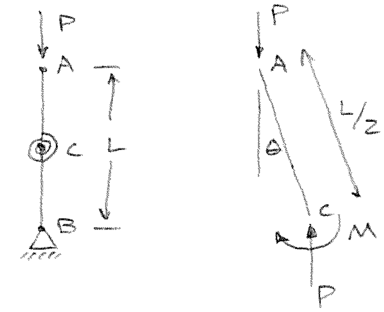
Buckling

- Consider design of column loaded axially
  - Might assume it's designed correctly if $\sigma = \frac{P}{A} < \sigma_{all}$ and $\delta = \frac{PL}{AE}$ is acceptable for application, but still might buckle
  - Buckling is manifestation of instability, a sudden change in configuration



- Consider a simplified model where column is made up a two rigid rods with torsional spring (constant K)
  - Couple produced by vertical loads $= P\frac{L}{2}\sin\theta$
  - Restoring moment by spring $= K2\theta$
  - Equilibrium when $P\frac{L}{2}\sin\theta = K2\theta$
    - In this case, $P = P_{cr}$ when system is in vertical configuration. Since θ is very small here, $\sin\theta \approx \theta \Rightarrow \frac{P_{cr}L}{2}\theta = K2\theta \Rightarrow P_{cr} = \frac{4K}{L}$
    - System is unstable when $P > P_{cr}$
- If $P > P_{cr}$, buckling occurs and a new equilibrium position is found

  $\Rightarrow P\frac{L}{2}\sin\theta = K2\theta$ ($\sin\theta \neq \theta$ now)

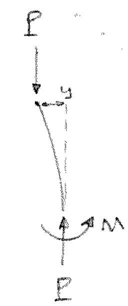
  $\Rightarrow \frac{PL}{4K} = \frac{\theta}{\sin\theta}$

    - For $\theta > 0$, $\theta > \sin\theta$, so a solution only exists here when $P > P_{cr}$

* So what factors do you think will be important to buckling?

- (column dimensions – I; length; stiffness; support cond's.)

- Euler's Formula for Pin-Ended Column



- $\Sigma M_{cut} = 0 = M(x) + Py \quad = M(x) = -Py$
- Recall $\frac{d^2y}{dx^2} = \frac{M}{EI} = \frac{-Py}{EI}$

$\Rightarrow \quad \frac{d^2y}{dx^2} + \frac{P}{EI}y = 0$

- linear, homogeneous, 2^{nd} order, constant coeff — since M depends on y; wasn't true for deflection prob's.
- Let $k^2 = \frac{P}{EI} \quad \Rightarrow \quad \frac{d^2y}{dx^2} + k^2 y = 0$

$\Rightarrow$ sol'n form: $y = A\sin kx + B\cos kx$

- B.C.'s: $y(x=0) = 0 = B\cos(0) = B$

$y(x=L) = 0 = A\sin kL$

- true when $kL = n\pi$ (also true when $A=0$, but that's trivial)

$\Rightarrow k^2L^2 = n^2\pi^2 = \frac{P}{EI}L^2$

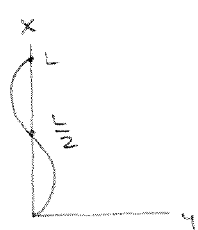
$\Rightarrow P = \frac{n^2\pi^2 EI}{L^2}$

- Smallest value of P that satisfies $A\sin kL = 0$ is when $n=1 \Rightarrow P_{cr} = \frac{\pi^2 EI}{L^2} \equiv$ Euler's Formula

$\Rightarrow \quad y = A\sin kx = A\sin\sqrt{\frac{P}{EI}}\,x = A\sin\sqrt{\frac{1}{\cancel{EI}}\frac{\pi^2\cancel{EI}}{L^2}}\,x = A\sin\frac{\pi}{L}x$

* still don't know y (since we don't know A), but we know the shape

- what do cases $n = 2, 3, \ldots$ represent?
  - Consider $n=2 \Rightarrow y = A\sin\frac{2\pi}{L}x$



  - satisfies B.C.'s @ $x = 0, L$
  - $P_{cr} = \frac{4\pi^2 EI}{L^2}$ (4x higher than $n=1$)
  - Will it ever happen? Only if we constrain $y=0$ @ $x = L/2$

  - Similar for higher n's

- What considerations can we make to design for buckling?
  - Maximize P_{cr} : $P_{cr} = \frac{\pi^2 E I}{L^2}$

    $\Rightarrow$ For selected material and length, maximize I

- Critical stress?

$$\sigma_{cr} = \frac{P_{cr}}{A} = \frac{\pi^2 E I}{A L^2}$$

  - Define r, radius of gyration $\Rightarrow r = \sqrt{\frac{I}{A}}$ $(I = A r^2)$

$$\Rightarrow \sigma_{cr} = \frac{\pi^2 E}{(A/I)L^2} = \frac{\pi^2 E}{(L/r)^2}$$

where $L/r \equiv$ slenderness ratio (only term that depends on column cross section)

<u>Recap</u>

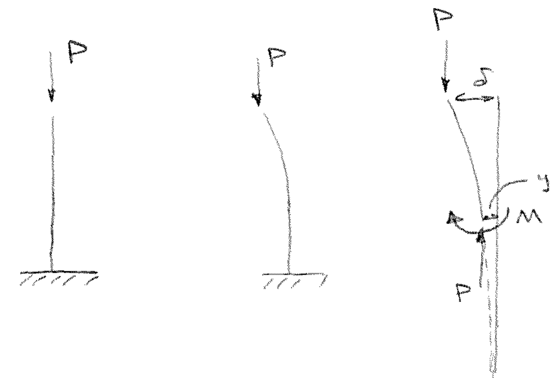
- Derived $P_{cr} = \frac{\pi^2 E I}{L^2}$ for pinned-pinned column

  $\equiv$ Euler load

  * material strength is not relevant as long as it is greater than $\sigma_{cr} = \frac{\pi^2 E}{(L/r)^2}$

- What factors affect buckling?
  - Length, L
  - Cross sectional I
  - Material stiffness, E

- What about support cond's. other than pinned-pinned?
  - General procedure
    - Obtain expression for bending moment $M(x)$
    - Sub into $EI\frac{d^2y}{dx^2} = M(x)$ + solve diff eq'n.
    - Apply B.C.'s



- Fixed Base - Free Top

$\Sigma M_{cut} = 0 = P(\delta - y) - M(x)$

$\Rightarrow M(x) = P(\delta - y)$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{P(\delta - y)}{EI} = \frac{P\delta}{EI} - \frac{Py}{EI}$$

$$\Rightarrow \frac{d^2y}{dx^2} + \underbrace{\frac{P}{EI}}_{k^2} y = \underbrace{\frac{P}{EI}}_{k^2}\delta$$

* need to come up w/rationale for chosen moment directio here relative to previous

* same as pinned-pinned, except non-homogeneous

$\Rightarrow y = y_H + y_P$

$= C_1 \sin kx + C_2 \cos kx + \delta$

(Apply B.C.'s)

$\Rightarrow y = \delta(1 - \cos kx)$

- Critical load?

$y(L) = \delta = \delta(1 - \cos kL)$

- $\cos kL = 0 \Rightarrow kL = n\frac{\pi}{2}$, $n = 1, 3, 5, \ldots$

$\Rightarrow k = \frac{n\pi}{2L}$

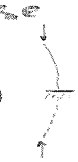
$\Rightarrow y = \delta\left(1 - \cos\frac{n\pi}{2L}x\right)$, $n = 1, 3, 5, \ldots$

$$\Rightarrow k^2 = \frac{n^2\pi^2}{4L^2} = \frac{P}{EI} \Rightarrow P = \frac{n^2\pi^2 EI}{4L^2}$$

$$\Rightarrow P_{cr} = \frac{\pi^2 EI}{4L^2}$$

$\Rightarrow$ Buckling load is 1/4 that of pinned-pinned case

$\Rightarrow$ equivalent to pinned-pinned with effective length, $L_e = 2L$ (can see this by inspection)



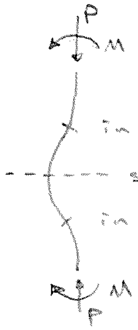
$$\Rightarrow P_{cr} = \frac{\pi^2 EI}{(2L)^2} \Rightarrow P_{cr} = \frac{\pi^2 EI}{L_e^2}$$

p-p: $L_e = L$

fixed-free: $L_e = 2L$

- Let's consider 2 other cases

  - Fixed - fixed



inflection - no curvature (@ $L/4$)

symmetry (@ $L/2$)

inflection

middle section

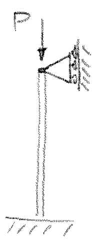
same as pinned-pinned

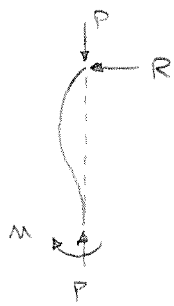
- Can show that $P_{cr} = \dfrac{4\pi^2 EI}{L^2} = \dfrac{\pi^2 EI}{(L/2)^2}$

  * 4 times stiffer than pinned-pinned

  - $L_e = L/2$

- Fixed- pinned



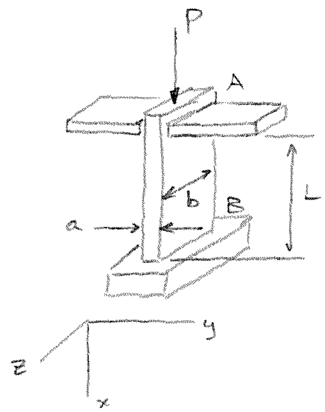


- Can show $P_{cr} = \dfrac{20.19\ EI}{L^2} = \dfrac{\pi^2 EI}{L_e^2}$

  where $L_e = 0.7\,L$

Example (S.P. 10.1, Beer 7, p. 701)

Given: aluminum column; end A constrained by smooth plates (can still slide and rotate)



Determine: (a) ratio of a/b for most efficient design against buckling

(b) most efficient cross section given $L = 20$ in, $E = 10.1 \times 10^6$ psi, $P = 5$ kips, and F. of S. = 2.5

* Most efficient design is when critical stresses corresponding to buckling for all possible buckling modes are equal

a) Which modes are relevant here?

- x-y plane: fixed-pinned $\Rightarrow L_e = 0.7L$
- x-z plane: fixed-free $\Rightarrow L_e = 2L$

- $$\sigma_{cr} = \frac{\pi^2 E}{(L_e/r)^2} \Rightarrow \frac{\pi^2 E}{(L_e/r_z)^2_{x-y}} = \frac{\pi^2 E}{(L_e/r_y)^2_{x-z}}$$

$$\Rightarrow \left(\frac{\overset{0.7L}{L_e}}{r_z}\right)_{x-y} = \left(\frac{\overset{2L}{L_e}}{r_y}\right)_{x-z}$$

- Recall $I = Ar^2 = abr^2$
  - $I_z = abr_z^2$ (r_z, I_z apply to x-y bending)

$$= \frac{1}{12} ba^3$$

$$\Rightarrow r_z^2 = \frac{1}{12}\frac{\cancel{b}a^{\cancel{3}2}}{\cancel{a}\cancel{b}} \Rightarrow r_z = \frac{a}{\sqrt{12}}$$

  - $I_y = abr_y^2 = \frac{1}{12}ab^3$

$$\Rightarrow r_y^2 = \frac{1}{12}\frac{\cancel{a}b^{\cancel{3}2}}{\cancel{a}\cancel{b}} \Rightarrow r_y = \frac{b}{\sqrt{12}}$$

$$\Rightarrow \frac{0.7L}{a/\sqrt{12}} = \frac{2L}{b/\sqrt{12}} \Rightarrow \frac{0.7}{a} = \frac{2}{b} \Rightarrow \underline{\frac{a}{b} = 0.35}$$

b) $$P_{cr} = \frac{\pi^2 EI}{L_e^2}\ ;\ I = Ar^2 \Rightarrow P_{cr} = \frac{\pi^2 EA}{(L_e/r)^2} = \frac{\pi^2 Eab}{(2L/b/\sqrt{12})^2} = \frac{\pi^2 Eab^3}{48L^2}$$

$$= \frac{\pi^2 E(0.35)b^4}{48L^2} \Rightarrow b^4 = \frac{(FS)P\ 48L^2}{\pi^2 E(0.35)} \Rightarrow \underline{b = 1.62\text{ in}}$$

$$\underline{a = 0.57\text{ in}}$$

- Let's make sure that Euler's formula is valid, i.e. that $\sigma_{cr} < \sigma_Y$
  - Assuming 6061 Al, $\sigma_Y = 35$ ksi
  - $\sigma_{cr} = \dfrac{P_{cr}}{A}$
    - we designed for P_{cr} to be 2.5(5 kips) in this problem

$$\Rightarrow \sigma_{cr} = \frac{(2.5)(5 \text{ kips})}{(0.57\text{ in})(1.62\text{ in})} = 13.54 \text{ ksi} \Rightarrow \text{OK}$$

- σ_{cr} increases as length decreases

$\Rightarrow$ How much shorter could column get for buckling to still be primary mode of failure?

$$\Rightarrow \sigma_{cr} = \frac{\pi^2 E}{(L_e/r)^2} = \sigma_y$$

$$= \frac{\pi^2 E}{\left((0.7)L/a/\sqrt{12}\right)^2} = \frac{\pi^2 E}{(0.7)^2 L^2}\frac{a^2}{12}$$

$$\Rightarrow L = \sqrt{\frac{\pi^2 E a^2}{(0.7)^2 \sigma_Y (12)}} = \frac{\pi a}{0.7}\sqrt{\frac{E}{12\sigma_Y}}$$

$$= \frac{\pi(0.57\text{ in})}{0.7}\sqrt{\frac{10.1\times10^6\text{ psi}}{12(35\times10^3\text{ psi})}} = \underline{12.54 \text{ in}}$$