The Millennium Prize Problems For Dummies

A UTM Math Club Presentation

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Introduction

What is the most difficult way to earn a million dollars?

What Are The Millennium Prize Problems?

Announced on May 24, 2000, the Millennium Prize Problems are a collection of 7 problems chosen by the Clay Mathematical Institute in order to both celebrate mathematics in the third millennium, and to "record some of the most difficult problems with which mathematicians were grappling at the turn of the second millennium."

The problems come from a wide range of mathematical subjects, like number theory, mathematical physics, geometry, PDEs, and topology.

Solving one of these problems will earn you \$1 million dollars.

The Problems

1. Birch and Swinnerton-Dyer Conjecture
2. Hodge Conjecture
3. Navier-Stokes Equation
4. P vs. NP
5. Riemann Hypothesis
6. Yang-Mills & The Mass Gap
7. Poincare Conjecture

Elliptic Curves

Elliptic curves, equations of the form

$$y^2 = x^3 + ax + b$$

are an important area of number theory and have had a key part in modern mathematical research, including the proof of Fermat's Last Theorem, Elliptic Curve Cryptography, and integer factorization.

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A particular area of interest is whether or not an elliptic curve has **rational solutions**.

Algebraic and Analytic Rank

It is known that the set of rational solutions of any elliptic curve C is a group, $C(\mathbb{Q})$, and has a finite generating set. If the number of solutions is infinite, then there must be an element in this generating set of infinite order. The number of such elements is called the **algebraic rank** of the elliptic curve, denoted r. We know that $r = 0$ if and only if $C(\mathbb{Q})$ if finite.

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Analytic rank is defined differently. We first consider our elliptic curve modulo a prime p, and denote its set of solution modulo p $C(\mathbb{F}_p)$. A theorem of Hasse tells us that

$$|C(\mathbb{F}_p) - p - 1| \leq 2\sqrt{p}$$

We'll let $C(\mathbb{F}_p) - p - 1 = a_p$.

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From here we can define an L-series using an Euler product that will allow us to collect the a_p values in a good way: Given an elliptic curve C we have

$$L(C,s) = \prod_{p|\Delta}(1 - a_p p^{-s} + p^{1-2s})^{-1} \cdot \prod_{p|\Delta}(1 - a_p p^{-s})^{-1}$$

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Then we define the **analytic rank** of C as

$$r_{an} = \text{ord}_{s=1} L(C,s)$$

Birch and Swinnerton-Dyer Conjecture (1965)

Bryan John Birch and Peter Swinnerton-Dyer studied the properties of L for $\Re(s) = 1$. They observed that for an elliptic curve C

$$L(C, 1) = \prod_p \frac{p}{|C(\mathbb{F}_p)|}$$

Upon further inspection, they noticed that the order of $L(C, 1)$ coincided with the number of generators of infinite order in $C(\mathbb{Q})$.

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Upon further inspection, they noticed that the order of $L(C, 1)$ coincided with the number of generators of infinite order in $C(\mathbb{Q})$. This led to the following conjecture:

Birch and Swinnerton-Dyer Conjecture

Let C be an elliptic curve over $\mathbb{Q}$. Then

$$r(C) = r_{an}(C)$$

Why Should I Care?

If this conjecture is true, we can easily compute if an elliptic curve has finite or infinite solutions over $\mathbb{Q}$ by computing $L(C, 1)$.

This means we can study local properties of an elliptic curve to decduce some of its global properties.

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It would also mean many arithmetic properties that have yet to be proven in general will hold true. For example:

- Every positive integer congruent to 5, 6, or 7, mod 8, is a congruent number.
- Every positive integer can be written as $s^2(t^3 - 91t - 182)$ for some $s, t \in \mathbb{Q}$.

Hodge Theory

Hodge Theory is a subfield of algebraic topology that studies the cohomology groups of smooth manifolds using PDEs.

It was conceived in the 1930s by Sir William Vallance Douglas Hodge for use in algebraic geometry, more specifically to study complex projective manifolds.

It is a very useful part of algebraic geometry, but also has applications in number theory.

Kahler Manifolds & Hodge Classes

Definition

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Definition

A homology class x in a homology group

$$H_k(X, \mathbb{C}) = H$$

where X is a Kahler manifold, is called a **Hodge class** if $k = 2p$, x is of type (p, p) in the decomposition of H, and lies in the image of the abelian group homomorphism

$$H_k(X, \mathbb{Q}) \to H$$

Algebraic Cycles and Classes

Let X be a Kahler manifold and Z a subvariety (subset) of X. An **algebraic cycle** on X is a formal sum of its subvarieties

$$\sum_i c_i Z_i$$

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where the c_i's are rational. The cohomology class of an algebraic cycle is just the sum of the cohomology classes of the subvarieties:

$$\sum_i c_i [Z_i]$$

we call this cohomology class **algebraic**.

Hodge Conjecture (1952)

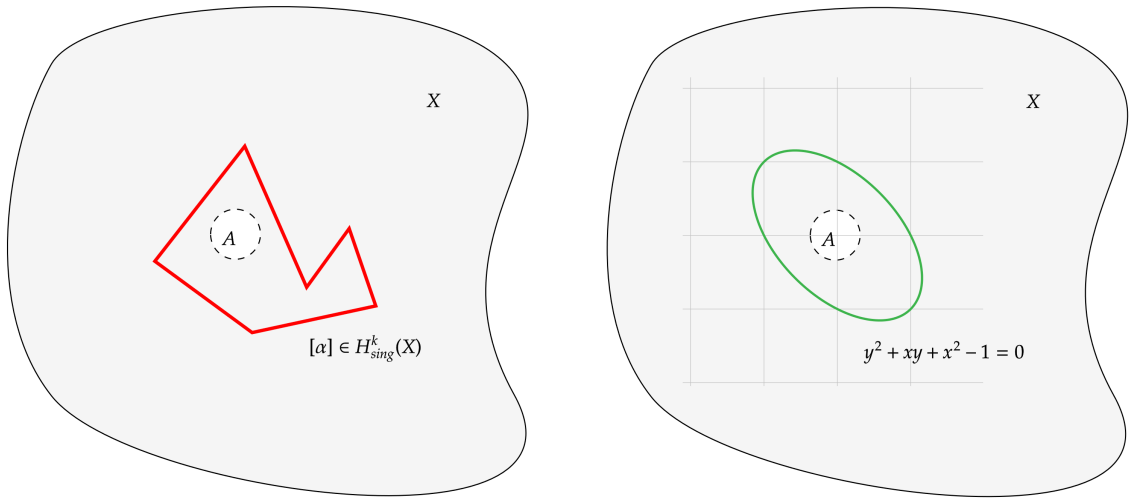
Hodge Conjecture

Let X be a Kahler manifold (projective complex manifold). Then every Hodge class on X is algebraic.

While special cases of the Hodge conjecture have been proven, a general proof remains elusive.

Why Should I Care?

If true, the Hodge conjecture would ensure that certain topological properties can be understood by studying much simpler shapes, like the zero set of a polynomial, inside of the space we're working on. These nicer structures can be studied using calculus or algebra, which make understanding the broader structure of these objects much easier.



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$$\nabla \cdot \vec{u} = 0 \tag{1}$$

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(1) is just the Conservation of Mass, while (2) is Newton's Second Law.

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1. Does a solution exist? If so,
2. Is that solution unique? If so,
3. Is that solution smooth and defined globally?

Time Complexity

In the realm of theoretical computer science, **time complexity** represents the amount of computer time it would take to run a particular algorithm.

Time complexity is often used to determine how the runtime of an algorithm increases as the amount of inputs increases. It is often represented by **Big-O Notation**.

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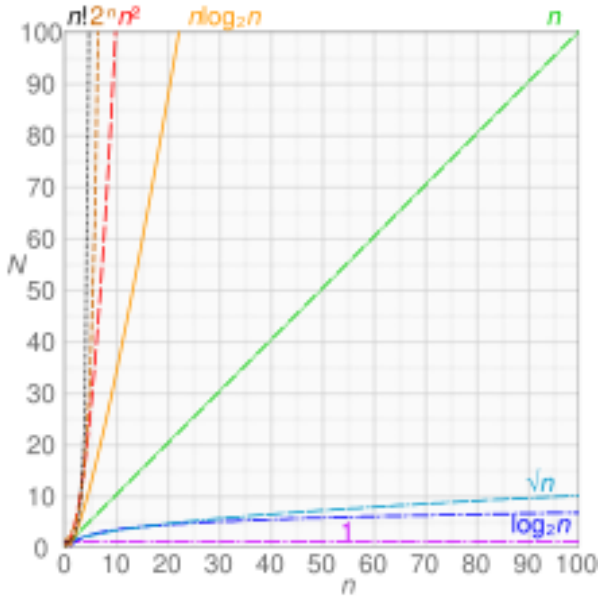
Time complexity is often used to determine how the runtime of an algorithm increases as the amount of inputs increases. It is often represented by **Big-O Notation**. Examples include

- Constant time $O(1)$
- Logarithmic time $O(\log n)$
- Linear time $O(n)$
- Quadratic time $O(n^2)$
- Polynomial time $O(2^{\log n})$
- Factorial time $O(n!)$



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Examples

Sorting a list using merge sort is a problem in P.

Finding a solution to a sudoku puzzle is a problem in NP.

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P vs. NP

What is the relationship between P and NP? Is $P = NP$?

Why Should I Care?

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With that said, it is widely accepted that $P \neq NP$, and many computer scientists have noted that if $P = NP$, there would still be issues with finding solutions to problems.

The Riemann-Zeta Function

Definition

The **Riemann-Zeta Function** $\zeta : \mathbb{C} \to \mathbb{C}$ is defined for all $s \in \mathbb{C}$ with $\Re(s) > 1$ and is given by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \cdots$$

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- $\zeta(-1) = 1 + 2 + 3 + \ldots$

The Riemann-Zeta Function

ζ converges only for complex numbers with real part greater than 1. However, Riemann (1859) showed that ζ is a mereomorphic function, and thus has a unique analytic continuation to all of $\mathbb{C}$. This continuation is given by the equation

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Using this equation we can show that $\zeta(-1) = \dfrac{-1}{12}$

The Zeroes of ζ

If $s = -2n$ is a negative even integer, we have that

$$\zeta(s) = 2^{-2n}\pi^{-2n-1}\sin(-n\pi)\Gamma(1+2n)\zeta(1+2n) = 0$$

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ζ does not vanish for any other values of s with a real component that is negative or greater than 1. Thus the non-trivial zeroes of ζ must be in the set

$$\{s \in \mathbb{C} : 0 < \Re(s) < 1\}$$

This is a special region called the **Critical Strip**.

The Riemann Hypothesis (1859)

Riemann knew that the zeroes inside the Critical Strip had to be symmetric around the line $s = \frac{1}{2} + it$. Explicit computations showed that many zeroes lied exactly on this line.

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George Hardy and John Littlewood (1914, 1921) showed that there an infinite number of such zeroes, and as of 2020, we know that at least 41% of non-trivial zeroes lie on this line.

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If the Riemann Hypothesis is true, then

$$\pi(x) = \mathrm{Li}(x) + O(\sqrt{x}\ln(x))$$

which gives us the best possible bound for the error in the Prime Number Theorem.

Why Should I Care?

Another interesting consequence is related to calculating gaps between the primes. Harald Cramer (1919), showed that if p_k is the k-th prime number, then if the Riemann Hypothesis is true,

$$p_{k+1} - p_k = O(\sqrt{p_k}\ln(p_k))$$

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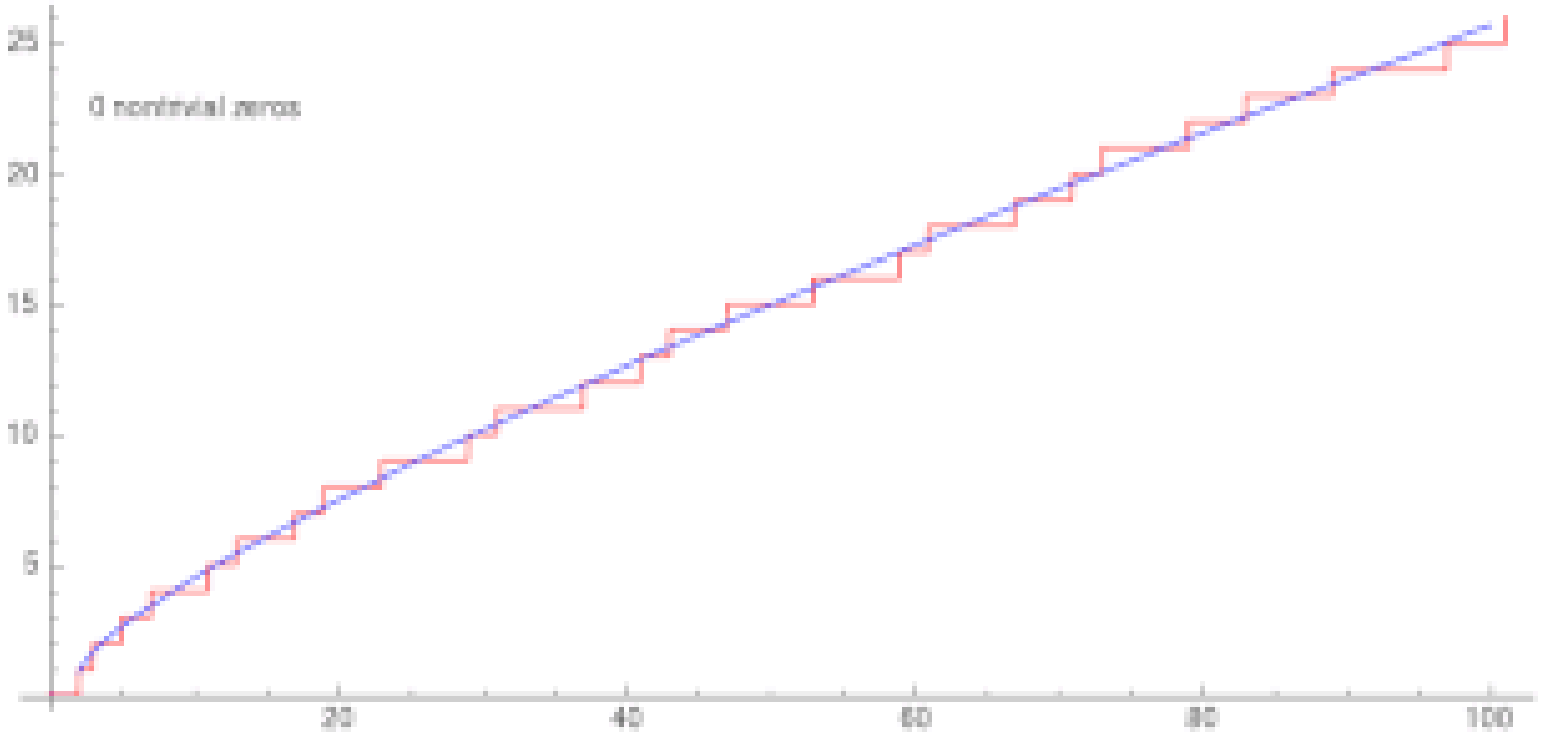
In short, if the Riemann Hypothesis is true, we get a lot of interesting information about the bounds on the gaps between primes, and their distribution.


0 nontrivial zeros
25
20
15
10
5
20
40
60
80
100

Quantum Field Theory

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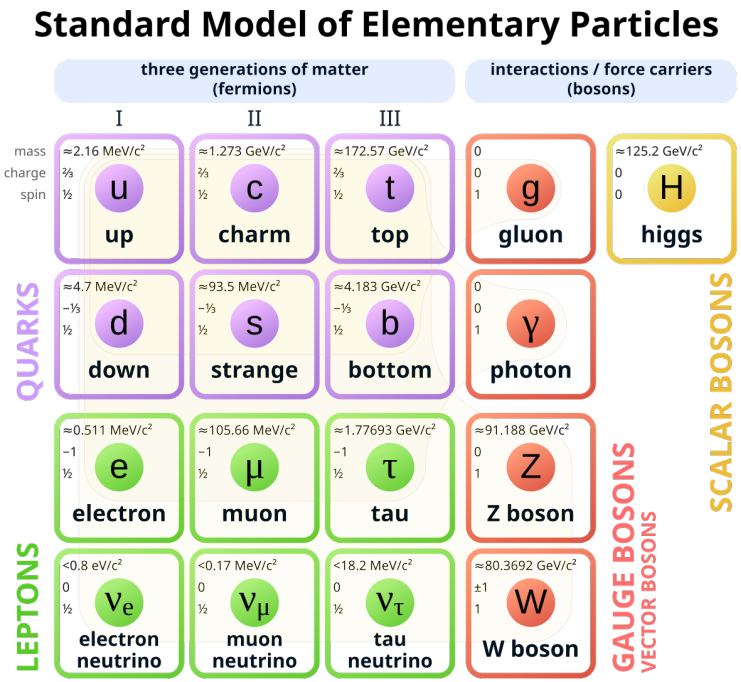

Yang-Mills Theory

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Problem: We don't have a great theoretical (mathematical) understanding of Yang-Mills Theory in 4D spacetime.

Wightman Axioms & The Mass Gap

The **Wightman Axioms**, developed by Arthur Wightman in the early 1950s, is a rigorous mathematical formulation of quantum field theory that provides a basis for rigorous study of quantum fields. They are:

1. W0: assumptions of relativistic quantum mechanics
2. W1: assumptions on the domain and continuity of the field
3. W2: transformation law of the field
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In a quantum system, there are many different energy states, the lowest of which is called the vacuum. The **mass gap** Δ_0 is the difference in energy between the vacuum and the next lowest energy state.

Yang-Mills Existence & Mass Gap (1950s)

Yang-Mills Existence & Mass Gap

For a gauge group G, there is a non-trivial Yang-Mills theory on $\mathbb{R}^4$ such that the theory

1. satisfies axiomatic properties as strong as the Wightman Axioms, and
2. there exists a mass gap $\Delta_0 > 0$.

While experimental and computational results have shown that a mass gap is likely to exist, no rigorous theoretical proof of 1 or 2 has been published.

Why Should I Care?

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Furthermore, if Yang-Mills theory does indeed satisfy the necessary axiomatic properties and have a mass gap, it would be the smallest non-trivial constructive quantum field theory in 4 dimensions.

Properties of Spheres

Spheres are cool.

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In addition to being closed and connected as topological objects, they have another special property:

If we pick any point along a sphere, and then draw any path that starts and ends at the chosen point, we can continuously deform this path and shrink it down to the point without the need to cut our path or the sphere.

Simply Connected

If two paths can be continuously deformed into one another, we say they're *equivalent*. The set of all non-equivalent paths which start and end at a point p in a space X is called the **Fundamental Group of X at p**, denoted $\pi(X, p)$.

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We say that X is **simply-connected** if any path that starts and ends at any point can be continuously deformed down to that point. More rigorously, for any $p \in X$,

$$\pi(X, p) = \{\mathrm{id}_p\}$$

Classifying Simply-Connected Spaces

Fact

Any compact 2-dimensional topological manifold without boundary that is simply connected is homeomorphic to the 2-sphere.

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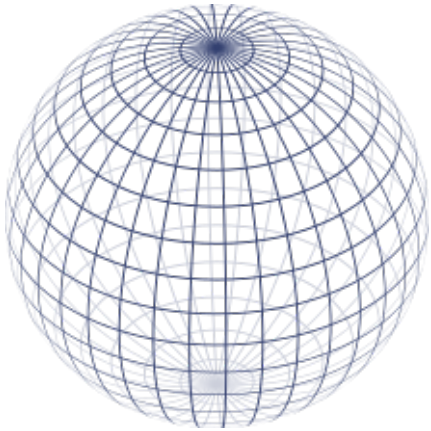

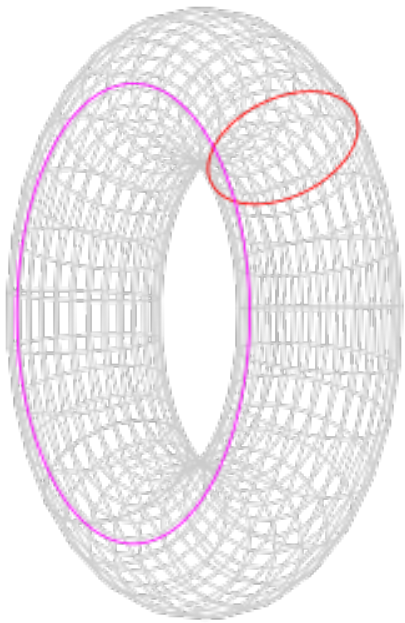

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A proof of a more generalized version of The Poincare Conjecture was proved for 4-dimensions and higher by Stephen Smale (1961), and Michael Freedman (1982), but a proof for 3-dimensions remained elusive.

Ricci Flow

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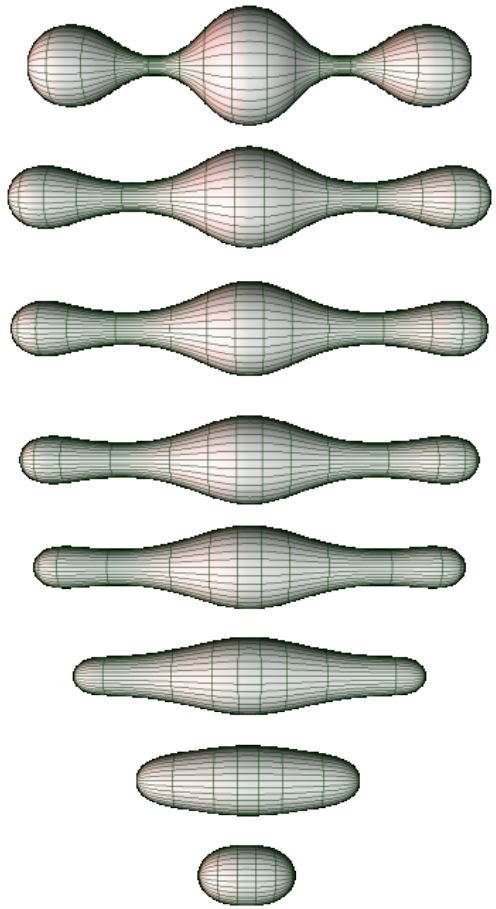
In essence, Ricci Flow is a process that acts like the Heat Equation, and tries to "smoothen" the surface we're working on so it has constant positive curvature; then it must be homeomophic to the 3-sphere.

The only problem is that Ricci Flow can create singularities.



The Solution

Between 2002 and 2003, Grigori Perelman published a solution to the Poincare Conjecture using Ricci Flow. While his solution was a rough sketch, the gaps were later filled in by others and a full proof was published by 2006.

Perelman was awarded the $1 million by CMI and was offered the Fields Medal, but he *declined both of them*.