# Intermediate Fluid Mechanics

## Lecture 8: Streamfunction

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# Chapter Overview

# Lecture Objectives

In this lecture we will:

- Briefly discuss elements of relevance related to the dimensionality of fluid flows.
- Introduce the concept of Streamfunction.

# Chapter Overview

# Spatial Dimensions of Flows

- One-Dimensional flow: This is a type of flow in which all flow characteristics vary only in one direction of the space.
- Two-Dimensional flow: This is a type of flow in which the flow characteristics vary in two cartesian directions only.
- Three-Dimensional flow: This is a flow in which the characteristics vary in three-cartesian directions.

# Two Dimensional Flows

An example of a two dimensional flow, is the flow through a pipe or a large-aspect-ratio rectangular duct.

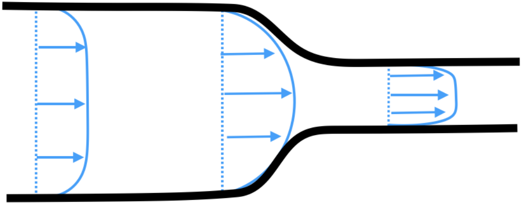

Figure: *(a) Example of a two-dimensional flow. (b) Example of a one-dimensional flow; in this case the flow is uniform in the vertical direction.*

In this case, the velocity is a function of both, the $x$- and $y$-directions.

# Quasi One-Dimensional Flows

Sometimes, it is convenient to make a simplification, called a one-dimensional approximation wherein one neglects the variation of the flow in one or more directions.

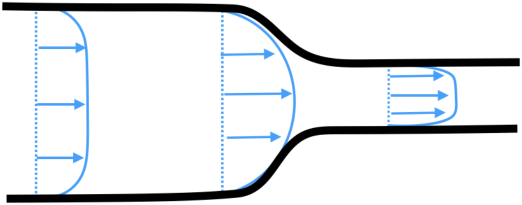

Figure: *(a) Example of a two-dimensional flow. (b) Example of a one-dimensional flow; in this case the flow is uniform in the vertical direction.*

InSubplot b, the $y$-dependence is being neglected, and it is assumed that at any $x$-location, the flow is well approximated by a uniform velocity having an equal magnitude to the $y$-averaged value.

# More Two Dimensional Flows

Other common examples of two-dimensional flow are the cases of a flow over an object with a constant cross-section and infinite length.

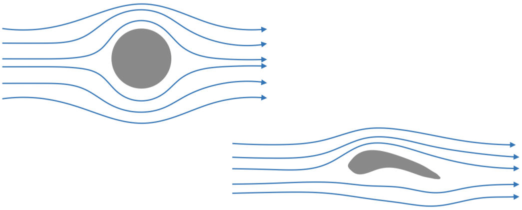

Figure: *(a) Example of a two-dimensional flow. (b) Example of a one-dimensional flow; in this case the flow is uniform in the vertical direction.*

Flow past a circular cylinder is the canonical example. Also, flow over airfoils with constant cross-sectional shape are generally assumed to be well-approximated as two-dimensional flows.

# Chapter Overview

# Streamfunction

In a two-dimensional plane flow, a streamfunction $\psi(x, y, t)$ may be defined such that

$$u \equiv \frac{\partial \psi}{\partial y} \qquad \text{and} \qquad v \equiv -\frac{\partial \psi}{\partial x} \tag{1}$$

$\Longrightarrow$ A velocity field defined this way is automatically incompressible.

This can be quickly verified using the definition of incompressible fluid,

$$\frac{\partial u_i}{\partial x_i} = 0 = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \frac{\partial^2 \psi}{\partial x\, \partial y} - \frac{\partial^2 \psi}{\partial y\, \partial x} = 0. \tag{2}$$

# Streamfunction (Continued ...)

Further, revisiting the concept of stremalines, which were defined as,

$$\frac{dx}{u} = \frac{dy}{v} \longrightarrow v\,dx - u\,dy = 0 \tag{3}$$

upon substitution of $u$ and $v$ by the stramfunction,

$$\frac{\partial \psi}{\partial x}\,dx + \frac{\partial \psi}{\partial y}\,dy = 0, \tag{4}$$

- $\Longrightarrow d\psi = 0$ along streamlines.
- $\Longrightarrow$ This is equivalent to say that the instantaneous streamlines are given by curves $\psi = cte$, whereby a different value of $\psi$ corresponds to a different streamline.
- The best way to draw streamlines is via a contour plot, by drawing the iso-contours of the streamfunction.

# Volume Flow Rate

Next, we will illustrate that the volume flow rate,

$$Q = \frac{\text{Volume Fluid}}{\text{Surface} \times \text{Unit time}} = \frac{\forall}{S\, t} \tag{5}$$

between a pair of streamlines is numerically equal to the difference in their $\psi$ values.

The volume flow rate $\dot{\forall}$ across a differential line element $d\vec{x}$ is

$$\begin{aligned}\dot{\forall} = \frac{\forall}{l\, t} &= v\, dx - u\, dy \\ \dot{\forall} &= \left(-\frac{\partial \psi}{\partial x}\right) dx - \left(\frac{\partial \psi}{\partial y}\right) dy \\ \dot{\forall} &= -d\psi \end{aligned} \tag{6}$$

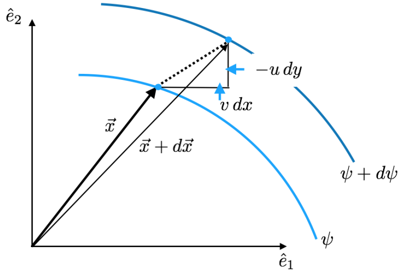


Figure: Graphical representaiton of the flow rate between two streamfucntions.

# Example: Streamlines in standing gravity wave

Standing gravity waves can occur on a water-air interface when the water is being driven below the interface by a piston or paddle and being reflected back due to the position of a solid wall.

In this situation (and assuming a two-dimensional flow), the velocity components are given by

$$u_1 = -v\, sin(wt)\, sin(kx_1)\, cosh[k(x_2 + h)] \tag{7}$$

$$u_2 = v\, sin(wt)\, cos(kx_1)\, sinh[k(x_2 + h)] \tag{8}$$

$$u_3 = 0 \tag{9}$$

Here $h$ is the depth of the water, $v$ is a velocity, w is the frequency and $k$ is the wavenumber.

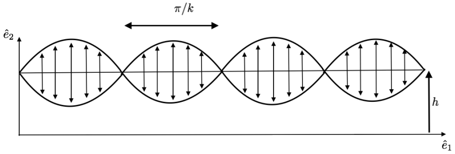


Figure: *Example of a standing wave.*

# Example: Streamlines in standing gravity wave

The streamlines will be represented by drawing the isocontours of $\psi$.

In this case, determining $\psi$ is a two stage process,

1. From the definition of $\psi$,

$$u_1 = \frac{\partial \psi}{\partial x_2} \tag{10}$$

and multiplying both sides by $dx_2$ and integrating gives,

$$\int u_1 \, dx_2 + f(x_1) = \psi(x_1, x_2). \tag{11}$$

Note that the integrand is an indefinite integral and so one needs to include an arbitrary function of $x_1$, *i.e.* $f(x_1)$, since $\psi$ is a function of both $x_1$ and $x_2$ (but the integral is only over $x_2$).

# Example: Streamlines in standing gravity wave

Substituting the functionality of the streamwise velocity component,

$$u_1 = -v\, sin(wt)\, sin(kx_1)\, cosh[k(x_2 + h)] \tag{12}$$

in the previous streamfunction expression,

$$\int u_1\, dx_2 + f(x_1) = \psi(x_1, x_2). \tag{13}$$

leads to,

$$\int -v\, sin(wt)\, sin(kx_1)\, cosh[k(x_2 + h)] dx_2 + f(x_1) = \psi \tag{14}$$

$$\frac{-v}{k}\, sin(wt)\, sin(kx_1)\, sinh[k(x_2 + h)] + f(x_1) = \psi. \tag{15}$$

At this point one needs to determine the function $f(x_1)$, hence another equation is required.

# Example: Streamlines in standing gravity wave

2. Using the other part of the definition for the streamfunction and substituting from the equations for $u_2$ and $\psi$,

$$u_2 = -\frac{\partial\psi}{\partial x_1} \tag{16}$$

$$v\, sin(wt)\, cos(kx_1)\, sinh[k(x_2 + h)] = -\Big[\frac{-v}{k}\, sin(wt)\, k\, cos(kx_1)\, sinh[k(x_2 + h)]\Big] \\ + \frac{df(x_1)}{dx_1}. \tag{17}$$

$$\tag{18}$$

It is possible to solve for $df/dx_1$, to find that

$$\frac{df}{dx_1} = 0 \longrightarrow f = C, \tag{19}$$

where $C$ is a constant.

# Example: Streamlines in standing gravity wave

Upon substitution of this value for $f$ back into equation for $\psi$ leads to,

$$\boxed{\psi = \frac{-v}{k}\, sin(wt)\, sin(kx_1)\, sinh[k(x_2 + h)] + C}. \tag{20}$$

- Since the actual magnitude of $\psi$ is unimportant (recall that it is the difference in $\psi$ that yields the volume flow rate), one is free to set the value of $\psi$ at one point in the flow.
- In this case, it is convenient to choose $\psi(x_1 = 0, x_2 = 0) = 0. \implies$ Hence, the value of $C$ that satisfies this criterion is $C = 0$.

Therefore, one has

$$\boxed{\psi = \frac{-v}{k}\, sin(wt)\, sin(kx_1)\, sinh[k(x_2 + h)]}. \tag{21}$$

# Example: Streamlines in standing gravity wave

Graphically,

$$\boxed{\psi = \frac{-v}{k}\, sin(wt)\, sin(kx_1)\, sinh[k(x_2 + h)]}. \tag{22}$$

results in:

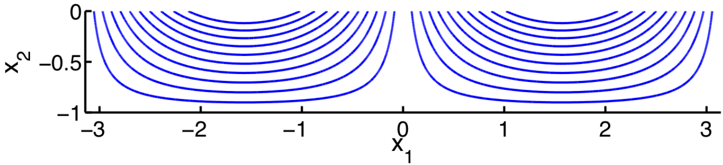


Figure: *Iso-contours of $\psi$ over the domain $-2\pi \leq x_1 \leq 2\pi$ and $-1 \leq y \leq 0$ for the parameter set $v = 1$, $k = 1$, $w = 1$, and $h = 1$. The resultant streamline pattern is shown below for a time $t = \pi/2$, whereby the amplitude of the wave will be greatest. Note, since the flow is unsteady, the streamline pattern changes at each instant in time. It can be observed that the streamlines form closed loops.*