Intermediate Fluid Mechanics

Lecture 20: Dimensional Analysis

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Chapter Overview

Lecture Objectives

- In this lecture we continue to investigate the concept of **Flow Similarity**.
- Specifically, we consider the case of **incomplete similarity**, which arises when more than one non-dimensional parameter significantly affects the dynamics of the flow.

Chapter Overview

Incomplete Similarity

From the last example in the previous lecture we realized that:

- It was not possible to achieve dynamic similarity by testing a $1:5$ scale model in a wind tunnel operating at standard atmospheric conditions.
- The reason being that one could not simultaneously match both the Reynolds number and the Mach number.

There are a few options one can try, ...

Incomplete Similarity (continued ...)

1- Change the fluid properties (by either using a different fluid or a different temperature) will affect the fluid viscosity and thereby the Reynolds number.

$$\frac{U_m\,L_m}{\nu_m} = \frac{U_p\,L_p}{\nu_p} \tag{1}$$

$$U_m = U_p\left(\frac{L_p}{L_m}\right)\left(\frac{\nu_m}{\nu_p}\right) \tag{2}$$

- In the previous example, $L_p/L_m = 5$ and $U_p = 30m/s$ are given.
- The fluid associated with the prototype is air at atmospheric conditions, so $\nu_p = 1.5 \times 10^{-5} m^2/s$.
- If we also run the model studies in air at atmospheric conditions then $\nu_m = \nu_p$, and U_m turns out to be too high to maintain similarity in the Mach number.

What shall one do?

Incomplete Similarity (continued ...)

$\Longrightarrow$ Therefore, to reduce U_m, one needs that $\nu_m < \nu_p$.

If we choose water at a temperature of $20^\circ C$ for the fluid in the model system, $\nu_m = 0.9 \times 10^{-6} m^2/s$.

In that case,

$$U_m = U_p \left(\frac{L_p}{L_m} \right) \left(\frac{\nu_m}{\nu_p} \right) = 9 m/s. \tag{3}$$

This is quite fast for a water channel but it is possible. Also, one can reduce the viscosity by (i) cooling the fluid, for the case of gases, or (ii) heating the fluid, for the case of liquids.

2- In the event that the fluid properties cannot be changed,

$\Longrightarrow$ One is constrained to operate at lower Reynolds number and extrapolate the measured drag coefficient from the model studies.

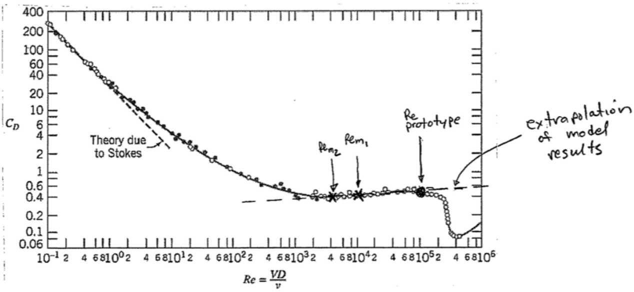

Figure: Drag coefficient for a sphere as a function of the Reynolds number

If the parameter of interest varies significantly with the parameter that is trying to be matched, then extrapolation of the model results will yield poor predictions of the behavior of the actual flow.

Incomplete Similarity (continued ...)

3- Geometric distortion may be used in some cases to relax the criterion of geometric similarity.

Let's consider the case one is interested in modeling dams, river systems, or other large-scale problems.

$\longrightarrow$ One may not be able to use a single scale factor to scale down all lengths.

Incomplete Similarity (continued ...)

Example 1:

In a river, since the vertical length (depth of the river) is so much smaller than the horizontal length, by the time we scale down the horizontal length to a managable size for conducting model experiments, the vertical length has become so small that viscous effects will dominate in the model system (unlike the situation in the actual dynamics).

$\Longrightarrow$ Different scale factors are typically used for the horizontal and vertical directions.

Incomplete Similarity (continued ...)

Example 2:

Flow over an object where the actual flow is incompressible, but the model flow is subsonic compressible (example previous lecture).

$\longrightarrow$ In this case, to match the Re, the speed in the model experiment must be so high that compressibility effects start to become important in the flow dynamics of the model.

$\Longrightarrow$ It can be shown theoretically that for some cases (such as airfoils), using a slightly distorted geometry in the subsonic, compressible flow will yield equivalent dynamics to that of the incompressible flow.

Incomplete Similarity (continued ...)

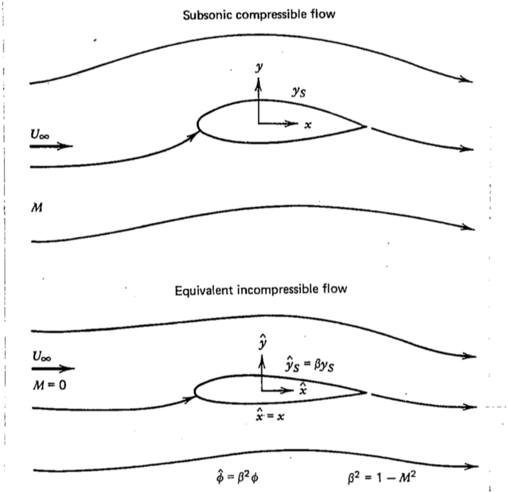

Figure: Flow dynamics around two different baldes.

Incomplete Similarity (continued ...)

4- In some cases, dynamic similarity cannot be achieved because two or more non-dimensional numbers cannot be matched simultaneously.

$\Longrightarrow$ We try to utilize *theoretical corrections* to the results from the model experiments.

• Consider for example the example of drag on a ship hull prototype, where $L_p = 100m$ and $U_p = 20m/s$.

$\longrightarrow$ We plan to build a scaled down model of the ship using a geometric scale 1:50 and test the model in a large towing tank to determine the drag force expected on the full-scale prototype.

Incomplete Similarity (continued ...)

There are two important sources of drag:

- Wave drag
- Skin-friction drag due to viscous effects, *i.e.*, shear stress acting over the surface of the hull.

Wave drag effects scale through the Froude number, while viscous effects scale through the Reynolds number.

$\Longrightarrow$ Therefore, we need to match both, Fr and Re in order to achieve dynamic similarity!

Incomplete Similarity (continued ...)

$$\text{viscous:} \qquad Re_m = Re_p \Rightarrow \frac{U_m L_m}{\nu_m} = \frac{U_p L_p}{\nu_p} \Rightarrow \frac{U_m}{U_p} = \frac{L_p}{L_m} = 50 \tag{4}$$

$$\text{wave:} \qquad Fr_m = Fr_p \Rightarrow \frac{U_m^2}{gL_m} = \frac{U_p^2}{gL_p} \Rightarrow \frac{U_m}{U_p} = \left(\frac{L_m}{L_p}\right)^{1/2} = 0.14. \tag{5}$$

$\Longrightarrow$ But,.... What's happening here?

- If we match Re, then we must operate the water channel at $U_m = 50U_p$
- If we match Fr, we must operate the water channel at $U_m = 0.14U_p$.

Question: What are we supposed to do? What if we change the fluid in the model?

Incomplete Similarity (continued ...)

Answer:

The Froude number is independent of the fluid, so we have $U_m/U_p = 0.14$.

Substituting this into the requirement that the Reynolds numbers match gives,

$$Re_m = Re_p \Rightarrow \frac{U_m L_m}{\nu_m} = \frac{U_p L_p}{\nu_p} \Rightarrow \frac{\nu_m}{\nu_p} = \left(\frac{U_m}{U_p}\right)\left(\frac{L_m}{L_p}\right) = (0.14)\left(\frac{1}{50}\right) = 2.8\times 10^{-3}. \tag{6}$$

$\Longrightarrow$ To match both *Fr* and *Re*, we need to find a fluid that has a viscosity three orders of magnitude (or 1000 times) lower than that of water.

Note: There is no practical solution to this because it is very expensive to achieve such small viscosities (*e.g.* super-cooled helium).

Incomplete Similarity (continued ...)

As an alternative:

- perform a model test in a tow tank with U_m determined based on matching Fr and not matching Re.
- measure the total drag force on the model, keeping in mind that

$$\underbrace{F_{D_T}}_{\substack{\text{total drag}\\\text{force}}} = \underbrace{F_{D_v}}_{\substack{\text{drag due to}\\\text{viscous effects}}} + \underbrace{F_{D_w}}_{\substack{\text{wave induced}\\\text{drag}}} \tag{7}$$

- estimate the viscous drag analytically using boundary layer theory or Fluent.
- obtain $F_{D_w} = F_{D_T} - F_{D_v}$ and calculate the wave drag coefficient from the model experiment, *i.e.*

$$C_{D_W}|_m = \frac{F_{D_W}|_m}{\rho_m L_m^3 g} \tag{8}$$

- Assume that $C_{D_W}|_m = C_{D_W}|_p$ since we properly matched the Froude number which governs the dynamics of this type of drag.

Chapter Overview

Dimensional Analysis using Buckingham's Pi Theorem

- To this point, we have approached the process of dimensional analysis by looking at the non-dimensional parameters that arise from non-dimensionalizing the governing differential equations.
- However, for many complicated flows, the precise form of the differential equations may not even be known.
- In this case, we seek to perform the dimensional analysis by considering the relevant dimensional variables and parameters in the flow.

Dimensional Homogeneity

The underlying principle in dimensional analysis is that of dimensional homogeneity,

$\Longrightarrow$ All terms in an equation must have the same dimension (*i.e.* units).

Dimensions: we define fundamental units as,

$$\text{mass} \equiv M \tag{9}$$
$$\text{length} \equiv L \tag{10}$$
$$\text{time} \equiv T \tag{11}$$
$$\text{temperature} \equiv \theta \tag{12}$$
$$\tag{13}$$

Dimensional Homogeneity (continued ...)

Let η be any variable or parameter, then the dimensions of η may be represented as

$$\text{Dimensions of } \eta : [\eta] = M^a L^b T^c \tag{14}$$

The corresponding example for velocity,

$$[u] = M^0 L^1 T^{-1} = L/T \tag{15}$$

Dimensional Homogeneity (continued ...)

In Dimensional Analysis, one needs to identify a quantity of interest in the fluid flow, *e.g.* pressure drop, drag force, power output, etc.

We then need to identify all of the variables and parameters that affect the quantity in which we are interested.

For example, pressure drop in a pipeline (Δp),

$$\Delta p = \mathcal{G}(d,\, l,\, e,\, U,\, \rho,\, \mu), \tag{16}$$

where d is the pipe diameter, l is the length of the pipe, e is the roughness height, U is the speed of the flow, ρ is the fluid density, μ is the fluid viscosity.

Dimensional Homogeneity (continued ...)

Note: It does take some experience to known which variables and parameters are important for a given scenario. This is one of the cost trade-offs, *i.e.* simplification in mathematics for physical intuition/experience.

One typically rearranges the functional form so that the right hand side is zero,

$$\mathcal{G}(\Delta p,\, d,\, l,\, e,\, U,\, \rho,\, \mu) = 0 \tag{17}$$

Buckingham-Pi Theorem (Bukingham, 1914)

Let $q_1,\, q_2,\, \cdots,\, q_n$ be n variables involved in a particular problem, so that a functional relationship exists of the form

$$\mathcal{G}(q_1,\, q_2,\, \cdots,\, q_n) = 0. \tag{18}$$

Buckingham's theorem states that the n variables can always be combined to form exactly $(n - r)$ independent non-dimensional variables, where r is the rank of the dimensional matrix of the problem.

Each non-dimensional variable or parameter is called a π-group (note the symbol π is used because the non-dimensional parameter can be written as a product of the variables $q_1,\, q_2,\, \cdots,\, q_n$ raised to some power).
Thus the previous equation can be rewritten as,

$$\phi(\pi_1,\, \pi_2, \cdots,\, \pi_{n-r}) = 0. \tag{19}$$

Buckingham-Pi Theorem: Example

The application of Buckingham's Pi Theorem is demonstrated by considering the previous example:

$$\mathcal{G}(\Delta p,\, d,\, l,\, e,\, U,\, \rho,\, \mu) = 0, \tag{20}$$

where $n = 7$, and $r = 3$.

$\Longrightarrow n - r = 4$ and we expect four non-dimensional π-groups.

In order to form the π groups, one first needs to select $r = 3$ '*repeating variables*', which will be represented in each of the π-groups.

Buckingham-Pi Theorem: Example

Guidelines in choosing '*repeating variables*':

- the repeated variables must have different dimensions.
- the set of repeating variables must contain all the fundamental dimensions: M, L, and T (and θ, if one of the variables has units of temperature.)
- Typically in fluids problems one choses a repeating variable in each of the categories: dynamic variable (*e.g.* characteristic velocity), geometric variable (*e.g.* characteristic length), and a fluid property.

Buckingham-Pi Theorem: Example

- For a pipeline problem, one normally chooses U_1, d_1, and ρ as '*repeating variables*'.
- A different choice of repeating variables will result in different set of non-dimensional π-groups.
- There is no correct or incorrect choice of '*repeating variables*'; although certain π-groups have already been identified and named accordingly.
- So, convention does guide our choice of '*repeating variables*'to some extent.

Buckingham-Pi Theorem: Example

Forming the π-groups

Each non-dimensional π-group is formed by combining the '*repeating variables*' with each of the remaining variables. The order does not matter.

Let's consider the case where we combine Δp with the repeating variables, U, d, ρ,

$$\pi_1 = U^a,\, d^b,\, \rho^c,\, \Delta p \tag{21}$$

One first must write each variable in terms of fundamental units,

$$M^0\, L^0\, T^0 = (L\, T^{-1})^a\, (L)^b\, (ML^{-3})^c\, (ML^{-1}\, T^{-2}) \tag{22}$$

$$M^0\, L^0\, T^0 = M^{c+1}\, L^{a+b-3c-1}\, T^{-a-2} \tag{23}$$

Buckingham-Pi Theorem: Example

In order for this equality to hold, the powers on M, L, and T must be identical,

$$M: \qquad 0 = c + 1 \qquad \Rightarrow c = -1 \tag{24}$$

$$T: \qquad 0 = -a - 2 \qquad \Rightarrow a = -2 \tag{25}$$

$$L: \qquad 0 = a + b - 3c - 1 \qquad \Rightarrow b = 0 \tag{26}$$

Therefore,

$$\pi_1 = U^{-2} d^0 \rho^{-1} \Delta p = \frac{\Delta p}{\rho U^2}. \tag{27}$$

Buckingham-Pi Theorem: Example

Similarly,

$$\pi_2 = U^a\, d^b\, \rho^c\, l = \frac{l}{d} \tag{28}$$

$$\pi_3 = U^a\, d^b\, \rho^c\, e = \frac{e}{d} \tag{29}$$

$$\pi_4 = U^a\, d^b\, \rho^c\, \mu = \frac{\mu}{\rho U d}. \tag{30}$$

Therefore, Buckingham's Pi theorem tells us that,

$$\frac{\Delta p}{\rho U^2} = \phi\left(\frac{l}{d}, \frac{e}{d}, \frac{\mu}{\rho U d}\right). \tag{31}$$

Question: How is this new information useful, if one still does not know the functional relationship for ϕ?