MAT311
Partial Differential Equations

The following is based on lecture notes taken during the Fall 2025 offering of MAT311: Partial Differential Equations, at the University of Toronto - Mississauga, and are based on the first six chapters of the textbook *Partial Differential Equations: An Introduction* by W.A. Strauss. The notes are broken up in sections based on the week they were taught, and not necessarily broken up based on the textbook chapters. The intention is for these notes to be a polished version of my own lecture notes that allows me to revise and look over the material multiple times, and thus should not be considered a primary source for learning about PDEs or their applications.

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Week 1

1.1 WHAT IS A PDE?

We begin our study of partial differential equations by asking ourselves an important question: what actually *is* a partial differential equation. If you've taken a course on ordinary differential equations, you'll find that PDEs are very similar, with one big difference. Before, we had an unknown function $u(x)$ that was dependent on a single variable. Now, in PDE world, we allow our unknown function to take on many variables:

$$u(x, y, \ldots)$$

This function can be real or complex valued. Moreover, instead of a single variable derivative, PDEs will now include *partial derivatives*, which can be displayed in several forms. The partial derivative of u with respect x can be written as

$$\frac{\partial u}{\partial x},\ \partial_x u,\ u_x$$

The order of the PDE corresponds to what partials are included in the equation. A **first order partial differential equation** is of the form

$$F(x, y, \ldots, u_x(x, y, \ldots), u_y(x, y, \ldots), \ldots, u(x, y, \ldots)) = 0$$

Where F is some function of the variables, the partial derivatives of u, and possibly u. Its a first order PDE becuase the first partial derivatives are used; if the second partial derivatives were used it would be a second order PDE. In general, an nth order equation involves up to the nth partial derivatives.

As with ODEs, we can also consider a PDE system, where there is more that one unknown equation:

$$u_1(x, y, \ldots), u_2(x, y, \ldots), \ldots$$

It should also be noted that there is *no general theory* of PDEs. They are, on average, much harder to prove, and there is very little that can be deduced about general PDEs. We'll often be looking at very specific types of PDEs and seeing how to solve them.

If they're so hard to solve, why should we even care about them? PDEs show up pretty much everywhere in the applied mathematical world, and are essential for modeling things in physics, finance, statistics, and much more.

Two examples of PDEs are shown below:

1. Given $u(x, y)$, $u_x + u_y = 0$
2. Given $u(t, x)$, $u_{tt} - u_{xx} + u^3 = 0$

1.1.1 LINEAR PDES

In ODEs, we discussed both linear and nonlinear equations at length. However, in the world of PDEs, nonlinear equations are much, *much* harder to solve. While we will discuss them briefly, we will mainly discuss linear PDEs. We define them now:

Definition 1.1. *We say a PDE for unknown function u is a* ***linear homogeneous PDE*** *if it is of the form*

$$\mathcal{L}(u) = 0$$

such that

1. $\mathcal{L}(u + v) = \mathcal{L}(u) + \mathcal{L}(v)$
2. $\mathcal{L}(cu) = c\mathcal{L}(u)$, *where c is a constant.*

If the PDE for u is of the form $\mathcal{L}(u) + g$ for some given function g, we say it is a ***linear inhomogeneous PDE****.*

What's nice about these equations is that if we have a single solution to the linear inhomogeneous PDE, we can rewrite the linear homogeneous one in terms of it: Suppose $\mathcal{L}(v) = g$. Then we have that

$$\begin{aligned} &\iff \mathcal{L}(v) = \mathcal{L}(u) \\ &\iff \mathcal{L}(v) - \mathcal{L}(u) = 0 \\ &\iff \mathcal{L}(v - u) = 0 && \text{(linearity of } \mathcal{L}) \end{aligned}$$

Similar to ODEs, the set of solutions to the $\mathcal{L}(u) = 0$ is a vector space, meaning

1. $\mathcal{L}(u) = 0,\ \mathcal{L}(v) = 0 \implies \mathcal{L}(u + v) = 0$
2. $\mathcal{L}(u) = 0 \implies \mathcal{L}(cv) = 0$, where c is a constant.

however, unlike ODEs, this vector space is almost always infinite dimensional. We will see this in action in the next section.

1.2 PDES THAT ARE ODES

The simplest examples of partial differential equations are those that are actually just ODEs in disguise. These can very quickly solved using ODE techniques. Let's look at some examples:

Example 1. *Consider the unknown $u(x, y)$. We'll find the general solution to*

$$u_x = 0$$

Intuitively, the general solution is just any function that is constant in x. To see this, we'll fix $y = y_0$. We can then rewrite it as

$$\frac{d}{dx}[u(x, y_0)] = 0$$

Then we get that $u(x, y_0)$ is constant as a function of x, so

$$u(x, y_0) = C(y_0)$$

because y_0 changes, we get that $u(x, y) = f(y)$, where f is a function of y.

Notice that f can be *any* function of y. This is very clearly an infinite dimensional vector space, as there is no real pattern to them at all.

Example 2. *We'll find the general solution $u(x, y)$ for*

$$u_{xx} = 0$$

We can just apply the same trick twice:

$$\begin{aligned} &\iff (u_x)_x = 0 \\ &\iff u_x(x, y) = f(y) \end{aligned}$$

Now fixing $y = y_0$,

$$\begin{aligned} &\iff \frac{d}{dx}[u(x, y_0)] = f(y_0) \\ &\iff u(x, y_0) = xf(y_0) + g(y_0) \\ &\iff u(x, y) = xf(y) + g(y) \end{aligned}$$

where f, g are arbitrary functions.

Notice that in the first example, we had a first order PDE, and its general solution has a single free constant, $f(y)$, while in the second example we had a second order PDE with two free constants, $f(y)$ and $g(y)$. In general, an nth order linear PDE with have n free constants.

Example 3. *Find the general solution $u(x, y)$ to*

$$u_{xx} + u = 0$$

Intuitively, this is just an ODE in x. Recall that the general solution to

$$v''(t) + v(t) = 0$$

is given by

$$v(t) = A\sin(t) + B\cos(t)$$

so our general solution is

$$u(x,y) = f(y)\sin(x) + g(y)\cos(x)$$

where f, g are arbitrary functions.

These tricks still work when dealing with higher-order partials of many variables:

Example 4. *Find the general solution $u(x,y)$ of*

$$u_{xy} = 0$$

We can treat this like nested partials. By Clairaut's Theorem, $u_{xy} = u_{yx}$, thus,

$$\begin{aligned} &(u_y)_x = 0 \\ \iff &u_y(x,y) = f(y) \end{aligned}$$

we can now treat this as an ODE in y, yielding

$$\begin{aligned} \iff &u(x,y) = \int f(y)\ dy + g(x) \\ \iff &u(x,y) = F(y) + g(x) \end{aligned}$$

where F, g are arbitrary functions.

For certain PDEs, it can be a good idea to reduce it down to an ODE and solve it using ODE techniques, though the methods will be much harder than the above.

1.3 CONSTANT COEFFICIENT 1ST ORDER LINEAR PDES (IN 2 VARIABLES)

A **constant coefficient 1st order linear PDE (in 2 variables)** are equations of the form

$$au_x + bu_y = 0$$

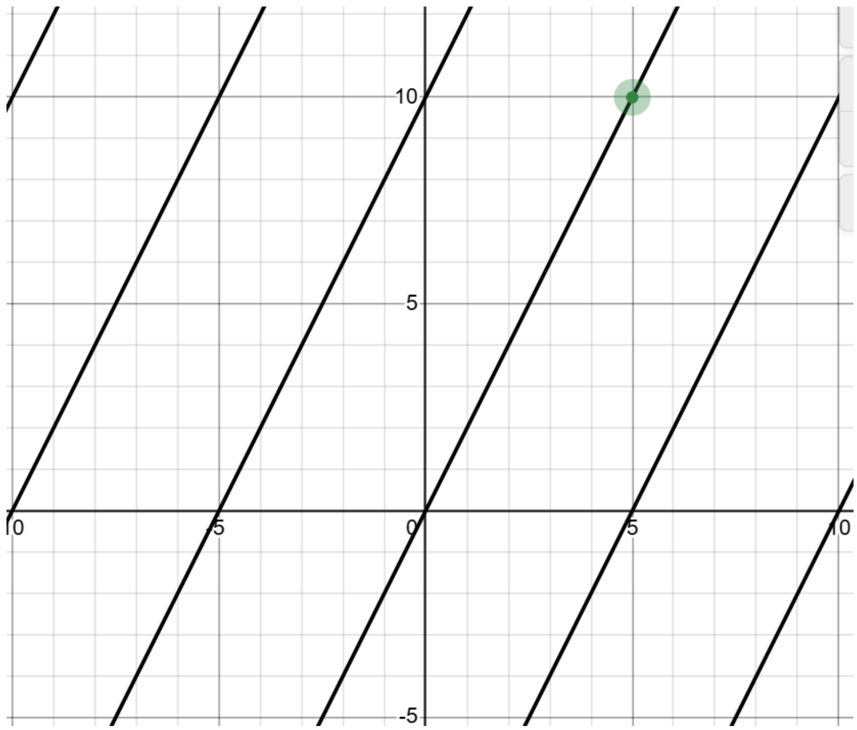
where $a, b \in \mathbb{R}$, and $u(x,y)$ is the unknown. This is a linear homogeneous equation. There are two main methods for solving them, which we will discuss now:

1.3.1 THE GEOMETRIC METHOD

Consider the vector $\vec{v} = (a,b) \in \mathbb{R}^2$. Then can rewrite the equation as

$$\vec{v} \cdot \vec{\nabla} u = 0$$

visually, this says that the function u remains constant in the direction of $\vec{v}$, which is given by the lines shown below



The lines parallel to $\vec{v}$ are of the form $bx - ay = c$, where c is some constant. In other words,

$$(b, -a) \perp \vec{v}$$

Because u is constant in the $\vec{v}$ direction, it must be constant along each line $bx - ay = c$ for each c. However, it can take a different value for each c. Thus, the value of $u(x, y)$ depends exclusively on what line the point (x, y) lies on, meaning

$$u(x, y) = f(bx - ay)$$

for any function f. The lines $bx - ay = c$ are called the **characteristics** of the PDE. In general, they are curves along which $u(x, y, \ldots)$ remains constant.

Let's check this works:

$$\begin{aligned} a\frac{\partial}{\partial x}[f(bx - ay)] + b\frac{\partial}{\partial y}[f(bx - ay)] &= af'(bx - ay) \cdot b + bf'(bx - ay) \cdot -a \\ &= abf'(bx - ay) - abf'(bx - ay) \\ &= 0 \end{aligned}$$

1.3.2 THE COORDINATE METHOD

We'll rewrite the equation in terms of new coordinates, which we define as

$$x' = ax + by$$
$$y' = bx - ay$$

Remark. *This seems like an arbitrary selection. One way we can find this is to define $x' = c_1x + c_2y, y' = c_3x + c_4y$ and then find the necessary constants that make things work*

This is non-singular whenever the determinant of the corresponding matrix is non-zero, which means $-a^2 - b^2 \neq 0$, which is true unless $a = b = 0$.

We now rewrite the PDE in terms of x', y':

$$\begin{aligned}\frac{\partial u}{\partial x} &= \frac{\partial u}{\partial x'}\frac{\partial x'}{\partial x} + \frac{\partial u}{\partial y'}\frac{\partial y'}{\partial y} \\ &= au_{x'} + bu_{y'} \\ \frac{\partial u}{\partial y} &= \frac{\partial u}{\partial x'}\frac{\partial x'}{\partial y} + \frac{\partial u}{\partial y'}\frac{\partial y'}{\partial y} \\ &= bu_{x'} - au_{y'}\end{aligned}$$

Thus,

$$\begin{aligned}0 &= au_x + bu_y \\ &= a(au_{x'} + bu_{y'}) + b(bu_{x'} - au_{y'}) \\ &= a^2u_{x'} + abu_{y'} + b^2u_{x'} - abu_{y'} \\ &= (a^2 + b^2)u_x'\end{aligned}$$

So we get that

$$\begin{aligned}&\Longrightarrow u_{x'} = 0 \\ &\Longrightarrow u(x', y') = f(y') \\ &\Longrightarrow u(x, y) = f(bx - ay)\end{aligned}$$

where f is arbitrary. To see the theory in action, let's do an example:

Example 5. *Find the solution to*

$$4u_x - 3u_y = 0$$

such that $u(0, y) = y^3$

Note the presence of a boundary condition. While before we were asking for a general class of functions that satisfy the PDE, this boundary condition restricts out set of valid answers to just a handful (usually just one).

We know that the general solution is of the form

$$u(x, y) = f(-3x - 4y)$$

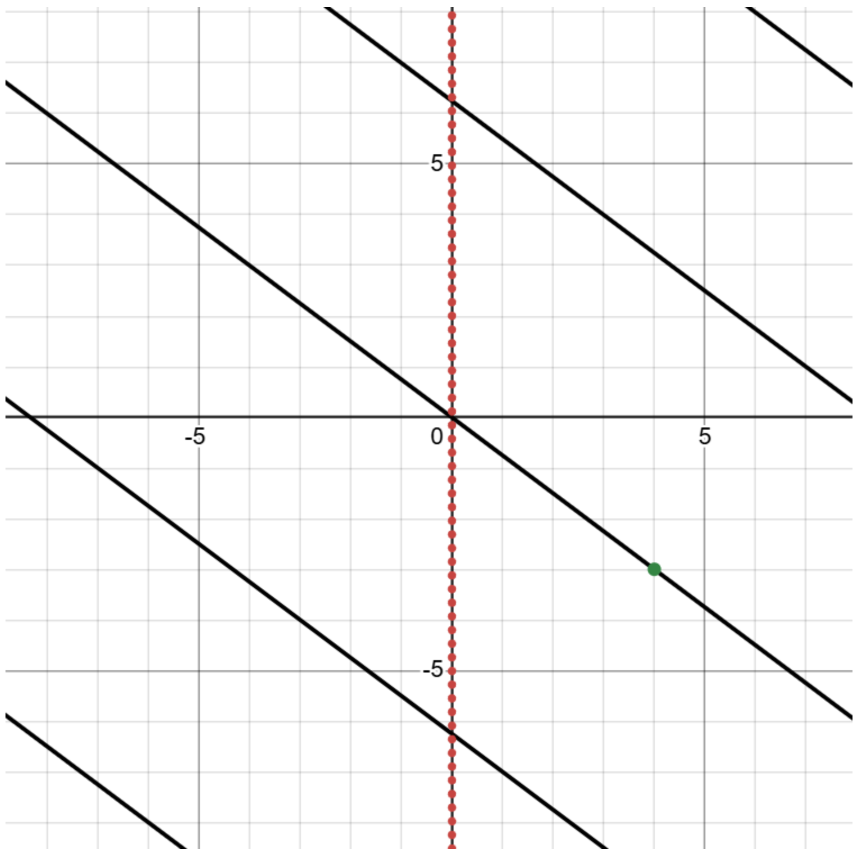
By the boundary condition, we also know that

$$y^3 = u(0, y) = f(-4y)$$

We want to determine $f(y)$. To do this we set $w = -4y$, meaning $y = \frac{-w}{4}$. Thus,

$$f(w) = \left(\frac{-w}{4}\right)^3 = \frac{-w^3}{64}$$
$$\implies u(x, y) = f(-3x - 4y) = \frac{(-3x - 4y)^3}{64}$$

What does this look like visually?



1.4 VARIABLE COEFFICIENTS

Before, we considered a, b to be real number constants, however, we can also consider situations where these are in fact functions of x and y. What's more interesting is that the general theory behind their solutions remains relatively unchanged, and relies on understand which curves. To see this, we'll first consider an example:

Example 6. *Find the general solution $u(x, y)$ of*

$$u_x + yu_y = 0$$

We use the geometric method. Rewriting as the dot product of the gradient vector yields

$$(\nabla u) \cdot (1, y) = 0$$

Before, it was clear what curves u was constant on: the lines $bx - ay = c$. But now the presence of y complicates things. In general, we want to find curves $(x, y(x))$ whose tangents are parallel to $(1, y)$ (as they should be orthogonal to the vector). The tangent vector of $(x, y(x))$ is $\left(1, \frac{dy}{dx}\right)$, and so we have that

$$\frac{dy}{dx}(x) = y(x) \implies y(x) = Ce^x$$

where C is any constant. We say that (x, Ce^x) is a ***characteristic curve****. The PDE is equivalent to saying that u is constant along such curves.*

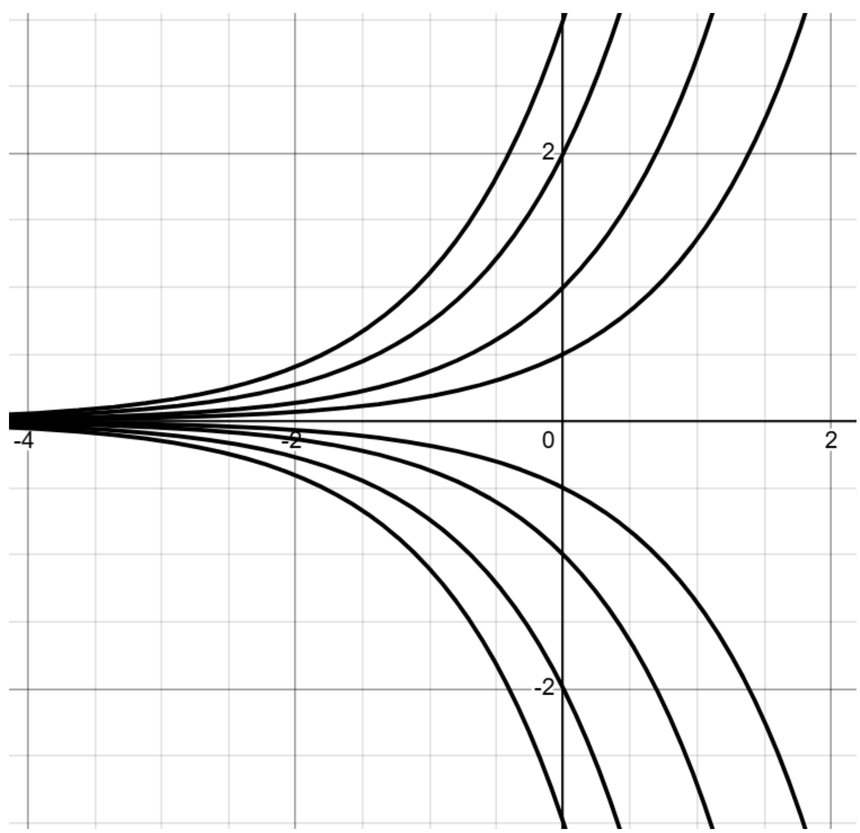
By the Chain Rule, we have that

$$\begin{aligned}\frac{d}{dx}[u(x, Ce^x)] &= u_x + u_y \frac{d}{dx}(Ce^x) \\ &= u_x + Ce^x u_y \\ &= u_x + y u_y \\ &= 0\end{aligned}$$

So we know that $u(x, Ce^x) = F(C)$ is some function of C. To solve for $u(x, y)$, we use the fact that $y = Ce^x$ and solve for C. Since $C = ye^{-x}$,

$$u(x, y) = F(ye^{-x})$$

where F is any function. Visually, the characteristic curves look like this:



It should be noted that every point in the plane must lie on a unique characteristic curve. Why? EXPLANATION REQUIRED

Example 7. *Find the general solution $u(x, y)$ to*

$$u_x + 2xy^2 u_y = 0$$

Like before, we rewrite as

$$(\nabla u) \cdot (1, 2xy^2) = 0$$

and by the same logic as previously, we require that

$$\frac{dy}{dx} = 2xy^2$$

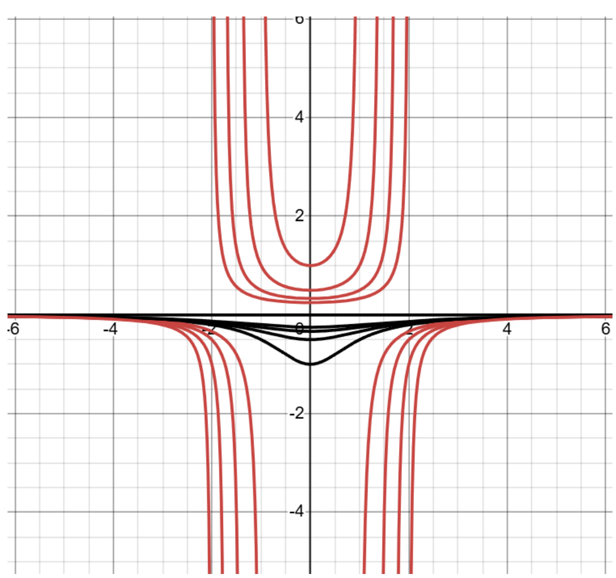
This is and ODE which can be solved with separation of variables. We first note the case that $y(x) = 0$. Then if $y \neq 0$:

$$\frac{dy}{dy^2} = 2x\ dx$$
$$\implies -\frac{1}{y} = x^2 + C$$
$$\implies y(x) = \frac{-1}{x^2 + C}$$

Note that this ODE may be difficult to solve, but this is fine because we only care about what C is, and we can easily solve for it. We have that

$$u(x, y) = F(C) = F\left(-\frac{1}{y} - x^2\right)$$

where F is arbitrary. What about the case where $y = 0$? Consider the picture:



Remark. *In these examples we took $a = 1$. For a general PDE with variable coefficients $f(x, y)u_x + g(x, y)u_y = 0$, we can always make it look like this by dividing by $f(x, y)$:*

$$u_x + \frac{g(x, y)}{f(x, y)} u_y = 0$$

In the case that $f(x, y) = 0$ at some point, we can just do the opposite and divide by $g(x, y)$, and the solution is symmetric.

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Week 2

2.1 THE ESSENTIAL PDES

In this section we introduce the main class of PDEs that will be studied in this class, which all come from physics. We will study them (usually) in their simplest form, using the least number of variables.

THE SIMPLE TRANSPORT EQUATION

We consider a 1-dimensional system in which a fluid is traveling through a pipe at a constant rate. If we let $u(t, x)$ represent the amount of fluid moving through the pipe at position x and at time t, then we get the PDE

$$u_t + cu_x = 0$$

where c is any constant. We call this the **simple transport equation**. This is a constant coefficient first order PDE, which we know to have solutions given by $u(t, x) = f(x - ct)$ for any function f.

Visually, this actually makes physical sense. For a solution $f(x - ct)$, we see that as t increases, the function literally moves at a constant rate along the x-axis.

THE WAVE EQUATION

We consider a flexible, elastic, and homogeneous string of length ℓ that moves up and down (transversely). Let $u(t, x)$ denote the height/displacement of the string. We then set

$$c = \sqrt{\frac{T}{\rho}}$$

where T is the tension in the string, and ρ is the density of the string. Then we get a PDE given by

$$u_{tt} - c^2 u_{xx} = 0$$

This is called the **wave equation**, and plays an important role in general relativity, fluid dynamics, and electromagnetism through Maxwell's Equations.

There are also many variations one the wave equation that are of relevance:

(i) Air resistance: given a constant $r > 0$ representing friction (or air resistance), we get

$$u_{tt} - c^2 u_{xx} + r u_t = 0$$

(ii) Elastic force in the transverse direction: suppose there was a force pushing back against the transverse movement of the string, given by $k > 0$. We get

$$u_{tt} - c^2 u_{xx} + k u = 0$$

This is called the **Klein-Gordon Equation**.

(iii) External forcing: Suppose $f(t, x)$ represents an external force in the system. Then we get

$$u_{tt} - c^2 u_{xx} = f(t, x)$$

The higher dimensional versions of these equations also show up often; the 3D version is seen in Maxwell's Equations. Furthermore, we can also consider c to be a function $c(t, x, y, z)$, which shows up in fluid dynamics and general relativity.

THE DIFFUSION/HEAT EQUATION

Suppose we have a chemical substance diffusing in a fluid, or say we are heating up an metal object. Given a function $u(t, x)$ representing this spread of the substance or heat with respect to position and time, then we have

$$u_t = k u_{xx}$$

where $k > 0$ is a constant related to the properties of the fluid or material. This is called the **heat/diffusion equation**. The higher dimensional forms are also relevant and play an interesting role in statistics through Brownian Motion.

THE LAPLACE EQUATION

Suppose u solves the heat or wave equation and is now settled and in a "stationary" state. We thus set time, and its related derivatives, to 0, and we get

$$u_{xx} = 0$$

The higher dimensional forms also exist and are extremely relevant to complex analysis. This is called the **Laplace Equation**, and the solutions to this equation are called **harmonic functions**.

2.2 INITIAL AND BOUNDARY CONDITIONS

Solutions to PDEs are generally a large class of functions; as we have seen previously, the set of solutions is an infinite dimensional vector space. To get a unique solution to a PDE, we often require additional conditions to be present. There are two common ways this can be achieved, which we discuss now.

INITIAL CONDITIONS

If we have, say, a time variable t, then we may specify the solution at a specific time t_0. Oftentimes we use $t_0 = 0$, but this t_0 may be any value.

Example 8. *Consider the heat equation $u_t = ku_{xx}$ with the initial condition that*

$$u(t_0, x) = \phi(x)$$

where ϕ is given. In this case, ϕ represents the distribution of heat at time t_0. If $t_0 = 0$, then this is the initial heat distribution.

Example 9. *Consider the wave equation $u_{tt} - c^2 u_{xx} = 0$ with the initial conditions that*

$$u(t_0, x) = \phi(x)$$
$$\frac{\partial u}{\partial t}(t_0, x) = \psi(x)$$

where ϕ, ψ are given. Note that we must specify the derivative with respect to t at t_0 because we use the second partial derivative of t in the wave equation. In this case, ϕ represents the height of the string at position x, while ψ represents its momentum.

BOUNDARY CONDITIONS

Suppose we have an unknown function $u(t, x_1, \ldots, x_{n-1}) : D \to \mathbb{R}$ or $\mathbb{C}$, where $D \subset \mathbb{R}^n$. As an explicit example, one can think of a 1-dimensional pipe with the heat equation, or a 2-dimensional disk (like that of the surface of a drum) with the wave equation. We will often require that some condition holds along the *boundary* of these domains, such as the ends of the pipe or the rim of the drum's surface. These conditions are called boundary conditions, and there are 3 standard ways we can define them:

(i) **Dirichlet Conditions** $u\big|_{\partial D} = \phi(x)$

(ii) **Neumann Conditions** $\left.\dfrac{\partial u}{\partial n}\right|_{\partial D} = \psi(x)$

(iii) **Robin Conditions** $\left.\left(\left.\dfrac{\partial u}{\partial n}\right|_{\partial D} + \alpha u\right)\right|_{\partial D} = \chi(x)$

where ϕ, ψ, χ are given.

Remark. *The term $\dfrac{\partial u}{\partial n}$ is the normal derivative, the directional derivative in the direction that is normal to the curve at the desired point.*

If the given function is the zero function, then we call the condition a **homogeneous condition**, otherwise, it is a **inhomogeneous condition**.

Example 10. *Consider the problem of the vibrating string. If we include a Dirichlet condition, then this can be thought of as holding the ends of the string fixed while the rest of the string vibrates. Think of this like the strings of a guitar.*

Including a Neumann condition can be thought of as tying the ends of the string to vertical poles and pulling the string tight, meaning only the ends of the string move freely.

Example 11. *Consider the problem of heat distribution on a metal rod. Including a Dirichlet condition may be thought of as fixing the rod's temperature along the boundary.*

Including a homogeneous Neumann condition may be thought of as insulating the rod, ensuring that no heat leaves the system.

Note that boundary conditions can also exist at infinity. For instance if $D = \mathbb{R} = \mathbb{R}^2$. For the 1-dimensional heat equation, we may require that

$$\lim_{x\to\infty} u(t,x) = 0$$

2.3 WELL-PROVED PROBLEMS

For a PDE problem to exist as a physical problem; that is, a problem that can exist in the real world and can be done in a physics class, three conditions must hold:

1. **Existence**: There is a solution to the PDE (under reasonable assumptions)
2. **Uniqueness**: There is at most one solution to the PDE
3. **Stability**: Small changes to the inputs must yield only small changes to the solution

2.4 CONSTANT COEFFICIENT 2ND ORDER PDE IN 2 VARIABLES

These types of PDEs will form the basis of our study of PDEs throughout the course, as they form the general class of PDEs from which the simple transport, wave, heat, and Laplace equations arise from.

Such equations are of the form

$$a_{11}u_{xx} + 2a_{12}u_{xy} + a_{22}u_{yy} + a_1u_x + a_2u_y + a_0u = 0$$

where $a_{11}, a_{12}, a_{22}, a_1, a_2, a_0$ are real coefficients. Like in ODEs where we cared more about the terms with first derivatives compared to other terms, we now care about the terms with second derivatives compared to the lower order terms.

Understanding the type of 2nd order PDE is of great importance and allows up to better understand how to approach solving them. There is a relatively simple algorithm for determining type, and how to simplify the equation:

Theorem 2.1. *Given a constant coefficient 2nd order PDE in two variables, then there exists a change of variables*

$$x' = b_1x + b_2y$$
$$y' = b_3x + b_4y$$

where b_1, b_2, b_3, b_4 are real constants with

$$\det\begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} \neq 0$$

such that

(i) ***Elliptic Case****: If $a_{12}^2 < a_{11}a_{22}$, then the PDE reduces to*

$$u_{x'x'} + u_{y'y'} + \text{ lower order terms} = 0$$

(ii) ***Hyperbolic Case****: If $a_{12}^2 > a_{11}a_{22}$, then the PDE reduces to*

$$u_{x'x'} - u_{y'y'} + \text{ lower order terms} = 0$$

(iii) ***Parabolic Case****: If $a_{12}^2 = a_{11}a_{22}$, then the PDE reduces to*

$$u_{x'x'} + \text{ lower order terms } = 0 \text{ OR lower order terms} = 0$$

Notice that the elliptic case is the 2D Laplace equation, while the second case is the wave equation.

Example 12. *Consider $u_{xx} - 5u_{xy} = 0$. We have that $a_{11} = 1, a_{12} = \frac{5}{2}, a_{22} = 0$, hence*

$$a_{12}^2 = \frac{25}{5} > 0 = a_{11}a_{22}$$

so this is hyperbolic.

While such equations won't be the main focus of the course, it should be noted that it is possible to formulate this idea for second order PDEs with variable coefficients. Say we have

$$yu_{xx} - 2u_{xy} + xu_{yy} = 0$$

If we specify a point (x, y), then we get constant coefficients and the theorem applies, meaning we can determine the PDE's type, *at that point*. This means the PDE will change type depending on the point in the plane.

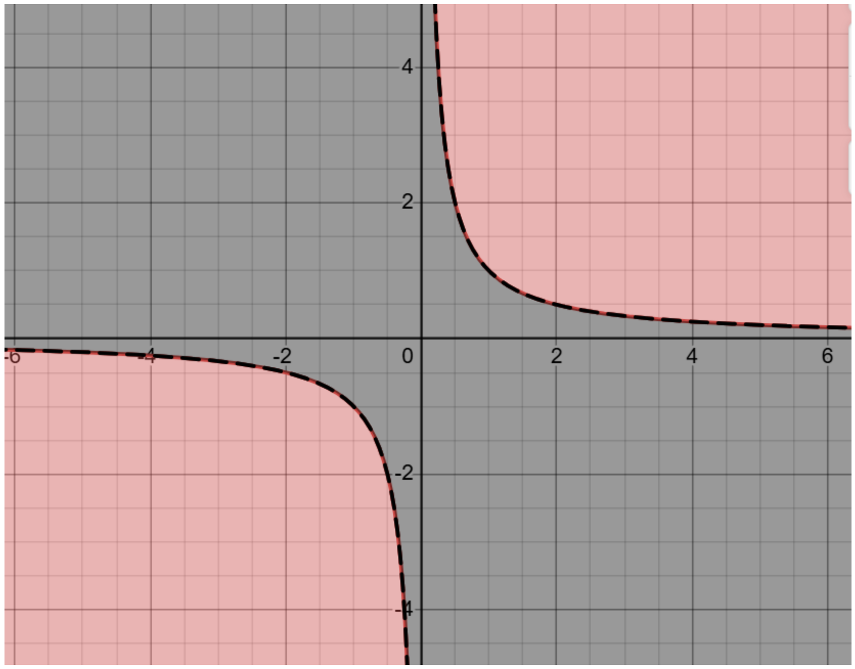
In the above case, we have that $a_{12}^2 = 1$, while $a_{11}a_{22} = yx$. Thus, we get that

$$1 = yx \implies \text{ parabolic}$$

$$1 > yx \implies \text{ hyperbolic}$$

$$1 < yx \implies \text{ elliptic}$$

This can be seen in the graph below, where the red area represents when the PDE is elliptic, while the black represents where it is hyperbolic; the intersection is where it is parabolic:



Let's find the change of variables for the equation

$$u_{xx} - 5u_{xy} = 0$$

First consider

$$x^2 - 5xy$$

Completing the square yields

$$\begin{aligned} x^2 - 5xy &= x^2 - 5xy + \left(\frac{5}{2}\right)^2 y^2 - \left(\frac{5}{2}\right)^2 y^2 \\ &= \left(x - \frac{5}{2}y\right)^2 - \left(\frac{5}{2}y\right)^2 \end{aligned}$$

Surprisingly, this same idea also works for partial derivatives:

$$\begin{aligned} u_{xx} - 5u_{yy} &= \frac{\partial}{\partial x}\frac{\partial}{\partial x}u - 5\frac{\partial}{\partial x}\frac{\partial}{\partial y}u \\ &= \left(\frac{\partial}{\partial x} - \frac{5}{2}\frac{\partial}{\partial y}\right)^2 u - \left(\frac{5}{2}\frac{\partial}{\partial y}\right)^2 u \end{aligned}$$

Now, we need (x', y') such that

$$\frac{\partial}{\partial x'} = \frac{\partial}{\partial x} - \frac{5}{2}\frac{\partial}{\partial y}\,, \quad \frac{\partial}{\partial y'} = \frac{5}{2}\frac{\partial}{\partial y}$$

We see that

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial x'} + \frac{\partial}{\partial y'} \ , \ \frac{\partial}{\partial y} = \frac{2}{5}\frac{\partial}{\partial y'}$$

Recalling the definition of x', y', we get by the Chain Rule that

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial x'}b_1 + \frac{\partial}{\partial y'}b_3$$
$$\frac{\partial}{\partial y} = \frac{\partial}{\partial x'}b_2 + \frac{\partial}{\partial y'}b_4$$

so we have that $b_1 = 1, b_2 = 0, b_3 = 1, b_4 = \frac{2}{5}$. Thus,

$$x' = x$$
$$y' = x + \frac{2}{5}y$$

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Week 3

3.1 THE WAVE EQUATION

We will now begin a relatively deep discussion on one of the standard types of second order PDE: the hyperbolic case. By the theorem from last week, we know such a PDE can always be reduced to one that looks like the wave equation, so this is what we will focus on.

3.1.1 A GENERAL SOLUTION

The wave equation is as follows:

$$u(x,t) : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \text{ or } \mathbb{C}$$
$$u_{tt} - c^2 u_{xx} = 0$$

as $x \in (-\infty, \infty)$, we can think of the physical problem of oscillating an infinitely long string of wire.

Such an equation will correspond to a polynomial

$$t^2 = c^2 x^2 = (t - cx)(t + cx)$$

as it can be factored as a difference of squares, we can do the same with the equation itself:

$$\iff \left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right) u = 0$$

From here there are a few methods that can solve it. For one, we can take $w = \left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right) u$ and solve the equation

$$\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right) w = 0$$

first before then solving for w. The method we will use here is an analog of the coordinate method from week 1, which is now referred to as **characteristic coordinates**.

We define

$$\xi = x + ct, \ \eta = x - ct$$

This transformation makes sense as its Jacobian has determinant $-2c \neq 0$. By the Chain Rule, we have that

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial}{\partial \eta}\frac{\partial \eta}{\partial x} = \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}$$
$$\frac{\partial}{\partial t} = \frac{\partial}{\partial \xi}\frac{\partial \xi}{\partial t} + \frac{\partial}{\partial \eta}\frac{\partial \eta}{\partial t} = c\frac{\partial}{\partial \xi} - c\frac{\partial}{\partial \eta}$$

So we can rewrite our equations as

$$\begin{aligned}\frac{\partial}{\partial t} - c\frac{\partial}{\partial x} &= c\frac{\partial}{\partial \xi} - c\frac{\partial}{\partial \eta} - c\left(\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}\right) \\ &= (-2c)\frac{\partial}{\partial \eta} \\ \frac{\partial}{\partial t} + c\frac{\partial}{\partial t} &= c\frac{\partial}{\partial \xi} - c\frac{\partial}{\partial \eta} + c\left(\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \xi}\right) \\ &= (2c)\frac{\partial}{\partial \xi}\end{aligned}$$

Thus, the wave equation becomes, in (ξ, η),

$$4c^2\left(\frac{\partial}{\partial \xi}\frac{\partial}{\partial \eta}\right)u = 0 \iff u_{\xi\eta} = 0$$

Recalling our solution to $u_{xy} = 0$ from week 1, we get that

$$u(\xi, \eta) = f(\xi) + g(\eta) \iff u(x,t) = f(x+ct) + g(x-ct)$$

where f, g are arbitrary C^2 functions.

What does this look like? Recalling the simple transport equation, we see that as t increases, f will move to the left at speed c, while g will move to the right at speed $-c$. This makes the f and g functions look like waves as time progresses. Here, we say that c is the **speed of propagation**. If you recall from physics, we denote the speed of light by c; the fact that we use c for both of these is not a coincidence.

Remark. *This is a very special case, and not all PDEs of second order are going to have nice, easy to derive general solutions.*

This method of factorization works in other settings, particularly on any hyperbolic constant coefficient PDE with only second order terms.

3.1.2 INITIAL VALUE PROBLEMS

In order for the problem to be well-posed, we require initial values to be added and our domain for time to be nonnegative.

$$u(x,t) : \mathbb{R} \times [0, \infty) \to \mathbb{R} \text{ or } \mathbb{C}$$
$$u_{tt} - c^2 u_{xx} = 0,\ u(x,0) = \varphi(x),\ u_x(x,0) = \psi(x)$$

Recall that our general solution is $u(x,t) = f(x+ct) + g(x-ct)$, but now we have the added initial conditions, which correspond to two equations:

$$\varphi(s) = f(s) + g(s) \tag{3.1}$$
$$\psi(s) = cf'(s) - cg'(x) \tag{3.2}$$

To find the specific solution, we must solve for f, g. To do this, we first derive φ to get

$$\varphi'(s) = f'(s) + g'(s) \tag{3.3}$$

Now, combining (3.2) and (3.3) in two separate ways yields

$$\psi(s) + c\varphi'(s) = 2cf'(s)$$
$$c\varphi'(s) - \psi(s) = 2cg'(s)$$

Dividing by $2c$ and integrating gives us

$$f(s) = \frac{1}{2}\varphi(s) + \frac{1}{2c}\int_0^s \psi(\tau)\ d\tau + A$$
$$g(s) = \frac{1}{2}\varphi(s) - \frac{1}{2c}\int_0^s \psi(\tau)\ d\tau + B$$

Notice that the constants make f, g not unique. This is ok, as long as u is unique. To check this, we can plug f, g into (3.1), which says

$$\varphi(s) = \varphi(s) + A + B \iff A + B = 0$$

Thus, our final solution is given by

$$\begin{aligned} u(x,t) &= f(x+ct) + g(x-ct) \\ &= \frac{1}{2}\left[\varphi(x+ct) + \varphi(x-ct)\right] + \frac{1}{2c}\int_0^{x+ct} \psi(\tau)\ d\tau - \frac{1}{2c}\int_0^{x-ct} \psi(\tau)\ d\tau \end{aligned}$$

as $t > 0$, $x - ct < x + ct$, so

$$= \frac{1}{2}\left[\varphi(x+ct) + \varphi(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(\tau)\ d\tau$$

This is indeed the unique solution for this problem that is continuous, as required.

Example 13. *Solve $u_{tt} - c^2 u_{xx} = 0$ such that $u(x,t) : \mathbb{R} \times [0,\infty) \to \mathbb{R}$ and*

$$u(x,0) = 0, u_t(x,0) = \cos(x)$$

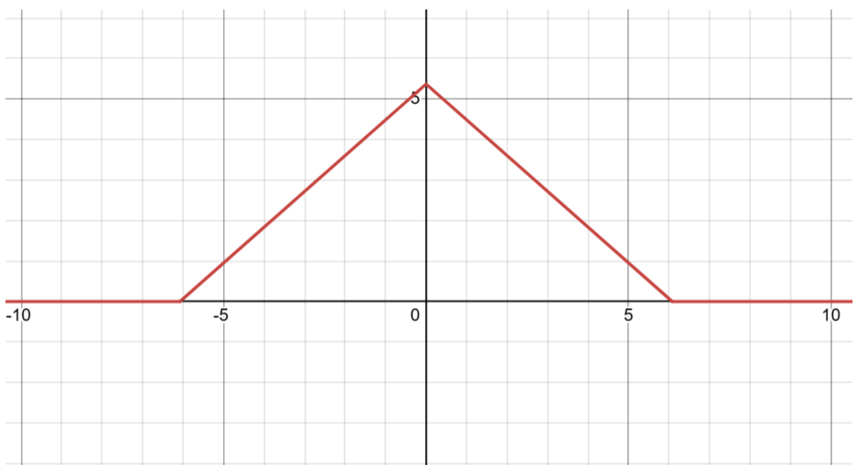
By the above, we have that

$$u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} \cos(\tau)\ d\tau$$
$$= \frac{1}{2c}[\sin(x-ct) - \sin(x+ct)]$$

Example 14 (The Plucked String Problem)**.** *Take $a, b > 0$, and consider the wave equation as described above, with the initial values*

$$\varphi(x) = u(x,0) = \begin{cases} b - \frac{b|x|}{a} & |x| \leq a \\ 0 & |x| > a \end{cases}$$
$$u_t(x,0) = 0$$

Graphically, this looks like the following:



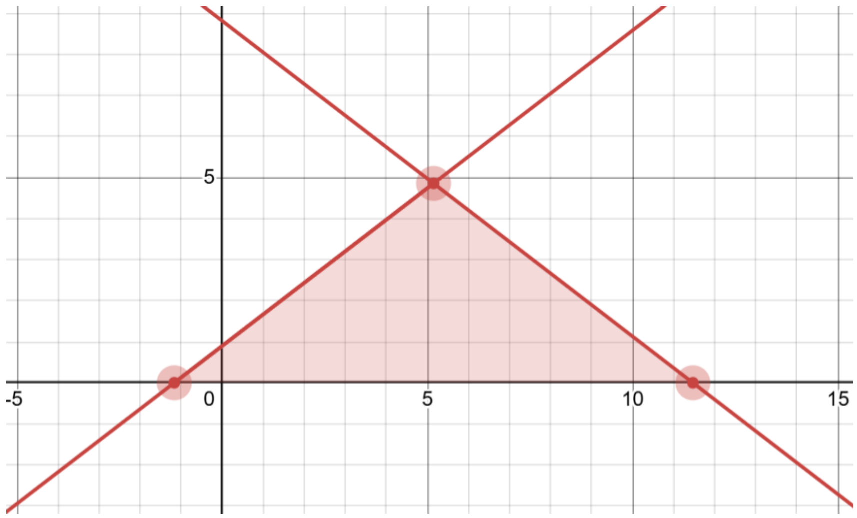
Then by the formula,

$$u(x,t) = \frac{1}{2}[\varphi(x+ct) + \varphi(x-ct)]$$

Now, the graph becomes two waves, moving in opposite directions, with half the amplitude (height) from before.

3.1.3 THE PATH LIGHT CONE

Suppose we wanted to compute the value $u(x_0, t_0)$ for some specific values of (x_0, t_0). The specific solution to the wave equation tells us that, to do this, we need to know the initial values of points ranging from $x_0 - ct_0$ to $x_0 + ct_0$. Graphically this can be described by a triangle, or cone:



We call this the **path light cone** or **domain of independence**; only points which lie in this region will affect our selected point.

From a physical perspective, if we were to consider this as light traveling as a wave through spacetime, this says that the speed of light is finite, as light can only affect future spacetime in a finite region.

As a consequence of this, for an interval I on which φ, ψ vanish, if $u(x,t)$ is such that its past light curve is inside I, then $u(x,t) = 0$. Thus, if φ, ψ are compactly supported, meaning they are nonzero in a bounded interval $[-A, A]$, then $u(x,t)$i s compactly supported at each time t.

3.1.4 ENERGY

Let $u(x,t) : \mathbb{R} \times [0, \infty) \to \mathbb{R}$ solve the wave equation, and suppose that $u(x,0), u_t(x,0)$ are compactly supported, meaning u is compactly supported for each t. WE define

$$E(t) = \frac{1}{2} \int_{-\infty}^{\infty} [(u_t(x,t))^2 + c^2(u_x(x,t))^2] \, dx$$

$E(t)$ is a well-defined function, since u_x, u_t are compactly supported. We call this function **energy**.

With this, we can prove a fundamental law of the physical world by simple algebra: the conservation of energy:

Theorem 3.1 (Law of Conservation of Energy)**.** *$E(t)$ is constant.*

Proof. We show that $E(t)$ has a derivative of 0. Using differentiation under the integral sign, we get that

$$\begin{aligned} \frac{dE}{dt} &= \frac{1}{2} \int_{-\infty}^{\infty} \left[\frac{\partial}{\partial t}(u_t)^2 + c^2 \frac{\partial}{\partial t}(u_x)^2 \right] \, dx \\ &= \int_{-\infty}^{\infty} [u_t u_{tt} + c^2 u_t u_{xt}] \, dt && \text{(Chain Rule)} \end{aligned}$$

as $u_{tt} = c^2 u_{xx}$,

$$= c^2 \int_{-\infty}^{\infty} [u_t u_{xx} + u_x u_{xt}]\, dx$$

Now integrate by parts:

$$\begin{aligned} &= c^2 \int_{-\infty}^{\infty} [-u_{xt} u_x + u_x u_{tx}]\, dx + c^2 u_t u_x \big|_{-\infty}^{\infty} \\ &= 0 \qquad\qquad (u_x, u_t \text{ compactly supported}) \end{aligned}$$

□

3.2 THE HEAT EQUATION

We now move to studying the heat equation. Recall that it is given by

$$u(x,t) : [0,\ell] \times [0,\infty) \to \mathbb{R}$$
$$u_t - k u_{xx} = 0$$

where $k > 0$ is constant. Solving this equation is much more complicated than the wave equation. However, we can still answer questions about the solutions and their properties in spite of not knowing the exact solutions. This is due to an important principle of such PDEs:

3.2.1 THE MAXIMUM PRINCIPLE

Theorem 3.2 (The Maximum Principle)**.** *Suppose $u(x,t)$ solves the heat equation for $x \in [0,\ell]$ and $t \in [0,T]$. Then $u(x,t)$ attains a maximum, and it is attained on one of the line $\{x = 0\}, \{x = \ell\}$, or $\{t = 0\}$.*

This follows from the Extreme Value Theorem, which holds because u is a continuous function on a closed and bounded set.

Intuitively, this just says that heat will flow from areas of high temperature to areas of low temperature. This matches with our expectations.

3.2.2 INITIAL VALUE PROBLEMS

We can consider an initial value problem for the heat equation, given by

$$\begin{aligned} u_t - k u_{xx} &= f(x,t) \\ u(x,0) &= \varphi(x) \\ u(0,t) &= g(t) \\ u(\ell,t) &= h(t) \end{aligned}$$

where f, φ, g, h are given functions. Note that φ represents the initial heat distribution along the bar.

The Maximum Principle can help us show that this IVP attains at most one solution.

Theorem 3.3. *There can only be at most one solution to the above IVP.*

We present two proofs:

Proof 1. Let $u_1(x,t), u_2(x,t)$ be solutions to the IVP. By linearity, we can set

$$w = u_1 - u_2$$

and this will solve the equation $u_t - ku_{xx} = 0$. In addition, we have that

$$\begin{aligned}
w(x,0) &= u_1(x,0) - u_2(x,0) = \varphi - \varphi = 0 \\
w(0,t) &= g - g = 0 \\
w(\ell,t) &= h - h = 0
\end{aligned}$$

Now, by the Maximum Principle, we know that w attains its maximum along one of the lines $\{x = 0\}, \{x = \ell\}, \{t = 0\}$, but this tells us that

$$\max_{[0,\ell]\times[0,T]} w = 0$$

so $w(x,t) \leq 0$. By the same logic for $-w$, we get that $w(x,t) \geq 0$. Thus,

$$w(x,t) = u_1(x,t) - u_2(x,t) = 0 \implies u_1(x,t) = u_2(x,t)$$

$\square$

The next proof uses what is called the **energy method**:

Proof 2. We define the energy of this system as

$$E(t) = \frac{1}{2}\int_0^\ell (u(x,t))^2 \, dx$$

where the $\frac{1}{2}$ is for a nicer calculation. Intuitively, heat dissipates over time, and so $\frac{dE}{dt}$ should be less than or equal to 0. If so, then we get

$$\frac{1}{2}\int_0^\ell (w(x,t))^2 \, dx \leq \frac{1}{2}\int_0^\ell (w(x,0))^2 \, dx = 0$$

As the integral vanishes for all $t \geq 0$, we thus get that $w(x,t) = 0$, which is what we want. Let's now prove this claim:

Suppose $u(x,t) : [0,\ell] \times [0,\infty) \to \mathbb{R}$ solves the homogeneous heat equation $u_t - ku_{xx} = 0$, with $u(0,t) = u(\ell,t) = 0$. Then

$$\begin{aligned}
\frac{dE}{dt} &= \frac{d}{dt}\frac{1}{2}\int_0^\ell (u(x,t))^2\ dx \\
&= \frac{1}{2}\int_0^\ell \frac{\partial}{\partial t}(u(x,t))^2\ dx \\
&= \int_0^\ell uu_t\ dx \\
&= k\int_0^\ell uu_{xx}\ dx && (u_t = ku_{xx}) \\
&= -k\int_0^\ell (u_x)^2\ dx + kuu_x\big|_0^\ell \\
&= -k\int_0^\ell (u_x)^2\ dx + 0 \\
&\le 0
\end{aligned}$$

as desired. $\square$

4

Week 4

4.1 THE HEAT EQUATION, CONTINUED

4.1.1 THE HEAT EQUATION ON AN INFINITE ROD

We will try and prove some formulas related to the heat equation, in the special case that we are dealing with a rod of infinite length. By doing this we are considering the case where $x \in \mathbb{R}$. We will also not consider 0 in the domain of t; the reason why will be obvious.

Remark. *The derivation we present here will seem somewhat random, as the tool used for deriving it, called the Fourier Transform, is not a core part of this course. The textbook presents another similar derivation that may yield better insights for some as to how we derive the following.*

We first define a function $S(x,t) : \mathbb{R} \times (0, \infty) \to \mathbb{R}$ by

$$S(x,t) = \frac{1}{\sqrt{4\pi kt}} e^{\frac{-x^2}{4kt}}$$

Note that $S(x,t) > 0$ for all x, t in the domain.

Proposition 4.1. *S solves the heat equation.*

Proof. We have

$$\begin{aligned}
\partial_t S &= \frac{1}{\sqrt{4\pi k}} \cdot \frac{-1}{2t^{3/2}} e^{\frac{-x}{4kt}} + \frac{1}{\sqrt{4\pi kt}} \frac{x^2}{4kt^2} e^{\frac{-x}{4kt}} \\
&= \frac{-1}{2} t^{-1} S + \frac{x^2}{4kt^2} S \\
\partial_x S &= \frac{1}{\sqrt{4\pi kt}} \cdot \frac{-x}{2kt} e^{\frac{-x^2}{4kt}} \\
&= \frac{-x}{2kt} S \\
\partial_{xx} S &= \frac{-1}{2kt} S - \frac{x}{2kt} \partial_x S \\
&= \frac{-1}{2kt} S + \frac{x^2}{4k^2t^2} S
\end{aligned}$$

Thus,

$$\begin{aligned}\partial_t S - k\partial_{xx} S &= S\left(\frac{-1}{2t} + \frac{x^2}{4kt^2} + \frac{1}{2t} - \frac{x^2}{4kt^2}\right)\\ &= 0\end{aligned}$$

as desired. □

Before we proceed, recall the Gaussian Integral:

$$\int_{-\infty}^{\infty} e^{-x^2}\ dx = \sqrt{\pi}$$

For each $t > 0$, we may consider

$$\int_{-\infty}^{\infty} S(x,t)\ dx$$

Informally, note that for fixed t, S will go to 0 exponentially fast as $x \to \pm\infty$. Thus, this integral is well-defined. We write it as

$$\frac{1}{\sqrt{4\pi kt}}\int_{-\infty}^{\infty} e^{-\left(\frac{x}{2\sqrt{kt}}\right)^2}\ dx$$

We now make a change of variables with

$$y = \frac{x}{2\sqrt{kt}}\ ,\ dy = \frac{1}{\sqrt{kt}}\ dx$$

Under this change, we get

$$\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-y}\ dy = \frac{1}{\sqrt{\pi}}\sqrt{\pi} = 1$$

We sometimes call this function S the **heat kernel**. Let's try and understand what happens to the heat kernel as $t \to 0$. We first make some observations:

1. When $x = 0$,

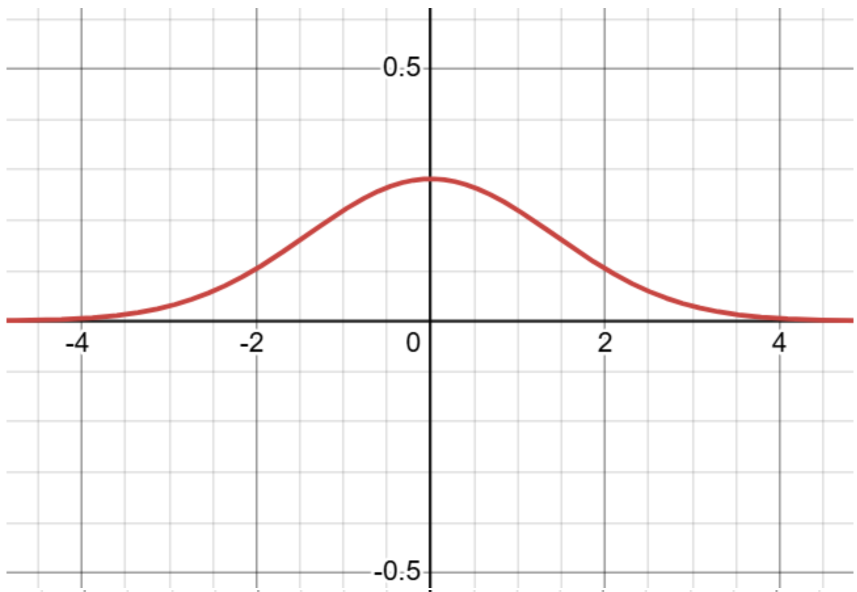
$$S(x,t) = \frac{1}{\sqrt{4\pi kt}} \xrightarrow{t\to 0} \infty$$

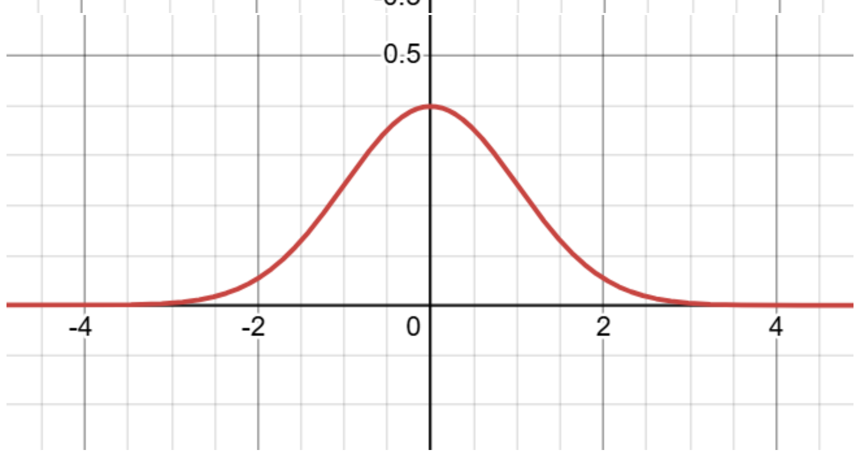
2. Suppose $|x| > \delta > 0$, where δ is small. In this case,

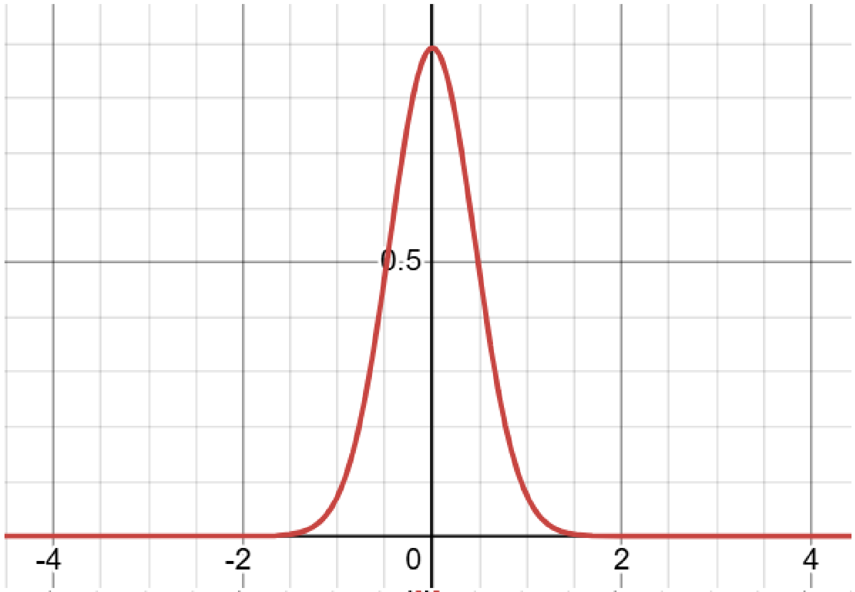
$$|S(x,t)| \leq \frac{1}{\sqrt{4\pi kt}} e^{\frac{-\delta^2}{4kt}}$$

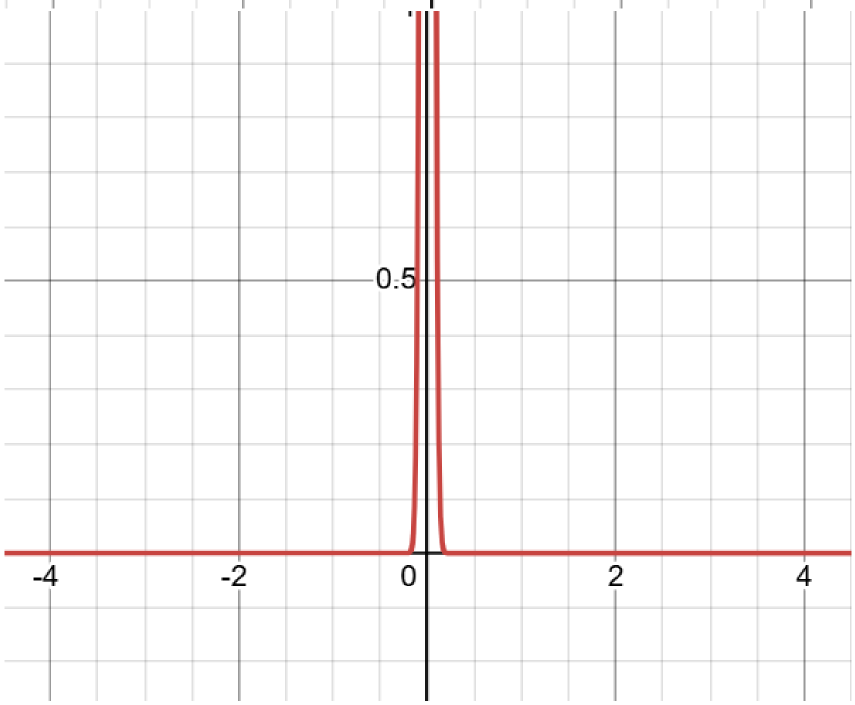
as $t \to 0$, the exponential is moving to 0 faster than the $\sqrt{4\pi kt}$ can blow S up to infinity, and so we are going to 0.

These observations help us understand what is going on: we start with a Gaussian-like curve with integral value 1. as t gets smaller, this curve gets thinner and thinner, and S is non-zero in a very small area around 0, while at 0 the value gets larger to preserve the area. Finally, at $t = 0$, we have a function that is 0 everywhere except at $x = 0$, where it has a value of infinity. This is shown below, where we show the graphs of S for $t = 1, \frac{1}{2}, \frac{1}{10}$, and $t = \frac{1}{1000}$.









Informally, when $t = 0$:

$$S(x,0) = \begin{cases} \infty & x = 0 \\ 0 & x \neq 0 \end{cases}$$

We call S an **approximation to the identity**.

S special because it is related to an important operation in PDE theory. For a continuous function $\phi(x)$ and $t > 0$, we can define

$$u(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\phi(y)\ dy$$

called the **convolution** of S and ϕ.

Proposition 4.2. *If $t > 0$, then u solves the heat equation.*

Proof. The result mainly follows from S solving the heat equation. We have by differentiation under the integral sign,

$$\partial_t u = \int_{-\infty}^{\infty} (\partial_t S)(x-y,t)\phi(y)\ dy$$
$$\partial_{xx} u = \int_{-\infty}^{\infty} (\partial_{xx} S)(x-y,t)\phi(y)\ dy$$

Thus,

$$\partial_t u - k\partial_{xx} = \int_{-\infty}^{\infty} (\partial_t S - k\partial_{xx} S)(x-y,t)\phi(y)\ dy = 0$$

as desired. □

The relevance of ϕ becomes clear once we notice that as we approach 0, the convolution goes to $\phi(x)$.

Proposition 4.3. $\lim_{t\to 0} u(x,t) = \phi(x)$.

Proof. We have that

$$u(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\phi(y)\ dy$$

Set $z = x - y,\ dz = -dy$. This flips the bounds of integration, producing a double negative. We get

$$= \int_{-\infty}^{\infty} S(z,t)\phi(x-z)\ dz$$

for $0 < t < 1$, we get

$$\approx \int_{z \text{ close to } 0} S(z,t)\phi(x)\ dz$$
$$= \phi(x)\int_{-\infty}^{\infty} S(z,t)\ dz$$
$$= \phi(x)$$

□

4.1.2 IVPS ON THE INFINITE ROD

We consider the IVP

$$\begin{cases} u_t - ku_{xx} = 0 & (x,t) \in \mathbb{R} \times (0,\infty) \\ u(x,0) = \phi(x) \end{cases}$$

Our work in the previous subsection shows that a solution to this IVP is

$$u(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\phi(y)\ dy$$

Note that this only deals with the *existence* of a solution, not uniqueness or whether it is stable or not. We will addresss uniqueness shortly.

Example 15. *Solve*

$$\begin{cases} u_t - ku_{xx} = 0 \\ u(x,0) = e^{-x} \end{cases}$$

We know a solution is

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4kt}} e^{-y}\ dy$$

We will simplify the term in the integral as follows:

$$\begin{aligned}
\exp\left(\frac{-(x-y)^2}{4kt} - y\right) &= \exp\left(\frac{-x^2+2xy-y^2}{4kt} - y\right) \\
&= \exp\left(\frac{-x^2+2xy-y^2-4kty}{4kt}\right) \\
&= \exp\left(\frac{-y^2-(4kt-2x)y-x^2}{4kt}\right) \\
&= \exp\left(\frac{-y^2-2(2kt-x)y-(2kt-x)^2+(2kt-x)^2-x^2}{4kt}\right) \\
&= \exp\left(\frac{-(y+(2kt-x)^2}{4kt} + \frac{4k^2t^2-4ktx}{4kt}\right) \\
&= \exp\left(\frac{-(y+(2kt-x)^2}{4kt}\right)\exp(kt-x) \\
&= \exp\left(-\left(\frac{y+(2kt-x)}{2\sqrt{kt}}\right)^2\right)\exp(kt-x)
\end{aligned}$$

Now setting $z = \dfrac{y+2kt-x}{2\sqrt{kt}},\ dz = \dfrac{dy}{\sqrt{4kt}}$, we get

$$
\begin{aligned}
u(x,t) &= \frac{\exp(kt-x)}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-z^2}\ dz \\
&= \exp(kt-x)
\end{aligned}
$$

4.1.3 UNIQUENESS ON THE INFINITE ROD

What does it mean when a solution to the IVP is *unique*? Given $\phi(x)$, suppose

$$u_1(x,t) = u_2(x,t)$$

solve the IVP defined at the start of last section. It follows that

$$
\begin{aligned}
(u_1)_t - k(u_1)_{xx} = (u_2)_t - k(u_2)_{xx} = 0 \\
u_1(x,0) = u_2(x,0) = \phi(x)
\end{aligned}
$$

so in general, is $u_1 = u_2$? No!

Let's rephrase the above. Define $w = u_1 - u_2$. Now if

$$
\begin{aligned}
w_t - kw_{xx} = 0 \\
w(x,0) = 0
\end{aligned}
$$

does $w = 0$? Yes! But with more mild assumptions added onto it.

Recall our solution from before

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} \exp\left(\frac{-(x,y)^2}{4kt}\right) \phi(y)\ dy$$

Let's assume that $\phi(x)$ is both continuous and *compactly supported*, meaning it is nonzero only on some closed interval $[-R,R]$. What happens when we take x to $\pm\infty$? Well, the exponential will go to 0, so the integral will also go to 0. Thus $u \to 0$

For any $T > 0$, define

$$A(x) = \max_{t\in[0,T]} u(x,t)$$

By the above logic, we get that $\lim_{x\to\pm\infty} A(x) = 0$. This is our extra assumption, and it acts like a boundary condition at $\pm\infty$.

Theorem 4.4 (Uniqueness of Heat Equation on the Infinite Rod)**.** *Suppose that $w(x,t)$: $\mathbb{R}\times(0,\infty)\to\mathbb{R}$ solves*

$$
\begin{aligned}
w_t - kw_{xx} = 0 \\
w(x,0) = 0
\end{aligned}
$$

and for each $T > 0$,

$$\lim_{x\to\pm\infty} \max_{t\in[0,T]} w(x,t) = 0$$

Then $w(x,t) = 0$ everywhere.

Proof. We apply the Maximum Principle. Let $T > 0$ and pick R to be very large. By the Maximum Principle,

$$\max_{[-R,R]\times[0,T]} w(x,t) = \max_{t=0} w(x,t) \text{ or } \max_{x=-R} w(x,t) \text{ or } \max_{x=R} w(x,t)$$

We know that $w(x,0) = 0$. Furthermore, by our extra assumption, the second and third max values will go to 0 as R goes to infinity. Thus,

$$\max_{t\in(0,T)} w(x,t) \leq 0$$

We repeat this idea for $-w$, which tells us that

$$\min_{t\in(0,T)} w(x,t) \geq 0$$

Combining, we get that

$$w(x,t) = 0\ ,\ t \in (0,T)$$

Taking $T \to \infty$ completes the proof. □

4.2 COMPARING THE HEAT AND WAVE EQUATIONS

There are several differences between the heat and wave equations that are of importance to us. We're focusing on qualatiative properties here, not quantitative ones.

1. The Speed of Propagation.

   We know that waves have a finite speed of propagation which is related to the value c. As $c \to \infty$, the speed of propagation increases, and in the limit, the light path curve will encompass the entire (x,t) plane.

   Contrast this with the heat equation, which actually has *infinite* speed of propagation.

   Theorem 4.5. *Let $\phi(x)$ be continuous with $\phi(x) \geq 0$ and suppose there is an x_0 such that $\phi(x_0) > 0$. Set*

   $$u(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\phi(y)\ dy$$
   $$S(x,t) = \frac{1}{\sqrt{4\pi kt}} e^{\frac{-x^2}{4kt}}$$

   which solves the heat equation $u_t - ku_{xx} = 0$ with $u(x,0) = \phi(x)$. Then if $t > 0$, $u(x,t) > 0$ for all x.

   Proof. $S(x,t) > 0$ when $t > 0$, so $S(x-y,t) > 0$ when $t > 0$. Now

   $$u(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\phi(y)\ dy$$

We know there is a point y_0 such that $\phi(y_0) > 0$, and ϕ is continuous. Thus there is an interval I on which ϕ is strictly positive. Then

$$\int_{-\infty}^{\infty} S(x-y,t)\phi(y)\ dy \geq \int_I S(x-y,t)\phi(y)\ dy > 0$$

□

what this means is simple: At $t = 0$, our rod has nonzero temperature in only a finite interval. But, the instant we let time begin to move in time, every point on the rod has some nonzero temperature.

Remark. *Hyperbolic PDEs have finite speed of propagation, whereas almost all other PDEs have an infinite speed of propagation.*

2. Singularities

   How many derivatives do the solutions to the wave and heat equations have?

   The wave equation has *singularity propagation*, meaning that while the wave will move, it will always look the same. Solutions to the wave equation are of the form

$$u(x,t) = f(x+ct) + g(x-ct)$$

   with f, g arbitrary, so

$$u(x,0) = f(x) + g(x)$$

   If $g = 0$, then $u(x,0) = f(x)$, meaning $u(x,t) = f(x+ct)$. Thus, if f is, say, C^2 but not C^3, so must 3. The differentiability of u thus corresponds to the differentiability of f, g.

   On the other hand, the heat equation *immediately* smoothens out the initial data once $t > 0$. Solutions are of the form

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{\frac{-(x-y)^2}{4kt}} \phi(y)\ dy$$

   Suppose ϕ is continuous. Then for $t > 0$,

$$\frac{\partial u}{\partial x} = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} \frac{\partial}{\partial x} e^{\frac{-(x-y)^2}{4kt}} \phi(y)\ dy = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} \frac{-2(x-y)}{4kt} e^{\frac{-(x-y)^2}{4kt}} \phi(y)\ dy$$

   Notice that even if $y \to \infty$, the exponential will dominate the other term in the integral, so it still converges. This argument is valid for derivatives with respect to t, as well as all other higher order derivatives. This tells u that u is a C^∞ function, meaning it is smooth.

3. Long-Time Behaviour.

   What happens to $u(x,t)$ as $t \to \infty$?

   For the wave equation, f, g will just translate in opposite directions as $t \to \infty$, so let's make it interesting. Consider the problem

$$\begin{cases} u_t - c^2 u_{xx} = 0 \\ u(x,0) = \phi(x), u_x(x,0) = \psi(x) \end{cases}$$

   and assume that ϕ, ψ are compactly supported. Then

$$u(x,t) = \frac{1}{2}[\phi(x+ct) + \psi(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\ ds$$

   Remember the light path curve figure from Week 3? Well, the integral of ψ corresponds to the bottom of that triangle, and $u(x,0)$ is nonzero is some bounded interval on that side of the triangle, with the rest of the triangle being as before. Notice that if t is large, we are going to be outside of this triangle, and so by causality, $x \pm ct$ will be so large that ϕ, ψ are going to vanish. Thus, if $t \gg 1$, we just get

$$u(x,t) = \frac{1}{2c}\int_{-\infty}^{\infty} \psi(s)\ ds$$

   Think of this as fixed displacement. We flick a string, and after a long enough time, the string looks flat, but raised or lowered by the above distance.

   For the heat equation, assume that ϕ has compact support. Then we have that

$$\lim_{t\to\infty} e^{\frac{-(x-y)^2}{4kt}} = e^0 = 1$$

   so $\lim_{t\to\infty} u(x,t) = 0$.

   Another way to see this is to define

$$w(x,t) := \sqrt{t}u(x,t) = \frac{1}{\sqrt{4\pi k}}\int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4kt}} \phi(y)\ dy$$

   and

$$\lim_{t\to\infty} w(x,t) = \frac{1}{\sqrt{4\pi k}}\int_{-\infty}^{\infty} \phi(y)\ dy$$

   and we see that

$$u(x,t) = \frac{1}{\sqrt{t}}\left(\frac{1}{\sqrt{4\pi k}}\int_{-\infty}^{\infty} \phi(y)\ dy\right) + o\left(\frac{1}{\sqrt{t}}\right)$$

   so as $t \to \infty$, $u \to 0$.

5

Week 5

5.1 WORKING ON THE HALF-LINE

We are now going to change our problems slightly to only consider $x \in (0, \infty)$, with boundary conditions applied at $\{x = 0\}$.

5.1.1 THE HEAT EQUATION

Dirichlet Boundary Conditions

We will consider the problem

$$\begin{cases} v(x,t) : (0,\infty) \times (0,\infty) \to \mathbb{R} \\ v_t - kv_{xx} = 0 \\ v(x,0) = \phi(x), \;\; v(0,t) = 0 \end{cases}$$

Recall that $v(0,t) = 0$ is a (homogeneous) Dirichlet boundary condition that means there is no heat at one end of the rod. Let's also assume that $\phi(0) = 0$.

The trick to solve this problem is to use **odd extension**.

Definition 5.1. *A function $\psi : \mathbb{R} \to \mathbb{R}$ is* ***odd*** *if $\psi(-x) = -\psi(x)$.*

Given a function $\phi(x) : (0,\infty) \to \mathbb{R}$, we may define the ***odd extension*** *of ϕ by*

$$\phi_{odd} : \mathbb{R} \to \mathbb{R}, \;\; \phi_{odd}(x) = \begin{cases} \phi(x) & x > 0 \\ -\phi(-x) & x < 0 \\ 0 & x = 0 \end{cases}$$

So what's the point of defining this? Suppose that $u(x,t) : \mathbb{R} \to \mathbb{R}$ solves

$$\begin{cases} u_t - ku_{xx} = 0 \\ u(x,0) = \phi_{odd}(x) \end{cases}$$

meaning that $u(x,t) = \int_{-\infty}^{\infty} S(x-y)\phi_{odd}(y)\,dy$. We claim that $u(x,t)$ is odd in x. To see this, first recall that u is smooth for $t > 0$. We have that

$$S(x,t) = \frac{1}{\sqrt{4\pi kt}} e^{\frac{-x^2}{4kt}}$$

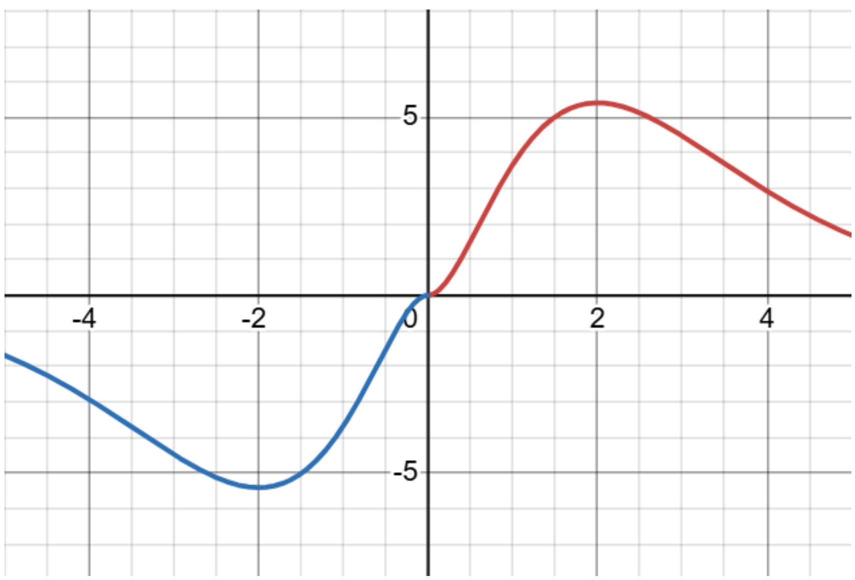
Notice that $S(-x,t) = S(x,t)$, as we square x. Thus,

$$\begin{aligned} u(-x,t) &= \int_{-\infty}^{\infty} S(-x-y,t)\phi_{odd}(y)\ dy \\ &= \int_{-\infty}^{\infty} S(x+y,t)\phi_{odd}(y)\ dy \end{aligned}$$

Set $z = -y, dz = dy$. This flips the integral, and so the minus signs cancel out, hence we get

$$\begin{aligned} &= \int_{-\infty}^{\infty} S(x-z,t)\phi_{odd}(y)\ dy \\ &= -\int_{-\infty}^{\infty} S(x-z,t)\phi_{odd}(z)\ dz \\ &= -u(x,t) \end{aligned}$$

as desired. Thus, $u(x,t)$ is also continuous when $t > 0$ and is odd in x. We also have that $u(0,t) = -u(0,t)$, so $u(0,t) = 0$. Graphically, this looks like some function which flips over the x-axis at $x = 0$:



So, if we define $v(x,t) := u(x,t)$, then v will solve the heat equation, and

$$\begin{aligned} v(x,0) &= u(x,0) = \phi_{odd}(x) = \phi(x) \\ v(0,t) &= u(0,t) = 0 \end{aligned}$$

as desired. Thus,

$$v(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\phi_{odd}(y)\ dy$$

We can solve this to be in terms of ϕ rather than ϕ_{odd} as follows:

$$\begin{aligned} v(x,t) &= \int_0^\infty S(x-y,t)\phi_{odd}(y)\ dy + \int_{-\infty}^0 S(x-y,t)\phi_{odd}(y)\ dy \\ &= \int_0^\infty S(x-y,t)\phi(y)\ dy + \int_{-\infty}^0 S(x-y,t)\phi(-y)\ dy \end{aligned}$$

Set $z = -y$ and $dz = -dy$. Thus,

$$\begin{aligned} &= \int_0^\infty S(x-y,t)\phi(y)\ dy - \int_0^\infty S(x+z,t)\phi(z)\ dz \\ &= \int_0^\infty [S(x-y,t) - S(x+y,t)]\phi(y)\ dy \end{aligned}$$

Neumann Boundary Conditions

We can use a similar idea for the heat equation with Neumann boundary conditions:

$$\begin{cases} u(x,t) : (0,\infty)\times(0,\infty) \to \mathbb{R} \\ u_t - ku_{xx} = 0 \\ u(x,0) = \phi(x),\ u_x(x,0) = 0 \end{cases}$$

where we think of the condition as saying there is insulation at the end of the rod.

The trick before was to use the fact that if f is odd and continuous, then $f(0) = 0$. Now, if f is *even* and smooth, then $f'(0) = 0$. Why? By Taylor's Theorem,

$$f(x) = f(0) + f'(0)x + O(x^2)$$

Thus,

$$f(-x) = f(0) - f'(0)x + O(x^2)$$

if $|x| \ll 1$, and $f(x) = f(-x)$, meaning f is even, then the remainder is very small, so we can ignore it. Thus,

$$f'(0) = -f'(0) \implies f'(0) = 0$$

From here, our derivations are similar to before, only this time we use the **even extension** of ϕ, given by

$$\phi_{even}(x) = \begin{cases} \phi(x) & x \geq 0 \\ \phi(-x) & x < 0 \end{cases}$$

at the end, we will see that

$$u(x,t) = \int_0^\infty [S(x-y),t) + S(x+y,t)]\phi(y)\ dy$$

Note the slight difference from Dirichlet conditions: this time we add in the integral instead of subtract.

5.1.2 THE WAVE EQUATION

Let's do the same thing for the wave equation:

Dirichlet Boundary Conditions

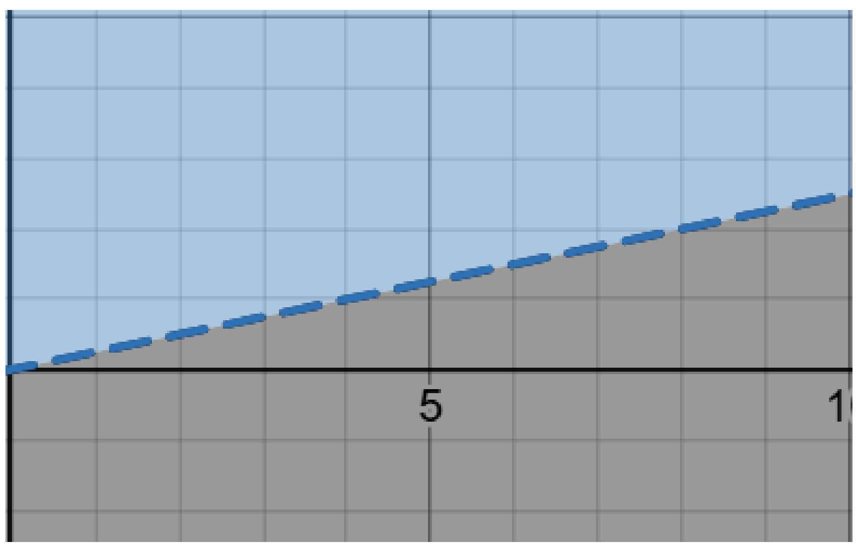
We find $v(x,t) : (0,\infty) \times (0,\infty) \to \mathbb{R}$ satisfying

$$\begin{cases} v_{tt} - c^2 v_{xx} = 0 \\ v(x,0) = \phi(x), \;\; v_t(x,0) = \psi(x) \\ v(0,t) = 0 \end{cases}$$

If we mimic the strategy for the heat equation by consider the odd extensions of ϕ, ψ, then we will see that

$$v(x,t) = \frac{1}{2}[\phi_{odd}(x+ct) + \phi_{odd}(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\; ds$$

Let's unpack this quickly to see what's going on. Consider the regions in the below figure, where the line is given by $x = ct$ in the (x,t) plane:



The region below the line, shaded in black, is where $x - ct < 0$, meaning $x > ct$. Here, we see that

$$\begin{aligned} \phi_{odd}(x+ct) &= \phi(x+ct) \\ \phi_{odd}(x-ct) &= \phi(x-ct) \\ \int_{x-ct}^{x+ct} \psi_{odd}(s)\; ds &= \int_{x-ct}^{x+ct} \psi(s)\; ds \end{aligned}$$

So in this region, v is just the usual solution to this form of the wave equation; one can see that this follows from causality of the wave equation.

In the region above the line, shaded in blue, $x < ct$. Thus,

$$\begin{aligned}
\phi_{odd}(x+ct) &= \phi(x+ct) \\
\phi_{odd}(x-ct) &= -\phi(ct-x) \\
\frac{1}{2c}\int_{x-ct}^{x+ct} \psi_{odd}(s)\ ds &= \frac{1}{2c}\int_0^{x+ct} \psi_{odd}(s)\ ds + \frac{1}{2c}\int_{x-ct}^{0} \psi_{odd}(s)\ ds \\
&= \frac{1}{2c}\int_0^{x+ct} \psi(s)\ ds - \frac{1}{2c}\int_{x-ct}^{0} \psi(-s)\ ds
\end{aligned}$$

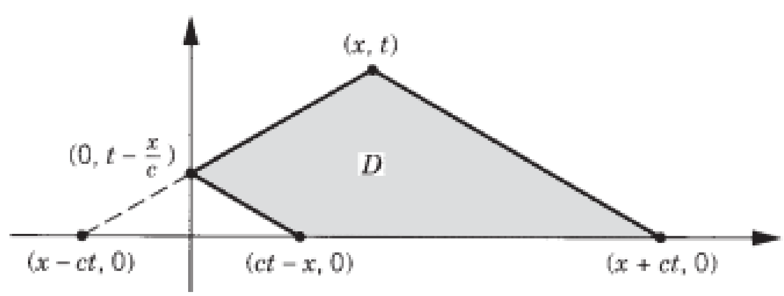
taking $\tau = -s, d\tau = -ds$, we get

$$= \frac{1}{2c}\int_0^{x+ct} \psi(s)\ ds - \frac{1}{2c}\int_0^{ct-x} \psi(\tau)\ d\tau$$

So in this region,

$$v(x,t) = \frac{1}{2}[\phi(x+ct) - \phi(ct-x)] + \frac{1}{2c}\int_{ct-x}^{ct+x} \psi(s)\ ds$$

What does this mean? Think of that path light cone from before, and suppose that one of its rays hits the t-axis in the (x,t) plane. What will happen is that ray will bounce off the axis at an angle, before shooting back towards the x-axis. As t increases, this corresponds to the path light cone perfectly bouncing off of the t-axis, and now translating to the right instead of the left. This situation will occur when, for example, we have an infinitely long string which we nail down at one end.



In a long term setting, recall that

$$\lim_{x\to\infty} u(x,t) = \frac{1}{2c}\int_{-\infty}^{\infty} \psi(s)\ ds$$

when ϕ, ψ compactly supported. What happens when we consider a Dirichlet boundary condition? Well, if $t \gg 1$, with x fixed, then we will lie in the blue region from before, and

$x+ct, ct-x$ will become both positive and very large, eventually reaching outside the support of ϕ and ψ. This means that

$$\int_{ct-x}^{xt+x} \psi(s)\ ds = 0$$

once $ct-x$ lies outside of ψ's support. Thus,

$$\lim_{t\to\infty} u(x,t) = 0$$

Visually, this just means that if we flick our string, one wave will move off to the right, and the other to the left, where it will reflect at the point $x=0$ (since it is fixed at 0), before following the other wave to the right. Thus, for very large values of t, the string will not have moved at all.

5.2 THE HEAT EQUATION, WITH A SOURCE

Before, we always set our PDE to be equal to 0. However, there are certain instances where we would rather have it be equal to some nonzero function of x, t. In this section, we will see how to solve the heat equation under these conditions.

5.2.1 NO BOUNDARY CONDITIONS

We will try and solve $u(x,t) : \mathbb{R}\times(0,\infty)\to\mathbb{R}$ such that

$$\begin{cases} u_t - ku_{xx} = f(x,t) \\ u(x,0) = \phi(x) \end{cases}$$

where f is some function. This represents heat *entering* the system, like putting a hot coal on top of our rod.

Deriving a solution will require us to use **Duhamel's Principle**, which tells us that, from the standard IVP for the heat equation (the one without f), we can get a formula for the IVP with f.

The solution is given by

$$u(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\phi(y)\ dy + \int_0^t \int_{-\infty}^{\infty} S(x-y,t-s)f(y,s)\ dy\ ds$$

where $S(x,t)$ is as before. Let's first check that this works. Without loss of generality, we can set $\phi(x)=0$. This is because, by linearity, we take u_1, u_2 so that

$$\begin{cases} (u_1)_t - k(u_1)_{xx} = 0 \\ u_1(x,0) = \phi(x) \end{cases} \qquad \text{and} \qquad \begin{cases} (u_2)_t - k(u_2)_{xx} = f(x,t) \\ u_2(x,0) = 0 \end{cases}$$

Then setting $u := u_1 + u_2$, u solves the problem we are dealing with.

To derive our solution, we use the following formula for differentiation of a definite integral:

$$\frac{d}{dt}\int_{a(t)}^{b(t)} h(t,x)\ dx = h(b(t),x)b'(t) - h(a(t),x)a'(t) + \int_{a(t)}^{b(t)} h_t(t,x)\ dx$$

Since the integral sends $t-s$ to 0, we get that

$$\begin{aligned} u_t &= \lim_{\tau\to 0}\int_{-\infty}^{\infty} S(x,y,\tau)f(y,t)\ dy + \int_0^t\int_{-\infty}^{\infty} \partial_t S(x-y,t-s)f(y,s)\ dy\ ds \\ &= f(x,t) + \int_0^t\int_{-\infty}^{\infty} \partial_t S(x-y,t-s)f(y,s)\ dy\ ds \\ u_x &= \int_0^t\int_{-\infty}^{\infty} \partial_x S(x-y,t-s)f(y,s)\ dy\ ds \\ u_{xx} &= \int_0^t\int_{-\infty}^{\infty} \partial_x^2 S(x-y,t-s)f(y,s)\ dy\ ds \end{aligned}$$

Thus, we get that

$$u_t - ku_{xx} = f(x,t) + \int_0^t\int_{-\infty}^{\infty} (\partial_t S - k\partial_x^2 S)(x-y,t-s)f(y,s)\ dy\ ds = f(x,t)$$

which is what we want.

The General Idea

We can generalize this idea to any equation of the form $u(x,t), f(x,t)$ for which we get

$$\begin{cases} \partial_t u - Lu = f \\ u(x,t) = \phi(x) \end{cases}$$

where L is a differential operator depending on x (it doesn't have to, but for now it will). Assume that for each $\phi(x)$ we have a function $P_\phi(x,t)$ such that

$$\partial_t P_\phi - LP_\phi = 0 \quad P_\phi(x,0) = \phi(x)$$

We get that P_ϕ solves the heat equation, and so

$$P_\phi(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\phi(y)\ dy$$

Using this, we can now solve our problem, with the solution given by

$$u(x,t) = \int_0^t P_{f(x,s)}(x,t-s)\ ds$$

We have that

$$\partial_t u = P_{f(x,s)}(x,0) + \int_0^t \partial_t P_f(x,t-s)\ ds = f(x,t) + \int_0^t \partial_t P_f(x,t-s)\ ds$$

$$Lu = \int_0^t L(P_f)(x,t-s)\ ds$$

Thus,

$$\partial_t u - Lu = f(x,t) + \int_0^t (\partial_t P_f - LP_f)(x, t-s)\ ds = f(x,t)$$

Now, solving for P might be hard, but once we find it, solving the problem with a source is actually pretty easy. For example, we can use this strategy for the wave equation.

5.2.2 DIRICHLET BOUNDARY CONDITIONS: THE HALF-LINE WITH A SOURCE

We consider the half-line problem from before, now with a source:

$$\begin{cases} v(x,t) : (0,\infty) \times (0,\infty) \to \mathbb{R} \\ v_t - kv_{xx} = f(x,t) \\ v(x,0) = \phi(x) \\ v(0,t) = 0 \end{cases}$$

There are two approaches to this: We can use odd reflections of ϕ and f in x, and use the formula we just derived, or, we can start with the formula when $f = 0$, which we solved last week, and use Duhamel's Principle. Regardless of method, we will get that

$$v(x,t) = \int_0^\infty [S(x-y,t) - S(x+y,t)]\phi(y)\ dy + \int_0^t \int_0^\infty [S(x-y,t-s) - S(x+y,t-s)] f(y,s)\ dy\ ds$$

5.2.3 AN INHOMOGENEOUS BOUNDARY CONDITION

We can use the methods for equations with sources to solve the heat equation with an *inhomogenous* boundary condition. Consider the problem

$$\begin{cases} v_t - kv_{xx} = 0 \\ v(x,0) = 0 \\ v(0,t) = h(t) \end{cases}$$

for $x \in (0,\infty)$. We can transform it into a problem we've already done by setting

$$w(x,t) := v(x,t) - h(t)$$

We have that

$$\begin{aligned} w_t - kw_{xx} &= v_t - kv_{xx} - h'(t) \\ &= -h'(t) \\ w(0,t) &= v(0,t) - h(t) \\ &= h(t) - h(t) \\ &= 0 \\ w(x,0) &= v(x,0) - h(0) \\ &= -h(0) \end{aligned}$$

Thus, we get the problem

$$\begin{cases} w_t - kw_{xx} = h'(t) \\ w(x,0) = -h(0) \\ w(0,t) = 0 \end{cases}$$

which is a problem that we just did previously. Thus,

$$w(x,t) = \int_0^\infty [S(x-y,t)-S(x+y,t)](-h(0))\,dx + \int_0^t \int_0^\infty [S(x-y,t-s)-S(x+y,t-s)](-h'(s))\,dy\,ds$$

6

Week 6

6.1 THE WAVE EQUATION, WITH A SOURCE

We now consider the wave equation with a source, which is the problem

$$\begin{cases} u(x,t) : \mathbb{R} \times (0,\infty) \to \mathbb{R} \\ u_{tt} - c^2 u_{xx} = f(x,t) \\ u(x,0) = \phi(x) \quad u_t(x,0) = \psi(x) \end{cases}$$

We claim that the unique solution to this problem is given by

$$u(x,t) = \frac{1}{2}[\phi(x+ct) + \phi(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\ ds + \frac{1}{2c}\iint_{\Delta(x,t)} f(y,s)\ dy\ ds$$

where $\Delta(x,t)$ is the path light cone from before. We may assume that $\phi, \psi = 0$ by the same logic as before. While there are many solving methods, like using modified versions of Duhamel's Principle, we will exploit the wave equations relatively simple structure to get a solution.

We will use characteristic coordinates. We define

$$\xi = x + ct, \quad \eta = x - ct$$

$$\begin{aligned} \frac{\partial}{\partial x} &= \frac{\partial}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial}{\partial \eta}\frac{\partial \eta}{\partial x} = \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta} \\ \frac{\partial}{\partial t} &= \frac{\partial}{\partial \xi}\frac{\partial \xi}{\partial t} + \frac{\partial}{\partial \eta}\frac{\partial \eta}{\partial t} = c\frac{\partial}{\partial \xi} - c\frac{\partial}{\partial \eta} \end{aligned}$$

so

$$x = \frac{1}{2}(\xi + \eta) \quad t = \frac{1}{2c}(\xi - \eta)$$

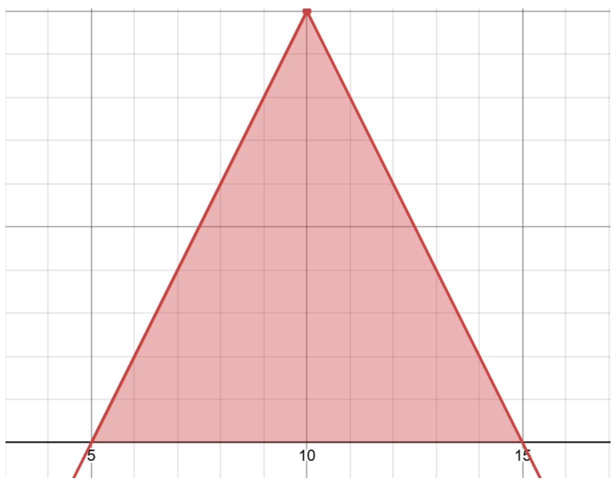
and we may rewrite our PDE as

$$
\begin{aligned}
u_{tt} - c^2 u_{xx} = f(x,t) &\iff \left(c\frac{\partial}{\partial \xi} - c\frac{\partial}{\partial \eta}\right)^2 u - c^2\left(\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}\right)^2 u = f \\
&\iff \left(c^2 \frac{\partial}{\partial \xi}^2 - 2c^2 \frac{\partial^2}{\partial \xi \partial \eta} + c^2 \frac{\partial}{\partial \eta}^2 - c^2 \frac{\partial}{\partial \xi}^2 - 2c^2 \frac{\partial^2}{\partial \xi \partial \eta} - c^2 \frac{\partial}{\partial \eta}^2\right) u = f(x,t) \\
&\iff -4c^2 \frac{\partial^2}{\partial \xi \partial \eta} u = f
\end{aligned}
$$

Thus, our solution in (ξ, η) will look something like

$$u(\xi, \eta) = -\frac{1}{4c^2} \int_?^\xi \int_?^\eta f \; d\xi \; d\eta + \cdots$$

Let's now consider the path light cone in these coordinates, which is shown below:



The left side is when η is constant, while the right side is when ξ is constant. We have that $u(x,0) = u_t(x,0) = 0$ along the line $\{t = 0\} \iff \{\xi = \eta\}$. Thus, $u|_{(\xi,\xi)} = 0$, and so do $u_\xi|_{(\xi,\xi)}, u_\eta|_{(\xi,\xi)}$.

We fix (x_0, t_0) where we originally wanted to evaluate $u(x,t)$. Set $\eta_0 = x_0 - ct_0, \xi_0 = x_0 + ct_0$. Then by the FTC, we get that

$$
\begin{aligned}
u(\xi_0, \eta_0) &= \int_{\eta_0}^{\xi_0} u_\xi(\xi, \eta_0) \; d\xi \\
&= \int_{\eta_0}^{\xi_0} \int_{\xi}^{\eta_0} u_{\xi\eta}(\xi, \eta) \; d\eta \; d\xi \\
&= -\frac{1}{4c^2} \int_{\eta_0}^{\xi_0} \int_{\xi}^{\eta_0} f \; d\eta \; d\xi && \text{(by the PDE)}
\end{aligned}
$$

Note that $\xi \in (\eta_0, \xi_0)$, with $\xi_0 > \eta_0$, meaning that $\xi > \eta_0$. Hence we flip the integral to give us

$$u(\xi_0, \eta_0) = \frac{1}{4c^2} \int_{\eta_0}^{\xi_0} \int_{\eta_0}^{\xi} f \; d\eta \; d\xi = \frac{1}{4c^2} \iint_{?} f \cdot A \; dx \; dt$$

Where the last equal sign is the change of variables back to (x, t). The A is the determinant of the transformation, given by

$$\begin{aligned} A &= \left| \det \begin{pmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \xi}{\partial t} \\ \frac{\partial \eta}{\partial x} & \frac{\partial \eta}{\partial t} \end{pmatrix} \right| \\ &= \left| \det \begin{pmatrix} 1 & c \\ 1 & -c \end{pmatrix} \right| \\ &= |-c-c| \\ &= 2c \end{aligned}$$

so we get

$$\frac{1}{2c} \iint_{?} f \; dx \; dt$$

What are we integrating? The triangle from before! The left side is when $\eta = \eta_0$, and the right side is when $\xi = \xi_0$. For the outside integral, we integrate over ξ, which goes from η_0 to ξ_0. For the inside integral, we integrate over η, which goes from η_0 to ξ.

In (x, t) coordinates, our triangle goes from 0 to t on the t-axis. For x, the left side of the triangle is given by the line $y - cs = x - ct$ and the right side is given by $y + cs = x + ct$. Solving for x means we are integrating from $x - c(t - s)$ to $x + c(t - s)$. Thus, our integral becomes

$$\int_0^t \int_{x-c(t-s)}^{x-c(t-s)} f(y, s) \; dy \; ds$$

Example 16. *Solve*

$$\begin{cases} u_{tt} - c^2 u_{xx} = \cos(x) \\ u(x, 0) = \sin(x) \quad u_t(x, 0) = 1 + x \end{cases}$$

Using the formula we've just derived, we get that

$$u(x, t) = \frac{1}{2}[\sin(x + ct) + \sin(x - ct)] + \frac{1}{2c} \int_{x-ct}^{x+ct} (1 + s) \; ds + \frac{1}{2c} \int_0^t \int_{x-c(t-s)}^{x+c(t-s)} \cos(y) \; dy \; ds$$

Like the heat equation, we can use this formula to solve for inhomogeneous boundary conditions. Reflection tricks will allow us to handle source terms on the half-line and homogeneous boundary conditions, and you can then go on to solve even more general problems.

Remark. *Everything up to this point will appear on the first term test.*

6.2 WORKING ON A FINITE INTERVAL

We now seek to solve both the wave and heat equations where the spatial variable x is defined on a finite interval $(0, \ell)$. Doing this requires us to introduce a new solving technique, and make some assumptions about our initial conditions that will be made rigorous in a later section.

6.2.1 THE WAVE EQUATION ON A FINITE INTERVAL WITH DIRICHLET B.C

We now wish to solve the wave equation on a finite interval. We will consider the equation with Dirichlet boundary conditions:

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 & 0 < x < \ell, \quad t > 0 \\ u(0,t) = u(\ell,t) = 0 \\ u(x,0) = \phi(x) \quad u_t(x,0) = \psi(x) \end{cases}$$

The fact that x is defined on a finite interval is going to make things more complicated, and thus we require a new solving method: **separation of variables**.

To begin, let's make an educated guess (commonly called an *Ansatz*) about the shape of our solution. Suppose that

$$u(x,t) = X(x)T(t)$$

Can we find a solution that looks like this? If so, then we have that

$$\begin{aligned} u_{tt} &= X(x)T''(t) \\ u_{xx} &= X''(x)T(t) \\ \implies X(x)T''(t) &- c^2 X''(x)T(t) = 0 \end{aligned}$$

In this case, we can push all the x terms on one side and all t terms on the other side. We get

$$X(x)T''(t) = c^2 X''(t)T(t)$$

and assuming X, T don't vanish,

$$-\frac{T''(t)}{c^2 T(t)} = -\frac{X''(x)}{X(x)}$$

Remark. *The minus sign is convention and will make the next part easier.*

So we have that a function of t is equal to a function of x. Clearly the only way this is possible is when these functions are both equal to a *constant* value.

$$-\frac{T''(t)}{c^2 T(t)} = -\frac{X''(x)}{X(x)} = \lambda \in \mathbb{R}$$

From this, we get two equations:

$$X''(x) + \lambda X(x) = 0$$

$$T''(t) + c^2\lambda T(t) = 0$$

Both of these are ODEs that we already know how to solve. The only thing we need to do now is check that they solve the boundary conditions from before. To do this, we take cases:

1. If $\lambda = 0$, then $X''(x) = 0$, and so
$$X(x) = Ax + B$$
for some constants A, B. Plugging in the boundary conditions, we get that
$$0 = X(0) = B \implies B = 0$$
$$0 = X(\ell) = A\ell + B \implies A\ell = 0 \implies A = 0$$
But then $X(x) = 0$, which results in the degenerative case that $u = 0$. We thus reject this case.
2. If $\lambda < 0$, then
$$X(x) = Ae^{-\sqrt{-\lambda}x} + Be^{\sqrt{-\lambda}x}$$
for some constants A, B. Plugging in the boundary conditions, we get that
$$0 = X(0) = A + B \implies A = -B$$
$$\begin{aligned} 0 = X(\ell) &= Ae^{-\sqrt{-\lambda}\ell} + Be^{\sqrt{-\lambda}\ell} \\ &= -Be^{-\sqrt{-\lambda}\ell} + Be^{\sqrt{-\lambda}\ell} \\ &= -B(e^{-\sqrt{-\lambda}\ell} - e^{\sqrt{-\lambda}\ell} \end{aligned}$$
We assume $B \neq 0$ (otherwise we get the degenerative case), and so we get that
$$\begin{aligned} e^{-\sqrt{-\lambda}\ell} - e^{\sqrt{-\lambda}\ell} = 0 &\implies -\sqrt{-\lambda} = \sqrt{-\lambda} \\ &\implies \lambda = 0 \end{aligned}$$
a contradiction. We again reject this case.
3. If $\lambda > 0$, then
$$X(x) = A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x)$$
for some constants A, B. Plugging in the boundary conditions, we get that
$$0 = X(0) = A \implies A = 0$$

$$0 = X(\ell) = B\sin(\sqrt{\lambda}\ell)$$

We again take $B \neq 0$, and so we have that

$$\begin{aligned}
\sin(\sqrt{\lambda}\ell) = 0 &\implies \sqrt{\lambda}\ell \text{ is a multiple of } \pi. \\
&\implies \sqrt{\lambda}\ell = n\pi \\
&\implies \lambda\ell^2 = (n\pi)^2 \\
&\implies \lambda = \left(\frac{n\pi}{\ell}\right)^2 \qquad\qquad (n = 1, 2, 3, \ldots)
\end{aligned}$$

We thus attain a solution when $\lambda > 0$. We denote

$$\begin{aligned}
\lambda_n &= \left(\frac{n\pi}{\ell}\right)^2 \\
X_n(x) &= \sin\left(\frac{n\pi x}{\ell}\right)
\end{aligned}$$

The λ_n's are called **eigenvalues**, while the $X_n(x)$'s are called **eigenfunctions**, which are related to the concept of eigenvectors (we will explain this later). It should also be noted that, without loss of generality, we're assume the constant is 1 since we can always absorb it into $T_n(t)$.

Now to deal with T. First fixing n, we get

$$T_n''(t) + c^2\left(\frac{n\pi}{\ell}\right)^2 T_n(t) = 0$$

This is the same ODE that we solved for when $\lambda > 0$. Thus,

$$T_n(t) = A_n\cos\left(\frac{n\pi c}{\ell}t\right) + B_n\sin\left(\frac{n\pi c}{\ell}t\right)$$

Combining, we get

$$u_n(x,t) = T_n(t)X_n(x) = \left(A_n\cos\left(\frac{n\pi c}{\ell}t\right) + B_n\sin\left(\frac{n\pi c}{\ell}t\right)\right)\sin\left(\frac{n\pi x}{\ell}\right)$$

PDEs are linear, so we can try adding up these u_n's and get

$$u(x,t) \stackrel{?}{=} \sum_{n=1}^{\infty}\left(A_n\cos\left(\frac{n\pi c}{\ell}t\right) + B_n\sin\left(\frac{n\pi c}{\ell}t\right)\right)\sin\left(\frac{n\pi x}{\ell}\right)$$

We're not even sure this sum makes sense, but, it actually does, and it is in fact the general solution for suitable choices of A_n and B_n.

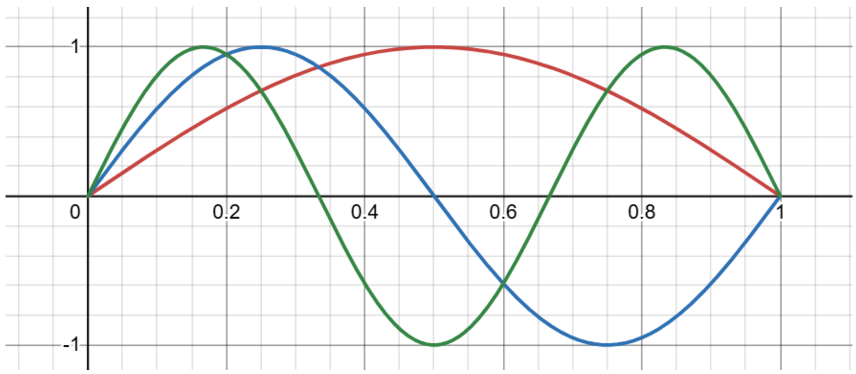
Under this assumption that it solves the wave equation with a Dirichlet boundary condition, we get a strange corollary. Given initial conditions $u(x,0) = \phi(x)$ and $u_t(x,0) = \psi(x)$, we'd get that

$$\phi(x) = \sum_{n=1}^{\infty} A_n\sin\left(\frac{n\pi x}{\ell}\right)$$

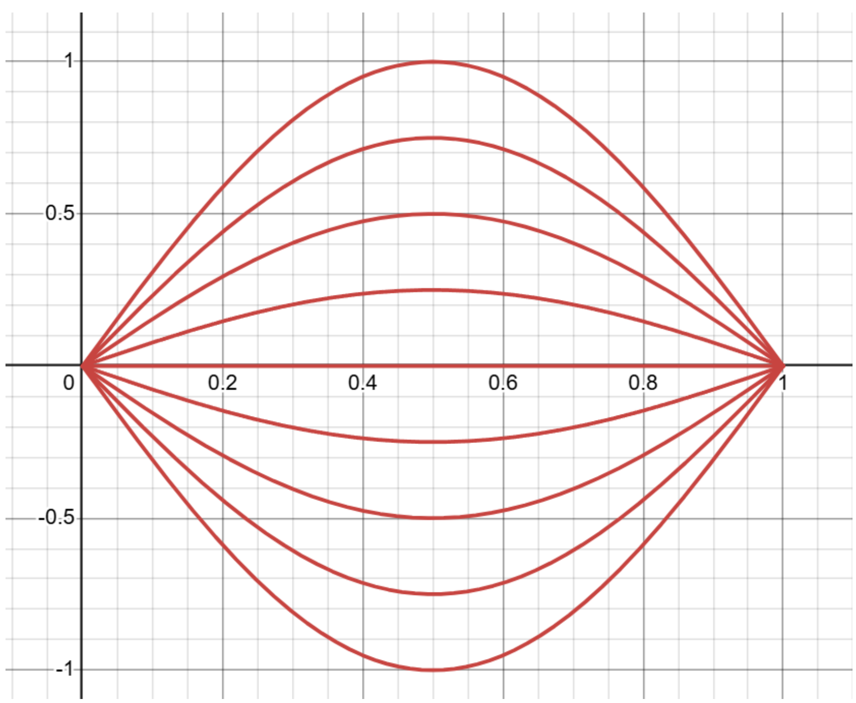
$$\psi(x) = \sum_{n=1}^{\infty} (u_n)_t(x,0) = \sum_{n=1}^{\infty} B_n \frac{\pi n c}{\ell} \sin\left(\frac{n\pi x}{\ell}\right)$$

This leads to a question: is it possible to represent functions $\phi(x), \psi(x)$ in terms of a sum of the above form? Is such a representation unique? We will answer these questions later.

For now, let's try and understand what this solution looks like, starting with the X_n's, the spatial components. On $(0, \ell)$, $X_n(x)$ will have $n-1$ zeroes, and oscillates more and more rapidly as we increase n. The figure below shows X_1, X_2, and X_3 when $\ell = 1$.



What about the time components T_n? This will oscillate from 1 to -1, and do so faster as n increases. Upon combining these values, $u_n(x,t)$ will look like the graph of $X_n(x)$, which will oscillate between $X_n(x)$ and $-X_n(x)$ faster and faster as t increases.



This is in contrast to the situation on the infinite or half-lines. Before, waves could "escape" by moving off to infinity. Now, they are stuck, and thus continue to affect the string forever.

6.2.2 THE HEAT EQUATION ON A FINITE INTERVAL WITH DIRICHLET B.C

Now let's try the same strategy for the heat equation problem

$$\begin{cases} u_t - ku_{xx} = 0 \\ u(0,t) = u(\ell,t) = 0 \\ u(x,0) = \phi(x) \end{cases}$$

The strategy is the exact same. By making an initial guess that u is some product of a function of x and a function of t, we get that

$$u_t - ku_{xx} = 0 \iff X(x)T'(t) - kX''(x)T(t) = 0$$

and so

$$\frac{T'(t)}{kT(t)} = \frac{X''(x)}{X(x)} = -\lambda$$

for a constant λ. We thus need to solve

$$X''(x) + \lambda X(x) = 0 \quad X(0) = X(\ell) = 0$$

$$T'(x) = K\lambda T(t) = 0$$

We've done some of this in our analysis of the wave equation already: For $n = 1, 2, 3, \ldots$,

$$X_n(x) = \sin\left(\frac{n\pi x}{\ell}\right), \quad \lambda_n = \left(\frac{n\pi}{\ell}\right)^2$$

What does change is our equation for T. Plugging in what we know gives

$$T'(t) + k\left(\frac{n\pi}{\ell}\right)^2 T_n(t) = 0 \implies T_n(t) = A_n \exp\left(-k\left(\frac{n\pi}{\ell}\right)^2 t\right)$$

and so

$$u_n(x,t) = A_n \exp\left(-k\left(\frac{n\pi}{\ell}\right)^2 t\right)\sin\left(\frac{n\pi x}{\ell}\right)$$

$$u(x,t) \stackrel{?}{=} \sum_{n=1}^{\infty} u_n(x,t)$$

With this solution, we get that $\phi(x) = u(x,0) \stackrel{?}{=} \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{\ell}\right)$. This looks different compared to the wave equation. While the X_n looks the same, the T_n is an exponential with negative exponent, meaning that it decreases in time, with the decrease going faster as $n \to \infty$. How fast is this decrease? Recall that for $x \in \mathbb{R}$, for a fixed x,

$$u(x,t) = \frac{1}{\sqrt{t}}\left(\frac{1}{\sqrt{4\pi k}}\int_{-\infty}^{\infty}\phi(y)\,dy\right) + o\left(\frac{1}{\sqrt{t}}\right)$$

so decay on the infinite rod will occur in polynomial time. Since we have an exponential on the finite rod, we get exponential decay, meaning heat dissipates much faster in the finite case compared to the infinite case.

AN ASIDE REGARDING EIGENFUNCTIONS

We call the λ_n's **eigenvalues** and the X_n's **eigenfunctions**, which act like eigenvectors. These terms may seem strange as they feel out of place here. However, they are at home in this context.

Recall from linear algebra that an $n \times n$ real valued matrix A is **self-adjoint** if $A = A^T$; A is also called a **symmetric** matrix. In other words, for any vectors $\vec{v}, \vec{w} \in \mathbb{R}^n$,

$$(A\vec{v}) \cdot \vec{w} = \vec{v} \cdot (A\vec{w})$$

The spectral theorem tells us that there is an orthonormal basis of eigenvectors for such operators. We thus get vectors

$$\vec{v_1}, \vec{v_2}, \ldots, \vec{v_n}$$

for which

$$\vec{v_1} \cdot \vec{v_j} = \begin{cases} 1 & i = j \text{ and } A\vec{v_i} = \lambda_i \vec{v_i} \text{ for some } \lambda_i \in \mathbb{R} \\ 0 & \text{otherwise} \end{cases}$$

Now, if we take $A = \partial_x^2$ to be the second derivative of a function with respect to x, and let $x \in (0, \ell)$, then for functions $f(x), g(x)$ with

$$f(0) = f(\ell) = 0 = g(0) = g(\ell)$$

the dot product is given by

$$(f, g) = \int_0^\ell f(x)g(x)\ dx$$

and one can see that $(\partial_x^2 f, g) = (f, \partial_x^2 g)$. Another version of the spectral theorem says that there is an orthonormal basis

$$x_1, \ldots, x_n$$

such that we get eigenvalues λ_i for each i, and

$$AX_n = \lambda_n x_n$$

Thus, $\phi(x) = \sum A_n X_n(x)$, exactly like we predicted before!

7

Week 7

7.1 WORKING ON A FINITE INTERVAL, CONTINUED

Last week we considered solutions to the wave and heat equations on finite intervals $(0, \ell)$ with Dirichlet boundary conditions. We begin this week with a study of the same problems but with Neumann boundary conditions.

7.1.1 THE WAVE EQUATION ON A FINITE INTERVAL WITH NEUMANN B.C

We seek to solve the problem

$$\begin{cases} u(x,t) : (0,\ell) \times (0,\infty) \to \mathbb{R} \\ u_{tt} - c^2 u_{xx} = 0 \\ u(x,0) = \phi(x), \quad u_t(x,0) = \psi(x) \\ u_x(0,t) = u_x(\ell,t) = 0 \end{cases}$$

Like before, we make a guess that $u(x,t) = T(t)X(x)$, and we wish to find $\lambda \in \mathbb{R}$ such that

$$X'' + \lambda X = 0$$

has a nonzero solution satisfying our boundary conditions: $X'(0) = X'(\ell) = 0$.

Remark. *We assume here that $\lambda \in \mathbb{R}$, but what if $\lambda \in \mathbb{C}$? Given possibly complex-valued functions $f(x), g(x)$ which satisfy the boundary conditions (which may be Dirichlet or Neumann, it doesn't matter), we get that*

$$f \cdot g = \int_0^\ell f(x)\overline{g(x)}\, dx$$

When boundary conditions are considered, integrating by parts shows that

$$\begin{aligned}
\int_0^\ell f''(x)\overline{g(x)}\ dx &= f'(x)g(x)\Big|_0^\ell - \int_0^\ell f'(x)\overline{g'(x)}\ dx \\
&= -\int_0^\ell f'(x)\overline{g'(x)}\ dx \\
&= -f(x)g'(x)\Big|_0^\ell + \int_0^\ell f(x)\overline{g''(x)}\ dx \\
&= \int_0^\ell f(x)\overline{g''(x)}\ dx
\end{aligned}$$

Now, given $X'' + \lambda X = 0$ for some $\lambda \in \mathbb{C}$ with X satisfying the boundary conditions, we get that

$$\begin{aligned}
\int_0^\ell X''\overline{X}\ dx &= -\lambda \int_0^\ell |X|^2\ dx && (X'' = -\lambda X) \\
&= \int_0^\ell X\overline{X''}\ dx && \text{(by the above work)} \\
&= -\overline{\lambda} \int_0^\ell |X|^2\ dx
\end{aligned}$$

Thus, $\lambda = \overline{\lambda}$, and so $\lambda \in \mathbb{R}$.

We again need to consider cases regarding the value of λ:

1. For $\lambda < 0$, the general solution to $X'' + \lambda X = 0$ is

$$X(x) = Ae^{\sqrt{-\lambda}x} - Be^{-\sqrt{-\lambda}x}$$

For constants A, B. We have that

$$X'(x) = A\sqrt{-\lambda}e^{\sqrt{-\lambda}x} - B\sqrt{-\lambda}e^{-\sqrt{-\lambda}x}$$

$$\begin{aligned}
& X'(0) = A\sqrt{-\lambda} - B\sqrt{-\lambda} = 0 \\
\implies & A = B
\end{aligned}$$

$$X'(\ell) = A\sqrt{-\lambda}e^{\sqrt{-\lambda}\ell} - A\sqrt{-\lambda}e^{-\sqrt{-\lambda}\ell}$$

Assuming that $A \neq 0$, we have

$$
\begin{aligned}
0 &= e^{\sqrt{-\lambda}\ell} - e^{-\sqrt{-\lambda}\ell} \\
\implies e^{-\sqrt{-\lambda}\ell} &= e^{\sqrt{-\lambda}\ell} \\
\implies &= -\sqrt{-\lambda} = \sqrt{-\lambda} \\
\implies \lambda &= 0
\end{aligned}
$$

So we reject this case as it leads to a contradiction.

2. For $\lambda = 0$, then $X'' = 0$, so

$$X(x) = Ax + B$$

for constants A, B. We have that $X'(x) = A \implies X'(0) = A$, so $A = 0$. Moreover, $X'(\ell) = A = 0$. This is different from our analysis of Dirichlet boundary conditions, as $\lambda = 0$ yields a valid solution. Without loss of generality we take

$$X(x) = 1$$

which is an eigenfunction with eigenvalue 0.

3. For $\lambda > 0$, we get the general solution

$$X(x) = A\sin(\sqrt{\lambda}x + B\cos(\sqrt{\lambda}x)$$

for constants A, B. We have that

$$X'(x) = \sqrt{\lambda}A\cos(\sqrt{\lambda}x) - B\sqrt{\lambda}\sin(\sqrt{\lambda}x)$$

$$
\begin{aligned}
X'(0) &= \sqrt{\lambda}A = 0 \\
\implies A &= 0
\end{aligned}
$$

$$X'(\ell) = -B\sqrt{\lambda}\sin(\sqrt{\lambda}\ell) = 0$$

$B = 0$ yields the degenerative case, so $\sin(\sqrt{\lambda}\ell) = 0$. We have seen this before, and can conclude that

$$\lambda = \left(\frac{\pi n}{\ell}\right)^2 \quad n = 1, 2, 3, \ldots$$

which is the same as the Dirichlet boundary conditions. There is a small change in X_n, since we use cos instead of sin:

$$X_n(x) = \cos\left(\frac{n\pi x}{\ell}\right)$$

It should be noted that the $\lambda = 0$ solution will be treated as the solution for $n = 0$, which makes sense given that $\lambda_0 = 0$, lining up with $X_0(x) = 1$ being the eigenfunction with eigenvalue 0.

For the function $T(t)$, we have that

$$T_n'' + c^2\lambda_n T_n(t) = 0$$

For $n = 0$, $T_0''(t) = 0$, so

$$T_0(t) = A_0 t + B_0$$

and when $n > 0$, we get

$$T_n'' + c^2\left(\frac{n\pi}{\ell}\right)T_n = 0$$

and so we have

$$T_n(t) = A_n \sin\left(\frac{n\pi ct}{\ell}\right) + B_n \cos\left(\frac{n\pi ct}{\ell}\right)$$

meaning our final guess for the general solution is given by

$$u(x,t) = \frac{1}{2}(A_0 t + B_0) + \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi ct}{\ell}\right) + B_n \cos\left(\frac{n\pi ct}{\ell}\right)$$

where that factor for $1/2$ is there for the sake of simpler calculations.

It should be noted that $\lim_{t\to\infty} u(x,t) = \infty$. What does this mean? Recalling that under Neumann conditions our string is held taught and fixed onto infinitely tall poles so that we can only move the endpoints, any action on the string, like flicking it upwards, will cause the string to just keep going in that direction forever.

7.1.2 THE HEAT EQUATION ON A FINITE INTERVAL WITH NEUMANN B.C

Now let's solve the heat equation on the same domain, with Neumann boundary conditions:

$$\begin{cases} u_t - ku_{xx} = 0 \\ u_x(0,t) = u_x(\ell,t) = 0 \\ u(x,0) = \phi(x) \end{cases}$$

Recalling that the spatial part $X(x)$ matches that of the wave equation, we get that

$$Xn(x) = \cos\left(\frac{n\pi x}{\ell}\right) \quad n = 0, 1, 2, 3, \ldots$$
$$\lambda_n = \left(\frac{n\pi}{\ell}\right)^2$$

The equation for $T_n(t)$ has changed:

$$T_n'(t) + k\lambda_n T_n(t) = 0$$

So we have cases: If $n = 0$, then $T_0'(t) = 0$, meaning $T_0(t) = A_0$ for some constant A_0. If $n > 0$, we get

$$T_n(t) = A_n \exp(-k\lambda_n t)$$

Thus, we get that

$$u(x,t) = A_0 + \sum_{n=1}^{\infty} A_n \exp\left(-kt\left(\frac{n\pi}{\ell}\right)^2\right)\cos\left(\frac{n\pi x}{\ell}\right)$$

Note that, like with Dirichlet conditions, as $t \to \infty$ the exponential goes to 0, meaning the sum will vanish and we are left with A_0. This makes sense, and means that our rod will, after a long enough time, have even temperature across its surface given by that constant.

OTHER BOUNDARY CONDITIONS

we can go through this exact same process for many different boundary conditions, though usually we wont be able to solve them exactly. Despite this, we will still get values that work, and we can still say things about them, which is what matters most in these contexts.

7.2 FOURIER TRANSFORM

In our previous derivations of the heat and wave equations on finite intervals, our results indicated that certain functions could be represented as the infinite sum of the sine and cosine functions. In this section, we demonstrate how this is true by attaining the necessary coefficients in these series.

7.2.1 FOURIER SINE SERIES

We begin with the Fourier Sine Series, which works for Dirichlet boundary conditions. Suppose we have a function $\phi(x) : (0, \ell) \to \mathbb{R}$. Can we find $\{A_n\}_{n=1}^{\infty}$ such that

$$\phi(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{\ell}\right)$$

If we can solve the wave equation with Dirichlet boundary conditions, then we must be able to do this.

To begin answering our question, we need an important result:

Theorem 7.1. *For positive integers $n \neq m$, we have*

$$\int_0^{\ell} \sin\left(\frac{n\pi x}{\ell}\right)\sin\left(\frac{m\pi x}{\ell}\right)\, dx = 0$$

Proof. Take $X_n = \sin\left(\frac{n\pi x}{\ell}\right)$. Then $X_n'' = -\lambda_n X_n$, where

$$\lambda_n = \left(\frac{n\pi}{\ell}\right)^2$$

We have that

$$\int_0^\ell X_n'' X_m \, dx = -\lambda_n \int_0^\ell X_n X_m \, dx$$

But using integration by parts, we get that

$$\begin{aligned}\int_0^\ell X_n'' X_m \, dx &= -\int_0^\ell X_n' X_m' \, dx \\ &= \int_0^\ell X_n X_m'' \, dx \\ &= -\lambda_m \int_0^\ell X_n X_m \, dx\end{aligned}$$

$n \neq m$, so $\lambda_m \neq \lambda_n$. Thus, the only possibility is that

$$\int_0^\ell X_n X_m \, dx = 0$$

as desired. □

What about the case that $n = m$? Note that

$$\sin^2\theta = \frac{1}{2} - \frac{1}{2}\cos(2\theta)$$

Using this fact, we get that

$$\begin{aligned}\int_0^\ell \sin^2\left(\frac{n\pi x}{\ell}\right) \, dx &= \int_0^\ell \frac{1}{2} \, dx - \frac{1}{2}\int_0^\ell \cos\left(\frac{2n\pi x}{\ell}\right) \, dx \\ &= \frac{\ell}{2} - \frac{1}{2}\left[\frac{\ell}{2n\pi}\sin\left(\frac{2n\pi x}{\ell}\right)\right]_0^\ell \\ &= \frac{\ell}{2} - \frac{1}{2}\left[\frac{\ell}{2n\pi}\sin(2n\pi) - \frac{\ell}{2n\pi}\sin(0)\right] \\ &= \frac{\ell}{2}\end{aligned}$$

With all this in hand, we can begin deriving our coefficients. We have that

$$\phi(x) \stackrel{?}{=} \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{\ell}\right)$$

To find the coefficients of this series, we can take the dot product of it and the sine value. This gives us

$$\begin{aligned}
\int_0^\ell \phi(x) \sin\left(\frac{m\pi x}{\ell}\right)\, dx &= \int_0^\ell \left(\sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{\ell}\right)\right) \sin\left(\frac{m\pi x}{\ell}\right)\, dx \\
&= \sum_{n=1}^{\infty} A_n \int_)^\ell \sin\left(\frac{n\pi x}{\ell}\right) \sin\left(\frac{m\pi x}{\ell}\right)\, dx \\
&= A_m \int_0^\ell \sin^2\left(\frac{m\pi x}{\ell}\right) \\
&= \frac{A_m \ell}{2}
\end{aligned}$$

Thus, we conclude that

$$A_m = \frac{2}{\ell}\int_0^\ell \phi(x) \sin\left(\frac{m\pi x}{\ell}\right)\, dx$$

This gives us the coefficients of ϕ in the wave equation! Recall the solution on a finite interval with Dirichlet boundary conditions given by

$$u(x,t) = \sum_{n=1}^{\infty} \left(A_n \cos\left(\frac{n\pi ct}{\ell}\right) + B_n \sin\left(\frac{n\pi ct}{\ell}\right)\right) \sin\left(\frac{n\pi x}{\ell}\right)$$

We have that

$$\phi(x) = u(x,0) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{\ell}\right)$$

meaning

$$A_n = \frac{2}{\ell}\int_0^\ell \phi(x) \sin\left(\frac{n\pi x}{\ell}\right)\, dx$$

Moreover, we can also find the coefficients for $\psi(x)$! We have

$$\begin{aligned}
\psi(x) = u_t(x,0) &= \sum_{n=1}^{\infty} \left(\frac{-A_n n\pi c}{\ell} \sin\left(\frac{n\pi c \cdot 0}{\ell}\right) + \frac{B_n n\pi c}{\ell} \cos\left(\frac{n\pi c \cdot 0}{\ell}\right)\right) \sin\left(\frac{n\pi x}{\ell}\right) \\
&= \sum_{n=1}^{\infty} \frac{B_n n\pi c}{\ell} \sin\left(\frac{n\pi x}{\ell}\right)
\end{aligned}$$

And so we get that

$$\begin{aligned}
\frac{B_n n\pi c}{\ell} &= \frac{2}{\ell}\int_0^\ell \psi(x) \sin\left(\frac{n\pi x}{\ell}\right)\, dx \\
\implies B_n &= \frac{2}{n\pi c}\int_0^\ell \psi(x) \sin\left(\frac{n\pi x}{\ell}\right)\, dx
\end{aligned}$$

7.2.2 FOURIER COSINE SERIES

We now find the coefficients of the Fourier Cosine Series. Recall that, using Neumann boundary conditions, saw that

$$\phi(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{\ell}\right)$$

What are the values of $A_0, A_1, A_2, \ldots$?
First, we need a result:

Theorem 7.2. *For $n \neq m$, we have that*

$$\int_0^\ell \cos\left(\frac{n\pi x}{\ell}\right)\cos\left(\frac{m\pi x}{\ell}\right)\, dx = 0$$

Proof. By setting $X_n = \cos\left(\frac{n\pi x}{\ell}\right)$, we again get that $X_n'' = -\lambda_n X_n$, where $\lambda_n = \left(\frac{n\pi}{\ell}\right)^2$. Repeating the method from the sine version of this theorem suffices. □

This works for $n, m \in \{0, 1, 2, 3, \ldots\}$. To find the coefficients we again take the dot product, but now of ϕ with cosine.

$$\begin{aligned}\int_0^\ell \phi(x)\cos\left(\frac{m\pi x}{\ell}\right)\, dx &= \int_0^\ell \left(\frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{\ell}\right)\right)\cos\left(\frac{m\pi x}{\ell}\right)\, dx \\ &= \frac{1}{2}\int_0^\ell A_0 \cos\left(\frac{m\pi x}{\ell}\right)\, dx + \sum_{n=1}^{\infty}\int_0^\ell A_n \cos\left(\frac{n\pi x}{\ell}\right)\cos\left(\frac{m\pi x}{\ell}\right)\, dx\end{aligned}$$

If $m = 0$, the infinite sum will vanish by our above theorem, leaving us with

$$\frac{\ell}{2}A_0$$

While if $m \neq 0$, the first term will vanish, also by the above theorem, and so will all terms in the infinite sum except $m = n$, leaving us with

$$A_m \int_0^\ell \cos^2\left(\frac{m\pi x}{\ell}\right)\, dx$$

Noting that

$$\int_0^\ell \cos^2\left(\frac{m\pi x}{\ell}\right)\, dx = \frac{\ell}{2}$$

We conclude that

$$A_m = \begin{cases} \frac{2}{\ell}\int_0^\ell \phi(x)\, dx & m = 0 \\ \frac{2}{\ell}\int_0^\ell \phi(x)\cos\left(\frac{m\pi x}{\ell}\right)\, dx & m \neq 0 \end{cases}$$

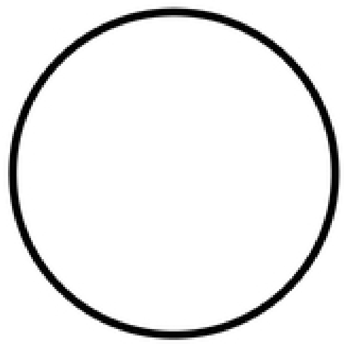
Which we can combine to just say that

$$A_n = \frac{2}{\ell}\int_0^\ell \phi(x)\cos\left(\frac{n\pi x}{\ell}\right)\, dx$$

The B_n coefficients can be derived from this like we did before. Note the value A_0, which is the average of the initial heat distribution. This makes sense, because in the limit, heat on a rod will become equally distributed; the heat at each point will correspond to the average value of heat on the rod.

7.2.3 FULL FOURIER SERIES: PERIODIC BOUNDARY CONDITIONS

Consider the heat equation, but instead of using a straight rod, we consider a *circular* rod.



Now, there is no "boundary" for us to work with. How can we deal with this?

Let's consider our domain of x to be $(-\ell, \ell)$. The condition now is that $u(x,t)$ is **periodic** in x with period 2ℓ. We can express this as

$$u(-\ell, t) = u(\ell, t)$$
$$u_x(\ell, t) = u_x(\ell, t)$$

Solving this will require separation of variables again. Upon doing this, we will get 3 equations:

$$X'' + \lambda X = 0 \quad x \in (-\ell, \ell)$$
$$X(\ell) = X(-\ell)$$
$$X'(\ell) = X'(-\ell)$$

with $\lambda \in \mathbb{R}$ (the proof of this is like what we did before.

There are a few options for X, we can have

$$X_n = \sin\left(\frac{n\pi x}{\ell}\right) \quad \lambda_n = \left(\frac{n\pi}{\ell}\right)^2 \quad n = 1, 2, 3, \ldots$$
$$X_n = \cos\left(\frac{n\pi x}{\ell}\right) \quad \lambda_n = \left(\frac{n\pi}{\ell}\right)^2 \quad n = 0, 1, 2, 3, \ldots$$

If we are doing the heat equation, we will have that

$$\phi(x) \stackrel{?}{=} \frac{1}{2}A_0 + \sum_{n=1}^{\infty}\left(A_n \cos\left(\frac{n\pi x}{\ell}\right) + B_n \sin\left(\frac{n\pi x}{\ell}\right)\right)$$

which is the **full Fourier series** for $x \in (-\ell, \ell)$. In order to find the coefficients A_n, B_n, it would be really nice if certain integrals cancelled out like before. In particular, it would be nice if

1. $\int_{-\ell}^{\ell} \sin\left(\frac{n\pi x}{\ell}\right) \cos\left(\frac{m\pi x}{\ell}\right)\ dx = 0$
2. $\int_{-\ell}^{\ell} \sin\left(\frac{n\pi x}{\ell}\right) \sin\left(\frac{m\pi x}{\ell}\right)\ dx = 0$
3. $\int_{-\ell}^{\ell} \cos\left(\frac{n\pi x}{\ell}\right) \cos\left(\frac{m\pi x}{\ell}\right)\ dx = 0$

Luckily they are all in fact true!

8

Week 8

8.1 FOURIER TRANSFORM, CONTINUED

8.1.1 FULL FOURIER SERIES, CONTINUED

Towards the end of last week, we introduced the full Fourier series

$$\phi(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty}\left(A_n \cos\left(\frac{n\pi x}{\ell}\right) + B_n \sin\left(\frac{n\pi x}{\ell}\right)\right)$$

This week we will find those coefficients A_n, B_n.

To do this we need to 3 integrals to vanish:

1. $\int_{-\ell}^{\ell} \sin\left(\frac{n\pi x}{\ell}\right)\cos\left(\frac{m\pi x}{\ell}\right)\ dx = 0$
2. $\int_{-\ell}^{\ell} \sin\left(\frac{n\pi x}{\ell}\right)\sin\left(\frac{m\pi x}{\ell}\right)\ dx = 0$
3. $\int_{-\ell}^{\ell} \cos\left(\frac{n\pi x}{\ell}\right)\cos\left(\frac{m\pi x}{\ell}\right)\ dx = 0$

To see why they vanish, recall that if a function $f(x)$ is *odd*, then

$$\int_{-\ell}^{\ell} f(x)\ dx = 0$$

For all n, m, we have that

$$\sin\left(\frac{n\pi x}{\ell}\right)\cos\left(\frac{m\pi x}{\ell}\right)$$

is odd; the product of an odd function, sin, and an even function, cos, is always odd. Thus, its integral from $-\ell$ to ℓ will vanish.

For the other two functions, they are either the product of two even functions, as seen with cos, or two odd functions, as seen with sin, meaning both are even functions. This is not a bad thing, though, as for any even function $f(x)$,

$$\int_{-\ell}^{\ell} f(x)\ dx = 2\int_{0}^{\ell} f(x)\ dx$$

Thus, we get that for $n \neq m$,

$$\int_{-\ell}^{\ell} \cos\left(\frac{n\pi x}{\ell}\right)\cos\left(\frac{m\pi x}{\ell}\right)\ dx = 2\int_{0}^{\ell} \cos\left(\frac{n\pi x}{\ell}\right)\cos\left(\frac{m\pi x}{\ell}\right)\ dx = 0$$

$$\int_{-\ell}^{\ell} \sin\left(\frac{n\pi x}{\ell}\right) \sin\left(\frac{m\pi x}{\ell}\right)\ dx = 2\int_{0}^{\ell} \sin\left(\frac{n\pi x}{\ell}\right) \sin\left(\frac{m\pi x}{\ell}\right)\ dx = 0$$

which follows from the Fourier cosine and sine series respectively. The last thing we need to do is consider when $n = m$ in both cases.

For cos, if $n \neq 0$, we get

$$\begin{aligned}\int_{-\ell}^{\ell} \cos^2\left(\frac{n\pi x}{\ell}\right)\ dx &= 2\int_{0}^{\ell} \cos^2\left(\frac{n\pi x}{\ell}\right)\ dx \\ &= 2\left(\frac{\ell}{2}\right) && \text{(derived previously)} \\ &= \ell\end{aligned}$$

and if $n = 0$, we get

$$\int_{-\ell}^{\ell} \cos(0)\ dx = \int_{-\ell}^{\ell} 1\ dx = 2\ell$$

For sin, we only consider when $n \neq 0$, giving us

$$\begin{aligned}\int_{-\ell}^{\ell} \sin^2\left(\frac{n\pi x}{\ell}\right)\ dx &= 2\int_{0}^{\ell} \sin^2\left(\frac{n\pi x}{\ell}\right)\ dx \\ &= 2\left(\frac{\ell}{2}\right) \\ &= \ell\end{aligned}$$

Upon taking dot products and simplifying, we will get that

$$\begin{aligned}A_n &= \frac{1}{\ell}\int_{-\ell}^{\ell} \phi(x)\cos\left(\frac{n\pi x}{\ell}\right)\ dx \\ B_n &= \frac{1}{\ell}\int_{-\ell}^{\ell} \phi(x)\sin\left(\frac{n\pi x}{\ell}\right)\ dx\end{aligned}$$

8.1.2 UNDERSTANDING CONVERGENCE

Throughout our study of these infinite sums, we've mad a bit of an assumption in saying that that they do in fact converge and thus are well-defined. While full rigour is not necessarily what we are going for in MAT311 (we leave that to those working in real and functional analysis), we should probably get some understanding of when these series converge. To do this we will state 3 theorems that give different flavours of convergence.

Theorem 8.1. *Suppose f is a C^1 function that satisfies either the Dirichlet, Neumann, or Periodic boundary conditions. Then we have that f is equal to the respective Fourier series (sine, cosine, or full), and*

$$\lim_{N\to\infty} \max_x \left| f(x) - \sum_{n=1}^{N} (\cdots)(x) \right| = 0$$

so $f(x) = \sum_{n=1}^{\infty} (\cdots)(x)$

This means that if f satisfies the boundary conditions of our PDE, and is C^1, then it is of the form we've previously defined and equal to the infinite sum (meaning said sum converges).

Theorem 8.2. *Suppose f is continuous and f' is piecewise continuous, all on some interval $[0, \ell]$. Then for $x \in (0, \ell)$,*

$$f(x) = \sum_{n=1}^{\infty} (\cdots)(x)$$

This is a weaker result but is still quite useful, the only downside being that we cannot use it to deal with the endpoints $x = 0, \ell$.

Here's one last theorem that gives another version of convergence of the sum.

Theorem 8.3. *Suppose that*

$$\int f^2 \, dx < \infty$$

Then

$$\lim_{N\to\infty} \int \left| f(x) - \sum_{n=1}^{N} (\cdots)(x) \right|^2 \, dx = 0$$

With these in mind, we can now compute some actual Fourier series.

Example 17. *Compute the Fourier cosine series of $f(x) = 1$ on $(0, \ell)$.*

For $n \neq 0$, the coefficients are given by

$$\begin{aligned} A_n &= \frac{2}{\ell} \int_0^{\ell} 1 \cdot \cos\left(\frac{n\pi x}{\ell}\right) \, dx \\ &= \frac{2}{\ell} \left[\frac{\ell}{n\pi} \sin\left(\frac{n\pi x}{\ell}\right) \right]_0^{\ell} \\ &= 0 \end{aligned}$$

while for $n = 0$,

$$A_0 = \frac{2}{\ell} \int_o^{\ell} 1 \, dx = 2$$

Thus, we get that

$$1 = \frac{1}{2} A_0 + 0 + \cdots = 1$$

This is a silly example, but it is still instructive.

Example 18. *Compute the Fourier sine series of $f(x) = 1$ on $(0, \ell)$.*

The coefficients are given by

$$\begin{aligned}
A_n &= \frac{2}{\ell} \int_0^\ell 1 \cdot \sin\left(\frac{n\pi x}{\ell}\right) \, dx \\
&= \frac{2}{\ell} \left[\frac{-\ell}{n\pi} \cos\left(\frac{n\pi x}{\ell}\right) \right]_0^\ell \\
&= \frac{-2}{n\pi} (\cos(n\pi) - 1) \\
&= \frac{-2}{n\pi} [(-1)^n - 1] \\
&= \begin{cases} 0 & n \text{ is even} \\ \frac{4}{n\pi} & n \text{ is odd} \end{cases}
\end{aligned}$$

Hence we have that

$$1 \stackrel{?}{=} \frac{4}{\pi} \left(\sin(x) + \frac{1}{3} \sin(3x) + \frac{1}{5} \sin(5x) + \cdots \right)$$

Theorem 8.2 tells us that this equality is true when $x \in (0, \ell)$, but not when $x = 0, \ell$. Indeed,

$$1 \neq \frac{4}{\pi} (0 + 0 + \cdots) = 0$$

so we get that 1 does not satisfy the Dirichlet boundary conditions in this case.

9

Week 9

9.1 MORE EXAMPLES OF FOURIER SERIES

Let's compute some additional examples of Fourier series.

Example 19. *compute the Fourier sine series for $f(x) = x$ on $[0, \ell]$.*

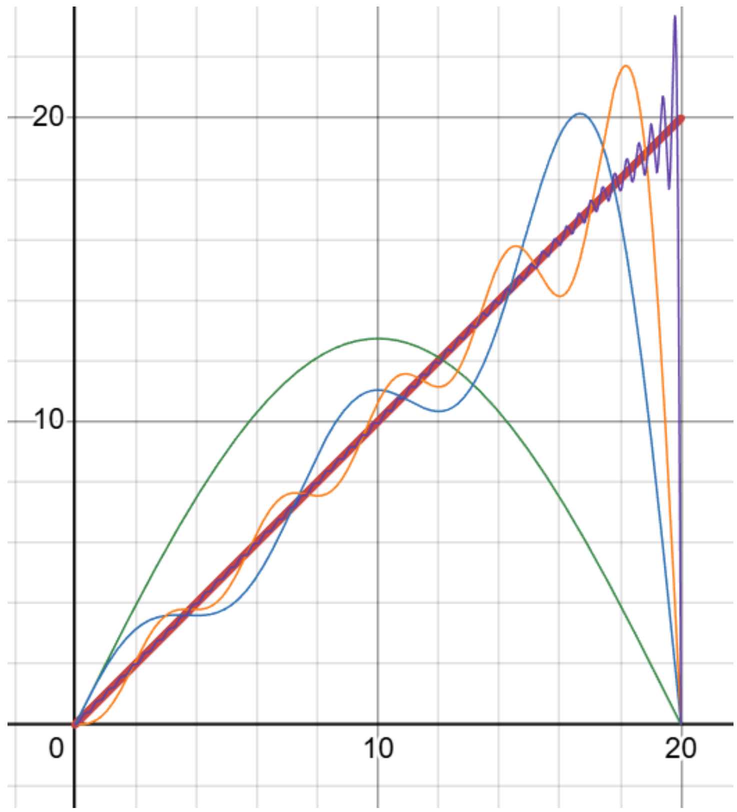
We have that

$$\begin{aligned}
A_m &= \frac{2}{\ell}\int_0^\ell x \sin\left(\frac{m\pi x}{\ell}\right)\, dx \\
&= \frac{-2}{\ell}\int_0^\ell x \left(\frac{\ell}{m\pi}\right)\frac{d}{dx}\left[\cos\left(\frac{m\pi x}{\ell}\right)\right]\, dx \\
&= \frac{-2x}{m\pi}\cos\left(\frac{m\pi x}{\ell}\right)\Bigg|_0^\ell + \frac{2}{m\pi}\int_0^\ell \frac{d}{dx}(x)\cos\left(\frac{m\pi x}{\ell}\right)\, dx \\
&= \frac{-2\ell}{m\pi}\cos(m\pi) + \frac{2\ell}{(m\pi)^2}\sin\left(\frac{m\pi x}{\ell}\right)\Bigg|_0^\ell \\
&= \frac{-2\ell}{m\pi}(-1)^m \\
&= \frac{2\ell}{m\pi}(-1)^{m+1}
\end{aligned}$$

This tells us that

$$x \stackrel{?}{=} \frac{2\ell}{\pi}\left[\sin\left(\frac{\pi x}{\ell}\right) - \frac{1}{2}\sin\left(\frac{2\pi x}{\ell}\right) + \frac{1}{3}\sin\left(\frac{3\pi x}{\ell}\right) - \cdots\right]$$

Recall that $f(x) = x$ is smooth on $x \in (0, \ell)$, so the series converges at these points. If $x = \ell$, the sum on the right vanishes and we do not get equality. We can see this in the figure below, where we show the series with 1 (green), 5 (blue), 10 (orange), and 100 (purple) terms, converging to the red line, which is $f(x)$. Notice how all of the functions vanish at ℓ, which in the graph is set to 20.



Example 20. *Compute the Fourier cosine series for $f(x) = x$ on $[0, \ell]$.*

For the $m = 0$ case, we get

$$A_0 = \frac{2}{\ell}\int_0^\ell x \cdot 1 \, dx = \frac{2}{\ell}\frac{x^2}{2}\bigg|_0^\ell = \frac{\ell^2}{\ell} = \ell$$

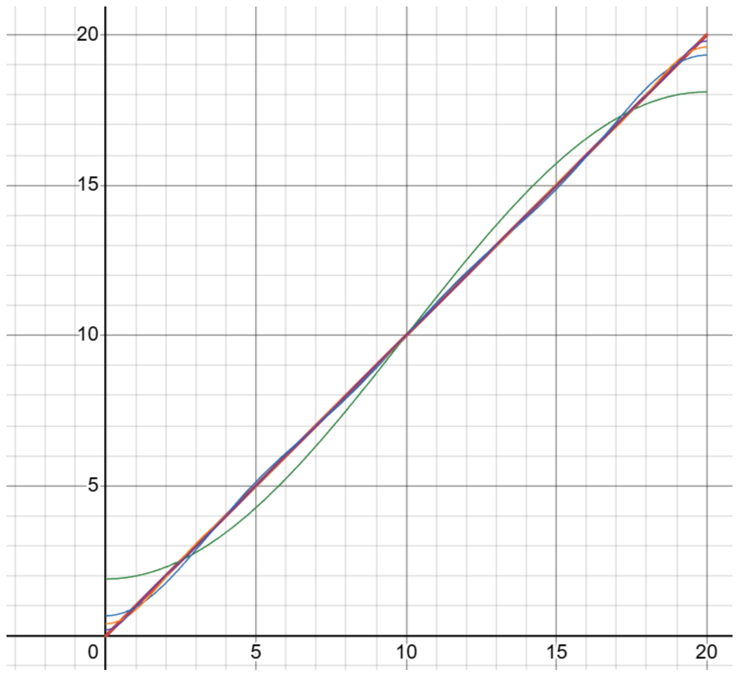
For $m > 0$, we get

$$\begin{aligned}
A_m &= \frac{2}{\ell}\int_0^\ell x\cos\left(\frac{m\pi x}{\ell}\right)\, dx \\
&= \frac{2}{\ell}\int_0^\ell x\frac{\ell}{m\pi}\frac{d}{dx}\left[\sin\left(\frac{m\pi x}{\ell}\right)\right]\, dx \\
&= \frac{2x}{m\pi}\sin\left(\frac{m\pi x}{\ell}\right)\bigg|_0^\ell - \frac{2}{m\pi}\int_0^\ell \frac{d}{dx}(x)\sin\left(\frac{m\pi x}{\ell}\right)\, dx \\
&= \frac{2\ell}{(m\pi)^2}\cos\left(\frac{m\pi x}{\ell}\right)\bigg|_0^\ell \\
&= \frac{2\ell}{(m\pi)^2}((-1)^m - 1) \\
&= \begin{cases} 0 & m \text{ is even} \\ -\dfrac{4\ell}{(m\pi)^2} & m \text{ is odd} \end{cases}
\end{aligned}$$

Thus,

$$x \stackrel{?}{=} \frac{1}{2}\ell - \frac{4\ell}{\pi^2}\left[\cos\left(\frac{\pi x}{\ell}\right) + \frac{1}{3^2}\cos\left(\frac{3\pi x}{\ell}\right) + \frac{1}{5^2}\cos\left(\frac{5\pi x}{\ell}\right) + \cdots\right]$$

This is in fact an equality when $x \in (0, \ell)$, but there's no guarantee it works on the endpoints. The figure below shows the sums with 1 (green), 3 (blue), 5 (orange), and 10 (purple) terms. Notice how quickly the convergence occurs compared to the sine series.



Example 21. *Find the full Fourier series for $f(x)$ on $(-\ell, \ell)$.*

As x is odd, we get that

$$A_0 = \frac{1}{\ell}\int_{-\ell}^{\ell} x\ dx = 0$$

For $m > 0$, we get that

$$A_m = \frac{1}{\ell}\int_{-\ell}^{\ell} x\cos\left(\frac{m\pi x}{\ell}\right)\ dx = 0$$

since x is odd and cos *is even.* sin *is odd, so we get that*

$$\begin{aligned} B_m &= \frac{1}{\ell}\int_{-\ell}^{\ell} x\sin\left(\frac{m\pi x}{\ell}\right)\ dx \\ &= \frac{2}{\ell}\int_{0}^{\ell} x\sin\left(\frac{m\pi x}{\ell}\right)\ dx \\ &= \frac{2\ell}{m\pi}(-1)^{m+1} \end{aligned}$$

which follows from the sine series calculation above. thus, the full Fourier series for $f(x) = x$ is just the sine series, taken on $(-\ell, \ell)$.

Example 22. *Solve $u_{tt} - c^2 u_{xx} = 0$ given*

$$\begin{cases} u(0,t) = u(0,\ell) = 0 & t > 0 \\ u(x,0) = x, u_t(x,0) = 0 \end{cases}$$

We know that

$$u(x,t) = \sum_{n=1}^{\infty} \left(A_n \cos\left(\frac{n\pi ct}{\ell}\right) + B_n \sin\left(\frac{n\pi ct}{\ell}\right)\right) \sin\left(\frac{n\pi x}{\ell}\right)$$

and that

$$x = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{\ell}\right), \quad 0 = \sum_{n=1}^{\infty} \frac{n\pi c}{\ell} B_n \sin\left(\frac{n\pi x}{\ell}\right)$$

Thus, we get that

$$A_n = \frac{2\ell}{\pi} \frac{(-1)^{n+1}}{n}, \quad B_n = 0$$

Hence, we conclude that

$$u(x,t) = \frac{2\ell}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \cos\left(\frac{n\pi ct}{\ell}\right) \sin\left(\frac{n\pi x}{\ell}\right)$$

We conclude this section with 2 remarks.

First, suppose we wish to expand a function $f(x)$ into its Fourier sine or cosine series, but it is defined on some interval (a,b) instead of $(0,\ell)$. To solve this, take

$$g(x) = f(x \pm c)$$

where g is defined on $x \in (0,\ell)$. In this case, we take

$$g(x) = f(x+a), \quad x \in (0, b-a)$$

We can then expand g into its sine or cosine series. For example, if we expand it into its sine series, we get

$$g(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{b-a}\right), \quad A_n = \frac{2}{b-a} \int_0^{b-a} g(x) \sin\left(\frac{n\pi x}{b-a}\right)\, dx$$

$$\implies f(x) = g(x-a) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi (x-a)}{b-a}\right), \quad A_n = \frac{2}{b-a} \int_0^{b-a} g(x-a) \sin\left(\frac{n\pi (x-a)}{b-a}\right)\, dx$$

Our second remark concerns the example of the full Fourier series for $f(x) = x$. The fact that it is just the sine series is not a coincidence, bur rather a general fact of odd functions.

Similarly, for even functions $\phi(x)$, we have that

$$\int_{-\ell}^{\ell} \sin\left(\frac{n\pi x}{\ell}\right)\ dx = 0$$

$$\frac{1}{\ell}\int_{-\ell}^{\ell} \phi(x)\cos\left(\frac{n\pi x}{\ell}\right) = \frac{2}{\ell}\int_{0}^{\ell} \phi(x)\cos\left(\frac{n\pi x}{\ell}\right)\ dx$$

Thus, the full Fourier series of $\phi(x)$ is just the cosine series.

Finally, recall that

$$\phi(x) = \frac{1}{2}[\phi(x)+\phi(-x)] + \frac{1}{2}[\phi(x)-\phi(x)]$$

where the first component is an even function and the second component is an odd function (check this). Thus, the full Fourier series of any function corresponds to the Fourier cosine series of the even part and the Fourier sine series of the odd part.

9.2 COMPLEX EXPONENTIALS & FOURIER SERIES

Recall that the sine and cosine functions may be written using complex exponentials:

$$\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2} \qquad \cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}$$

This means that any expansions involving sine and cosine, like in Fourier series, can be written instead as expansions in $e^{\pm\frac{in\pi x}{\ell}}$. This can be tedious, but we can use the same methods from before, and in fact, these methods become much easier to do once we incorporate complex exponentials!

Let's derive the full Fourier series using $\{e^{\frac{i\pi n x}{\ell}}\}_{n=-\infty}^{\infty}$. The new orthogonality condition is that for $n \neq m$

$$\int_{-\ell}^{\ell} e^{\frac{in\pi x}{\ell}}\overline{e^{\frac{im\pi x}{\ell}}}\ dx = \int_{-\ell}^{\ell} e^{\frac{in\pi x}{\ell}}e^{-\frac{im\pi x}{\ell}}\ dx$$

We get that

$$\int_{-\ell}^{\ell} e^{\frac{i(n-m)\pi x}{\ell}}\ dx = \int_{-\ell}^{\ell} \frac{\ell}{(n-m)}\frac{d}{dx}e^{\frac{i(n-m)\pi x}{\ell}}\ dx = 0$$

because $e^{\frac{ik\pi x}{\ell}}$ is periodic over $(-\ell,\ell)$ whenever k is an integer. Moreover, we have that

$$\int_{-\ell}^{\ell} e^{\frac{in\pi x}{\ell}}\overline{e^{\frac{in\pi x}{\ell}}}\ dx = \int_{-\ell}^{\ell} 1\ dx = 2\ell$$

Now, the full Fourier series for $f(x)$ on $(-\ell,\ell)$ is given by

$$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{\frac{in\pi x}{\ell}}$$

We see that

$$\int_{-\ell}^{\ell} f(x)e^{-\frac{in\pi x}{\ell}}\,dx = \sum_{n=-\infty}^{\infty} C_n \int_{-\ell}^{\ell} e^{\frac{in\pi x}{\ell}} e^{-\frac{in\pi x}{\ell}}\,dx = 2\ell C_n$$

Thus,

$$C_m = \frac{1}{2\ell}\int_{-\ell}^{\ell} f(x)e^{-\frac{im\pi x}{\ell}} \quad m = 0, 1, 2, 3, \ldots$$

The = sign has the same conditions as for regular Fourier series.

Example 23. *Compute the full Fourier series for $f(x) = x$ on $(-\ell, \ell)$ using the complex exponential.*

We have that, for $n = 0$,

$$C_0 = \frac{1}{2\ell}\int_{-\ell}^{\ell} x\,dx = 0$$

and for $n \neq 0$, we get

$$\begin{aligned}
C_n &= \frac{1}{2\ell}\int_{-\ell}^{\ell} xe^{-\frac{in\pi x}{\ell}}\,dx \\
&= \frac{1}{2\ell}\int_{-\ell}^{\ell} x\left(-\frac{\ell}{in\pi}\right)\frac{d}{dx}(e^{-\frac{in\pi x}{\ell}})\,dx \\
&= \frac{-x}{2in\pi}e^{-\frac{in\pi x}{\ell}}\bigg|_{-\ell}^{\ell} + \frac{1}{2in\pi}\int_{-\ell}^{\ell} e^{-\frac{in\pi x}{\ell}}\,dx \\
&= \frac{-\ell}{2in\pi}e^{-in\pi} - \frac{\ell}{2in\pi}e^{in\pi} + \frac{1}{2in\pi}\int_{-\ell}^{\ell} e^{-\frac{in\pi x}{\ell}}\,dx \\
&= \frac{-\ell}{2in\pi}e^{-inx}(1 + e^{2\pi in)} \\
&= \frac{-\ell}{in\pi}e^{in\pi} \\
&= \frac{-\ell}{in\pi}(-1)^n
\end{aligned}$$

9.3 THE FULL VERSION OF THE HEAT EQUATION ON A FINITE INTERVAL

To wrap up our study of Fourier series, let's go all in and solve the heat equation in its most general of forms: for $x \in (0, \ell)$,

$$\begin{cases} u_t - ku_{xx} = f(x,t) \\ u(0,t) = h(t), u(\ell,t) = j(t) \\ u(x,0) = \phi(x) \end{cases}$$

To solve this, we could try separation of variables. Setting $u(x,t) = X(x)T(t)$, we get

$$X(x)T'(t) - kX''(x)T(t) = f(x,t)$$

and there isn't much else that we can do. Instead, let's try a different approach.

Remark. *What we are going to do works on $(0, \ell)$, but is not guaranteed to work on the end points.*

We write

$$u(x,t) = \sum_{n=1}^{\infty} u_n(t) \sin\left(\frac{n\pi x}{\ell}\right)$$
$$f(x,t) = \sum_{n=1}^{\infty} f_n(t) \sin\left(\frac{n\pi x}{\ell}\right)$$

We wish to derive equations for each $A_n(t)$. A naive approach would be to apply the PDE to this series, giving us

$$\left(\frac{\partial}{\partial t} - k\frac{\partial^2}{\partial x^2}\right)\left[\sum_{n=1}^{\infty} u_n(t) \sin\left(\frac{n\pi x}{\ell}\right)\right] = \sum_{n=1}^{\infty} f_n(t) \sin\left(\frac{n\pi x}{\ell}\right)$$

where $f_n(t)$ are the coefficients in f's Fourier sine series. This seems like a good approach, but there are issues along the boundaries, and so this will not always work. For example, we have previously seen that

$$1 = \sum_{n \text{ odd}} \frac{4}{n\pi} \sin(nx)$$

but deriving gives us

$$0 = \sum_{n \text{ odd}} \frac{4}{\pi} \cos(nx)$$

which is clearly not true since the right hand side is not going to 0.

Not all hope is lost, as we still have one more trick up our sleeves. We will instead try to find the coefficients *directly*. Instead of applying the PDE to the sum, let's plug the PDE into it:

$$\int_0^{\ell} (u_t - ku_{xx}) \sin\left(\frac{n\pi x}{\ell}\right)\, dx = \int_0^{\ell} f(x,t) \sin\left(\frac{n\pi x}{\ell}\right) = f_n(t)$$

Let's break up the right side into two equations.

$$\frac{2}{\ell}\int_0^{\ell} u_t \sin\left(\frac{n\pi x}{\ell}\right)\, dx - \frac{2k}{\ell}\int_0^{\ell} u_{xx} \sin\left(\frac{n\pi x}{\ell}\right)\, dx$$

We compute each of these individually: Using reverse differentiation under the integral sign, we get

$$\begin{aligned}\frac{2}{\ell}\int_0^{\ell} u_t \sin\left(\frac{n\pi x}{\ell}\right)\, dx &= \frac{d}{dt}\frac{2}{\ell}\int_0^{\ell} u \sin\left(\frac{n\pi x}{\ell}\right)\, dx \\ &= \frac{d}{dt}u_n\end{aligned}$$

Moreover, applying IBP twice gives us

$$\begin{aligned}
\frac{-2k}{\ell}\int_0^\ell u_{xx}\sin\left(\frac{n\pi x}{\ell}\right)\,dx &= \frac{-2k}{\ell}u_x\sin\left(\frac{n\pi x}{\ell}\right)\Big|_0^\ell + \frac{2k}{\ell}\int_0^\ell u_x\left(\frac{n\pi}{\ell}\right)\cos\left(\frac{n\pi x}{\ell}\right)\,dx \\
&= \frac{2kn\pi}{\ell^2}u\cos\left(\frac{n\pi x}{\ell}\right)\Big|_0^\ell + \frac{2kn\pi}{\ell^2}\left(\frac{n\pi}{\ell}\right)\int_0^\ell u\sin\left(\frac{n\pi x}{\ell}\right)\,dx \\
&= \frac{(-1)^n 2kn\pi}{\ell^2}j(t) - \frac{2kn\pi}{\ell^2}h(t) + k\left(\frac{n\pi}{\ell}\right)^2\left(\frac{2}{\ell}\int_0^\ell u\sin\left(\frac{n\pi x}{\ell}\right)\,dx\right) \\
&= \frac{(-1)^n 2kn\pi}{\ell^2}j(t) - \frac{2kn\pi}{\ell^2}h(t) + k\left(\frac{n\pi}{\ell}\right)^2 u_n
\end{aligned}$$

Combining, we get an ODE,

$$\frac{d}{dt}u_n(t) + k\left(\frac{n\pi}{\ell}\right)^2 u_n(t) = f_n(x,t) + \frac{2kn\pi}{\ell^2}(h(t) - j(t)(-1)^n)$$

which we can solve using an integrating factor. We write

$$\frac{d}{dt}u_n(t) + k\left(\frac{n\pi}{\ell}\right)^2 u_n(t) = e^{-tk\left(\frac{n\pi}{\ell}\right)^2}\frac{d}{dt}[e^{tk\left(\frac{n\pi}{\ell}\right)^2}u_n(t)]$$

and setting $P_n(t) = f_n + \frac{2kn\pi}{\ell^2}(h(t) - j(t)(-1)^n)$, we get

$$\begin{aligned}
&\implies \frac{d}{dt}[e^{tk\left(\frac{n\pi}{\ell}\right)^2}u_n(t)] = e^{tk\left(\frac{n\pi}{\ell}\right)^2}P_n(t) \\
&\implies e^{tk\left(\frac{n\pi}{\ell}\right)^2}u_n(t) = u_n(0) + \int_0^t e^{sk\left(\frac{n\pi}{\ell}\right)^2}P_n(s)\,ds \\
&\implies u_n(t) = e^{-tk\left(\frac{n\pi}{\ell}\right)^2}u_n(0) + e^{-tk\left(\frac{n\pi}{\ell}\right)^2}\int_0^t e^{sk\left(\frac{n\pi}{\ell}\right)^2}P_n(s)\,ds
\end{aligned}$$

where $u_n(0) = \frac{2}{\ell}\int_0^\ell \phi(x)\sin\left(\frac{n\pi x}{\ell}\right)\,dx$.

If we take $f = h = j = 0$, then $P_n(t) = 0$, and get a much more simplified equation.

This process works for any case where separation of variables was used, including Neumann B.C, periodic B.C, the wave equation, Schrodinger's Equation, etc.

10

Week 10

10.1 THE LAPLACE EQUATION

We now pivot towards studying the Laplace equation. In dimensions 1,2, and 3, the Laplace equation is written as

$$u_{xx} = 0 \quad u_{xx} + u_{yy} = 0 \quad u_{xx} + u_{yy} + u_{zz} = 0$$

We may also write this equation as

$$\Delta u = 0$$

where Δ is the **Laplacian** operator.

This equation deals with situations where a solution is in an *equilibrium/stationary* state, that is, they are independent of time. Take for example the heat equation in 1 dimension,

$$u_t - ku_{xx} = 0$$

If our solution is in a stationary state, then it is independent of time, meaning the u_t is gone, leaving us with the 1 dimensional Laplace equation.

In a sense, the Laplace equation is the most relevant PDE in mathematical physics, playing an important role in electrostatics, steady fluid flow, and Brownian motion, among other areas. Studying this equation in detail - like we did for the heat and wave equations - is somewhat outside the scope of the course, but we can still look at it and understand some interesting properties.

10.1.1 THE MAXIMUM/MINIMUM PRINCIPLE

Like the heat equation, the Laplace equation has a Max/Min Principle, though, it is a little more complex compared to before.

Our domain is now not a square of values, but rather any open set $D \subset \mathbb{R}^2$ with a "nice" boundary ("nice" just means that it doesn't cause us any problems). We suppose that, inside of D,

$$\Delta u = 0$$

and that u extends continuously to ∂D. We know that $D \cup \partial D$ is a closed and bounded subset of $\mathbb{R}^2$, so the Extreme Value Theorem tells us that u attains a maximum and minimum on D. Take a wild guess where they are...

Theorem 10.1 (The Maximum/Minimum Principle for Laplace Equations)**.** *Let $D \subset \mathbb{R}^2$ and u be as above, then*

$$\max_D u = \max_{\partial D} u \quad \textit{and} \quad \min_D u = \min_{\partial D} u$$

One may notice that this is somewhat related to the 2nd Derivative Test: If a critical point likes within the interior, the $\nabla u = 0$. If that point is a strict extremum, then $\nabla^2 u$ has all positive or negative eigenvalues (depending on if it's a max or min). Now $\Delta = \operatorname{tr} \nabla^2 u$. However, if we have a strict max/min, this value will be positive or negative, and not 0, meaning the point cannot be an extremum.

Uniqueness of the Dirichlet Problem

The Maximum/Minimum Principle can allow us to prove, like in studying the heat equation, that the Laplace equation has unique solutions in certain scenarios. Let's consider the Laplace equation with Dirichlet boundary conditions. We let D be a bounded, open subset of $\mathbb{R}^2$ and let

$$h : \partial D \to \mathbb{C}$$

be a smooth function, and $F : D \to \mathbb{C}$ be a smooth too. We seek to find a solution to

$$\begin{cases} u : D \to \mathbb{C} \\ \Delta u = F \\ u|_{\partial D} = h \end{cases}$$

For simplicity, we can take $F = 0$.

Proposition 10.2. *The above problem has at most 1 solution.*

Proof. Suppose that both u_1, u_2 solve the problem. We define

$$w = u_1 - u_2$$

We see that

$$\begin{aligned} \Delta w &= \Delta(u_1 - u_2) \\ &= \Delta u_1 - \Delta u_2 \\ &= 0 \\ w|_{\partial D} &= (u_1 - u_2)|_{\partial D} \\ &= u_1|_{\partial D} - u_2|_{\partial D} \\ &= 0 \end{aligned}$$

so the problem for w is

$$\begin{cases} \Delta w = 0 \\ w|_{\partial D} = 0 \end{cases}$$

so the Maximum/Minimum Principle applies. We get that

$$\max_D w = \max_{\partial D} w = 0$$
$$\min_D w = \min_{\partial D} w = 0$$

hence $w = 0$, meaning $u_1 = u_2$. □

10.1.2 INVARIANCE UNDER RIGID MOTION

Another important property of the Laplace equation is that is invariant under *rigid motions*, that is, the symmetries of the plane. There are 2 such symmetries:

(i) Translation: $(x, y) \mapsto (x + a, y + b)$ for constants a, b.

(ii) Rotation: $(x, y) \mapsto (x \cos\alpha + y \sin\alpha, x \sin\alpha + y \cos\alpha)$ for some $\alpha \in [0, 2\pi)$.

What's special about this is that Δ is the only differential operator with this property. This makes is perfect for studying *isotropic* situations in engineering, where there is no preferred direction.

The fact that we have rotational invariance suggests that the Laplace equation would have a simpler form when converted to *polar coordinates*. Let's compute it. Given polar coordinates (r, θ), we have

$$x = r\cos\theta, \quad y = r\sin\theta$$
$$r = \sqrt{x^2 + y^2}, \quad \theta = \arctan\left(\frac{y}{x}\right)$$