Title: **Methods for First-Order ODEs**

Authors: **William Franzen, Noah Harbor, Brandon Mitchell, Logan Reed**

Abstract

Sometimes, it can be difficult or impossible to find the solution to a differential equation. In such cases, it can be useful to approximate the solution using numerical methods such as Euler's or Runge-Kutta. Specifically, Euler's, improved Euler's, Midpoint, Runge-Kutta, and Taylor's methods will be explored and compared. These methods' accuracies and the things going for and against them will be discussed.

Method Comparison

Euler's method, the Midpoint method, and the fourth-order Runge-Kutta method were used to approximate the solution of the differential equation $y' = y + 2x - x^2$, with initial value $y(0) = 1$, over the interval $0 \leq x \leq 2$. These methods were all executed with h = 0.05, with the addition of a more precise variation of the Euler's method, which used h = 0.001. These approximations were then compared to the exact solution, $y(x) = x^2 + e^x$, and the absolute value of the error was determined.

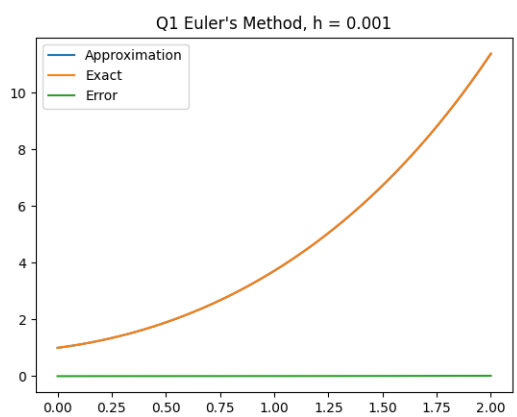


Figure 1: Euler's Method, h=0.001

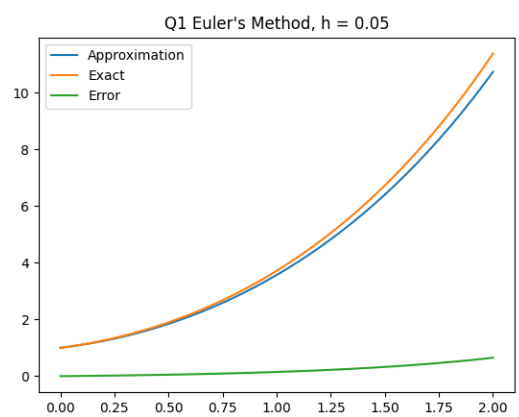


Figure 2: Euler's Method, h=0.05

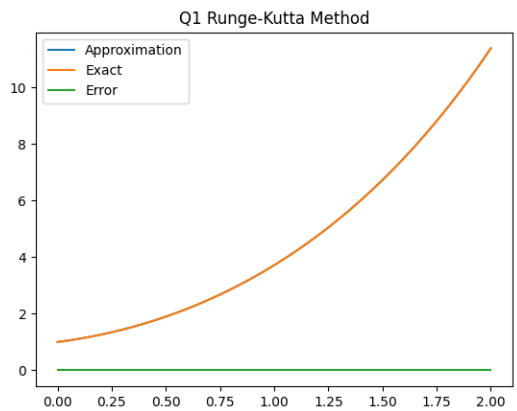


Figure 3: Runge-Kutta Method, h=0.05

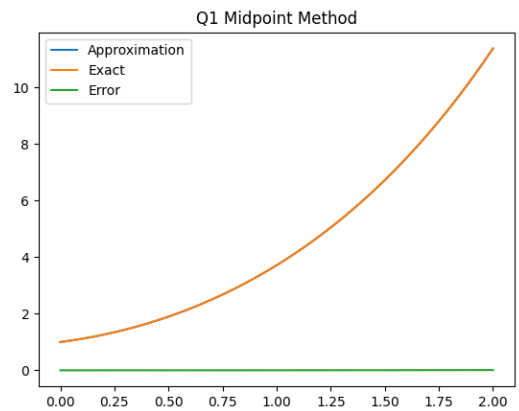


Figure 4: Midpoint Method, h=0.05

All of these methods produced results with small amounts of error. However, Euler's method had some noticeable errors in the later iterations. This result is as expected, because Euler's method is a first order method, meaning the error is proportional to h. The Midpoint method is of order two, and the Runge-Kutta Method is of order 4, which yields errors proportional to h^2 and h^4 respectively. When the step size is the same for each method, and less than one, the error will decrease as the order increases. When using the smaller h value 0.001, Euler's method becomes more accurate. This also makes sense, as the error is directly proportional to the step size.

To further analyze the differences in accuracies for each method, the solutions to the initial value problem, $y' = -y + x + 1$ with $y(-2) = -2 + e^2$ on the interval $-2 \leq x \leq 3$ were approximated using Euler's method, the Midpoint method, the fourth-order Runge-Kutta method, and Taylor's method of order two. The graphs shown below represent each method approximation, the exact solution of the initial value function, and the absolute value of the error of the approximation.

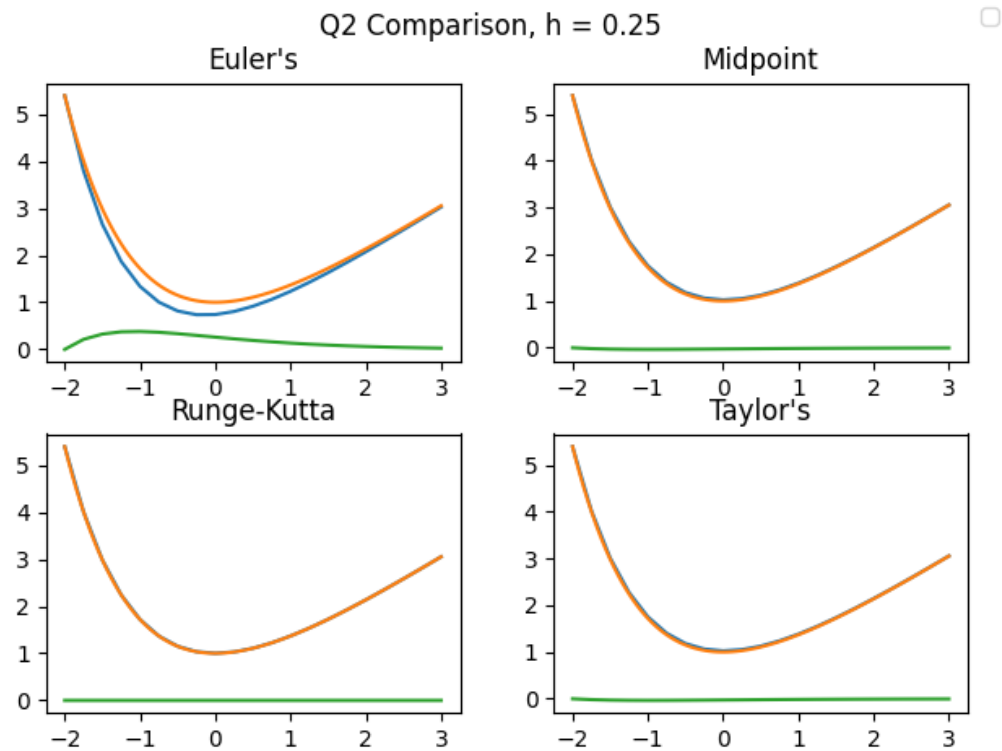


Figure 5: Method Comparison, h=0.25

Once again, it is shown that Euler's method is the least accurate method of approximation. The errors presented by the other three methods are very nearly zero. The errors of the Midpoint and Taylor's methods were very similar, which is as expected, since both methods are order two. Runge-Kutta again was the most accurate method used. The table below shows the error values for each method.

Table 1:Absolute Errors

X	Euler's	Midpoint	Runge-Kutta	Taylor's
-2	0	0	0	0
-1.75	0.212810601807	0.018097401283	0.000057713542	0.018097401283
-1.5	0.325345014689	0.028232865044	0.000089895154	0.028232865044
-1.25	0.373084915725	0.033033554555	0.000105016151	0.033033554555
-1	0.380338297156	0.034356071554	0.000109049428	0.034356071554
-0.75	0.363542368135	0.033498342773	0.000106160257	0.033498342773
-0.5	0.333628034342	0.031355572569	0.000099213727	0.031355572569

It is important to acknowledge that accuracy may not be the only determining factor in which method is best to approximate an initial value problem. For the Runge-Kutta method, five equations must be solved to determine the next y value. This process could be very time consuming, expensive, and tedious, depending on the specific problem being evaluated and the

resources available. Similarly, the Taylor method requires partial derivatives of both x and y for the function. This method might not be the best suited for complicated functions, especially when the Midpoint method could be utilized more quickly and produce a similar error. Finally, any of these methods, but specifically Euler's method, can experience a reduction in truncation errors if the step size is shortened. Once again, this might not be the best solution, depending on the situation, as more iterations are required. Overall, the desired degree of accuracy should be weighed against the time, skill, and resources required to complete the approximation.

Taylor's Method of Order Two

Taylor's method of order two was used to approximate the solution to the IVP $y' = -y + x + 1$ with $y(-2) = -2 + e^2$ on the interval $-2 \leq x \leq 3$. The method was ran three times with these values. However, the h, or step, value was varied from 0.2 to 0.1 to 0.05.

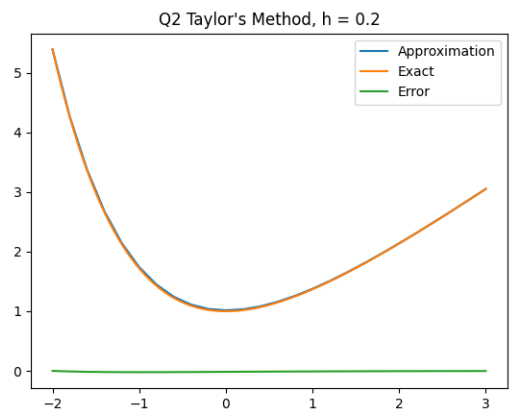


Figure 6: Taylor's Method, h=0.2

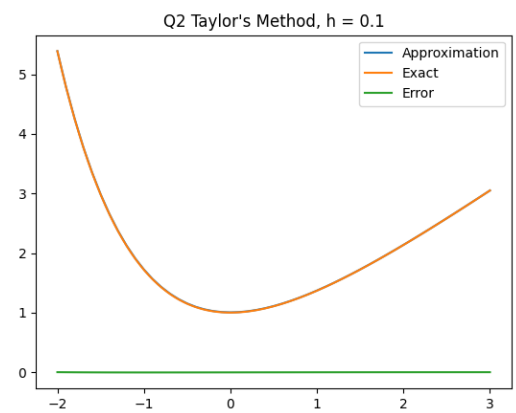


Figure 7: Taylor's Method, h=0.1

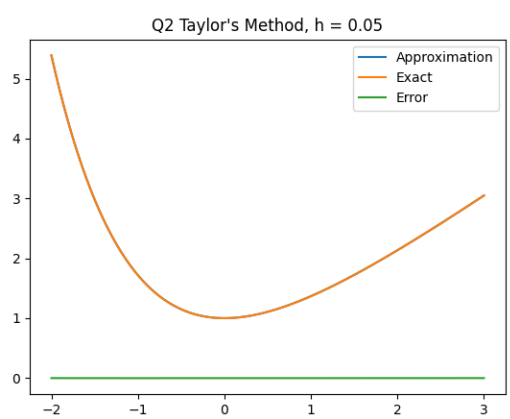


Figure 8: Taylor's Method, h=0.05

Unfortunately, the graphs do not do a great job in showing the change in error for each step size. For each step size, it appears to be zero. However, when the error is examined directly, it becomes clear that the lower the h value, the lower the error. Below are the error values at the first five points for each h value. Only the first five are included for brevity, but the values can be found in the output of the program.

Table 2: Taylor's Method, h = 0.2

-2	0.0
-1.8	0.00937853671018729
-1.6	0.015368896525854048
-1.4	0.018889116310520482
-1.2	0.020636125450711695

Table 3: Taylor's Method, h = 0.1

-2	0.0
-1.9	0.001201327252969619
-1.8	0.002174207013730722
-1.7	0.002951220913942620
-1.6	0.003560820045120039

Table 4: Taylor's Method, h = 0.05

-2	0.0
-1.95	0.000152033518487471
-1.9	0.000289240640757526
-1.85	0.000412705775843846
-1.8	0.000523442831318554

As one can see, the graphs with a lower h value have lower error values, typically by an order of magnitude. This makes sense as a function consists of many small lines. If more lines are used, the resulting function will be more accurate overall as it can better fit to the solution curve. However, this increased accuracy comes at a cost. Four times as many points are calculated for h = 0.05 when compared to h = 0.2. These extra calculations could be time consuming to do by hand or potentially computationally expensive.

Fourth-Order Runge-Kutta Method and Asymptotes

The Runge-Kutta method was used to approximate the following IVP: $y' = y^2;\ y(0) = 1$ for the interval $0 \leq x \leq 2$ where the interval $h = 0.25$. The exact solution is $y(x) = \frac{1}{1-x}$. You can see the approximation compared to the exact solution in Figure 9.

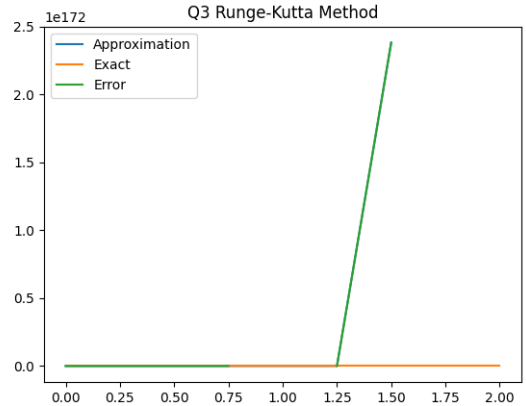


Figure 9: Runge-Kutta Method, Complete View

The IVP in question grows exponentially. As such, the approximation eventually shot way up to the point of an overflow error. This is why the approximation stops at 1.50. Note the scale of the axis. To better show what's going on, a zoomed in version can be seen in Figure 10.

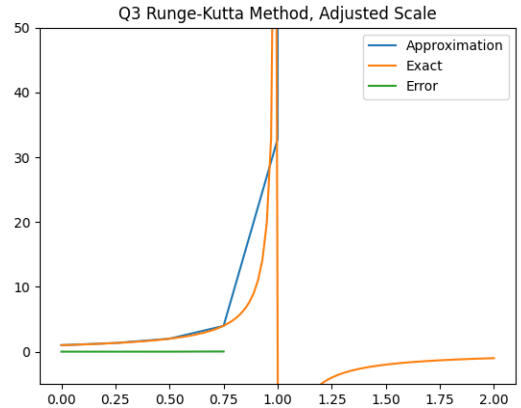


Figure 10: Runge-Kutta Method, Adjusted Scale

The asymptote is much more visible at this scale. Due to the asymptote and the IVP growing exponentially, the error part of the graph doesn't show up very well. Runge-Kutta and the other methods are meant to solve ordinary differential equations. As such, there shouldn't be any discontinuities in the solution being approximated. These methods rely on finding a tangent line and using that to estimate the next point, so discontinuities like this in the solution could result in odd behavior.

This same IVP was tested with Euler's method. The same exponential increase can be seen, though it takes place a little later. This could be due to the fact that Euler's method is less accurate so it has a bit of a harder time tracking the asymptote.

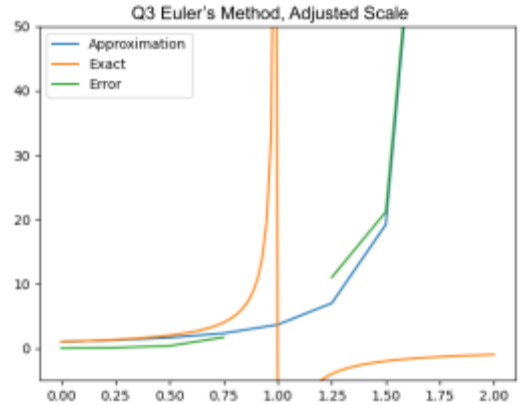


Figure 11: Euler's Method, Adjusted Scale

Trigonometric Functions and Euler's

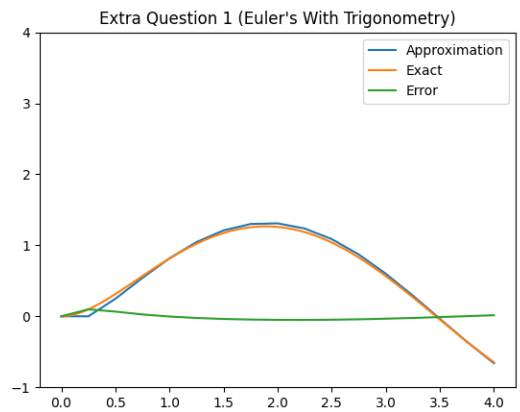


Figure 12: Extra Question 1- Euler's With Trigonometry

Euler's method was tested on a trigonometric IVP, $y' = 4sin(x) - 3y,\ y(0) = 0$, with a step size of 0.25. The exact solution is $2/5 * (e^{-3x} + 3sin(x) - cos(x)$. The estimated values were very close to the original values, but the errors are visible on the graph when they are compared to each other. Overall, Euler's method appears to be a reliable method to solve trigonometric IVPs.

Logarithmic Functions and Euler's

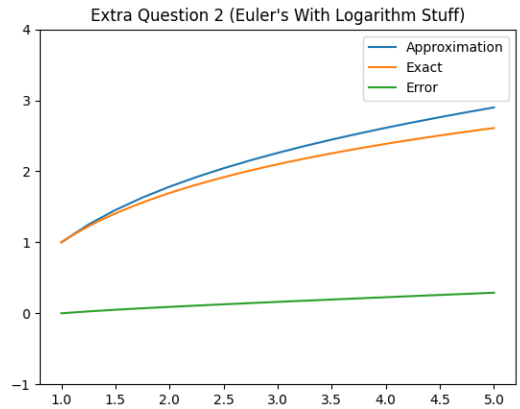


Figure 13: Extra Question 2- Euler's With Logarithmic Functions

Euler's method was tested on an IVP that used logarithm functions, $y' = (y - ln(x))/x,\ y(1) = 1$, with a step size of 0.25. The solution is $ln(x) + 1$. For this logarithmic function, the estimated values in the earlier steps started very close to the exact function. However, after each iterative step, the approximations became more detached from the expected values. The error term appears to change at a near-linear rate. Euler's method appears to be good for early iterations in a logarithmic function, but it slowly becomes less reliable after each iteration.

Midpoint Method and Step Size

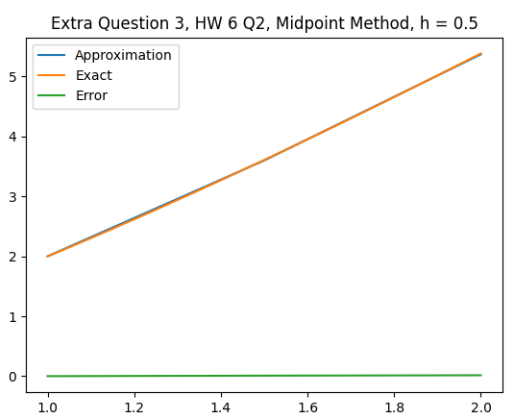


Figure 14: Midpoint Method, h=.05, HW 6 Q2

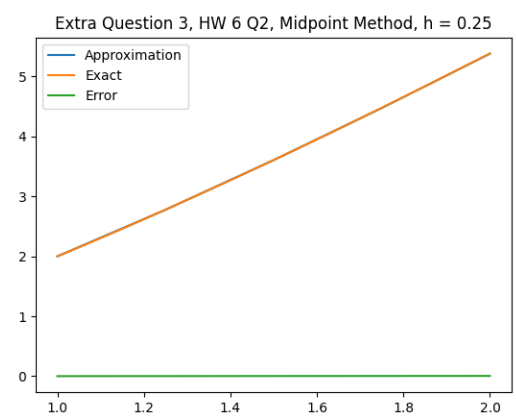


Figure 15: Midpoint Method, h=.25, HW 6 Q2

The Midpoint method was tested on the IVP $y' = 1 + (y/x),\ y(1) = 2$. The solution is $y = xln(x) + 2x$. This is the second problem on homework assignment six. Two different h values were used, 0.25 and 0.5, to see how step size affected the accuracy of the Midpoint

method. Since the graphs do not display the differences clearly, the values can be seen in Table 5 and Table 6 below.

Table 5: Midpoint Method, h = 0.25

1	0.0
1.25	0.001151661364984679
1.5	0.0021370561016405
1.75	0.003026152585514197
2	0.003854578680108744

Table 6: Midpoint Method, h = 0.5

1	0.0
1.5	0.008197662162246377
2	0.01486578969131891

As expected, the error varied between the two outputs when the step size varied. However, they are still relatively close. Since this problem came from the homework, it may not be the best IVP to use when doing such a comparison. An IVP with sharp changes in y-values may do a better job of showcasing how the error of the Midpoint method varies with step size.

Euler's vs Improved Euler's

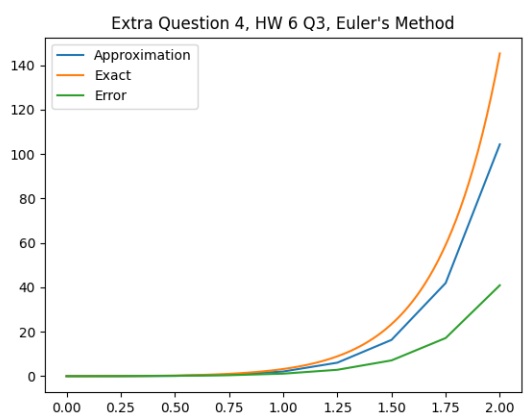


Figure 16: Euler's Method, HW 6 Q3

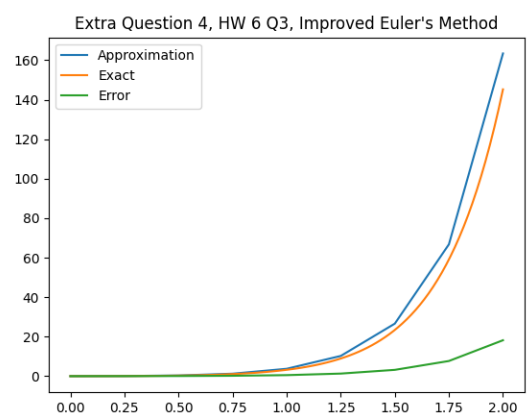


Figure 17: Improved Euler's Method, HW 6 Q3

Euler's method and improved Euler's method were tested on the IVP $y' = xe^{3x} - 2y,\ y(0) = 0$ with a step size of 0.25. The solution is $1/5 * xe^{3x} - 1/25 * e^{2x} + 1/25 * e^{-2x}$. This was taken from question three on homework six. In the early iterative steps for this function, the estimated values were very close to the exact values. However, towards the end, the estimated values started to diverge from the exact values. When the error terms of the two methods are compared, it can be seen that the improved Euler's

method had a smaller error term than Euler's method. Another interesting point is that the improved Euler's method seems to underestimate the function, whereas Euler's method tends to overestimate the function. This could be important to keep in mind when applying these approximation techniques to real life problems.

Asymptote Explorations

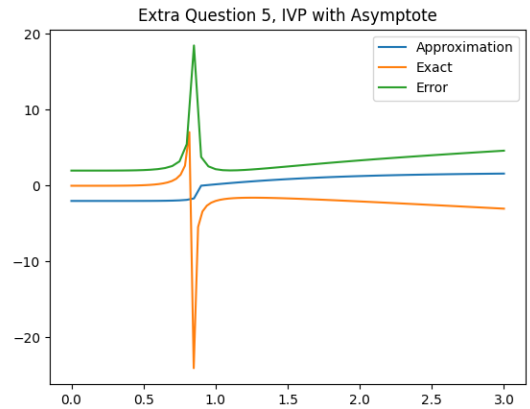
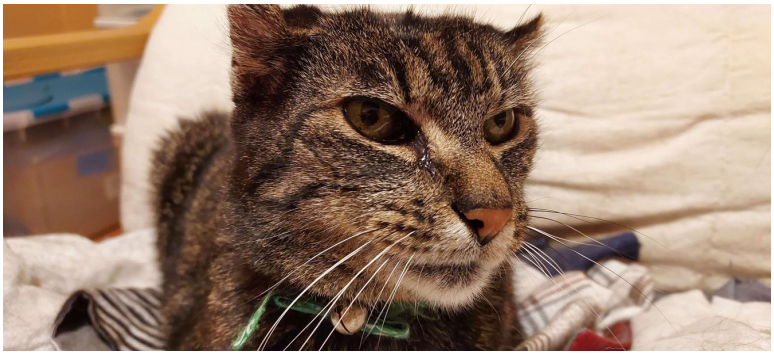
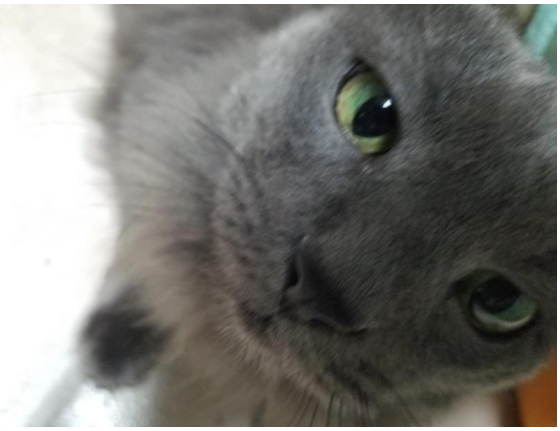


Figure 18: Extra Question 5- Euler's with an Asymptote

After messing around with problem three and seeing how a discontinuous function can affect the graph, Euler's method was tested on an IVP $y' = 2x / (y + x^{2y})$, $y(0) = -2$ with a step value of 0.5. The solution $2x^5/(-2x^4 + 1)$ has an asymptote at around 0.82. Overall, the estimated values were far off from the exact values. The error term is significantly large at every point in the graph. The same behavior from last time is not seen. Perhaps this is because the step value doesn't intersect with the asymptote. Nonetheless, it does behave oddly around the asymptote. Overall, Euler's method doesn't appear to be very reliable on noncontinuous functions.

Cats





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