Chapter 1

Introduction

For any set S and group G, there is a unique group up to isomorphism satisying the universal property, called the free group generated by S. If $|S| = n$, then we call this group the free group of rank n, and denote it as F_n.

Free groups were first studied as an example of Fuchsian groups, which are discrete subgroups of $PSL_2(\mathbb{Z})$, but they did not become relevant on their own until the late 1800s. Recall that we may write any group using its group presentation. von Dyck, who was first to express groups using a presentation, showed that free groups have the simplest possible one: The free group F_n has presentation

$$\langle a_1, a_2, \dots a_n \rangle$$

meaning it consists of n generators with no relations between them; this is often used as an alternate definition of F_n. It is thus the case that every group is the quotient of a free group. The algebraic properties of free groups were investigated by Nielsen, who also gave them their name, whilst a comprehensive overview of free groups was given by Reidemeister in his 1932 book on combinatorial group theory.

The simple nature of free groups is equal parts useful and deceiving. Consdier The Word Problem.

The Word Problem: For a group G, let g be an element expressed as a product of G's generators. Find an algorithm that shows, in finitely many steps, whether or not g is the identity element.

While it is known that this problem is undecidable for arbitary groups, it is answerable if G is a free group. Recall that a free group has no relations between the generators, apart from the obvious one that $a_i a_i^{-1} = 1$. Thus, given an arbitrary element of a free group, one can just check if any of these pairs exist, and eliminate them one at a time until none remain; if generators are still present, the element is not the identity, and if no generators are left, the element is the identity. Thus, while The Word Problem is undecidable, it is easily solvable in the case of free groups.

This does not mean that free groups are a trivial class of groups. In fact, understanding their algebraic properties has been an area of study for over a hundred years. To demonstrate this, we consider two questions.

First, suppose we are given the presentation of an arbitrary group G. Can we easily check whether or not G is a free group? This question was answered in the negative in the 1950s, where it was shown that such a question is undecidable.

The second question is one that is often asked when a property of a structure is first defined: do the substructures of this structure also have this property? In particular, are the subgroups of a free group also free groups? This was answered in the positive in the 1920s, first in a restricted form by Nielsen, then the the full form independently by Dehn and Schreier However, such proofs were not trivial, and required high level topological or algebraic methods to get to a rigorous proof. Understanding the proofs would require understanding very high level results in group theory, far beyond the level of an undegraduate course in abstract algebra.

Answers to the both of these questions do exist in simpler forms within the field of Combinatorial/Geometric Group Theory, which in part seeks to understand a group's algebraic properties by looking at how it acts on a space. This extra assumption on how a group acts upon a space makes these results much easier to understand at the undergraduate level.

An answer to the first question can be found thanks to the Ping-Pong Lemma, which describes a sufficient condition for a group to be free, so long as it acts in a certain way on a set X:

Theorem 1.1 (The Ping-Pong Lemma). *Let $\{g_1, g_2, \ldots, g_n\}$ generate a group G, which acts on a set X. If*

(i) X contains n subsets $X_1, \ldots, X_n$ such that $X_i \cap X_j = \varnothing$, and

(ii) $g_i^k(X_j) \subset X_i$ for all nonzero powers k and $i \neq j$.

Then G is isomorphic to F_n.

More interestingly, an answer to question two is a corollary of a result proven by Serre in 1969, who showed that free groups may be classified by how they act on trees.

Theorem 1.2 (Serre's Theorem). *G is a free group if and only if it acts freely on a tree.*

Corollary 1.3 (Nielsen-Schreier Theorem). *Every subgroup of a free group is itself free.*

In this paper, we present full proofs and explanations of both results. First, we prove the Ping-Pong Lemma both in the case of $n = 2$, and in the general case for arbitrary n. We also apply the Ping-Pong Lemma, and another result, to show that free groups of any rank, including one of infinite rank, lies within the matrix group $SL_2(\mathbb{Z})$. Afterwards, we prove both sides of Serre's Theorem. In proving the forwards direction, we define a special graph for our free group to act on called the Cayley Graph, and then show that free groups acts freely on their corresponding Cayley Graphs. To prove the reverse direction, we show that every group acting on a tree induces a special tiling of the tree, before finding a free generating set using this tiling.

Chapter 2

Free Groups

2.1 The Universal Property

For us to work with free groups, we first need to understand what it means for a group to be "free."

Definition 2.1. Let S be a set. Then a group F_S is called the **free group generated by** S if the following holds: for any function f from S to a group G, there is a unique homomorphism $\varphi : F_S \to G$ such that the below diagram commutes (where ι denotes inclusion).

$$\begin{tikzcd} & F_S \arrow[d, dashed, "\varphi"] \\ S \arrow[ur, "\iota"] \arrow[r, "f"'] & G \end{tikzcd}$$

We call S the **generating set** of F_S.

The condition that F_S must satisfy is called the **universal property**, and plays an important role in developing free structures in category theory. A priori, it is not obvious that such a group exists for every such set. However, not only is there such a group, but this group is unique up to isomorphism.

Theorem 2.2. *For any set S, there is a free group F_S generated by S.*

F_S is also unique up to isomorphism; if F_S^1, F_S^2 are both free groups generated by S, with inclusion maps given by $\iota_1 : S \to F_S^1, \iota_2 : S \to F_S^2$, then there are unique isomorphisms $\hat{\iota_2} : F_S^1 \to F_S^2$ and $\hat{\iota_1} : F_S^2 \to F_S^1$ such that

$$\hat{\iota_2} \circ \iota_1 = \iota_2, \quad \hat{\iota_1} \circ \iota_2 = \iota_1, \quad \hat{\iota_1}^{-1} = \hat{\iota_2}$$

Proof. We let F_S be the set of reduced words with letters given by elements of S, with the operation given by concatenation and reduction. Now, for any $f : S \to G$, where G is a group, we define φ as follows: we send the empty word $\varnothing$ to 1_G, the identity element of G, and for any $s \in S$,

$$\varphi(s) = f(s)$$

where we think of s as a word consisting of a single letter in F_S. φ can then be uniquely extended to all other words by the fact that it is a homomorphism.

We now show uniqueness. Setting $G = F_S^1$ and $f = \iota_1$ in the diagram from Definition 2.1 gives us

$$\begin{tikzcd} & F_S^1 \arrow[d, dashed, "\mathrm{id}_{F_S^1}"] \\ S \arrow[ur, "\iota_1"] \arrow[r, "\iota_1"'] & F_S^1 \end{tikzcd}$$

Similarly, setting $G = F_S^2$ and $f = \iota_2$, we get

$$\begin{tikzcd} & F_S^1 \arrow[d, dashed, "\hat{\iota_2}"] \\ S \arrow[ur, "\iota_1"] \arrow[r, "\iota_2"'] & F_S^2 \end{tikzcd}$$

from which we get that $\hat{\iota_2} \circ \iota_1 = \iota_2$. Swapping F_S^1 and F_S^2 and their respective maps gives us $\hat{\iota_1}$ such that $\hat{\iota_1} \circ \iota_2 = \iota_1$.

Notice that $\hat{\iota_1} \circ \hat{\iota_2} : F_S^1 \to F_S^1$ satisfies

$$\hat{\iota_1} \circ \hat{\iota_2} \circ \iota_1 = \hat{\iota_1} \circ \iota_2 = \iota_1$$

satisfying the same equation that $\mathrm{id}_{F_S^1}$ does. By uniqueness, we get that $\hat{\iota_1} \circ \hat{\iota_2} = \mathrm{id}_{F_S^1}$. A similar argument shows that $\hat{\iota_2} \circ \hat{\iota_1} = \mathrm{id}_{F_S^2}$. This completes the proof. $\square$

The presence of a set S seems to complicate our definition, as we must define free groups in terms of this generating set. However, we can greatly reduce the number of sets we must consider by making a key observation. Suppose S, T are sets such that $|S| = |T|$, and let F_S, F_T be the corresponding free groups. Recall that we construct each free group by sending each element of the generating set to a letter in the free group, and define the homomorphisms φ by how they map these letters. As S, T are the same size, there is a bijective map $h : S \to T$. More importantly, this map lifts to one on the free groups

$$\psi : F_S \to F_T, \quad \psi(\iota_S(s)) = \iota_T(h(s))$$

As h is a bijection on the generating sets, and each free group is uniquely determined by the letters these sets map to, ψ is an isomorphism of the groups. We have just proven

Theorem 2.3. *Every free group generated by sets of the same cardinality are isomorphic.*

Because of this, we only need to consider free groups generated by sets of the form $\{1, 2, \dots, n\}$; the size of this set is called the free group's **rank**. We will thus denote these groups by F_n, called the **free group of rank** n. For the purposes of this paper, we will restrict our study to free groups of countable rank, meaning we consider F_n for all $n \in \mathbb{N}$, as well as F_∞, the free group with generators x_i for all natural i.

The construction of free groups makes them naturally important in the study of arbitrary groups. The first to notice this fact was Walther von Dyck, one of the founders of Combinatorial Group Theory, and the first to study groups via generators and relations. It was through this study that he found the usefulness of free groups:

Theorem 2.4. *Every group G is the quotient of a free group.*

Proof. Let G have a presentation given by $\langle g_1, g_2, \ldots, g_n | R\rangle$. Let $S = \{1, 2, \ldots, n\}$ and let F_n be the free group generated by S. Consider the map

$$f : S \to G, \quad f(i) = g_i$$

Then by the universal property, let $\varphi : F_n \to G$ be the unique homomorphism associated with f. It follows that

$$\ker \varphi = \{g = g_1^{e_1} \cdots g_n^{e_n} : g = 1\} = R$$

In other words, $\ker G$ is the set of all relations on G. Thus, we get that

$$G = F_n / \ker \varphi$$

□

Because of von Dyck's Theorem, we are able to define free groups in a very simple manner. By construction, the relations within a free group F_n are strictly those saying that $x_i x_i^{-1} = 1$ for any $i \in \{1, 2, \ldots, n\}$. Thus, free groups have the simplest possible presentation, and can be defined as just a set of generators with no relations:

Definition 2.5. The **free group of rank** n is the group F_n, the set of all reduced words in symbols $\{x_1, \ldots, x_n\}$, with operation given by concatenation and reduction. Its presentation is

$$\langle a_1, a_2, \ldots, a_n\rangle$$

and the identity is the empty word $\varnothing$.

Becuase of this result, we are able to translate the algebraic definition of a group, that which uses the universal property, to one using reduced words, which are much easier to work with.

2.2 The Rank of Subgroups of F_2

Rank is a group theoretic analogue of dimension for vector spaces.

Definition 2.6. The **rank** of a group G is the size of the smallest possible generating set for G.

$$\operatorname{rank}(G) = \min\{|X| : X \subseteq G, \langle X\rangle = G\}$$

For a vector space V, any subspace W will have at most dimension $\dim V$, which makes intuitive sense. However, despite being an analogue of dimension for groups, this property does not hold for arbitrary groups; it hold for special types of groups, such as finite abelian groups.

The subgroups of free groups can oftentimes have larger rank than the free group itself. In particular, F_2, not only contains subgroups of larger rank, but they contain subgroups of every possilbe rank larger than 2, including subgroups of countably infinite rank!

Theorem 2.7. *(a) For all $n > 2$, $F_n \subset F_2$.*

(b) $F_\infty \subset F_2$, *where*

$$F_\infty = \langle a_1, a_2, a_3, \ldots \rangle$$

is the free group of countably infinite rank.

Proof. (a) We write $F_2 = \langle a, b \rangle$, and define

$$w_1 = a$$
$$w_2 = b^{-1}ab$$
$$w_3 = b^{-2}ab^{-2}$$
$$\vdots$$
$$w_n = b^{-n}ab^n$$

With the operations of concatenation and reduction, these form the generating set of a group. It thus suffices to show that there are no relations between these generators. First notice that for any w_i and any integer k:

$$\begin{aligned} w_i^k &= (b^{-i}ab^i)(b^{-i}ab^i)\cdots(b^{-i}ab^i) \\ &= b^{-i}a(b^ib^{-i})a(b^ib^{-i})a\cdots a(b^ib^{-i})ab^i \\ &= b^{-i}a^kb^i \end{aligned}$$

Now, let $w = w_{i_i}^{k_1} \cdots w_{i_l}^{e_l}$ be an arbitrary word. We have that

$$\begin{aligned} w &= w_{i_i}^{k_1} \cdots w_{i_l}^{e_l} \\ &= (b^{-i_1}a^{e_1}b^{i_1})(b^{-i_2}a^{e_2}b^{i_2})\cdots(b^{-i_l}a^{e_l}b^{i_l}) \\ &= b^{-i_1}a^{e_1}b^{i_i-i_2}a^{e_2}b^{i_2-i_3}\cdots b^{i_{l-1}-i_l}a^{e_l}b^{i_l} \end{aligned}$$

and this is fully reduced. Moreover, it clearly cannot be expressed as the reduced product of any other set other product of the generators. Therefore, we conclude that there are no relations between the w_i's, and hence they generate the free group F_n, so $F_n \subset F_2$.

(b) The proof is identical to (a), with the generating set of F_∞ being the set of w_i for all $i \in \mathbb{N}$. $\square$

One might ask if this result generalizes to free groups of rank higher than 2? In particular, for any $k \in \mathbb{N}$, is it true that $F_n \subset F_k$ for $n > k$? If so, we'd get a chain of inclusions

$$F_2 \supset F_3 \supset F_4 \supset \cdots$$

The answer is no, and follows from a result of Schreier:

Theorem 2.8 (Schreier Index Formula)**.** *If H is a subgroup of F_n with index e, then H has rank*

$$1 + e(n - 1)$$