Folding Free Groups

A UTM Math Club Presentation

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What is a Free Group?

A **word** in a, b is any finite string of a, b, a^{-1}, b^{-1}.

$$\varnothing, a, b, ababab, a^{-1}abbb^{-1}aa, aaaaa$$

We say it is **reduced** if no symbol is next to its inverse.

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The set of reduced words in symbols a, b is the **free group of rank 2**, denoted F_2, with multiplication given by concatenation then reduction.

$$abba^{-1} \cdot ababa^{-1}b = abbbaba^{-1}b$$

If we used n symbols, we'd get F_n, the **free group of rank** n.

Subgroups of a Free Group

Theorem (Nielsen-Schreier)

Every subgroup of a free group is a free group

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Consider F_2 and let $H = \langle abab^{-1}, ab^2, bab, ba^3b^{-1} \rangle$.

1. H is free (by Nielsen-Schreier), but what is its rank?
2. How can I check if a word $g \in F_2$ is in H or not?
3. What is the index of H?
4. Is H normal in F_2?

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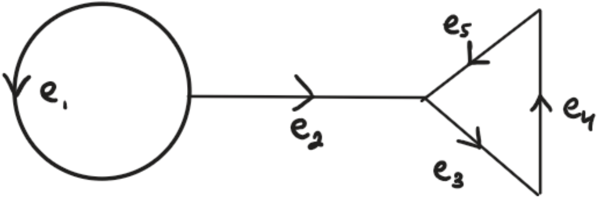
Edge Paths on Graphs

Consider a directed graph Γ. Each edge e in Γ has a **initial vertex** and a **terminating vertex**. An **edge path** in Γ is any string of edges $e_0 \cdots e_k$ such that e_{i-1}'s terminal vertex is e_i's initial vertex.



Fundamental Group of Γ

An edge path is a **loop** if e_0's initial vertex is e_k's terminal vertex; if this vertex is v, we say it is **based at** v. A loop is **tight** if for all i, $e_{i-1} \neq e_i^{-1}$.

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We define $\pi_1(\Gamma, v)$ to be the set of all tight loops based at v.

Definition

$\pi_1(\Gamma, v)$ is a group, called the **Fundamental Group of Γ at** v, where the multiplication is concatenation then tightening.

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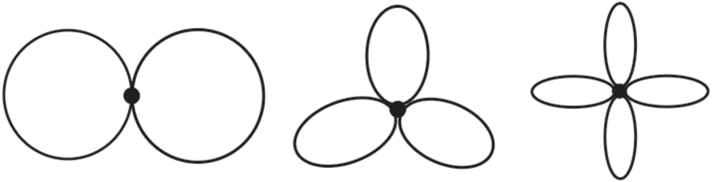
Theorem

If Γ is a directed graph with finitely many edges, then for all vertices of Γ, $\pi_1(\Gamma, v) \cong F_n$, where n is $1 + \#$ of edges - $\#$ of vertices.

Example: R_n

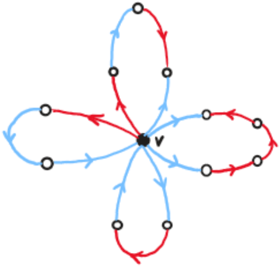
The n-rose R_n is a graph with one vertex and n loops. By the previous theorem, it is isomorphic to F_n.

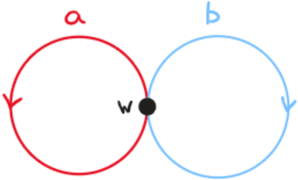


F_2, H as Fundamental Groups

We'll let $F_2 \cong \pi_1(R_2, w)$, with edges labeled a, b.

H is the loops in R_2 made by tightening concatenations of the loops $abab^{-1}, ab^2, bab, ba^3b^{-1}$. Let Γ_H be the graph with 4 loops given by these generators connected at one vertex. Then $H \cong \pi_1(\Gamma_H, v)$.





Mapping Γ_H to R_2

We get a canonical graph map between Γ_H and R_2, inducing a homomorphism between their fundamental groups

$$\rho : \pi_1(\Gamma_H, v) \to \pi_1(R_2, w)$$

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ρ is surjective onto H. If it is injective, then we get an isomorphism, hence $H \cong F_4$.

Question

Is ρ injective? If not, how can we make it injective?

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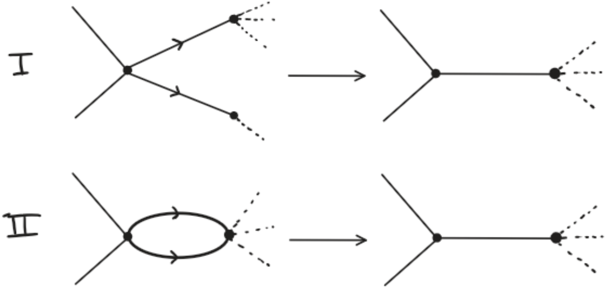
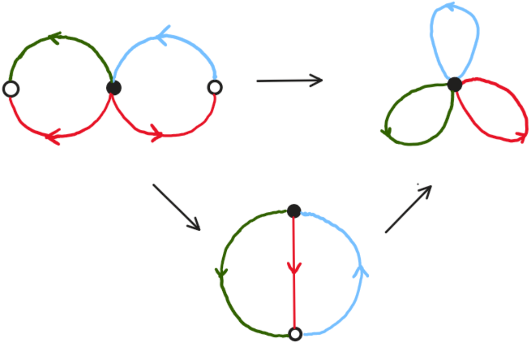
What is a Fold?

Let e_1, e_2 be distinct edges in Γ with the same initial vertex v. The **folded graph** $\Gamma_{e_1=e_2}$ is made by removing e_1, e_2, replacing them with an edge e with the same initial vertex. There are two types of folds:



Factoring Through Folds

If we have a graph map $\Gamma \to \Delta$, and we can make a fold in Γ, we can first fold, then map the folded graph to Δ:



If $\Gamma \to \Delta$ cannot be factored then we call the map an **immersion**.

Theorem

If $\Gamma \to \Delta$ *is an immersion, then the induced homomorphism* $\rho : \pi_1(\Gamma, v) \to \pi_1(\Delta, w)$ *is injective.*

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Proof.

It suffices to show that ρ maps tight paths to tight paths; If so, then for tights loops α, β at w:

$$\rho(\alpha) = \rho(\beta) \implies \rho(\alpha\beta^{-1}) = 1 \implies \alpha\beta^{-1} = 1 \implies \alpha = \beta$$

Let $\alpha = e_0 \cdots e_k$ be a tight path in Γ and consider $e_i e_{i+1}$. e_i^{-1}, e_{i+1} have the same initial vertex, so they're mapped to different edges by ρ, hence

$$\rho(e_{i+1}) \neq \rho(e_i^{-1}) = \rho(e_i)^{-1}$$

so $\rho(\alpha)$ remains tight, as desired. $\square$

The Big Picture

Theorem (Stallings' Theorem)

If $\Gamma \to \Delta$ is a graph map between finite graphs, we get a factorization

$$\Gamma = \Gamma_0 \to \Gamma_1 \to \cdots \to \Gamma_k \to \Delta$$

where $\Gamma_{i-1} \to \Gamma_i$ is a fold and $\Gamma_k \to \Delta$ is an immersion.

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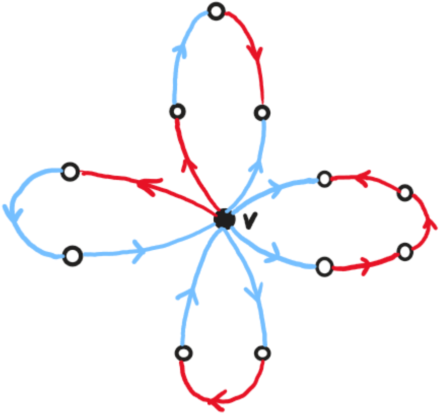
Application 1: Determining Rank

We have a map $\Gamma_H \to R_2$. We'll fold Γ_H until we can't anymore:



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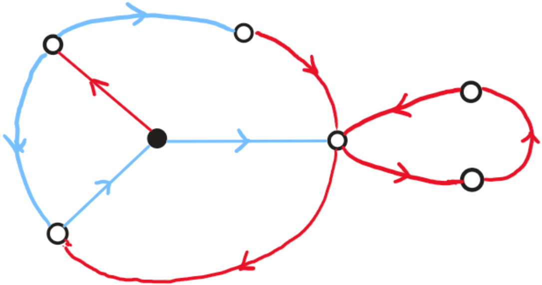

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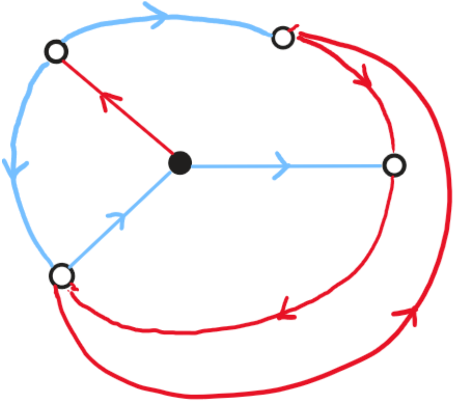

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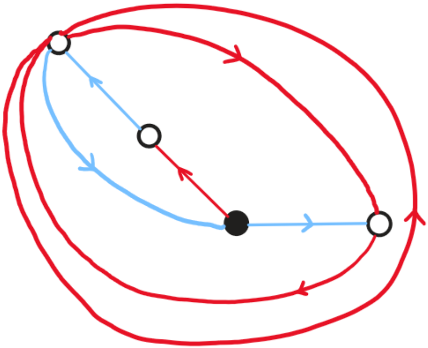

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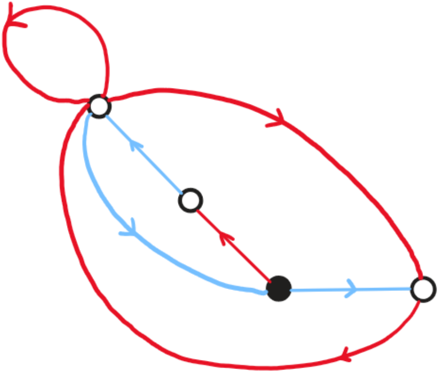

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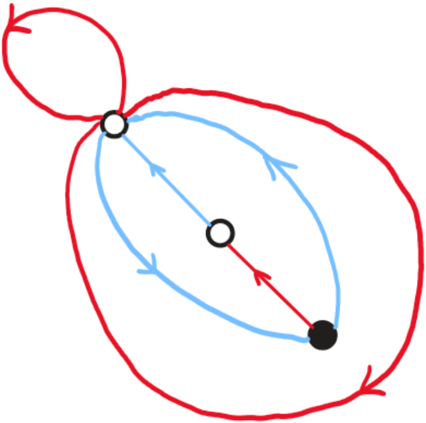

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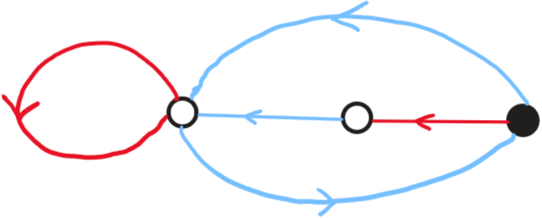

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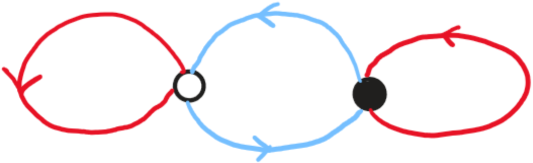

We call this folded graph Δ_H.

We now have an immersion $\Delta_H \to R_2$. As $\rho : \pi_1(\Gamma_H, v) \to \pi_1(R_2, w)$ surjects onto H, the homomorphism

$$\hat{\rho} : \pi_1(\Delta_H, v) \to \pi_1(R_2, w)$$

is surjective onto H, too. It is an immersion so it is injective. Thus,

$$H \cong \pi_1(\Delta_H, v) \cong F_3$$

Application 2: Membership

Consider $g = ab^{-1}a^{-1}b^{-1}a \in F_2$. Is $g \in H$?

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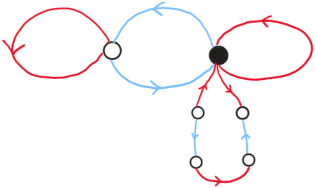
Let $\Delta_{H,g}$ be Δ_H with the g path attached to v as a loop:



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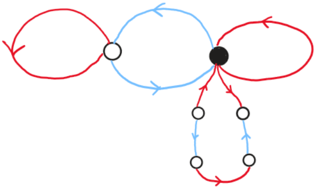

We can fold $\Delta_{H,g}$ until we get an immersion. If the folds end at Δ_H, then

$$\langle H, g\rangle \cong \pi_1(\Delta_{H,g}, v) \cong \pi_1(\Delta_H, v) \cong H$$

so $g \in H$. Otherwise $g \notin H$. To determine membership, it suffices to check if g is a loop in Δ_H.

Application 3: Determining Index

We consider two lemmas:

Lemma

If $\Delta \to R_n$ is an immersion and $H \cong \pi_1(\Delta, v)$, then if g is a tight path from v to v', the set of all such tight paths is Hg.

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We say that an immersion is a **covering** if each vertex has $2n$ adjacent edges.

Lemma

If $\Delta \to R_n$ is a covering, then all tight loops in R_n are tight paths in Δ.

Theorem

Let Δ be a finite graph and $\Delta \to R_n$ a covering. Then $\pi_1(\Delta, v)$ has finite index given by the number of vertices.

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Proof.

Assume Δ has k vertices, and let $v = v_1$. Then for each $v_i, i = 1, \ldots, n$, the set of tight paths from v_1 to v_i is $\pi_1(\Delta, v)g_i$, where g_i is a tight path from v_1 to v_i.

As we have a covering, all tight loops in R_n are tight paths in Δ that start at v_1. Such a path must end at some vertex, say v_i, hence this loop is in $\pi_1(\Delta, v)g_i$. So

$$\pi_1(\Delta, v)g_1 \cup \cdots \cup \pi_1(\Delta, v)g_k = \pi_1(R_n, w) \cong F_n$$

□

Because $\Delta_H \to R_2$ is a covering, and Δ_H has 2 vertices, we get that H has index 2.

What if $\Delta \to R_n$ is not a covering? We can make it one by inductively adding the necessary edges, producing a new graph $\tilde{\Delta}$.

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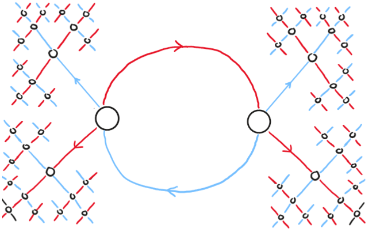

Because $\tilde{\Delta} \to R_n$ is a covering, we proceed as usual but now the index of $\pi_1(\tilde{\Delta}, w)$ is ∞.

Theorem

If Δ is a finite graph and $\Delta \to R_n$ an immersion, then $\pi_1(\Delta, v)$ has finite index in $\pi_1(R_n, w)$ iff the map is a covering; its index is the number of vertices in Δ.

Application 4: Normality

Let $\Delta \to R_n$ be a covering and $H \cong \pi_1(\Delta, v)$. As Hg is the set of tight paths from v to v' (where g is a tight path from v to v'), $g^{-1}H$ is the set of all paths from v' to v. Thus, $g^{-1}Hg$ is the set of tight loops at v'.

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If H is normal, then $g^{-1}Hg = H$, meaning every tight loop at v' can be traced as a tight loop at v; Δ looks the same when viewed either from v' or v. We say Δ is **vertex transitive**.

Theorem

If $\Delta \to R_n$ is a covering and Δ is vertex transitive, then $\pi_1(\Delta, v) \trianglelefteq \pi_1(R_n, w)$.

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Δ_H is vertex transitive, so

$$H \trianglelefteq F_n \iff F_3 \trianglelefteq F_2$$