Intermediate Fluid Mechanics (ME EN 5700/6700)
MidTerm Exam, Fall 2025

Name: **UiD:**

(*Note: The exam is closed notes, book, and notebook. No calculator is needed. Only the provided equation sheet is needed.*)

1. Continuum Hypothesis

(a) [2 points] Explain why the continuum hypothesis is important in the study of fluid dynamics

The continuum hypothesis is important in fluid mechanics because it allows us to examine the dynamics and kinematics of a fluid at the macroscopic level without considering the molecular nature of matter.

(b) [2 points] For a gas, when is the continuum hypothesis valid [Hint: think length scales]?

The continuum hypothesis is valid in a gas when the mean free path of the gas is much ~~larger~~ than the smallest length scale of the fluid flow phenomena of interest.
smaller

2. Write the following vector quantities in index notation.

(a) [2 points] $\vec{u} \times \left(\vec{\nabla} \times \vec{u}\right)$ $\longrightarrow = \varepsilon_{lmi}\varepsilon_{ijk}\, u_m \partial_j u_k$

$\vec{\nabla} \times \vec{u} = \varepsilon_{ijk}\, \partial_j u_k$

$$u_i \frac{\partial u_k}{\partial x_l} \epsilon_{klm} \epsilon_{imn}$$

(b) [2 points] $\vec{\nabla} \cdot \left(\vec{\nabla} \times \vec{u}\right)$

$$\frac{\partial}{\partial x_i} \frac{\partial u_j}{\partial x_k} \epsilon_{jki}$$

(c) [2 points] $\vec{\nabla} \times \vec{\nabla}\phi$

$$\frac{\partial}{\partial x_i} \frac{\partial \phi}{\partial x_j} \epsilon_{ijk}$$

3. Navier-Stokes equations

(a) [5 points] Cauchy's equation of motion is given by: $\rho \frac{Du_i}{Dt} = \rho g_i + \frac{\partial \tau_{ij}}{\partial x_j}$ and represents a system of equations that govern a material's motion. What information about the material is missing that would be required to solve this set of equations?

The equation set is un-closed (more unknowns than equations) and requires a constitutive model relating the stress tensor τ_{ij} to the velocity field u_i.

(b) [8 points] Simplify the following form of the Navier-Stokes equation by assuming the flow is incompressible (show your work).

$$\rho \frac{Du_i}{Dt} = \rho\, g_i - \frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_j}\left[\mu \frac{\partial u_i}{\partial x_j} + \frac{\mu}{3}\frac{\partial u_m}{\partial x_m}\delta_{ij}\right] + F_i \tag{1}$$

For incompressible flow the conservation of mass equation is given by

$$\frac{\partial u_i}{\partial x_i} = 0 \tag{2}$$

and as a result the second part of the viscous term is zero reducing the equation to

$$\rho \frac{Du_i}{Dt} = \rho\, g_i - \frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_j}\left[\mu \frac{\partial u_i}{\partial x_j}\right] + F_i \tag{3}$$

This can be simplified further by realizing that for incompressible flow both ρ and μ are constants resulting in

$$\frac{Du_i}{Dt} = g_i - \frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_j^2} + F_i \tag{4}$$

(c) [2 points] What is the no-slip condition and how does it relate (or becomes useful) to the equation from part b?

The no-slip condition states that the flow velocity at an interface (e.g., surface of a solid object) must be equal to the velocity of the interface. This forms the basis for boundary conditions at interfaces in the equations in part b.

4. Consider the 2D velocity field shown below and given by: $u = x^2$ and $v = -2xy$

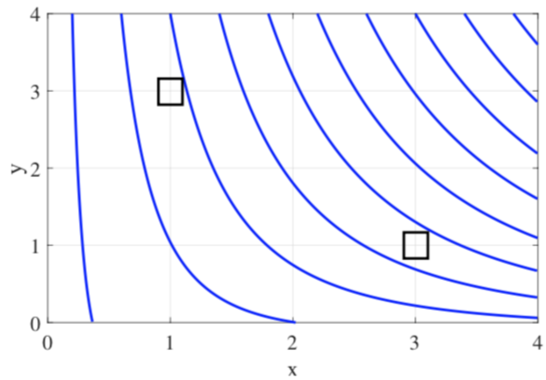

Figure 1: Streamlines of the flow indicated above.

(a) [5 points] Is the flow incompressible (show why or why not)?

For an incompressible flow in 2D,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{5}$$

and for this flow we have,

$$\frac{\partial (x^2)}{\partial x} + \frac{\partial(-2xy)}{\partial y} = 2x - 2x = 0. \tag{6}$$

It is therefore incompressible.

(b) [10 points] Calculate the equation for the streamlines in the flow

By definition in 2D the streamline is given by

$$\frac{dx}{dy} = \frac{u}{v} \quad (2p) \tag{7}$$

which for our velocity field gives,

$$\frac{dx}{dy} = \frac{x^2}{-2xy} = \frac{x}{-2y} \tag{8}$$

this can then be integrated,

$$\frac{dx}{x} = \frac{dy}{-2y} \Longrightarrow \int \frac{dx}{x} = \int \frac{dy}{-2y} \rightarrow (3p) \tag{9}$$

$$ln\left(\frac{x}{x_0}\right) = -\frac{1}{2}\, ln\left(\frac{y}{y_0}\right) \tag{10}$$

$$x = x_0 \left(\frac{y}{y_0}\right)^{-1/2} \longrightarrow 5p. \tag{11}$$

(c) [5 points] Determine the components in the strain-rate tensor, e_{ij} (show your work).

By definition

$$e_{ij} = \frac{1}{2}\Big(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\Big) \tag{12}$$

and for a 2D flow, only four components can exist:

$$e_{11} = \Big(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial x}\Big) = \frac{\partial u}{\partial x} = \frac{\partial (x^2)}{\partial x} = 2x \tag{13}$$

$$e_{12} = \Big(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\Big) = \frac{1}{2}\Big(\frac{x^2}{\partial y} + \frac{\partial(-2xy)}{\partial x}\Big) = -y \tag{14}$$

$$e_{22} = \Big(\frac{\partial v}{\partial y} + \frac{\partial v}{\partial y}\Big) = \frac{\partial v}{\partial y} = \frac{\partial(-2xy)}{\partial y} = -2x \tag{15}$$

(d) [5 points] Determine the components in the vorticity vector, ω_i (show your work).

For a 2D flow only 1 component of the vorticity vector is possible

$$\omega_z = \Big(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\Big) \tag{16}$$

and for our flow,

$$\omega_z = \Big(\frac{\partial(-2xy)}{\partial x} - \frac{\partial(x^2)}{\partial y}\Big) = -2y \tag{17}$$

(e) [10 points] Consider two initially square particles with centroid located at positions $(x, y) = (1, 3)$ and $(x, y) = (3, 1)$. How will the fluid particles look after an infinitesmial time Δt later in each case? [Support your reasoning with calculations]

5p. for the numbers

5p for the correct explantion or drawing.

(f) [6 points] Derive expressions for the components of the acceleration (a_x, a_y) of a generic fluid particle.

For a generic fluid particle the acceleration can be calculated from the material derivative so that we have

$$a_x = \frac{Du}{Dt} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = 0 + x^2\,\frac{\partial (x^2)}{\partial x} + (-2xy)\frac{\partial (x^2)}{\partial y} = 2x^3. \tag{18}$$

$$a_y = \frac{Dv}{Dt} = \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = 0 + x^2\,\frac{\partial (-2xy)}{\partial x} + (-2xy)\frac{\partial (-2xy)}{\partial y} = \tag{19}$$

$$= x^2\,(-2y) + (-2xy)(-2x) = 2x^2 y. \tag{20}$$

5. Consider a planar flow in a channel with porous walls. A constant vertical velocity V_w exists at the top and bottom walls as shown. Flow is driven through the channel by a pressure gradient $\partial P/\partial x = -K$.

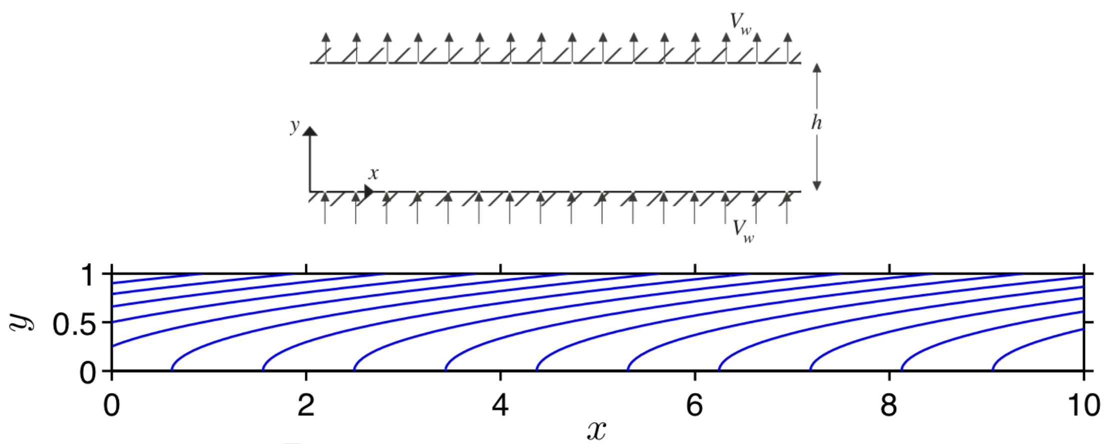

Figure 2: Flow in a channel with porus walls.

(a) [6 points] Use the continuity equation to find the vertical velocity v. [State assumptions]

Assumptions: 1) fully-developed, 2) incompressible

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \rightarrow 1p$$

3p.

fully developed therefore $\partial/\partial x = 0$ so that $\rightarrow v = const.$ Applying our boundary condition that $v(y=0) = V_w \rightarrow const = V_w \rightarrow \underbrace{v = V_w}_{2p.}$.

(b) [10 points] Simplify the Navier-Stokes equation (x-component ONLY) for this flow and provide the appropriate boundary conditions. [State assumptions]

$$\rho\frac{\partial u}{\partial t}+\rho u\frac{\partial u}{\partial x}+\rho v\frac{\partial u}{\partial y}=-\frac{\partial P}{\partial x}+\mu\frac{\partial^2 u}{\partial x^2}+\mu\frac{\partial^2 u}{\partial y^2}$$

assuming: 1) steady state (SS), 2) fully developed (FD) flow, 3) incompressible flow; one obtains that

$$\rho\underbrace{\frac{\partial u}{\partial t}}_{\text{SS}}+\rho u\underbrace{\frac{\partial u}{\partial x}}_{\text{FD}}+\rho v\frac{\partial u}{\partial y}=-\frac{\partial P}{\partial x}+\mu\underbrace{\frac{\partial^2 u}{\partial x^2}}_{FD}+\mu\frac{\partial^2 u}{\partial y^2}$$

missing -3

using $v = V_w$ and $-\partial P/\partial x = K$ we have:

$$\rho V_w\frac{\partial u}{\partial y}=K+\mu\frac{\partial^2 u}{\partial y^2}$$ } 8p.

With boundary conditions: $u(y=0)=0$ and $u(y=h)=0$ } 2p.

(c) [5 points] For the case where $V_w = 0$, explain the role of viscous diffusion in the fluid dynamics.

In this case, the channel acts like a basic "solid wall" channel. The walls act as momentum sinks. The impact of this sink diffuses vertically (away from the walls) towards the centerline due to molecular interactions between neighboring fluid particles. Fluid near the wall slows down and fluid at the channel centerline speeds up to satisfy conservation of mass as a result of this process.

(d) [5 points] How would you expect the streamline pattern to change if the viscosity of the fluid was increased? [Suggestion: write the acceleration term in (b) from the Lagrangian viewpoint, then consider the motion of a fluid particle.]

$$\rho\frac{Du}{Dt}=K+\mu\frac{\partial^2 u}{\partial y^2}$$

If the viscosity increases, then the rate of change of stream wise momentum (LHS in above equation) will increase. The vertical velocity (V_w) remains constant and therefore, the streamlines will appear to flatten out. If μ is large enough the streamlines will flatten out and become horizontal.

(e) [8 points] Solve for the horizontal velocity component.

$$\mu\frac{\partial^2 u}{\partial y^2} - \rho V_w \frac{\partial u}{\partial y} = -K \quad \text{with } u(y=0) = u(y=h) = 0$$

Homogeneous part:

$$\mu\frac{\partial^2 u}{\partial y^2} - \rho V_w \frac{\partial u}{\partial y} = 0$$

$$\text{characteristic polynomial: } \mu\lambda^2 - \rho V_w \lambda = 0 \rightarrow \lambda_1 = 0,\ \lambda_2 = \frac{\rho V_w}{\mu}$$

$$\text{homogeneous solution: } u_h = c_1 + c_2 \text{exp}\left(\frac{V_w y}{\nu}\right)$$

$$\text{particular solution: } u_p = c_3 y + c_4 \text{ (guess this) plug it into ODE } \rightarrow c_4 = 0,\ c_3 = \frac{K}{\rho V_w}$$

$$\text{The total solution is: } u = u_h + u_p = c_1 + c_2 \text{exp}\left(\frac{V_w y}{\nu}\right) + \left(\frac{V_w y}{\nu}\right) y$$

$$\text{From the BCs: } -c_1 = c_2 = \frac{Kh}{\rho V_w}\left(\frac{1}{1 - exp(V_w y/\nu}\right) \rightarrow$$

h

$$u = \frac{Kh}{\rho V_w}\left[\frac{y}{h} - \frac{1 - exp(V_w y/\nu)}{1 - exp(V_w h/\nu)}\right]$$