Constructing the Jones Polynomial

An Introduction to Knot Invariants

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UTM MCS Reading Course Showcase
April 7, 2025

Table of Contents

Knot Diagrams & Crossing Structure

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Dealing with knots as 3D objects can be challenging, so we instead consider projections of a knot onto a 2D plane; this is called its **knot diagram**.

To do this, we project the knot onto the plane so that the knot has no "degenerate" intersections. At each intersection, we assign a **crossing structure** to indicate which strand is above the other. If the knot is oriented, its diagram will denote the orientation with an arrow.

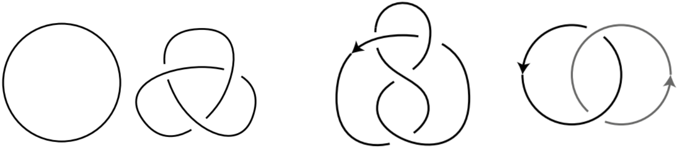


Figure: Some Examples of Knot Diagrams

The Reidemeister Moves

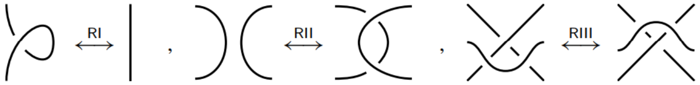
We need to define when two knot diagrams are equivalent.

The Reidemeister Moves

We need to define when two knot diagrams are equivalent. This is where the Reidemeister Moves come in:

Definition

The **Reidemeister Moves** are given by **planar isotopy** (any move which doesn't change the crossing structure), as well as the moves **RI, RII, RIII**, shown below.



RI RII RIII

Reidemeister's Theorem

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Showing two knots are equivalent via Reidemeister moves can be challenging...

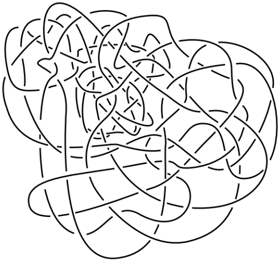

Figure: This knot is equivalent to the unknot...

What is a Knot Invariant?

Knot invariants are a much more efficient way of determining if two knots are not equal:

A knot invariant is a function that assigns to each knot a value. If two knots are assigned to different values, then they have to be not equal.

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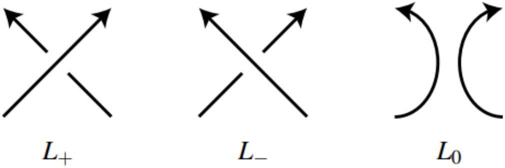
A knot invariant is a function that assigns to each knot a value. If two knots are assigned to different values, then they have to be not equal.

To prove that a function is a knot invariant, it suffices to show that it outputs the same value on two knots related by a single Reidemeister Move.

Changing a Crossing

Let L be an oriented knot and let c be some crossing in L. Then the knots L_+, L_-, and L_0 are the same as L except at c, where c has been changed as follows:



The Jones Polynomial

Definition

Let L be a knot, with L_+, L_-, L_0 as before, and let $\mathscr{O}$ denote the unknot. Then the **Jones Polynomial**, $J(L)$, is calculated using the following relations:

$$t^{-1}J(L_+) - tJ(L_-) = (t^{1/2} - t^{-1/2})J(L_0) \tag{1}$$

$$J(\mathscr{O}) = 1 \tag{2}$$

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Lemma

If L is a knot made of k disjoint unknots, then

$$J(L) = (-t^{-1/2} - t^{1/2})^{k-1}$$

Example: Hopf Link

$t^{-1} J \Bigg($  $\Bigg) = t J \Bigg($  $\Bigg) + \left(t^{\frac{1}{2}} - t^{-\frac{1}{2}}\right) J \Bigg($  $\Bigg)$

$= t J \Bigg($   $\Bigg) + \left(t^{\frac{1}{2}} - t^{-\frac{1}{2}}\right) J \Bigg($  $\Bigg)$

$$= t\left(-t^{-\frac{1}{2}} - t^{\frac{1}{2}}\right) + \left(t^{\frac{1}{2}} - t^{-\frac{1}{2}}\right),$$

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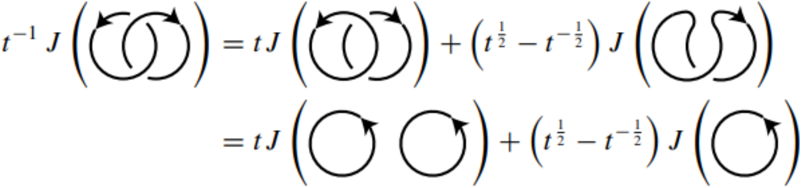

$$= t\left(-t^{-\frac{1}{2}} - t^{\frac{1}{2}}\right) + \left(t^{\frac{1}{2}} - t^{-\frac{1}{2}}\right),$$

Multiplying both sides by t and simplifying gives us $-t^{\frac{5}{2}} - t^{\frac{1}{2}}$.

Crossing Signs

If our knot is oriented, we can say if a crossing is an **over** or **under crossing** and give it a sign; under crossings are assigned a -1, whereas over crossings are assigned a $+1$.

The Writhe

Definition

Let D be an oriented knot diagram. Then the **writhe** of D, denoted $\omega(D)$, is the sum of the signs of all crossings in D.

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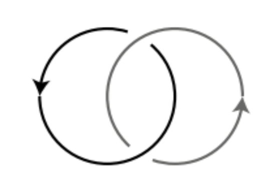
Example

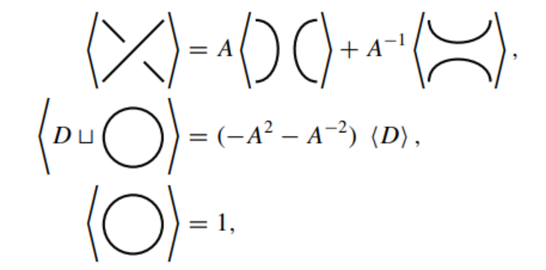
Let D be the Hopf Link. Then $\omega(D) = -2$. If we reverse the orientation of the gray circle, then $\omega(D) = 2$.



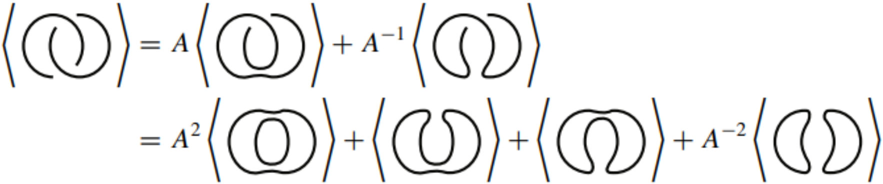
The Kauffman Bracket

Definition

Let D be a knot diagram. The **Kauffman Bracket** of D, denoted $\langle D \rangle$, is given by the following relations:



where each knot diagram on line 1 is the same except around a particular crossing that is changed as shown.



$$
\begin{aligned}
&= A^2(-A^2 - A^{-2}) + 1 + 1 + A^{-2}(-A^2 - A^{-2}) \\
&= -A^4 - A^{-4}.
\end{aligned}
$$

Modifying the Kauffman Bracket

Is the Kauffman Bracket an invariant of oriented knots?

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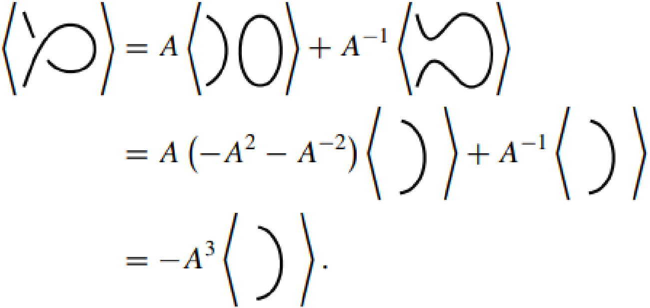
Is the Kauffman Bracket an invariant of oriented knots? No, it is not preserved under R1:



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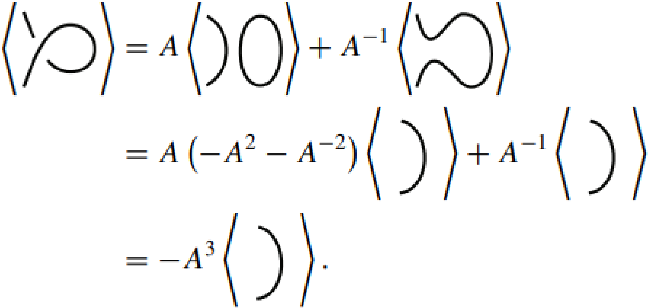

However, we can modify the Kauffman Bracket to make it an invariant.

Modifying the Kauffman Bracket

Theorem

Let $F(D) = (-A)^{-3\omega(D)}\langle D\rangle$*. Then* F *is invariant under the Reidemeister Moves and thus is an invariant of oriented knots.*

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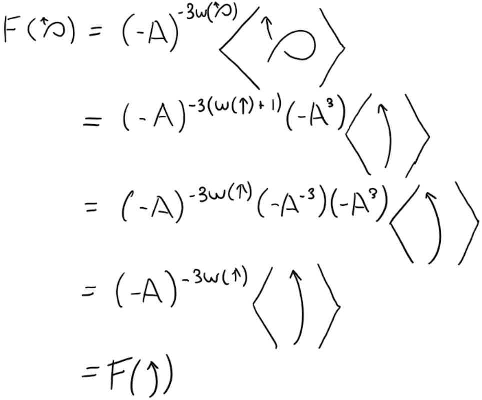

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That looks familar...Setting $A^2 = t^{-1/2}$, we get

$$\begin{aligned} &= -t^{\frac{1}{2}} - t^{\frac{5}{2}} \\ &= J(L) \end{aligned}$$

Constructing the Jones Polynomial

Theorem

Let L be an oriented knot and D its diagram. Then

$$J(L) = F(D)|_{A^2 = t^{-1/2}}$$

and hence the Jones Polynomial is a well-defined invariant of oriented knots.

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This is a fairly rudimentary way to construct the Jones Polynomial. There does exist a more general construction that creates many more knot invariants using representation theory, particularly group representations of the braid group $\mathfrak{B}_n$. If you want to see it, check out *An Introduction to Quantum and Vassilev Knot Invariants* (2019) by Jackson & Moffatt.

Acknowledgments

Images involving knots were taken from *An Introduction to Quantum and Vassiliev Knot Invariants* (2019), by David M. Jackson & Iain Moffatt.

I would like to thank Professor Thind for supervising me during this course. His guidance and insights were extremely helpful in building my understanding of the material.