Run these examples in MATLAB. Save the code and the output (results/figures) in a Word document. Submit it as one pdf file in Canvas. Please refer to your textbook for any missing text.

1. Fourier series of a square wave.

EXAMPLE 4.2 **Fourier series of a square wave**

Consider the square wave of Figure 4.4. This signal is common in physical systems. For example, this signal appears in many electronic oscillators as an intermediate step in the generation of a sinusoid.

We now calculate the Fourier coefficients of the square wave. Because

$$x(t) = \begin{cases} V, & 0 < t < T_0/2 \\ -V, & T_0/2 < t < T_0 \end{cases}$$

from (4.23), it follows that

$$\begin{aligned} C_k &= \frac{1}{T_0}\int_{T_0} x(t)e^{-jk\omega_0 t}\,dt \\ &= \frac{V}{T_0}\int_0^{T_0/2} e^{-jk\omega_0 t}\,dt - \frac{V}{T_0}\int_{T_0/2}^{T_0} e^{-jk\omega_0 t}\,dt \\ &= \frac{V}{T_0(-jk\omega_0)}\left[e^{-jk\omega_0 t}\Big|_0^{T_0/2} - e^{-jk\omega_0 t}\Big|_{T_0/2}^{T_0} \right]. \end{aligned}$$

The values at the limits are evaluated as

$$\omega_0 t\Big|_{t=T_0/2} = \frac{2\pi}{T_0}\frac{T_0}{2} = \pi;\ \omega_0 T_0 = 2\pi.$$

Therefore,

$$\begin{aligned} C_k &= \frac{jV}{2\pi k}(e^{-jk\pi} - e^{-j0} - e^{-jk2\pi} + e^{-jk\pi}) \\ &= \begin{cases} -\dfrac{2jV}{k\pi} = \dfrac{2V}{k\pi}\angle -90^\circ, & k \text{ odd} \\ 0, & k \text{ even} \end{cases}, \end{aligned} \tag{4.24}$$

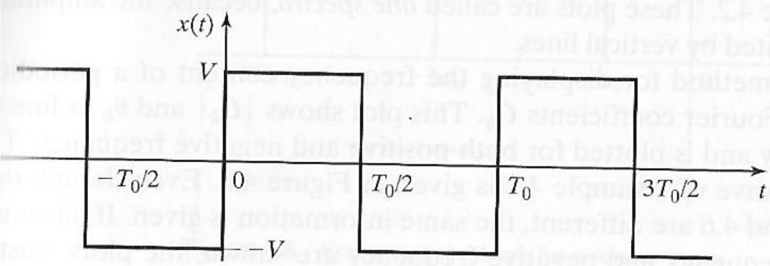

Figure 4.4 Square wave with amplitude V.

with $C_0 = 0$. The value of C_0 is seen by inspection, since the square wave has an average value of zero. Also, C_0 can be calculated from (4.24) by L'Hôpital's rule, Appendix B.

The exponential form of the Fourier series of the square wave is then

$$x(t) = \sum_{\substack{k=-\infty \\ k\,\text{odd}}}^{\infty} \frac{2V}{k\pi} e^{-j\pi/2} e^{jk\omega_0 t}. \tag{4.25}$$

The combined trigonometric form is given by

$$x(t) = \sum_{\substack{k=1 \\ k\,\text{odd}}}^{\infty} \frac{4V}{k\pi} \cos(k\omega_0 t - 90^\circ) \tag{4.26}$$

from (4.13). Hence, the first harmonic has an amplitude of $4V/\pi$, the third harmonic $4V/3\pi$, the fifth harmonic $4V/5\pi$, and so on. The calculation of C_1 is verified with the MATLAB program

```
syms C1 ker t
w0=2*pi; k=1;
ker=exp (-j*k*w0*t);
C1=int (ker, 0, 0.5) +int (-ker, 0.5, 1)
double (C1)
```

Equation (4.26) is easily converted to the trigonometric form, because $\cos(a - 90^\circ) = \sin a$. Hence,

$$x(t) = \sum_{\substack{k=1 \\ k\,\text{odd}}}^{\infty} \frac{4V}{k\pi} \sin k\omega_0 t.$$

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2. Fourier Series for an impulse train

EXAMPLE 4.5 **Fourier series for an impulse train**

The Fourier series for the impulse train shown in Figure 4.10 will be calculated. From (4.23),

$$C_k = \frac{1}{T_0}\int_{T_0} x(t)e^{-jk\omega_0 t}dt$$

$$= \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} \delta(t)e^{-jk\omega_0 t}\,dt = \frac{1}{T_0}e^{-jk\omega_0 t}\Big|_{t=0} = \frac{1}{T_0}.$$

This result is based on the property of the impulse function

[eq(2.41)] $$\int_{-\infty}^{\infty} f(t)\delta(t-t_0)\,dt = f(t_0),$$

provided that $f(t)$ is continuous at $t = t_0$. The exponential form of the Fourier series is given by

$$x(t) = \sum_{k=-\infty}^{\infty} \frac{1}{T_0}e^{jk\omega_0 t}. \quad (4.27)$$

A line spectrum for this function is given in Figure 4.11. Because the Fourier coefficients are real, no phase plot is given. From (4.13), the combined trigonometric form for the train of impulse functions is given by

$$x(t) = \frac{1}{T_0} + \sum_{k=1}^{\infty}\frac{2}{T_0}\cos k\omega_0 t.$$

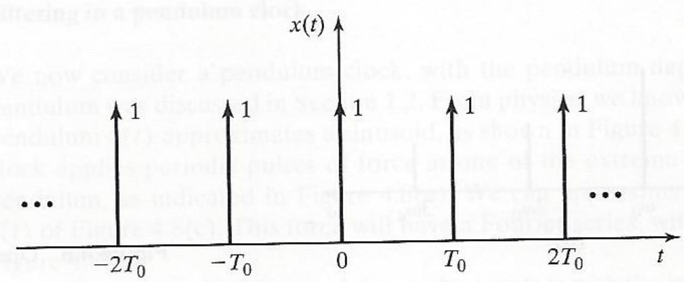

Figure 4.10 Impulse train.

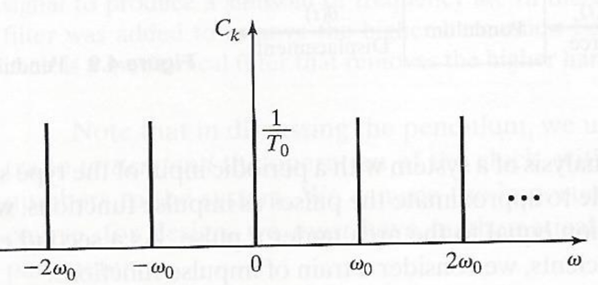

Figure 4.11 Frequency spectrum for impulse train.

Note that this is also the trigonometric form.

A comparison of the frequency spectrum of the square wave (Figure 4.6) with that of the train of impulse functions (Figure 4.11) illustrates an important property of impulse functions. For the square wave, the amplitudes of the harmonics decrease by the factor $1/k$, where k is the harmonic number. Hence, we expect that the higher harmonics can be ignored in most situations and that a finite sum of the harmonics is usually adequate to represent a square wave. This statement cannot be applied to the impulse train. The amplitudes of the harmonics remain constant for all harmonic frequencies. Hence, usually all harmonics must be considered for a train of impulse functions. This point is considered further in the next section.

Given in Table 4.3 are the Fourier coefficients of seven periodic signals that are important in engineering applications. Since the coefficient C_0 is the average, or dc, value of the signal, this value is not unique for a particular form of a periodic signal. For example, if we add a constant value to a sawtooth signal, the result is still a sawtooth signal, with only the average value C_0 changed. This point is covered in greater detail in Section 4.6.

A MATLAB program that verifies the first three coefficients of the triangular wave in Table 4.3 is

```
syms Ck ker t
for k=1 : 3
    w0=2*pi;
    ker=exp (-j*k*w0*t);
    Ck=int (2*t*ker, 0, 0.5) + int (2* (1 - t) * ker, 0.5, 1);
    simplify (Ck)
end
```

This program can also be written in the general variable k, but the results must be simplified. The coefficients of the remaining signals in Table 4.3 can be derived by altering this program in an appropriate manner.

As a final example in this section, we consider the important case of a train of rectangular pulses.

3. Generate a sinc function in Matlab.

4. Rectangular function and its Fourier Transform

EXAMPLE 5.2 **A MATLAB program to create a rectangular pulse and its Fourier Transform**

```
MATLAB program for new example in Section 5.1
>> syms t w
% Create and plot x(t) = rect(t/2).
>> x=4*(heaviside(1-t)-heaviside(-1-t))
>> ezplot(x,-4,4)
% Compute and plot X(ω), the Fourier transform of x(t) = rect(t/2).
>> X=fourier(x)
>> figure(2), ezplot(X,-6,6),hold
```

5. The time-scaling property of the Fourier Transform

EXAMPLE 5.4 **The time-scaling property of the Fourier transform**

We now find the Fourier transform of the rectangular waveform

$$g(t) = \mathrm{rect}(2t/T_1).$$

From the result of Example 5.1,

$$[\mathrm{eq}(5.4)] \qquad V\,\mathrm{rect}\,(t/T) \;\overset{\mathscr{F}}{\longleftrightarrow}\; TV\,\mathrm{sinc}\,(T\omega/2),$$

it is seen that $g(t)$ is simply the particular case where $T = T_1$, the time-scaling factor $a = 2$, and $V = 1$, as shown in Figure 5.6(a). Applying the time-scaling property to the transform pair obtained in (5.4) results in

$$g(t) = f(2t),$$

where

$$f(t) = \mathrm{rect}\,(t/T_1).$$

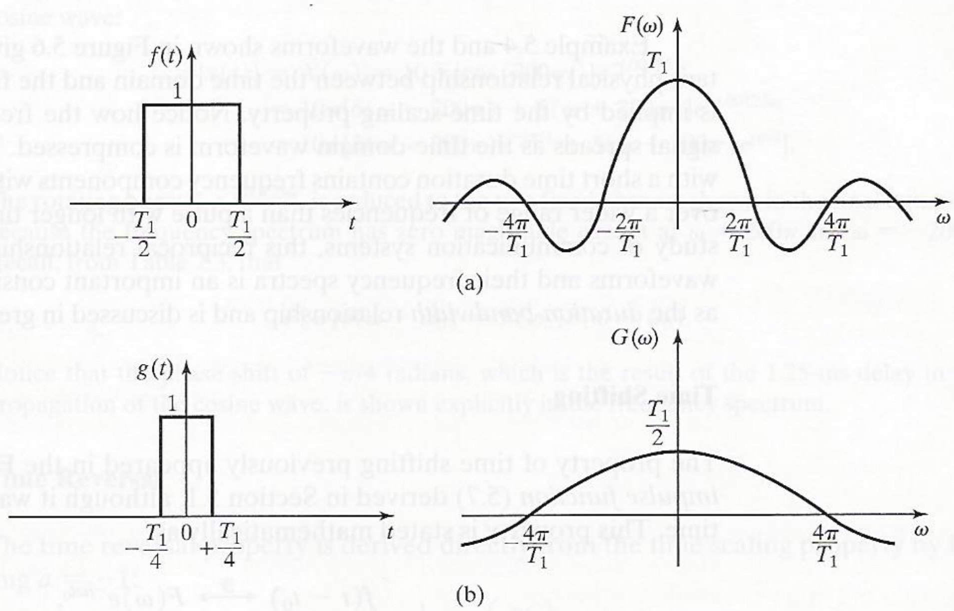

Figure 5.6 Rectangular pulses and their frequency spectra.

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Therefore, from (5.12),

$$G(\omega) = \frac{1}{2} F(\omega/2) = \frac{T_1}{2} \operatorname{sinc}\left(\frac{\omega T_1}{4}\right).$$

This result is illustrated in Figure 5.6(b).

The following MATLAB program illustrates the duration–bandwidth relationship:

```
% This program displays the relationship of the magnitude frequency
spectra of two
% rectangular pulses to illustrate the time-scaling property
syms t w
% Create a rectangular pulse with amplitude and pulse-width=1.
T=0.5;
x1=(heaviside (T-t)-heaviside(-T-t));
% Plot the rectangular pulse, x1(t).
figure(1), ezplot(x1, [-3,3]), grid, title('x1=rect (t/1)')
%
% Create and plot a rectangular pulse with amplitude=1 and pulse-
% width = 0.2.
T=0.1
x2=(heaviside(T-t)-heaviside(-T-t));
figure (2), ezplot (x2, [-1,1]), grid, title ('x2 = rect (t/0.2)')
%
% Compute and plot the Fourier transforms of the two rectangular pulses.
X1=fourier(x1);
X2=fourier(x2);
ezplot (X1,-20,20), grid, title ('X1=F(x1)')
ezplot (X2,-100,100), grid, title ('X2=F(x2)')
```

Example 5.4 and the waveforms shown in Figure 5.6 give insight into an important physical relationship between the time domain and the frequency domain, which is implied by the time-scaling property. Notice how the frequency spectrum of the signal spreads as the time-domain waveform is compressed. This implies that a pulse with a short time duration contains frequency components with significant magnitudes over a wider range of frequencies than a pulse with longer time duration does. In the study of communication systems, this reciprocal relationship between time-domain waveforms and their frequency spectra is an important consideration. This is known as the *duration-bandwidth* relationship and is discussed in greater detail in Chapter 6.