_4)s^2}{s^2(s^2+1)} &= \frac{s^3 - s^2 + 1}{s^2(s^2+1)}.
\end{aligned}$$

Match coefficients of like powers of s:

$$\begin{aligned}
C_3 &= 1 \\
C_4 + 1 &= -1 \qquad \Longrightarrow C_4 = -2
\end{aligned}$$

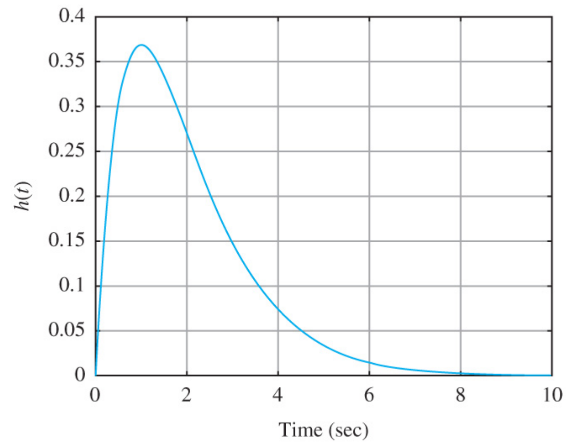


Figure 3.45: Impulse response for Problem 3.10

$$Y(s) = \frac{1}{s^2} + \frac{s}{s^2+1} - 2\frac{1}{s^2+1}$$
$$y(t) = t + \cos t - 2\sin t.$$

We can verify the answer using MATLAB:
> > dsolve('D2y+y-t=0','y(0)=1','Dy(0)=-1','t')
ans =
-2*sin(t)+cos(t)+t

10. Using the convolution integral, find the step response of the system whose impulse response is given below and shown in Figure 3.45:

$$h(t) = \{ \begin{array}{cc} te^{-t} & t \geq 0 \\ 0 & t < 0. \end{array} .$$

Solution: There are only two cases to consider.

Case (a): For the case $t \leq 0$, the situation is illustrated in the following Figure part (c). There is no overlap between the two functions ($u(t-\tau)$ and $h(\tau)$) and the output is zero

$$y_1(t) = 0.$$

Case (b): For the case $t \geq 0$, the situation is displayed in the following Figure part (d). The output of the system is given by

$$y_2(t) = \int_0^t h(\tau)u(t-\tau)d\tau = \int_0^t (\tau e^{-\tau})(1)d\tau = 1-(t+1)e^{-t}.$$

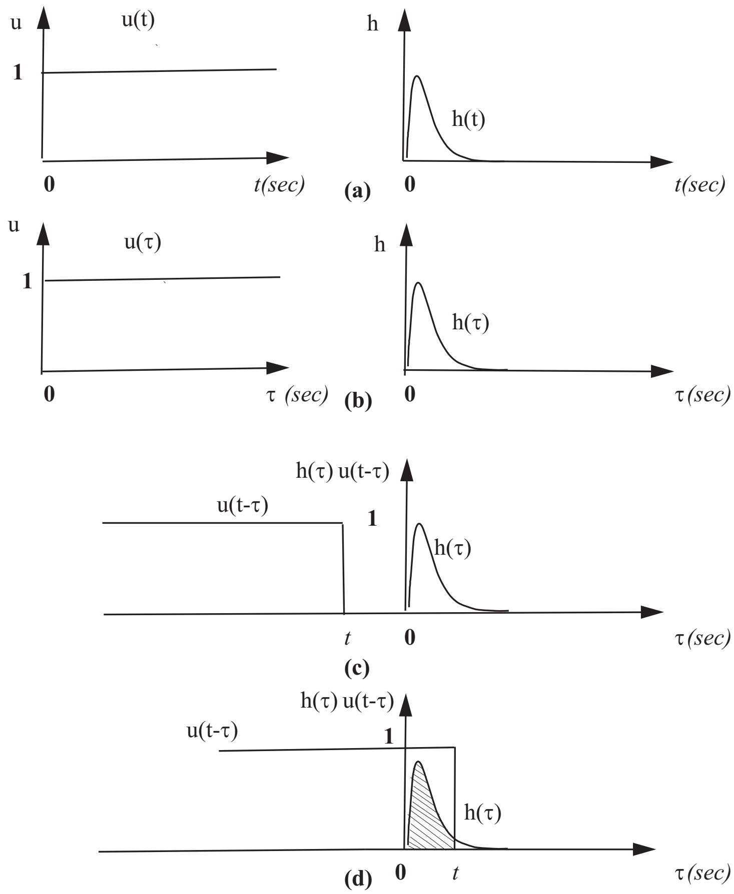


Illustration of convolution.

The output of the system is the composite of the two segments computed above as shown in the following figure.

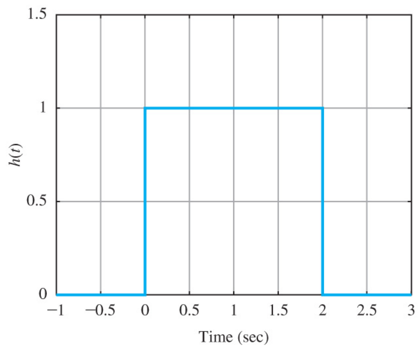
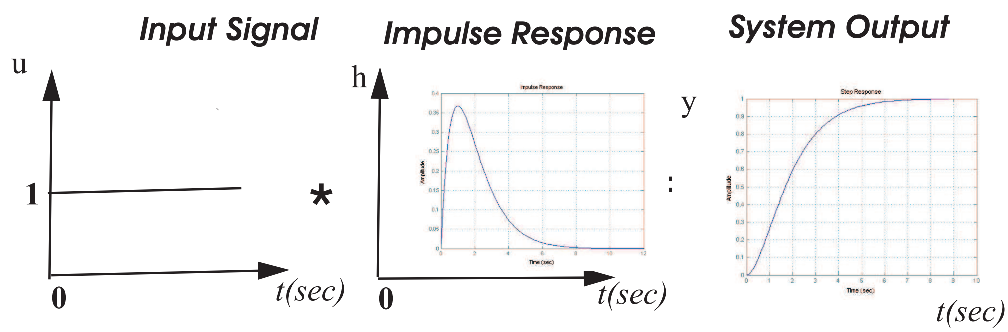


Figure 3.46: Impulse response for Problem 3.11.



System output response.

11. Using the convolution integral, find the step response of the system whose impulse response is given below and shown in Figure 3.46:

$$h(t) = \left\{ \begin{array}{cc} 1 & 0 \leq t \leq 2 \\ 0 & t < 0 \text{ and } t > 2 \end{array} \right. .$$

Solution: There are three cases to consider as shown in the following figure**.**

Case (a): For the case $t \leq 0$, the situation is illustrated in the following Figure part (c). There is no overlap between the two functions ($u(t-\tau)$ and $h(\tau)$) and the output is zero

$$y_1(t) = 0$$

Case (b): For the case $0 \geqslant t \geq 2$, the situation is displayed in the following Figure part (d) and shows partial overlap. The output of the system is given by

$$y_2(t) = \int_0^t h(\tau)u(t-\tau)d\tau = \int_0^t (1)(1)d\tau = t.$$

Case (c): For the case $t \geq 2$, the situation is displayed in the following Figure part (e) and shows total overlap. The output of the system is given by

$$y_3(t) = \int_0^t h(\tau)u(t-\tau)d\tau = \int_0^2 (1)(1)d\tau = 2.$$

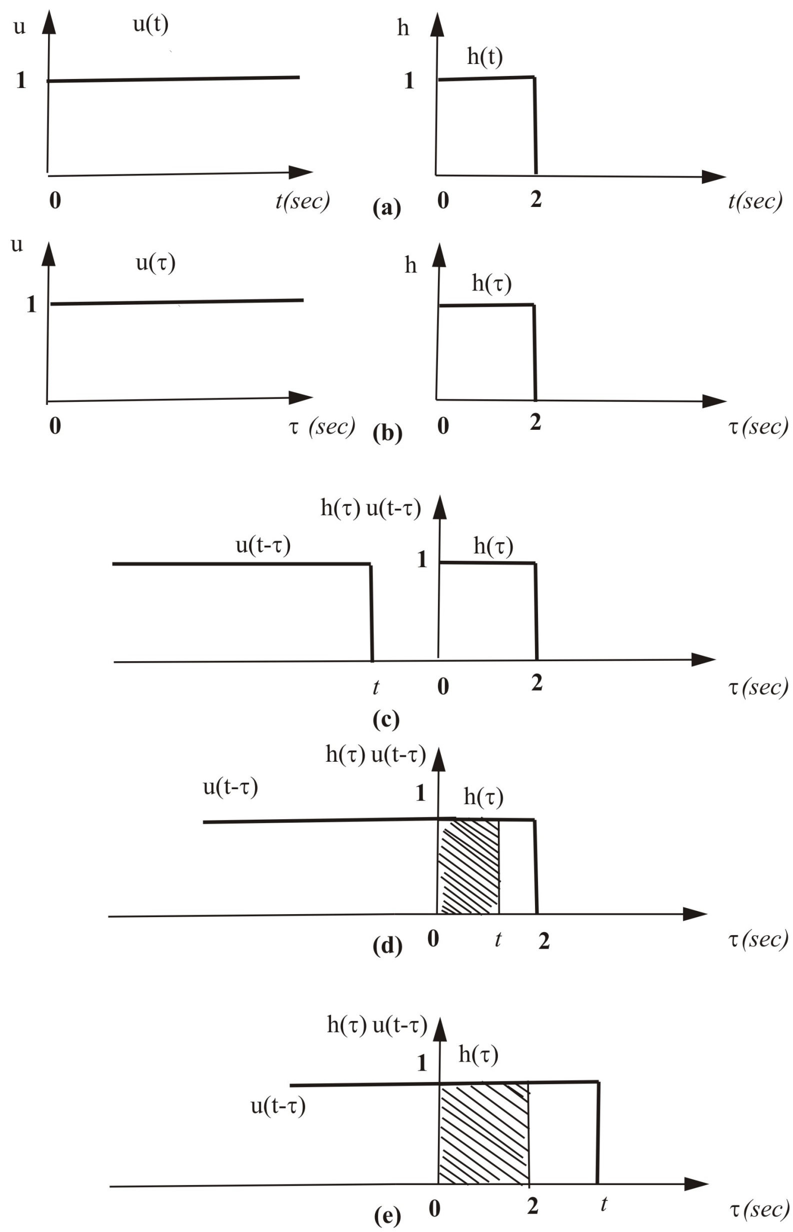


Illustration of convolution.

The output of the system is the composite of the three segments computed above as shown in the following figure.

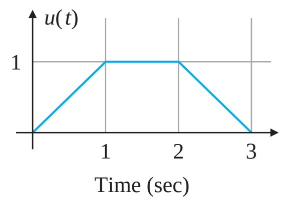
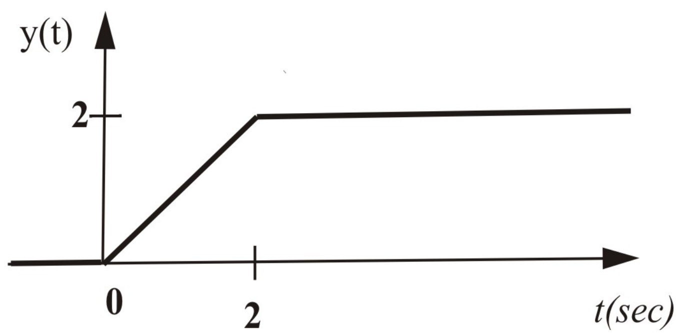


Figure 3.47: Plot of input for Problem 3.12



System output response.

12. Consider the standard second-order system

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}.$$

a) Write the Laplace transform of the signal in Fig. 3.47. b). What is the transform of the output if this signal is applied to $G(s)$. c) Find the output of the system for the input shown in Fig. 3.47.

Solution:

(a) The input signal may be written as:

$$u(t) = t - (t-1) * 1(t-1) - (t-2) * 1(t-2) + (t-3) * 1(t-3),$$

where $1(t-\tau)$ denotes a delayed unit step. The Laplace transform of the input signal is:

$$U(s) = \frac{1}{s^2}(1 - e^{-s} - e^{-2s} + e^{-3s}).$$

We can verify this in MATLAB:

```
>> ilaplace(1/s^2*(1-exp(-s)-exp(-2*s)+exp(-3*s)))

ans =

t-heaviside(t-1)*(t-1)-heaviside(t-2)*(t-2)+heaviside(t-3)*(t-3)
```

(b) The Laplace transform of the output if this input signal is applied is:

$$Y(s) = G(s)U(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}(\left(\frac{1}{s^2}\right))(1 - e^{-s} - e^{-2s} + e^{-3s}).$$

(c) However to make the mathematical manipulation easier, consider only the response of the system to a (unit) ramp input:

$$Y_1(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}(\left(\frac{1}{s^2}\right)).$$

Partial fractions yields the following:

$$Y_1(s) = \frac{1}{s^2} - \frac{\frac{2\zeta}{\omega_n}}{s} + \frac{\frac{2\zeta}{\omega_n}(s + 2\zeta\omega_n - \frac{\omega_n}{2\zeta})}{(s + \omega_n\zeta)^2 + (\omega_n\sqrt{1-\zeta^2})^2}.$$

Use the following Laplace transform pairs for the case $0 \le \zeta < 1$:

$$L^{-1}\{\left\{\frac{s + z_1}{(s+a)^2 + \omega^2}\right\}\} = \sqrt{\frac{(z_1 - a)^2 + \omega^2}{\omega^2}}e^{-at}\sin(\omega t + \phi),$$

where

$$\phi \equiv \tan^{-1}(\left(\frac{\omega}{z_1 - a}\right)).$$

$$L^{-1}\{\left\{\frac{1}{s^2}\right\}\} = t \qquad \text{unit ramp}$$

$$L^{-1}\{\left\{\frac{1}{s}\right\}\} = 1(t) \qquad \text{unit step}$$

and the following Laplace transform pairs for the case $\zeta = 1$:

$$L^{-1}\{\left\{\frac{1}{(s+a)^2}\right\}\} = te^{-at}.$$

$$L^{-1}\{\left\{\frac{s}{(s+a)^2}\right\}\} = (1 - at)e^{-at}.$$

$$L^{-1}\{\left\{\frac{1}{s^2}\right\}\} = t \qquad \text{unit ramp,}$$

$$L^{-1}\{\left\{\frac{1}{s}\right\}\} = 1(t) \qquad \text{unit step,}$$

the following is derived:

$$y_1(t) = \{ \begin{array}{ll} t - \frac{2\zeta}{\omega_n} + \frac{e^{-\zeta\omega_n t}}{\omega_n\sqrt{1-\zeta^2}}\sin(\omega_n\sqrt{1-\zeta^2}t + \tan^{-1}\frac{2\zeta\sqrt{1-\zeta^2}}{2\zeta^2-1}) & \begin{array}{c} 0 \le \zeta < 1 \\ t \ge 0 \end{array} \\ t - \frac{2}{\omega_n} + \frac{2}{\omega_n}e^{-\omega_n t}(\frac{\omega_n}{2}t + 1) & \begin{array}{c} \zeta = 1 \\ t \ge 0 \end{array} \end{array} .$$

Since $u(t)$ consists of a ramp and three delayed ramp signals, using superposition (the system is linear), then:

$$y(t) = y_1(t) - y_1(t-1) - y_1(t-2) + y_1(t-3) \qquad\qquad t \ge 0.$$

13. A rotating load is connected to a field-controlled DC motor with negligible field inductance. A test results in the output load reaching a speed of 1 rad/sec within 1/2 sec when a constant input of 100 V is applied to the motor terminals. The output steady-state speed from the same test is found to be 2 rad/sec. Determine the transfer function $\Theta(s)/V_f(s)$ of the motor.

Solution:

Equations of motion for a DC motor:

$$J_m\ddot{\theta}_m + b\dot{\theta}_m = K_m i_a,$$

$$K_e\dot{\theta}_m + L_a\frac{di_a}{dt} + R_a i_a = v_a,$$

but since there's negligible field inductance $L_a = 0$.

Combining the above equations yields:

$$R_a J_m\ddot{\theta}_m + R_a b\dot{\theta}_m = K_t v_a - K_t K_e\dot{\theta}_m.$$

Applying Laplace transforms yields the following transfer function:

$$\frac{\Theta(s)}{V_f(s)} = \frac{\frac{K_t}{J_m R_a}}{s(s + \frac{K_t K_e}{R_a J_m} + \frac{b}{J_m})} = \frac{K}{s(s+a)},$$

where $K = \frac{K_t}{J_m R_a}$ and $a = \frac{K_t K_e}{R_a J_m} + \frac{b}{J_m}$.

K and a are found using the given information:

$$\begin{aligned} V_f(s) &= \frac{100}{s} \text{ since } V_f(t) = 100V, \\ \dot{\theta}(\left(\frac{1}{2}\right)) &= 2 \text{ rad/sec.} \end{aligned}$$

For the given information we need to utilize $\dot{\theta}_m(t)$ instead of $\theta_m(t)$:

$$s\Theta(s) = \frac{100K}{s(s+a)}.$$

Using the Final Value Theorem and assuming that the system is stable:

$$\lim_{s \to 0} \frac{100K}{s+a} = \lim_{s \to 0} \dot{\theta}(\left(\frac{1}{2}\right)) = 2 = \frac{100K}{a}.$$

Take the inverse Laplace transform:

$$\begin{aligned} L^{-1}\{\frac{100K}{a}\frac{a}{s(s+a)}\} &= \frac{100K}{a}(1-e^{-at}) = 2(1-e^{-at}) = 1, \\ 0.5 &= e^{-\frac{a}{2}} \quad \text{yields} \quad a = 1.39, \\ K &= \frac{2}{100}a \quad \text{yields} \quad K = 0.0278, \\ \frac{\Theta(s)}{V_f(s)} &= \frac{0.0278}{s(s+1.39)}. \end{aligned}$$

14. For the system in Fig. 2.54, compute the transfer function from the motor voltage to position θ_2.

Solution:

From Problem 2.20:

$$\begin{gathered} L\frac{di_a}{dt} + R_a i_a + k_e \dot{\theta}_1 = v_a \\ k_t i_a = J_1 \ddot{\theta}_1 + b(\dot{\theta}_1 - \dot{\theta}_2) + k(\theta_1 - \theta_2) + B\dot{\theta}_1 \\ J_2 \ddot{\theta}_2 + b(\dot{\theta}_2 - \dot{\theta}_1) + k(\theta_2 - \theta_1) = 0 \end{gathered}$$

So we have:

$$\begin{gathered} LsI_a(s) + R_a I_a(s) + sk_e\Theta_1(s) = V_a(s), \\ k_t I_a(s) = s^2 J_1 \Theta_1(s) + b[\Theta_1(s) - \Theta_2(s)]s + k[\Theta_1(s) - \Theta_2(s)] + Bs\Theta_1(s), \\ s^2 J_2 \Theta_2(s) + b[\Theta_2(s) - \Theta_1(s)]s + k[\Theta_2(s) - \Theta_1(s)] = 0, \end{gathered}$$

we have:

$$
\begin{aligned}
\frac{\Theta_2(s)}{V_a(s)} &= \frac{k_t(bs+k)}{\det\begin{bmatrix} sk_e & 0 & Ls+R_a \\ J_1s^2+Bs+bs+k & -bs-k & -k_t \\ -bs-k & J_2s^2+bs+k & 0 \end{bmatrix}}, \\
&= \frac{k_t(bs+k)}{\begin{matrix}(Ls+Ra)[J_1J_2s^4+(J_1b+BJ_2+bJ_2)s^3+(J_1k+Bb+KJ_2)s^2+Bks] \\ +k_ek_tJ_2s^3+k_ek_tbs^2+kk_ek_ts\end{matrix}}, \\
&= \frac{k_t(bs+k)}{\begin{matrix}J_1J_2s^5+J_2[J_1R_a+L(b+B)]s^4 \\ +[J_2k_ek_tJ_1L(b+k)+LJ_2k+R_a(b+B)J_2-Lb^2]s^3 \\ +[L(b+B)(b+k)-2bkL+J_1R_a(b+k)+R_aJ_2k-b^2R_a]s^2 \\ +[k_ek_t(b+k)+kL(b+k)-bk^2+R_a(b+B)(b+k)-2bkR_a]s+kR_ab\end{matrix}}.
\end{aligned}
$$

15. Compute the transfer function for the two-tank system in Fig. 2.58 with holes at A and C.

Solution:

From Problem 2.26 but with $s = a$ tank area we have:

$$
\begin{bmatrix} \Delta\dot{h}_1 \\ \Delta\dot{h}_2 \end{bmatrix} = \frac{1}{6a}\begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} \Delta h_1 \\ \Delta h_2 \end{bmatrix} + \frac{\omega_{in}}{a}\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{-10}{3a} \\ 0 \end{bmatrix},
$$

$$
\begin{aligned}
\Delta\dot{h}_1 &= \frac{-\Delta h_1 + 6\omega_{in} - 20}{6a}, \\
\Delta\dot{h}_2 &= \frac{1}{6a}(\Delta h_1 - \Delta h_2), \\
s\Delta h_1(s) &= \frac{-\Delta h_1(s) + 6\omega_{in}(s)}{6a}, \\
s\Delta h_2(s) &= \frac{1}{6a}[\Delta h_1(s) - \Delta h_2(s)], \\
\Delta h_2(s) &= \frac{\omega_{in}(s)}{6a[a(\frac{1}{6a}+s)]^2}, \\
\frac{\Delta h_2(s)}{\omega_{in}(s)} &= \frac{1}{6[a(\frac{1}{6a}+s)]^2}.
\end{aligned}
$$

16. For a second-order system with transfer function

$$
G(s) = \frac{2}{s^2+s-2},
$$

determine the following:

(a) The DC gain;

(b) The final value to a unit step input.

Solution:

(a) If we blindly compute the DC gain $G(0) = \frac{2}{-2} = -1$. This answer is **not** correct as explained in part (b). This is because the DC gain is not defined for an unstable system and the output of the system is unbounded.

(b) $\lim_{t\to\infty} y(t) = ?$

The poles of the system are: $s^2 + s - 2 = 0 \Longrightarrow s = 1, -2$.

Since the system has an *unstable* pole, the Final Value Theorem is *not* applicable. The output is unbounded:

$$y(t) = (\frac{2}{3}e^t - \frac{2}{3}e^{-2t})1(t).$$

17. Consider the continuous rolling mill depicted in Fig. 3.48. Suppose that the motion of the adjustable roller has a damping coefficient b, and that the force exerted by the rolled material on the adjustable roller is proportional to the material's change in thickness: $F_s = c(T - x)$. Suppose further that the DC motor has a torque constant K_t and a back-emf constant K_e, and that the rack-and-pinion has effective radius of R.

(a) What are the inputs to this system? The output?

(b) Without neglecting the effects of gravity on the adjustable roller, draw a block diagram of the system that explicitly shows the following quantities: $V_s(s)$, $I_0(s), F(s)$ (the force the motor exerts on the adjustable roller), and $X(s)$.

(c) Simplify your block diagram as much as possible while still identifying output and each input separately.

Solution:

(a)

$$\begin{array}{lll} & & \text{Inputs: Input voltage} \longrightarrow v_s(t) \\ \text{thickness} & \longrightarrow & \text{T} \\ & & \text{Outputs : gravity} \longrightarrow \text{mg} \\ \text{thickness} & \longrightarrow & x \end{array}$$

(b) Dynamic analysis of adjustable roller:

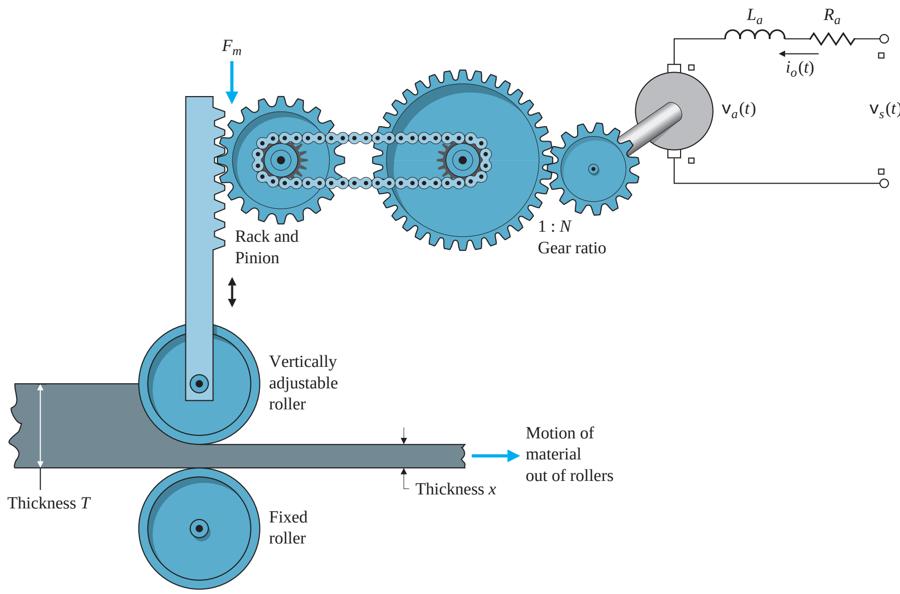


Figure 3.48: Continuous rolling mill

$$m\ddot{x} = c(T - x) - mg - b\dot{x} - F_m,$$

$$\Longrightarrow (s^2 m + sb + c)X(s) + F_m(s) + \frac{mg - cT}{s} = 0. \qquad (1)$$

Torque in rack and pinion:

$$T_{RP} = RF_m = NT_{motor},$$

$$\text{but} \quad T_{motor} = K_t I_f i_o,$$

$$F_m = \frac{NK_t I_f}{R} i_o \qquad (2)$$

DC motor circuit analysis:

$$v_s(t) = R_a i_o + L_a \frac{di_o}{dt} + v_a(t),$$

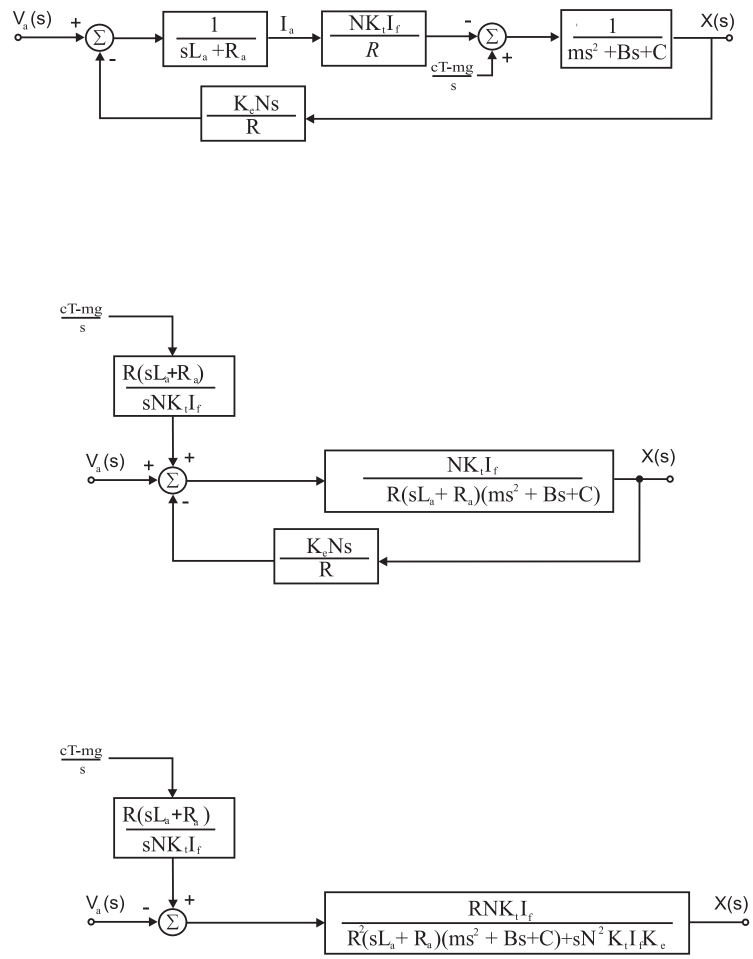
$$v_a(t) = u_e \dot{\theta},$$

$$\frac{\theta R}{N} = x,$$

$$I_o(s) = \frac{V_s(s) - \frac{K_e N}{R} sX(s)}{R_a + sL_a} \qquad (3)$$

Combining (1), (2), and (3):

$$0 = (s^2 m + sb + c)X(s) + \frac{mg - cT}{s} + \frac{NK_t I_f}{R}\left[\left[\frac{V_s(s) - \frac{K_e N}{R} sX(s)}{sL_a + R_a}\right]\right].$$



Block diagrams for rolling mill.

Problems and Solutions for Section 3.2: System Modeling Diagrams

18. Consider the block diagram shown in Fig. 3.49. Note that a_i and b_i are constants. Compute the transfer function for this system. This special structure is called the "control canonical form" and will be discussed further in Chapter 7.

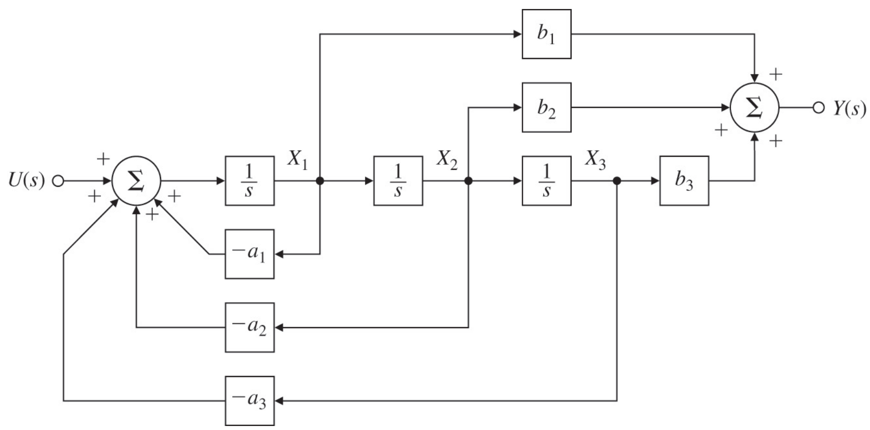
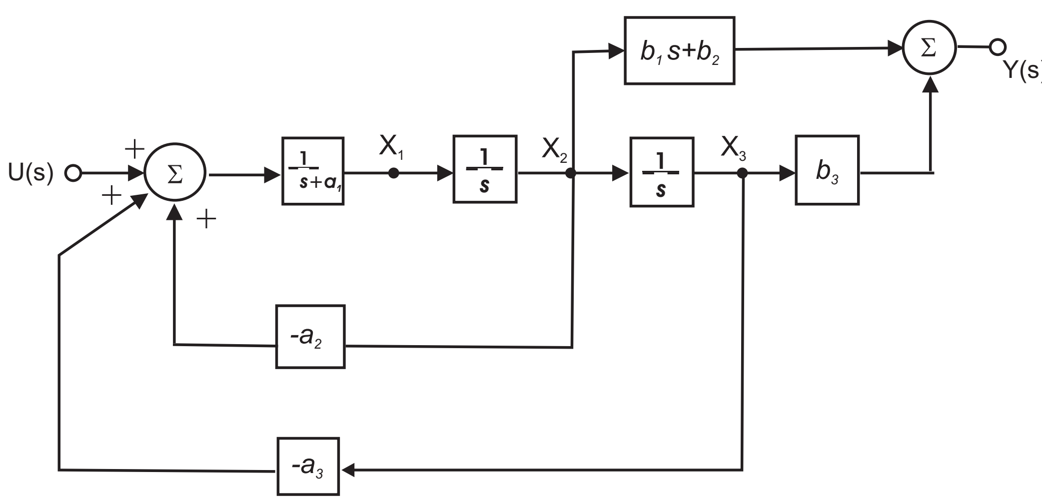


Figure 3.49: Block diagram for Problem 3.18.

Solution:

We move the pickoff point at X_1 to the right past the second integrator to get $b_1 s + b_2$ as shown in the next figure.



Block diagram reduction for Problem 3.18.

We then move the pickoff point at X_2 past the third integrator to get $s(b_1 s + b_2) + b_3$. We then have a block with the transfer function $b_1 s^2 +$

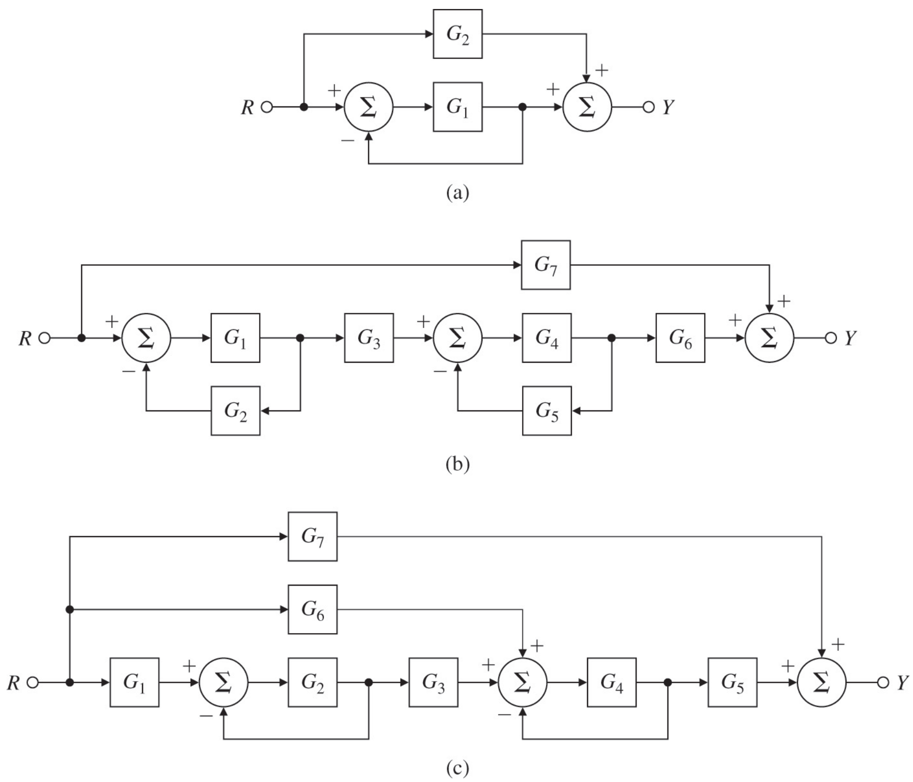


Figure 3.50: Block diagrams for Problem 3.19

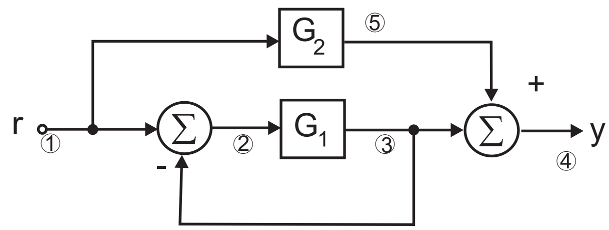
$b_2 s + b_3$ at the output. Meanwhile we apply the feedback rule to the first inner loop to get $\frac{1}{s+a_1}$ as shown in the figure and repeat for the second and third loops. We finally have:

$$\frac{Y(s)}{U(s)} = \frac{b_1 s^2 + b_2 s + b_3}{s^3 + a_1 s^2 + a_2 s + a_3}.$$

Example W.3.1 (Appendix W3.2.3 on the web) shows that we can obtain the same answer using Mason's rule.

19. Find the transfer functions for the block diagrams in Fig. 3.50.

Solution:

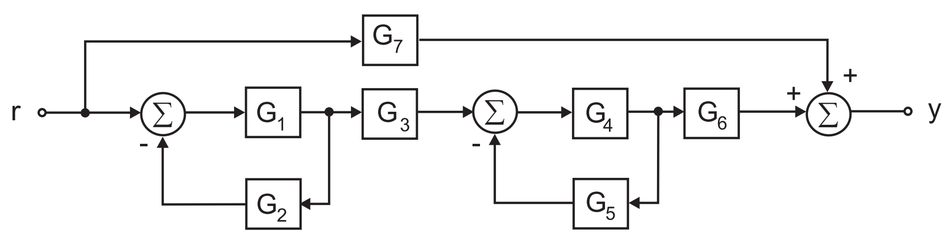


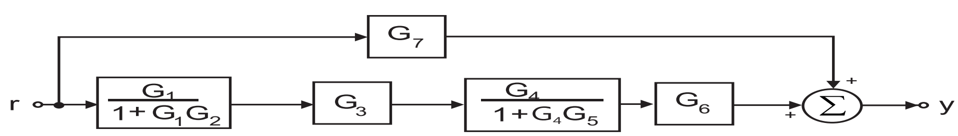
(a)

Block diagram for Fig. 3.50 (a)

$$\frac{Y}{R} = \frac{G_1}{1+G_1} + G_2.$$

(b)

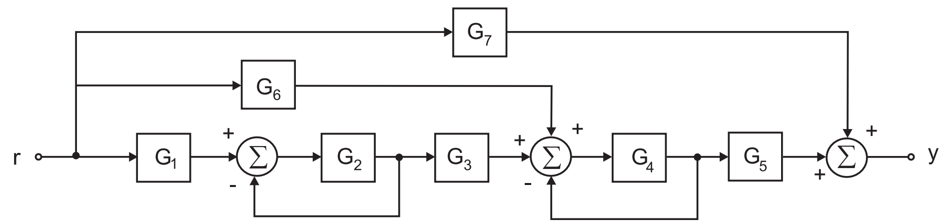


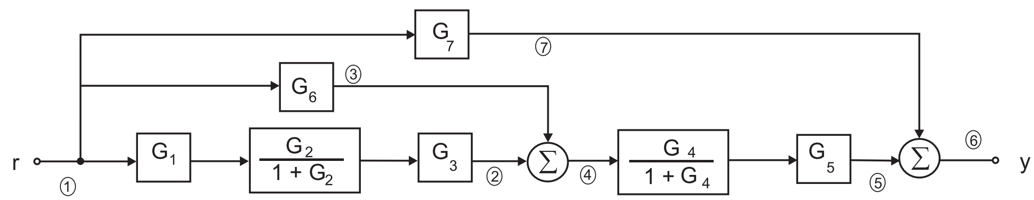


Block diagram for Fig. 3.50 (b): reduced

$$\frac{Y}{R} = G_7 + \frac{G_1G_3G_4G_6}{(1+G_1G_2)(1+G_4G_5)}.$$

(c)





Top: Block diagram for Fig. 3.50 (c) ; Bottom: Block diagram for Fig 3.50 (c) reduced.

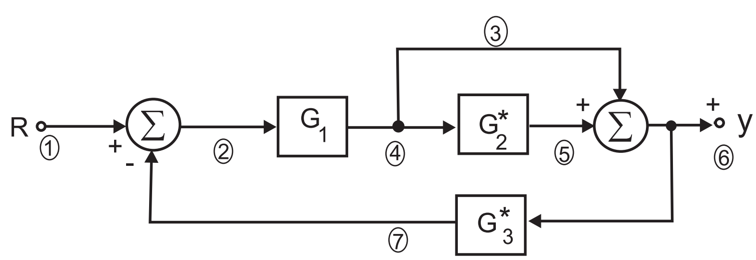
$$\frac{Y}{R} = G_7 + \frac{G_6 G_4 G_5}{1 + G_4} + \frac{G_1 G_2 G_3}{1 + G_2} \times \frac{G_4 G_5}{1 + G_4}.$$

20. Find the transfer functions for the block diagrams in Fig. 3.51, using the ideas of block diagram simplification. The special; structure in Fig. 3.51 (b) is called the "observer canonical form" and will be discussed in Chapter 7.

Solution:

Part (a): Transfer functions found using the ideas of Figs. 3.9 and 3.10**:**

(a)



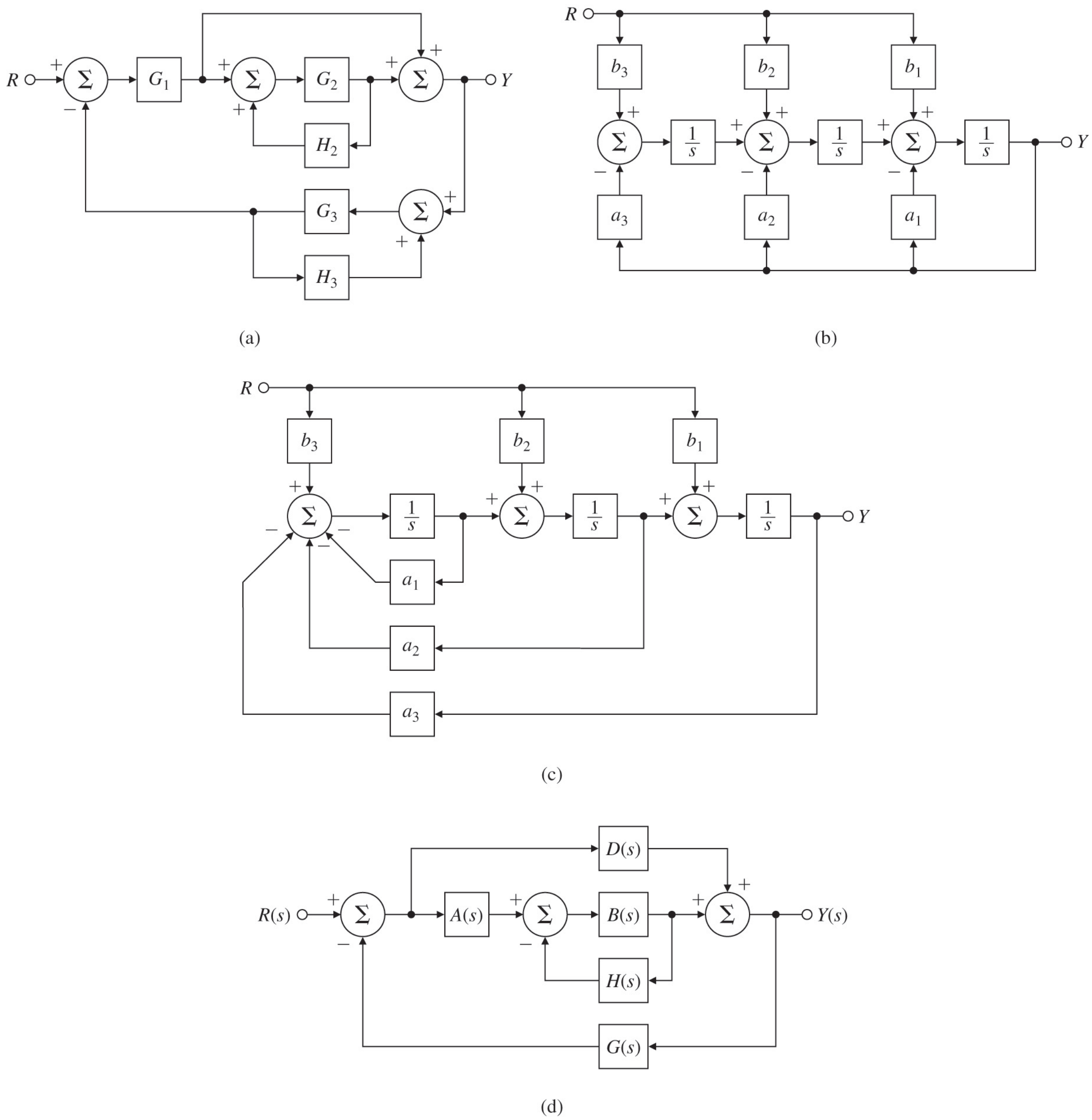
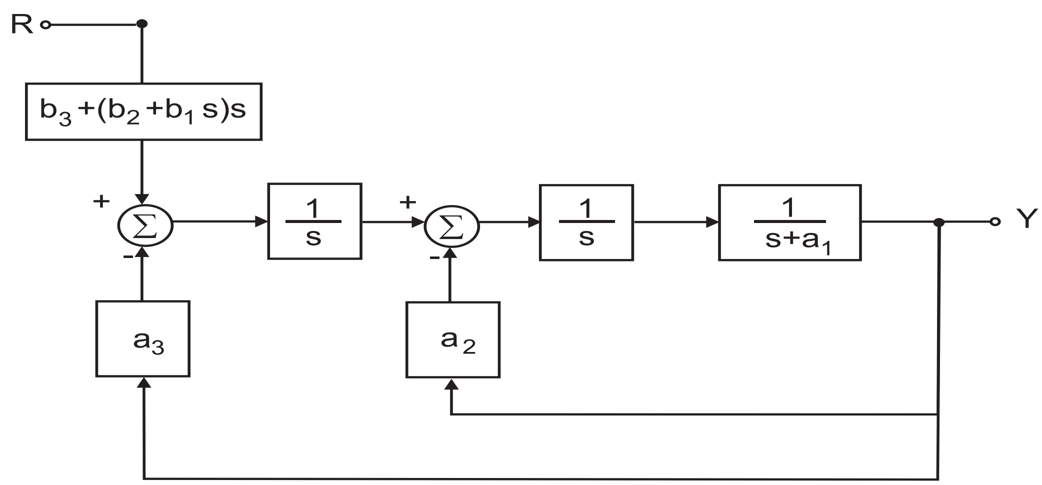


Figure 3.51: Block diagrams for Problem 3.20

$$G_2^* = \frac{G_2}{1 - G_2 H_2},$$
$$G_3^* = \frac{G_3}{1 - G_3 H_3},$$

$$\frac{Y(s)}{R(s)} = \frac{G_1(1 + G_2^*)}{1 + G_1(1 + G_2^*)G_3^*} = \frac{G_1(1 - G_2H_2)(1 - G_3H_3) + G_1G_2(1 - G_3H_3)}{(1 - G_2H_2)(1 - G_3H_3) + G_1G_3(1 - G_2H_2) + G_1G_2G_3}.$$

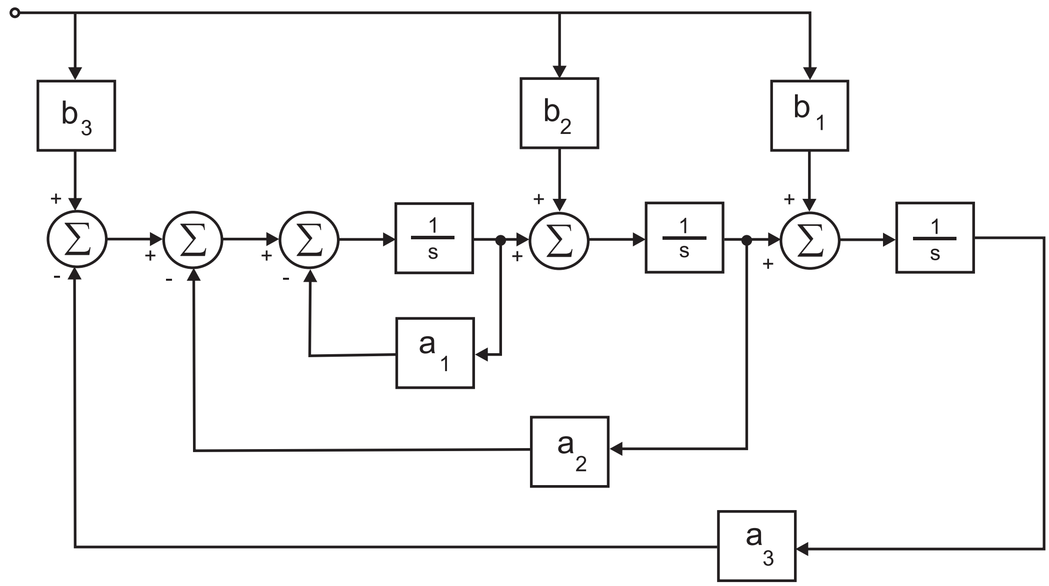
(b) We move the summer on the right hand side past the integrator to get $b_1 s$ and repeat to get $(b_2 + b_1 s)s$. Meanwhile we apply the feedback rule to the first inner loop to get $\frac{1}{s+a_1}$ as shown in the next figure and repeat for the second and third loops to get:



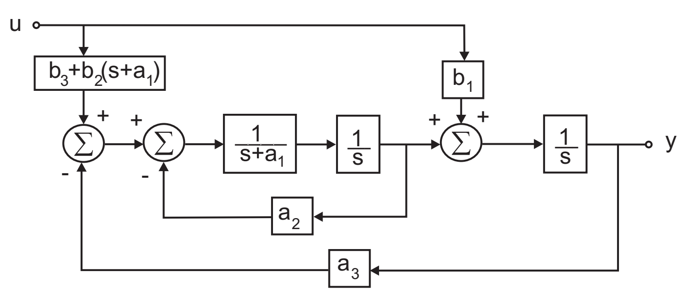
(b) Simplified block diagram for Fig. 3.51(b).

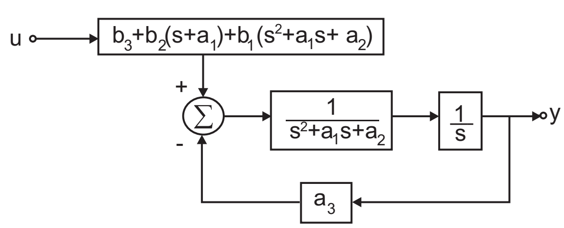
$$\frac{Y(s)}{R(s)} = \frac{b_1 s^2 + b_2 s + b_3}{s^3 + a_1 s^2 + a_2 s + a_3}.$$

(c) Applying block diagram reduction: reduce the innermost loop, shift b_2 to the b_3 node by mutiplying by $(s+a_1)$, reduce the next innermost loop and continue systematically to obtain:



(c) Block diagram for Fig. 3.51(c).





(c) Simplified block diagram for Fig. 3.51(c).

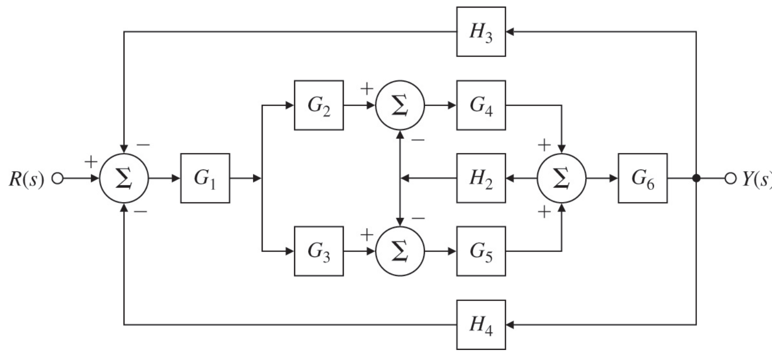
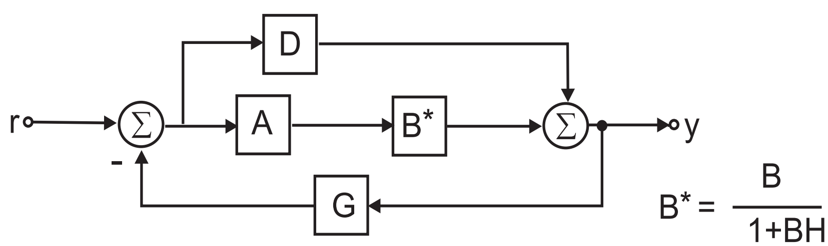


Figure 3.52: Block diagram for Problem 3.21

$$\frac{Y(s)}{R(s)} = \frac{b_1 s^2 + (a_1 b_1 + b_2)s + a_1 b_2 + a_2 b_1 + b_3}{s^3 + a_1 s^2 + a_2 s + a_3}.$$

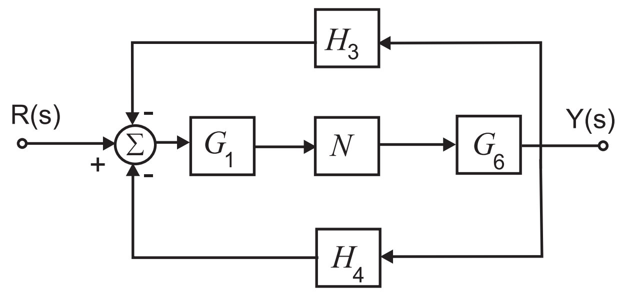
(d)



$$\frac{Y(s)}{R(s)} = \frac{D + AB^*}{1 + G(D + AB^*)} = \frac{D + DBH + AB}{1 + BH + GD + GBDH + GAB}.$$

21. Use block-diagram algebra to determine the transfer function between $R(s)$ and $Y(s)$ in Fig. 3.52.

Solution:



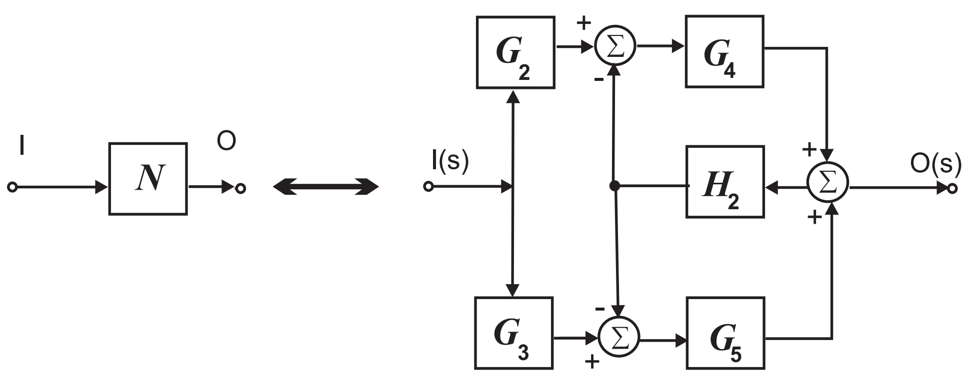
Block diagram for Fig. 3.52: reduced.

$$\begin{aligned} Q &= R - PH_3 - PH_4 \\ &= R - P(H_3 - H_4) \\ P &= G_1 N G_6 = Y \end{aligned}$$

We find that:

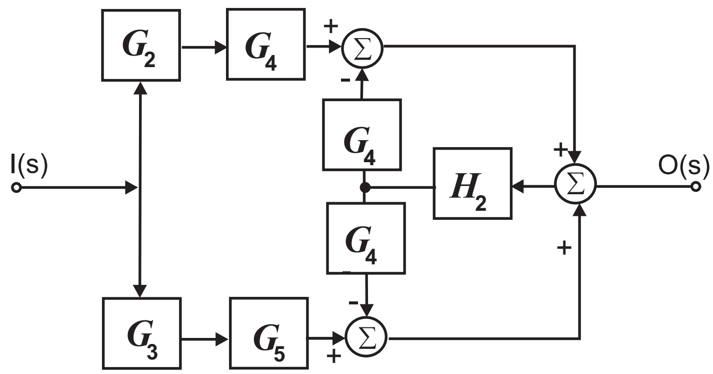
$$\frac{Y}{R} = \frac{G_1 N G_6}{1 + (H_3 + H_4) G_1 N G_6}$$

Now, what is N?



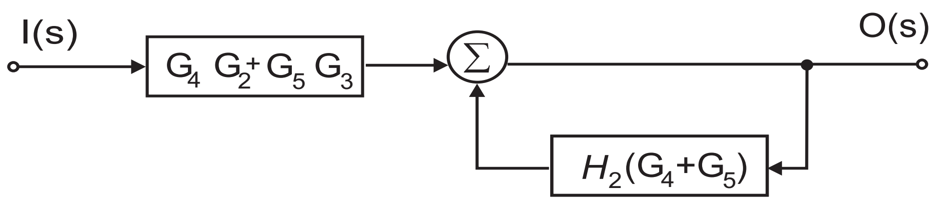
Block diagrams for Fig. 3.52.

Move G_4 and G_3 :



Block diagram for Fig. 3.52: reduced.

Combine symmetric loops as in the first step:



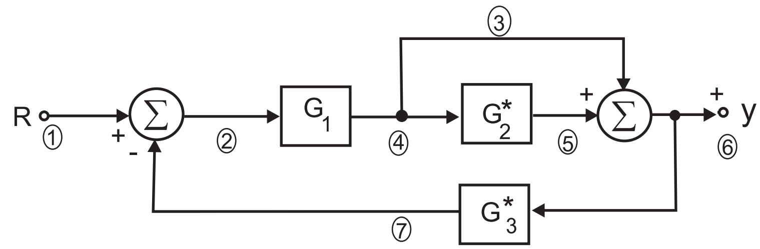
Block diagram for Fig. 3.52: reduced.

The result is:

$$
\begin{aligned}
N &= \frac{O}{I} = \frac{G_4G_2 + G_5G_3}{1 + H_2(G_4 + G_5)}, \\
\frac{Y(s)}{R(s)} &= \frac{G_1(G_4G_2 + G_5G_3)G_6}{1 + H_2(G_4 + G_5) + (H_3 + H_4)G_1(G_4G_2 + G_5G_3)G_6}.
\end{aligned}
$$

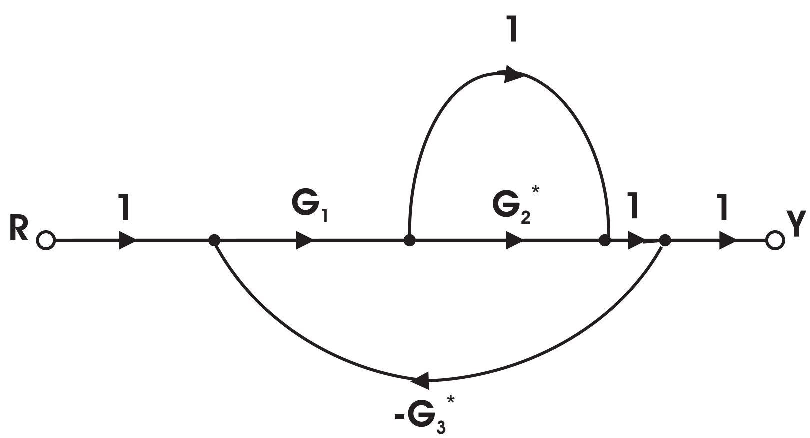
22. ▲ Find the transfer functions for the block diagrams in Fig. 3.51, using Mason's rule.

Solution: Transfer functions are found using Mason's rule,



Block diagram for Fig. 3.51 (a).

$$
\begin{aligned}
G_2^* &= \frac{G_2}{1 - G_2 H_2}, \\
G_3^* &= \frac{G_3}{1 - G_3 H_3}.
\end{aligned}
$$



Flow graph for Fig. 3.51(a).

(a) Mason's rule for Fig. 3.51(a):

Forward Path Gains

1 2 4 5 6	$p_1 = G_1 G_2^*$
1 2 4 3 6	$p_2 = G_1$

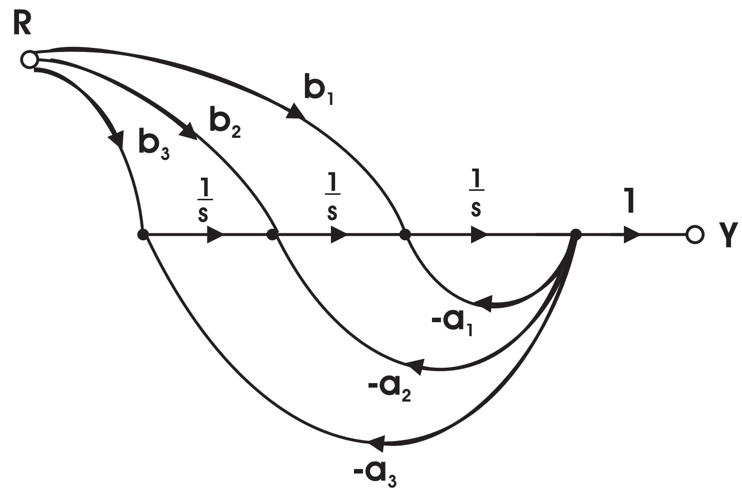
Loop Path Gains

2 4 3 7 2	$\ell_1 = -G_1 G_3^*$
2 4 5 7 2	$\ell_2 = -G_1 G_2^* G_3^*$

$$\begin{aligned}\frac{Y}{R} &= \frac{p_1+p_2}{1-\ell_1-\ell_2} = \frac{G_1(1+G_2^*)}{1+G_1G_3^*(1+G_2^*)} \\ &= \frac{G_1(1-G_2H_2)(1-G_3H_3)+G_1G_2(1-G_3H_3)}{1+(1-G_2H_2)(1-G_3H_3)+G_1G_3(1-G_2H_2)+G_1G_2G_3}.\end{aligned}$$

This is the same answer as in Problem 3.20(a).

(b) Mason's rule for Fig. 3.51(b):



Flow graph for Fig. 3.51(b).

Forward path gains:

$$p_1 = \frac{b_3}{s^3}, \quad p_2 = \frac{b_2}{s^2}, \quad p_3 = \frac{b_1}{s}$$

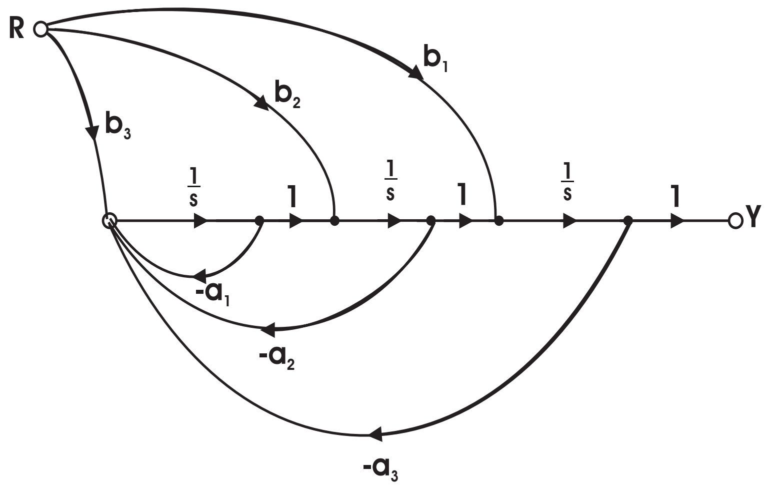
Loop path gains:

$$\ell_1 = -\frac{a_3}{s^3}, \quad \ell_2 = -\frac{a_2}{s^2}, \quad \ell_3 = -\frac{a_1}{s}$$

$$\frac{Y}{R} = \frac{p_1+p_2+p_3}{1-\ell_1-\ell_2-\ell_3} = \frac{\frac{b_3}{s^3}+\frac{b_2}{s^2}+\frac{b_1}{s}}{1+\frac{a_3}{s^3}+\frac{a_2}{s^2}+\frac{a_1}{s}} = \frac{b_1s^2+b_2s+b_3}{s^3+a_1s^2+a_2s+a_3}$$

This is the same answer as in Problem 3.20(b).

(c) Mason's Rule for Fig. 3.51(c):



Flow graph for Fig. 3.51(c).

Forward path gains:

$$p_1 = \frac{b_3}{s^3}, \quad p_2 = \frac{b_2}{s^2}[1 + \frac{a_1}{s}], \quad p_3 = \frac{b_1}{s}[1 + \frac{a_1}{s} + \frac{a_2}{s^2}]$$

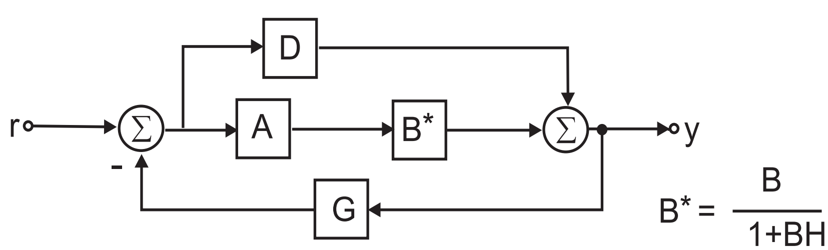
Loop path gains:

$$\ell_1 = -\frac{a_3}{s^3}, \quad \ell_2 = -\frac{a_2}{s^2}, \quad \ell_3 = -\frac{a_1}{s}$$

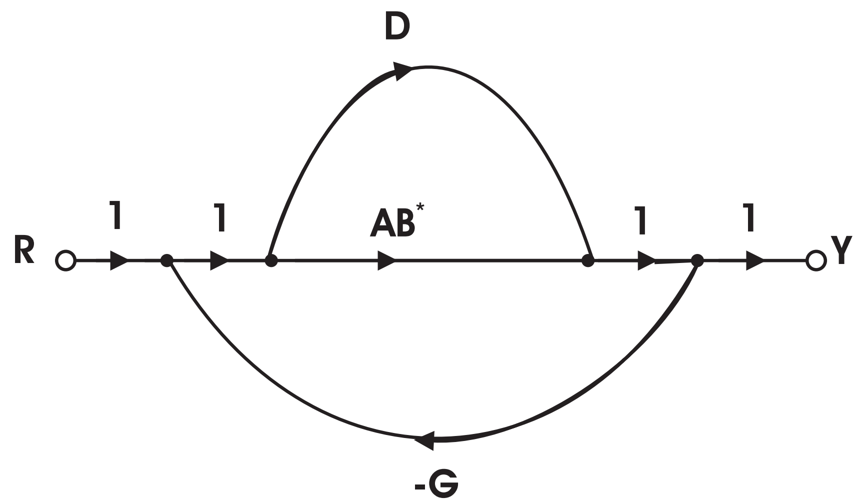
$$\frac{Y}{R} = \frac{p_1 + p_2 + p_3}{1 - \ell_1 - \ell_2 - \ell_3} = \frac{b_1 s^2 + (a_1 b_1 + b_2)s + a_1 b_2 + a_2 b_1 + b_3}{s^3 + a_1 s^2 + a_2 s + a_3}$$

This is the same answer as in Problem 3.20(c).

(d) Mason's rule for Fig. 3.51(d): The system is tightly connected, easy to apply Mason's rule.



Block diagram for Fig. 3.51(d).



Flow graph for Fig. 3.51(d).

Forward path gains:

$$p_1 = D, \quad p_2 = AB^*$$

Loop path gains:

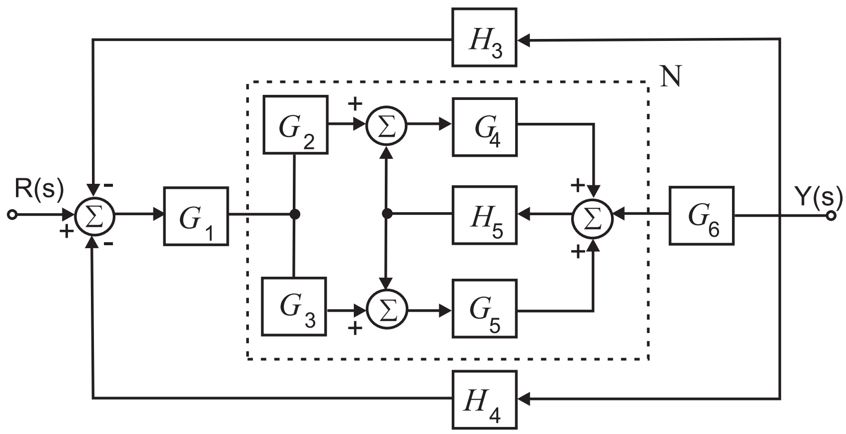
$$\ell_1 = -GD, \quad \ell_2 = -AB^*G$$

$$\frac{Y}{R} = \frac{p_1 + p_2}{1 - \ell_1 - \ell_2} = \frac{D + AB^*}{1 + G(D + AB^*)} = \frac{D + DBH + AB}{1 + BH + GD + GBDH + GAB}$$

This is the same answer as in Problem 3.20(d).

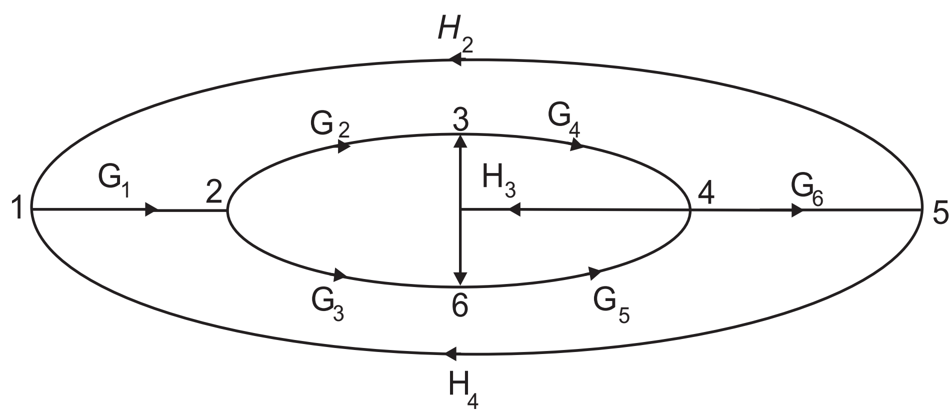
23. ▲ Use Mason's rule to determine the transfer function between $R(s)$ and $Y(s)$ in Fig. 3.52.

Solution:



Block diagram for Fig. 3.52.

The associated signal flow graph is,



Flow graph for Fig. 3.52.

Forward Path	**Gain**
1 2 3 4 5	$p_1 = G_1G_2G_4G_6$
1 2 6 4 5	$p_2 = G_1G_3G_5G_6$
Loop Path	**Gain**
1 2 3 4 5 1	$\ell_1 = -G_1G_2G_4G_6H_3$
1 2 3 4 5 1	$\ell_2 = -G_1G_2G_4G_6H_4$
1 2 6 4 5 1	$\ell_3 = -G_1G_3G_5G_6H_3$
1 2 6 4 5 1	$\ell 4 = -G_1G_3G_5G_6H_4$
3 4 3	$\ell_5 = -G_4H_2$
3 4 3	$\ell_6 = -G_5H_2$

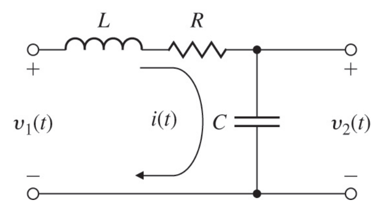


Figure 3.53: Circuit for Problem 3.24

and the determinants are

$$\begin{aligned}
\Delta &= 1 + [(H_3 + H_4)G_1(G_2G_4 + G_3G_5)G_6 + H_2(G_4 + G_5)], \\
\Delta_1 &= 1 - (0), \\
\Delta_2 &= 1 - (0), \\
\Delta_3 &= 1 - (0), \\
\Delta_4 &= 1 - (0).
\end{aligned}$$

Applying the rule, the transfer function is

$$\begin{aligned}
\frac{Y(s)}{R(s)} &= \frac{1}{\Delta}\sum G_i\Delta_i = \frac{p_1 + p_2}{1 - \ell_1 - \ell_2 - \ell_3 - \ell_4 - \ell_5 - \ell_6} \\
&= \frac{G_1(G_4G_2 + G_5G_3)G_6}{1 + H_2(G_4 + G_5) + (H_3 + H_4)G_1(G_4G_2 + G_5G_3)G_6}.
\end{aligned}$$

Problems and Solutions for Section 3.3: Effect of Pole Locations

24. For the electric circuit shown in Fig. 3.53, find the following:

    (a) The time-domain equation relating *i(t)* and $v_1(t)$;

    (b) The time-domain equation relating *i(t)* and $v_2(t)$;

    (c) Assuming all initial conditions are zero, the transfer function $V_2(s)/V_1(s)$ and the damping ratio ζ and undamped natural frequency ω_n of the system;

    (d) The values of R that will result in $v_2(t)$ having an overshoot of no more than 25%, assuming $v_1(t)$ is a unit step, $L = 10\ mH$, and $C = 4\ \mu F$.

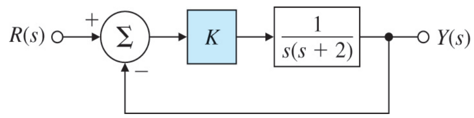


Figure 3.54: Unity feedback system for Problem 3.25

Solution:

(a)

$$v_1(t) = L\frac{di}{dt} + Ri + \frac{1}{C}\int i(t)dt.$$

(b)

$$v_2(t) = \frac{1}{C}\int i(t)dt.$$

(c)

$$\frac{V_2(s)}{V_1(s)} = \frac{\frac{1}{sC}}{sL + R + \frac{1}{sC}} = \frac{1}{s^2LC + sRC + 1}.$$

(d) For 25% overshoot $\zeta \approx 0.4$,

$$\begin{aligned} 0.4 &\approx \zeta = \frac{R}{2\sqrt{\frac{L}{C}}} \\ R &= 2\zeta\sqrt{\frac{L}{C}} = (2)(0.4)\sqrt{\frac{10\times10^{-3}}{4\times10^{-6}}} = 40\Omega. \end{aligned}$$

25. For the unity feedback system shown in Fig. 3.54, specify the gain K of the proportional controller so that the output $y(t)$ has an overshoot of no more than 10% in response to a unit step.

Solution:

$$\begin{aligned} \frac{Y(s)}{R(s)} &= \frac{K}{s^2 + 2s + K} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}, \\ \omega_n &= \sqrt{K}, \\ \zeta &= \frac{2}{2\omega_n} = \frac{1}{\sqrt{K}}. \qquad (1) \end{aligned}$$

In order to have an overshoot of no more than 10%:

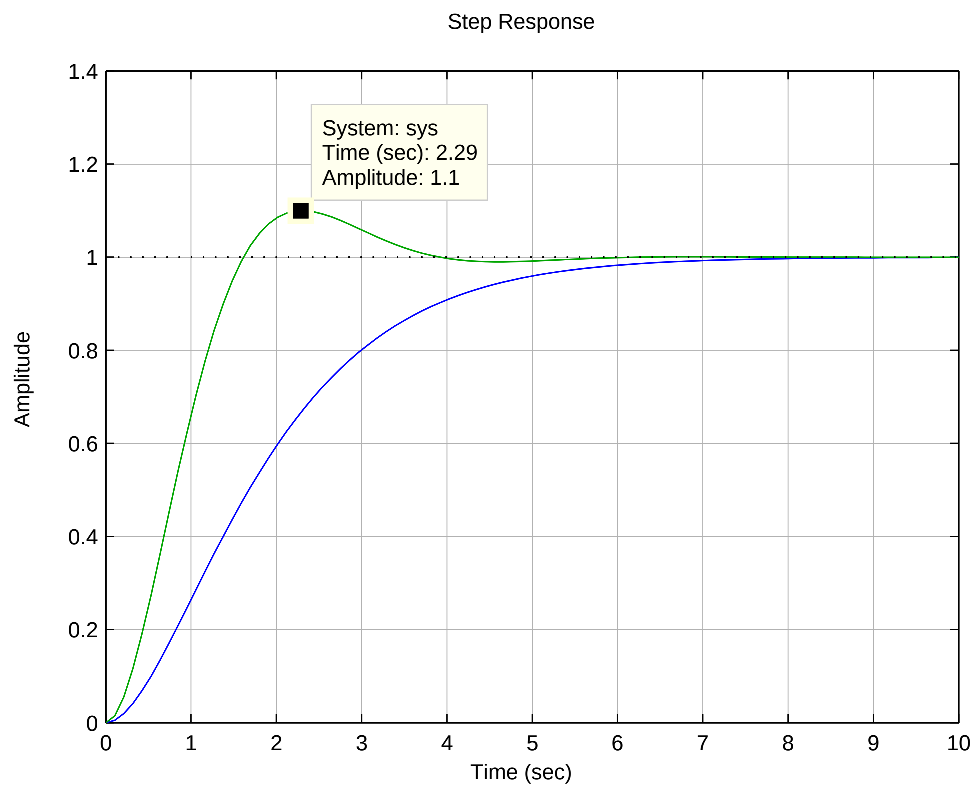
$$M_p = e^{-\pi\zeta/\sqrt{1-\zeta^2}} \leq 0.10.$$

Solving for ζ :

$$\zeta = \sqrt{\frac{(\ln M_p)^2}{\pi^2 + (\ln M_p)^2}} \geq 0.591.$$

Using (1) and the solution for ζ:

$$K = \frac{1}{\zeta^2} \leq 2.86,$$
$$\therefore 0 < K \leq 2.86.$$



Step responses for $K = 1$ (blue) and $K = 2.86$ (green).

26. For the unity feedback system shown in Fig. 3.55, specify the gain and pole location of the compensator so that the overall closed-loop response to a unit-step input has an overshoot of no more than 25%, and a 1% settling time of no more than 0.1 sec. Verify your design using MATLAB.

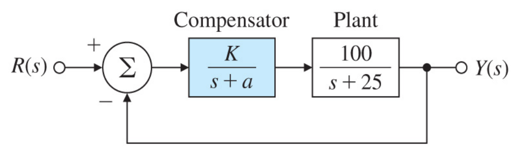


Figure 3.55: Unity feedback system for Problem 3.26

Solution:

$$\frac{Y(s)}{R(s)} = \frac{100K}{s^2 + (25+a)s + 25a + 100K} = \frac{100K}{s^2 + 2\zeta\omega_n s + \omega_n^2}.$$

Using the given information:

$$\begin{aligned} R(s) &= \frac{1}{s} \qquad \text{unit step}, \\ M_p &\leq 25\%, \\ t_s &\leq 0.1\,\text{sec}\,. \end{aligned}$$

Solve for ζ :

$$\begin{aligned} M_p &= e^{-\pi\zeta/\sqrt{1-\zeta^2}}, \\ \zeta &= \sqrt{\frac{(\ln M_p)^2}{\pi^2 + (\ln M_p)^2}} \geq 0.4037. \end{aligned}$$

Solve for ω_n:

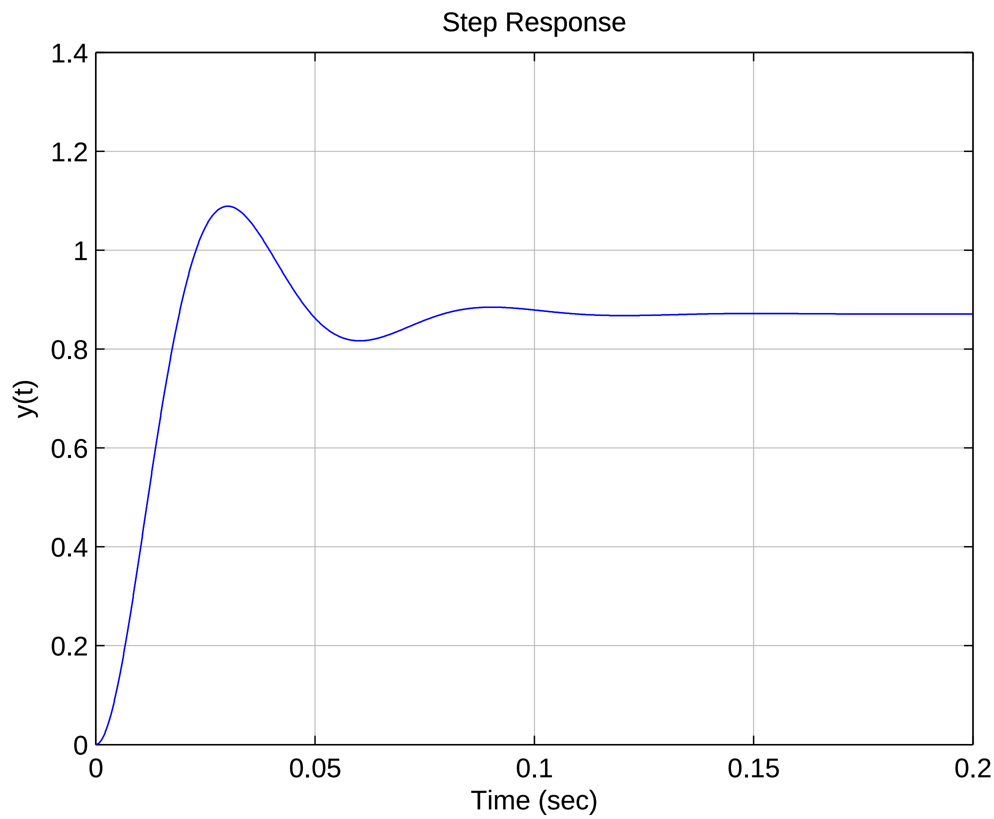
$$e^{-\zeta\omega_n t_s} = 0.01 \qquad \text{For a 1\% settling time.}$$

$$t_s \leq \frac{4.605}{\zeta\omega_n} = 0.1,$$
$$\Longrightarrow \omega_n \approx 114.07.$$

Now find a and K :

$$2\zeta\omega_n = (25 + a),$$
$$a = 2\zeta\omega_n - 25 = 92.10 - 25 = 67.10,$$
$$\omega_n^2 = (25a + 100K),$$
$$K = \frac{\omega_n^2 - 25a}{100} \approx 113.34.$$

The step response of the system using MATLAB is shown next.

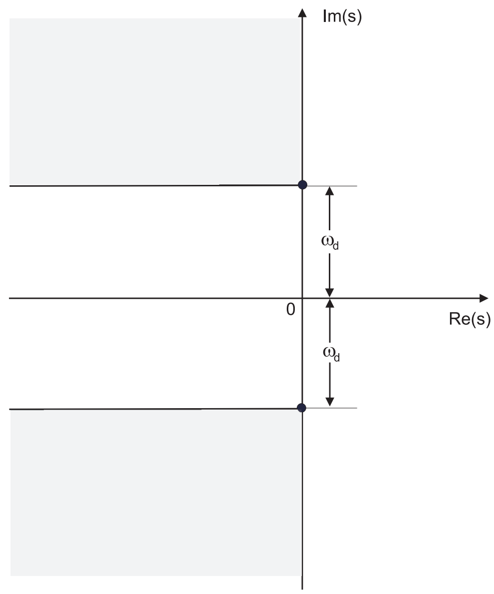


Step response for Problem 3.26.

Problems and Solutions for Section 3.4: Time-Domain Specifications

27. Suppose you desire the peak time of a given second-order system to be less than t_p'. Draw the region in the s-plane that corresponds to values of the poles that meet the specification $t_p < t_p'$.

Solution:



s-plane region to meet peak time constraint: shaded.

$$\omega_d t_p = \pi \Longrightarrow t_p = \frac{\pi}{\omega_d} < t_p^{'},$$
$$\frac{\pi}{t_p'} < \omega_d.$$

28. A certain servomechanism system has dynamics dominated by a pair of complex poles and no finite zeros. The time-domain specifications on the rise time (t_r), percent overshoot

(M_p), and settling time (t_s) are given by,

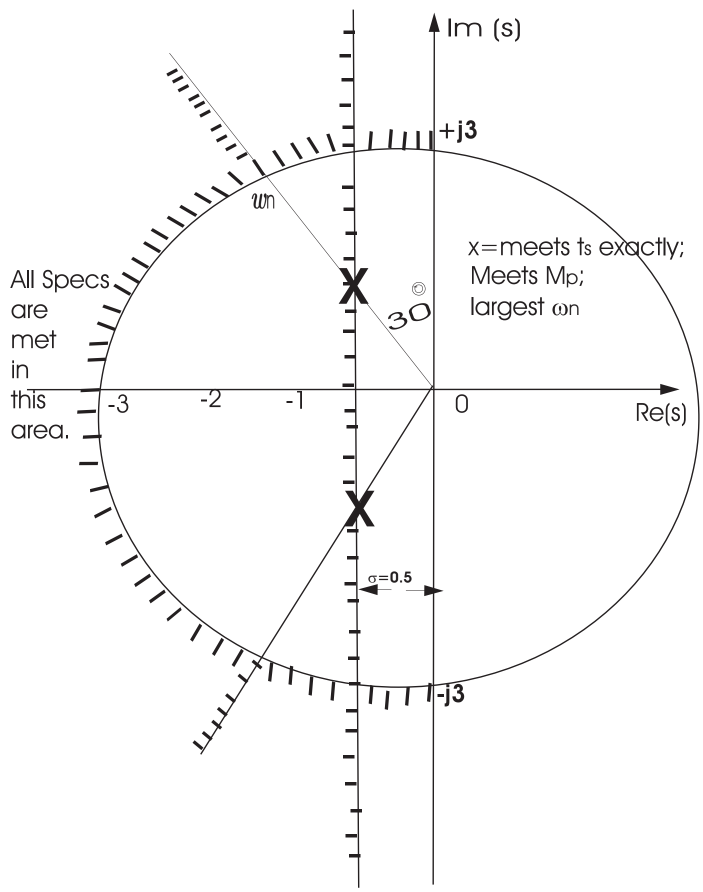
$$\begin{aligned} t_r &\leq 0.6\,\text{sec}\,, \\ M_p &\leq 17\%, \\ t_s &\leq 9.2\,\text{sec}\,. \end{aligned}$$

(a) Sketch the region in the s-plane where the poles could be placed so that the system will meet *all* three specifications.

(b) Indicate on your sketch the specific locations (denoted by $\times$) that will have the smallest rise-time and also meet the settling time specification *exactly*.

Solution:

(a)-(b)



s-plane region to meet the specifications.

29. A feedback system has the following response specifications,

- Percent overshoot $M_p \leq 16\%$.
- Settling time $t_s \leq 6.9$ sec.
- Rise time $t_r \leq 1.8$ sec.

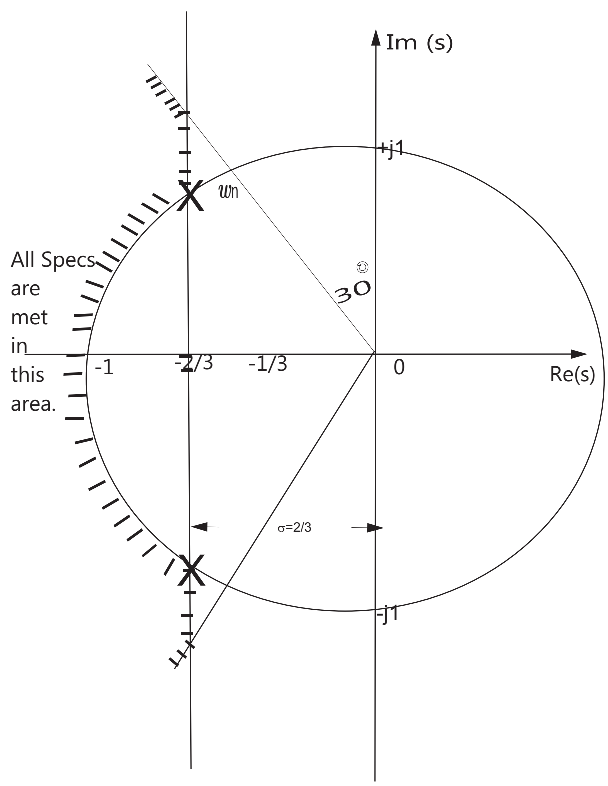
(a) Sketch the region of acceptable closed-loop poles in the s-plane for the system assuming the transfer function can be approximated as simple second-order.

(b) What is the expected overshoot if the rise time and settling time specifications are met *exactly*?

Solution:

(a) $M_p \leq 16\% \Rightarrow \zeta \geqslant 0.5$, $\sigma \geqslant \frac{4.6}{6.9} = \frac{2}{3}$, $\omega_n \geqslant \frac{1.8}{t_r} = 1$.

(b) At the points where the ω_n and σ lines meet ($\times$) we have $\zeta = 2/3 = 0.66$. From Fig. 3.23 $M_p \approx 6\%$.



s-plane region to meet the specifications.

30. Suppose you are to design a unity feedback controller for a first-order plant depicted in Fig. 3.56. (As you will learn in Chapter 4, the configuration shown is referred to as a proportional-integral controller.) You are to design the controller so that the closed-loop poles lie within the shaded regions shown in Fig. 3.56.

(a) What values of ω_n and ζ correspond to the shaded regions in Fig. 3.57? (A simple estimate from the figure is sufficient.)

(b) Let $K_\alpha = \alpha = 2$. Find values for K and K_I so that the poles of the closed-loop system lie within the shaded regions.

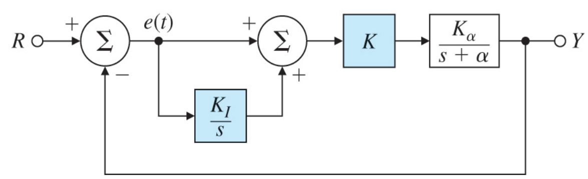


Figure 3.56: Unity feedback system for Problem 3.30

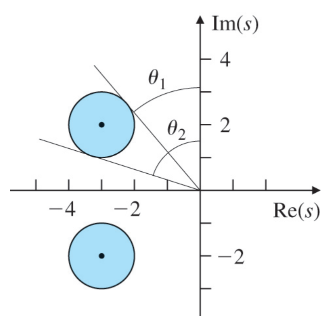


Figure 3.57: Desired closed-loop pole locations for Problem 3.30

(c) Prove that no matter what the values of K_α and α are, the controller provides enough flexibility to place the poles anywhere in the complex (left-half) plane.

Solution:

(a) The values could be worked out mathematically but working from the diagram:

$$\begin{aligned}\sqrt{3^2+2^2} &= 3.6 \Longrightarrow 2.6 \le \omega_n \le 4.6,\\ \theta &= \sin^{-1}\zeta,\\ \zeta &= \sin\theta.\end{aligned}$$

From the figure:

$$\begin{aligned}\theta &\approx 34^\circ \qquad \zeta_1 = 0.554,\\ \theta &\approx 70^\circ \qquad \zeta_2 = 0.939,\end{aligned}$$

$$\Longrightarrow 0.6 \le \zeta \le 0.9 \qquad (\textit{approximately})$$

(b) Closed-loop pole positions:

$$\begin{aligned}s(s+\alpha)+(Ks+KK_I)K_\alpha &= 0,\\ s^2+(\alpha+KK_\alpha)s+KK_IK_\alpha &= 0.\end{aligned}$$

For this case:

$$s^2+(2+2K)s+2KK_I = 0 \qquad (*)$$

Choose roots that lie in the center of the shaded region,

$$\begin{gathered}(s+(3+j2))(s+(3-j2)) = s^2+6s+13 = 0,\\ s^2+(2+2K)s+2KK_I = s^2+6s+13,\\ 2+2K = 6 \Longrightarrow K = 2,\\ 13 = 4K_I \Longrightarrow K_I = \frac{13}{4}.\end{gathered}$$

(c) For the closed-loop pole positions found in part (b), in the (*) equation the value of K can be chosen to make the coefficient of s take on any value. For this value of K a value of K_I can be chosen so that the quantity KK_IK_α takes on any value desired. This implies that the poles can be placed anywhere in the complex plane.

31. The open-loop transfer function of a unity feedback system is

$$G(s) = \frac{K}{s(s+2)}.$$

The desired system response to a step input is specified as peak time $t_p = 1$ sec and overshoot $M_p = 5\%$.

(a) Determine whether both specifications can be met simultaneously by selecting the right value of K.

(b) Sketch the associated region in the s-plane where both specifications are met, and indicate what root locations are possible for some likely values of K.

(c) Pick a suitable value for K, and use MATLAB to verify that the specifications are satisfied.

Solution:

(a)

$$T(s) = \frac{Y(s)}{R(s)} = \frac{G(s)}{1+G(s)} = \frac{K}{s^2+2s+K} = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}.$$

Equate the coefficients of like powers of s:

$$\begin{array}{r} 2 = 2\zeta\omega_n \\ K = \omega_n^2 \end{array} \quad (*)$$
$$\Longrightarrow \quad \omega_n = \sqrt{K} \qquad \zeta = \frac{1}{\sqrt{K}}.$$

We would need:

$$\begin{array}{rcl} \frac{M_p\%}{100} = 0.05 = e^{\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}} & \Longrightarrow & \zeta = 0.69, \\ t_p = 1\,\text{sec} = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} & \Longrightarrow & \omega_n = 4.34. \end{array}$$

But the combination ($\zeta = 0.69$, $\omega_n = 4.34$) that we need is <u>not</u> possible by varying K alone. Observe that from equations (*) $\zeta\omega_n = 1 \neq 0.69 \times 4.34$.

(b) Now we wish to have:

$$\begin{array}{r} M_p^* = r \times 0.05 = e^{\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}} \\ t_p^* = r \times 1\,\text{sec} = \frac{\pi}{\omega_d} \end{array} \quad (**)$$

where $r \equiv$relaxation factor.

Recall the conditions of our system:

$$\begin{aligned}\omega_n &= \sqrt{K},\\ \zeta &= \frac{1}{\sqrt{K}},\end{aligned}$$

replace ω_n and ζ in the system (**):

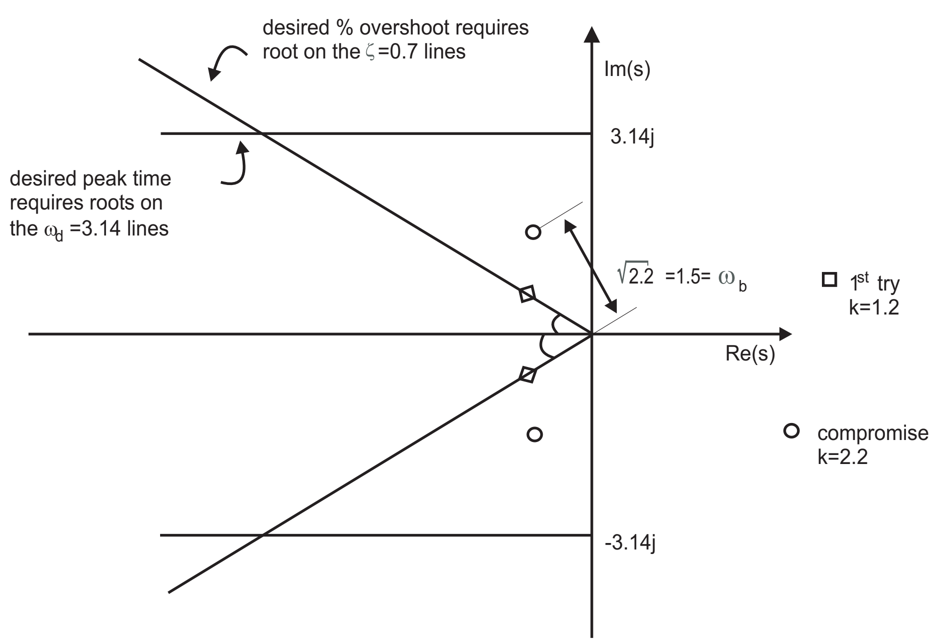
$$\Longrightarrow \quad \begin{array}{c} -\frac{\pi}{\sqrt{K-1}} = r \times 0.05 \\ 1\,\text{sec} = \frac{\pi}{\sqrt{K-1}} \end{array}$$

$$\begin{aligned}\Longrightarrow r \times 0.05 = e^{-r} \qquad &\Longrightarrow \quad r \cong 2.21.\\ K = 1 + \frac{\pi^2}{r^2} \qquad &\Longrightarrow \quad K = 3.02.\end{aligned}$$

then with $K = 3.02$ we will have:

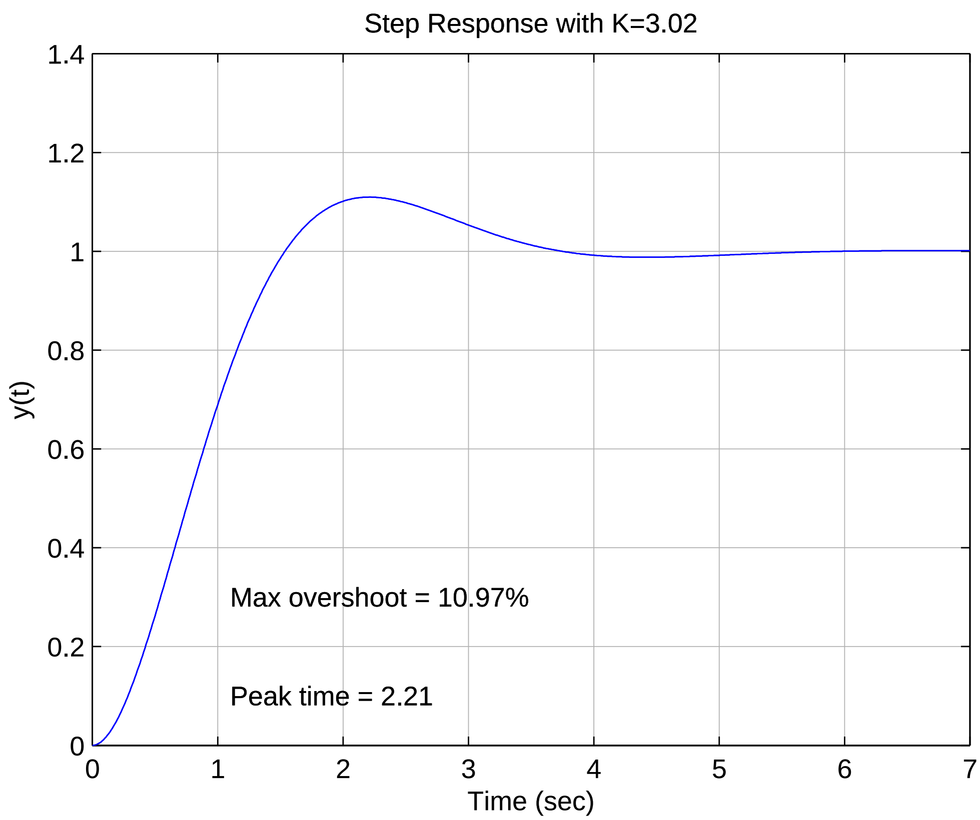
$$\begin{aligned}M_p^* &= rM_p = 2.21 \times 0.05 = 0.11.\\ t_p^* &= rt_p = 2.21 \times 1\,\text{sec} = 2.21\,\text{sec}\,.\end{aligned}$$

Note: * denotes actual location of closed-loop roots.



s-plane regions.

```
% Problem 3.31 FPE8e
K=3.02;
num=[K];
den=[1, 2, K];
sys=tf(num,den);
t=0:.01:7;
y=step(sys,t);
plot(t,y);
yss = dcgain(sys);
Mp = (max(y) - yss)*100;
% Finding maximum overshoot
msg_overshoot = sprintf('Max overshoot = %3.2f%%', Mp);
% Finding peak time
idx = max(find(y==(max(y))));
tp = t(idx);
msg_peaktime = sprintf('Peak time = %3.2f', tp);
xlabel('Time (sec)');
ylabel('y(t)');
msg_title = sprintf('Step Response with K=%3.2f',K);
title(msg_title);
text(1.1, 0.3, msg_overshoot);
text(1.1, 0.1, msg_peaktime);
grid on;
```



Problem 3.31: Closed-loop step response.

32. A simple mechanical system is shown in Fig. 3.58 (a). The parameters are k =spring constant, b =viscous friction constant, m =mass. A step of 2 Newtons force is applied as $f = 2 \times 1(t)$ and the resulting step response is shown in Fig. 3.58 (b). What are the values of the system parameters k, b, and m?

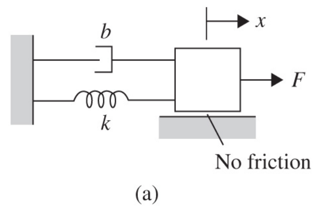
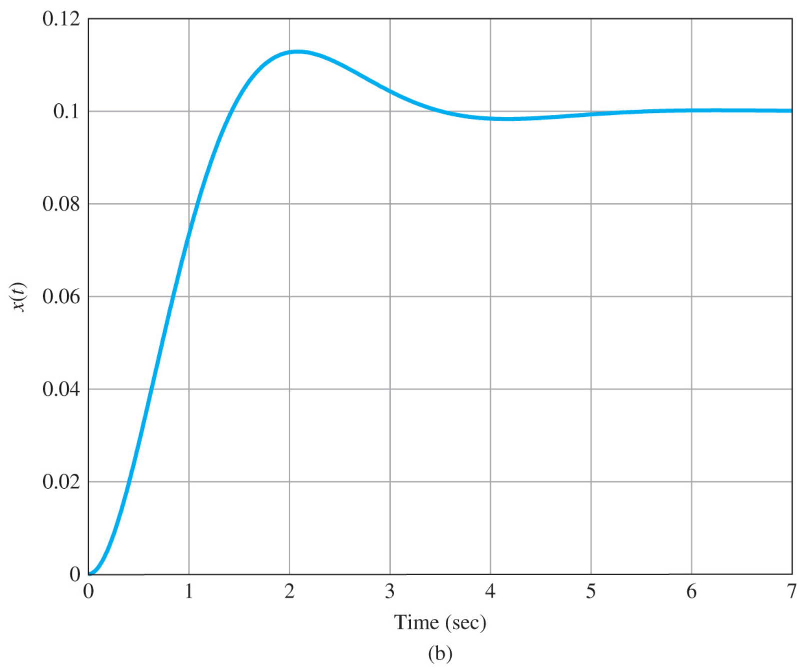


Figure 3.58: (a) Mechanical system for Problem 3.32



(b) Step response for Problem 3.32

Solution: The equation of motion is

$$m\ddot{x} + b\dot{x} + kx = F.$$

The transfer function is

$$\frac{X(s)}{F(s)} = G(s) = \frac{\frac{1}{m}}{s^2 + \frac{b}{m}s + \frac{k}{m}}.$$

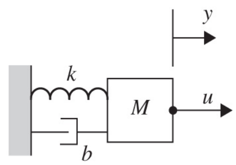


Figure 3.59: Simple mechanical system for Problem 3.33

In this case

$$\begin{aligned} 2G(0) &= 0.1, \\ 2(\left(\frac{1}{k}\right)) &= 0.1, \\ k &= 20. \end{aligned}$$

We observe that

$$\omega_n^2 = \frac{k}{m} = \frac{20}{m}, \quad 2\zeta\omega_n = \frac{b}{m}.$$

From the plot

$$\begin{aligned} t_r &= 1\,\text{sec} = \frac{1.8}{\omega_n} \Rightarrow \omega_n = 1.8 = \sqrt{\frac{20}{m}} \Rightarrow m = 6.17, \\ M_p &= \frac{y(t_p) - y(\infty)}{y(\infty)} = (0.113 - 0.1)/0.1 \times 100\% = 13.1\% \Rightarrow \zeta = 0.543, \\ b &= 2\zeta\omega_n m = 2(0.543)(1.8)(6.17) = 12.06. \end{aligned}$$

33. A mechanical system is shown in Fig. 3.59. The mass $M = 20$ kg and the control force, u, is proportional to the reference input, $u = Ar$.

(a) Derive the transfer function from R to Y.

(b) Determine the values of the parameters k, b, A such that the system has a rise time of $t_r = 1$ sec and overshoot of $M_p = 16\%$, and zero-steady-state error to a step in r.

Solution: (a) The equation of motion is

$$M\ddot{y} + b\dot{y} + ky = u$$

The transfer function is

$$\begin{aligned} (Ms^2 + bs + k)Y(s) &= U(s) = AR(s), \\ \frac{Y(s)}{R(s)} &= T(s) = \frac{A}{Ms^2 + bs + k} = \frac{A}{M(s^2 + \frac{b}{M}s + \frac{k}{M})}, \\ &= \frac{A}{M(s^2 + 2\zeta\omega_n s + \omega_n^2)}. \end{aligned}$$

(b)

$$
\begin{aligned}
T(0) &= 1 \Rightarrow \frac{A}{k} = 1, \\
t_r &= \frac{1.8}{\omega_n} = \frac{1.8}{\sqrt{\frac{k}{M}}} = 1 \Rightarrow \frac{k}{M} = 3.24, \\
M_p &= 16\% \Rightarrow \zeta = 0.5, \\
2\zeta\omega_n &= \frac{b}{M} \Rightarrow \omega_n = \frac{b}{M} = \sqrt{\frac{k}{M}} = 1.8
\end{aligned}
$$

Let us choose

$$A = k = 1, \quad M = \frac{1}{3.24} = 0.3086, \quad b = 0.5556.$$

34. The equations of motion for the DC motor shown in Fig. 2.33 were given in Eqs. (2.62-63) as

$$J_m \ddot{\theta}_m + (b + \frac{K_t K_e}{R_a})\dot{\theta}_m = \frac{K_t}{R_a} v_a.$$

Assume that

$$
\begin{aligned}
J_m &= 0.01 \ kg \cdot m^2, \\
b &= 0.001 \ N \cdot m \cdot sec, \\
K_e &= 0.02 \ V \cdot sec, \\
K_t &= 0.02 \ N \cdot m/A, \\
R_a &= 10 \ \Omega.
\end{aligned}
$$

(a) Find the transfer function between the applied voltage v_a and the motor speed $\dot{\theta}_m$.

(b) What is the steady-state speed of the motor after a voltage $v_a = 10$ V has been applied?

(c) Find the transfer function between the applied voltage v_a and the shaft angle θ_m.

(d) Suppose feedback is added to the system in part (c) so that it becomes a position servo device such that the applied voltage is given by

$$v_a = K(\theta_r - \theta_m),$$

where K is the feedback gain. Find the transfer function between θ_r and θ_m.

(e) What is the maximum value of K that can be used if an overshoot $M_p < 20\%$ is desired?

(f) What values of K will provide a rise time of less than 4 sec? (Ignore the M_p constraint.)

(g) Use MATLAB to plot the step response of the position servo system for values of the gain $K = 0.5$, 1, and 2. Find the overshoot and rise time for each of the three step responses by examining your plots. Are the plots consistent with your calculations in parts (e) and (f)?

Solution:

$$J_m\ddot{\theta}_m + (b + \frac{K_tK_e}{R_a})\dot{\theta}_m = \frac{K_t}{R_a}v_a.$$

(a)

$$\begin{aligned} J_m\Theta_m s^2 + (b + \frac{K_tK_e}{R_a})\Theta_m s &= \frac{K_t}{R_a}V_a(s) \\ \frac{s\Theta_m(s)}{V_a(s)} &= \frac{\frac{K_t}{R_aJ_m}}{s + \frac{b}{J_m} + \frac{K_tK_e}{R_aJ_m}}. \end{aligned}$$

$$\begin{aligned} J_m &= 0.01kg\cdot m^2, \\ b &= 0.001N\cdot m\cdot \sec, \\ K_e &= 0.02V\cdot\sec, \\ K_t &= 0.02N\cdot m/A, \\ R_a &= 10\Omega. \end{aligned}$$

$$\frac{s\Theta_m(s)}{V_a(s)} = \frac{0.2}{s+0.104}.$$

(b) Final Value Theorem

$$\dot{\theta}(\infty) = \frac{s(10)(0.2)}{s(s+0.104)}|_{s=0} = \frac{2}{0.104} = 19.23.$$

(c)

$$\frac{\Theta_m(s)}{V_a(s)} = \frac{0.2}{s(s+0.104)}.$$

(d)

$$\begin{aligned} \Theta_m(s) &= \frac{0.2K(\Theta_r - \Theta_m)}{s(s+0.104)}. \\ \frac{\Theta_m(s)}{\Theta_r(s)} &= \frac{0.2K}{s^2 + 0.104s + 0.2K}. \end{aligned}$$

(e)

$$\begin{aligned}
M_p &= e^{-\pi\zeta/\sqrt{1-\zeta^2}} = 0.2 \qquad (20\%), \\
\zeta &= 0.4559. \\
Y(s) &= \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}. \\
2\zeta\omega_n &= 0.104, \\
\omega_n &= \frac{0.104}{2(0.4559)} = 0.114 rad/\sec, \\
\omega_n^2 &= 0.2K, \\
K &< 6.50 \times 10^{-2}.
\end{aligned}$$

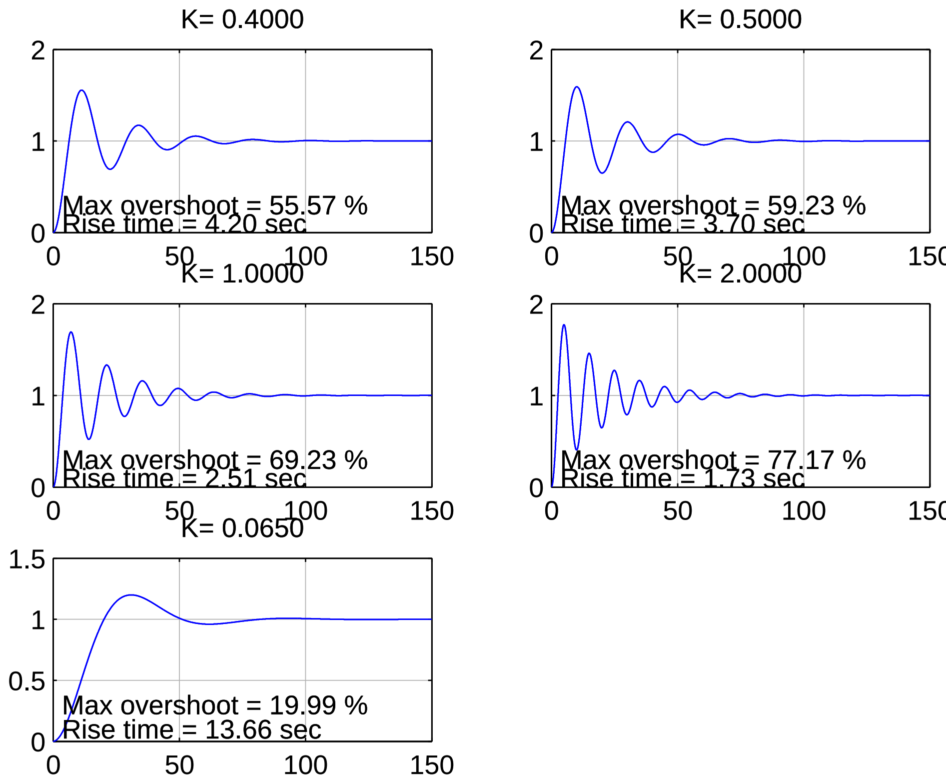
(f)

$$\begin{aligned}
\omega_n &\geq \frac{1.8}{t_r} \\
\omega_n^2 &= 0.2K \\
K &\geq 1.01.
\end{aligned}$$

(g) MATLAB

```
% Problem 3.34 FPE8e
clear all
close all
K1=[0.5 1.0 2.0 6.5e-2];
t=0:0.01:150;
for i=1:1:length(K1)
K = K1(i);
titleText = sprintf(' K= %1.4f ', K);
wn = sqrt(0.2*K);
num=wn^2;
den=[1 0.104 wn^2];
zeta=0.104/2/wn;
sys = tf(num, den);
y= step(sys, t);
% Finding maximum overshoot
if zeta < 1
Mp = (max(y) - 1)*100;
overshootText = sprintf(' Max overshoot = %3.2f %', Mp);
else
overshootText = sprintf(' No overshoot');
end
```

```
% Finding rise time
idx_01 = max(find(y <0.1));
idx_09 = min(find(y >0.9));
t_r = t(idx_09) - t(idx_01);
risetimeText = sprintf(' Rise time = %3.2f sec', t_r);
% Plotting
subplot(3,2,i);
plot(t,y);
grid on;
title(titleText);
text( 0.5, 0.3, overshootText);
text( 0.5, 0.1, risetimeText);
end
%%%%%%%%%%%%%%%%%%
% Function for computing rise time
function tr = risetime(t,y)
% A. Emami 2006
% normalize y to 1:
y = y/y(length(y));
idx1 = min(find(y >=0.1));
idx2 = min(find(y >=0.9));
if ~isempty(idx1) & ~isempty(idx2);
tr = t(idx2) - t(idx1);
else
tr = 0
end
```



Problem 3.34: Closed-loop step responses for several values of K.

For part (e) we concluded that $K < 6.50 \times 10^{-2}$ in order for $M_p <$ 20%. This is consistent with the above plots. For part (f) we found that $K \geq 1.01$ in order to have a rise time of less than 4 seconds. We actually see that our calculations is slightly off and that K can be $K \geq 0.5$, but since $K \geq 1.01$ is included in $K \geq 0.5$, our answer in part (f) is consistent with the above plots.

35. You wish to control the elevation of the satellite-tracking antenna shown in Figs. 3.60 and 3.61. The antenna and drive parts have a moment of inertia J and a damping B; these arise to some extent from bearing and aerodynamic friction, but mostly from the back emf of the DC drive motor. The equations of motion are

$$J\ddot{\theta} + B\dot{\theta} = T_c,$$

Figure 3.60: Satellite Antenna

where T_c is the torque from the drive motor. Assume that

$$J = 600,000 kg \cdot m^2 \quad B = 20,000\ N \cdot m \cdot sec.$$

(a) Find the transfer function between the applied torque T_c and the antenna angle θ.

(b) Suppose the applied torque is computed so that θ tracks a reference command θ_r according to the feedback law

$$T_c = K(\theta_r - \theta),$$

where K is the feedback gain. Find the transfer function between θ_r and θ.

(c) What is the maximum value of K that can be used if you wish to have an overshoot $M_p < 10\%$?

(d) What values of K will provide a rise time of less than 80 sec? (Ignore the M_p constraint.)

(e) Use MATLAB to plot the step response of the antenna system for K = 200, 400, 1000, and 2000. Find the overshoot and rise time of the four step responses by examining your plots. Do the plots to confirm your calculations in parts (c) and (d)?

Solution:

$$J\ddot{\theta} + B\dot{\theta} = T_c$$

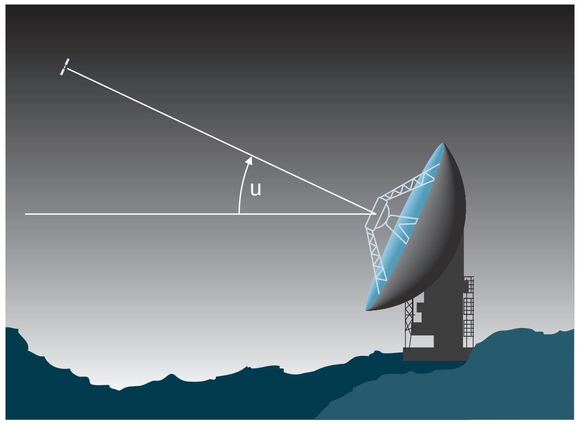


Figure 3.61: Schematic of antenna for Problem 3.35

(a)

$$\begin{aligned}
J\Theta s^2 + B\Theta s &= T_c(s), \\
\frac{\Theta(s)}{T_c(s)} &= \frac{1}{s(Js+B)}, \\
J &= 600,000 kg \cdot m^2, \\
B &= 20,000 N \cdot m \cdot \sec, \\
\frac{\Theta(s)}{T_c(s)} &= \frac{1.667 \times 10^{-6}}{s(s+\frac{1}{30})}.
\end{aligned}$$

(b)

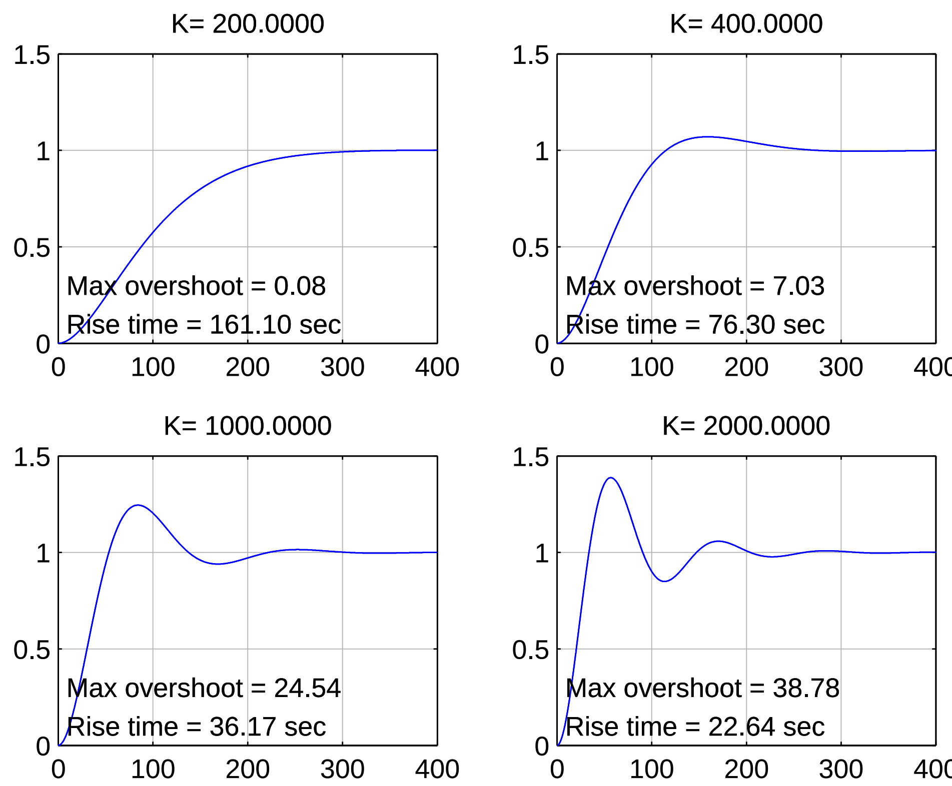
$$\begin{aligned}
\Theta(s) &= \frac{1.667 \times 10^{-6} K(\Theta_r - \Theta)}{s(s+\frac{1}{30})}, \\
\frac{\Theta(s)}{\Theta_r(s)} &= \frac{1.667K \times 10^{-6}}{s^2 + \frac{1}{30}s + 1.667K \times 10^{-6}}.
\end{aligned}$$

(c)

$$\begin{aligned}
M_p &= e^{-\pi\zeta/\sqrt{1-\zeta^2}} = 0.1 \qquad (10\%), \\
\zeta &= 0.591. \\
Y(s) &= \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}, \\
2\zeta\omega_n &= \frac{1}{30}, \\
\omega_n &= \frac{\frac{1}{30}}{2(0.591)} = 0.0282 rad/\sec, \\
\omega_n^2 &= 1.667K \times 10^{-6}, \\
K &< 477.
\end{aligned}$$

(d)

$$\begin{aligned}
\omega_n &\geq \frac{1.8}{t_r}, \\
\omega_n^2 &= 1.667K \times 10^{-6}, \\
K &\geq 304.
\end{aligned}$$



(e) Problem 3.35: Step responses for several values of K.

(e) The results compare favorably with the predictions made in parts (c) and (d). For $K < 477$ the overshoot was less than 10, the rise-time was less than 80 seconds.

36. (a) Show that the second-order system

$$\ddot{y} + 2\zeta\omega_n\dot{y} + \omega_n^2 y = 0, \quad y(0) = y_o, \quad \dot{y}(0) = 0,$$

has the response

$$y(t) = y_o \frac{e^{-\sigma t}}{\sqrt{1-\zeta^2}} \sin(\omega_d t + \cos^{-1}\zeta).$$

Prove that, for the underdamped case ($\zeta < 1$), the response oscillations decay at a predictable rate (see Fig. 3.62) called the **logarith-**

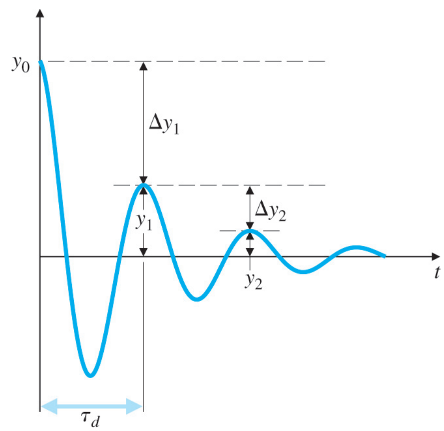


Figure 3.62: Definition of logarithmic decrement

mic decrement

$$\begin{aligned}\delta &= \ln \frac{y_o}{y_1} = \sigma\tau_d \\ &= \ln \frac{\Delta y_1}{y_1} \cong \ln \frac{\Delta y_i}{y_i},\end{aligned}$$

where

$$\tau_d = \frac{2\pi}{\omega_d} = \frac{2\pi}{\omega_n\sqrt{1-\zeta^2}},$$

is the damped natural period of vibration.
The damping coefficient in terms of the logarithmic decrement is then

$$\zeta = \frac{\delta}{\sqrt{4\pi^2+\delta^2}}.$$

Solution:

(a) We take Laplace transform of both sides of the equation to obtain

$$(s^2Y(s) - sy(0) - \dot{y}(0)) + 2\zeta\omega_n(sY(s) - y(0)) + \omega_n^2 Y(s) = 0.$$

We complete the squares to obtain,

$$Y(s) = y_0\frac{(s+2\zeta\omega_n)}{s^2+2\zeta\omega_n s+\omega_n^2} = y_0\frac{(s+2\zeta\omega_n)}{(s+\zeta\omega_n)^2+(\omega_n\sqrt{1-\zeta^2})^2}.$$

and rewrite $Y(s)$ as,

$$Y(s) = y_0 \frac{(s+\zeta\omega_n)}{(s+\zeta\omega_n)^2 + (\omega_n\sqrt{1-\zeta^2})^2} + y_0 \frac{\zeta\omega_n}{\omega_n\sqrt{1-\zeta^2}} \frac{\omega_n\sqrt{1-\zeta^2}}{(s+\zeta\omega_n)^2 + (\omega_n\sqrt{1-\zeta^2})^2}.$$

Using Items #19, #20 from theTable of Laplace Transorms we obtain,

$$y(t) = y_0[e^{-\zeta\omega_n t}\cos(\omega_d t) + e^{-\zeta\omega_n t}\frac{\zeta}{\sqrt{1-\zeta^2}}\sin(\omega_d t)],$$

$$y(t) = y_0 \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t}[\sqrt{1-\zeta^2}\cos(\omega_d t) + \zeta\sin(\omega_d t)],$$

We use the following trigonometric identity

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta,$$

where

$$\alpha = \omega_d t, \quad \cos\beta = \zeta, \quad \sin(\beta) = \sqrt{1-\zeta^2},$$

to obtain the compact answer

$$y(t) = y_0 \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin(\omega_d t + \cos^{-1}\zeta).$$

(b)

$$\dot{y}(t) = \frac{y_0}{\sqrt{1-\zeta^2}}[-\sigma e^{-\sigma t}\sin(\omega_d t + \cos^{-1}\zeta) + e^{-\sigma t}\omega_d\cos(\omega_d t + \cos^{-1}\zeta)].$$

$$\dot{y}(t) = \frac{y_0}{\sqrt{1-\zeta^2}} e^{-\sigma t}\sin\omega_d t.$$

$$\begin{aligned}
\dot{y}(t) &= 0 \qquad \Longrightarrow t = \frac{n\pi}{\omega_d} \qquad (\ n \text{ is any integer}) \\
t_{\max} &= 2\frac{n\pi}{\omega_d} = n\tau_d, \\
y(t)|_{t_{max}} &\equiv y_n = y_0 \frac{e^{-\sigma n\tau_d}}{\sqrt{1-\zeta^2}} \sin(\cos^{-1}\zeta).
\end{aligned}$$

Note:

$$\sin(\cos^{-1}\zeta) = \sqrt{1-\zeta^2}$$

$$y_n = \frac{y_0\sqrt{1-\zeta^2}}{\sqrt{1-\zeta^2}} e^{-\sigma n\tau_d} = y_0 e^{-\sigma n\tau_d} \qquad (*)$$

(Proof of the first line)

$$\delta = \ln \frac{y_0}{y_n} = \sigma \tau_d,$$

From (*)

$$y_1 = y_0 e^{-\sigma \tau_d} \Longrightarrow \ln \frac{y_0}{y_n} = \sigma \tau_d.$$

(Proof of the second line)

$$\begin{aligned} \Delta y_n &= y_{n-1} - y_n, \\ \Delta y_n &= y_0 e^{-(n-1)\sigma \tau_d} - y_0 e^{-\sigma n \tau_d} = y_0 e^{-\sigma n \tau_d}(e^{\sigma \tau_d} - 1), \end{aligned}$$

$$\begin{aligned} \Longrightarrow \quad \frac{\Delta y_n}{y_n} &= \frac{y_0 e^{-\sigma n \tau_d}}{y_0 e^{-\sigma n \tau_d}}(e^{\sigma \tau_d} - 1) = e^{\sigma \tau_d} - 1, \\ \Longrightarrow \quad \frac{\Delta y_n}{y_n} &= \frac{\Delta y_i}{yi} \qquad \text{for all } i, n. \end{aligned}$$

Problems and Solutions for Section 3.5: Effects of Zeros and Additional Poles

37. In aircraft control systems, an ideal pitch response (q_o) versus a pitch command (q_c) is described by the transfer function

$$\frac{Q_o(s)}{Q_c(s)} = \frac{\tau \omega_n^2 (s + 1/\tau)}{s^2 + 2\zeta \omega_n s + \omega_n^2}.$$

The actual aircraft response is more complicated than this ideal transfer function; nevertheless, the ideal model is used as a guide for autopilot design. Assume that t_r is the desired rise time and that

$$\begin{aligned} \omega_n &= \frac{1.789}{t_r}\ , \\ \frac{1}{\tau} &= \frac{1.6}{t_r}\ , \\ \zeta &= 0.89\ . \end{aligned}$$

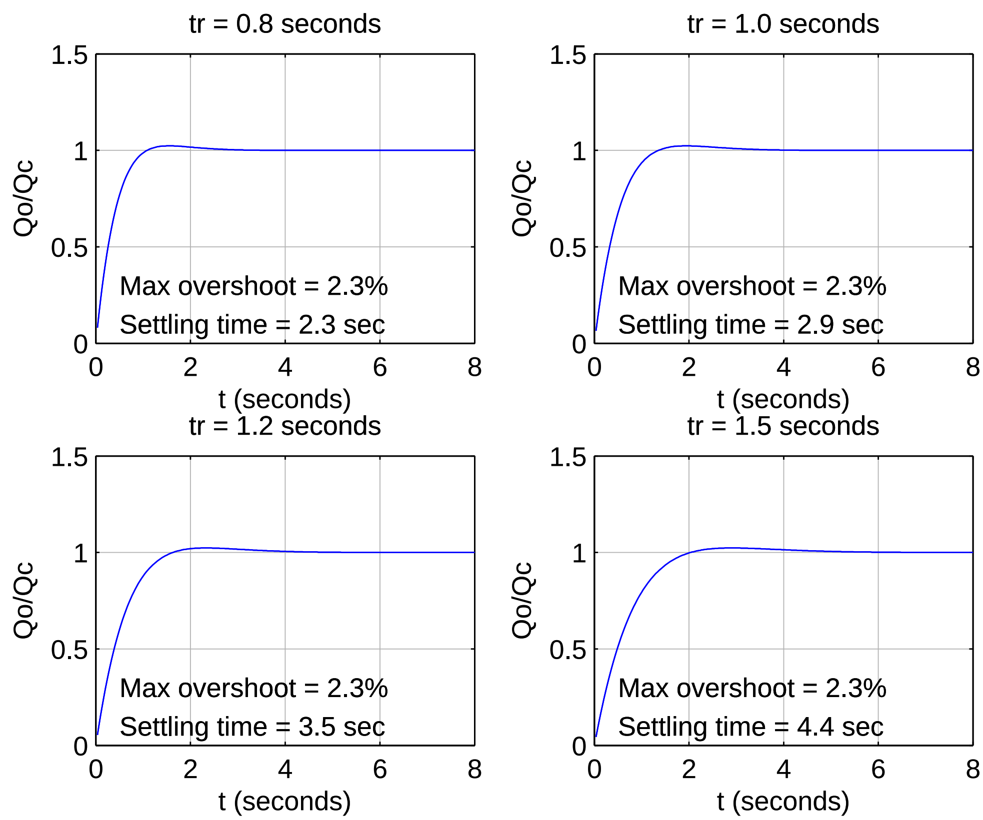
Show that this ideal response possesses a fast settling time and minimal overshoot by plotting the step response for $t_r = 0.8$, 1.0, 1.2, and 1.5 sec.

Solution:

The following program statements in MATLAB produce the following plots:

```
% Problem 3.37 FPE8e
tr = [0.8 1.0 1.2 1.5];
t=[1:240]/30;
tback=fliplr(t);
```

```
clf;
for I=1:4,
    wn=(1.789)/tr(I); %Rads/second
    tau=tr(I)/(1.6); %tau
    zeta=0.89; %
    b=tau*(wn^2)*[1 1/tau];
    a=[1 2*zeta*wn (wn^2)];
    y=step(b,a,t);
    subplot(2,2,I);
    plot(t,y);
    titletext=sprintf('tr=%3.1f seconds',tr(I));
    title(titletext);
    xlabel('t (seconds)');
    ylabel('Qo/Qc');
    ymax=(max(y)-1)*100;
    msg=sprintf('Max overshoot=%3.1f%%',ymax);
    text(.50,.30,msg);
    yback=flipud(y);
    yind=find(abs(yback-1) >0.01);
    ts=tback(min(yind));
     msg=sprintf('Settling time =%3.1f sec',ts);
    text(.50,.10,msg);
     grid on;
end
```



Problem 3.37: Ideal pitch response.

38. Approximate each of the transfer functions given below with a second order transfer function.

$$
\begin{aligned}
G_1(s) &= \frac{(0.5s+1)(s+1)}{(0.55s+1)(0.95s+1)(s^2+s+1)}, \\
G_2(s) &= \frac{(0.5s+1)(s+1)}{(0.55s+1)(0.95s+1)(s^2+0.2s+1)}, \\
G_3(s) &= \frac{(-0.5s+1)(s+1)}{(0.95s+1)(0.05s+1)(s^2+s+1)}, \\
G_4(s) &= \frac{(0.5s+1)(s+1)}{(0.55s+1)(0.05s+1)(s^2+s+1)}, \\
G_5(s) &= \frac{(0.5s+1)(0.02s+1)}{(0.55s+1)(0.95s+1)(s^2+s+1)}.
\end{aligned}
$$

Solution: Approximated by standard second-order system; $\zeta = 0.5$ and

$M_p = 17\%$.

$$G_1(s) = \frac{(0.5s+1)(s+1)}{(0.55s+1)(0.95s+1)(s^2+s+1)} \approx \frac{1}{s^2+s+1},$$

Approximated by standard second-order, $\zeta = 0.1$ (light damping).

$$G_2(s) = \frac{(0.5s+1)(s+1)}{(0.55s+1)(0.95s+1)(s^2+0.2s+1)} \approx \frac{1}{s^2+0.2s+1},$$

RHP zero causes undershoot.

$$G_3(s) = \frac{(-0.5s+1)(s+1)}{(0.95s+1)(0.05s+1)(s^2+s+1)} \approx \frac{-0.5s+1}{s^2+s+1},$$

Extra zero causes more overshoot (as compared to a standard second-order system).

$$G_4(s) = \frac{(0.5s+1)(s+1)}{(0.55s+1)(0.05s+1)(s^2+s+1)} \approx \frac{(s+1)}{(s^2+s+1)},$$

This one *cannot* be approximated with a second-order. Extra pole causes longer rise-time (as compared to a standard second-order system).

$$G_5(s) = \frac{(0.5s+1)(0.02s+1)}{(0.55s+1)(0.95s+1)(s^2+s+1)} \approx \frac{1}{(0.95s+1)(s^2+s+1)}.$$

39. A system has the closed-loop transfer function

$$\frac{Y(s)}{R(s)} = T(s) = \frac{2700(s+25)}{(s+1)(s+45)(s+60)(s^2+8s+25)},$$

where R is a step of size 7.

(a) Give an expression for the form of the output time history as a sum of terms showing the shape of each component of the response.

(b) Give an estimate of the settling time of this step response.

Solution:

(a) $T(0) = 1$, $R(s) = \frac{7}{s}$:

$$\begin{aligned} Y(s) &= \frac{7}{s} + \frac{K_1}{s+1} + \frac{K_2}{s+45} + \frac{K_3}{s+60} + \frac{K_4 s + K_5}{(s+4)^2+9}, \\ y(t) &= 7 \times 1(t) + K_1 e^{-t} + K_2 e^{-45t} + K_3 e^{-60t} + K_6 e^{-4t}\sin(3t+\phi). \end{aligned}$$

(b) Settling time is set by the pole at -1:

$$t_s = \frac{4.6}{\sigma} = 4.6\,\text{sec}\,.$$

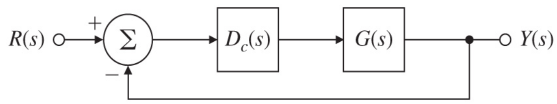


Figure 3.63: Unity feedback system for Problem 3.40

40. Consider the system shown in Fig. 3.63, where

$$G(s) = \frac{1}{s(s+3)} \quad and \quad D_c(s) = \frac{K(s+z)}{s+p}.$$

Find K, z, and p so that the closed-loop system has a 10% overshoot to a step input and a settling time of 1.5 sec (1% criterion).

Solution:

For the 10% overshoot:

$$\begin{aligned} M_p &= e^{-\pi\zeta/\sqrt{1-\zeta^2}} = 10\%, \\ \implies & \zeta = \sqrt{\frac{(\ln M_p)^2}{\pi^2 + (\ln M_p)^2}} = 0.6. \end{aligned}$$

For the 1.5sec (1% criterion):

$$\omega_n = \frac{4.6}{\zeta t_s} = \frac{4.6}{(0.6)(1.5)} = 5.11.$$

The closed-loop transfer function is:

$$\frac{Y(s)}{R(s)} = \frac{K\frac{s+z}{s+p} \times \frac{1}{s(s+3)}}{1 + K\frac{s+z}{s+p} \times \frac{1}{s(s+3)}} = \frac{K(s+z)}{s(s+3)(s+p) + K(s+z)}.$$

Method I.

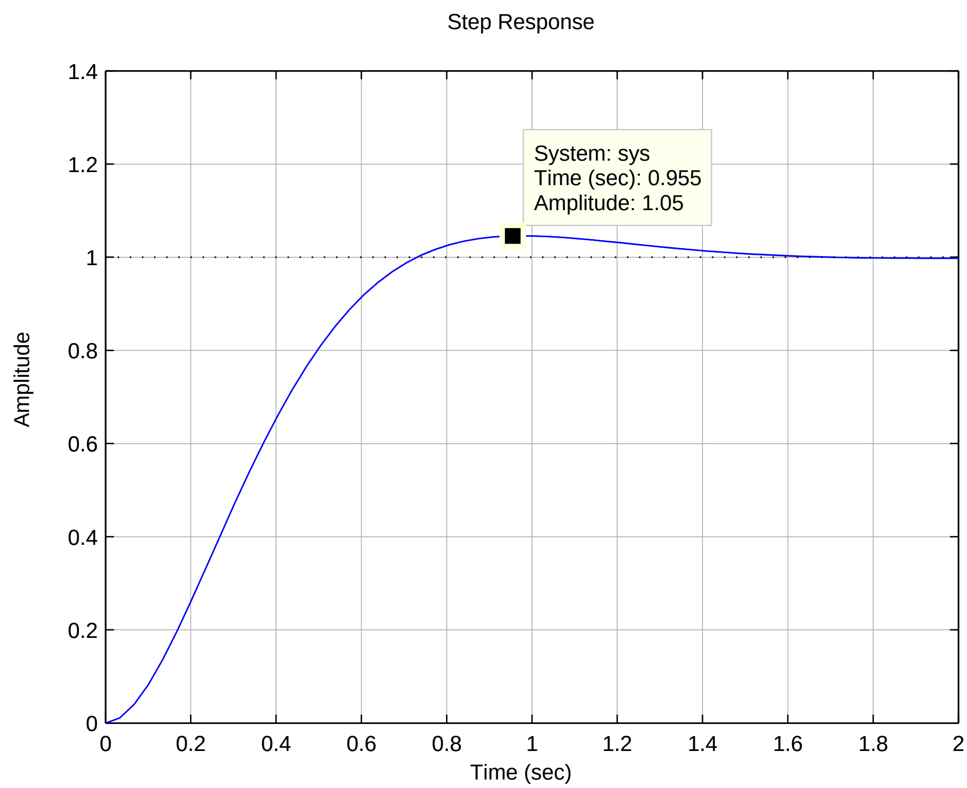
From inspection, if $z = 3$, $(s+3)$ will cancel out and we will have a standard form transfer function. As perfect cancellation is impossible, assign z a value that is very close to 3, say 3.1. But in determining the K and p, assume that $(s+3)$ and$(s+3.1)$ cancelled out each other. Then:

$$\frac{Y(s)}{R(s)} = \frac{K}{s^2 + ps + K}$$

As the additional pole and zero will affect the system response, pick some larger damping ratio.

Let $\zeta = 0.7$

$$\begin{aligned}\omega_n &= \frac{4.6}{\zeta t_s} = \frac{4.6}{(0.7)(1.5)} = 4.38, \text{ so let } \omega_n = 4.5, \\ p &= 2\zeta\omega_n = 2 \times 0.7 \times 4.5 = 6.3, \\ K &= \omega_n^2 = 20.25.\end{aligned}$$



Step response: Method I.

Method II.

There are 3 unknowns (z, p, K) and only 2 specified conditions. We can arbitrarily choose p large such that complex poles will dominate in the system response.

Try $\quad p = 10z$

Choose a damping ratio corresponding to an overshoot of 5% (instead of 10%, just to be safe).

$$\zeta = 0.707.$$

From the formula for settling time (with a 1% criterion)

$$\omega_n = \frac{4.6}{\zeta t_s} = \frac{4.6}{0.707 \times 1.5} = 4.34,$$

adding some margin, let $\omega_n = 4.88$.The characteristic equation is

$$Q(s) = s^3 + (3+p)s^2 + (3p+K)s + Kz = (s+a)(s^2 + 2\zeta\omega_n s + \omega_n^2).$$

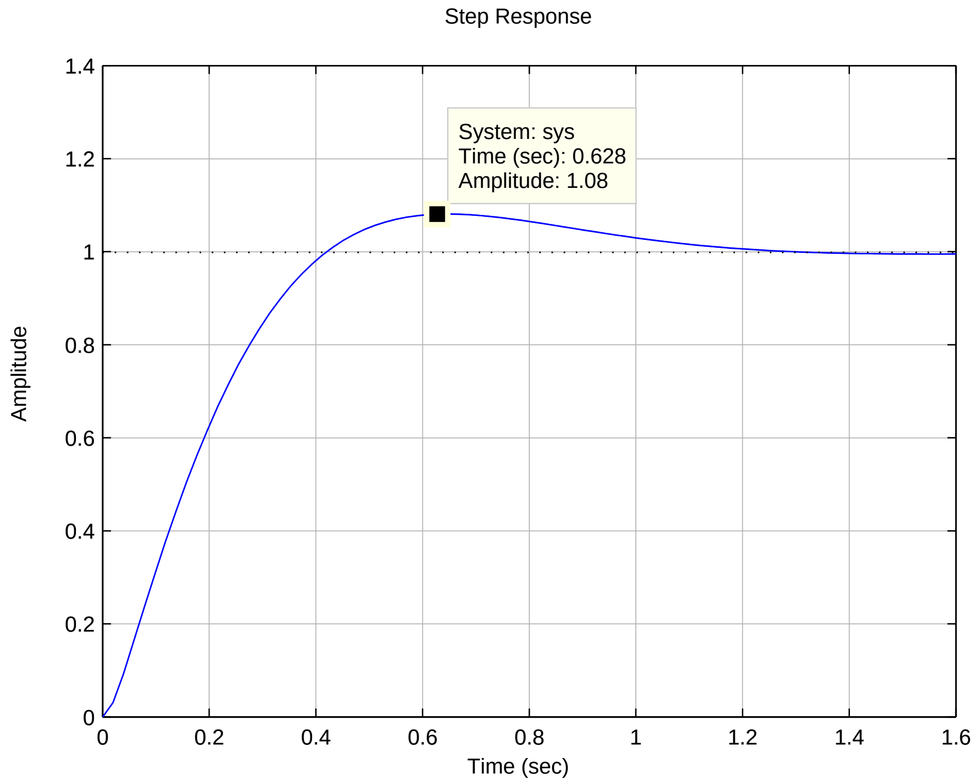
We want the characteristic equation to be the product of two factors, a couple of conjugated poles (dominant) and a non-dominant real pole far form the dominant poles.

Equate the coefficients of like powers of s in the expressions of the characteristic equation.

$$\begin{aligned} \omega_n^2 a &= Kz, \\ 2\zeta\omega_n a + \omega_n^2 &= 30z + K, \\ 2\zeta\omega_n + a &= 3 + 10z. \end{aligned}$$

Solving the three equations we get

$$\begin{aligned} z &= 5.77, \\ p &= 57.7, \\ K &= 222.45, \\ a &= 53.79. \end{aligned}$$



Step response: Method II.

41. ▲ Sketch the step response of a system with the transfer function

$$G(s) = \frac{s/2+1}{(s/40+1)[(s/4)^2+s/4+1]}.$$

Justify your answer on the basis of the locations of the poles and zeros. (Do not find inverse Laplace transform.) Then compare your answer with the step response computed using MATLAB.

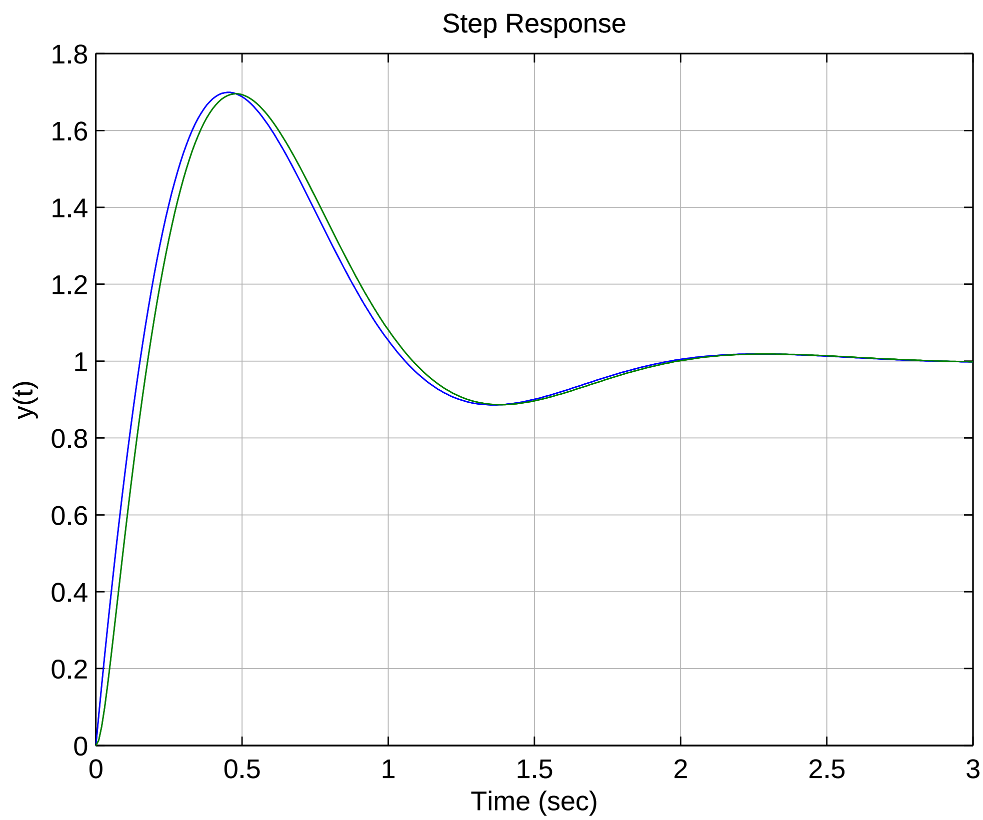
Solution:

From the location of the poles, we notice that the real pole is a factor of 20 away from the complex pair of poles. Therefore, the response of the system is *dominated* by the complex pair of poles.

$$G(s) \approx \frac{(s/2+1)}{[(s/4)^2+s/4+1]}.$$

This is now in the same form as equation (3.72) where $\alpha = 1$, $\zeta = 0.5$ and $\omega_n = 4$. Therefore, Fig. 3.24 suggests an overshoot of over 70%. The step response is the same as shown in Fig. 3.27, for $\alpha = 1$, with more than 70% overshoot and settling time of 3 seconds. The MATLAB plots below confirm this.

```
% Problem 3.41 FPE 8e

num=[1/2, 1];

den1=[1/16, 1/4, 1];

sys1=tf(num,den1);

t=0:.01:3;

y1=step(sys1,t);

den=conv([1/40, 1],den1);

sys=tf(num,den);

y=step(sys,t);

plot(t,y1,t,y);

xlabel('Time (sec)');

ylabel('y(t)');

title('Step Response');

grid on;
```



Problem 3.41: Comparison of step responses: third-order system (green), second-order approximation (blue).

42. A closed-loop transfer function is given below

$$H(s) = \frac{[((\frac{s}{10}))^2 + 0.1((\frac{s}{10})) + 1][\frac{s}{2} + 1][\frac{s}{0.1} + 1]}{[((\frac{s}{4}))^2 + ((\frac{s}{4})) + 1][((\frac{s}{10}))^2 + 0.09((\frac{s}{10})) + 1][\frac{s}{0.02} + 1]}.$$

Estimate the percent overshoot, M_p, and the transient settling time, t_s for this system.

Solution:

$$H(s) \approx \frac{\frac{s}{2} + 1}{((\frac{s}{4}))^2 + ((\frac{s}{4})) + 1}.$$

$$\zeta = 0.5, \omega_n = 4; z = 2; \alpha = 1, M_p = 70\%, t_s = \frac{4.6}{\zeta\omega_n} = 2.3\,\text{sec}.$$

43. A transfer function, $G(s)$, is given below.

$$G(s) = \frac{[((\frac{s}{100}))^2 + 0.01((\frac{s}{100})) + 1]}{[((\frac{s}{10}))^2 + ((\frac{s}{10})) + 1][\frac{s}{5} + 1][((\frac{s}{100}))^2 + 0.1((\frac{s}{100})) + 1]}.$$

If a step input is applied to this plant, what do you estimate the rise-time, settling time, and overshoot to be? Give a brief statement of your reasons, in each case.

Solution:

Second-order term has $\zeta = 0.5, \omega_n = 10$; Extra pole with $\alpha = 1, \omega_n t_r = 2.3$ or $t_r = 0.23\,\text{sec}$.

Settling time is about the same as standard second-order: $t_s \approx \frac{4.6}{\sigma} = 0.92\,\text{sec}$.

Extra pole with $\alpha = 1, M_p = 8\%$.

44. Three closed-loop transfer functions are given below.

$$\begin{aligned}
\frac{Y(s)}{R(s)} &= T_1(s) = \frac{2}{(s^2 + 2s + 2)}, \\
\frac{Y(s)}{R(s)} &= T_2(s) = \frac{2(s+3)}{2(s^2 + 2s + 2)}, \\
\frac{Y(s)}{R(s)} &= T_3(s) = \frac{6}{(s+3)(s^2 + 2s + 2)}.
\end{aligned}$$

In each case, provide estimates of the rise-time, settling time, and percent overshoot to a unit step input in r.

Solution:

(a) $t_r = \frac{1.8}{\omega_n} = \frac{1.8}{\sqrt{2}} = 1.2728\,\text{sec}$, $t_s = \frac{4.6}{\zeta\omega_n} = 4.6\,\text{sec}$, $M_p = 5\%$ for $\zeta = 0.5$.

(b) $t_r = \frac{1.8}{\omega_n} = \frac{1.8}{\sqrt{2}} = 1.2728\,\text{sec}$, $t_s = \frac{4.6}{\zeta\omega_n} = 4.6\,\text{sec}$, $M_p = 5^{+}\%$.(a bit larger than (a)).

(c) $t_r = \frac{2.3}{\omega_n} = \frac{2.3}{\sqrt{2}} = 1.6263\,\text{sec}$ (larger than (a)), $t_s = \frac{4.6}{\zeta\omega_n} = 4.6\,\text{sec}$ (same as before) $M_p = 5^{-}\%$ (a bit less than (a)).

45. Five transfer functions with unity DC gain are given below.

(a) Which transfer function (s) will meet an overshoot specification of $M_p \leq 5\%$?

(b) Which transfer function (s) will meet a rise time specification of $t_r \leq 0.5$ sec?

(c) Which transfer function (s) will meet a settling time specification of $t_s \leq 2.5$ sec?

$$
\begin{aligned}
G_1(s) &= \frac{40}{(s^2+4s+40)}, \\
G_2(s) &= \frac{40}{(s+1)(s^2+4s+40)}, \\
G_3(s) &= \frac{120}{(s+3)(s^2+4s+40)}, \\
G_4(s) &= \frac{20(s+2)}{(s+1)(s^2+4s+40)}, \\
G_5(s) &= \frac{36040/401 \times (s^2+s+401)}{(s^2+4s+40)(s^2+s+901)}.
\end{aligned}
$$

Solution:

(a) $G_2(s)$, $G_3(s)$ and $G_4(s)$ meet the spec and $G_1(s)$ and $G_5(s)$ do not.

(a) $G_1(s)$, $G_3(s)$ and $G_5(s)$ meet the spec and $G_2(s)$ and $G_4(s)$ do not.

(a) $G_1(s)$, and $G_3(s)$ meet the spec and $G_2(s)$, $G_4(s)$ and $G_5(s)$ do not.

46. Consider the two nonminimum phase systems,

$$G_1(s) = -\frac{2(s-1)}{(s+1)(s+2)}, \tag{1}$$

$$G_2(s) = \frac{3(s-1)(s-2)}{(s+1)(s+2)(s+3)}. \tag{2}$$

(a) Sketch the unit step responses for $G_1(s)$ and $G_2(s)$, paying close attention to the transient part of the response.

(b) Explain the difference in the behavior of the two responses as it relates to the zero locations.

(c) Consider a stable, strictly proper system (that is, m zeros and n poles, where $m < n$). Let $y(t)$ denote the step response of the system. The step response is said to have an undershoot if it initially starts off in the "wrong" direction. Prove that a stable, strictly proper system has an undershoot if and only if its transfer function has an *odd* number of *real* RHP zeros.

Solution:

(a) For $G_1(s)$:

$$
\begin{aligned}
Y_1(s) &= \frac{1}{s}G_1(s) = \frac{-2(s-1)}{s(s+1)(s+2)}, \\
H(s) &= k\frac{\prod^j (s-z_j)}{\prod^l (s-p_l)}, \\
R_{p_i} &= \lim_{s\to p_i}[(s-p_i)H(s)] = \lim_{s\to p_i} k\frac{\prod^j (s-z_j)}{\prod^l_{l\neq i}(s-p_l)} = k\frac{\prod^j (p_i-z_j)}{\prod^l_{l\neq i}(p_i-p_l)}.
\end{aligned}
$$

Each factor $(p_i - z_j)$ or $(p_i - p_l)$ can be thought of as a complex number (a magnitude and a phase) whose pictorial representation is a vector pointing to p_i and coming from z_j or p_l respectively.

The method for calculating the residue at a pole p_i is:

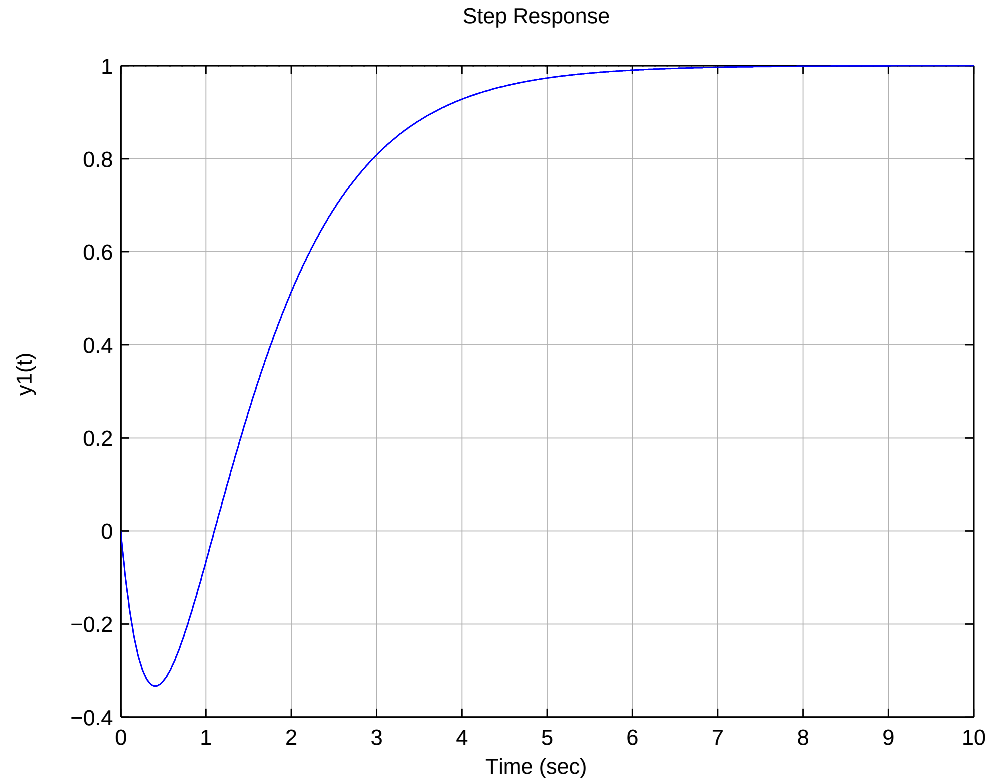
(1) Draw vectors from the rest of the poles and from all the zeros to the pole p_i.

(2) Measure magnitude and phase of these vectors.

(3) The residue will be equal to the gain, multiplied by the product of the vectors coming from the zeros and divided by the product of the vectors coming from the poles.

In our problem:

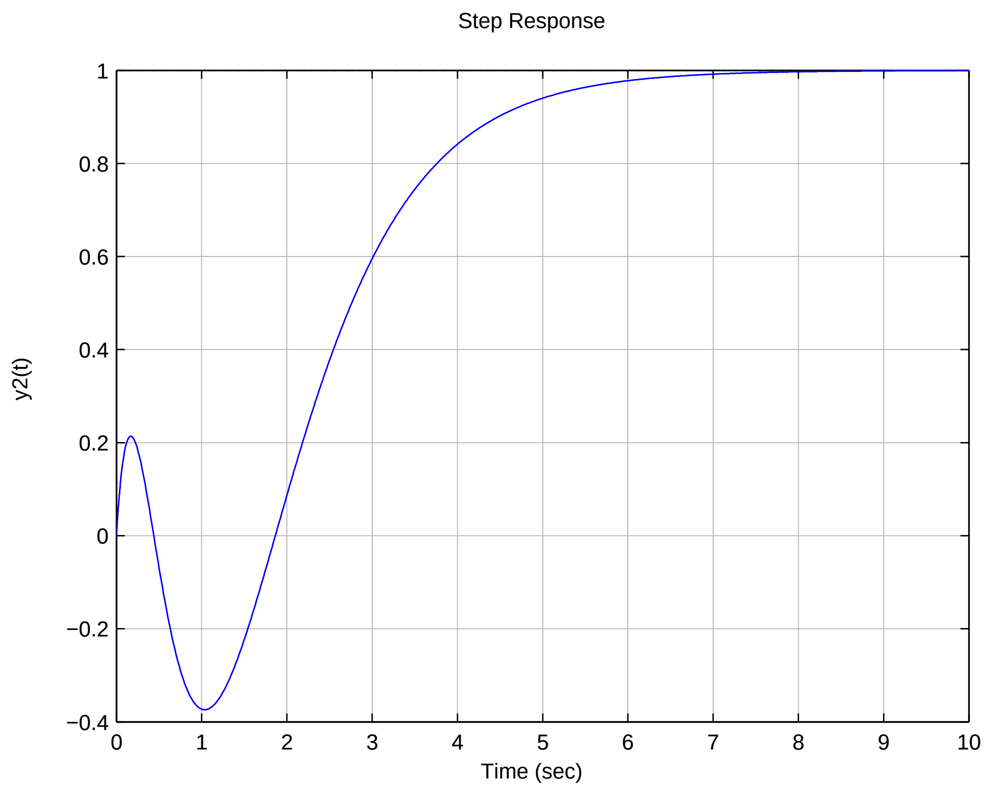
$$
\begin{aligned}
Y_1(s) &= \frac{-2(s-1)}{s(s+1)(s+2)} = \frac{R_0}{s} + \frac{R_{-1}}{(s+1)} + \frac{R_{-2}}{(s+2)} = \frac{1}{s} - \frac{4}{s+1} + \frac{3}{s+2}, \\
y_1(t) &= 1 - 4e^{-t} + 3e^{-2t}.
\end{aligned}
$$



Problem 3.46: Step response for a non-minimum phase system with one *real* RHP zero.

For $G_2(s)$:

$$
\begin{aligned}
Y_2(s) &= \frac{3(s-1)(s-2)}{s(s+1)(s+2)(s+3)} = \frac{1}{s} + \frac{-9}{(s+1)} + \frac{18}{(s+2)} + \frac{-10}{(s+3)}, \\
y_2(t) &= 1 - 9e^{-t} + 18e^{-2t} - 10e^{-3t}.
\end{aligned}
$$



Problem 3.46: Step response of a non-minimum phase system with two *real* zeros in the RHP.

(b) The first system presents an "undershoot". The second system, on the other hand, starts off in the right direction.

The reasons for this initial behavior of the step response will be analyzed in part c.

In $y_1(t)$: dominant at $t = 0$ the term $-4e^{-t}$

In $y_2(t)$: dominant at $t = 0$ the term $18e^{-2t}$

(c) The following concise proof is from Reference [1] (see also References [2]-[3]).

Without loss of generality assume the system has unity DC gain ($G(0) = 1$). Since the system is stable, $y(\infty) = G(0) = 1$,and it is reasonable to assume $y(\infty) \neq 0$. Let us denote the pole-zero excess as $r = n - m$.Then, $y(t)$ and its $r - 1$ derivatives are zero at $t = 0$, and $y^r(0)$ is the first non-zero derivative. The system has an undershoot if $y^r(0)y(\infty) < 0$. The transfer function may be re-written as

$$G(s) = \frac{\prod_{i=1}^{m}(1 - \frac{s}{z_i})}{\prod_{i=1}^{m+r}(1 - \frac{s}{p_i})}$$

.

The *numerator* terms can be classified into three types of terms:
(1). The first group of terms are of the form $(1 - \alpha_i s)$ with $\alpha_i > 0$.
(2). The second group of terms are of the form $(1+\alpha_i s)$ with $\alpha_i > 0$.
(3). Finally, the third group of terms are of the form, $(1+\beta_i s+\alpha_i s^2)$ with $\alpha_i > 0$, and β_i could be negative.
However, $\beta_i^2 < 4\alpha_i$, so that the corresponding zeros are complex.
All the *denominator* terms are of the form (2), (3), above. Since,

$$y^r(0) = \lim_{s\to\infty} s^r G(s)$$

it is seen that the *sign* of $y^r(0)$ is determined entirely by the number of terms of group 3 above. In particular, if the number is *odd*, then $y^r(0)$ is *negative* and if it is even, then $y^r(0)$ is positive. Since $y(\infty) = G(0) = 1$,then we have the desired result.

References

[1] Vidyasagar, M., "On Undershoot and Nonminimum Phase Zeros," *IEEE Trans. Automat. Contr.*, Vol. AC-31, p. 440, May 1986.

[2] Clark, R., N., *Introduction to Automatic Control Systems*, John Wiley, 1962.

[3] Mita, T. and H. Yoshida, "Undershooting phenomenon and its control in linear multivariable servomechanisms, " *IEEE Trans. Automat. Contr.*, Vol. AC-26, pp. 402-407, 1981.

47. Find the relationships for the impulse response and the step response corresponding to Equation (3.65) for the cases where,

(a) the roots are repeated.

(b) the roots are both real. Express your answers in terms of hyperbolic functions (sinh, cosh) to best show the properties of the system response.

(c) the value of the damping coefficient, ζ, is negative.

Solution:

(a) In this case we have $\zeta = 1$

$$H(s) = \frac{Y(s)}{U(s)} = \frac{\omega_n^2}{(s+\omega_n)^2}.$$

For the impulse response, $U(s) = 1$, and using Item #8 from Table A.2 we find

$$h(t) = \omega_n^2 t e^{-\omega_n t}.$$

We can then integrate the impulse response to obtain the step response. Alternatively, for a unit step input, $U(s) = \frac{1}{s}$ and

$$Y(s) = \frac{\omega_n^2}{(s+\omega_n)^2}\frac{1}{s}.$$

Using Item #15 from Table A.2 we find

$$y(t) = 1 - e^{-\omega_n t}(1 + \omega_n t).$$

(b) We re-write $H(s)$ as follows

$$H(s) = \frac{Y(s)}{U(s)} = \frac{\omega_n^2}{(s+\zeta\omega_n + \omega_n\sqrt{\zeta^2-1})(s+\zeta\omega_n - \omega_n\sqrt{\zeta^2-1})},$$

where $|\zeta| > 1$. For the impulse response, $U(s) = 1$ and using Item #13 from Table A.2,

$$\begin{aligned}
h(t) &= -\frac{\omega_n^2}{2\omega_n\sqrt{\zeta^2-1}}(e^{-(\zeta\omega_n+\omega_n\sqrt{\zeta^2-1})t} - e^{-(\zeta\omega_n-\omega_n\sqrt{\zeta^2-1})t}),\\
&= \frac{\omega_n}{2\sqrt{\zeta^2-1}}e^{-\zeta\omega_n t}(e^{+(\omega_n\sqrt{\zeta^2-1})t} - e^{-(\omega_n\sqrt{\zeta^2-1})t}),\\
&= \frac{\omega_n}{\sqrt{\zeta^2-1}}e^{-\zeta\omega_n t}\sinh(\omega_n\sqrt{\zeta^2-1}t).
\end{aligned}$$

We can then integrate the impulse response to obtain the unit step response. Alternatively, for a unit step input, $U(s) = \frac{1}{s}$ and using partial fraction expansion

$$Y(s) = \frac{1}{s} + \frac{\frac{1}{2\sqrt{\zeta^2-1}(\zeta+\sqrt{\zeta^2-1})}}{s+\zeta\omega_n+\omega_n\sqrt{\zeta^2-1}} - \frac{\frac{1}{2\sqrt{\zeta^2-1}(\zeta-\sqrt{\zeta^2-1})}}{s+\zeta\omega_n-\omega_n\sqrt{\zeta^2-1}},$$

and using Item #7 from Table A.2

$$\begin{aligned}
y(t) &= 1 + \frac{1}{2\sqrt{\zeta^2-1}(\zeta+\sqrt{\zeta^2-1})}e^{-(\zeta\omega_n+\omega_n\sqrt{\zeta^2-1})t}\\
&\quad -\frac{1}{2\sqrt{\zeta^2-1}(\zeta-\sqrt{\zeta^2-1})}e^{-(\zeta\omega_n-\omega_n\sqrt{\zeta^2-1})t},\\
&= 1 + \frac{1}{2\sqrt{\zeta^2-1}}e^{-\zeta\omega_n t}(\frac{1}{(\zeta+\sqrt{\zeta^2-1})}e^{-\omega_n\sqrt{\zeta^2-1}t} - \frac{1}{(\zeta-\sqrt{\zeta^2-1})}e^{+\omega_n\sqrt{\zeta^2-1}t}),\\
&= 1 + \frac{1}{2\sqrt{\zeta^2-1}}e^{-\zeta\omega_n t}((\zeta-\sqrt{\zeta^2-1})e^{-\omega_n\sqrt{\zeta^2-1}t} - (\zeta+\sqrt{\zeta^2-1})e^{+\omega_n\sqrt{\zeta^2-1}t}),\\
y(t) &= 1 - e^{-\zeta\omega_n t}(\cosh(-\omega_n\sqrt{\zeta^2-1}t) + \frac{\zeta}{\sqrt{\zeta^2-1}}\sinh(\omega_n\sqrt{\zeta^2-1}t)).
\end{aligned}$$

Notice that unlike the expression for the impulse response on FPE 8e page 128 (Eq. 3.66) and the step response on FPE 8e page 133 (Eq. 3.70), these responses *do not oscillate* due to the behavior of the cosh and sinh functions.

(c) Now we have the remaining case where ζ is negative and $|\zeta| < 1$, since we already dealt with the case of $|\zeta| > 1$ in the previous part (b). The impulse response and the step responses are exactly the same as given in on pages 128 (Eq. 3.66) and 133 (Eq. 3.70)

$$\begin{aligned} h(t) &= \frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\sigma t} \sin(\omega_d t), \\ y(t) &= 1 - \frac{e^{-\sigma t}}{\sqrt{1-\zeta^2}} \cos(\omega_d t - \beta), \end{aligned}$$

except now ζ is negative and the exponential terms become unbounded and the system is unstable.

48. Consider the following second-order system with an extra pole:

$$H(s) = \frac{\omega_n^2 p}{(s+p)(s^2 + 2\zeta\omega_n s + \omega_n^2)}.$$

Show that the unit step response is

$$y(t) = 1 + Ae^{-pt} + Be^{-\sigma t}\sin(\omega_d t - \theta),$$

where

$$\begin{aligned} A &= \frac{-\omega_n^2}{\omega_n^2 - 2\zeta\omega_n p + p^2}, \\ B &= \frac{p}{\sqrt{(p^2 - 2\zeta\omega_n p + \omega_n^2)(1-\zeta^2)}}, \\ \theta &= \tan^{-1}\frac{\sqrt{1-\zeta^2}}{-\zeta} + \tan^{-1}\frac{\omega_n\sqrt{1-\zeta^2}}{p - \zeta\omega_n}. \end{aligned}$$

(a) Which term dominates $y(t)$ as p gets large?

(b) Give approximate values for A and B for small values of p.

(c) Which term dominates as p gets small? (Small with respect to what?)

(d) Using the preceding explicit expression for $y(t)$ or the step command in MATLAB, and assuming that $\omega_n = 1$ and $\zeta = 0.7$, plot the step response of the preceding system for several values of p ranging from very small to very large. At what point does the extra pole cease to have much effect on the system response?

Solution:

Second-order system:

$$H(s) = \frac{\omega_n^2 p}{(s+p)(s^2 + 2\zeta\omega_n s + \omega_n^2)}.$$

Unit step response:

$$Y(s) = \frac{1}{s}H(s), \quad y(t) = L^{-1}\{Y(s)\},$$

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = (s + \sigma + j\omega_d)(s + \sigma - j\omega_d),$$

where

$$\sigma = \zeta\omega_n, \quad \omega_d = \omega_n\sqrt{1-\zeta^2}.$$

Thus from partial fraction expansion:

$$Y(s) = \frac{k_1}{s} + \frac{k_2}{s+p} + \frac{k_3}{s+\sigma+j\omega_d} + \frac{k_4}{s+\sigma-j\omega_d},$$

solving for k_1, k_2, k_3, and k_4:

$$\begin{aligned}
k_1 &= H(0) \Longrightarrow k_1 = 1, \\
k_2 &= \frac{\omega_n^2 p}{s(s+\sigma+j\omega_d)(s+\sigma-j\omega_d)}|_{s=-p} \Longrightarrow k_2 = \frac{-\omega_n^2}{\omega_n^2 - 2p\zeta\omega_n + p^2}, \\
k_3 &= (s+\sigma+j\omega_d)Y(s)|_{s=-\sigma-j\omega_d} \\
\Longrightarrow & k_3 = \frac{p}{2\sqrt{(1-\zeta^2)(p^2 - 2p\zeta\omega_n + \omega_n^2)}} e^{-i\theta} = |k_3|e^{-i\theta} \\
k_4 &= k_3^*
\end{aligned}$$

where

$$\theta = \tan^{-1}(\left(\frac{\sqrt{1-\zeta^2}}{-\zeta}\right)) + \tan^{-1}(\left(\frac{\omega_n\sqrt{1-\zeta^2}}{p-\zeta\omega_n}\right)).$$

Thus

$$Y(s) = \frac{1}{s} + \frac{k_2}{s+p} + |k_3|(\frac{e^{-i\theta}}{s+\sigma+j\omega_d} + \frac{e^{+i\theta}}{s+\sigma-j\omega_d}).$$

Inverse Laplace:

$$y(t) = 1 + k_2 e^{-pt} + |k_3|(e^{-i\theta}e^{-(\sigma+j\omega_d)t} + e^{+i\theta}e^{-(\sigma-j\omega_d)t}),$$

or

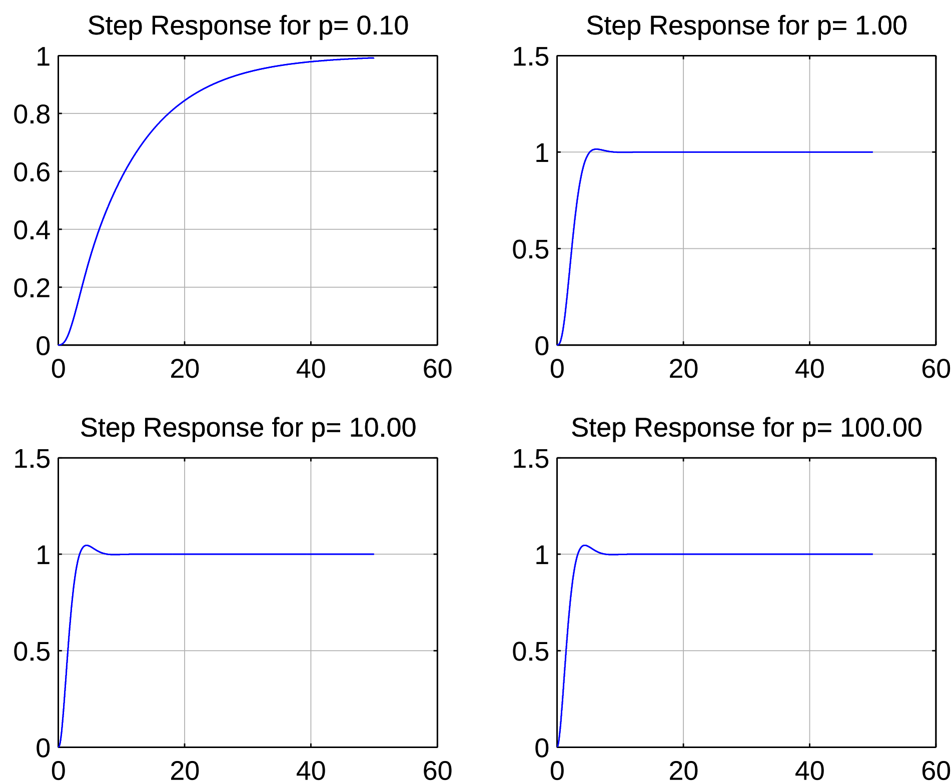
$$y(t) = 1 + \underbrace{\frac{-\omega_n^2}{\omega_n^2 - 2p\zeta\omega_n + p^2}}_{A} e^{-pt} + \underbrace{\frac{p}{\sqrt{(1-\zeta^2)(p^2 - 2p\zeta\omega_n + \omega_n^2)}}}_{B} e^{-\sigma t}\cos(\omega_d t + \theta).$$

(a) As p gets large the B term dominates.

(b) For small p: $A \approx -1, B \approx 0$.

(c) As p gets small A dominates.

(d) The effect of a change in p is not noticeable above $p \approx 10$.



Problem 3.48: Step responses for several values of p.

49. Consider the second order unity DC gain system with one finite zero,

$$H(s) = \frac{\omega_n^2(s+z)}{z(s^2 + 2\zeta\omega_n s + \omega_n^2)}.$$

(a) Show that the unit-step response is

$$y(t) = 1 - \frac{1}{z}\frac{e^{-\sigma t}}{\sqrt{1-\zeta^2}}\sqrt{\omega_n^2 + z^2 - 2\zeta\omega_n}\cos(\omega_d t - \beta_1),$$

where

$$\beta_1 = \tan^{-1} \frac{\zeta z - \omega_n}{\sqrt{1-\zeta^2} z}.$$

(b) Derive an expression for the overshoot, M_p, for this system.

(c) For a given value of overshoot, M_p, how do we solve for ζ and ω_n?

Solution:

(a). We write the transfer function in partial fraction form,

$$H(s) = \frac{\omega_n^2}{(s^2 + 2\zeta\omega_n s + \omega_n^2)} + \frac{1}{z} \frac{\omega_n^2 s}{(s^2 + 2\zeta\omega_n s + \omega_n^2)}.$$

The step response of the first term is as given in Chapter 3, and that of the second term is simply the derivative of that (i.e., the impulse response) scaled by $1/z$:

$$\begin{aligned}
y(t) &= y_1 + \frac{1}{z}\frac{dy_1}{dt}, \\
y(t) &= 1 - \frac{e^{-\sigma t}}{\sqrt{1-\zeta^2}} \cos(\omega_d t - \beta) + \frac{1}{z}[\frac{\sigma e^{-\sigma t}}{\sqrt{1-\zeta^2}} \cos(\omega_d t - \beta) + \frac{\omega_d e^{-\sigma t}}{\sqrt{1-\zeta^2}} \sin(\omega_d t - \beta)], \\
y(t) &= 1 - \frac{e^{-\sigma t}}{\sqrt{1-\zeta^2}}(-1 + \frac{\sigma}{z}) \cos(\omega_d t - \beta) + \frac{e^{-\sigma t}}{\sqrt{1-\zeta^2}} \frac{\omega_d}{z} \sin(\omega_d t - \beta) \\
\text{where } \beta &= \tan^{-1} \frac{\zeta}{\sqrt{1-\zeta^2}} = \sin^{-1}(\zeta).
\end{aligned}$$

Now as in Chapter 3 we combine the last two terms to yield,

$$\begin{aligned}
y(t) &= 1 + \frac{1}{z} \frac{e^{-\sigma t}}{\sqrt{1-\zeta^2}} (\sqrt{\omega_n^2 + z^2 - 2\zeta\omega_n}) \cos(\omega_d t - (\beta + \beta_2)), \\
\text{where } \beta_2 &= \tan^{-1} \frac{\omega_n \sqrt{1-\zeta^2}}{\zeta\omega_n - z}.
\end{aligned}$$

Using the trigonometric identity,

$$\tan^{-1} A + \tan^{-1} B = \tan^{-1} \frac{A+B}{1-AB},$$

we combine the last two terms in the argument of the cosine term,

$$\beta_1 = \beta + \beta_2 = \tan^{-1}(\left(\frac{\frac{\zeta}{\sqrt{1-\zeta^2}} + \frac{\omega_n\sqrt{1-\zeta^2}}{\zeta\omega_n - z}}{1 - \frac{\zeta}{\sqrt{1-\zeta^2}} \frac{\omega_n\sqrt{1-\zeta^2}}{\zeta\omega_n - z}}\right)) = \tan^{-1} \frac{\zeta z - \omega_n}{\sqrt{1-\zeta^2} z} = -\tan^{-1} \frac{-\zeta z + \omega_n}{\sqrt{1-\zeta^2} z}.$$

Hence we have the final desired result,

$$y(t) = 1 - \frac{1}{z} \frac{e^{-\sigma t}}{\sqrt{1-\zeta^2}} (\sqrt{\omega_n^2 + z^2 - 2\zeta\omega_n}) \cos(\omega_d t + \beta_1).$$

(b) At peak time t_p, we have that

$$\frac{dy(t_p)}{dt} = 0,$$

$$-\frac{\sigma}{z}\frac{e^{-\sigma t}}{\sqrt{1-\zeta^2}}(\sqrt{\omega_n^2+z^2-2\zeta\omega_n})\cos(\omega_d t-\beta_1)-$$

$$\frac{\omega_d}{z}\frac{e^{-\sigma t}}{\sqrt{1-\zeta^2}}(\sqrt{\omega_n^2+z^2-2\zeta\omega_n})\sin(\omega_d t-\beta_1) = 0,$$

$$\begin{aligned}
\cos(\omega_d t-\beta_1-\beta_3) &= 0, \\
\beta_3 &= \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}, \\
\beta_1+\beta_3 &= \tan^{-1}\frac{\frac{\zeta z-\omega_n}{\sqrt{1-\zeta^2}z}+\frac{\sqrt{1-\zeta^2}}{\zeta}}{1-\frac{\zeta z-\omega_n}{\sqrt{1-\zeta^2}z}\frac{\sqrt{1-\zeta^2}}{\zeta}} = \tan^{-1}(\left(\frac{z-\omega_n}{\omega_n}\right)), \\
t_p &= \frac{1}{\omega_d}[\tan^{-1}(\left(\frac{z-\zeta\omega_n}{\omega_n\sqrt{1-\zeta^2}}\right))+\frac{3}{2}\pi], \\
M_p &= y(t_p)-1, \\
M_p &= \frac{1}{z}\sqrt{z^2-z\zeta\omega_n+\omega_n^2}e^{-\sigma t_p}.
\end{aligned}$$

(c) For a given overshoot M_p, the values of ω_n and ζ have to be found by trial and error. In general, they will be different than the standard second order system values unless z is large that is the zero is far away.

50. The block diagram of an autopilot designed to maintain the pitch attitude θ of an aircraft is shown in Fig. 3.64. The transfer function relating the elevator angle δ_e and the pitch attitude θ is

$$\frac{\theta(s)}{\delta_e(s)} = G(s) = \frac{50(s+1)(s+2)}{(s^2+5s+40)(s^2+0.03s+0.06)},$$

where θ is the pitch attitude in degrees and δ_e is the elevator angle in degrees. The autopilot controller uses the pitch attitude error e to adjust the elevator according to the transfer function

$$\frac{\delta_e(s)}{e(s)} = D_c(s) = \frac{K(s+3)}{s+10}.$$

Using MATLAB, find a value of K that will provide an overshoot of less than 10% and a rise time faster than 0.5 sec for a unit-step change in θ_r. After examining the step response of the system for various values of K,

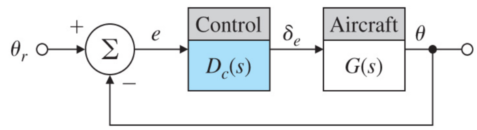


Figure 3.64: Block diagram of autopilot for Problem 3.50

comment on the difficulty associated with making rise-time and overshoot measurements for complicated systems.

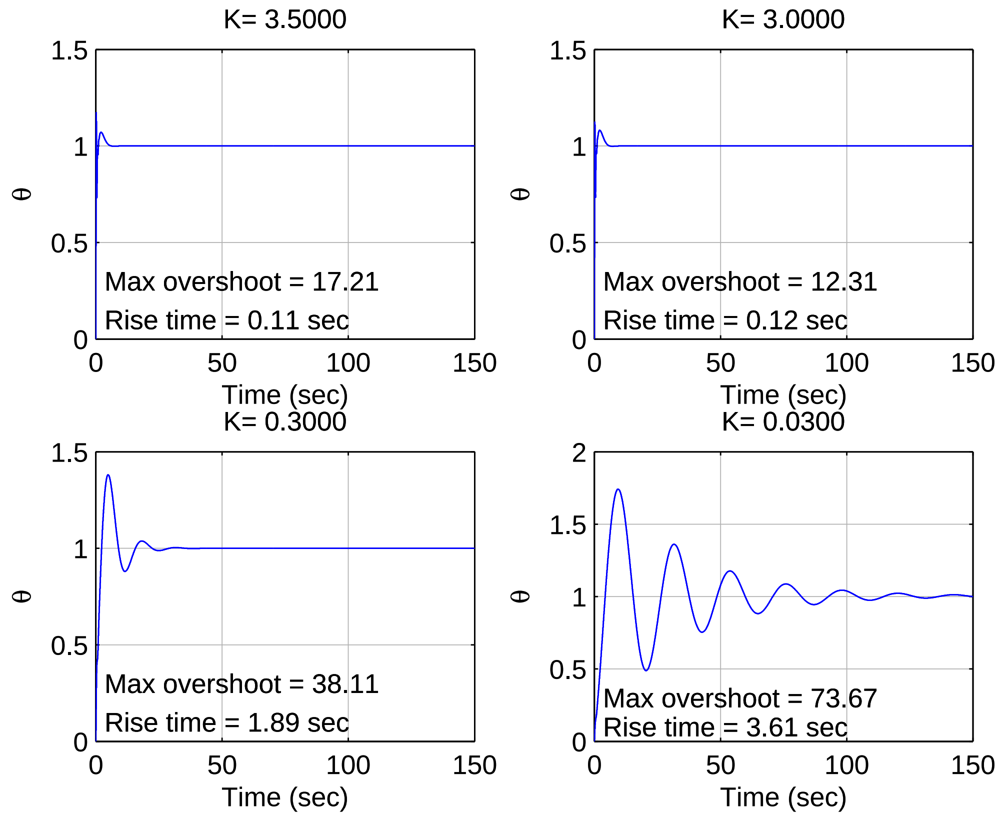
Solution:

$$
\begin{aligned}
G(s) &= \frac{\Theta(s)}{\delta_e(s)} = \frac{50(s+1)(s+2)}{(s^2+5s+40)(s^2+0.03s+0.06)}, \\
D_c(s) &= \frac{\delta_e(s)}{e(s)} = \frac{K(s+3)}{(s+10)},
\end{aligned}
$$

where

$$
\begin{aligned}
e(s) &= \Theta_r - \Theta, \\
\frac{\Theta(s)}{\Theta_r(s)} &= \frac{G(s)D_c(s)}{1+G(s)D_c(s)}, \\
&= \frac{50K(s+1)(s+2)(s+3)}{(s^2+5s+40)(s^2+0.03s+0.06)(s+10)+K(s+3)}, \\
&= \frac{50K(s^3+6s^2+11s+6)}{s^5+15.03s^4+(50K+90.51)s^3+(300K+403.6)s^2+(17.4+550K)s+(24+300K)}.
\end{aligned}
$$

Output must be normalized to the final value of $\frac{\Theta(s)}{\Theta_r(s)}$ for easy computation of the overshoot and rise-time. In this case the design criterion for overshoot cannot be met easily which is indicated in the sample plots.



Problem 3.50: Step responses for an autopilot for various values of K.

Problems and Solutions for Section 3.7: Stability

51. A measure of the degree of instability in an unstable aircraft response is the amount of time it takes for the *amplitude* of the time response to double (see Fig. 3.65), given some nonzero initial condition.

    (a) For a first-order system, show that the **time to double** is

    $$\tau_2 = \frac{\ln 2}{p},$$

    where p is the pole location in the RHP.

    (b) For a second-order system (with two complex poles in the RHP), show that

    $$\tau_2 = \frac{\ln 2}{-\zeta\omega_n}.$$

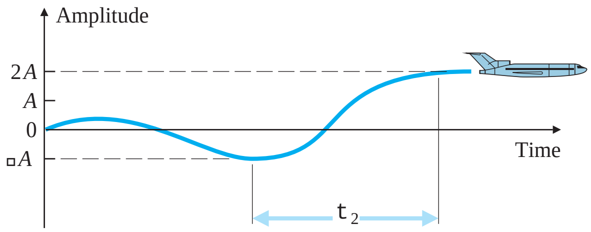


Figure 3.65: Time to double

Solution:

(a) First-order system, $H(s)$ could be:

$$
\begin{aligned}
H(s) &= \frac{k}{(s-p)}, \\
h(t) &= L^{-1}[H(s)] = ke^{pt}, \\
h(\tau_0) &= ke^{p\tau_0}, \\
h(\tau_0+\tau_2) &= 2h(\tau_0) = ke^{p(\tau_0+\tau_2)},
\end{aligned}
$$

$$
\begin{aligned}
\Longrightarrow \qquad & 2ke^{p\tau_0} = ke^{p\tau_0}e^{p\tau_2}, \\
\Longrightarrow \qquad & \tau_2 = \frac{\ln 2}{p}.
\end{aligned}
$$

(b) Second-order system:

$$y(t) = y_0 \frac{e^{\omega_n |\zeta| t}}{\sqrt{1-|\zeta^2|}} \sin(\omega_n \sqrt{1-|\zeta^2|}t + \cos^{-1}\zeta),$$

where

$$\cos^{-1}\zeta = \cos^{-1}|\zeta| + \pi$$

$$\Longrightarrow y(t) = y_0 \frac{e^{\omega_n |\zeta| t}}{\sqrt{1-|\zeta^2|}} (-1)\sin(\omega_n \sqrt{1-|\zeta^2|}t + \cos^{-1}|\zeta|)$$

Note: Instead of working with a negative ζ, everything is changed to $|\zeta|$.

$$\begin{aligned}
|t_0| &= -y_0 \frac{e^{\omega_n|\zeta|t}}{\sqrt{1-|\zeta^2|}}, \\
|\tau_0| &= -y_0 \frac{e^{\omega_n|\zeta|\tau_0}}{\sqrt{1-|\zeta^2|}}, \\
|\tau_0+\tau_2| &= -y_0 \frac{e^{\omega_n|\zeta|(\tau_0+\tau_2)}}{\sqrt{1-|\zeta^2|}} = 2|\tau_0|
\end{aligned}$$

$$\begin{aligned}
&\Longrightarrow \qquad e^{\omega_n|\zeta|\tau_2} = 2 \\
&\Longrightarrow \qquad \tau_2 = \frac{\ln 2}{\omega_n|\zeta|} = \frac{\ln 2}{-\omega_n\zeta} \qquad (\zeta \leq 0)
\end{aligned}$$

Note: This problem shows that $\sigma = \omega_n|\zeta|$ (the real part of the poles) is inversely proportional to the time to double.

The further away from the imaginary axis the poles lie, the faster the response is (either increasing faster for RHP poles or decreasing faster for LHP poles).

52. Suppose that unity feedback is to be applied around the listed open-loop systems. Use Routh's stability criterion to determine whether the resulting closed-loop systems will be stable.

(a) $KG(s) = \frac{4(s+2)}{s(s^3+2s^2+3s+4)}$

(b) $KG(s) = \frac{2(s+4)}{s^2(s+1)}$

(c) $KG(s) = \frac{4(s^3+2s^2+s+1)}{s^2(s^3+2s^2-s-1)}$

Solution:

(a)

$$1 + KG = s^4 + 2s^3 + 3s^2 + 8s + 8 = 0.$$

The Rouh array is,

$$\begin{array}{lllll}
s^4 & : & 1 & 3 & 8 \\
s^3 & : & 2 & 8 & \\
s^2 & : & a & b & \\
s^1 & : & c & & \\
s^0 & : & d & &
\end{array}$$

where

$$\begin{aligned} a &= \frac{2\times 3 - 8\times 1}{2} = -1 \qquad b = \frac{2\times 8 - 1\times 0}{2} = 8, \\ c &= \frac{3a-2b}{a} = \frac{-8-16}{-1} = 24, \\ d &= b = 8. \end{aligned}$$

2 sign changes in the first column $\Longrightarrow$ 2 roots not in the LHP$\Longrightarrow$ unstable.

(b)

$$1 + KG = s^3 + s^2 + 2s + 8 = 0.$$

The Routh's array is,

$$\begin{array}{ccll} s^3 & : & 1 & 2 \\ s^2 & : & 1 & 8 \\ s^1 & : & -6 & \\ s^0 & : & 8 & \end{array}$$

There are two sign changes in the first column of the Routh array. Therefore, there are two roots not in the LHP.

(c)

$$1 + KG = s^5 + 2s^4 + 3s^3 + 7s^2 + 4s + 4 = 0.$$

The Routh array is,

$$\begin{array}{cclll} s^5 & : & 1 & 3 & 4 \\ s^4 & : & 2 & 7 & 4 \\ s^3 & : & a_1 & a_2 & \\ s^2 & : & b_1 & b_2 & \\ s^1 & : & c_1 & & \\ s^0 & : & d_1 & & \end{array}$$

where

$$\begin{aligned} a_1 &= \frac{6-7}{2} = \frac{-1}{2} \qquad a_2 = \frac{8-4}{2} = 2 \\ b_1 &= \frac{-7/2-4}{-1/2} = 15 \qquad b_2 = \frac{-4/2-0}{-1/2} = 4 \\ c_1 &= \frac{30+2}{15} = \frac{32}{15} \\ d_1 &= 4 \end{aligned}$$

2 sign changes in the first column $\Longrightarrow$ 2 roots not in the LHP $\Longrightarrow$ unstable.

53. Use Routh's stability criterion to determine how many roots with positive real parts the following equations have:

    (a) $s^4 + 8s^3 + 32s^2 + 80s + 100 = 0.$

    (b) $s^5 + 10s^4 + 30s^3 + 80s^2 + 344s + 480 = 0.$

    (c) $s^4 + 2s^3 + 7s^2 - 2s + 8 = 0.$

    (d) $s^3 + s^2 + 20s + 78 = 0.$

    (e) $s^4 + 6s^2 + 25 = 0.$

Solution:

(a)

$$s^4 + 8s^3 + 32s^2 + 80s + 100 = 0$$

The Routh array is,

$$\begin{array}{lcccc} s^4 & : & 1 & 32 & 100 \\ s^3 & : & 8 & 80 & \\ s^2 & : & 22 & 100 & \\ s^1 & : & 80 - \dfrac{800}{22} = 43.6 & & \\ s^0 & : & 100 & & \end{array}$$

$\Longrightarrow$ No sign changes in the first column of the array: No roots not in the LHP.

(b)

$$s^5 + 10s^4 + 30s^3 + 80s^2 + 344s + 480 = 0$$

$$\begin{array}{lcccc} s^5 & : & 1 & 30 & 344 \\ s^4 & : & 10 & 80 & 480 \\ s^3 & : & 22 & 296 & \\ s^2 & : & -54.5455 & 480 & \\ s^1 & : & 489.60 & & \\ s^0 & : & 480 & & \end{array}$$

$\Longrightarrow$ Two sign changes in the first column of the array: 2 roots not in the LHP.

(c)

$$s^4 + 2s^3 + 7s^2 - 2s + 8 = 0$$

There are roots in the RHP (not all coefficients are >0). The Routh array is,

$$\begin{array}{lcccc}
s^4 & : & 1 & 7 & 8 \\
s^3 & : & 2 & -2 & \\
s^2 & : & 8 & 8 & \\
s^1 & : & -4 & & \\
s^0 & : & 8 & &
\end{array}$$

$\Longrightarrow$ Two sign changes in the first column of the array: 2 roots not in the LHP.

(d) The Routh array is,

$$\begin{array}{lccc}
s^3 & : & 1 & 20 \\
s^2 & : & 1 & 78 \\
s^1 & : & -58 & \\
s^0 & : & 78 &
\end{array}$$

There are two sign changes in the first column of the Routh array. Therefore, there are two roots not in the LHP.

(e)

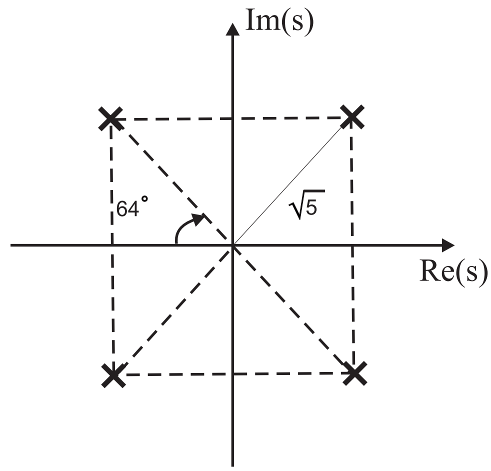
$$a(s) = s^4 + 6s^2 + 25 = 0$$

Two coefficients (those of s^3 and s) are missing so there are roots outside the LHP.
Create a new row by $\frac{da(s)}{ds}$.
The Routh array with the new row is,

$$\begin{array}{lcccl}
s^4 & : & 1 & 6 & 25 \\
s^3 & : & 4 & 12 & \longleftarrow \textbf{new row} \\
s^2 & : & 3 & 25 & \\
s^1 & : & 12 - \dfrac{100}{3} = -21.3 & & \\
s^0 & : & 25 & &
\end{array}$$

$\Longrightarrow$ Two sign changes in the first column of the array: 2 roots not in the LHP.
check:

$$\begin{aligned}
a(s) &= 0 \Longrightarrow s^2 = -3 \pm 4j = 5e^{j(\pi \pm 0.92)} \\
s &= \sqrt{5}e^{j(\frac{\pi}{2} \pm 0.46) + n\pi j} \qquad n = 0, 1
\end{aligned}$$



Problem 3.53: s-plane pole locations.

54. Find the range of K for which all the roots of the following polynomial are in the LHP:

$$s^5 + 5s^4 + 10s^3 + 10s^2 + 5s + K = 0.$$

Use MATLAB to verify your answer by plotting the roots of the polynomial in the s-plane for various values of K.

Solution:

$$s^5 + 5s^4 + 10s^3 + 10s^2 + 5s + K = 0.$$

The Routh array is,

$$\begin{array}{lclll} s^5 & : & 1 & 10 & 5 \\ s^4 & : & 5 & 10 & K \\ s^3 & : & a_1 & a_2 & \\ s^2 & : & b_1 & K & \\ s^1 & : & c_1 & & \\ s^0 & : & K & & \end{array}$$

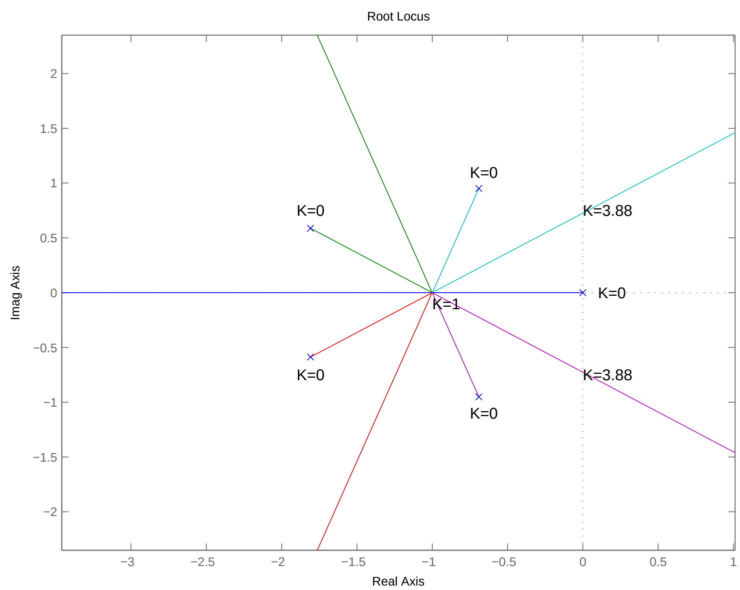
where

$$\begin{aligned} a_1 &= \frac{5(10) - 1(10)}{5} = 8 \qquad a_2 = \frac{5(5) - 1(K)}{5} = \frac{25 - K}{8} \\ b_1 &= \frac{(a_1)(10) - (5)(a_2)}{a_1} = \frac{55 + K}{8} \\ c_1 &= \frac{(b_1)(a_2) - (a_1)(K)}{b_1} = \frac{-(K^2 + 350K - 1375)}{5(55 + K)} \end{aligned}$$

For stability: all elements in first the first column of the Routh array must be positive. That results in the following set of constraints:

$$
\begin{aligned}
(1)b_1 &= \frac{55+K}{8} > 0 \Longrightarrow K > -55 \\
(2)c_1 &= \frac{-(K^2+350K-1375)}{5(55+K)} > 0, \frac{-(K-3.88)(K+354)}{5(55+K)} > 0 \Longrightarrow -55 < K < 3.88 \\
(3)d_1 &= K > 0
\end{aligned}
$$

Combining (1), (2), and (3) $\Longrightarrow 0 < K < 3.88$. If we plot the roots of the polynomial for various values of K we obtain the following plot (called the root locus, see Chapter 5).



Problem 3.54: Roots of the polynomial in the s-plane for various values of K.

55. The transfer function of a typical tape-drive system is given by

$$KG(s) = \frac{K(s+4)}{s[(s+0.5)(s+1)(s^2+0.4s+4)]},$$

where time is measured in milliseconds. Using Routh's stability criterion, determine the range of K for which this system is stable when the characteristic equation is $1 + KG(s) = 0$.

Solution:

$$1 + KG(s) = s^5 + 1.9s^4 + 5.1s^3 + 6.2s^2 + (2 + K)s + 4K = 0.$$

The Routh array is,

$$\begin{array}{llll}
s^5 : & 1.0 & 5.1 & 2+K \\
s^4 : & 1.9 & 6.2 & 4K \\
s^3 : & a_1 & a_2 & \\
s^2 : & b_1 & 4K & \\
s^1 : & c_1 & & \\
s^0 : & 4K & &
\end{array}$$

where

$$\begin{aligned}
a_1 &= \frac{(1.9)(5.1) - (1)(6.2)}{1.9} = 1.837 \qquad a_2 = \frac{(1.9)(2+K) - (1)(4K)}{1.9} = 2 - 1.1K \\
b_1 &= \frac{(a_1)(6.2) - (a_2)(1.9)}{a_1} = 1.138(K + 3.63) \\
c_1 &= \frac{(b_1)(a_2) - (4K)(a_1)}{b_1} = \frac{-(1.25K^2 + 9.61K - 8.26)}{1.138(K + 363)} = \frac{-(K + 8.47)(K - 0.78)}{0.91(K + 3.63)}
\end{aligned}$$

For stability we must have all the elements in the first column of the routh array to be positive, and that results in the following set of constraints:

$$\begin{array}{llll}
(1) & b_1 & = & K + 3.63 > 0 \Longrightarrow K > -3.63, \\
(2) & c_1 & > & 0 \Longrightarrow -8.43 < K < 0.78, \\
(3) & d_1 & > & 0 \Longrightarrow K > 0.
\end{array}$$

Intersection of (1), (2), and (3) $\Longrightarrow 0 < K < 0.78$.

56. Consider the closed-loop magnetic levitation system shown in Figure 3.66.

(a) Compute the transfer function from the input (R) to the output (Y).

(b) Assume $K_o = 1$. Determine the conditions on the system parameters (a, K, z, p), to guarantee closed-loop system stability.

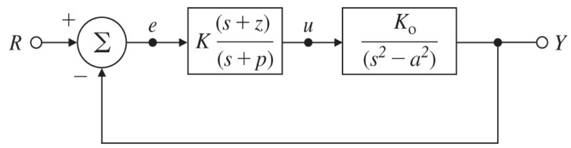


Figure 3.66: Magnetic levitation system for Problem 3.56

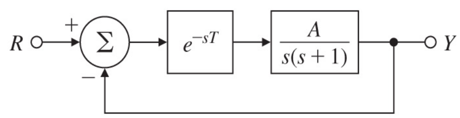


Figure 3.67: Control system for Problem 3.57

Solution:(a) The transfer function is

$$\frac{Y}{R} = \frac{\frac{K(s+z)}{s+p}\frac{K_\circ}{s^2-a^2}}{1+\frac{K(s+z)}{s+p}\frac{K_\circ}{s^2-a^2}} = \frac{KK_\circ(s+z)}{s^3+ps^2+(KK_\circ-a^2)s+KK_\circ z-pa^2}$$

(b) With $K_\circ = 1$ we have $Den(s) = s^3 + ps^2 + (K - a^2)s + Kz - pa^2$; constructing the Routh array we obtain

$$\begin{array}{lcc}
s^3: & 1 & K-a^2 \\
s^2: & p & Kz-pa^2 \\
s^1: & \frac{-Kz+pa^2+Kp-pa^2}{p} = \frac{-Kz+Kp}{p} & \\
s^0: & Kz-pa^2 &
\end{array}$$

Therefore, for stability, all the elements in the first column to be positive and we obtain the following set of constraints:

$$\begin{array}{rcll}
p & > & 0, & \\
Kp-Kz & > & 0 & \text{if} \quad K>0 \Rightarrow p>z, \\
Kz-pa^2 & > & 0 & \text{if} \quad K>0 \Rightarrow z>\frac{pa^2}{K}.
\end{array}$$

57. Consider the system shown in Fig. 3.67.

 (a) Compute the closed-loop characteristic equation.

 (b) For what values of (T, A) is the system stable? *Hint*: An approximate answer may be found using

$$e^{-Ts} \cong 1 - Ts$$

or

$$e^{-Ts} \cong \frac{1 - \frac{T}{2}s}{1 + \frac{T}{2}s}$$

for the pure delay. As an alternative, you could use the computer MATLAB (Simulink) to simulate the system or to find the roots of the system's characteristic equation for various values of T and A.

Solution:

(a) The characteristic equation is,

$$s(s+1) + Ae^{-Ts} = 0$$

(b) Using $e^{-Ts} \cong 1 - Ts$, the characteristic equation is,

$$s^2 + (1 - TA)s + A = 0$$

The Routh's array is,

$$\begin{array}{lcll} s^2 & : & 1 & A \\ s^1 & : & 1 - TA & 0 \\ s^0 & : & A & \end{array}$$

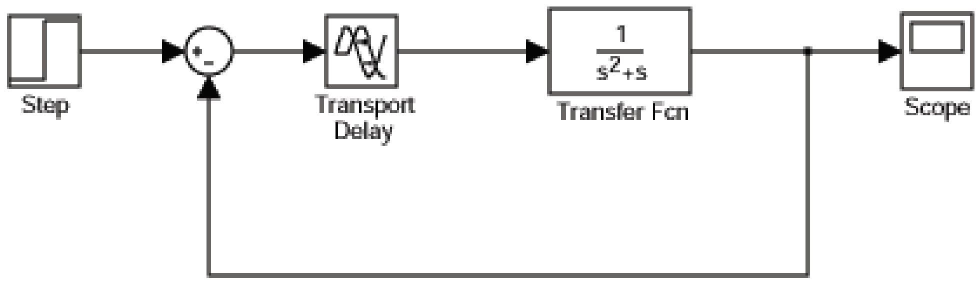
For stability we must have $A > 0$ and $TA < 1$.
Using $e^{-Ts} \cong \frac{(1-\frac{T}{2}s)}{(1+\frac{T}{2}s)}$, the characteristic equation is,

$$s^3 + (1 + \frac{2}{T})s^2 + (\frac{2}{T} - A)s + \frac{2}{T}A = 0$$

The Routh's array is,

$$\begin{array}{lcll} s^3 & : & 1 & (\frac{2}{T} - A) \\ s^2 & : & (1 + \frac{2}{T}) & \frac{2A}{T} \\ s^1 & : & \frac{(1 + \frac{2}{T})(\frac{2}{T} - A) - \frac{2A}{T}}{(1 + \frac{2}{T})} & 0 \\ s^0 & : & \frac{2A}{T} & \end{array}$$

For stability we must have all the coefficients in the first column be positive. The following Simulink diagram simulates the closed-loop system.



Problem 3.57: Simulink simulation diagram.

58. Modify the Routh criterion so that it applies to the case in which all the poles are to be to the left of $-\alpha$ when $\alpha > 0$. Apply the modified test to the polynomial

$$s^3 + (6+K)s^2 + (5+6K)s + 5K = 0,$$

finding those values of K for which all poles have a real part less than -1.

Solution:

Let $p = s+\alpha$ and substitute $s = p-\alpha$ to obtain a polynomial in terms of p. Apply the standard Routh test to the polynomial in p.

For the example $p = s+1$ or $s = p-1$. Substitute this in the polynomial,

$$(p-1)^3 + (6+K)(p-1)^2 + (5+6K)(p-1) + 5K = 0$$

or

$$p^3 + (3+K)p^2 + (4K-4)p + 1 = 0.$$

The Routh's array is,

$$\begin{array}{lclc}
p^3 & : & 1 & 4K-4 \\
p^2 & : & 3+K & 1 \\
p^1 & : & \dfrac{(3+K)(4K-4)-1}{3+K} & 0 \\
p^0 & : & 1 &
\end{array}$$

For stability, all the elements in the first column must be positive We must have $K > -3$ and $4K^2 + 8K - 13 > 0$. The roots of the second-order polynomial in K are $K = 1.06$ and $K = -3.061$. The second-order polynomial remains positive if $K > 1.06$ or $K < -3.061$. Therefore, we must have $K > 1.06$.

59. Suppose the characteristic polynomial of a given closed-loop system is computed to be

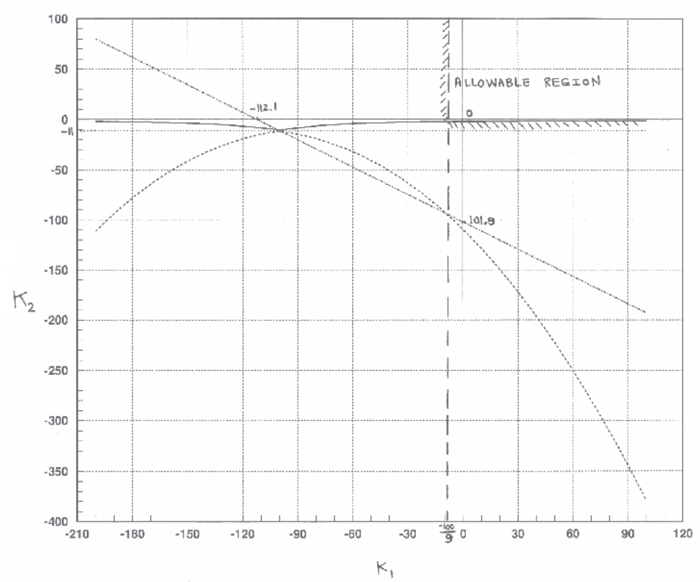
$$s^4+(11+K_2)s^3+(121+K_1)s^2+(K_1+K_1K_2+110K_2+210)s+11K_1+100=0.$$

Find constraints on the two gains K_1 and K_2 that guarantee a stable closed-loop system, and plot the allowable region(s) in the (K_1, K_2) plane. You may wish to use the computer to help solve this problem.

Solution: The Routh array is,

$$\begin{array}{lllll}
s^4 & : & 1 & 121+K_1 & 11K_1+100 \\
s^3 & : & 11+K_2 & K_1+K_1K_2+110K_2+210 & 0 \\
s^2 & : & \frac{(11K_2+10K_1+1121)}{K_2+11} & 11K_1+100 & \\
s^1 & : & \multicolumn{3}{l}{\frac{10(111K_2^2+K_1^2K_2+199K_1K_2+12342K_2+K_1^2+189K_1+22331)}{(11K_2+10K_1+1121)}} \\
s^0 & : & 11K_1+100 & &
\end{array}$$

For stability, the elements in the first column must all be positive. This means that $K_2 > -11$ and $K_1 > -\frac{100}{11}$. The region of stability is shown in the following figure.



Problem 3.59: s-plane region for stability.

60. Overhead electric power lines sometimes experience a low-frequency, high-amplitude vertical oscillation, or **gallop**, during winter storms when the line conductors become covered with ice. In the presence of wind, this ice can assume aerodynamic lift and drag forces that result in a gallop up to several meters in amplitude. Large-amplitude gallop can cause clashing conductors and structural damage to the line support structures caused by the large dynamic loads. These effects in turn can lead to power outages. Assume that the line conductor is a rigid rod, constrained to vertical motion only, and suspended by springs and dampers as shown in Fig. 3.68. A simple model of this conductor galloping is

$$m\ddot{y} + \frac{D(\alpha)\dot{y} - L(\alpha)v}{(\dot{y}^2 + v^2)^{1/2}} + T(\left(\frac{n\pi}{\ell}\right))y = 0,$$

where

$$\begin{aligned}
m &= \text{ mass of conductor },\\
y &= conductor's\ vertical\ displacement,\\
D &= aerodynamic\ drag\ force,\\
L &= aerodynamic\ lift\ force,\\
v &= wind\ velocity,\\
\alpha &= aerodynamic\ angle\ of\ attack = -\tan^{-1}(\dot{y}/v),\\
T &= conductor\ tension,\\
n &= number\ of\ harmonic\ frequencies,\\
\ell &= length\ of\ conductor.
\end{aligned}$$

Assume that $L(0) = 0$ and $D(0) = D_0$ (a constant), and linearize the equation around the value $y = \dot{y} = 0$. Use Routh's stability criterion to show that galloping can occur whenever

$$\frac{\partial L}{\partial \alpha} + D_0 < 0.$$

Solution:

$$m\ddot{y} + [\left[\frac{D(\alpha)\dot{y} - L(\alpha)v}{\sqrt{\dot{y}^2 + v^2}}\right]] + T(\left(\frac{n\pi}{l}\right))^2 y = 0,$$

Let $x_1 = y$ and $x_2 = \dot{y} = \dot{x}_1$

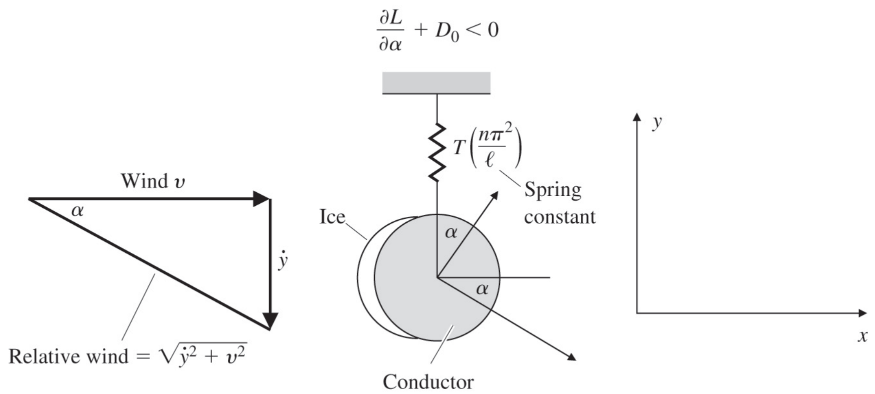


Figure 3.68: Electric power-line conductor

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -\frac{1}{m}[\left[\frac{D(\alpha)x_2 - L(\alpha)v}{\sqrt{x_2^2+v^2}}\right]] - \frac{T}{m}(\left(\frac{n\pi}{l}\right))^2 x_1 = 0 \\
\alpha &= -\tan^{-1}(\left(\frac{x_2}{v}\right)) \\
\dot{x}_1 &= f_1(x_1, x_2) \\
\dot{x}_2 &= f_2(x_1, x_2)
\end{aligned}$$

$$\begin{aligned}
\dot{x}_1 &= \dot{x}_2 = 0 \quad \text{implies} \quad x_2 = 0 \\
x_2 &= 0 \quad \text{implies} \quad \alpha = 0 \\
\alpha &= 0 \quad \text{implies} \ -\frac{T}{m}(\left(\frac{n\pi}{l}\right))^2 x_1 = 0 \quad \text{implies } x_1 = 0.
\end{aligned}$$

$$\frac{\partial f_1}{\partial x_1} = 0, \qquad \frac{\partial f_2}{\partial x_2} = 1, \qquad \frac{\partial f_2}{\partial x_1} = -\frac{T}{m}(\left(\frac{n\pi}{l}\right))^2$$

$$\begin{aligned}
\frac{\partial f_2}{\partial x} = \frac{\partial}{\partial x_2}\{-\frac{1}{m}[\left[\frac{D(\alpha)x_2 - L(\alpha)v}{\sqrt{x_2^2+v^2}}\right]]\} \\
= -\frac{1}{m}\{\frac{1}{\sqrt{x_2^2+v^2}}[\frac{\partial D}{\partial \alpha}\frac{\partial \alpha}{\partial x_2}x_2 + D(\alpha) - \frac{\partial L}{\partial \alpha}\frac{\partial \alpha}{\partial x_2}] - \\
-[\left[\frac{D(\alpha)x_2 - L(\alpha)v}{\sqrt{x_2^2+v^2}}\right]][\left[\frac{-x_2}{(x_2^2+v^2)^{\frac{3}{2}}}\right]]\}
\end{aligned}$$

Now,

$$\frac{\partial \alpha}{x_2} = \frac{\partial}{\partial x_2}(-\tan^{-1}(\left(\frac{x_2}{v}\right))) = \frac{-1}{1+\frac{x_2^2}{v^2}}(\left(\frac{1}{v}\right))$$

so,

$$\begin{aligned}\frac{\partial f_2}{\partial x_2} = \frac{-1}{m}\{\frac{1}{\sqrt{x_2^2+v^2}}[\frac{-\frac{\partial D}{\alpha}x_2}{v(1+\frac{x_2^2}{v^2})} + D(\alpha) + \frac{\frac{\partial L}{\partial \alpha}v}{v(1+\frac{x_2^2}{v^2})}] \\ -[\left[\frac{D(\alpha)x_2 - L(\alpha)v}{\sqrt{x_2^2+v^2}}\right]][\left[\frac{-x_2}{(x_2^2+v^2)^{\frac{3}{2}}}\right]]\} \\ \frac{\partial f_2}{\partial x_2}|_{x_2=0} = -\frac{1}{m}\{\frac{1}{v}[D_0 + \frac{\partial L}{\partial \alpha}]\} = -\frac{1}{mv}(D_0 + \frac{\partial L}{\partial \alpha})\end{aligned}$$

For no damping (or negative damping) δx_2 term must be ≤ 0 so this implies $D_0 + \frac{\partial L}{\partial \alpha} < 0$.