Intermediate Fluid Mechanics

Lecture 2: Kinematics of Fluid Motion

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ME 5700/6700

August 21, 2025

Chapter Overview

Lecture Objectives

In this section we will study the kinematics of fluid motion. That is, we will examine:

- the displacement,
- velocity
- acceleration
- deformation
- rotation

of the fluid without worrying about the forces acting on the fluid.

Chapter Overview

Eulerian vs. Lagrangian viewpoints

As engineers it is possible to study the kinematics of fluid motion using two different viewpoints:

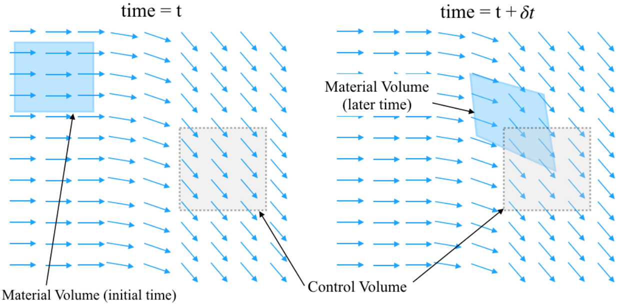

Figure: *(a) One can draw an imaginary (control) volume in space (that remains fixed in space) and observe the flow moving in and out of the volume (i.e. Eulerian Framework); (b) One can tag a selected volume of fluid molecules (material volume) and follow that material through the flow field.*

Eulerian vs. Lagrangian viewpoints

Lagrangian (material) description:

- In this case one solves a set of equations to determine the flow properties of a given set of fluid elements ($\sim$ particles).
- This approach tracks fluid elements in space and time, and through the study of the trajectory of these elements it is possible to extract information about the flow.
- In this case, quantities are specified in reference to their original location at time t_0 ($\vec{\xi} = \vec{x}(t_0)$).
- In this description, the independent variables are $\vec{\xi}$ and t.
- Alternatively, the dependent variables (*e.g.* density and velocity) are associated with the same material ($\vec{\xi}$),

$$\vec{x} = \vec{x}(\vec{\xi}, t), \quad \vec{u} = \vec{u}(\vec{\xi}, t), \quad \rho = \rho(\vec{\xi}, t), ... \tag{1}$$

Eulerian vs. Lagrangian viewpoints

Eulerian (spatial) description:

- In this case one solves a set of equations to determine the flow properties at any point in the flow field at any given time.
- Fluid characteristic quantities (density, velocity, ...) are specified at a spatial point $\vec{x}$ in the flow.
- In this case, the independent variables are $\vec{x}$ and t.
- The dependent variables (also termed as '*field*' variables) are expressed as,

$$\vec{u} = \vec{u}(x, y, z, t), \quad \rho = \rho(x, y, z, t). \tag{2}$$

Examples:

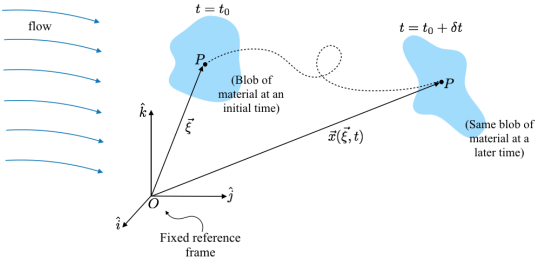

Figure: *Trajectory of a blob of fluid as represented and measured when using a Lagrangian framework. The fluid element labeled as P is located at position $\vec{\xi}$ at time t_0. In a time δt, the flow has moved the fluid element P to a new position $\vec{x}$. In this way, the vector $\vec{\xi}$ defines or labels the fluid material. At time t_0 each fluid element has a unique position vector $\vec{\xi}$ associated with it.*

Example

$\Longrightarrow$ Can you think of examples of both Lagrangian & Eulerian fluid measurements?

Examples:

Example

$\Longrightarrow$ Can you think of examples of both Lagrangian & Eulerian fluid measurements?

Some cases are:

- Ocean buoy,
- Anemometer on ship,
- pitot-tube on an airplane,
- hot-wire anemometer in the wind tunnel,...
- Meteorological Balloon

Acceleration of a fluid element (or fluid particle)

The velocity of a fluid particle is the time rate of change of that particle's position,

$$\vec{u} = \frac{d\vec{x}(\vec{\xi}, t)}{dt} \tag{3}$$

- $\Longrightarrow$ expressed this way corresponds to the **Lagrangian** framework.
- As traditionally understood, the velocity of a particle corresponds to how 'fast' does it change its position.
- Now the question is, <u>how do we express/compute velocity in an Eulerian framework?</u>

Note:

Realize that in an Eulerian framework one aims to know the velocity of the fluid at a given point in space at a given time (t), regardless of which fluid particle is occupying that space at time t.

Chapter Overview

Acceleration of a fluid element (or fluid particle) – Material Derivative

Consider a variable scalar field such as Temperature,

$$T = T(x, y, z, t) \quad \textbf{`Eulerian Description'} \tag{4}$$

Then, from Calculus, the total derivative (using the chain rule) can be written as,

$$dT = \underbrace{\frac{\partial T}{\partial t} dt}_{I} + \underbrace{\frac{\partial T}{\partial x} dx + \frac{\partial T}{\partial y} dy + \frac{\partial T}{\partial z} dz}_{II} \tag{5}$$

- **Term I:** local change in temperature with time at a fixed point $\vec{x}$ in space.
- **Term II:** change in temperature at $\vec{x}$ due to particles with different temperature advecting from $\vec{x}$ to $\vec{x} + d\vec{x}$ at time t.

Acceleration of a fluid element (or fluid particle) – Material Derivative

This means that,

$$\underbrace{\frac{dT}{dt}}_{\substack{\text{time rate of change}\\ \text{following fluid particle}}} = \frac{\partial T}{\partial t} + \frac{\partial T}{\partial x}\frac{dx}{dt} + \frac{\partial T}{\partial y}\frac{dy}{dt} + \frac{\partial T}{\partial z}\frac{dz}{dt} \tag{6}$$

$$= \frac{\partial T}{\partial t} + \frac{\partial T}{\partial x}u + \frac{\partial T}{\partial y}v + \frac{\partial T}{\partial z}w \tag{7}$$

$$= \underbrace{\frac{\partial T}{\partial t}}_{\substack{\text{local time}\\ \text{rate of change}}} + \underbrace{\vec{u}\cdot\nabla T}_{\text{advection}}. \tag{8}$$

Note on Mathematics I

Note that given the following two vectors,

$$\vec{u} = u\,\hat{i} + v\,\hat{j} + w\,\hat{k} \tag{9}$$

$$\vec{\nabla} = \frac{\partial}{\partial x}\,\hat{i} + \frac{\partial}{\partial y}\,\hat{j} + \frac{\partial}{\partial z}\,\hat{k}, \tag{10}$$

their corresponding dot product leads to,

$$\vec{u}\cdot\vec{\nabla} = u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}. \tag{11}$$

Therefore, it is important to realize that

$$\vec{u}\cdot\vec{\nabla} \neq \vec{\nabla}\cdot\vec{u}. \tag{12}$$

The Dot Product:

Recall from linear algebra that the dot product is the projection of one vector on to another. Mathematically, the dot product is represented by,

$$\vec{a} \cdot \vec{b} = a\,b\,cos\theta, \tag{13}$$

where $a = |\vec{a}|$, and $b = |\vec{b}|$ are the magnitudes or lengths of the vectors and θ is the angle between them. The dot product may also be written as,

$$\vec{a} \cdot \vec{b} = \sum_{i=1}^{3} a_i\, b_i = a_1\, b_1 + a_2\, b_2 + a_3\, b_3, \tag{14}$$

where $\vec{a} = a_1\,\hat{i} + a_2\,\hat{j} + a_3\,\hat{k}$ and $\vec{b} = b_1\,\hat{i} + b_2\,\hat{j} + b_3\,\hat{k}$.

Note that if $\vec{a}$ and $\vec{b}$ are perpendicular, then $\vec{a} \cdot \vec{b} = 0$, since $cos\,90^\circ = 0$.

General form of the material Derivative:

Therefore, the acceleration of a fluid element or Material Derivative will be given in general form as,

$$\frac{d\,\square}{dt} = \frac{\partial\,\square}{\partial t} + (\vec{u} \cdot \vec{\nabla})\,\square \tag{15}$$

Note:

Realize that you will often encounter a capital 'D' to refer to the material derivative, instead of a lower case 'd', *i.e.* $\frac{D\,\square}{Dt} \equiv \frac{d\,\square}{dt}$.

Material Derivative of velocity:

Let's now apply the Material Derivative to the velocity vector $\vec{u}$,

$$\underbrace{\underbrace{\frac{d\vec{u}}{dt}}_{I}}_{Lagrangian} = \underbrace{\underbrace{\frac{\partial \vec{u}}{\partial t}}_{II} + \underbrace{(\vec{u} \cdot \vec{\nabla})\vec{u}}_{III},}_{Eulerian} \tag{16}$$

- Term *I* represents the local acceleration of individual fluid particles,
- Term *II* is the local time rate of change of velocity at a fixed point,
- Term *III* is the advection of momentum.

Example:

Think what each term means when studying the flow field in a pipe contraction.

Material Derivative of velocity in component form:

Let's now take the Material Derivative of the velocity field,

$$\underbrace{\underbrace{\frac{d\vec{u}}{dt}}_{I}}_{Lagrangian} = \underbrace{\underbrace{\frac{\partial \vec{u}}{\partial t}}_{II} + \underbrace{(\vec{u} \cdot \vec{\nabla})\vec{u}}_{III}}_{Eulerian}, \tag{17}$$

and write it in component form:

$$\frac{du}{dt} = \frac{\partial u}{\partial t} + (\vec{u} \cdot \vec{\nabla})u \tag{18}$$

$$\frac{dv}{dt} = \frac{\partial v}{\partial t} + (\vec{u} \cdot \vec{\nabla})v \tag{19}$$

$$\frac{dw}{dt} = \frac{\partial w}{\partial t} + (\vec{u} \cdot \vec{\nabla})w \tag{20}$$

Example:

Consider a steady flow that exits a garden hose and impinges on a fence. Can you describe the velocity evolution of the fluid particle, in both the Lagrangian and Eulerian framework?

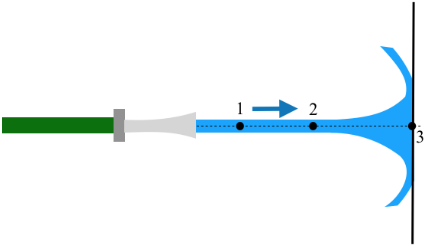

Figure: *Point 1 represents the fluid particle at time t; Point 2 represents the fluid particle at time $t + \delta t$, Point 3 represents the fluid particle at the stagnation point, where the velocity is zero.*

Example: (continued)

- The fluid particle located at position 1 at time t moves to location 2 in a time δt.
- In this new location $u_2 < u_1$, (*i.e.* the momentum of the fluid particle decreases as it approaches the stagnation point.)
- However, the flow is steady, meaning that if we sit at any given location, the velocity does not appear to change with time (unless someone turns off the water).
- Therefore, in this case advection is non-zero because the velocity of the fluid particle is changing as it passes through any given point in space.

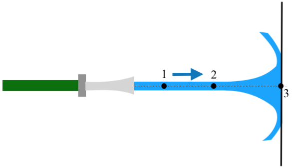

Figure: *Point 1 represents the fluid particle at time t; Point 2 represents the fluid particle at time $t + \delta t$, Point 3 represents the fluid particle at the stagnation point, where the velocity is zero.*

Example: (continued)

Mathematically, this translates by saying that the material derivative is different than zero,

$$\frac{D\vec{u}}{Dt} \neq 0, \tag{21}$$

but given that the material derivative can be rewritten in an Eulerian framework as,

$$\frac{D\vec{u}}{Dt} = \frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \nabla)\vec{u} \neq 0 \tag{22}$$

and that

$$\frac{\partial \vec{u}}{\partial t} = 0, \tag{23}$$

it hence means that

$$(\vec{u} \cdot \nabla)\vec{u} \neq 0, \tag{24}$$

as explained above.

Why is the Material Derivative important?

- The conservation laws (*e.g.* Newton's 2nd law), are usually written for systems of fluid particles (*i.e.* they are written in a Lagrangian framework). e.g.$\vec{F} = m\vec{a}$.
- In fluid mechanics, it is too hard to track the motion of individual fluid particles, especially from an experimental stand point.
- We prefer to sit at a fixed point in space and observe the flow as it passes by us.
- Therefore, there is a need to understand the dynamics of the flow in terms of the Eulerian framework.
- The Material Derivative allows us to convert between the Lagrangian & Eulerian frameworks for this purpose.

$$\frac{D\,\square}{Dt} : \quad \text{Lagrangian} \Longleftrightarrow \text{Eulerian} \tag{25}$$

Note:

Non-linearity of the advection term for momentum, $(\vec{u} \cdot \nabla)\vec{u}$

When writing the components of the advection term,

$$\text{x-component:} \quad \underbrace{u\frac{\partial u}{\partial x}} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} \tag{26}$$

$$\text{y-component:} \quad u\frac{\partial v}{\partial x} + \underbrace{v\frac{\partial v}{\partial y}} + w\frac{\partial v}{\partial z} \tag{27}$$

$$\text{z-component:} \quad u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + \underbrace{w\frac{\partial w}{\partial z}} \tag{28}$$

The marked terms are non-linear terms given that the velocity is multiplied by its own derivative. This will cause problems when one tries to solve the governing equations in the Eulerian framework.