Aerospace Propulsion

Lecture 7

Compressible Flows: Part I

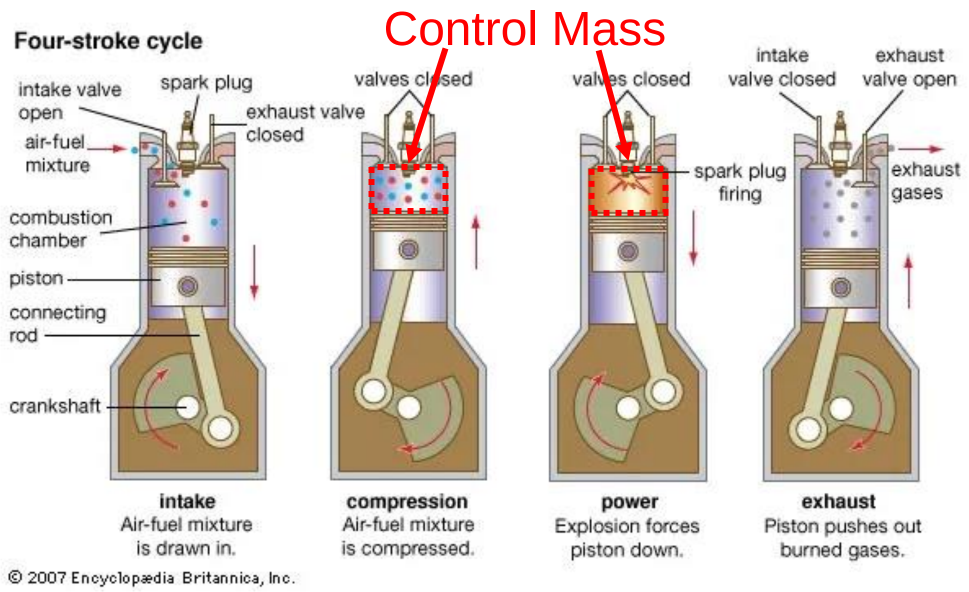
Compressible Flows: Part I

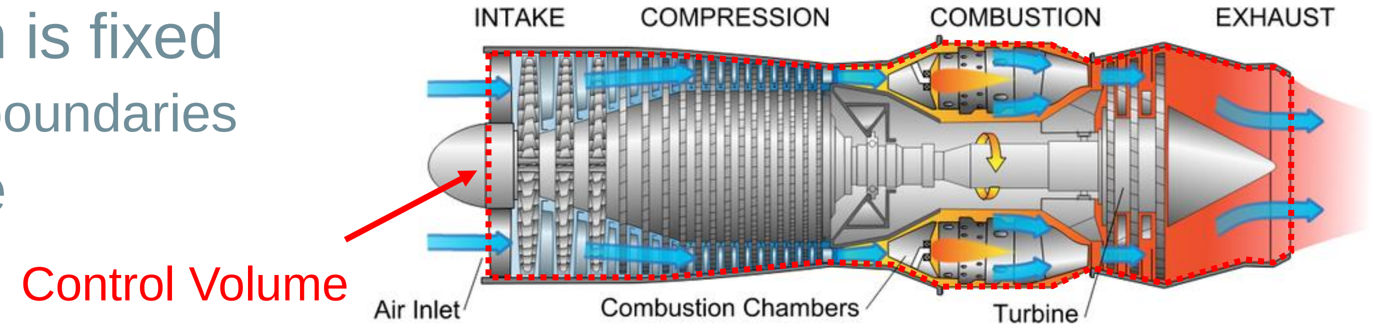
- Fluid Conservation Principles
- Speed of Sound
- Isentropic Flow
- Stagnation Properties

Fluid Conservation Principles

- Control Mass (Closed system)
  - Mass within system is fixed
    - Mass does not cross boundaries
  - Example: Gasoline Engine

- Control Volume (Open system)
  - Volume within system is fixed
    - Mass freely crosses boundaries
  - Example: Gas turbine





Fluid Conservation Principles

- Reynold's transport theorem
  - $$\underbrace{\frac{DN}{Dt}}_{\text{Total rate of change of } N} = \underbrace{\frac{\partial}{\partial t}\iiint_{CV} \eta\rho dV}_{\text{Change of N in control volume}} + \underbrace{\oiint_{CS} \eta\rho\left(\vec{\boldsymbol{V}}\cdot\hat{n}\right) dA}_{\text{Flux of N across boundaries}}$$

V is volume
$\vec{V}$ is velocity

for now...

* $\hat{n}$ points out

  - N = extensive property
  - $\eta = N/m$

Fluid Conservation Principles

- Mass
  - Generally: $\frac{Dm}{Dt} = 0$
  - $N = m$
  - $\eta = \frac{m}{m} = 1$
  - $0 = \frac{\partial}{\partial t} \oiiint_{CV} \rho dV + \oiint_{CS} \rho \left(\vec{\boldsymbol{V}} \cdot \hat{\boldsymbol{n}} \right) dA$
  - $0 = \oiint_{CS} \rho \left(\vec{\boldsymbol{V}} \cdot \hat{\boldsymbol{n}} \right) dA$ (steady state)

Fluid Conservation Principles

- Momentum
  - Generally: $\frac{Dm\vec{V}}{Dt} = \vec{\boldsymbol{F}}_{\boldsymbol{net}}$

Newtons First Law

  - $N = m\vec{\boldsymbol{V}}$
  - $\eta = \vec{\boldsymbol{V}}$
  - $\vec{\boldsymbol{F}}_{\boldsymbol{net}} = \frac{\partial}{\partial t}\oiiint_{CV} \rho\vec{\boldsymbol{V}}dV + \oiint_{CS} \rho\vec{\boldsymbol{V}}\left(\vec{\boldsymbol{V}} \cdot \hat{\boldsymbol{n}}\right) dA$
  - $\vec{\boldsymbol{F}}_{\boldsymbol{net}} = \oiint_{CS} \rho\vec{\boldsymbol{V}}\left(\vec{\boldsymbol{V}} \cdot \hat{\boldsymbol{n}}\right) dA$ (steady state)

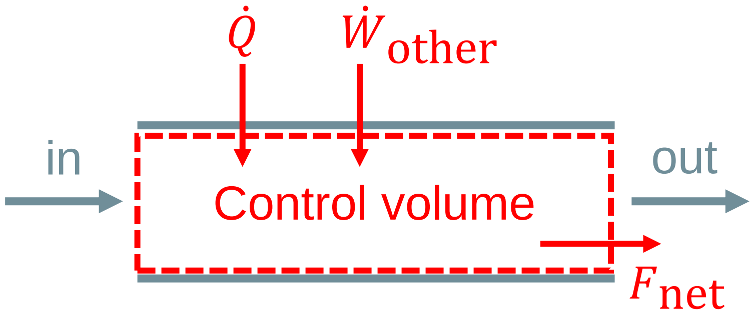
Fluid Conservation Principles

First Law of Thermo

- Energy
  - Generally: $\frac{DE}{Dt} = \dot{Q} - \dot{W} = \dot{Q} - \overbrace{\oiint_{CS} p\left(\vec{\boldsymbol{V}} \cdot \hat{\boldsymbol{n}}\right) dA}^{\text{Pressure work on surface}} - \overbrace{\dot{W}_{\text{other}}}^{\text{Shaft, viscous, etc.}}$
  - $N = E$
  - $\eta = e = u + \frac{1}{2}\vec{\boldsymbol{V}} \cdot \vec{\boldsymbol{V}}$
  - $\dot{Q} - \dot{W}_{\text{other}} = \frac{\partial}{\partial t}\oiiint_{CV} \rho\left(\underbrace{u + \frac{\vec{\boldsymbol{V}} \cdot \vec{\boldsymbol{V}}}{2}}_{e}\right) dV + \oiint_{CS} \rho\left(\underbrace{u + \frac{\vec{\boldsymbol{V}} \cdot \vec{\boldsymbol{V}}}{2}}_{e} + \boxed{\frac{p}{\rho}}\right)\left(\vec{\boldsymbol{V}} \cdot \hat{\boldsymbol{n}}\right) dA$
  - $\dot{Q} - \dot{W}_{\text{other}} = \oiint_{CS} \rho\left(u + \frac{\vec{\boldsymbol{V}} \cdot \vec{\boldsymbol{V}}}{2} + \frac{p}{\rho}\right)\left(\vec{\boldsymbol{V}} \cdot \hat{\boldsymbol{n}}\right) dA$ (steady state)

Fluid Conservation Principles

- Example: Steady pipe flow
  - Mass
    - $\dot{m}_{out} = \dot{m}_{in}$
  - Momentum
    - $F_{net} = (\dot{m}V)_{out} - (\dot{m}V)_{in}$
  - Energy
    - $\dot{Q} + \dot{W}_{other} = \left[\dot{m}\left(u + \frac{V^2}{2} + \frac{p}{\rho}\right)\right]_{out} - \left[\dot{m}\left(u + \frac{V^2}{2} + \frac{p}{\rho}\right)\right]_{in}$



Mass (Steady)

$0 = \oiint \rho(\vec{V}\cdot\hat{n})\,dA$

$-(\rho V A)_{in} + (\rho V A)_{out} = 0$

$-\dot{m}_{in} + \dot{m}_{out} = 0$

Momentum (Steady)

$\oiint \rho\vec{V}(\vec{V}\cdot\hat{n})\,dA = F_{net}$

$-(\rho V^2 A)_{in} + (\rho V^2 A)_{out} = F_{net}$

Energy (Steady)

$\dot{Q} + \dot{W}_{other} = \oiint \rho\left(u + \frac{V^2}{2} + \frac{p}{\rho}\right)(\vec{V}\cdot\hat{n})\,dA$

$\dot{Q} + \dot{W}_{other} = -\left[\rho\left(u + \frac{V^2}{2} + p/\rho\right)VA\right]_{in} + \left[\rho\left(u + \frac{V^2}{2} + \frac{p}{\rho}\right)VA\right]_{out}$

$\dot{Q} + \dot{W}_{other} = -\left[\dot{m}\left(u + \frac{V^2}{2} + p/\rho\right)\right]_{in} + \left[\dot{m}\left(u + \frac{V^2}{2} + p/\rho\right)\right]_{out}$

Speed of Sound

$$\gamma = \frac{c_p}{c_v}$$

- Sound waves are infinitesimal pressures waves
  - *Reversible and adiabatic (therefore also isentropic)
- Isentropic speed of sound a
  - $a \equiv \sqrt{\frac{\partial p}{\partial \rho}_s}$ } definition
- For an Ideal Gas (constant specific heats)
  - $a = \sqrt{\gamma \frac{p}{\rho}} = \sqrt{\gamma R T}$
- Mach Number: $M = V/a$

Isentropic
ideal gas
constant cp

$$\left(\frac{p}{p_0}\right) = \left(\frac{\rho}{\rho_0}\right)^\gamma$$

$$\Rightarrow \boxed{p = c\rho^\gamma}$$

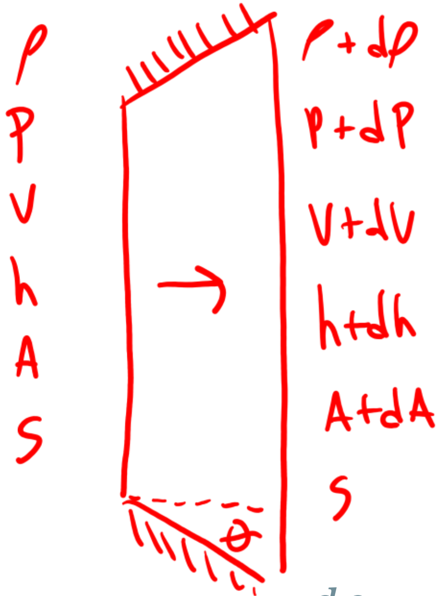
$$\left(\frac{\partial p}{\partial \rho}\right)_s = c\gamma\rho^{\gamma-1} = c\gamma\frac{\rho^\gamma}{\rho} = \frac{\gamma p}{\rho}$$

$$a = \sqrt{\left(\frac{\partial p}{\partial \rho}\right)_s} = \sqrt{\frac{\gamma p}{\rho}} = \sqrt{\gamma R T}$$

Ideal gas Law: $p = \rho R T$

Isentropic Flow

- "One-dimensional" flow, no friction, no heating
  - Mass



Fluid element

- $\frac{d\rho}{\rho} + \frac{dV}{V} + \frac{dA}{A} = 0$

Mass (Steady)

$$\oiint \rho(\vec{V}\cdot\hat{n})\,dA = 0$$

$$(\rho+d\rho)(V+dV)(A+dA) - \rho VA = 0$$

$$\boxed{\rho VA} + \rho V dA + \rho dV A + \underbrace{\rho dV dA}_{\text{H.O.T.}} + d\rho VA + d\rho V dA + d\rho dV A + d\rho dV dA \boxed{-\rho VA} = 0$$

$$\rho V dA + \rho dV A + d\rho VA = 0$$

chain rule

$$d(\rho VA) = 0$$

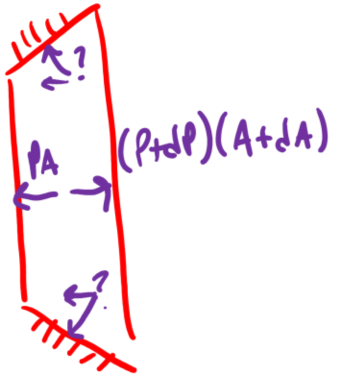
$$d\dot{m} = 0$$

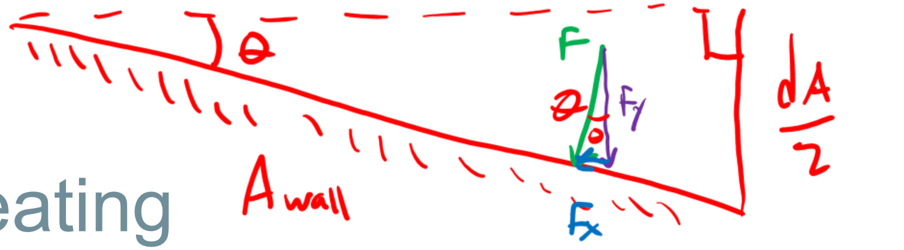
divide ρVA

$$\frac{dA}{A} + \frac{dV}{V} + \frac{d\rho}{\rho} = 0$$

Recall, $\frac{dA}{A} + \frac{dV}{V} + \frac{d\rho}{\rho} = 0$

Isentropic Flow





- "One-dimensional" flow, no friction, no heating
  - Momentum (Steady, x-direction)

$$\oiint \rho \vec{V}(\vec{V}\cdot\hat{n})\,dA = \Sigma F_x$$

$$(\rho+d\rho)(V+dV)(V+dV)(A+dA) - \rho V^2 A = \Sigma F_x$$

Expand, neglect HOT

$$\boxed{\rho V^2 A} + \rho V^2 dA + 2\rho V A\,dV + V^2 A\,d\rho \boxed{-\rho V^2 A} = \Sigma F_x$$

$$\rho V^2 A\left(\frac{dA}{A} + \frac{2\,dV}{V} + \frac{d\rho}{\rho}\right) = \Sigma F_x$$

$$\rho V^2 A\left(\frac{dV}{V}\right) = \Sigma F_x$$

  - $dp + \rho V dV = 0$

$$\rho V^{\cancel{2}} \cancel{A} \frac{dV}{\cancel{V}} = -\cancel{A}\,dP$$

Assume:

- θ is very small
- pressure varies linearly across the wall
- Average pressure on each wall $P + \frac{dP}{2}$

Trig:

$$\sin\theta = \frac{dA/2}{A_{wall}} \Rightarrow A_{wall} = \frac{dA}{2\sin\theta}$$

$$\sin\theta = \frac{F_x}{F} \Rightarrow F_x = F\sin\theta$$

$$\Sigma F_x = PA - (P+dP)(A+dA) + \underbrace{\underbrace{\underbrace{\left(P+\frac{dP}{2}\right)}_{\text{Pressure}}\underbrace{\left(\frac{dA}{2\sin\theta}\right)}_{A_{wall}}\underbrace{(2)}_{\text{2 walls}}}_{F}\sin\theta}_{F_x}$$

$$\Sigma F_x = \boxed{PA - PA} - A\,dP \boxed{-P\,dA} \underbrace{-\,dP\,dA}_{\text{HOT}} \boxed{+P\,dA} \underbrace{+\,dP\,dA/2}_{\text{HOT}}$$

$$\Sigma F_x = -A\,dP$$

Isentropic Flow

- "One-dimensional" flow, no friction, no heating
  - Energy (Steady)

$$\oiint \rho\left(e + P/\rho\right)(\vec{V}\cdot\hat{n})\,dA = 0$$

$$\oiint \rho e(\vec{V}\cdot\hat{n})\,dA + \oiint P(\vec{V}\cdot\hat{n})\,dA = 0$$

$$(\rho + d\rho)(e + de)(V + dV)(A + dA) - \rho e V A + (P + dP)(V + dV)(A + dA) = 0$$

Expand, neglect HOT, use continuity

$$de = -\frac{P}{\rho}\left[\frac{dP}{P} - \frac{d\rho}{\rho}\right] \quad (1)$$

For open system, prefer enthalpy

$$e = u + \tfrac{1}{2}V^2, \quad h \equiv u + P/\rho \quad \Big\} \quad e = h - P/\rho + \tfrac{1}{2}V^2$$

$$de = dh - \frac{dP}{\rho} + \frac{P\,d\rho}{\rho^2} + V\,dV \quad (2)$$

(1) = (2)

$$-\frac{P}{\rho}\left[\frac{dP}{P} - \frac{d\rho}{\rho}\right] = dh - \frac{dP}{\rho} + \frac{P\,d\rho}{\rho^2} + V\,dV$$

$$dh + V\,dV = 0$$

- $dh + VdV = 0 \Rightarrow d\left(\underbrace{\boxed{\frac{\gamma}{\gamma - 1}R}}_{c_p}T + \frac{1}{2}V^2\right) = 0$

Isentropic Flow

- “One-dimensional” flow, isentropic, **no work or heat**
  - $d\left(h + \frac{1}{2}V^2\right) = d\left(\frac{\gamma}{\gamma-1}RT + \frac{1}{2}V^2\right) = 0$
- First Law (open system)
  - $d\left(h + \frac{1}{2}V^2\right) = \delta q - \delta w$
  - Complicated accounting
    - If we add heat/work, does it change the kinetic energy or enthalpy?

  - We can avoid the distinction by creating a new property that accounts for enthalpy and kinetic energy

Stagnation Properties

$C_p = \frac{R\gamma}{\gamma - 1}$

$\gamma = \frac{C_p}{C_v}$ $R = C_p - C_v$

- Thermodynamic properties of a gas brought to rest isentropically and adiabatically
- Stagnation Enthalpy
  - $h_t = h + \frac{1}{2}V^2$
- Stagnation Temperature
  - $T_t = T\left(1 + \frac{\gamma - 1}{2}M^2\right)$

Constant Cp

$h_t = h + \frac{1}{2}V^2$

$h_t - h = \frac{1}{2}V^2$

$C_p(T_t - T) = \frac{1}{2}V^2$

$T_t = T + \frac{V^2}{2C_p}$

$T_t = T + \frac{V^2}{2}\frac{\gamma - 1}{R\gamma}$

$T_t = T\left(1 + \frac{V^2}{2}\frac{\gamma - 1}{\gamma}\frac{1}{RT}\right)$

$T_t = T\left(1 + \frac{V^2}{2}\frac{\gamma - 1}{a^2}\right)$

$T_t = T\left(1 + M^2\frac{\gamma - 1}{2}\right)$

ideal gas

$a^2 = \gamma R T$

Stagnation Properties

- Thermodynamic properties of a gas brought to rest isentropically and adiabatically
- Stagnation Pressure
  - $p_t = p\left(1 + \frac{\gamma - 1}{2} M^2\right)^{\frac{\gamma}{\gamma - 1}}$
- Stagnation Density
  - $\rho_t = \rho\left(1 + \frac{\gamma - 1}{2} M^2\right)^{\frac{1}{\gamma - 1}}$

isentropic flow equation

- What happens to these quantities in an isentropic process?

No work, No heat → all remain Constant

Stagnation Properties

- Recall we said for the First Law (open system)
  - $d\left(h + \frac{1}{2}V^2\right) = \delta q - \delta w$
- Using the stagnation concept, we instead write
  - $dh_t = \delta q - \delta w$
- We no longer need to worry which term the heat and work modify (worry about it later)