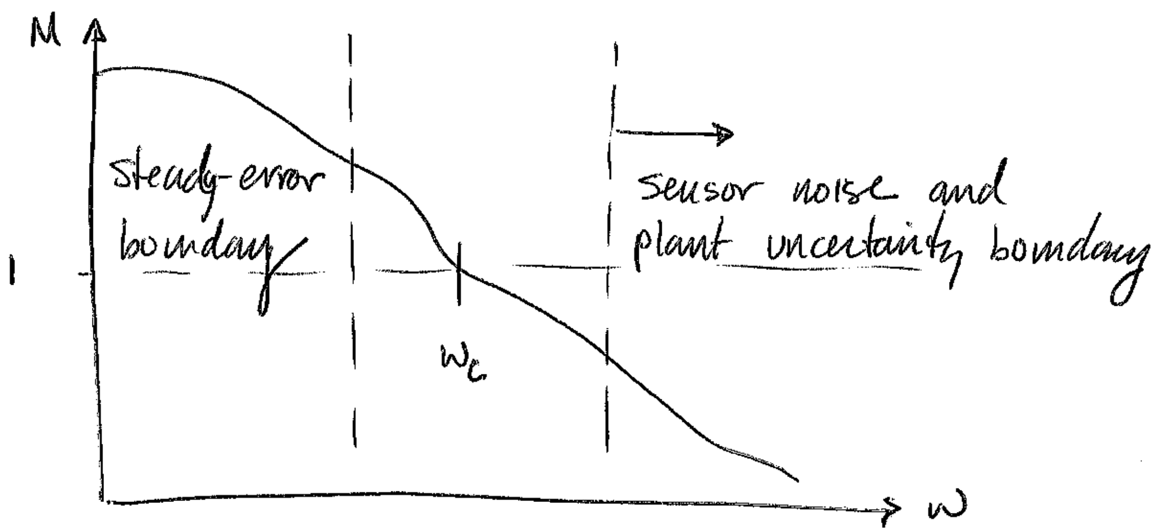
Sensitivity in Frequency Response

Goal: Develop conditions on the Bode plot of the open-loop transfer function DG to ensure good performance.

Recall from before:

1. feedback helps to reduce errors and sensitivity.
2. Reducing high frequency gains helps reduce noise effects. i.e. lead compensator.

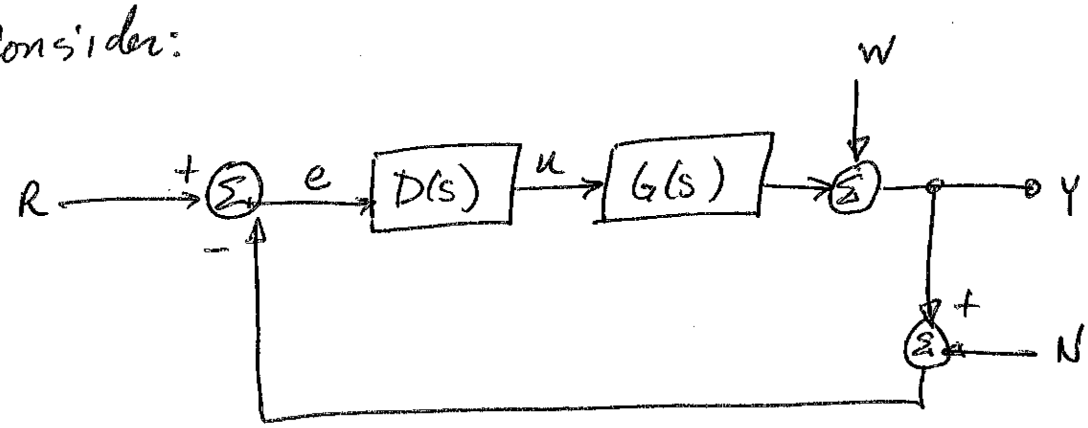
Basic Design Criteria in Bode form



the Sensitivity functions

These functions are used to show the sensitivity of closed-loop systems to aid control designers.

Consider:



N = Measurement noise

W = Disturbance

Main objective is to keep $e = r - y$ small!

Also, keep output y small due to W (disturbance).

From the block diagram:

$$Y = W + GDe$$

but $\quad e = R - Y - N$, then

$$Y = W + GD(R - Y - N)$$

or:

$$Y = W + GDR - GDY - GDN$$

$$(1+GD)Y = W + GD(R-N)$$

solve for Y:

$$\boxed{Y = (1+GD)^{-1}W + (1+GD)^{-1}GD(R-N)}$$

Now we look at the tracking error:

$$E = R - Y \quad \text{(definition)}$$

$$E = R - (1+GD)^{-1}W - (1+GD)^{-1}GD(R-N)$$

$$E = R - (1+GD)^{-1}W - (1+GD)^{-1}GDR + (1+GD)^{-1}GDN$$

$$E = \underbrace{\left(R - (1+GD)^{-1}GDR\right)}_{(1+GD)^{-1}R} - (1+GD)^{-1}W + (1+GD)^{-1}GDN$$

$$\boxed{E = (1+GD)^{-1}(R-W) + (1+GD)^{-1}GDN}$$

$$E = (1+GD)^{-1}(R-N) + (1+GD)^{-1}GDN$$

is the transfer function from r to e

and we define the sensitivity function:

$$S(s) = (1+GD)^{-1} \Rightarrow$$ want to be small, then $e \rightarrow$ small!

which is the transfer function from N to $-e$!

the complementary sensitivity function is:

$$T(s) = (1+GD)^{-1}GD$$

which is the transfer function between input r and y $\Longrightarrow$ also the closed loop system transfer function: Can also be written as:

$$T(s) = (1+(GD)^{-1})^{-1}$$

And we can also see:

$$S(s) + T(s) = 1$$

this is the relationship that establishes the inherent restrictions imposed by nature and the fundamental tradeoff available to the control system designer.

What we want is:

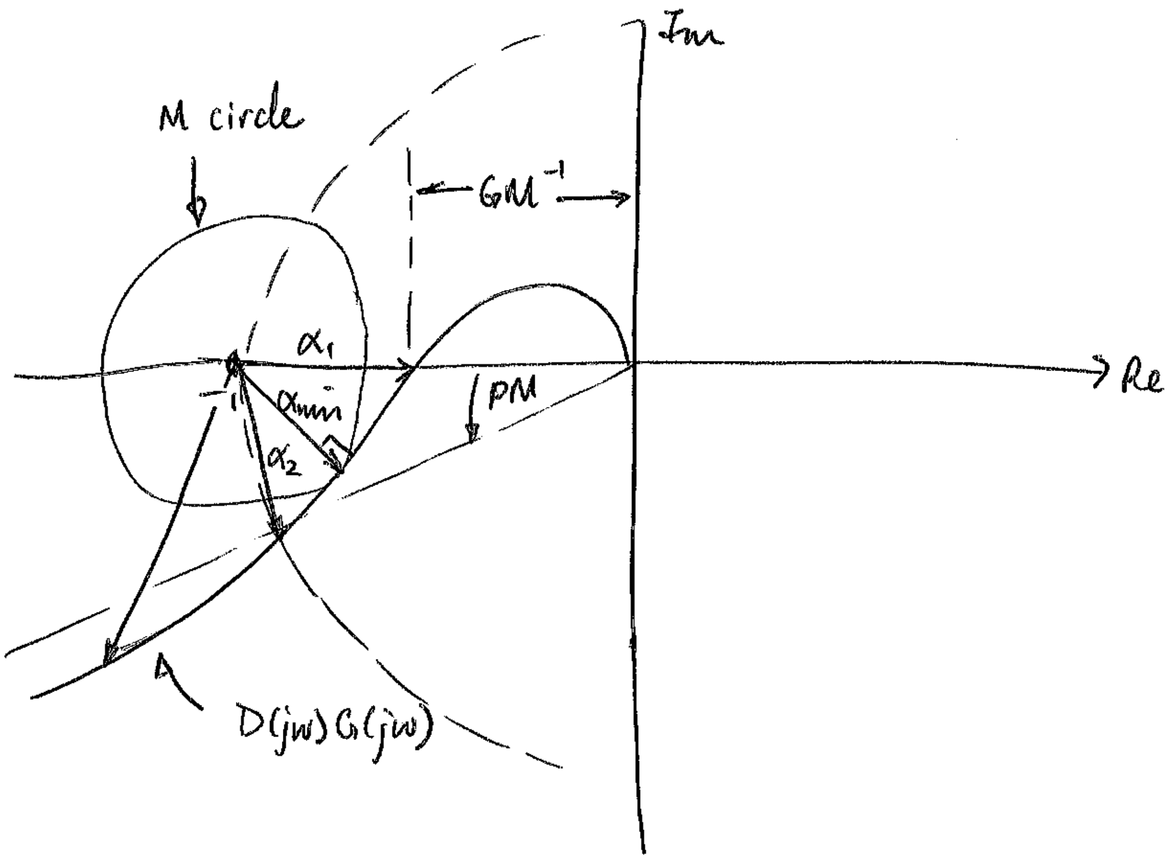
$S(s)$ to be as small as possible.

but we also have to consider the effects of $T(s)$, since

$$T(s) + S(s) = 1$$

<u>Relationship with Nyquist</u>:

$S(s) = (1+GD)^{-1}$ is the inverse distance from $G(j\omega)D(j\omega)$ to the (-1) point.



$$\alpha_{min} = \frac{1}{\max_{w} |S(jw)|} = \min_{w} \frac{1}{|S(jw)|}$$

then we can see that:

$$GM = \frac{1}{1-\alpha_1}$$

and

$$PM = 2\sin^{-1}\left(\frac{\alpha_2}{2}\right)$$

Time Delays

Refers to a delay time between the output and input (i.e. lag)

* Time delays always reduce the stability of a system

Laplace Transform:

time function: $f(t-\lambda)$

where λ = delay time $\lambda \geq 0$

then:

$$F(s) = F(s)e^{-s\lambda}$$

Ex.

$$G(s) = \frac{e^{-5s}}{(10s+1)(60s+1)}$$

Now what about the root locus?

$$1 + KG(s) = 0$$

$$1 + K \frac{e^{-5s}}{(10s+1)(60s+1)} = 0$$

$$\Rightarrow \quad 600s^2 + 70s + 1 + Ke^{-5s} = 0$$

Can't use stand "hand" plotting techniques because of e^{-5s} term, so we approximate using a technique called Padé Approximat of e^{-5s}:

by letting

$$e^{-s} = 1 - s + \frac{s^2}{2} - \frac{s^3}{3!} + \frac{s^4}{4!} + \ldots$$

(McLauren Series)

then we substitute $s = T_d s$ $\quad T_d$ = delay time

and match the term as follows:

$$e^{-s} - \frac{b_0 s + b_1}{a_0 s + 1} = \varepsilon$$

we choose $\varepsilon \cong 0$ (small) then solve for coefficients b_0, b_1 .. etc.

<u>In frequency Response:</u>

look at $G_D(s) = e^{-sT}$ in terms of magnitude and phase!

Magnitude:

$$|G_D(j\omega)| = |e^{-j\omega T}| = |\cos\omega T - j\sin\omega T| = 1$$

for all ($\forall$) ω

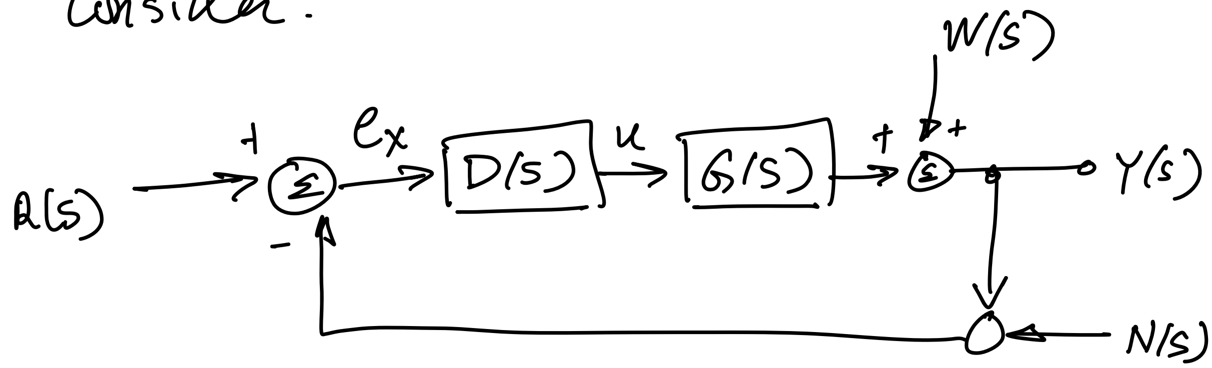
this is expected since the delay is only a shift in time (i.e. phase).

$$\angle G_D(j\omega) = -\omega T$$

which grows as $\omega \to \infty$!

Sensitivity - summary

Consider:



$W \equiv$ disturbance

$N \equiv$ measurement noise

we want to keep error: $e = r - y$ as small as possible! Also, minimize impact of W on the output y.

From block diagram:

$$Y(s) = W(s) + D(s)\,G(s)\,e_x(s)$$

but $e_x(s) = R(s) - Y(s) - N(s)$

$$\Rightarrow \quad Y(s) = W(s) + D(s)G(s)\left[R(s) - Y(s) - N(s)\right]$$

or solving for $Y(s)$, we get:

$$Y(s) = \left[\frac{1}{1 + D(s)G(s)}\right] W(s) + \left[\frac{1}{1 + D(s)G(s)}\right] G(s)D(s)\left(R(s) - N(s)\right)$$

Noting that the error is: $E(s) = R(s) - Y(s)$

we can sub in $Y(s)$ to get:

$$E(s) = \underbrace{\left[\frac{1}{1 + D(s)G(s)}\right]}\left(R(s) - W(s)\right) + \underbrace{\left[\frac{1}{1 + D(s)G(s)}\right]} G(s)D(s)N(s)$$

we call this the "sensitivity" Transfer function

$$S(s) = \left(1 + D(s)G(s)\right)^{-1}$$

we want this to be as small as possible over all frequencies to minimize noise and reject disturbances.

Note that the closed-loop T.F. from R to Y is:

$$\frac{Y(s)}{R(s)} = T(s) = \frac{D(s)G(s)}{1+D(s)G(s)} = \left(1+D(s)G(s)\right)^{-1} D(s)G(s)$$

But $$\left[1+\frac{1}{D(s)G(s)}\right]^{-1} = \left(1+D(s)G(s)\right)^{-1} D(s)G(s)$$

and:

$$S(s)+T(s) = \frac{1}{1+D(s)G(s)} + \frac{D(s)G(s)}{1+D(s)G(s)} = 1$$

So, we are constrained by:

$$\boxed{S(s)+T(s) = 1}$$

Summary

* When designing $D(s)$ for given $G(s)$, $W(s)$ and $N(s)$, we constrained by

$$S(s) + T(s) = 1$$

which is a fundamental tradeoff.

* We want $S(s)$ to be as small as possible, but we need to consider effects of $T(s)$