An Introduction to Homology

A UTM Math Club Presentation

E. Gu and B. Papandrea

University of Toronto - Mississauga
Department of Mathematical and Computational Sciences

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Preface

This talk is based on notes from a Fall 2025 reading course on Algebraic Topology with Professor Micheal Groechenig, based on the textbook *Algebraic Topology* (2002) by Allen Hatcher.

The content of this talk is somewhere at the level of senior undergraduate to first-year graduate courses in algebraic topology. We will try to make this easy to understand, but it is very high-level stuff.

Unless otherwise specified, all functions throughout this talk are assumed to be continuous.

Motivation

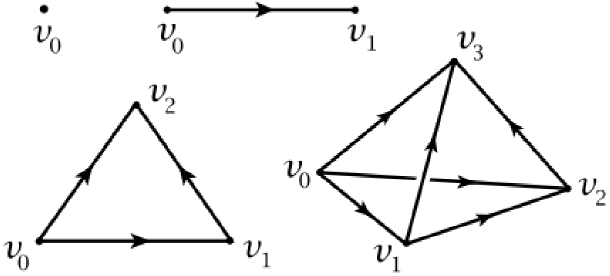
What is homology?

Homology attempts to find *algebraic* invariants for *topological* spaces, by associating algebraic structures with the space.

At an intuitive level, the invariants are related to the “holes” in a surface or space, and more generally can tell us about the overall structure of our topological space.

Δ-Complexes

We first restrict our set of spaces to those which can be constructed using triangles. An n-**simplex** is an n-dimension equivalent of a triangle, which we denote $[v_0, v_1, \ldots, v_n]$.



The standard n-simplex is denoted Δ^n, and has vertices given by unit vectors along the coordinate axes in $\mathbb{R}^n$

Δ-Complexes

A Δ-**Complex** on a space X is a set of maps $\sigma_\alpha : \Delta^n \to X$ such that each $x \in X$ lies in he image of exactly one σ_α, restricting σ_α to a face gives another map σ_β in the Δ-complex, and $A \subset X$ is open if and only if $\sigma_\alpha^{-1}(A)$ is open in Δ^n for all α.

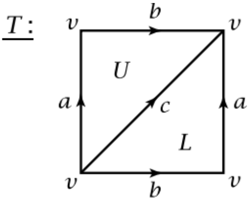


Figure: A Δ-complex of the torus T

Simplicial Homology

Let $\Delta_n(X)$ be the group generated by $e^n_\alpha = \sigma_\alpha(\Delta^n)$ in X; elements are called n-**chains** and are written as

$$\sum_\alpha n_\alpha \sigma_\alpha$$

There is a map $\partial_n : \Delta_n(X) \to \Delta_{n-1}(X)$, called a **boundary homomorphism**, defined as a signed sum of faces:

$$\partial_n(\sigma_\alpha) = \sum_i (-1)^i \sigma_\alpha|_{[v_0,\ldots,\hat{v}_i,\ldots,v_n]}$$

Example

$\partial_2[v_0, v_1, v_2] = [v_1, v_2] - [v_0, v_2] + [v_0, v_1]$

Simplicial Homology

$\partial_{n+1}\partial_n = 0$ (Trust us...), meaning the sequence of maps

$$\cdots \xrightarrow{\partial_{n+2}} \Delta_{n+1}(X) \xrightarrow{\partial_{n+1}} \Delta_n(X) \xrightarrow{\partial_n} \Delta_{n-1}(X) \xrightarrow{\partial_{n-1}} \cdots$$

is such that $\operatorname{im}\partial_{n+1} \subset \ker\partial_n$. We call this a **chain complex**.

Definition

For a given Δ-complex on a space X, the n**-th Simplicial Homology Group** $H_n^{\Delta}(X)$ is the quotient group $\ker\partial_n / \operatorname{im}\partial_{n+1}$

Simplicial Homology of the Torus

Using the Δ-complex for the torus T described previously, we have a single vertex v, three edges a, b, c, and two triangles U, L. So $\Delta_0(T) = \mathbb{Z}$, $\Delta_1(T) = \mathbb{Z}^3$, and $\Delta_2(T) = \mathbb{Z}^2$. $\Delta_n(T) = 0$ for all $n > 2$.

Notice that $\partial_1(a) = v - v = 0 = \partial_1(b) = \partial_1(c)$. We also have that $\partial_2(U) = a + b - c = \partial_2(L)$. $\partial_0 = \partial_3 = 0$.

Thus, we get that

$$H_0^\Delta(T) = \ker \partial_0 / \operatorname{im} \partial_1 = \mathbb{Z}/\{0\} = \mathbb{Z}$$

$$H_1^\Delta(T) = \ker \partial_1 / \operatorname{im} \partial_2 = \mathbb{Z}^3/\mathbb{Z} = \mathbb{Z}^2$$

$$H_2^\Delta(T) = \ker \partial_2 / \operatorname{im} \partial_3 = \mathbb{Z}/\{0\} = \mathbb{Z}$$

$$H_n^\Delta(X) = 0 \ \forall n > 2$$

Singular Homology

Our maps σ_α, as defined previously, restrict our focus to a very small set of spaces. We can expand our scope by simply only requiring that each map σ_α be *continuous*, allowing them to have things like singularities. Surprisingly, *everything we did before still works!*

Instead of $\Delta_n(X)$ we use $C_n(X)$, and our boundary homomorphism $\partial_n : C_n(X) \to C_{n-1}(X)$ is defined exactly the same, with $\partial_n \partial_{n-1} = 0$ still. Thus, $\ker \partial_n / \operatorname{im} \partial_{n+1}$ is still a group, now called the n**-th Singular Homology Group**, and denoted $H_n(X)$.

Properties of Singular Homology

(i) If $X = \bigoplus_\alpha X_\alpha$, $H_n(X) \cong \bigoplus_\alpha H_n(X_\alpha)$.

(ii) If $X \neq \varnothing$ is path-connected, then $H_0(X) \cong \mathbb{Z}$. If X has n connected components, then $H_0(X) \cong \mathbb{Z}^n$.

(iii) If $X = \{x\}$, then $H_n(X) = 0$ for all $n > 0$, and $H_0(X) = \mathbb{Z}$.

(iv) $H_1(X) \cong \pi_1(X)^{\mathsf{ab}}$

(v) For any Δ-complex on X, $H_n(X) \cong H_n^\Delta(X)$.

Reduced Homology

Sometimes, we would like for $H_0(X) = 0$. We can do this by augmenting our chain complex. Consider the map

$$\varepsilon : C_0(X) \to \mathbb{Z}, \quad \varepsilon\left(\sum n_i \sigma_i\right) = \sum n_i$$

By construction, $\varepsilon(\operatorname{im} \partial_1) = 0$, so we get a chain complex

$$\cdots \to C_1(X) \to C_0(X) \xrightarrow{\varepsilon} \mathbb{Z} \to 0$$

The homology groups from this chain complex are called the n-**th Reduced Homology Groups**, $\widetilde{H}_n(X)$. We have that

$$H_0(X) \cong \widetilde{H}_0(X) \oplus \mathbb{Z}, \quad H_n(X) \cong \widetilde{H}_n(X) \ \forall n > 0$$

Homotopy Invariance

This is great and all, but what do we do with this? Like any good algebraist, we want an *invariant*. And indeed, it is one!

Turns out, homology is invariant for spaces that are "almost the same". What does that even mean?

Homotopy Equivalence

You probably thought of *homeomorphic* spaces - the same under bending, twisting, stretching, etc. That works, but is actually a bit too strong!

We care about spaces that are **homotopy equivalent**; intuitively, this is similar to spaces that are homeomorphic, but we also allow *contraction* and *expansion*.

For instance, a solid disk (or even all of $\mathbb{R}^n$) is homotopy equivalent to a single point, but they are not homeomorphic.

Homotopy Equivalence

Definition

We say two maps $f, g : X \to Y$ are **homotopic** if there exists a map $H : X \times [0, 1] \to Y$ such that:

$$H(x, 0) = f(x) \qquad H(x, 1) = g(x)$$

for all $t \in [0, 1]$.

Definition

We say two *spaces* X and Y are **homotopy equivalent** if there exists a pair of maps $f : X \to Y$ and $g : Y \to X$ such that:

$$g \circ f \sim \mathrm{id}_X \qquad f \circ g \sim \mathrm{id}_Y$$

where $\sim$ denotes two maps being homotopic.

Chain Maps

We'll need the following tool.

Definition

Given a map $f : X \to Y$, a **chain map** is an induced homomorphism $f_\# : C_n X \to C_n Y$ defined as such:

$$f_\#(\sigma) = f \circ \sigma : \Delta^n \to Y \qquad f_\#(\Sigma_i n_i \sigma_i) = \sum_i n_i f_\#(\sigma_i)$$

This takes a chain in X and maps it through f to get a chain in Y.

Chain Maps

Importantly, chain maps commute with the boundary operator, that is, $f_\# \circ \partial = \partial \circ f_\#$.

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n+1}(X) \arrow[r, "\partial"] \arrow[d, "f_\sharp"] & C_n(X) \arrow[r, "\partial"] \arrow[d, "f_\sharp"] & C_{n-1}(X) \arrow[r] \arrow[d, "f_\sharp"] & \cdots \\
\cdots \arrow[r] & C_{n+1}(Y) \arrow[r, "\partial"] & C_n(Y) \arrow[r, "\partial"] & C_{n-1}(Y) \arrow[r] & \cdots
\end{tikzcd}$$

This implies that $f_\#$ respects cycles and boundaries. Therefore:

Proposition

A chain map $f_\# : C_n(X) \to C_n(Y)$ induces a homomorphism $f_* : H_n(X) \to H_n(Y)$ between the homology groups for all n.

Chain Maps

Properties of Chain Maps

(i) $(g \circ f)_* = g_* \circ f_*$
(ii) $\mathrm{id}_* = \mathrm{id}$

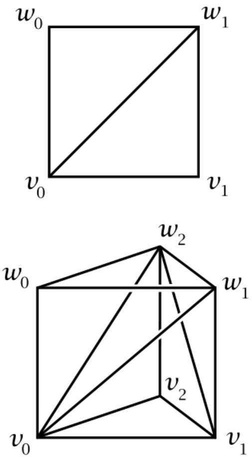
Theorem

If $f, g : X \to Y$ are homotopic, then $f_* = g_*$.

As a corollary, if X and Y are two homotopy equivalent spaces, then $H_n(X)$ is isomorphic to $H_n(Y)$ for all n.

Therefore, homotopy equivalent spaces indeed have the same homology groups, and these groups make sense as an invariant!

Proof by Prisms



Exact Sequences

Definition

A sequence

$$\cdots \to A_{n+1} \xrightarrow{\alpha_{n+1}} A_n \xrightarrow{\alpha_n} A_{n-1} \to \cdots$$

is **exact** if $\ker \alpha_n = \operatorname{im} \alpha_{n+1}$ for all n

These are all chain complexes whose homology groups are all trivial.

Properties of Exact Sequences

(i) If $0 \to A \xrightarrow{\alpha} B$, then α is injective.
(ii) If $A \xrightarrow{\alpha} B \to 0$, then α is surjective.
(iii) If $0 \to A \xrightarrow{\alpha} B \to 0$, then α is an isomorphism.
(iv) If $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$, then α injective, β surjective, and $C \simeq B/\operatorname{im}\alpha \simeq \operatorname{coker}\alpha$

The Big Idea

Given a space X and subset $A \subseteq X$, we hope to find a nice relationship between the homology groups of X, A, and X/A.

Theorem

Let $A \subseteq X$ be closed, nonempty, and a deformation retract of a neighbourhood in X. Let $i : A \to X$ be inclusion and $j : X \to X/A$ the standard quotient map. Then the sequence

$$\cdots \widetilde{H}_n(A) \xrightarrow{i_*} \widetilde{H}_n(X) \xrightarrow{j_*} \widetilde{H}_n(X/A) \xrightarrow{\partial} \widetilde{H}_{n-1}(A) \xrightarrow{i_*} \cdots \to \widetilde{H}_0(X/A) \to 0$$

is exact.

If (X, A) satisfy this theorem, we call it a **good pair**.

Relative Homology

Given $A \subseteq X$, sometimes it's actually okay to ignore the structure of A when studying X's homology. In doing so, we study the **relative** homology of X with respect to A. Define

$$C_n(X, A) = C_n(X)/C_n(A)$$

These form a chain complex, like before, with similar boundary homomorphisms, giving us $H_n(X, A)$. These fit into a long exact sequence:

$$\cdots \to H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X, A) \xrightarrow{\partial} H_{n-1}(A) \to \cdots$$

Notice that the long exact sequence for relative homology groups is similar to the sequence for quotients - it's in fact slightly more general. To derive the sequence, consider the following commutative diagrams:

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n+1}(X) \arrow[r, "\partial"] \arrow[d, "f_\sharp"] & C_n(X) \arrow[r, "\partial"] \arrow[d, "f_\sharp"] & C_{n-1}(X) \arrow[r] \arrow[d, "f_\sharp"] & \cdots \\
\cdots \arrow[r] & C_{n+1}(Y) \arrow[r, "\partial"] & C_n(Y) \arrow[r, "\partial"] & C_{n-1}(Y) \arrow[r] & \cdots
\end{tikzcd}$$

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\cdots \arrow[r] & A_{n+1} \arrow[r, "\partial"] \arrow[d, "i"] & A_n \arrow[r, "\partial"] \arrow[d, "i"] & A_{n-1} \arrow[r] \arrow[d, "i"] & \cdots \\
\cdots \arrow[r] & B_{n+1} \arrow[r, "\partial"] \arrow[d, "j"] & B_n \arrow[r, "\partial"] \arrow[d, "j"] & B_{n-1} \arrow[r] \arrow[d, "j"] & \cdots \\
\cdots \arrow[r] & C_{n+1} \arrow[r, "\partial"] \arrow[d] & C_n \arrow[r, "\partial"] \arrow[d] & C_{n-1} \arrow[r] \arrow[d] & \cdots \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Long Exact Sequences

The map ∂ shown in the long exact sequence can thus be obtained via the following lemma:

Consider this commutative diagram, where the rows are (short) exact.

$$\begin{tikzcd}
 & A \arrow[r, "f"] \arrow[d, "a"] & B \arrow[r, "g"] \arrow[d, "b"] & C \arrow[r] \arrow[d, "c"] & 0 \\
0 \arrow[r] & A' \arrow[r, "f'"] & B' \arrow[r, "g'"] & C' &
\end{tikzcd}$$

Long Exact Sequences

Then there exists a homomorphism d so that the sequence is exact:

$$
\begin{tikzcd}
& \ker a \arrow[r] \arrow[d] & \ker b \arrow[r] \arrow[d] & \ker c \arrow[d] \arrow[dddll, rounded corners, "d"] & \\
& A \arrow[r, "f"] \arrow[d, "a"] & B \arrow[r, "g"] \arrow[d, "b"] & C \arrow[r] \arrow[d, "c"] & 0 \\
0 \arrow[r] & A' \arrow[r, "f'"] \arrow[d] & B' \arrow[r, "g'"] \arrow[d] & C' \arrow[d] & \\
& \operatorname{coker} a \arrow[r] & \operatorname{coker} b \arrow[r] & \operatorname{coker} c &
\end{tikzcd}
$$

This lemma is known as the **Snake Lemma**!

Equivalence of Homologies

This is great and all, but we've been working with singular homology - what about simplicial homology?

We should reasonably expect that the studies of either homologies ultimately agree. Thankfully, this is indeed the case!

There is a canonical homomorphism $H_n^\Delta(X, A) \to H_n(X, A)$ given by sending an n-simplex to its corresponding map $\sigma : \Delta^n \to X$.

Theorem

The homomorphisms $H_n^\Delta(X, A) \to H_n(X, A)$ are isomorphisms for all n and pairs (X, A).

Equivalence of Homologies

Consider the following diagram:

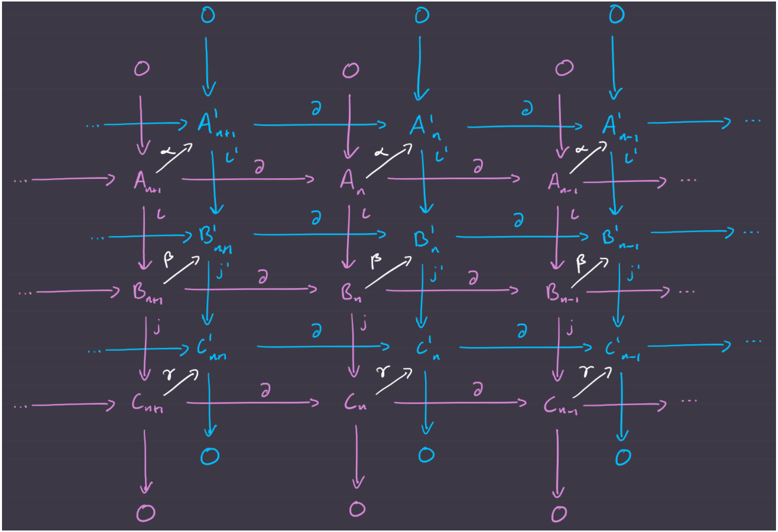
$$\begin{array}{ccccccccc}
H^{\Delta}_{n+1}(X^k, X^{k-1}) & \longrightarrow & H^{\Delta}_{n}(X^{k-1}) & \longrightarrow & H^{\Delta}_{n}(X^k) & \longrightarrow & H^{\Delta}_{n}(X^k, X^{k-1}) & \longrightarrow & H^{\Delta}_{n-1}(X^{k-1}) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
H_{n+1}(X^k, X^{k-1}) & \longrightarrow & H_{n}(X^{k-1}) & \longrightarrow & H_{n}(X^k) & \longrightarrow & H_{n}(X^k, X^{k-1}) & \longrightarrow & H_{n-1}(X^{k-1})
\end{array}$$

Here, X^k represents all simplicies in X of dimension k or less - this is called the k-**skeleton** of X.

The first and fourth columns are isomorphisms by a technical argument, the second and fifth columns are isomorphisms by induction, so the third column is an isomorphism by the **Five Lemma**.

Time Check!

3D commutative diagram jumpscare



A **CW-Complex** is very similar to a Δ-complex, but instead of simplices we use n-balls.

Lemma

If X is a CW-Complex,

(i) $H_k(X^n, X^{n-1}) = 0$ *for all* $k \neq n$. *If* $k = n$, *this is a free abelian group with basis in one-to-one correspondence with the* n*-cells of* X.

(ii) $H_k(X^n) = 0$ *for all* $k > n$.

(iii) $H_k(X^n) \hookrightarrow H_k(X)$ *is an isomorphism for all* $k < n$ *and surjective when* $k = n$.

Cellular Homology

The long exact sequences of $(X^{n+1}, X^n), (X^n, X^{n-1})$, and (X^{n-1}, X^{n-2}) thus fit into a diagram:

$$\begin{tikzcd}
 & & & & 0 & & \\
 & 0 \arrow[dr] & & H_n(X^{n+1}) \approx H_n(X) \arrow[ur] & & & \\
 & & H_n(X^n) \arrow[ur] \arrow[dr, "j_n"] & & & & \\
\cdots \arrow[r] & H_{n+1}(X^{n+1}, X^n) \arrow[ur, "\partial_{n+1}"] \arrow[rr, "d_{n+1}"] & & H_n(X^n, X^{n-1}) \arrow[dr, "\partial_n"'] \arrow[rr, "d_n"] & & H_{n-1}(X^{n-1}, X^{n-2}) \arrow[r] & \cdots \\
 & & & & H_{n-1}(X^{n-1}) \arrow[ur, "j_{n-1}"'] & & \\
 & & & 0 \arrow[ur] & & &
\end{tikzcd}$$

The sequence going along the middle of the diagram is a chain complex, and we thus get the n-**th Cellular Homology Group** $H_n^{\mathrm{CW}}(X)$.

The Mayer-Vietoris Sequence

A **Mayer-Vietoris sequence** is very similar to the long exact sequence discussed earlier, but with slight modifications.

Suppose $A, B \subseteq X$ are subspaces such that the interiors of A and B cover all of X.

We'll denote by $C_n(A + B) \leq C_n(X)$ the subgroup consisting of chains which are sums of chains in A and chains in B; this forms a chain complex.

The Mayer-Vietoris Sequence

Consider the short exact sequence given by:

$$0 \to C_n(A \cap B) \xrightarrow{\varphi} C_n(A) \oplus C_n(B) \xrightarrow{\psi} C_n(A+B) \to 0$$

where $\varphi(x) = (x, -x)$ and $\psi(x, y) = x + y$. Applying the same construction from before, we obtain a long exact sequence:

$$\cdots \to H_n(A \cap B) \xrightarrow{\Phi} H_n(A) \oplus H_n(B) \xrightarrow{\Psi} H_n(X) \xrightarrow{\partial} H_{n-1}(A \cap B) \to \cdots$$

This is our **Mayer-Vietoris sequence**, and is sometimes more convenient to use than the usual long exact sequence.