Phillip J. Curtiss, Associate Professor
Department of Computer Science, Montana Tech

The *Quickhull* algorithm finds the points in $\Re^2$, from a collection of n points provided, that form a *convex hull* - that is a regular polygon that *encloses* the provided points that do not form the convex hull.

The algorithm accomplishes this through using a *divide and conquer* strategy whereby:

- the n points provided are sorted based on the value of their x-coordinate
- it finds the extreme points P_1 and P_n by examining the x-coordinate of all the provided points - these points will form the starting points of the convex hull. Any points co-linear with this line segment and not the end-points will not be part of the convex hull.
- this line segment divides the remaining set of points into two (2) groups, a group of points "to the left" of the line segment (upper hull), and a set of points "to the right" of the line segment (lower hull). The two sets of points can now be treated in exactly the same manner for further dividing the problem into smaller instances:
  - in the upper hull, locate a point P_{max} such that its distance from the line $\overrightarrow{P_1P_n}$ is the largest. This forms a triangle from the line segments $\overrightarrow{P_1P_{max}}$, $\overrightarrow{P_{max}P_n}$, and $\overrightarrow{P_1P_n}$ or $\triangle P_1P_{max}P_n$, such that points in the interior of this triangle cannot be point lying on the convex hull.
  - Each of these new line segments that in the point P_{max} will further divide the set of points in the upper hull into two smaller groups of points. The same procedure is done in this new smaller upper and smaller lower hull until there are no points in the upper hull of such a division.
  - When this occurs, the points forming the line segment $\overrightarrow{P_1P_{max}}$ for the specific smaller instance of the problem must be points on the convex hull.
- The procedure of dividing and locating points forming the convex hull continues until there are no more points to divide, resulting in a set of points comprising the convex hull for the original instance of the problem.

Computing Leftness or Rightness

Whether a point is to the left or right of a line segment can be formally computed by considering the sign of the *determinant* of the three (3) points - the two end points of the line segment, and the third candidate point. Given three points $q_1(x_1, y_1)$, $q_2(x_2, y_2)$, and $q_3(x_3, y_3)$ each in $\Re^2$, one can compute the determinant by forming a matrix of the coordinates such that the first column has the x-coordinates for the three points, the second column has the y-coordinates for the three points, and the last column has all 1s, as follow:

$$
\begin{aligned}
2S = \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} &= x_1 \cdot [(y_2 \cdot 1) - (1 \cdot y_3)] - y_1 \cdot [(x_2 \cdot 1) - (1 \cdot x_3)] + 1 \cdot [(x_2 \cdot y_3) - (y_2 \cdot x_3)] \\
&= x_1y_2 - x_1y_3 - y_1x_2 + y_1x_3 + x_2y_3 - y_2x_3 \\
&= x_1y_2 + x_2y_3 + x_3y_1 - x_1y_3 - x_2y_1 - x_3y_2 \\
&= (y_3 - y_1) \cdot (x_2 - x_1) - (y_2 - y_1) \cdot (x_3 - x_1)
\end{aligned}
$$

Where S is the area of the triangle formed by the points q_1, q_2, and q_3. If $S > 0$, then the point $q_3(x_3, y_3)$ is to the left of the line segment $\overrightarrow{q_1q_2}$.

Computing the Distance

Once you have the area, S, of the triangle formed by three points, $\triangle q_1q_2q_3$, the magnitude of the area must also be equal to $\frac{1}{2}bh$ of the right angle formed by the normal projection of point q_3 onto the line segment

$\overrightarrow{q_1 q_2}$. Therefore, $h = 2S/b$, with h being the shortest distance from $q_3(x_3, y_3)$ to the line segment $\overrightarrow{q_1 q_2}$. The base, b, is just the distance between the two points q_1 an q_2, or $b = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

Therefore, the distance for q_3 to the line segment $\overrightarrow{q_1 q_2}$ is given by,

$$h = \left\| \frac{(y_3 - y_1) \cdot (x_2 - x_1) - (y_2 - y_1) \cdot (x_3 - x_1)}{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}} \right\|$$

Concluding Remarks

Even though you will need some way to determine if a given point is on the "left" side of a line segment dividing the upper and lower hulls for each instance, you are not required to use the method described here. Similarly, in terms of finding the maximum point away from the line segment dividing the upper and lower hulls for each instance, you are not required to use the method described herein.

Which ever method you use, make sure you prove that the methodology is going to work with the information provided at each smaller instance of the *quickhull* algorithm and does not depend on information that is not tracked by the algorithm.