Digital Control – Quick overview

Z Transform:

Mathematical tool for analyzing linear discrete systems. Analogous to the Laplace transform for linear continuous systems.

Recall the Laplace transform:

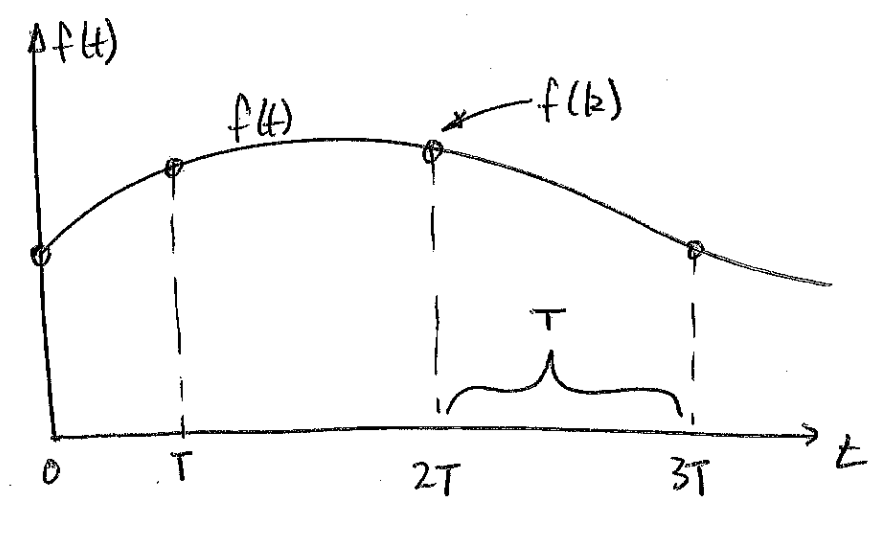
$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty f(t)e^{-st}dt$$

and w/ zero initial conditions:

$$\mathcal{L}\{\dot{f}(t)\} = sF(s)$$

This relationship allows us to find a transfer function form a differential Equation.

Consider A continuous sampled signal



then the z-transform is defined by

$$\mathcal{Z}\{f(k)\} = F(z) = \sum_{K=0}^{\infty} f(k) z^{-k}$$

where $f(k)$ is the sampled $f(t)$ and $K = 0, 1, 2, \ldots$

then we have a similar property in z-transform:

$$\mathcal{Z}\{f(k-1)\} = z^{-1} F(z)$$

which allows us to find the transfer function for the discrete system.

Ex: Consider the general 2nd order difference equation:

$$y(k) = -a_1 y(k-1) - a_2 y(k-2) + b_0 u(k) + b_1 u(k-1) + b_2 u(k-2)$$

can be converted over to z-domain

$$Y(z) = (-a_1 z^{-1} - a_2 z^{-2})\, Y(z) + (b_0 + b_1 z^{-1} + b_2 z^{-2})\, U(z)$$

then

$$\boxed{\frac{Y(z)}{U(z)} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2}}{1 + a_1 z^{-1} + a_2 z^{-2}}}$$

z-transform Inversion

Table in text relates z-transforms to discrete-time functions.

Given a z-transform, we can expand it using partial fraction expansion and use tables to find the time function just like Laplace equations.

Another Approach (Long divison)

Given
$$Y(z) = \frac{N(z)}{D(z)}$$

Use long division to divide numerator by denominator and then compute the resulting discrete time-function with:

$$\mathcal{Z}\{f(k)\} = F(z) = \sum_{k=0}^{\infty} f(k) z^{-k}$$

Example:

Consider the 1st order system:

$$y(k) = \alpha y(k-1) + u(k)$$

$$y(k) - \alpha y(k-1) = u(k)$$

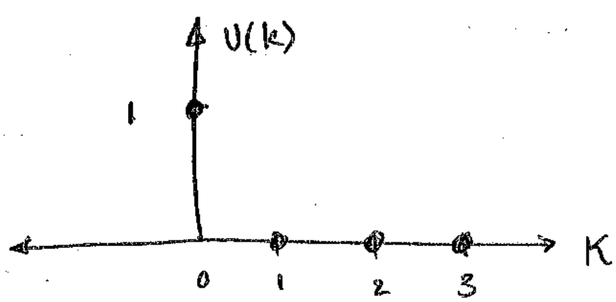
$$(1 - \alpha z^{-1}) Y(z) = U(z)$$

$$\Rightarrow \quad \frac{Y(z)}{U(z)} = \frac{1}{1-\alpha z^{-1}}$$

Assume the input $U(z)$ is a unit pulse, then

$$u(0) = 1$$

$$u(k) = 0 \qquad k \neq 0$$



then the z-transform of $U(z) = 1$

so:

$$Y(z) = \frac{1}{1-\alpha z^{-1}}$$

Now to find the time series, we expand using long divison :

$$Y(z) = 1 - \alpha z^{-1} \overline{\big)\, 1} \qquad \text{quotient: } 1 + \alpha z^{-1} + \alpha^2 z^{-2} + \alpha^3 z^{-3} + \dots$$

$$1 \overset{\oplus}{-} \alpha z^{-1}$$

$$\alpha z^{-1} + 0$$

$$\ominus\ \alpha z^{-1} \overset{\oplus}{-} \alpha^2 z^{-2}$$

$$\alpha^2 z^{-2} \quad + 0$$

$$\ominus \alpha^2 z^{-2} \overset{\oplus}{-} \alpha^3 z^{-3}$$

$$\alpha^3 z^{-3} + 0$$

So :

$$Y(z) = 1 + \alpha z^{-2} + \alpha^2 z^{-2} + \alpha^3 z^{-3} + \dots$$

Comparing to $\mathcal{Z}\{f(k)\} = F(z) = \sum_{k=0}^{\infty} f(k) z^{-k}$

$K=0 \Rightarrow y(0) = 1$

$K=1 \Rightarrow y(1) = \alpha$

$y(2) = \alpha^2$

$y(3) = \alpha^3$

$\vdots$

Relationship between s and z

Recall from Laplace analysis:

$$f(t) = e^{-at}$$

$$F(s) = \frac{1}{s+a}$$

where $s = -a$ was the pole in s-plane.

Now, in z-domain:

$$F(z) = \mathcal{Z}\{e^{-akT}\}$$

$$F(z) = \frac{z}{z - e^{-aT}}$$

the pole in z-domain is: $z = e^{-aT}$

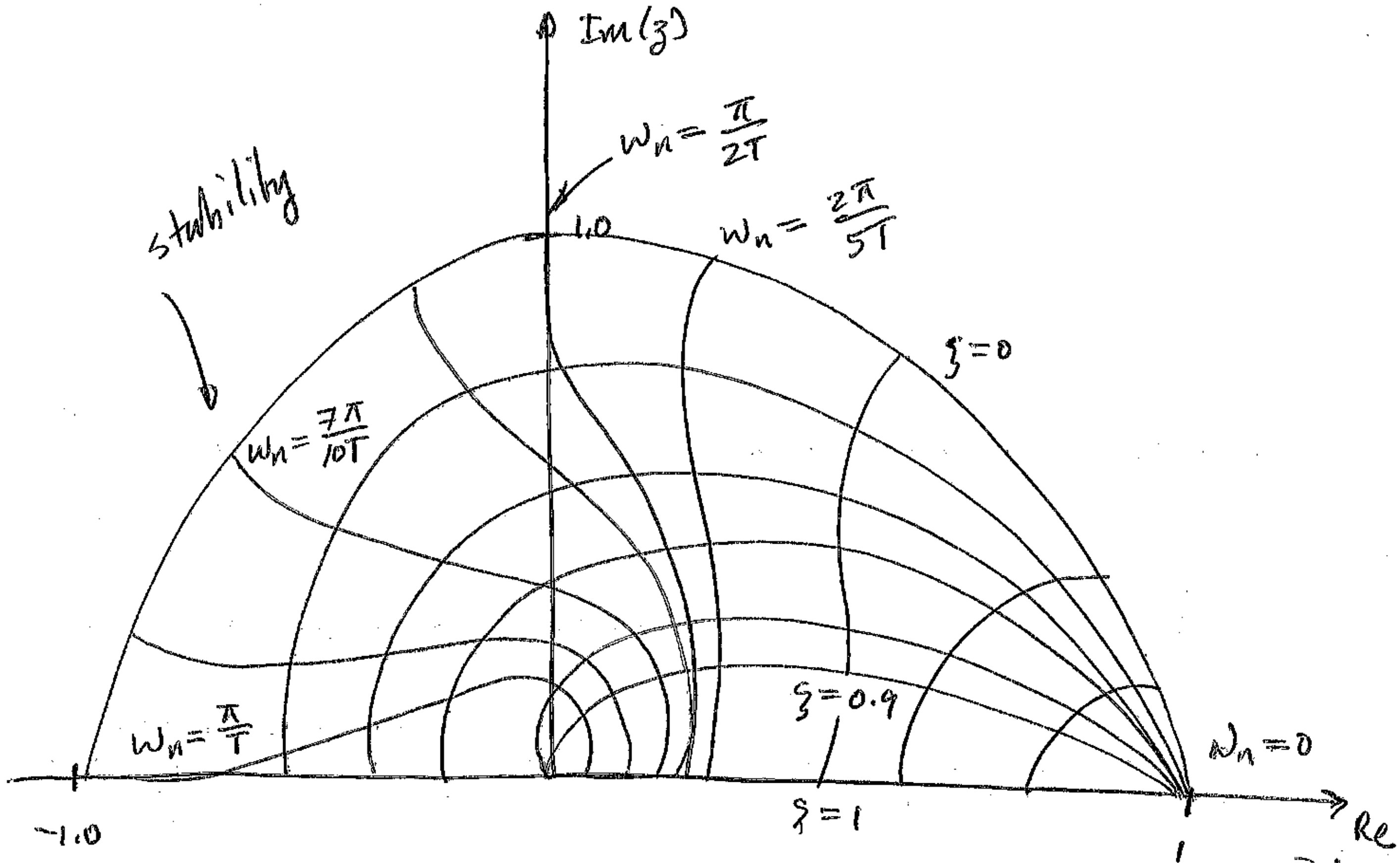
In general, the equivalent characteristics in the z-plane are related to those in the s-plane by the expression:

$$z = e^{sT}$$

where T is the sampling period.

For a 2nd order System : $s = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$

the mapping $z = e^{sT}$ T = sampling period



1. Stability boundary is a unit circle $|z| = 1$ $s = \pm j\omega \Rightarrow z = \underbrace{e^{\pm j\omega T}}_{\text{circle}}$
2. Small region around $z = +1$ is the same as $s = 0$
3. The negative real axis always represents a frequency of $\omega_s/2$ where $\omega_s = 2\pi/T$ = sample rate in rad/s
4. Vertical line in s-plane mapp into circles in z-plane (constant real part)
5. Horizontal lines in s-plane map into radial lines in z-plane (constant imaginary part).

6. Frequencies greater than $\omega_s/2$ is called the Nyquist rate. This is the aliasing or folding frequency. As a result we should always sample at least twice as fast as the highest frequency component in order to represent that signal with samples.

The time sequence associated with points in the z-plane are shown in the attached plot. (see slide)

Final Value Theorem

In Laplace:

$$\lim_{t \to \infty} x(t) = x_{ss} = \lim_{s \to 0} s X(s)$$

if poles of $s X(s)$ are in LHP.

Discrete:

$$\lim_{k \to \infty} x(k) = x_{ss} = \lim_{z \to 1} (1 - z^{-1}) X(z)$$

if poles of $(1 - z^{-1}) X(z)$ are inside unit circle.

Figure 8.6 shows the mapping of lines of constant damping ζ and natural frequency ω_n from the s-plane (Fig. 3.11) to the upper half of the z-plane, using Eq. (8.19). The mapping has several important features (See Problem 8.20):

1. The stability boundary is the unit circle $|z| = 1$.
2. The small vicinity around $z = +1$ in the z-plane is essentially identical to the vicinity around $s = 0$ in the s-plane.
3. The z-plane locations give response information normalized to the sample rate, rather than to time as in the s-plane.
4. The negative real z-axis always represents a frequency of $\omega_s/2$, where $\omega_s = 2\pi/T$ = sample rate in radians per second.
5. Vertical lines in the left half of the s-plane (the constant real part or time constant) map into circles within the unit circle of the z-plane.
6. Horizontal lines in the s-plane (the constant imaginary part of the frequency) map into radial lines in the z-plane.
7. Frequencies greater than $\omega_s/2$, called the **Nyquist rate**, appear in the

Nyquist rate $= \omega_s/2$

FIGURE 8.6
Natural frequency (solid color) and damping loci (light color) in the z-plane; the portion below the Re(z)-axis (not shown) is the mirror image of the upper half shown.

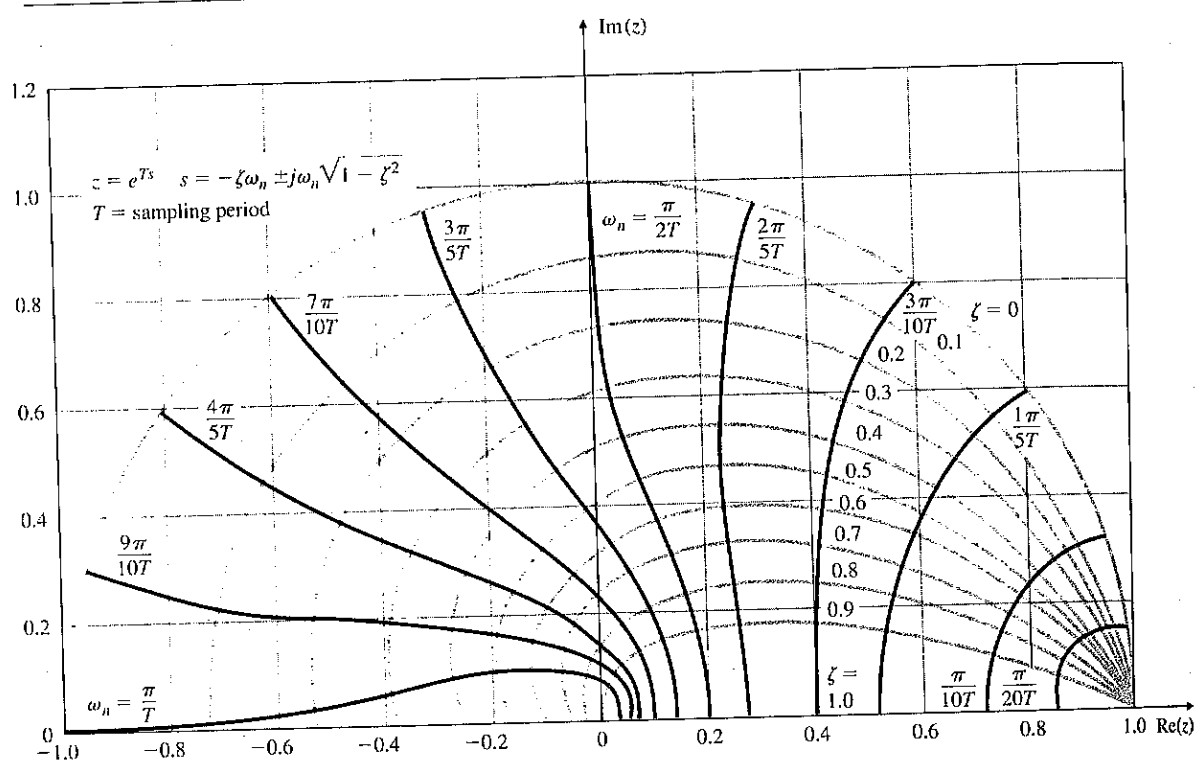
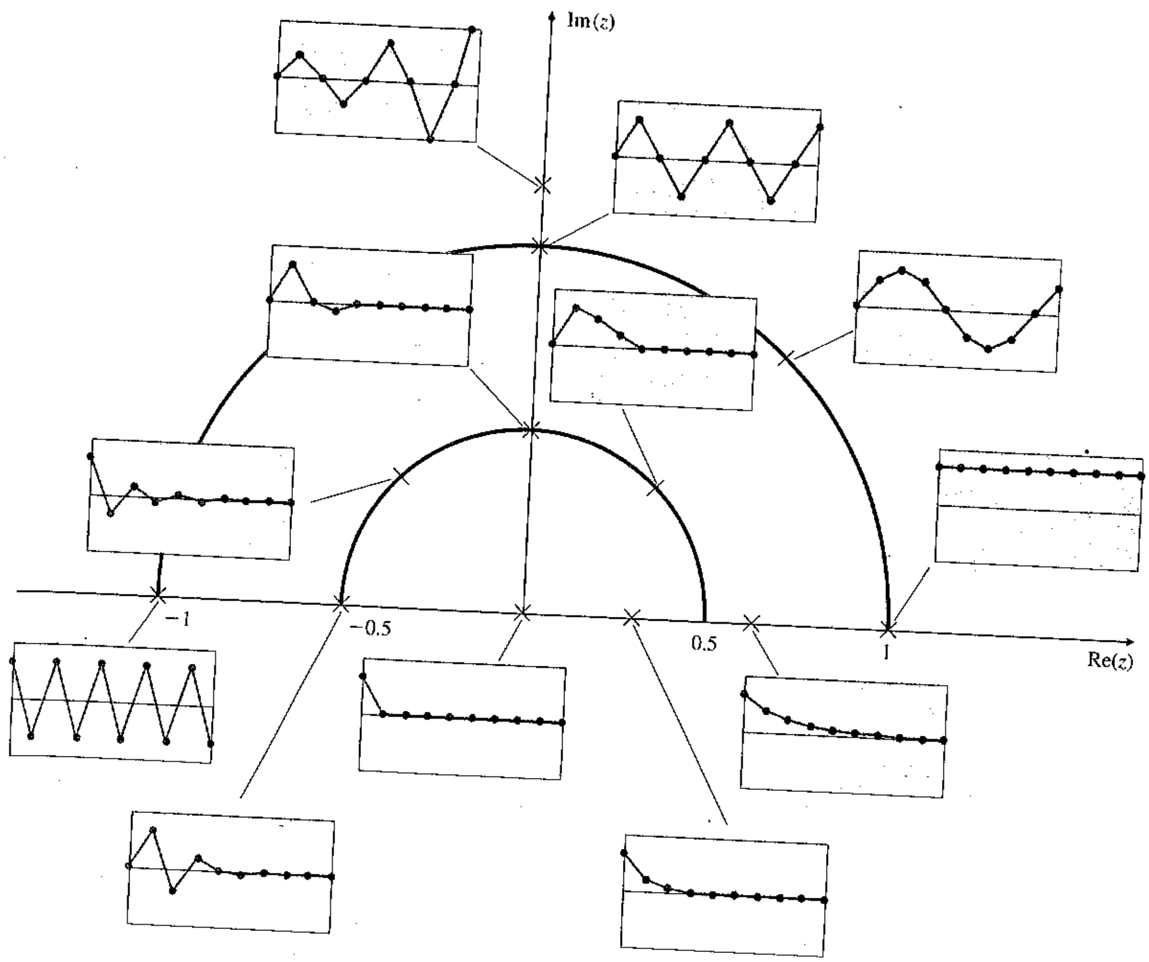


FIGURE 8.7
Time sequences associated with points in the z-plane



z-plane on top of corresponding lower frequencies because of the circular character of the trigonometric functions imbedded in Eq. (8.19). This overlap is called **aliasing** or **folding**. As a result it is necessary to sample at least twice as fast as a signal's highest frequency component in order to represent that signal with the samples. (We will discuss aliasing in greater detail in Section 8.6.3.)

To provide insight into the correspondence between z-plane locations and the resulting time sequence, Fig. 8.7 sketches time responses that would result from poles at the indicated locations. This figure is the discrete companion of Fig. 3.9.

example:

Find DC gain of transfer function:

$$G(z) = \frac{X(z)}{U(z)} = \frac{0.58(1+z)}{(z+0.16)}$$

we let $u(k) = 1$ for $k \geq 0$, (step function)

$$U(z) = \frac{1}{1-z^{-1}}$$

and

$$X(z) = \frac{0.58(1+z)}{(1-z^{-1})(z+0.16)}$$

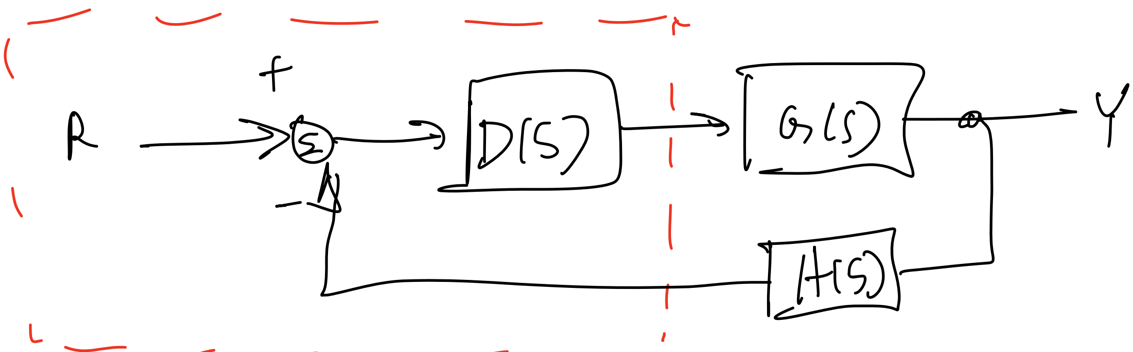
Final value theorem:

$$X_{ss} = \lim_{z \to 1} \left[\frac{0.58(1+z)}{\cancel{(1-z^{-1})}(z+0.16)} \right] \cancel{(1-z^{-1})}$$

$$\boxed{X_{ss} = 1}$$

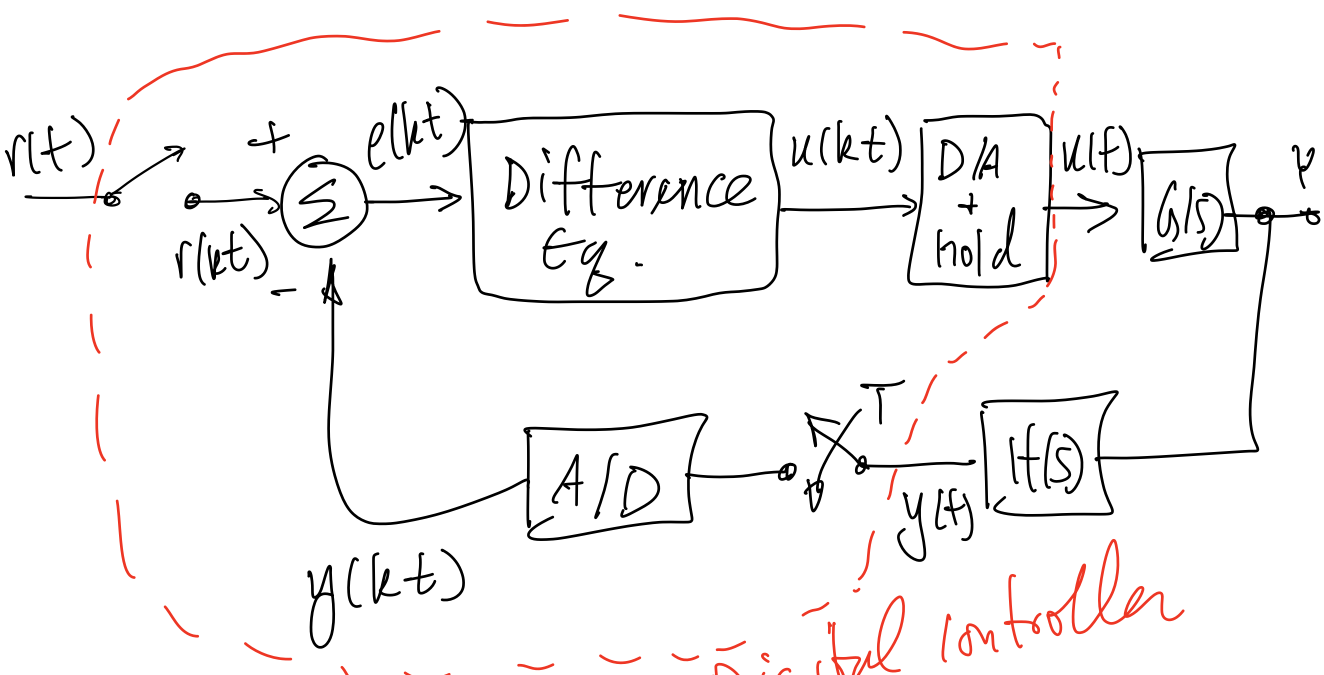
Digitization

Consider the continuous-time control system:



continuous controller

We want to implement controller using a digital computer, so we need to discretize:



Digital controller

Simplest way to do this: Euler's method

This is also known as forward rectangular rule:

Let $\dot{x} \cong \dfrac{x(k+1) - x(k)}{T}$

$T \equiv$ sample period.

$k \equiv$ integer and $t_k = kT$

$x(k) =$ value of $x(t)$ at t_k

$x(k+1) =$ value of $x(t)$ at t_{k+1}

Example

Let controller $D(s)$ be:

$$D(s) = \frac{U(s)}{E(s)} = K_0 \frac{s+a}{s+b}$$

rewrite:

$$(s+b)\,U(s) = K_0(s+a)\,E(s)$$

$$\Rightarrow \quad \dot{u} + bu = K_0(\dot{e} + ae)$$

Apply Euler's method:

$$\frac{u(k+1)-u(k)}{T} + bu(k) = K_0\left[\frac{e(k+1)-e(k)}{T} + ae(k)\right]$$

solve for $u(k+1)$, the input at next time step:

old input

old error

$$\underbrace{u(k+1)}_{\text{current input}} = u(k) + T\left\{-bu(k) + K_0\left[\frac{e(k+1)-e(k)}{T} + ae(k)\right]\right\}$$

Let's code this up as a program:

READ current $y(k+1)$, $r(k+1)$

$e(k+1) = r(k+1) - y(k+1)$

Compute $u(k+1) =$ equation above

SEND out $u(k+1)$ to system

Update:

$$u(k) = u(k+1)$$

$$e(k) = e(k+1)$$

REPEAT!

But for higher powers of s in controller, we need to consider other ways, for example: Z

Design by Emulation

Design process:

1. Design continuous compensator from chapters 1 – 6.
2. Digitizing the continuous compensation
3. Use discrete analysis, simulation or experimentation to verify the design.

Step 1: Design continuous compensators

– we already know how to do this.

Step 2: Digitization Procedures

– we will examine 3 methods

1. Tustin's Method
2. Matched-pole/zero (MPZ) Method
3. Modified Matched Pole zero (MMPZ)

Step 3: Analysis: we will examine various analysis tools.

Digitization Procedures:

Finding the best $D(z)$ to match the continuous $D(s)$ compensator.

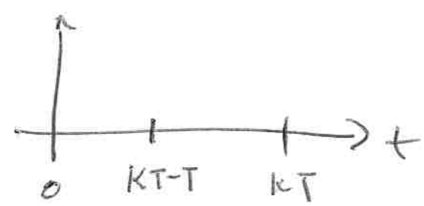
* Note : $D(z)$ is only an approximation of $D(s)$!!

Tustin's Method

Consider :

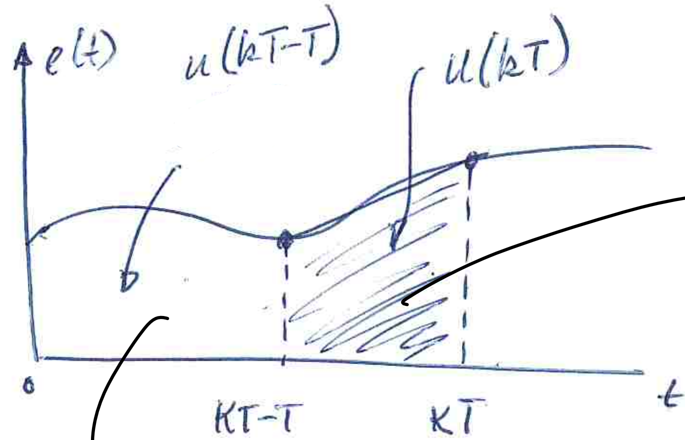
$$D(s) = \frac{U(s)}{E(s)} = 1/s \qquad \xrightarrow{e} \boxed{D(s)} \longrightarrow u$$

this approach is a numerical integration technique.



$$\Rightarrow \qquad U(s) = \frac{1}{s} E(s)$$

$$u(kT) = \int_0^{kT-T} e(t)\,dt + \int_{kT-T}^{kT} e(t)\,dt$$



OR

$$u(kT) = u(kT-T) + \text{area under } e(t) \text{ over the last } T$$

For Tustin's Method, we approximate e(t) by trapezodial approximation as shown above.

Rewrite $u(kT) = u(k)$ current

$u(kT-T) = u(k-1)$ previous

then the trapezodial approximation yields:

$$u(k) = u(k-1) + \underbrace{\frac{T}{2}\left[e(k-1) + e(k)\right]}_{\text{Trapezoid approx.}}$$

Now we take the z-transform of difference equation.

$$\frac{U(z)}{E(z)} = \frac{T}{2}\left(\frac{1+z^{-1}}{1-z^{-1}}\right) = \frac{1}{\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}$$

$$\Rightarrow \quad \frac{U(z)}{E(z)} = D(z) = \frac{1}{\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}$$

for $\boxed{D(s) = \frac{1}{s} \Rightarrow s = \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right) + a}$

But when $D(s) = \frac{a}{s+a}$, then

$$D(z) = \frac{a}{\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right) + a}$$

In fact, to use Tustin's or the Bilinear approximation, we simply substitute:

$\boxed{s = \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}$ into $D(s)$

the result is $D(z)$!

Example of Tustin's Method

$T_s = 15\,Hz$ (sample rate)

Given $$D(s) = 70\left(\frac{s+2}{s+10}\right)$$

Use Tustin's Method to get $D(z)$

solution

Tustin's Method calls for

$$s = \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right) \quad \text{where} \quad T = \frac{1}{T_s}$$

$$T = \frac{1}{15} \Rightarrow \frac{2}{T} = 30$$

so: $$s = 30\left(\frac{1-z^{-1}}{1+z^{-1}}\right)$$

when we substitute into $D(s)$, we get:

$$D(z) = \frac{56 - 49z^{-1}}{1 - 0.5z^{-1}}$$

Now we can write the difference equation:

$$D(z) = \frac{U(z)}{E(z)} = \frac{56 - 49z^{-1}}{1 - 0.5z^{-1}}$$

$$\Rightarrow u(k) = 0.223u(k-1) + 70e(k) - 59.12e(k-1)$$

or equivalently:

$$u(k+1) = 0.5u(k) + 56[e(k+1) - 0.875e(k)]$$

<u>Matlab solution:</u>

```
NumDs = [1  2]*70
denDs = [1  10]
T= 1/15
[NumDz, denDz] = c2dm(numDs, denDs, T, 'tustin')
```

$$\Rightarrow \boxed{D(z) = \frac{56 - 49z^{-1}}{1 - 0.5z^{-1}}}$$

Matched Pole/Zero Method

This technique applies the relation

$$z = e^{sT}$$

to the poles and zeros of a transfer function.

Case 1 Numerator & Denominator Power are equal

consider: $D(s) = K_c \dfrac{s+a}{s+b}$

then $\boxed{D(z) = K_d \dfrac{z - e^{-aT}}{z - e^{-bT}}}$ ← sub in $z = e^{sT}$

Find K_d by causing DC gain of $D(s)$ to equal $D(z)$ using Final Value Theorem:

$$\lim_{s \to 0} K_c \frac{s+a}{s+b} = \lim_{z \to 1} K_d \frac{z - e^{-aT}}{z - e^{-bT}}$$

$$K_c \frac{a}{b} = K_d \frac{(1 - e^{-aT})}{(1 - e^{-bT})}$$

$$\Rightarrow \boxed{K_d = K_c \frac{a}{b} \frac{(1 - e^{-bT})}{(1 - e^{-aT})}}$$

Case 2 Higher order terms, i.e. denominator higher order than numerator.

consider:

$$D(s) = K_c \frac{s+a}{s(s+b)}$$

then we add $(z+1)$ term to numerator until numerator / Denominator term are equal in power.

$$D(z) = K_d \frac{(z+1)(z-e^{-aT})}{(z-1)(z-e^{-bT})}$$

Now compute K_d by dropping pole at $s=0$ and $z=1$

$$\Rightarrow \lim_{s\to 0} K_c \left(\frac{s+a}{s+b}\right) = \lim_{z\to 1} K_d \frac{(z+1)(z-e^{-aT})}{(z-e^{-bT})}$$

$$\Rightarrow \boxed{K_d = K_c \frac{a}{2b}\left(\frac{1-e^{-bT}}{1-e^{-aT}}\right)}$$

Modified Match Pole/zero Method (MMPZ)

This modified technique changes the numerator term to one less power than the denominator term.

Consider Case 2 of MPZ

$$D(s) = K_c \frac{s+a}{s(s+b)}$$

let $$\boxed{D(z) = K_d \frac{(z - e^{-bT})}{(z-1)(z-e^{-bT})}}$$

we do not add the extra $\overset{(z+1)}{(1+z^{-1})}$ term!

then $$\boxed{K_d = K_c \frac{a}{b}\left(\frac{1-e^{-bT}}{1-e^{-aT}}\right)}$$

This method does not perform as well as MPZ because of the issue with using data that is one cycle old.