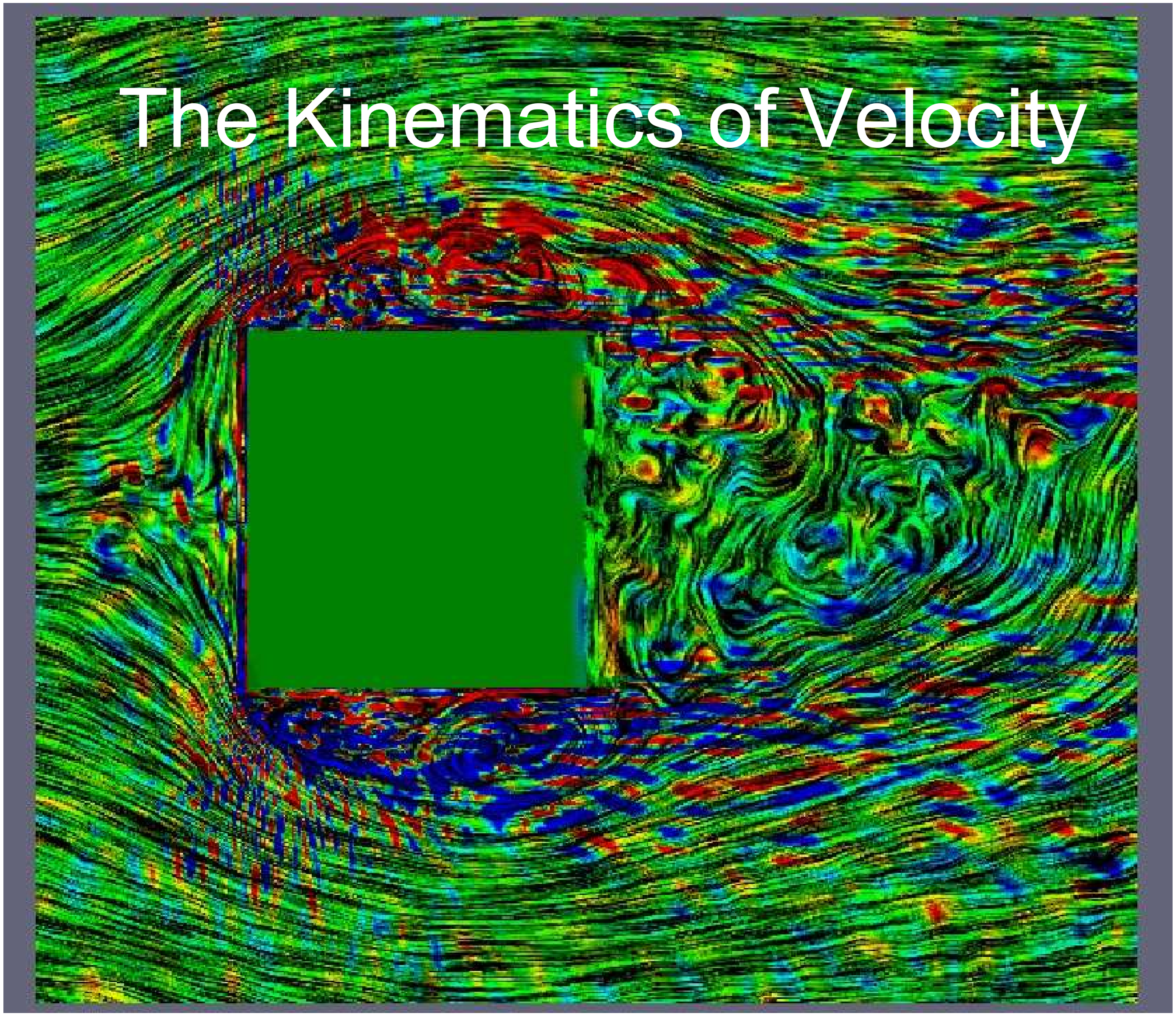
Kinematics

- Kinematic Concepts
  - Velocity: Fluid lines, particle paths, streamlines, etc.
  - Vorticity: Vortex lines, sheets and tubes
- Helmholtz´s Vortex Theorems
- Kelvin´s Circulation Theorem
- Some applications and examples

The Kinematics of Velocity

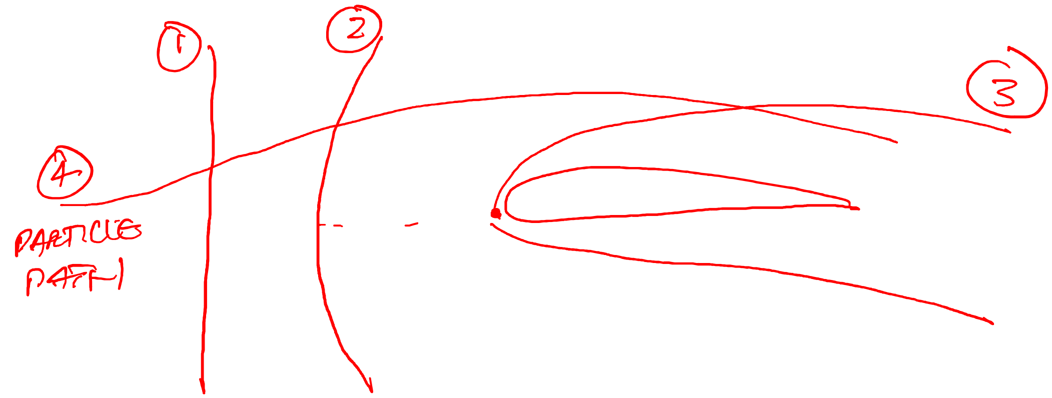


Kinematic Concepts - Velocity

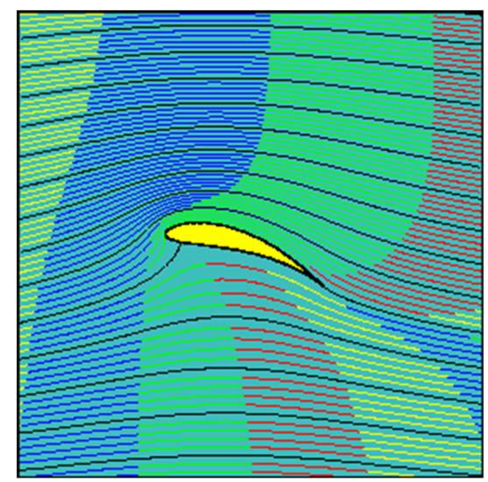
1. <u>Fluid Line</u>. ANY CONTINOUS STRING OF FLUID PARTICLES.

   MOVES WITH FLOW

   CANNOT BE BROKEN



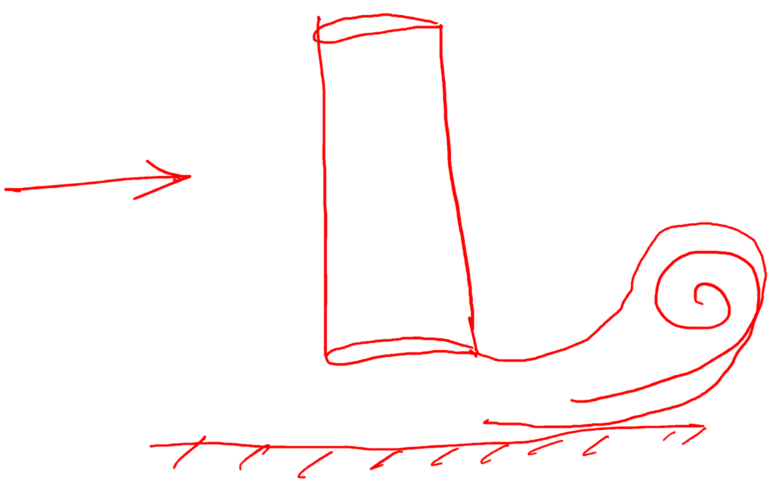
2. <u>Particle Path</u>. PATH TRAVERESED BY FLUID PARTICLE

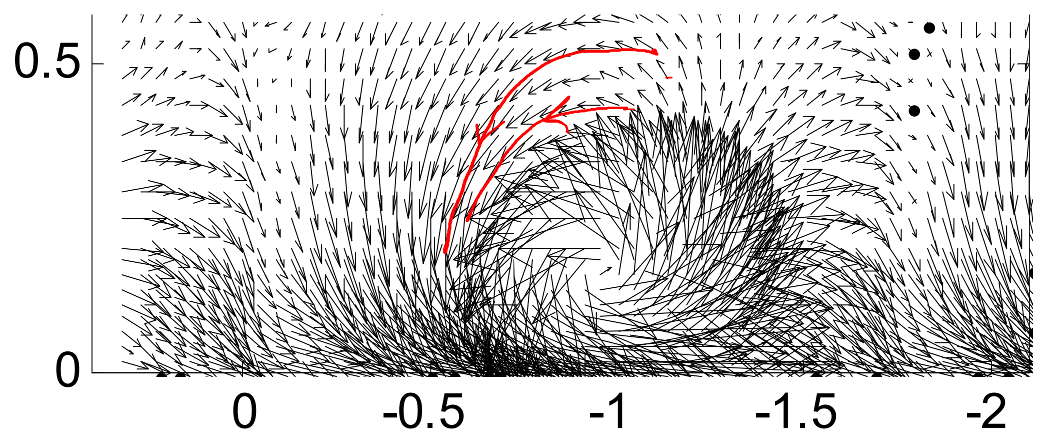


Kinematic Concepts - Velocity

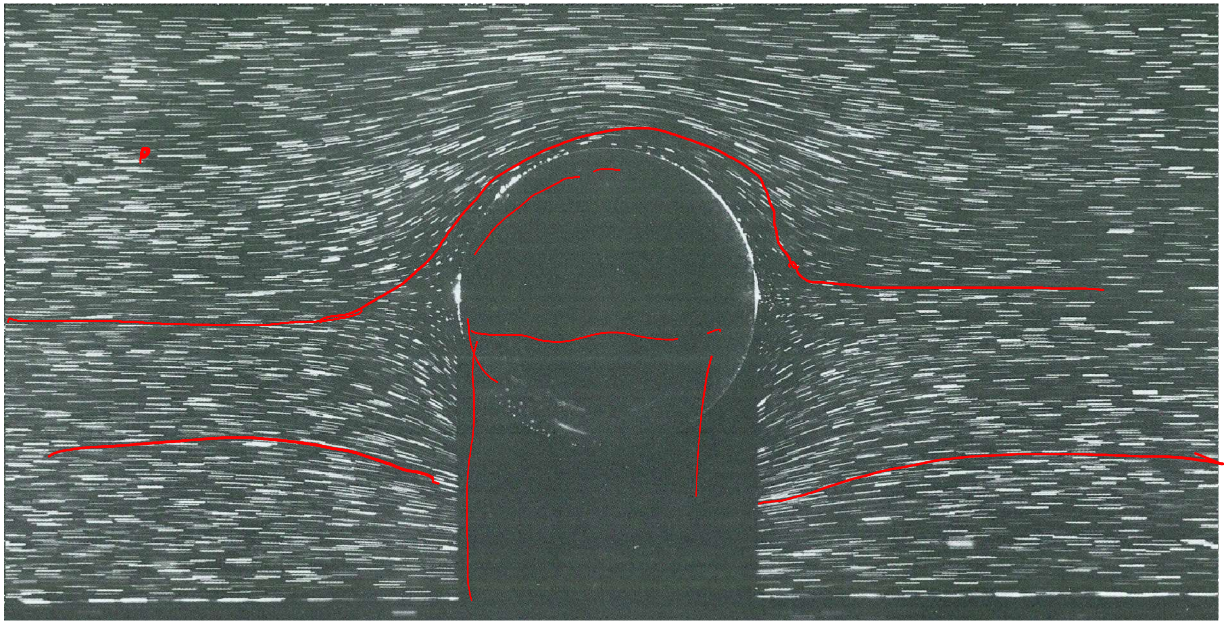
3. Streamline. Line tangent to the velocity vector. No flow across a streamline.

In a steady flow : streamlines = particle paths





Frame of Reference



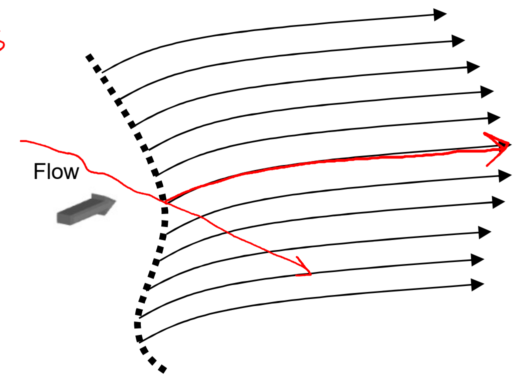
8. Sphere moving through a tube at $R=0.10$, relative motion. A free sphere is falling steadily down the axis of a tube of twice its diameter filled with glycerine. The camera is moved with the speed of the sphere to show the flow relative to it. The photograph has been rotated to show flow from left to right. Tiny magnesium cuttings are illuminated by a thin sheet of light, which casts a shadow of the sphere. *Coutanceau 1968*

Kinematic Concepts - Velocity

3. Streamsurface.

SURFACE TANGENT TO $\vec{V}$ EVERYWHERE

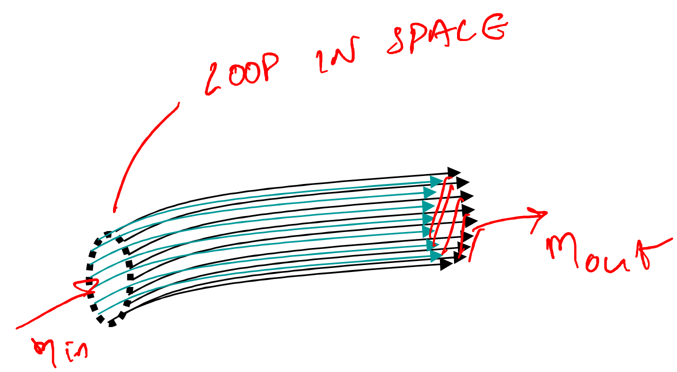
NO FLOW THROUGH A STREAMLINE

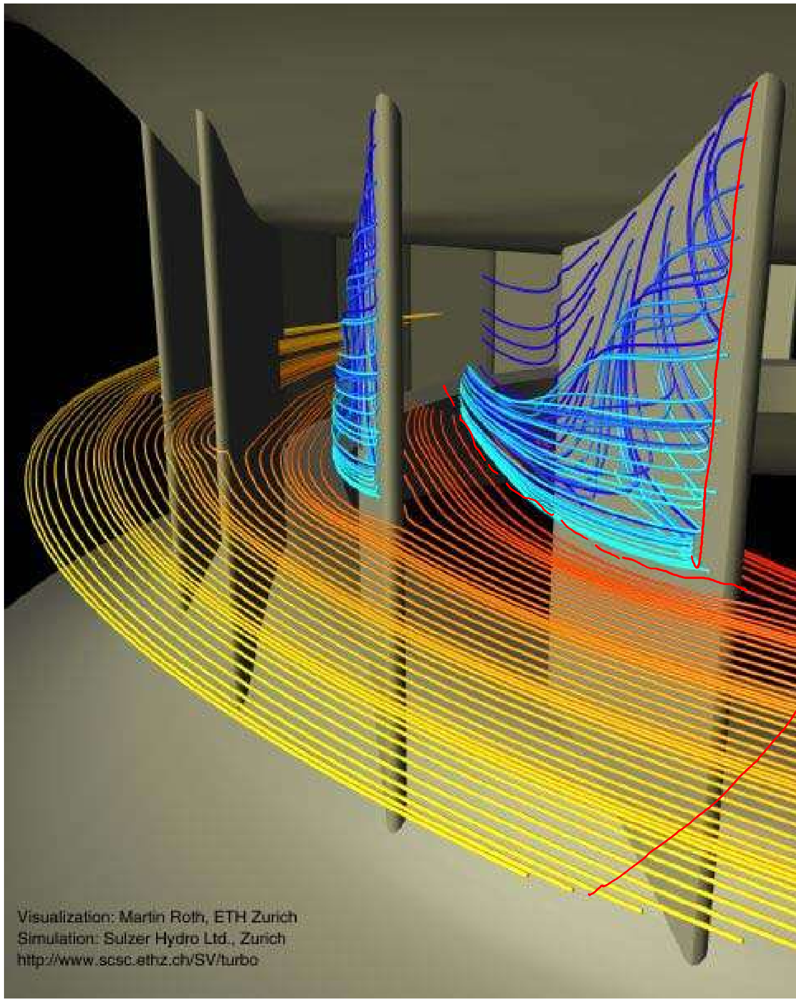


4. Streamtube. STREAM SURFACE

~~REDUCED INT~~

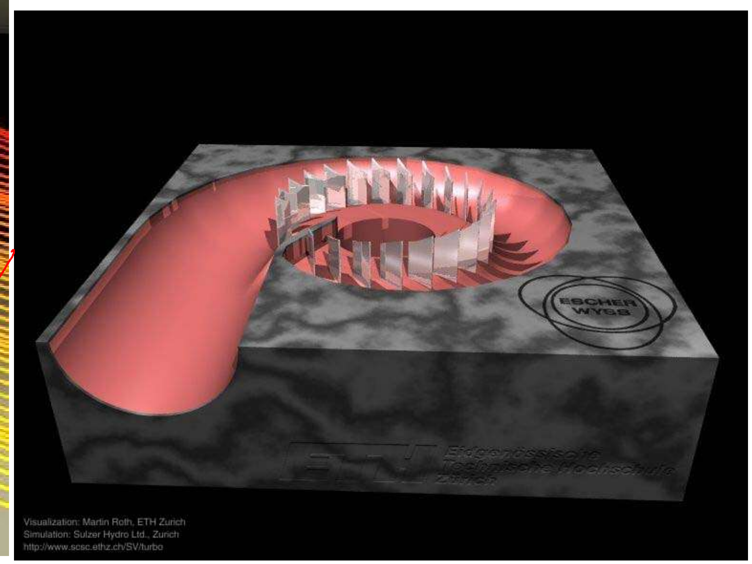
ROLLED INTO A TUBE





Francis turbine simulation ETH Zurich

http://www.cg.inf.ethz.ch/~bauer/turbo/research_gallery.html

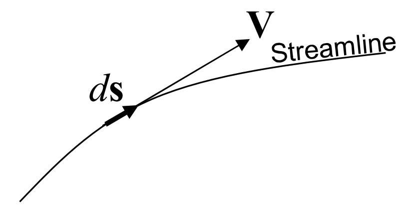


Mathematical Description

1. Streamlines $\vec{ds} \times \vec{V} = 0$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ dx & dy & dz \\ u & v & w \end{vmatrix} = 0$$

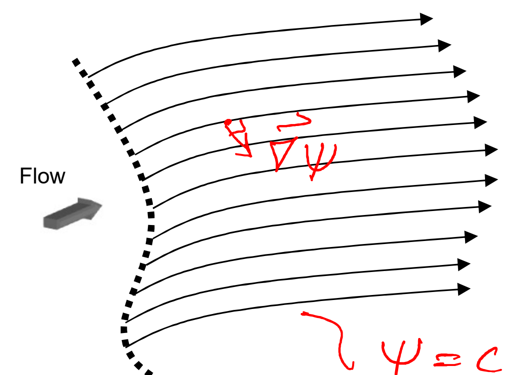
$$\Rightarrow \frac{dy}{dx} = \frac{v}{u} \; ; \; \frac{dz}{dx} = \frac{w}{u} \; \cdots$$



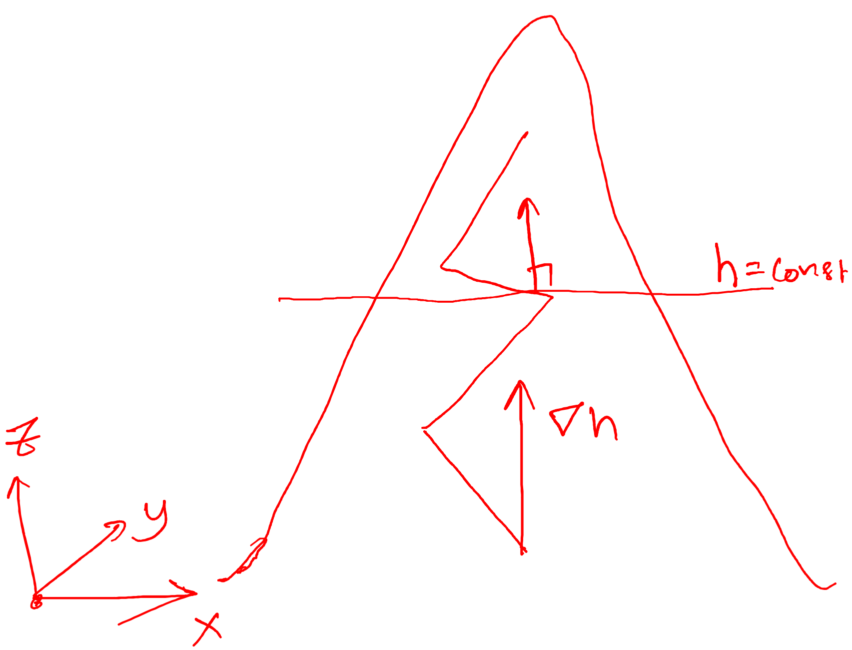
2. Streamsurfaces

DEFINE $\psi = \psi(x, y, z, t) \Rightarrow$ CONTOUR SURFACE OF ψ are STREAMSURFACE

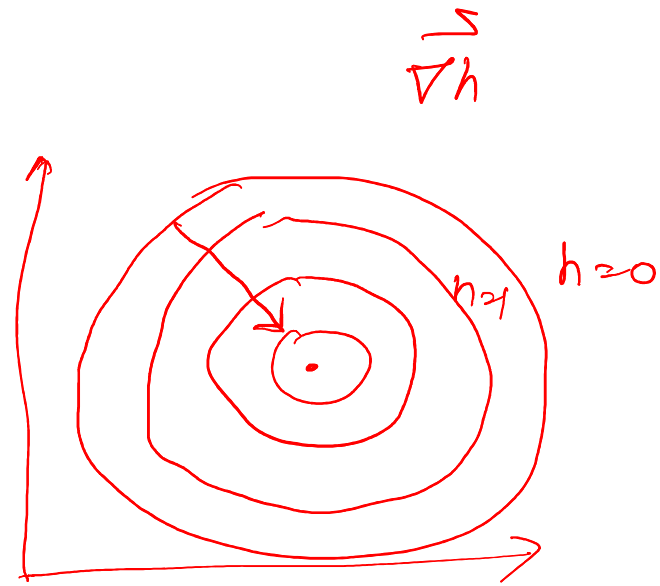
ψ = STREAM FUNCTION



$\psi_1 = \text{const}$



h=const
∇h
z
y
x

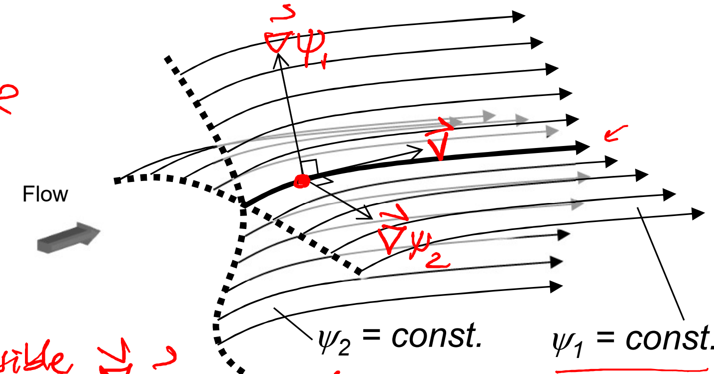


$\vec{\nabla} h$
h=0
h=1

Mathematical Description

3. Relationship between streamlines and streamsurfaces

$\vec{V}$, $\nabla\psi_1$, $\nabla\psi_2$ ARE PERPENDICULAR TO EACH OTHER



$$\boxed{\alpha \vec{V} = \nabla\psi_1 \times \nabla\psi_2}$$

where $\alpha = \alpha(x, y, z, t)$. Now if the flow is incompressible $\vec{\nabla}\cdot\vec{V} = 0$

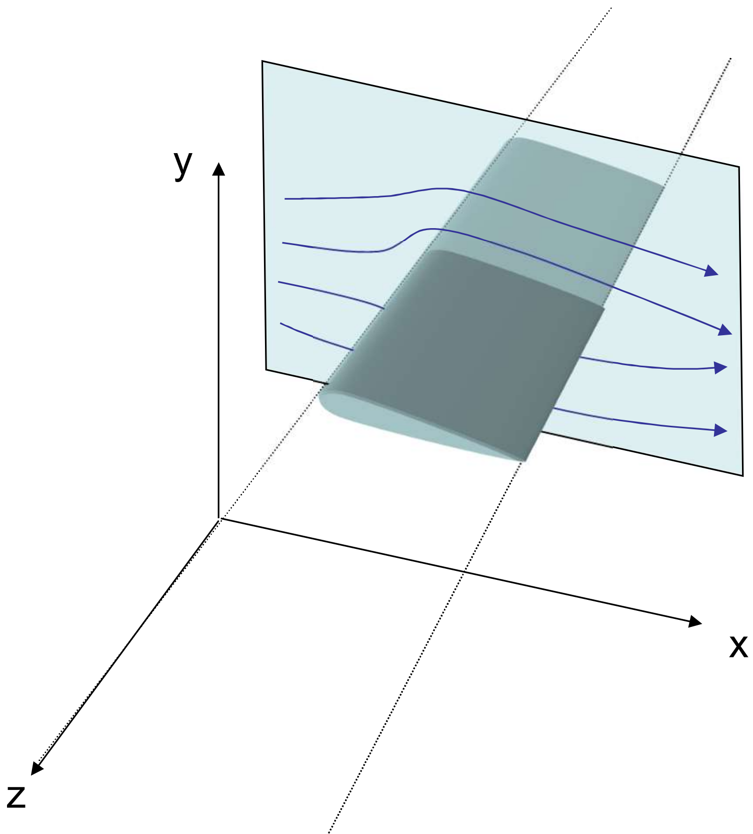
otherwise $\frac{1}{\rho}\frac{D\rho}{Dt} = -\vec{\nabla}\cdot\vec{V}$; But if the flow is steady $\nabla\cdot(\rho\vec{V}) = 0$

Now $\nabla\cdot(\alpha\vec{V}) = \nabla\cdot(\nabla\psi_1 \times \nabla\psi_2) = 0$; If the flow were incompressible, $\alpha = 1$

If the flow were steady and compressible, $\alpha = \rho$

Example: 2D – Flow Over An Airfoil

$\alpha \vec{V} = \vec{\nabla}\psi_1 \times \vec{\nabla}\psi_2$

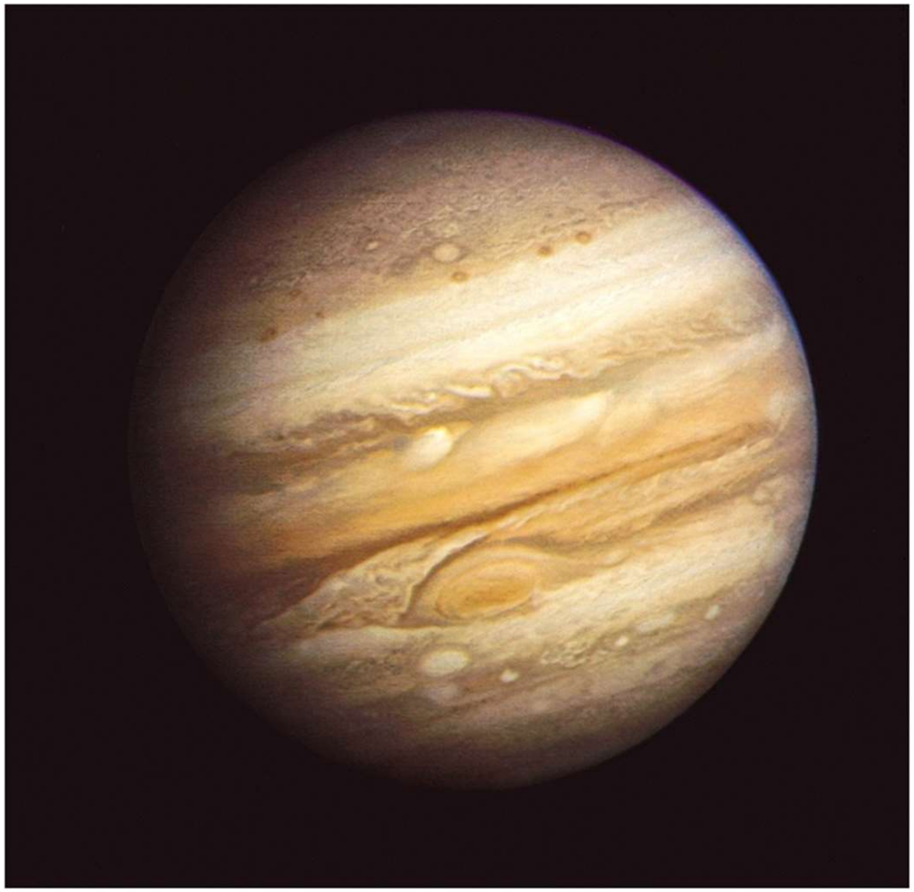


Find consistent relations for the steamfunctions (explicit or in terms of the velocity field).

$\psi_2 = z$; $\nabla\psi_2 = \vec{k}$

$$\alpha \vec{V} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial\psi_1/\partial x & \partial\psi_1/\partial y & \partial\psi_1/\partial z \\ 0 & 0 & 1 \end{vmatrix}$$

$\alpha u = \partial\psi_1/\partial y$; $\alpha v = -\dfrac{\partial\psi_1}{\partial x}$

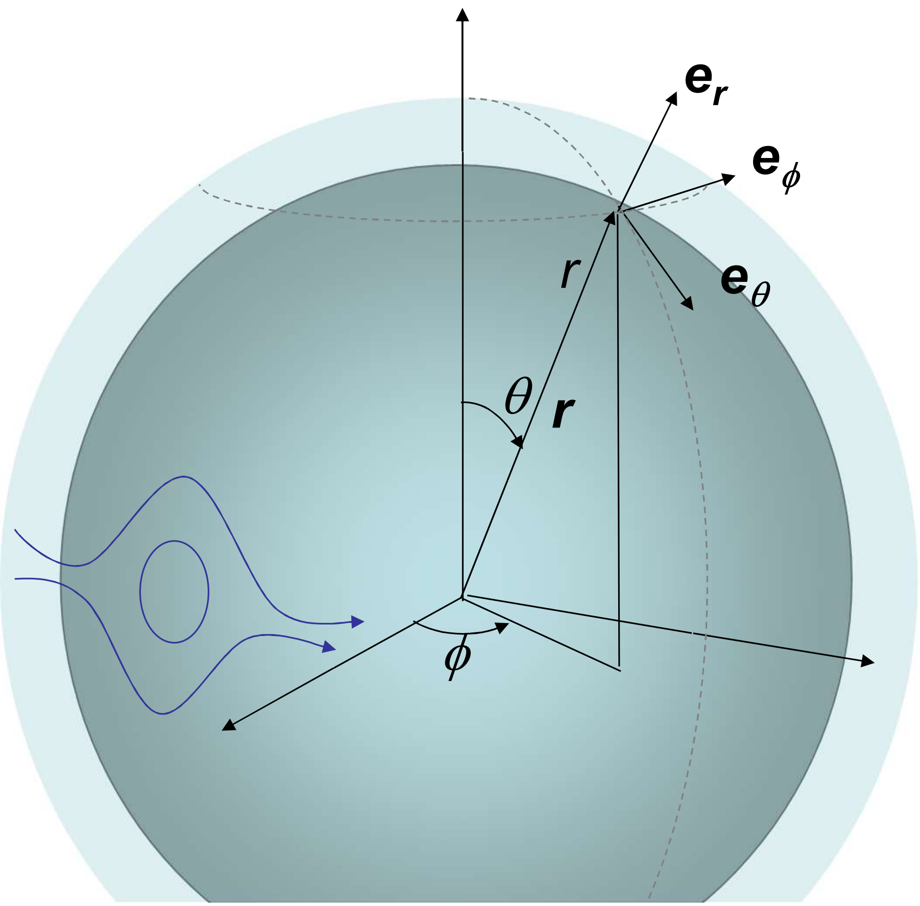


Jupiter

Example: Spherical Flow

$\alpha \vec{V} = \nabla \psi_1 \times \nabla \psi_2$

Flow takes place in spherical shells (no radial velocity).



Find a set of streamfunctions.

$\psi_2 = r \Rightarrow \nabla \psi_2 = \vec{e_r}$

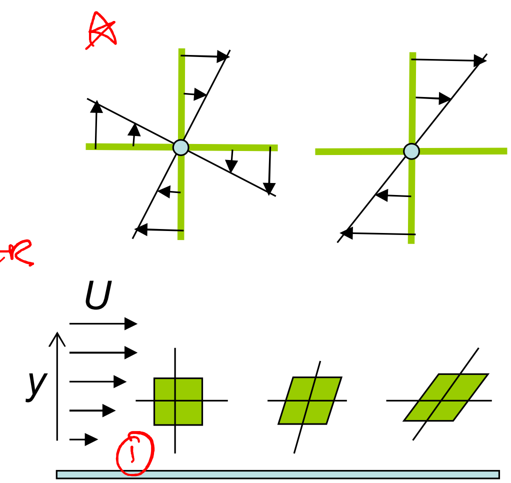
$$\alpha \vec{V} = \begin{vmatrix} \vec{e_r} & \vec{e_\theta} & \vec{e_\phi} \\ \frac{\partial \psi_1}{\partial r} & \frac{1}{r}\frac{\partial \psi_1}{\partial \theta} & \frac{1}{r\sin\theta}\frac{\partial \psi_1}{\partial \phi} \\ 1 & 0 & 0 \end{vmatrix}$$

$$\alpha V_\theta = \frac{1}{r\sin\theta}\frac{\partial \psi_1}{\partial \phi}$$

$$\alpha V_\phi = -\frac{1}{r}\frac{\partial \psi_1}{\partial \theta}$$

Vorticity Ω

- $\vec{\Omega} = \vec{\nabla} \times \vec{V}$
- SUM OF ROTATION RATE OF PERPENDICULAR FLUID LINES
- $\vec{\Omega}$ & $\vec{V}$ NOT GENERALLY PERPENDICULAR TO EACH OTHER
- $\vec{\nabla} \cdot \vec{\Omega} \equiv 0$ always true

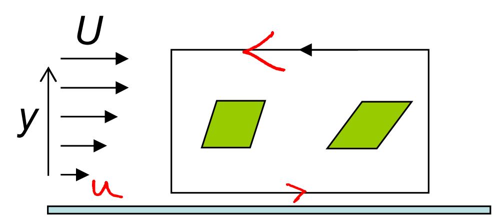


No spin, but a net rotation rate

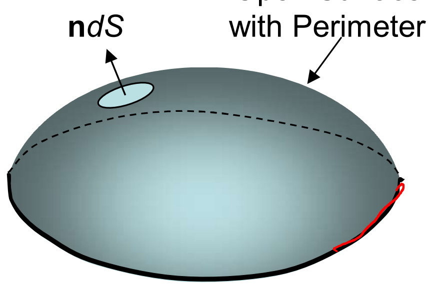
Circulation Γ

- $\Gamma \equiv \oint \vec{V} \cdot d\vec{s}$
- MACROSCOPIC ROTATION OF THE FLUID
- $\Gamma_c \neq 0$ DOES NOT IMPLY SPIN.
- STOKES'S THEOREM

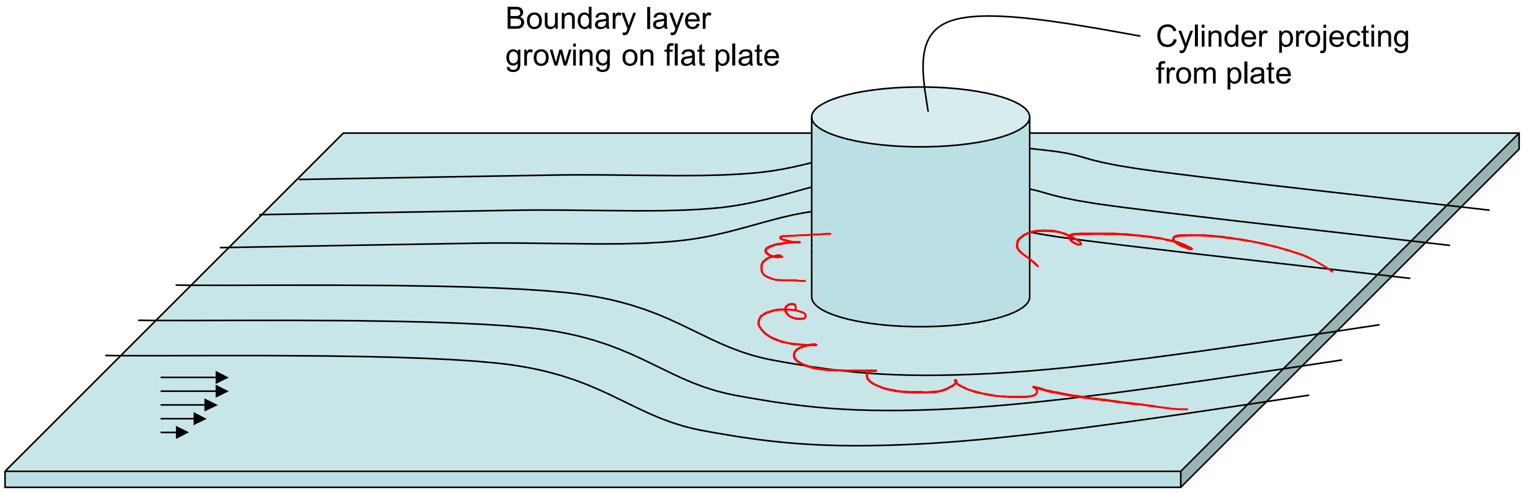
$\rightarrow \oint_S \vec{\Omega} \cdot \vec{n}\, dS \equiv 0$

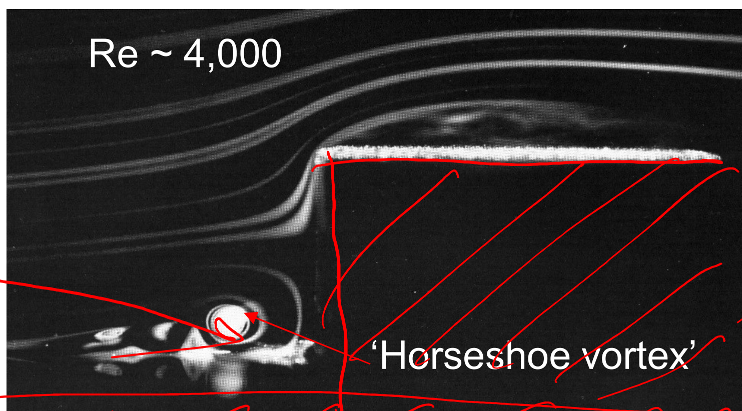


$$\int_S \boldsymbol{\Omega}.\mathbf{n}\,dS = \oint_C \mathbf{V}.d\mathbf{s} = \Gamma_c$$

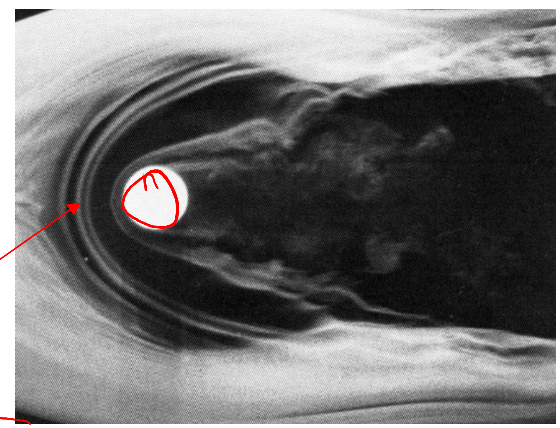


Flow Past a Cylinder on a Plate





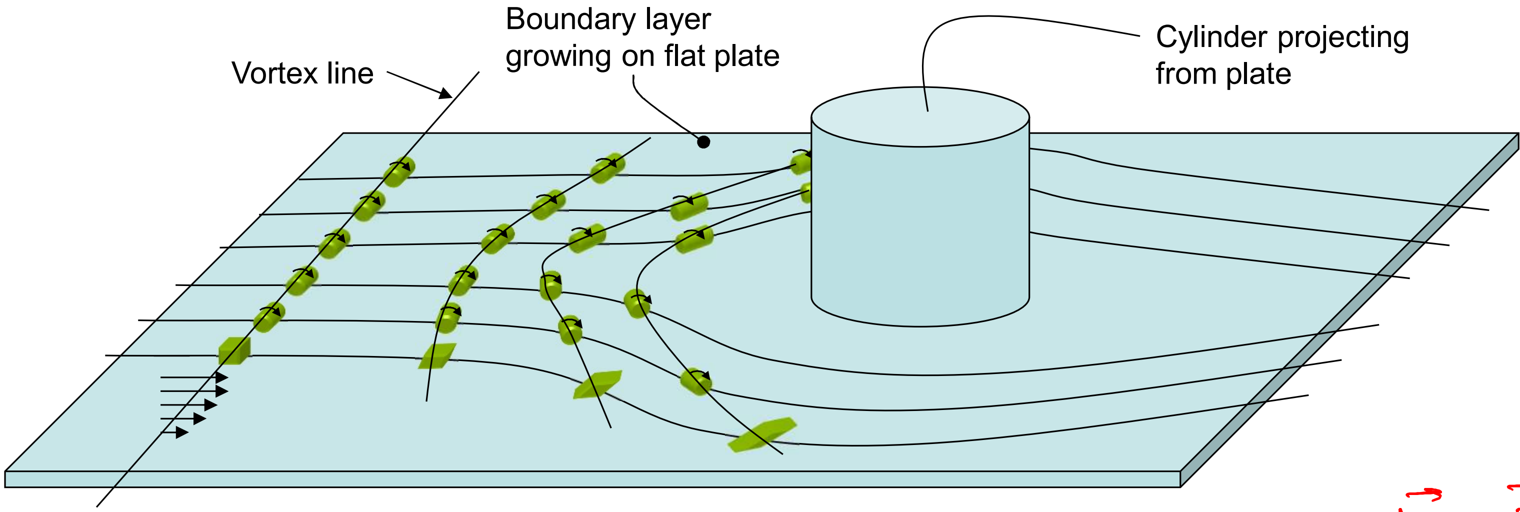
Side view



Top view

Pictures are from *"An Album of Fluid Motion"* by Van Dyke

Kinematic Concepts - Vorticity



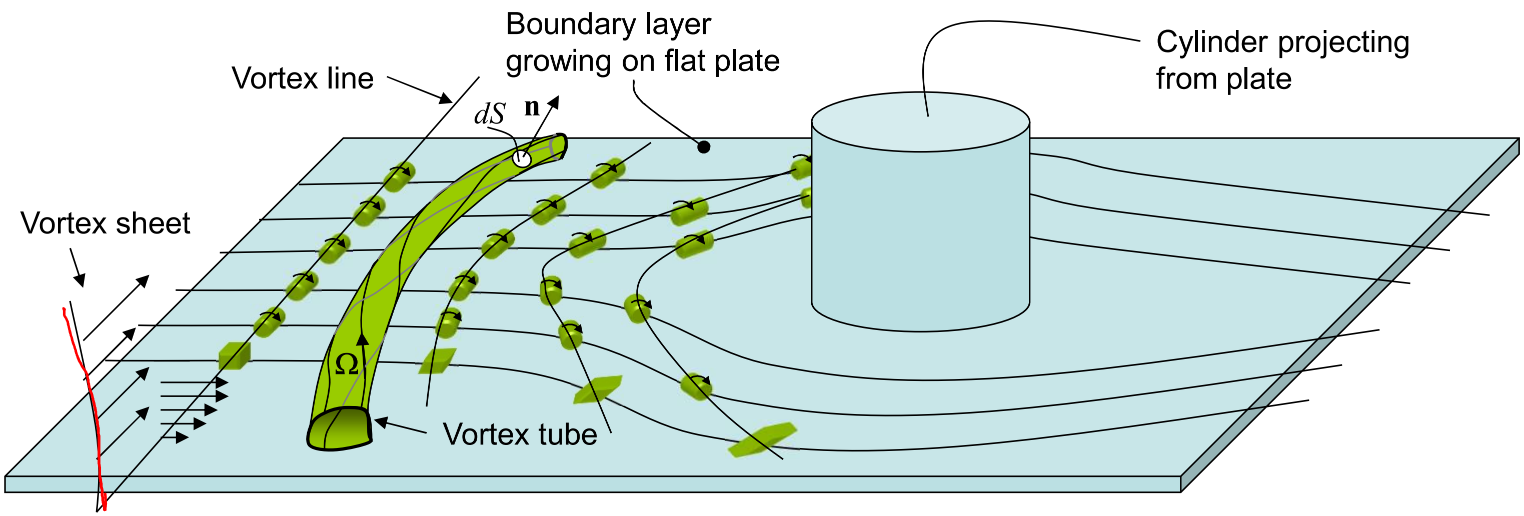
$\vec{ds} \times \vec{\Omega} = 0$

Vortex Line: A LINE EVERYWHERE TANGENT TO THE VORTICITY VECTOR

A LINE THAT THREADS TOGETHER THE AXES OF ROTATION OF FLUID PARTICLES

NOT A FUNCTION OF FRAME OF REFERENCE

Kinematic Concepts - Vorticity



Vortex sheet: SURFACE FORMED BY ALL THE VORTEX LINES PASSING THROUGH A CURVE IN SPACE. NO FLUX THROUGH A SURFACE.
VORTICITY

$$\vec{\Omega} \cdot n\,d\vec{S} = 0$$

Vortex tube: VORTEX SHEET ROLLED INTO A TUBE

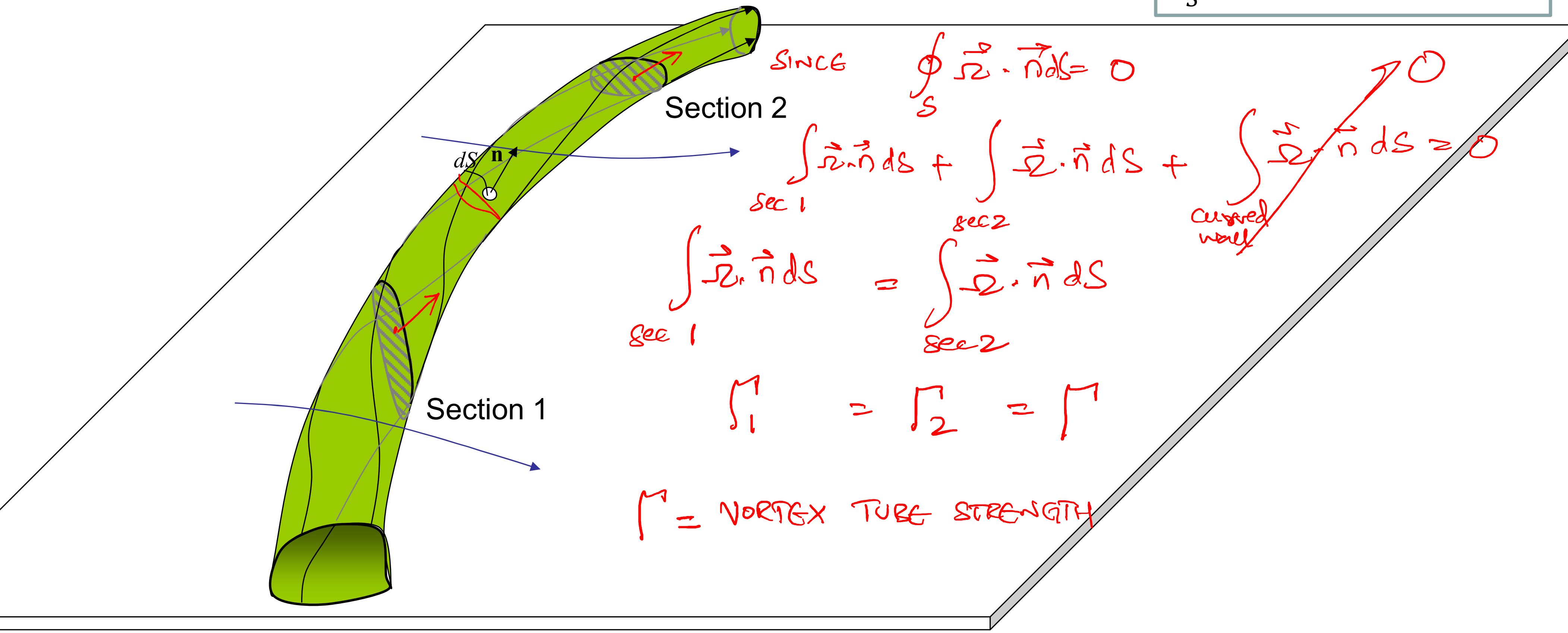
Vortex Tube

$$\int_S \boldsymbol{\Omega}\,.\,\mathbf{n}dS = \int_C \mathbf{V}.\,d\mathbf{s} = \Gamma_c$$

$$\oint_S \boldsymbol{\Omega}.\,\mathbf{n}dS = 0$$

$\vec{\nabla}\cdot\vec{\Omega} = 0$

- Vorticity behaves as though it were the velocity field of an incompressible fluid
- The flux of vorticity thru' an surface is equal to the circulation around the edge of that surface



SINCE $\oint_S \vec{\Omega}\cdot\vec{n}\,dS = 0$

$$\int_{sec\ 1} \vec{\Omega}\cdot\vec{n}\,dS + \int_{sec\ 2} \vec{\Omega}\cdot\vec{n}\,dS + \cancel{\int_{\text{curved wall}} \vec{\Omega}\cdot\vec{n}\,dS}^{\,0} = 0$$

$$\int_{sec\ 1} \vec{\Omega}\cdot\vec{n}\,dS = \int_{sec\ 2} \vec{\Omega}\cdot\vec{n}\,dS$$

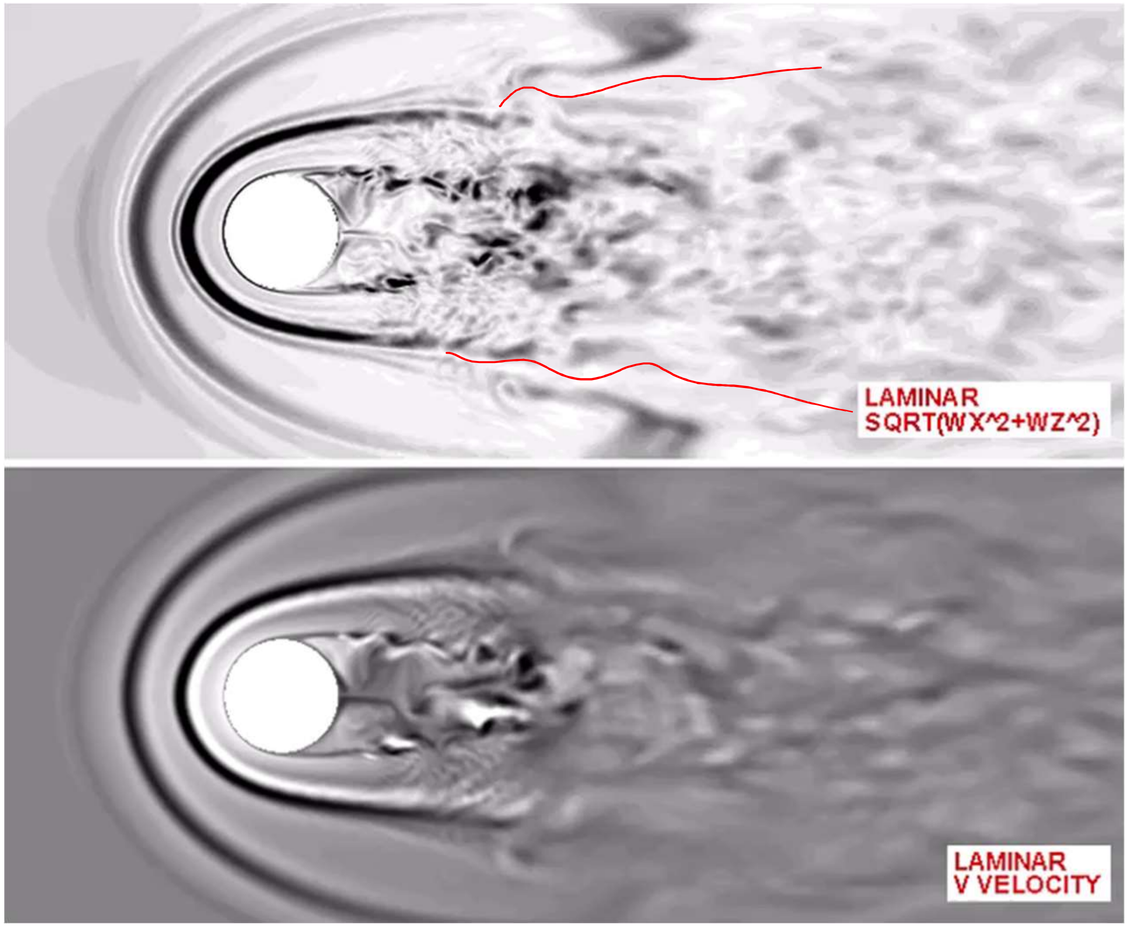
$$\Gamma_1 = \Gamma_2 = \Gamma$$

Γ = VORTEX TUBE STRENGTH

Implications
(Helmholtz' Vortex Theorems, Part 1)

- The strength of a vortex tube (defined as the circulation around it) is constant along the tube.
- The tube, and the vortex lines from which it is composed, can therefore never end. They must extend to infinity or form loops.
- The average vorticity magnitude inside a vortex tube is inversely proportional to the cross-sectional area of the tube

Large Eddy Simulation Re=5000



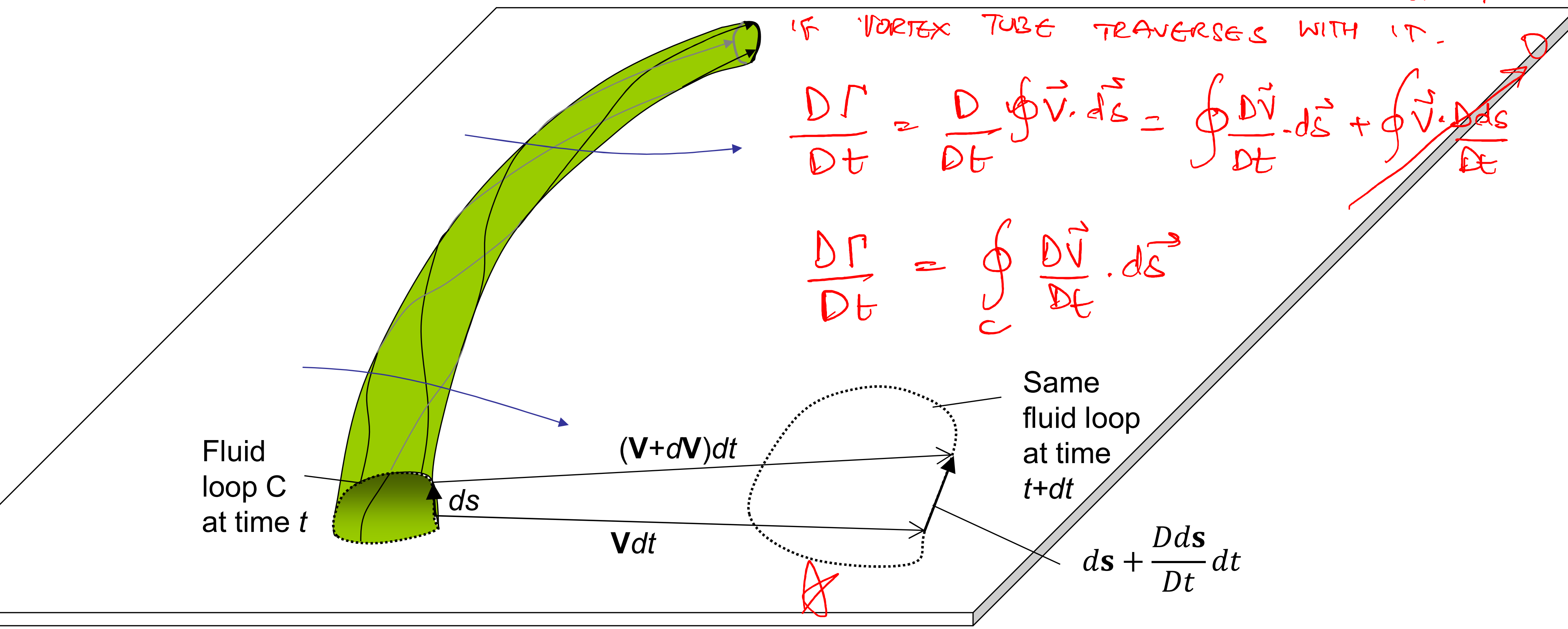
George Constantinescu IIHR, U. Iowa

But, does the vortex tube travel along with the fluid, or does it have a life of its own?

CIRCULATION AROUND A FLUID TUBE WILL BE CONSTANT IF VORTEX TUBE TRAVERSES WITH IT.

$$\frac{D\Gamma}{Dt} = \frac{D}{Dt}\oint \vec{V}\cdot d\vec{s} = \oint \frac{D\vec{V}}{Dt}\cdot d\vec{s} + \cancel{\oint \vec{V}\cdot \frac{Dd\vec{s}}{Dt}}^{\,0}$$

$$\frac{D\Gamma}{Dt} = \oint_C \frac{D\vec{V}}{Dt}\cdot d\vec{s}$$



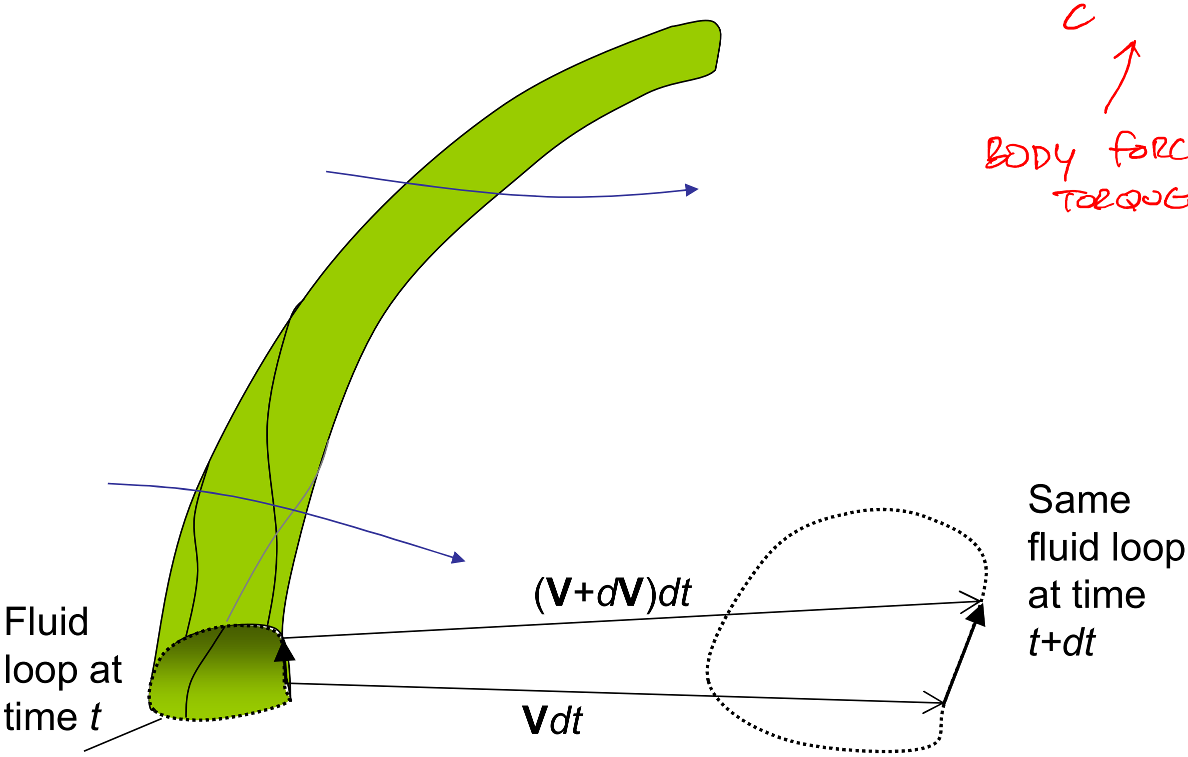
So $$\frac{D\Gamma}{Dt} = \int_C \frac{D\mathbf{V}}{Dt}.d\mathbf{s}$$ and $$\frac{D\mathbf{V}}{Dt} = \mathbf{f} - \frac{\nabla p}{\rho} + \overbrace{\frac{1}{\rho}\left[(\nabla.\boldsymbol{\tau}_x)\mathbf{i} + (\nabla.\boldsymbol{\tau}_y)\mathbf{j} + (\nabla.\boldsymbol{\tau}_z)\mathbf{k}\right]}^{\vec{f}_v}$$

$$\frac{D\Gamma}{Dt} = \oint_C \vec{f}.d\vec{s} - \oint_C \frac{\nabla p}{\rho}.d\vec{s} + \oint_C \vec{f}_v.d\vec{s}$$

BODY FORCE TORQUE

PRESSURE FORCE TORQUE

VISCOUS FORCE TORQUE



Body Force Torque $\oint_C \mathbf{f}.d\mathbf{s}$

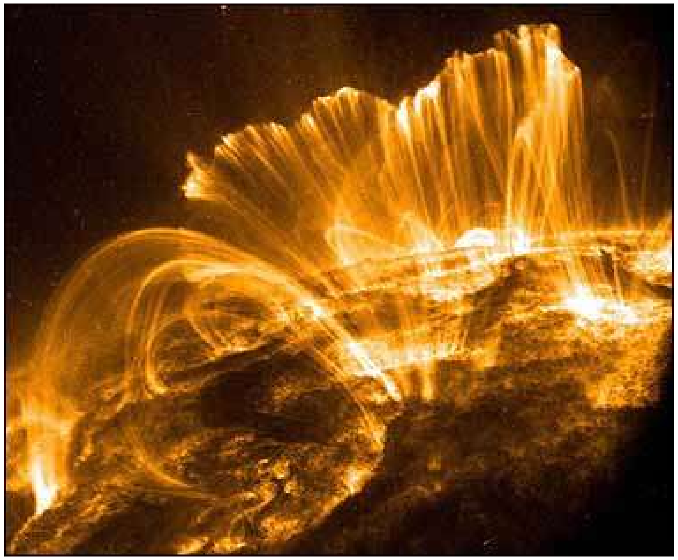
STOKES THEOREM $\oint_C \vec{f} \cdot d\vec{s} = \int_S \underline{\nabla \times \vec{f}} \cdot n \, dS$

FOR GRAVITY : $\vec{f} = -g\hat{k} \Rightarrow \vec{\nabla} \times \vec{f} = 0$

ZERO FOR MOST PRACTICAL SIMULATIONS

EXCEPTIONS : MHD DRIVES

SOLAR WIND

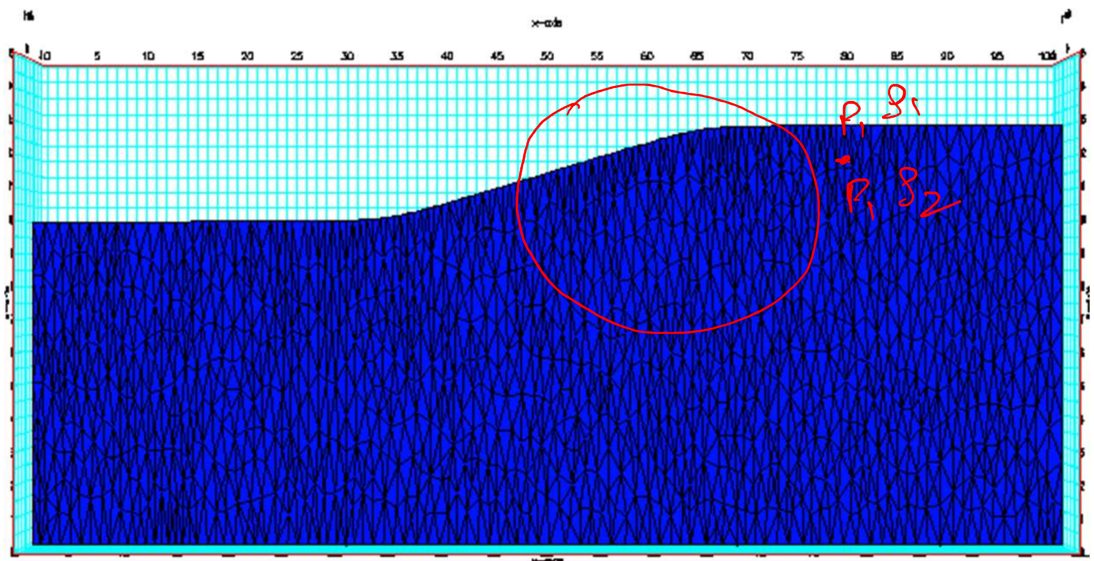


Pressure Force Torque

$$-\oint_C \frac{\nabla p}{\rho} . d\mathbf{s}$$

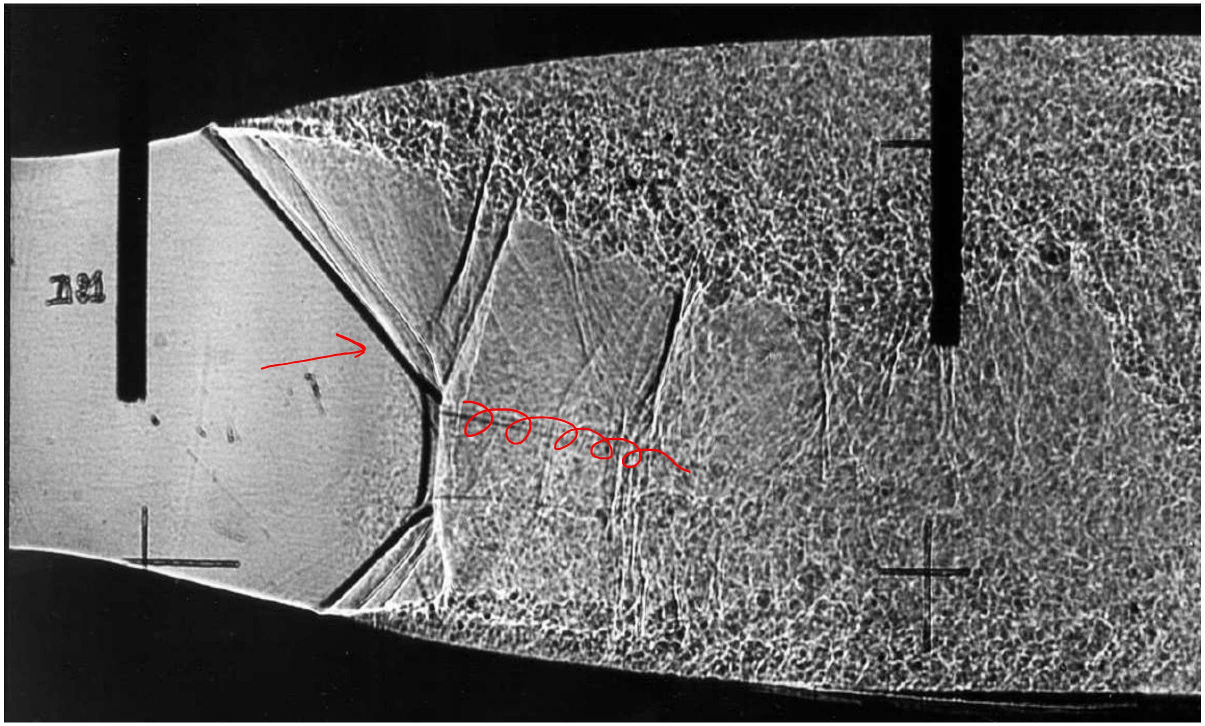
$$= \int_S \nabla \times \left(\frac{\nabla p}{\rho}\right) . \vec{n}\, dS = \int \left(\frac{1}{\rho^2} \nabla\rho \times \nabla p\right) . \vec{n}\, dS$$

If $\underbrace{p = p(\rho)}_{\text{BAROTROPIC FLUID}}$, Then ∇p is parallel to $\nabla\rho$ and pressure force torque is zero.



Earth Science and Engineering Imperial College UK

Shock in a CD Nozzle

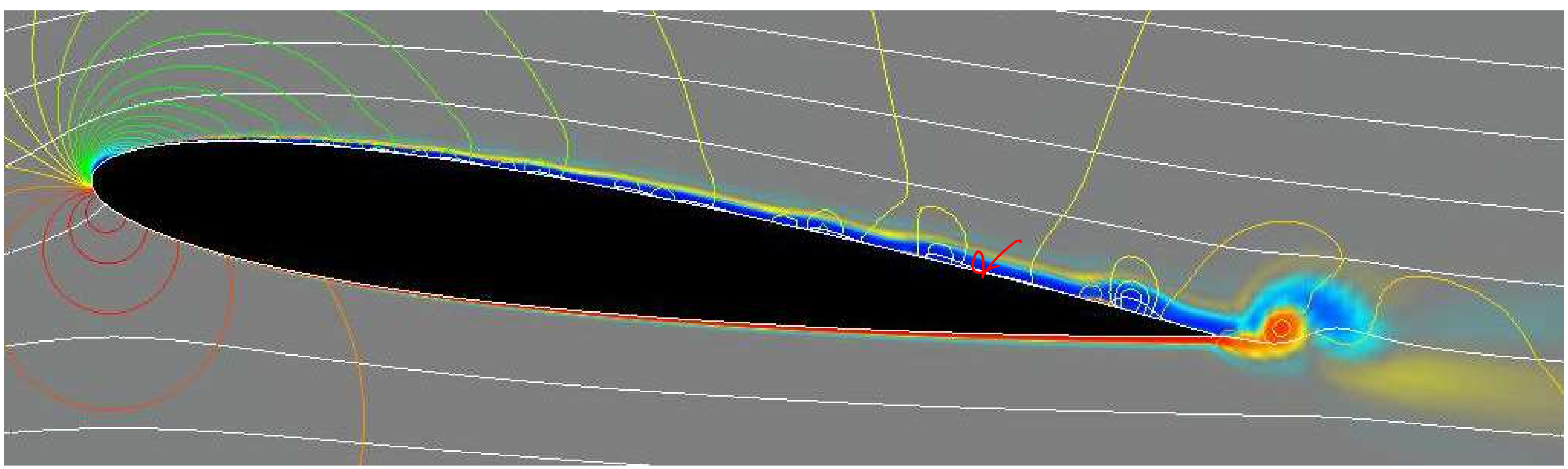


Bourgoing & Benay (2005), ONERA, France

Schlieren visualization
Sensitive to in-plane index of ref. gradient

Viscous Force Torque $\oint_C \mathbf{f}_v . d\mathbf{s}$

- NON-ZERO WHENEVER YOU HAVE VISCOUS EFFECTS (e.g. BL or wakes)
- VISCOUS TORQUES CAN BE VERY SMALL AT HIGH REYNOLDS NUMBERS AND CAN OFTEN BE IGNORED OVER SHORT DISTANCES



Implications

In the absence of body-force torques, pressure torques and viscous torques…

- the circulation around a fluid loop stays constant: $\frac{D\Gamma}{Dt} = 0$
  Kelvin's Circulation Theorem
- a vortex tube travels with the fluid material (as though it were part of it), or
  - a vortex line will remain coincident with the same fluid line
  - the vorticity convects with the fluid material, and doesn't diffuse
  - fluid with vorticity will always have it
  - fluid that has no vorticity will never get it

$$\frac{D\Gamma}{Dt} = \underbrace{\oint_C \mathbf{f}.d\mathbf{s}}_{\text{'Body force torque'}} - \underbrace{\oint_C \frac{\nabla p}{\rho}.d\mathbf{s}}_{\text{'Pressure force torque'}} + \underbrace{\oint_C \mathbf{f}_v.d\mathbf{s}}_{\text{Viscous force 'torque'}}$$

Helmholtz' Vortex Theorems, Part 2

Vorticity Transport Equation

- The kinematic condition for convection of vortex lines with fluid lines is found as follows

$$\frac{D}{Dt}(\mathbf{\Omega} \times \mathbf{ds}) = 0$$

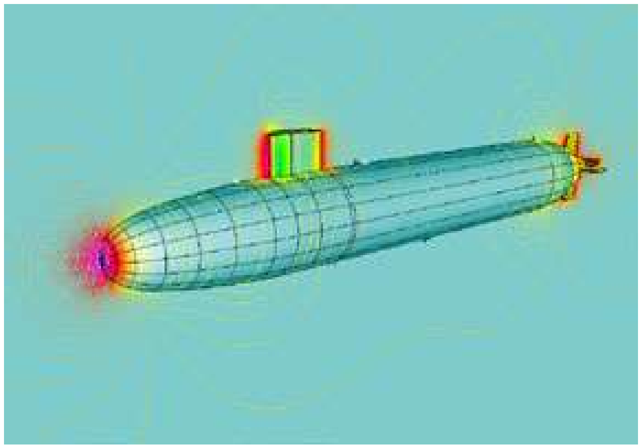
After a lot of math we get....

$$\boxed{\frac{D\mathbf{\Omega}}{Dt} = \mathbf{\Omega}.\nabla\mathbf{V}}$$

Example: Irrotational Flow?

Consider a vehicle moving at constant speed in homogeneous medium (i.e. no free surfaces) under the action of gravity, moving into a stationary fluid.

Apparent uniform flow (**V** = const.)



http://www.lcp.nrl.navy.mil/~ravi/par3d.html

WHERE IS THIS FLOW IRROTATIONAL?

IRROTATIONAL : $\Gamma = 0$ or $\Omega = 0$

FAR AHEAD, THERE IS NO VORTICITY, FLOW IS IRROTATIONAL.

NO BODY FORCE TORQUES AND PRESSURE FORCE & TORQUES UNLIKELY

VISCOUS FORCE TORQUES SIGNIFICANT ONLY IN BL, & WAKES

FLOW MUST BE IRROTATIONAL EVERYWHERE ELSE.

Example: Irrotational Flow?

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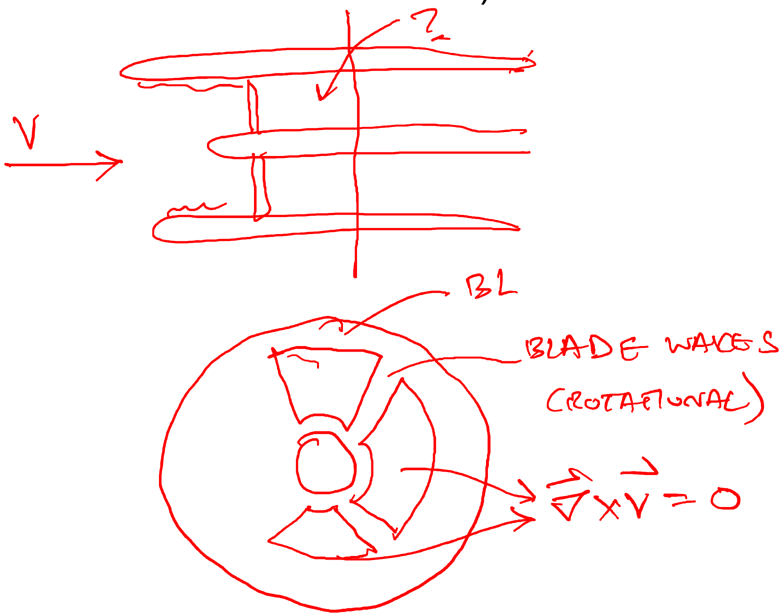
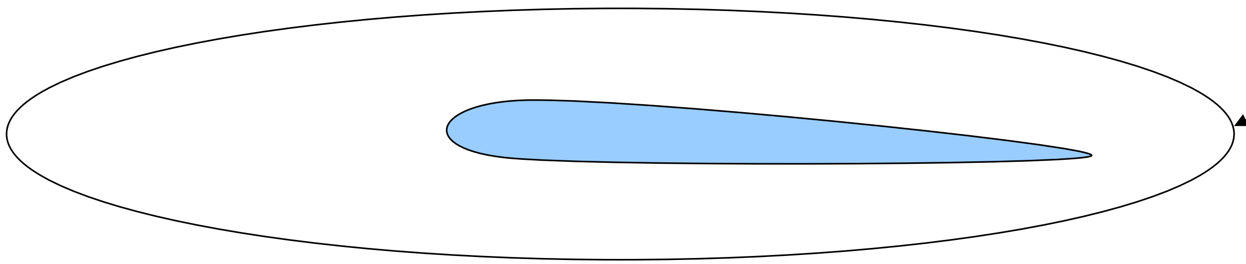


Example: The Starting Vortex

Consider a stationary arifoil in a stationary medium.

V = 0

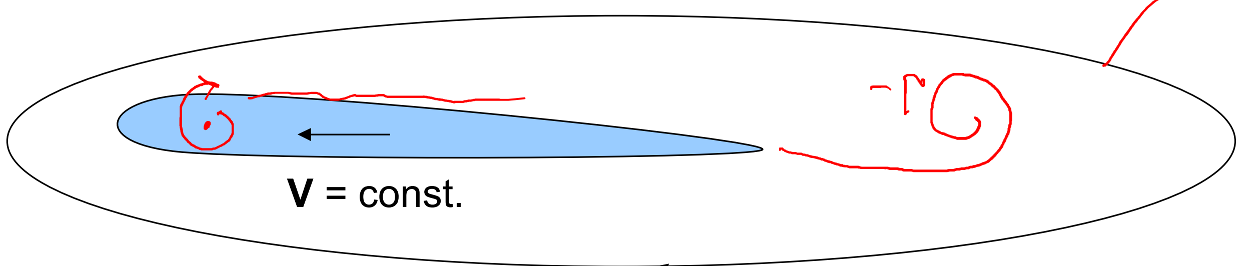


Fluid loop C

$$\Gamma_C = \oint_C \mathbf{V}.\mathbf{ds} = 0$$

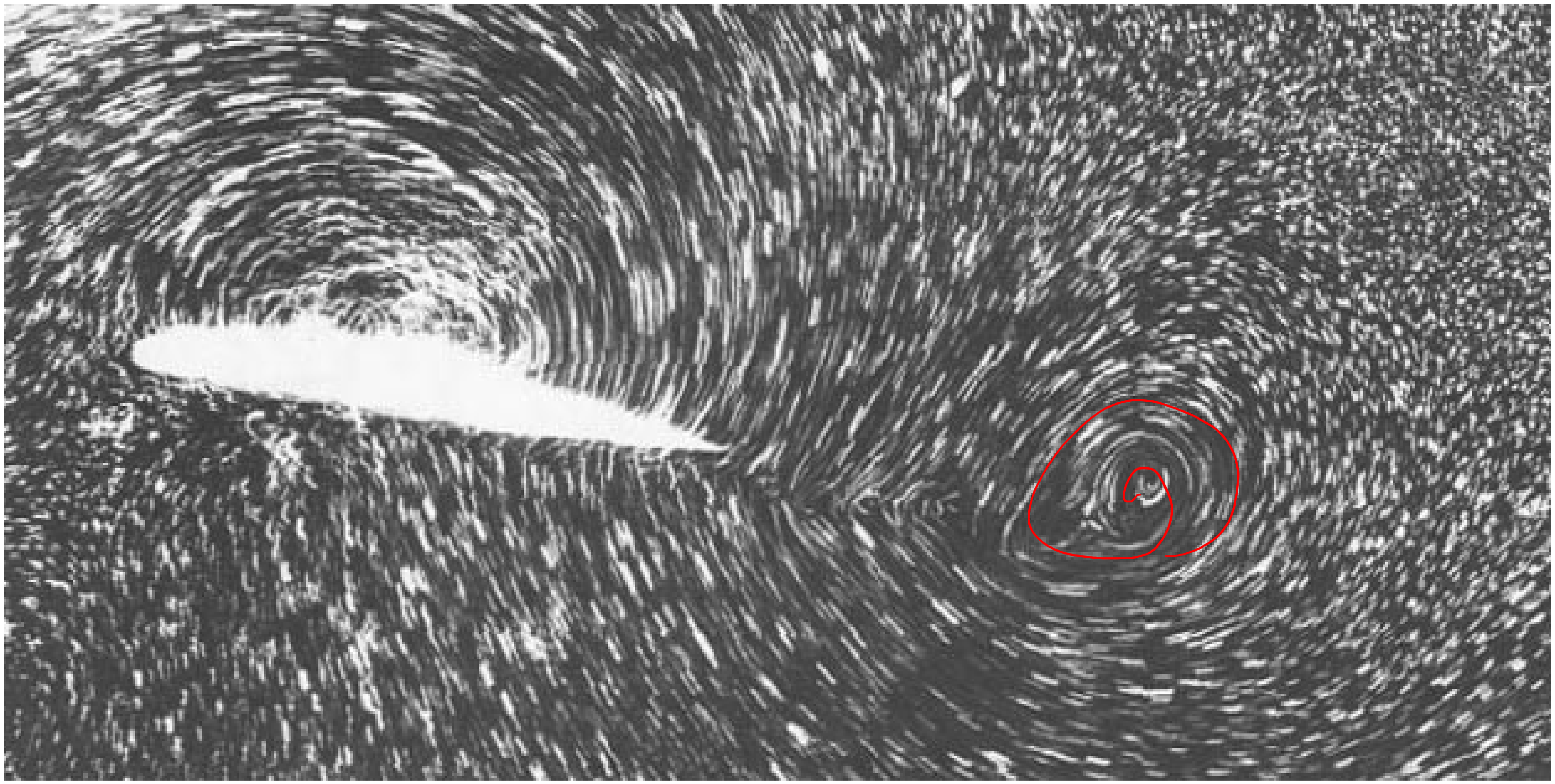
Now suppose the airfoil starts moving to left. (Using the fact that that a lifting airfoil in motion has a circulation about it).

$\Gamma_c = 0$



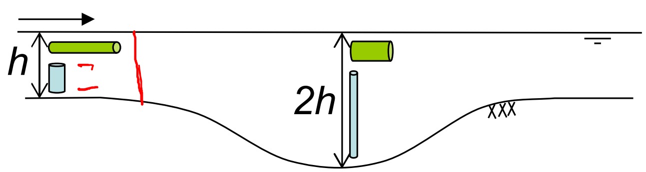
Fluid loop C

Starting Vortex



Pictures are from *“An Album of Fluid Motion”* by Van Dyke

Example: Flow over a depression in a river bed



A river flows over a depression locally doubling the depth. The river contains turbulence that is too weak to change the overall flow pattern. An turbulent eddy convects from upstream over the depression. Estimate its strength in the depression if the eddy is initially (a) vertical, and (b) horizontal.

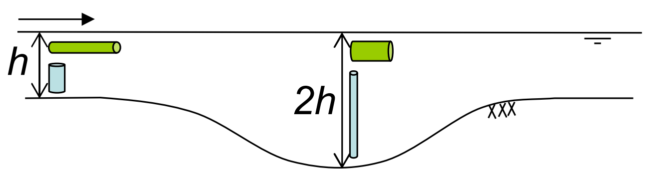
Solution: Need to assume that the viscous torques are not significant for the eddy so that the ~~fluid~~ vortex tube it occupies remains coincident with the same fluid tube.

(a) HEIGHT DOUBLES

X-SECTION HALVES (CONS. OF MASS)

VERTICAL VORTICITY WILL DOUBLE (HELMHOLTZ THEOREMS Part II)

Example: Flow over a depression in a river bed



A river flows over a depression locally doubling the depth. The river contains turbulence that is too weak to change the overall flow pattern. An turbulent eddy convects from upstream over the depression. Estimate its strength in the depression if the eddy is initially (a) vertical, and (b) horizontal.

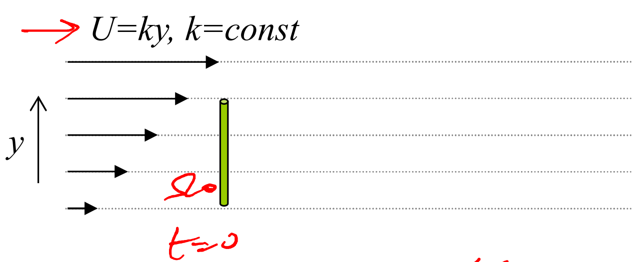
Solution: Need to assume that the viscous torques are not significant for the eddy so that the ~~fluid~~ vortex tube it occupies remains coincident with the same fluid tube.

 b) X-SECTION DOUBLES

VORTICITY HALVES

The River Avon nearing flood stage, Salisbury, Wiltshire, UK

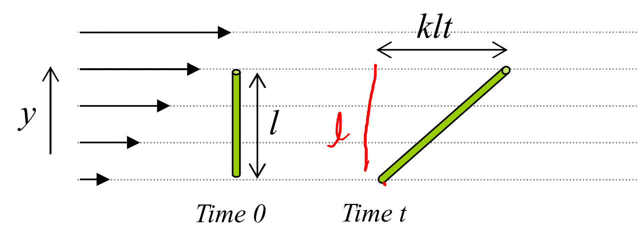
Example: Evolution of turbulence in a shear flow



Turbulence is convected and distorted in a shear flow, as shown. Consider an eddy that at time t=0 is vertically aligned and has a vorticity Ω_o. Estimate the vorticity magnitude and angle of an eddy, as functions of time.

Solution: Need to assume that the viscous torques are not significant for the eddy so that the vortex tube it occupies remains coincident with the same fluid tube. Also need to assume that the eddy is too weak to influence the flow that is convecting it.

Consider a segment of the eddy of length l. In time t the top of the eddy will convect further than the bottom by an amount equal to the difference in the velocity (kl) times the time.



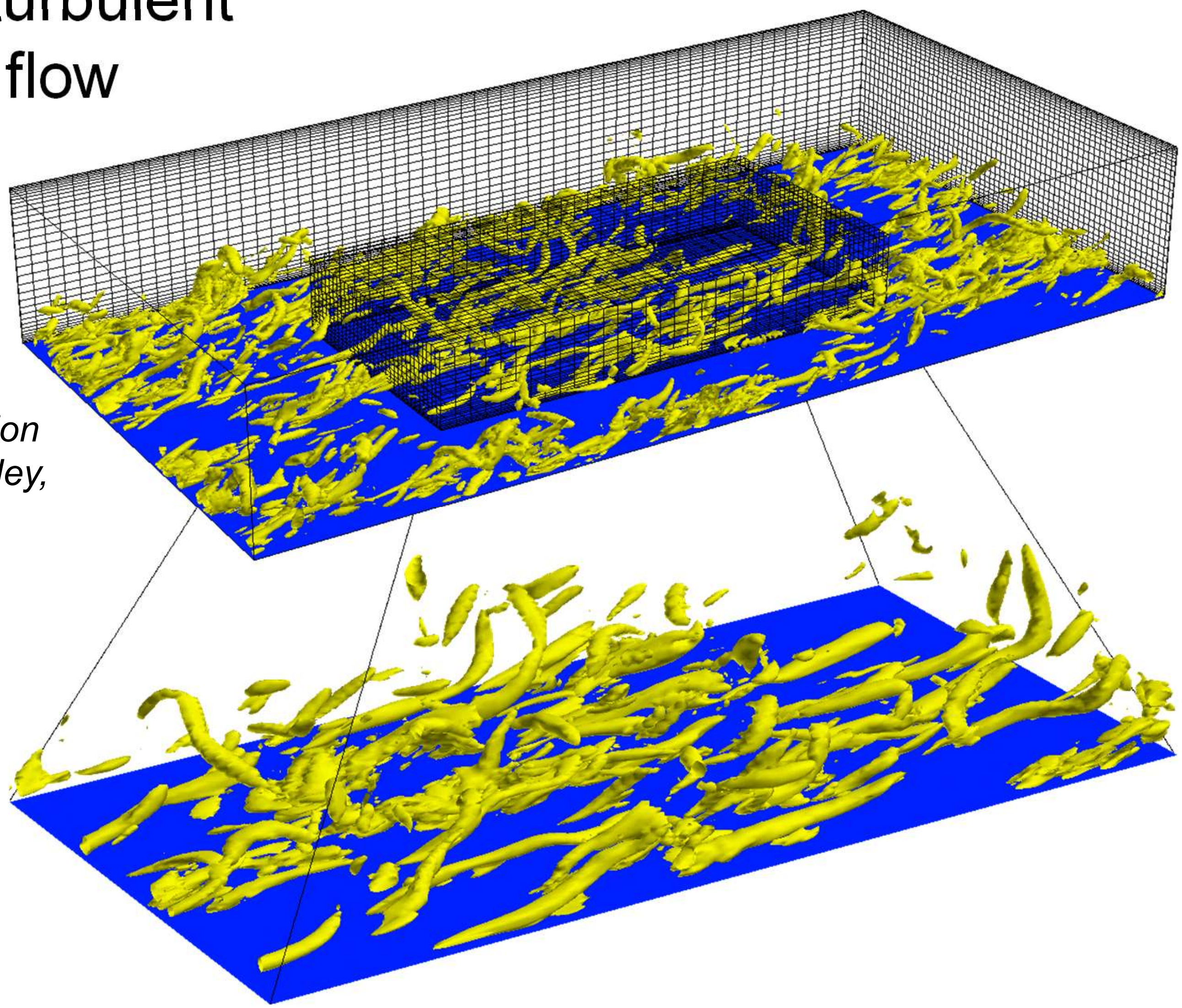
The angle of the fluid tube containing the eddy will thus be $\arctan(klt/l) = \arctan(kt)$

The fluid tube also grows longer by the factor $\sqrt{(klt)^2 + l^2}\,/\,l = \sqrt{(kt)^2 + 1}$

The cross sectional area of the tube thus reduces by this factor, and therefore the vorticity increases by this factor, i.e.

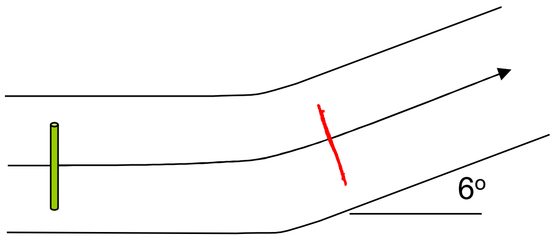
$$\Omega = \Omega_o\sqrt{(kt)^2 + 1}$$

Eddies in a turbulent channel flow



DNS Simulation Thomas Bewley, Edward Hammond & Parviz Moin Stanford University

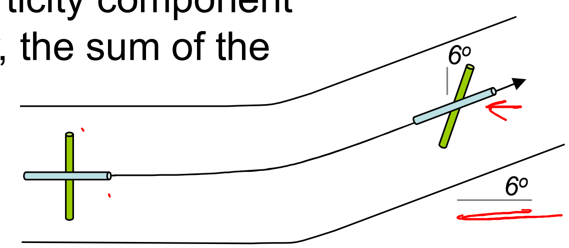
Example: Flow around a corner in a channel



Air flows through a duct with a 6° corner. An initially vertical eddy is introduced into the otherwise uniform flow upstream and convects around the corner. Estimate its orientation downstream.

Solution: Need to assume that the viscous torques are not significant for the eddy so that the ~~fluid~~ vortex tube it occupies remains coincident with the same fluid tube, and so that no vorticity is generated in the turn. Also need to assume that the eddy is too weak to influence the flow that is convecting it.

Consider two initially perpendicular fluid lines, one in the streamwise direction and one vertical, coincident with the eddy. Since there is no vorticity component perpendicular to the page, and no torques to generate any, the sum of the rotation rates of these two fluid lines must remain zero.



The streamwise fluid line will follow the streamline and thus rotate counterclockwise by 6°. The vertical fluid line and thus the eddy must therefore rotate clockwise by 6°, as shown.

Turbulent flow in a pipe through a 180° bend.

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