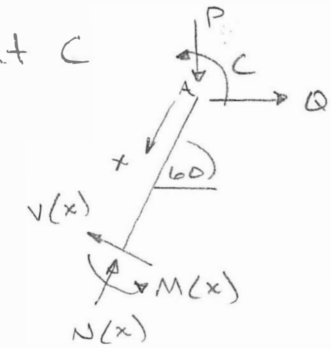
Example : more Castigliano (prob 11.105, Beer 7, p. 820, but also find θ_A)

- Given / find : see slide

- Sol'n : Add dummy load Q and dummy moment C



  - $\delta_{A,P}$ (vert) $= \frac{1}{EI}\int M \frac{\partial M}{\partial P}\,dx + \frac{1}{AE}\int N \frac{\partial N}{\partial P}\,dx$
  - $\delta_{A,Q}$ (horiz) $= \frac{1}{EI}\int M \frac{\partial M}{\partial Q}\,dx + \frac{1}{AE}\int N \frac{\partial N}{\partial Q}\,dx$
  - $\theta_A = \frac{1}{EI}\int M \frac{\partial M}{\partial C}\,dx + \frac{1}{AE}\int N \frac{\partial N}{\partial C}\,dx$

- Find $M(x)$, $N(x)$

- Section AB

  - $\Sigma F_{60} = 0 = N(x) - P\sin 60 + Q\cos 60 \Rightarrow N(x) = \frac{\sqrt{3}}{2}P - \frac{1}{2}Q$

    $\Rightarrow \frac{\partial N}{\partial P} = \frac{\sqrt{3}}{2}$; $\frac{\partial N}{\partial Q} = -\frac{1}{2}$; $\frac{\partial N}{\partial C} = 0$

  - $\Sigma M_{cut} = 0 = M(x) - Px\cos 60 - Qx\sin 60 + C$

    $\Rightarrow M(x) = \frac{1}{2}Px + \frac{\sqrt{3}}{2}Qx - C$

    $\Rightarrow \frac{\partial M}{\partial P} = \frac{x}{2}$; $\frac{\partial M}{\partial Q} = \frac{\sqrt{3}}{2}x$; $\frac{\partial M}{\partial C} = -1$

$$\Rightarrow \delta_{A,P1} = \frac{1}{AE}\int_0^L \left(\frac{\sqrt{3}}{2}P - \frac{1}{2}\cancelto{0}{Q}\right)\frac{\sqrt{3}}{2}\,dx + \frac{1}{EI}\int_0^L\left(\frac{1}{2}Px + \frac{\sqrt{3}}{2}\cancelto{0}{Q}x - \cancelto{0}{C}\right)\frac{x}{2}\,dx$$

$$= \frac{1}{AE}\,\frac{3}{4}Px\Big|_0^L + \frac{1}{EI}\,\frac{1}{4}P\,\frac{x^3}{3}\Big|_0^L = \underline{\frac{3}{4}\frac{PL}{AE} + \frac{1}{12}\frac{PL^3}{EI}}$$

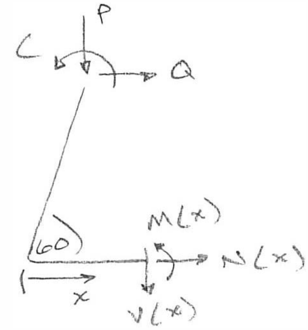
$$\Rightarrow \delta_{A,Q1} = \frac{1}{AE}\int_0^L\left(\frac{\sqrt{3}}{2}P - \frac{1}{2}\cancelto{0}{Q}\right)\left(-\frac{1}{2}\right)dx + \frac{1}{EI}\int_0^L\left(\frac{1}{2}Px + \frac{\sqrt{3}}{2}\cancelto{0}{Q}x - \cancelto{0}{C}\right)\left(\frac{\sqrt{3}}{2}x\right)dx$$

$$= -\frac{1}{AE}\,\frac{\sqrt{3}}{4}Px\Big|_0^L + \frac{1}{EI}\,\frac{\sqrt{3}}{4}P\,\frac{x^3}{3}\Big|_0^L = \underline{-\frac{\sqrt{3}}{4}\frac{PL}{AE} + \frac{\sqrt{3}}{12}\frac{PL^3}{EI}}$$

$$\Rightarrow \theta_{C1} = \frac{1}{AE}\int_0^L\left(\frac{\sqrt{3}}{2}P - \frac{1}{2}\cancelto{0}{Q}\right)(0)\,dx + \frac{1}{EI}\int_0^L\left(\frac{1}{2}Px + \frac{\sqrt{3}}{2}\cancelto{0}{Q}x - \cancelto{0}{C}\right)(-1)\,dx$$

$$= -\frac{1}{EI}\,\frac{1}{2}P\,\frac{x^2}{2}\Big|_0^L = \underline{-\frac{1}{4}\frac{PL^2}{EI}}$$

· Section BC



· $\Sigma F_x = 0 = N(x) + Q \Rightarrow N(x) = -Q$

$\Rightarrow \frac{\partial N}{\partial P} = 0 \; ; \; \frac{\partial N}{\partial Q} = -1 \; ; \; \frac{\partial N}{\partial C} = 0$

· $\Sigma M_{cut} = 0 = M(x) + C + P(x - L\overbrace{\cos 60}^{1/2}) - QL\underbrace{\sin 60}_{\sqrt{3}/2}$

$\Rightarrow M(x) = \frac{\sqrt{3}}{2} QL - P(x - \frac{L}{2}) - C$

$\Rightarrow \frac{\partial M}{\partial P} = \frac{L}{2} - x \; ; \; \frac{\partial M}{\partial Q} = \frac{\sqrt{3}}{2} L \; ; \; \frac{\partial M}{\partial C} = -1$

$\Rightarrow \delta_{A,P2} = \frac{1}{AE}\int_0^L (-\cancel{Q}^{0})(0)\,dx + \frac{1}{EI}\int_0^L \left[\frac{\sqrt{3}}{2}\cancel{Q}^{0} L - P(x - \frac{L}{2}) - \cancel{C}^{0}\right](\frac{L}{2} - x)\,dx$

$= \frac{1}{EI}\int_0^L P(x^2 - Lx + \frac{L^2}{4})\,dx = \frac{P}{EI}\left[\frac{x^3}{3} - L\frac{x^2}{2} + \frac{L^2}{4}x\right]_0^L$

$= \frac{P}{EI}\left(\frac{L^3}{3} - \frac{L^3}{2} + \frac{L^3}{4}\right) = \underline{\frac{1}{12}\frac{PL^3}{EI}}$

$\Rightarrow \delta_{Q,P2} = \frac{1}{AE}\int_0^L (-\cancel{Q}^{0})(-1)\,dx + \frac{1}{EI}\int_0^L \left[\frac{\sqrt{3}}{2}\cancel{Q}^{0} L - P(x - \frac{L}{2}) - \cancel{C}^{0}\right](\frac{\sqrt{3}}{2} L)\,dx$

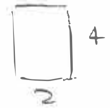
$= -\frac{1}{EI}\int_0^L \frac{\sqrt{3}}{2} PL(x - \frac{L}{2})\,dx = -\frac{\sqrt{3}}{2}\frac{PL}{EI}\left[\frac{x^2}{2} - \frac{L}{2}x\right]_0^L$

$= -\frac{\sqrt{3}}{2}\frac{PL}{EI}\left(\frac{L^2}{2} - \frac{L^2}{2}\right) = \underline{0}$

$\Rightarrow \theta_{C2} = \frac{1}{AE}\int_0^L (-\cancel{Q}^{0})(0)\,dx + \frac{1}{EI}\int_0^L + P(x - \frac{L}{2})(+1)\,dx$

$= \frac{P}{EI}\left[\frac{x^2}{2} - \frac{L}{2}x\right]_0^L = \frac{P}{EI}\left(\frac{L^2}{2} - \frac{L^2}{2}\right) = \underline{0}$

$\Rightarrow \delta_{vert} = \frac{3}{4}\frac{PL}{AE} + \frac{1}{12}\frac{PL^3}{EI} + \frac{1}{12}\frac{PL^3}{EI} = \boxed{\frac{3}{4}\frac{PL}{AE} + \underbrace{\frac{1}{6}\frac{PL^3}{EI}}_{\text{dominant for large } L}}$

* If  $A = 8$, $I = \frac{1}{12} bh^3 = 10.7$

$\Rightarrow \delta_{horiz} = \boxed{-\frac{\sqrt{3}}{4}\frac{PL}{AE} + \frac{\sqrt{3}}{12}\frac{PL^3}{EI}}$ * note direction

$\Rightarrow \theta_c = \boxed{-\frac{1}{4}\frac{PL^2}{EI}}$ * note direction