1. a) $\frac{\partial T}{\partial t} = -B\omega \sin(\omega t)$

b) $\frac{DT}{Dt} = \frac{\partial T}{\partial t} + \vec{V}\cdot\vec{\nabla} T$

$\vec{V}\cdot\vec{\nabla} = \left(\frac{C}{r^2}\vec{e}_r + D\sin(\omega t+\theta)\vec{e}_\phi\right)\cdot\left(\vec{e}_r\frac{\partial}{\partial r} + \frac{\vec{e}_\theta}{r}\frac{\partial}{\partial\theta} + \frac{\vec{e}_\phi}{r\sin\theta}\frac{\partial}{\partial\phi}\right)$

$\vec{V}\cdot\vec{\nabla} = \frac{C}{r^2}\frac{\partial}{\partial r} + \frac{D\sin(\omega t+\theta)}{r\sin\theta}\frac{\partial}{\partial\phi} + 0$

so $\vec{V}\cdot\vec{\nabla} T = \frac{C}{r^2}(2Ar\sin\theta) + \frac{D\sin(\omega t+\theta)}{r\sin\theta}\cdot 0$

So, $\frac{DT}{Dt} = -B\omega\sin(\omega t) + \frac{C}{r^2}(2Ar\sin\theta)$

c) Acceleration $\frac{D\vec{V}}{Dt} = \frac{\partial\vec{V}}{\partial t} + \vec{V}\cdot\nabla\vec{V}$

where $\frac{\partial\vec{V}}{\partial t} = D\cos(\omega t+\theta)\cdot\omega\,\vec{e}_\phi$

and $\vec{V}\cdot\nabla\vec{V} = \frac{C}{r^2}\frac{\partial\vec{V}}{\partial r} + \frac{D\sin(\omega t+\theta)}{r\sin\theta}\frac{\partial\vec{V}}{\partial\phi}$

$= \frac{C}{r^2}\left(\frac{-2C}{r^3}\right)\vec{e}_r + \frac{D\sin(\omega t+\theta)}{r\sin\theta}\left[\frac{C}{r^2}\frac{\partial\vec{e}_r}{\partial\phi} + D\sin(\omega t+\theta)\frac{\partial\vec{e}_\phi}{\partial\phi}\right]$

So $\vec{V}\cdot\vec{\nabla}\vec{V} = \frac{-2C^2}{r^5}\vec{e}_r + \frac{D\sin(\omega t+\theta)}{r\sin\theta}\left[\frac{C}{r^2}\sin\theta\,\vec{e}_\phi + D\sin(\omega t+\theta)\left(-\sin\theta\,\vec{e}_r - \cos\theta\,\vec{e}_\theta\right)\right]$

so $\frac{D\vec{V}}{Dt} = D\omega\cos(\omega t+\theta)\vec{e}_\phi - \frac{2C^2}{r^5}\vec{e}_r + \frac{D\sin(\omega t+\theta)}{r\sin\theta}\left[\frac{C}{r^2}\sin\theta\,\vec{e}_\phi + D\sin(\omega t+\theta)\left(-\sin\theta\,\vec{e}_r - \cos\theta\,\vec{e}_\theta\right)\right]$

d) $A, B, C, D = 1$ $\quad (r,\theta,\phi) = (1, \pi/4, \pi/2)$

$\vec{\nabla}\times\vec{V} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \vec{e}_r & r\vec{e}_\theta & r\sin\theta\,\vec{e}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi} \\ V_r & rV_\theta & r\sin\theta V_\phi \end{vmatrix}$

$= \frac{1}{r^2\sin\theta}\begin{vmatrix} \vec{e}_r & r\vec{e}_\theta & r\sin\theta\,\vec{e}_\phi \\ \partial/\partial r & \partial/\partial\theta & \partial/\partial\phi \\ \frac{C}{r^2} & 0 & r\sin\theta D\sin(\omega t+\theta) \end{vmatrix}$

$\vec{\nabla}\times\vec{V} = \frac{1}{r^2\sin\theta}\left[\vec{e}_r\left(Dr\cos\theta\sin(\omega t+\theta) + Dr\sin\theta\cos(\omega t+\theta)\right) - r\vec{e}_\theta\left(D\sin\theta\sin(\omega t+\theta) - 0\right)\right]$

At $(r,\theta,\phi) = (1, \pi/4, \pi/2)$, $\sin\theta = \frac{1}{\sqrt{2}}$, $\cos\theta = \frac{1}{\sqrt{2}}$

so $\vec{\nabla}\times\vec{V} = \vec{e}_r\left(D\sin(\omega t+\pi/4) + D\cos(\omega t+\pi/4)\right) - \vec{e}_\theta D\sin(\omega t+\pi/4)$

And Angular velocity $= \frac{1}{2}\vec{\nabla}\times\vec{V}$

e) $\vec{\nabla}\cdot\vec{V} = \frac{1}{r^2}\frac{\partial r^2 V_r}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial V_\theta\sin\theta}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial V_\phi}{\partial\phi}$ (second term crossed out, $= 0$)

$= \frac{1}{r^2}\frac{\partial C}{\partial r} + 0 + \frac{1}{r\sin\theta}\frac{\partial(D\sin(\omega t+\theta))}{\partial\phi}$

$\therefore \vec{\nabla}\cdot\vec{V} = 0$

So time rate of change of volume of fluid particle at $(1, \pi/4, \pi/2)$ is zero

2) a) circumferentially averaged angular velocity : $\vec{\omega} = \frac{1}{2}\vec{\nabla}\times\vec{V}$

$\vec{\omega} = \frac{1}{2}\vec{\nabla}\times\vec{V}$

$= \frac{1}{2r}\begin{vmatrix} \vec{e}_r & r\vec{e}_\theta & \vec{e}_z \\ \partial/\partial r & \partial/\partial\theta & \partial/\partial z \\ 0 & r\frac{\Omega_0 z}{\tan\alpha} & 0 \end{vmatrix}$

$= \frac{1}{2r}\left[\vec{e}_r\left(-\frac{\Omega_0 r}{\tan\alpha}\right) + \vec{e}_z\left(\frac{\Omega_0 z}{\tan\alpha}\right)\right]$

$\vec{\omega} = \frac{-\Omega_0}{2r\tan\alpha}\vec{e}_r + \frac{\Omega_0 z}{2r\tan\alpha}\vec{e}_z$

r should have canceled out here

$\vec{\omega} = \frac{\Omega_0}{2r\tan\alpha}\left(-r\vec{e}_r + z\vec{e}_z\right)$

b) $\vec{a} = \frac{D\vec{V}}{Dt} = \frac{\partial\vec{V}}{\partial t} + \vec{V}\cdot\vec{\nabla}(\vec{V})$

$= \left(\frac{\Omega_0 z}{\tan\alpha}\frac{1}{r}\frac{\partial}{\partial\theta}\right)\left(\frac{\Omega_0 z}{\tan\alpha}\vec{e}_\theta\right)$

$= \frac{\Omega_0^2 z^2}{r\tan\alpha^2}\frac{\partial\vec{e}_\theta}{\partial\theta}$

$\vec{a} = -\frac{\Omega_0^2 z^2}{r\tan\alpha^2}\vec{e}_r$

c) $\tau = \mu\frac{\partial U}{\partial z}$. ON PLATE $\frac{\partial U}{\partial z} = \frac{\Omega_0}{\tan\alpha}$

so $\tau = \frac{\mu\Omega_0}{\tan\alpha}$

3) a) $\psi = -\frac{1}{4\pi|r-r_0|} = \frac{-1}{4\pi s}$ where $s = |r - r_0|$

Now

$\nabla\psi = \frac{\partial\psi}{\partial x_1}\vec{i} + \frac{\partial\psi}{\partial x_2}\vec{j} + \frac{\partial\psi}{\partial x_3}\vec{k}$

$= \frac{-1}{4\pi s^2}\left(\frac{\partial s}{\partial x_1}\vec{i} + \frac{\partial s}{\partial x_2}\vec{j} + \frac{\partial s}{\partial x_3}\vec{k}\right)$

Note $s^2 = |r - r_0|^2 = (x_1 - x_{01})^2 + (x_2 - x_{02})^2 + (x_3 - x_{03})^2$

or $2s\frac{\partial s}{\partial x_1} = 2(x_1 - x_{01})$

$\frac{\partial s}{\partial x_1} = \frac{1}{s}(x_1 - x_{01})$

Similarly $\frac{\partial s}{\partial x_2} = \frac{1}{s}(x_2 - x_{02})$; $\frac{\partial s}{\partial x_3} = \frac{1}{s}(x_3 - x_{03})$

$\therefore \nabla\psi = -\frac{1}{4\pi s^3}\left[(x_1 - x_{01})\vec{i} + (x_2 - x_{02})\vec{j} + (x_3 - x_{03})\vec{k}\right]$

$\nabla\psi = -\frac{1}{4\pi|r-r_0|^3}\left[\vec{r} - \vec{r}_0\right]$ where $\vec{r} = x_1\vec{i} + x_2\vec{j} + x_3\vec{k}$, $\vec{r}_0 = x_{01}\vec{i} + x_{02}\vec{j} + x_{03}\vec{k}$

b) $\nabla^2\psi = \delta(r - r_0)$; $\nabla^2\phi = 0$ (see question)

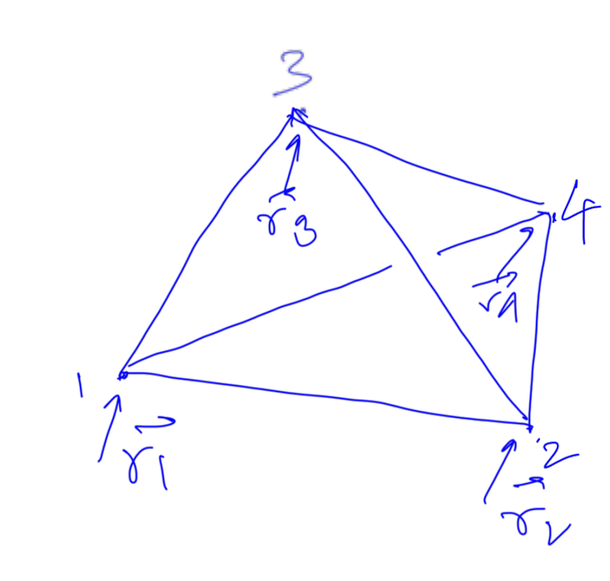
$\int_R \psi\nabla^2\phi - \phi\nabla^2\psi\, d\tau = \oint \psi\frac{\partial\phi}{\partial n} - \phi\frac{\partial\psi}{\partial n}\, dS$

$-\int_R \phi\,\delta(r - r_0) = \oint -\frac{1}{4\pi|r-r_0|}\frac{\partial\phi}{\partial n} - \phi\nabla\psi\cdot n\, dS$

$= -\phi(r_0) = \oint -\frac{1}{4\pi|r-r_0|}\frac{\partial\phi}{\partial n} - \frac{\phi(r-r_0)\cdot\vec{n}}{4\pi|r-r_0|^3}$

$\Rightarrow \phi(r_0) = \frac{1}{4\pi|r-r_0|}\oint \frac{\partial\phi}{\partial n} - \frac{(r-r_0)\cdot\vec{n}}{(r-r_0)^2}\phi\, dS$

4)



Or

Velocities at vertices ① – ④ are $\vec{V}_1, \vec{V}_2, \vec{V}_3, \vec{V}_4$

curl $\vec{V} \equiv -\lim \frac{1}{d\tau}\oint \vec{V}\times\vec{n}\, dS$

$\approx -\frac{1}{R}\sum_{i=1}^{4}\vec{V}^{(i)}\times\vec{A}^{(i)}$

WHERE $\vec{V}^{(i)}$ is Average velocity vector on FACE OPPOSITE VERTEX ⓘ

Eg. $\vec{V}^{(4)} = \frac{1}{3}\left(\vec{V}_1 + \vec{V}_2 + \vec{V}_3\right)$

And $\vec{A}^{(i)}$ is the area vector (outward normal) on FACE OPPOSITE VERTEX ⓘ

Eg $\vec{A}^{(4)} = \frac{1}{2}\left(\vec{r}_2 - \vec{r}_1\right)\times\left(\vec{r}_3 - \vec{r}_1\right)$

And R is the volume of the tetrahedron by vector triple product.

$= -\frac{1}{3}\vec{A}^{(4)}\cdot\left(\vec{r}_4 - \vec{r}_2\right)$