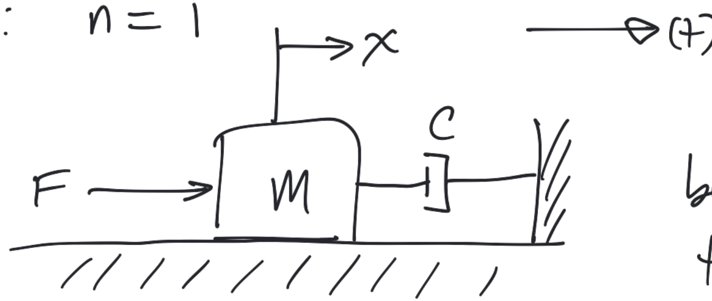
First- and Second-order Systems

$$T(s) = \frac{N(s)}{D(s)}$$ polynomials in s

First-order system:

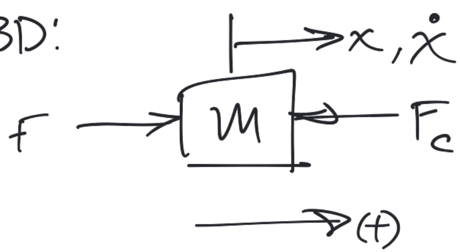
Means: $n = 1$

Ex



Find model between Force to the velocity of the mass

FBD:



sum forces:

$\sum F = m\ddot{x}$, let $\dot{x} = v$

$F - c\dot{x} = m\ddot{x}$, we have $\dot{x} = v$

$\Rightarrow m\ddot{x} + c\dot{x} = F \Rightarrow m\dot{v} + cv = F$

$\Rightarrow \dot{v} + \frac{c}{m} v = \frac{1}{m} F$

Take L.T. w/ zero I.C.

$$sV(s) + \frac{c}{m} V(s) = \frac{1}{m} F(s)$$

Find the transfer function: $T(s) = \frac{V(s)}{F(s)}$

$\Rightarrow V(s)\left[s + \frac{c}{m}\right] = \frac{1}{m} F(s)$

$$\Rightarrow \frac{V(s)}{F(s)} = \frac{1/m}{s + c/m} = \frac{b_0}{s + a_0}$$

③

In general, 1st-order system transfer function can be written as follows:

$$T(s) = \frac{V(s)}{F(s)} = \frac{b_0}{s+a_0} \cdot 1 = \frac{b_0}{s+a_0} \cdot \frac{a_0}{a_0}$$

$$= \frac{a_0}{s+a_0} \cdot \underbrace{\frac{b_0}{a_0}}_{K} = K \frac{1/\tau}{s+1/\tau}$$

$$\Rightarrow T(s) = K \frac{1/\tau}{s+1/\tau} \quad \text{where } K = \frac{b_0}{a_0} \text{ and } \frac{1}{\tau} = a_0$$

τ is the time constant, the time it takes the output to reach 63% of final value when the input is a step input.

Ex: $T(s) = \frac{3}{s+8}$ we want K and τ

$$\Rightarrow T(s) = K \frac{1/\tau}{s + 1/\tau} \quad ; \quad \begin{aligned} b_0 &= 3 \\ a_0 &= 8 \end{aligned}$$

$$K = \frac{b_0}{a_0} \qquad \frac{1}{\tau} = a_0 \Rightarrow \tau = \frac{1}{a_0}$$

Sub int numbers, we get:

$$K = \frac{3}{8} \text{ and } \tau = 1/8$$

$$\Rightarrow T(s) = 3 \cdot \frac{8}{s+8}$$

Ex

$m = 1$

$$T(s) = \frac{s+3}{s+9}$$

$n = 1$ First order

$$T(s) = \underbrace{\frac{s}{s+9}}_{\text{1st order system}} + \underbrace{\frac{3}{s+9}}_{\text{1st order system}}$$

And the time constant $\Rightarrow \tau = 1/9$

(6)

$$\frac{V(s)}{F(s)} = T(s) = K\,\frac{1/\tau}{s+1/\tau} \Rightarrow \frac{V(s)}{F(s)} = K\,\frac{1/\tau}{s+1/\tau}$$

$$(s+1/\tau)\,V(s) = K\left(1/\tau\right)F(s) \Rightarrow sV(s) + \frac{1}{\tau}V(s) = \frac{K}{\tau}F(s)$$

we can take the Laplace inverse:

$$\Rightarrow \underbrace{\dot{v}(t) + \frac{1}{\tau}v(t)}_{\substack{\text{Natural response} \\ \text{"free response"}}} = \underbrace{\frac{K}{\tau}f(t)}_{\text{Input}}$$

Input $\Rightarrow$ response due to forcing function "Forced response"

if we solve O.D.E.

$$\text{solution } v(t) = \underbrace{\text{free response} + \text{forced response}}_{\text{Transient + steady-state}}$$

Solution :

$$\dot{v}(t) + \frac{1}{\tau} v(t) = \frac{K}{\tau} f(t)$$

$$sV(s) + \frac{1}{\tau} V(s) = \frac{K}{\tau} F(s)$$

Suppose the input is a unit step,

$$u(t) = \begin{cases} 0 & t \leq 0 \\ 1 & t > 0 \end{cases}$$

Assume zero I. C.

The solution $v(t)$:

$$v(t) = K\left(1 - e^{-t/\tau}\right)$$

L.T. Approach

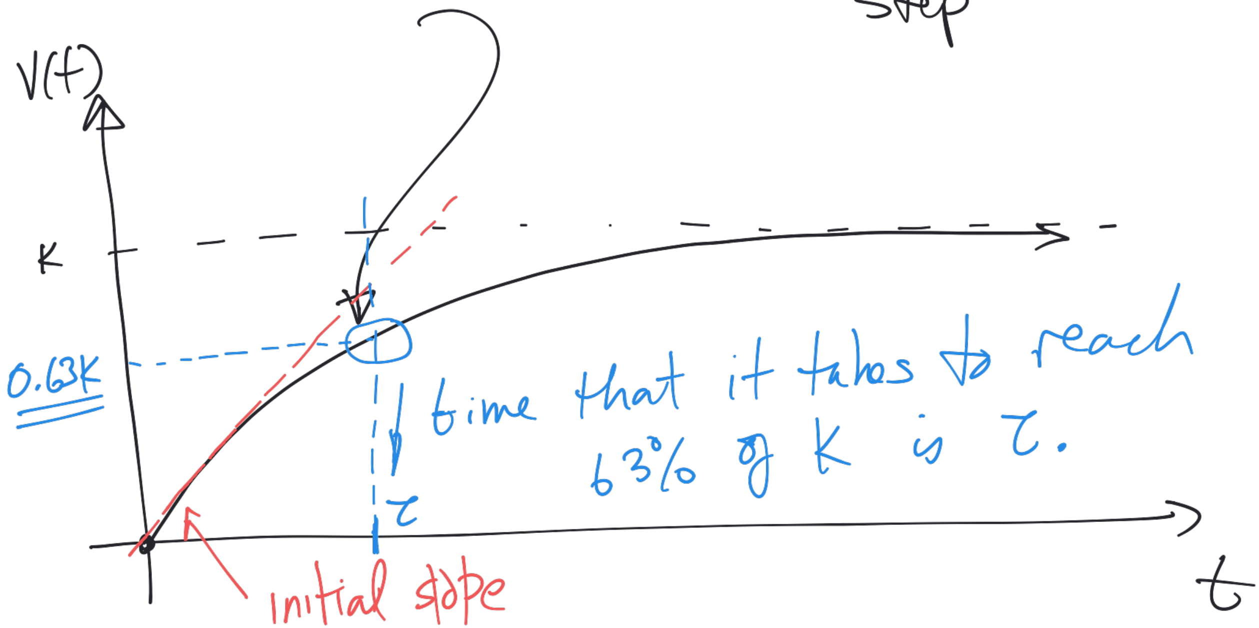
$$\mathcal{L}\{u(t)\} = \frac{1}{s}$$

$$v(t) = \mathcal{L}^{-1}\{\quad\}$$

τ = time constant.

$$V(t) = K\left(1 - e^{-t/\tau}\right)$$

input was a unit step



when $t=0 \Rightarrow V(0)=0$; $t \to \infty$ $V(\infty) = K$

Intial slope: $\frac{d}{dt}V(t)\Big|_{t=0} = \frac{K}{\tau}e^{-t/\tau}\Big|_{t=0} \Rightarrow K/\tau$